

**A mathematical perspective on  
the phenomenology of  
non-perturbative Quantum Field Theory**

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Monograph

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## **Abstract**

This monograph aims to build some new mathematical structures originated from Dyson–Schwinger equations for the description of non-perturbative aspects of gauge field theories whenever bare or running coupling constants are strong enough.

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# PREFACE

*The strength of Mathematics is its ability to create models  
which are absolutely vital for producing physical parameters.  
A mathematician is like a surrealist painter who can design  
the purest portraits from known and unknown universes.*

Recent discoveries in Science and Technology from the smallest to the largest scales have approved clearly the importance of advanced research activities in Basic Science which can bring a new package of fundamental knowledge for the analysis of complicated systems in natural phenomena. To obtain a comprehensive description of those complexities requires to build any possible interrelations among different fields in Mathematics (as the purest mental production of human beings). The resulting connections can lead us to achieve some new theoretical methodologies which are essential tools for scientists to build advanced practical models in dealing with complexities of the nature. The designed models together with some computational algorithms will lead scientists to solution procedures.

This research work has a multidisciplinary foundation in the context of Mathematics, High Energy Theoretical Physics and Theoretical Computer Science. It plans to discover some new knowledge about the most complicated or unknown parts of Quantum Field Theories whenever the coupling constants are strong enough in terms of building some new advanced mathematical structures. The outstanding consequence of this research work is to provide a new mathematical interpretation of the phenomenology of Quantum Field Theory with strong coupling constants under discrete, analytic and logical settings. If we study simultaneously these different but related settings, then our mathematical outputs will be useful for the better understanding of the behavior of quantum physical systems in non-perturbative situations. We study the phenomenology of non-perturbative aspects of gauge field theories in terms of some new combinatorial, geometric and categorical tools. We apply graph limits to formulate a new analytic generalization for solutions of Dyson–Schwinger equations which is useful for a complete theory of renormalization for these non-perturbative equations. We also show the use of Tutte polynomials for the combinatorial representation of solutions of Dyson–Schwinger equations. Then we propose a new

concept of complexity for the study of Dyson–Schwinger equations under different running coupling constants. The analytic behavior of these equations are also discussed via some new noncommutative geometric tools. We finally apply graphon models of Dyson–Schwinger equations to build a new theory of ordered algebraic sub-structures for the analysis of entanglement in strongly coupled gauge field theories and a new topos model for the logical description of topological regions of Feynman diagrams which encode non-perturbative parameters. All these observations enable us to describe non-perturbative phenomenology and intrinsic foundations of strongly coupled gauge field theories under discrete, analytic and logical platforms.

The Lagrangian approach to Quantum Field Theory, which is on the basis of the Feynman path integral formalism, has made extraordinary theoretical and experimental progress for the study of elementary particles and their interactions at the highest level of energies and the smallest scales under a perturbative setting. This approach encodes physical information of a quantum system with infinite degrees of freedom in terms of Green’s functions as infinite formal expansions of ill-defined iterated integrals and powers of coupling constants.

Quantum Electrodynamics (QED) concerns interactions among matter (electron, positron) and light (photons). Quantum Flavourdynamics (QFD) concerns weak interactions inside the nucleus of an atom which change the flavour or type of quarks to describe  $\beta^-$  decay and  $\beta^+$  decay under  $W, Z$  bosons. Quantum Chromodynamics (QCD) concerns strong interactions of quarks and gluons inside the nucleus of an atom to build composite hadrons such as protons and neutrons. Standard Model, as the most successful experienced model, has provided a practical platform to collect quantum field theories corresponding to electromagnetic, weak and strong interactions into a united Quantum Field Theory model. The modified versions of the Standard Model in the context of Noncommutative Geometry have also provided a new updated (theoretical) model which is (minimally) coupled to gravity as the weakest fundamental force in the nature. The constructions of gauge field theories in Theoretical and Experimental High Energy Physics, as updated Quantum Field Theory models, are on the basis of the modified Standard Model of elementary particles. These gauge theories can analyze electroweak and strong interactions of elementary particles in the scale of distances down to the order of  $10^{-16}$  centimeters while neutrino masses have also been accounted. In addition, under a more theoretical setting, String Theory as the other class of Quantum Field Theory models, which does not have ultraviolet divergencies, has been introduced and developed to deal with gravity in terms of Quantum Field Theories with matrix fields and higher generalizations of matrix models. The classical one-loop Feynman diagram should be replaced with its stringy counterpart which is a torus. Other more general Feynman diagrams should be replaced with Riemann surfaces and world sheets. This mathematical theory is capable of describing

### Quantum Gravity in Space-Time.

The first fundamental challenge in perturbative setting is the appearance of so complicated nested (sub-)divergencies which live in each term of Green's functions. These ill-defined terms, known as Feynman integrals, can be theoretically reduced to some finite values as the result of the renormalization machinery and many loop techniques. However some extra parameters (i.e. counterterms) should be added to the original Lagrangian of the physical theory during the extraction of finite values. The discovery of a co-multiplication structure hidden inside of the (Bogoliubov)–Zimmermann's forest formula has led us to understand the Bogoliubov–Parasiuk–Hepp–Zimmermann perturbative renormalization in the language of the Connes–Kreimer Hopf algebra of Feynman diagrams and the Riemann–Hilbert problem. Thanks to this setting, a geometric interpretation of Dimensional Regularization on the basis of flat equi-singular connections had been formulated by Connes and Marcolli. This study was lifted immediately onto a universal categorical setting where it is possible to associate a category of Lie group representations to each renormalizable Quantum Field Theory. Thanks to this platform, nowadays there exists a diverse spectrum of advanced mathematical techniques and tools to deal with ill-defined iterated Feynman integrals in physical theories to generate finite values from infinities.

The second fundamental challenge in perturbative setting is dealing with complicated infinities originated from Green's functions which encode quantum motions in physical theories with strong couplings. The lack of a rigorous mathematical methodology for the study of aspects beyond perturbation boundary has made so many difficulties to understand completely Quantum Field Theory. In physical theories with strong (running or bare) couplings, it is already impossible to study the full behavior of quantum systems under perturbation series and in this situation, we need to concern non-perturbative methods such as numerical methods, Borel summation method, theory of instantons and lattice model. In addition, the self-similar nature of Green's functions makes an alternative way for us to study non-perturbative aspects in the context of fixed point equations of Green's functions. The resulting equations, which are known as Dyson–Schwinger equations, contain an infinite collection of coupled integral equations depended on the coupling constants. In QCD with higher energies and short distances, the strength of couplings is small enough where we can expect the asymptotic freedom behavior of the physical system. In this situation, we can consider Dyson–Schwinger equations via some perturbative tools such as many loop computational techniques. However in QCD with relatively lower energies and long distances, the strength of couplings is more than or equal to 1 where the physical system these equations behave non-perturbatively. In this situation, we need to consider non-linear Dyson–Schwinger equations via non-perturbative methods of computations. Work on the phenomenol-

ogy of strong running couplings in QCD has been considered under a physical perspective to provide some computational methods in dealing with non-perturbative parameters. Thanks to the applications of the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams to Quantum Field Theory, we already have a combinatorial reformulation for Dyson–Schwinger equations in the language of Hochschild cohomology theory. The unique solution of each equation DSE determines a free commutative connected graded Hopf subalgebra of the renormalization Hopf algebra. This mathematical approach to Dyson–Schwinger equations has already provided some new combinatorial and geometric tools for the computation of some non-perturbative parameters. The foundations of a differential Galois theory and a Tannakian formalism for the study of non-perturbative aspects of Quantum Field Theories have been designed and developed by the author on the basis of the Connes–Marcolli universal category of flat equi-singular vector bundles. The author applied these platforms to clarify a new motivic method for the study of (systems of ) Dyson–Schwinger equations in terms of sub-categories of mixed Tate motives. In addition, the author has shown a new method for identifying the amount of non-computability of non-perturbative parameters in the context of the renormalization Hopf algebra of the Halting problem in the theory of computation.

This research work proposes some new applications of mathematical tools originated from Combinatorics, Functional Analysis, Noncommutative Geometry, Category Theory and Logic to deal with infinite graphs generated by solutions of Dyson–Schwinger equations. These mathematical platforms can provide some new techniques for the computation of non-perturbative parameters. In addition, they suggest a new methodology for the description of the intrinsic foundations of strongly coupled gauge field theories (such as quantum entanglement and quantum logic) under a non-perturbative setting. These investigations will help us to understand the indeterministic nature of non-perturbative Quantum Field Theory models.

Generally speaking, the achievements of this research work can improve our knowledge about the phenomenology of non-perturbative Quantum Field Theory in terms of studying an individual Dyson–Schwinger equations or studying these equations with respect to each other.

In the first level, we focus on the mathematical foundations of Dyson–Schwinger equations to bring some new computational tools in dealing with non-perturbative parameters generated by large Feynman diagrams. At this level, we consider each Dyson–Schwinger equation as an individual object in the vector space  $\mathcal{S}^{\Phi, g}$  generated by all Dyson–Schwinger equations derived from Green’s functions of a given Quantum Field Theory  $\Phi$  under different scales  $\lambda g$  of the bare coupling constant  $g$  where  $0 < \lambda \leq 1$ . We equip this infinite dimensional vector space with a topological structure defined via the graphon representations of Feynman diagrams. Under a combinatorial setting, we discuss the structure of a new model for large Feynman diagrams

in the language of combinatorial polynomials and random graphs. Furthermore, we discuss the complexity of non-perturbative parameters generated by Dyson–Schwinger equations in the context of theory of computation. In this direction we try to show the importance of a new multi-scale non-perturbative Renormalization Group for the description of the Kolmogorov complexity in dealing with Dyson–Schwinger equations. Under a geometric setting, we explain the dynamics of non-perturbative aspects in a physical theory with respect to the mathematical structures originated from Dyson–Schwinger equations. We build a Noncommutative Geometry model for each Dyson–Schwinger equation which leads us to interpret quantum motions in the context of theory of spectral triples and noncommutative differential forms. Under a Functional Analysis setting, we discuss the evolution of fixed point equations of Green’s functions by defining a new generalized version of the Fourier transformation on the Banach algebra  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  with respect to a new Haar measure  $\mu_{\text{Haar}}$  integration theory on the space of solutions of Dyson–Schwinger equations.

In the second level, we focus on the mathematical foundations of non-perturbative Quantum Field Theory where we must deal with all possible Dyson–Schwinger equations under different running couplings. We explain the construction of a new Hopf algebra structure  $\mathcal{S}_{\text{graphon}}^{\Phi}$  on the topological vector space of graphons which contribute to representations of Feynman diagrams and their finite or infinite formal expansions. The resulting topological Hopf algebra is capable to encode large Feynman diagrams generated by solutions of Dyson–Schwinger equations in different rescalings of the bare coupling constant  $g$ . Therefore we can embed the space  $\mathcal{S}^{\Phi,g}$  into  $\mathcal{S}_{\text{graphon}}^{\Phi}$ . This Hopf algebra leads us to formulate a new topological Hopf algebraic renormalization theory for large Feynman diagrams via a topological generalization of the Connes–Kreimer BPHZ renormalization program. We then define a new multi-scale Renormalization Group on the collection  $\mathcal{S}^{\Phi,g}$  to control the simultaneously rescaling of the momentum parameter and the bare coupling constant in Dyson–Schwinger equations of a given gauge field theory  $\Phi$ . This new Renormalization Group is useful to study the computational complexity of a strongly coupled equation  $\text{DSE}(g)$  in terms of the computational complexities of a sequence of Dyson–Schwinger equations under weaker running couplings. Furthermore, we explain the foundations of a differential calculus theory on  $\mathcal{S}^{\Phi,g}$  (as a separable Banach space with respect to the cut-distance topology) where thanks to the Feynman graphon models of Dyson–Schwinger equations and the theory of Gâteaux derivative, we formulate a new theory of differentiation and integration on  $\mathcal{S}^{\Phi,g}$ . Under an ordered algebraic setting, we organize Feynman graphon models of solutions of Dyson–Schwinger equations into some lattices of substructures to provide a new interpretation for the quantum entanglement in interacting gauge field theories. We explain mathematically the information flow among elementary (virtual) particles in QFT models via a new class

of topological Hopf algebras generated by Dyson–Schwinger equations. We then lift this lattice models onto the level of Tannakian sub-categories of the Connes–Marcolli category of flat equi-singular vector bundles to show the importance of this universal category for the study of the geometry of quantum entanglement. Under a categorical setting, we organize topological Hopf subalgebras derived from solutions of Dyson–Schwinger equations into a new small category to build a new topos model which can encode the logical evaluations of non-perturbative situations. The structure of this topos model is depended on the strength of running couplings to show the importance of the phenomenology of strong running couplings in the determination of truth values of propositions about Dyson–Schwinger equations.

Thanks to these two levels of observations, we expect to provide a new insight into the complicated problems of non-perturbative situations where the strength of the coupling constants do really change the mathematics and the logics of quantum theory models.

# Chapter 1

## Introduction

- *Physical backgrounds*
- *Mathematical backgrounds*
- *Recent progress and objectives*

## 1.1 *Physical backgrounds*

Modern Theoretical and Experimental High Energy Physics have been established on the basis of Quantum Field Theory models under the Lagrangian setting which is (minimally) coupled to gravity via the incorporation of massive neutrinos. The foundations of Quantum Field Theory were initiated in terms of the interpretation of the quantized version of Electrodynamics in the language of the Feynman path integral formalism under a perturbation setting. The appearance of gauge field theories which include Quantum Electrodynamics (QED), Electroweak theory, Quantum Chromodynamics (QCD) and Quantum Gravity have developed rigorously our theoretical knowledge about the fundamental properties of elementary particles before we could reach to appropriate empirical information. Thanks to these backgrounds, mathematicians and theoretical physicists have already made outstanding achievements for the description of interactions of elementary particles under different settings in the context of advanced mathematical models. For example, mathematical tools in Noncommutative Geometry, Algebraic Geometry, Combinatorics and Category Theory have been applied to formulate Standard Model and other extended theories which include supersymmetry, gravitational interactions or extended objects such as strings and brane theory. We can also address tensor models as higher dimensional generalizations of matrix models which aim to achieve a theory of random geometries in dimensions higher than two. This class of theories helps us for the construction of discrete approaches to quantizing gravity. [27, 28, 32, 40, 46, 55, 103, 104, 112, 117, 128, 138, 140, 147, 156, 157, 186, 194, 200]

The Lagrangian formalism enables us to understand Quantum Field Theory by working on Green's functions as infinite formal expansions of Feynman integrals or their corresponding diagrams where the amount of some fundamental parameters such as the strength of the bare coupling constants or the domain of momenta make the resulting series divergent or asymptotic free. In perturbative physical theories we expect to have some convergent series.

For example, in  $\phi^4$  model the partition function is given by

$$Z[B] := \int \mathcal{D}\phi e^{-L(\phi)+\int B\phi} \quad (1.1)$$

such that

$$L(\phi) = \int d^4x \left( \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r_0\phi^2 + \frac{1}{4!}u_0\phi^4 \right), \quad (1.2)$$

and  $B$  is an external field. If we set

$$Z_0 := \int \mathcal{D}\phi e^{-L_0(\phi)+\int B\phi}, \quad L_0(\phi) := \int d^4x \left( \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r_0\phi^2 \right), \quad (1.3)$$

then we can develop  $Z$  as a series in  $u_0$  around  $Z_0$  to achieve

$$Z = \int \mathcal{D}\phi \left(1 - \frac{u_0}{4!} \int_{x_1} \phi^4(x_1) + \frac{1}{2} \left(\frac{u_0}{4!}\right)^2 \int_{x_1, x_2} \phi^4(x_1) \phi^4(x_2) + \dots\right) e^{-L_0(\phi) + \int B\phi}. \quad (1.4)$$

This expansion can be represented in the language of Feynman diagrams which leads us to a combinatorial formulation for Green's functions. For  $n$  elementary particles we have

$$G_n(x_1, \dots, x_n) = \langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{\int e^{-S[\phi]} \phi(x_1) \dots \phi(x_n) \mathcal{D}\phi}{\int e^{-S[\phi]} \mathcal{D}\phi} \quad (1.5)$$

such that  $S[\phi] = S_0[\phi] + gS_{\text{int}}[\phi]$  where  $g$  is the (bare) coupling constant. The fluctuations generated by the  $\phi^4$  term around the Gaussian integral  $Z_0$  are large where they determine iterated integrals with the general form

$$\int^{\Lambda} d^d q_1 \dots d^d q_L \prod_i (\text{propagator}(q_i)) \quad (1.6)$$

such that the ultra-violet regulator  $\Lambda$  provides a cut-off at the upper bound type of integral. The dependency of these integrals to the parameter  $\Lambda$  makes a rigorous challenge for the computation of universal quantities. Theory of perturbative renormalization provides the machinery to reparametrize the perturbative expansion in such a way that the sensitive dependence on  $\Lambda$  has been eliminated. In this situation, the renormalization group enables us to partially resum the perturbative expansions to achieve some universal computational results. [32, 40, 147, 197]

In general speaking, it is possible to investigate the situations beyond perturbation theory in terms of some expressions such as

$$P(g) = X_0 + X_1 g + X_2 g^2 + \dots + X_n g^n + \dots \quad (1.7)$$

such that  $g$  is the coupling constant and each term  $X_i$  represents the class of Feynman diagrams which contribute to the  $i$ -order of perturbative expansion. It is obvious that physical theories with very small  $g$  could be encoded by only some beginning finite number of terms from the above expansion while physical theories with strong coupling  $g$  produce infinite number of terms. These non-perturbative aspects have been concerned in Theoretical Physics via Dyson–Schwinger equations as a quantized version of the Euler–Lagrange equations of motion originated from the principal of the least action. These equations, which can be determined by fixed point equations of Green's functions, have been studied under analytic and numerical methods in Theoretical and Mathematical Physics such that we can address standard techniques such as Borel summation, theory of instantons, lattice models, etc in dealing with these equations to generate some estimations for non-perturbative parameters. In physical theories with strong

couplings such as Hadron Physics, we should deal with hadrons such as protons and neutrons as the composite particles build up from quarks and gluons (as elementary particles under strong interaction). In general, QCD, as a nonabelian gauge theory with the symmetry group  $SU(3)$ , has provided a modern understanding of the complicated nature of hadrons and nuclei where we study the strong interactions of quarks and gluons under confinement and chiral symmetry breaking. The appearance of nuclear weak force enables us to describe any change in quark's flavour via  $W$  bosons. In fact, the weak force is not only responsible for interactions between particles, but it also allows heavy particles to decay by emitting or absorbing some of the force carriers. We can describe QCD as a matrix-valued modification of electromagnetic theory in terms of replacing photons by gluons and electrons by quarks while quantum fluctuations of the fields could determine the force law. The quantization of Chromodynamics involves the regularization and renormalization of ultraviolet divergencies which generate a mass-scale where mass-dimensionless quantities become dependent on a mass-scale. The current quark-masses are the only evident scales in QCD. The main experimental confirmations of QCD have been investigated at high energies and high momentum transfers (or short distances) where the QCD coupling is small and correspondingly the forces are weak. This situation, which is known as asymptotic freedom property, enables us to detect the composite structure of hadrons by scattering high energy electrons. The most difficult challenge in this model can be observed when perturbation theory fails to describe the short range static potential obtained from quenched lattice simulations where the difference between the non-perturbatively determined potential and perturbation theory at short distance has been parameterized by a linear term. In addition, there are also so many difficulties for the study of the asymptotic behavior of QCD-perturbative series beyond the two-loop level where the original effort is to find a way to subtract perturbative contributions to a given physical process in order to isolate non-perturbative terms. In the domain of relatively low energies and momentum transfers such as  $Q^2 \sim 1 - 5 \text{ GeV}^2$  while the proton's mass is approximately  $1 \text{ GeV}$ , the QCD coupling constants are larger where many loops perturbative calculations should be applied. Because of the nontrivial vacuum structure of QCD, in the domain of lower energies and momentum transfers (or large distances) such as  $Q^2 \leq 1 \text{ GeV}^2$ , the QCD coupling constants are stronger than one. Under this condition, the analytic calculations do not useful but there are some methods such as the chiral effective theory, lattice calculations, large  $N$  limits and Dyson–Schwinger equations to provide some algorithmic computations. This situation, which is actually the failure to directly observe coloured excitations in a detector, is the origin of the concept of confinement as a fundamental challenge that we do not see free quarks or gluons in nature but rather we only see colourless. The analytic description of confinement is one difficult task for the understanding of continuum

QCD. The phenomenology of confinement can be studied in the context of Dyson–Schwinger equations. [9, 40, 48, 136, 137, 138, 157]

The situations beyond perturbation boundary deal with divergencies originated from Green’s functions under strong running coupling constants where we might need to consider Feynman diagrams with many numbers or infinite number of loops. What does this class of diagrams means and how can we deal with infinite formal expansions of this class of diagrams? Applying advanced mathematical structures is helpful for the better understanding of this class of divergencies. It is obvious that having strong mathematical modelings is the initial step to make fundamental progress in dealing with the complicated behavior of elementary particles inside of the nuclei. We can address the mathematical foundations of the (modified) Standard Model as a good theoretical methodology which led scientists to obtain strong experimental investigations about elementary particles. String Theory is another powerful mathematical platform which aims to provide a unified theory for the study of elementary particles with respect to all fundamental forces. The basic philosophy of this research work is to build and develop the mathematical foundations of QFT models with strong couplings where as the consequence, we expect to provide some new mathematical tools for the better understanding of physical parameters beyond perturbation theory. Our study suggests a new interpretation from the phenomenology of strong couplings in the context of combinatorial, geometric and categorical settings.

## 1.2 *Mathematical backgrounds*

The contributions of mathematical tools to Quantum Field Theories have been extraordinary developed when the (Bogoliubov–)Zimmermann forest formula was reinterpreted by Kreimer in the context of (co)algebraic combinatorial tools. This reinterpretation had been concerned by Connes and Kreimer to build a new modern formulation for the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) perturbative renormalization in Quantum Field Theory on the basis of the theory of Hopf algebras and the Riemann–Hilbert problem. The Connes–Kreimer approach has become the main foundation in many research efforts for the study of complicated Feynman integrals, Green’s functions and Renormalization Group methods where it has led the Theoretical Physics’s community to achieve some new mathematical tools for the description of physical parameters in (renormalizable) gauge field theories under algebraic, combinatorial and geometric settings. It is now possible to encapsulate the machinery of perturbative renormalization in terms of a connected graded free commutative non-cocommutative (finite type) Hopf algebraic structure  $H_{FG}(\Phi)$  on Feynman diagrams of a physical theory  $\Phi$  which has a Lie algebraic nature determined by the insertion opera-

tor. The compatibility of the fundamental identities such as Slavnov–Taylor and Ward identities in QCD and QED with the renormalization coproduct have been shown in the language of Hopf ideals. The phenomenology of counterterms has been concerned underlying a geometric treatment to provide some alternative methods for the computation of these physical values in the language of (singular) differential equations. In this setting, a new class of equi-singular flat connections can govern the values of counterterms with respect to the  $\beta$ -function. This setting has been lifted onto a universal Tannakian formalism where a renormalizable Quantum Field Theory is studied via a category of geometric objects which can be recovered by the neutral Tannakian category of finite dimensional representations of the affine group scheme  $\mathbb{G}_\Phi := \text{Hom}(H_{\text{FG}}(\Phi), -)$ . [2, 3, 15, 16, 27, 28, 30, 33, 31, 35, 42, 43, 46, 77, 78, 105, 113, 114, 115, 116, 143, 178, 179, 193, 200]

One of the most fundamental result in this direction is actually the discovery of a very deep interrelationship between Feynman integrals and the theory of motives where a motivic renormalization machinery has been formulated to deal with divergencies in the language of Picard–Fuchs equations and other powerful tools. The theory of motives in Algebraic Geometry aims to concern the existence of a universal cohomology theory for algebraic varieties defined over a base field  $k$  while taking values into an abelian tensor category. The construction of a category of motives (mixed motives) related to general varieties is a difficult task. The noncommutative version of motivic objects provides the motivic cohomology applied in the construction of a universal cohomology theory. The structure of mixed Tate motives as elements of the subring  $\mathbb{Z}[\mathbb{L}]$  of the Grothendieck group  $K_0(\text{Var}_k)$  of  $k$ -varieties has been considered where  $\mathbb{L} := [\mathbb{A}^1]$  is the Grothendieck class of the affine line. The application of motives enables us to develop a unified setting underlying different cohomology theories such as Betti, de Rham,  $l$ -adic, crystalline and étale. For this purpose, the construction of an abelian tensor category, which provides a linearization of the category of algebraic varieties, has been studied to provide some fundamental requirements of standard conjectures of Grothendieck. The importance of motives in Quantum Field Theory have been discussed in different settings. The Bloch–Esnault–Kreimer approach which informs interesting applications of Hodge type structures in the calculation processes of Feynman integrals underlying graph polynomials [21, 22, 115, 194]. The Aluffi–Marcolli approach, which builds the motivic version of Feynman rules characters, applied Kirchhoff–Symanzik polynomials to formulate a new version of algebro-geometric (dimensionally regularized) Feynman rules characters. These abstract characters send classes in the Grothendieck ring of conical immersed affine varieties to the classes in the Grothendieck ring of varieties spanned by the classes  $[X_\Gamma]$ . This formalism, which is on the basis of the deletion–contraction operators and the Tutte–Grothendieck polynomial, enables us to relate Feynman diagrams with periods of algebraic varieties. This framework provides a motivic

treatment in the study of perturbative renormalization process at the level of the universal motivic Feynman rules character [7, 8]. The Connes–Marcolli approach deals with the geometric interpretation of counterterms on the basis of flat equi-singular connections such that these geometric objects have been organized in a categorical structure  $\mathcal{E}^\Phi$  which is recovered by the neutral Tannakian category of finite dimensional representations of the affine group scheme  $\mathbb{G}_\Phi$ . This category has been embedded (as a sub-category) into the universal category  $\mathcal{E}^{\text{CM}}$  of flat equi-singular vector bundles with the corresponding universal affine group scheme  $\mathbb{U}$ . Objects of this universal category, which has Tannakian nature, address mixed Tate motives which contribute to divergencies of renormalizable physical theories. In addition,  $\mathcal{E}^{\text{CM}}$  determines the universal singular frame as the unique loop with values in  $\mathbb{U}$  which provides the universal counterterm. The Lie algebra of the universal affine group scheme leads us to formulate a particular shuffle type Hopf algebra. [46, 134, 135]

A single Feynman diagram reports only a small piece of information about a finite number of possible interactions among (virtual) elementary particles where its on-shell part (i.e. incoming and outgoing particles) obeys the mass-energy equation and the conservation of momenta while its off-shell part (i.e. virtual particles) obeys no special rules or measurements. The iterated integral corresponding to a given Feynman diagram might contain nested or overlapping sub-divergencies. Infinite formal expansions of Feynman diagrams (as polynomials with respect to coupling constants), which are organized in Green’s functions, have an essential role to encapsulate various possible chains of interactions which could or might happen among elementary particles in a physical theory. These formal expansions can be studied in terms of the self-similar nature of Green’s functions which allows us to formulate fixed point equations known as Dyson–Schwinger equations. Numerical methods, large  $N$  limit, Borel resummation, lattice models and theory of instantons can provide some approximations for strongly coupled Dyson–Schwinger equations ([136, 137, 138]) while the real time dynamics of these non-perturbative equations require other advanced mathematical tools. In addition, these non-perturbative equations in physical theories with strong couplings have also been considered in the context of the Connes–Kreimer renormalization Hopf algebra to provide some new advanced mathematical tools for the computation of their solutions. This Hopf algebraic formalism is one of the original motivations of this research program. In this direction we need to formulate the Hochschild cohomology of (commutative) bialgebras as the dual notion of the Hochschild cohomology of (commutative) algebras. For a given commutative Hopf algebra  $H$ , consider linear maps  $T : H \rightarrow H^{\otimes n}$  as  $n$ -cochains where the coboundary operator is defined by

$$\mathbf{b}T := (\text{id} \otimes T)\Delta + \sum_{i=1}^n (-1)^i \Delta_i T + (-1)^{n+1} T \otimes \mathbb{I} \quad (1.8)$$

such that  $\Delta_i$  is the coproduct  $\Delta$  of  $H$  applied to the  $i$ -th factor in  $H^{\otimes n}$ . The Kreimer's renormalization coproduct on Feynman diagrams can be reformulated recursively in terms of the linear operator  $B^+$  on Feynman diagrams, known as the grafting operator, as the following way

$$\Delta_{\text{FG}} B^+ = (\text{id} \otimes B^+) \Delta_{\text{FG}} + B^+ \otimes \mathbb{I}. \quad (1.9)$$

The operator  $B^+$ , as a linear homogeneous endomorphism of degree one, replaces a vertex in a given Feynman diagram with a whole graph in terms of the type of the targeting vertex and the types of external edges of the second graph. Thanks to (1.8) and (1.9), the grafting operator could determine some generators of the first rank Hochschild cohomology group of the Connes–Kreimer Hopf algebra of Feynman diagrams. For each primitive Feynman diagram  $\gamma$ ,  $B_\gamma^+$  is a Hochschild one cocycle. [42, 65, 69, 108]

The first importance of the grafting operator is its role for the rooted tree representation of any complicated Feynman diagram in terms of its primary components (or its primitive Feynman subdiagrams). It is possible to embed the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams of a given gauge field theory into a decorated version of the Connes–Kreimer Hopf algebra of non-planar rooted trees. The grafting operator acts on each forest  $t_1 \dots t_n$  to deliver a rooted tree by adding a new vertex  $r$  as the root and  $n$  new edges which connect the roots of  $t_n$ s to  $r$ . It is shown that the pair  $(H_{\text{CK}}, B^+)$  has a universal property with respect to the Hochschild cohomology theory. This pair is actually the initial object for a particular category of objects  $(H, T)$  consisting of a commutative Hopf algebra  $H$  and a Hochschild one cocycle  $T$  on  $H$ . The Hopf algebra homomorphisms which commute with the cocycles are morphisms of this category. Decorations on trees enable us to update  $H_{\text{CK}}$  with respect to each physical theory. For a given physical theory  $\Phi$ , each 1PI Feynman diagram without any sub-divergencies is a primitive element in the renormalization Hopf algebra  $H_{\text{FG}}(\Phi)$ . We can encode this class of objects via vertices in rooted trees where edges can determine the positions of those primitive subgraphs inside of a more complicated Feynman diagram. It is shown the existence of an injective Hopf algebraic homomorphism from  $H_{\text{FG}}(\Phi)$  to the Hopf algebra  $H_{\text{CK}}(\Phi)$  of decorated non-planar rooted trees. [16, 27, 28, 67, 68, 88, 105, 107, 109]

The second importance of the grafting operator is its fundamental role in the reconstruction of Dyson–Schwinger equations under a combinatorial setting. For a given family  $\{\gamma_n\}_{n \geq 1}$  of primitive (1PI) Feynman diagrams with the corresponding Hochschild one cocycles  $\{B_{\gamma_n}^+\}_{n \geq 1}$ , a class of combinatorial Dyson–Schwinger equations in  $H_{\text{FG}}(\Phi)[[\lambda g]]$  is defined by

$$X = \mathbb{I} + \sum_{n \geq 1} (\lambda g)^n \omega_n B_{\gamma_n}^+(X^{n+1}) \quad (1.10)$$

such that  $g$  is the bare coupling constant. This class of equations accepts a unique solution  $X = \sum_{n \geq 0} (\lambda g)^n X_n$  as the formal expansion of finite Feynman diagrams where for each  $n > 0$ , we have

$$X_n = \sum_{j=1}^n \omega_j B_{\gamma_j}^+ \left( \sum_{k_1 + \dots + k_{j+1} = n-j, k_i \geq 0} X_{k_1} \dots X_{k_{j+1}} \right). \quad (1.11)$$

$X_0$  is the empty graph, each  $X_n$  is an object in the Hopf algebra  $H_{\text{FG}}(\Phi)$ , and the unique solution  $X$  lives in a completion of  $H_{\text{FG}}(\Phi)[[\lambda g]]$  with respect to the  $n$ -adic topology. The Cartier–Quillen–Milnor–Moore theorem shows us that the unique solution of each Dyson–Schwinger equation DSE can determine the generators of a Faa di Bruno type Hopf subalgebra  $H_{\text{DSE}}$  of the Connes–Kreimer renormalization Hopf algebra. It is a free commutative unital counital (non-)cocommutative connected graded finite type Hopf subalgebra where its coproduct on generators  $X_n$  does not depend on the parameters  $\omega_j$ . The Mellin transform allows us to deform these combinatorial type of equations to their original integral versions. It is possible to lift this formalism onto the level of systems of Dyson–Schwinger equations where we deal with a system  $(S)$  of a finite collection of equations with the general form

$$(S) : \quad \forall i \in I, \quad x_i = \sum_{j \in J_i} B_{(i,j)}^+ (f^{(i,j)}(x_k, k \in I)) \quad (1.12)$$

such that  $I := \{1, \dots, n\}$ ,  $J_i$  is a graded connected set,  $B_{(i,j)}^+$ s are Hochschild one cocycles and  $f^{(i,j)}$ s are formal series in  $\mathbb{K}[[\alpha_1, \dots, \alpha_n]]$ . It is shown that the system  $(S)$  has a unique solution such that under some conditions it can determine the Hopf subalgebra  $H_{(S)}$  originated from the Hopf subalgebras  $H_1, \dots, H_n$  generated by combinatorial Dyson–Schwinger equations in the system. [19, 36, 64, 73, 74, 75, 108, 111]

The main skeleton of a combinatorial Dyson–Schwinger equation is actually a family of Hochschild one cocycles. There exists a surjective map from the first rank Hochschild cohomology group to the space of primitive Feynman diagrams of the renormalization Hopf algebra. It means that each family  $\{\gamma_n\}_{n \geq 1}$  of primitive Feynman diagrams determines the corresponding family  $\{B_{\gamma_n}^+\}_{n \geq 1}$  of Hochschild one cocycles. It is important to note that each 1PI Feynman diagram, which is free of sub-divergencies, is a primitive element but they are not the only primitives in the renormalization Hopf algebra. In other words, there are primitive Feynman diagrams in higher degrees which can determine Hochschild one cocycles. [19, 42, 109]

### 1.3 Recent progress and objectives

The combinatorial reformulation of Dyson–Schwinger equations in terms of the renormalization Hopf algebra and Hochschild cohomology theory has ap-

plied recently for the construction of some new mathematical treatments in dealing with the computations of non-perturbative parameters. These efforts have already developed our knowledge about the mathematical foundations of the phenomenology of non-perturbative parameters under different settings. [34, 64, 75, 76, 108, 110, 111, 118, 121, 142, 167, 171, 172, 173, 187, 190, 195]

The Milnor–Moore theorem ([144]) allows us to determine the infinite dimensional complex graded pro-unipotent Lie group  $\mathbb{G}_\Phi(\mathbb{C})$  which is actually the complex points of the affine group scheme  $\mathbb{G}_\Phi = \text{Hom}(H_{\text{FG}}(\Phi), -)$ . This Lie group, which is the projective limit of linear algebraic groups  $G_n$  embedded as Zariski closed subsets in some  $\text{GL}_{m_n}$ s, is rich enough to encode (dimensionally regularized) Feynman rules characters with respect to the scale and angle dependence of amplitudes [31]. In addition, we can also determine the infinite dimensional complex graded pro-unipotent Lie group

$$\mathbb{G}_{\text{DSE}}(\mathbb{C}) := \text{Hom}(H_{\text{DSE}}, \mathbb{C}) \quad (1.13)$$

for each given Dyson–Schwinger equation DSE. There exists a natural injective Hopf algebra homomorphism  $\rho : H_{\text{DSE}} \rightarrow H_{\text{FG}}(\Phi)$ . If we apply  $\text{Spec}$  as a contravariant functor, then we can obtain a surjective morphism  $\tilde{\rho} : \text{Spec}(H_{\text{FG}}(\Phi)) \rightarrow \text{Spec}(H_{\text{DSE}})$  between spaces of prime ideals in the commutative algebras  $H_{\text{FG}}(\Phi)$  and  $H_{\text{DSE}}$  equipped with the Zariski topology. This map can be lifted onto the surjective group homomorphism

$$\bar{\rho} : \mathbb{G}_\Phi(\mathbb{C}) \longrightarrow \mathbb{G}_{\text{DSE}}(\mathbb{C}). \quad (1.14)$$

The Hopf subalgebra  $H_{\text{DSE}}$  is actually a Hopf ideal in  $H_{\text{FG}}(\Phi)$  and we can consider its corresponding quotient Hopf algebra  $H_{\text{FG}}(\Phi)/H_{\text{DSE}}$ . In the dual setting, the associated sub-group scheme of this quotient Hopf algebra can determine a Lie sub-group of  $\mathbb{G}_\Phi(\mathbb{C})$ . However the existence of Lie (sub)groups  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$  corresponding to Dyson–Schwinger equations have been applied to bring a new geometric setting for the study of non-perturbative parameters in the context of differential systems together with singularities. The construction of a category of flat equi-singular  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$ -connections with respect to each equation DSE has been addressed to encode the BPHZ renormalization of the unique solution  $X_{\text{DSE}}$  in the context of differential Galois theory. In other words, the Connes–Marcolli geometric interpretation of counterterms and the Connes–Marcolli universal Tannakian machinery in dealing with renormalizable physical theories have been developed for the study of Dyson–Schwinger equations. We can study the geometry of any given equation DSE in terms of finite dimensional representations of the Lie group  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$  which is organized in the neutral Tannakian category  $\text{Rep}_{\mathbb{G}_{\text{DSE}}^*}$ . This class of categories has been embedded as subcategories into the Connes–Marcolli universal category  $\mathcal{E}^{\text{CM}}$ . Thanks to these backgrounds, we already have the construction of a differential

Galois theory for the computation of fundamental non-perturbative parameters such as global  $\beta$ -functions and non-perturbative counterterms in the language of Picard–Fuchs equations. In addition, it is now possible to identify a class of mixed Tate motives with respect to each Dyson–Schwinger equation. [160, 162, 163, 164]

On the first hand, under the Hopf algebraic renormalization platform, Dyson–Schwinger equations are the fundamental tools to determine Hopf subalgebras in Quantum Field Theory. On the second hand, ”substructure” is a mutual fundamental concept in Galois Theory and Theory of Computation where intermediate algorithms have been studied recently in terms of algebraic methods in Galois Theory. On the third hand, the Manin’s program for the interpretation of the Halting problem in the context of the Connes–Kreimer BPHZ renormalization machinery has initiated the foundations of a brilliant interrelationship between the amount of computability and the computation of counterterms encoded via the renormalization Hopf algebra of the Halting problem. The combination of these achievements have already led us to describe intermediate algorithms in terms of (systems of) Dyson–Schwinger equations where we can understand the amount of non-computability of non-perturbative renormalized values and their corresponding non-perturbative counterterms in the language of the Halting problem, Operad Theory and the theory of Hall sets. Results in this direction show a new multidisciplinary bridge between Theoretical Computer Science and non-perturbative Quantum Field Theory. [63, 131, 132, 133, 167, 198, 199]

Applications of the theory of graphons to Quantum Field Theory is another output of the renormalization Hopf algebraic platform. The foundations of a new combinatorial interpretation of Feynman diagrams and their infinite formal expansions have been studied recently where we embed (large) Feynman diagrams into a compact topological space obtained by an enrichment of the Connes–Kreimer renormalization Hopf algebra with respect to the cut-distance topology. The immediate consequence of this new topological-combinatorial setting is the formulation of a generalization of the BPHZ renormalization for solutions of Dyson–Schwinger equations. In this direction, thanks to some tools in Measure Theory, a new differential calculus machinery on Feynman diagrams was built which has led us to study the evolution of Dyson–Schwinger equations in terms of their partial sums. [168, 169]

Applications of non-commutative differential graded algebras to Quantum Field Theory were also considered to study the geometry of quantum motions where some new models of gauge theories have been obtained. In this direction, the structure of a non-perturbative version of the Connes–Kreimer renormalization group has been described in the language of integrable systems. In addition, recently, we have formulated a new class of infinite dimensional spectral triples which encode the geometry of Dyson–Schwinger equations. These observations enable us to clarify the essential

role of Noncommutative Geometry in the future of non-perturbative Quantum Field Theory models. [52, 53, 54, 56, 161, 166, 171]

Combinatorial Dyson–Schwinger equations, Connes–Kreimer–Marcolli Hopf algebraic renormalization platform and the theory of graphons for sparse graphs are the main motivational topics for us in this research to study non-perturbative Quantum Field Theory. Our main attempt in this work is to develop mathematical structures originated from Dyson–Schwinger equations to discover some new information about the complicated behavior of Quantum Field Theories under strong coupling constants. This work aims also to bring some new mathematical tools to deal with the computation of non-perturbative parameters.

Under a combinatorial setting, we plan to apply the theory of graphon representations of Feynman diagrams together with other combinatorial and topological tools to provide a new Hopf algebraic machinery for the renormalization of solutions of Dyson–Schwinger equations under different running coupling constants. In a more general platform, we consider the cut-distance topological vector space  $\mathcal{S}^{\Phi, g}$  generated by all fixed point equations of Green’s functions of a given strongly coupled gauge field theory  $\Phi$  under different running coupling constants  $\lambda g$  with respect to the bare coupling constant  $g$  such that  $0 < \lambda \leq 1$ . It is possible to topologically complete the space  $\mathcal{S}^{\Phi, g}$  to achieve a new Banach space where we need to work on equivalence classes of Dyson–Schwinger equations up to the weakly isomorphic as an equivalence relation. Equations  $DSE_1, DSE_2$  are called weakly isomorphic (or weakly equivalent) if their corresponding solutions  $X_{DSE_1}, X_{DSE_2}$  have weakly isomorphic Feynman graphon models. In other words,

$$DSE_1 \approx DSE_2 \iff W_{X_{DSE_1}} \sim W_{X_{DSE_2}}. \quad (1.15)$$

Up to this equivalence relation, we can define the distance

$$d(X_{DSE_1}, X_{DSE_2}) := d_{\text{cut}}([W_{X_{DSE_1}}], [W_{X_{DSE_2}}]) \quad (1.16)$$

between Dyson–Schwinger equations such that  $d_{\text{cut}}$  is the metric structure on unlabeled Feynman graphon classes. Each unlabeled Feynman graphon class contains all weakly isomorphic Feynman graphons and all relabeled Feynman graphons generated by invertible measure preserving transformations of the base measure space  $(\Omega, \mu_\Omega)$  of our graphon model. We then address some new applications of combinatorial polynomials such as Kirchhoff–Symanzik and Tutte polynomials to formulate a new parametric representation theory for solutions of Dyson–Schwinger equations. This study is useful for the construction of algebro-geometric Feynman rules on the topological Hopf algebra  $\mathcal{S}_{\text{graphon}}^\Phi$  of Feynman graphons. We also concern the concept of complexity for the description of non-perturbative parameters where we explain the construction of a new multi-scale Renormalization Group machinery on

$\mathcal{S}^{\Phi,g}$  which is useful on two levels. Firstly, it can provide a mathematical machinery for the approximation of Dyson–Schwinger equations in strong couplings via equations in weaker couplings. Secondly, it helps us to initiate a new version of the Kolmogorov complexity in dealing with Dyson–Schwinger equations.

Under a geometric setting, we show some new applications of Noncommutative Geometry, Measure Theory and Functional Analysis to study the geometry of non-perturbative Quantum Field Theory. We build the Gâteaux differential calculus on the space  $\mathcal{S}^{\Phi,g}$  which provides a global differential geometry for the geometric description of strongly coupled gauge field theories. In addition, we explain the concept of evolution on  $\mathcal{S}^{\Phi,g}$  with respect to a generalized version of the Fourier transformation. We also build the theory of spectral triples for solutions of Dyson–Schwinger equations which provides a local differential geometry in dealing with the quantum motions in strongly coupled gauge field theories.

Under a foundational setting, we apply Feynman graphon models to consider some intrinsic foundations of Quantum Field Theory models under strong running coupling constants such as quantum entanglement and logical concepts. In this direction, we offer a new mathematical methodology for the description of quantum entanglement in interacting quantum physical theories in the context of the theory of lattices and intermediate substructures in the Theory of Computation. This mathematical formalism enables us to explain information flow in physical theories with strong couplings on the basis of lattices of topological Hopf algebras and Lie subgroups. We lift this mathematical modeling onto a categorical setting to show that the universal category  $\mathcal{E}^{\text{CM}}$  is suitable to encode the quantum entanglement process. At this level, we expect to show a new application of motives in dealing with information flow. In addition, we put forward the construction of a new topos of presheaves on a particular base category which encodes the logical evaluations of propositions about the cut-distance topological regions of Feynman diagrams. We try to show that how this topos model concerns the strength of running coupling constants for the logical study of gauge field theories. For this purpose, objects of the base category of the non-perturbative topos encode all Dyson–Schwinger equations of a given gauge field theory under different running coupling constants. The non-perturbative topos can evaluate the logical propositions about cut-distance topological regions which are generated by solutions of Dyson–Schwinger equations.

## Chapter 2

# A theory of renormalization for Dyson–Schwinger equations

- *Quantum Field Theory*
- *Hochschild cohomology of the renormalization Hopf algebra*
- *Renormalization Hopf algebra of Feynman graphons and filtration of large Feynman diagrams*
  - *Graphons*
  - *Feynman graphons*
- *A generalization of the BPHZ renormalization machinery for large Feynman diagrams via Feynman graphons*

Having no comprehensive physical description of infinite formal expansions of Feynman diagrams which contribute to polynomials with respect to strong running coupling constants such as (1.7), it is indeed difficult to analyze a renormalization program for these infinite expansions. The major discourse in this situation is to find some meaningful mathematical insights associated to fixed point equations of Green's functions. Then these mathematical reasonings serve to compel technical terms and models for the explanation of the renormalization process for Dyson–Schwinger equations.

The original task in this chapter is to explain a renormalization program on the space  $\mathcal{S}^{\Phi, g}$  which consists of Dyson–Schwinger equations with respect to running couplings  $\lambda g$  in a given physical theory  $\Phi$  with the bare coupling constant  $g$ . For this purpose, we apply the theory of graphons for sparse graphs to build a new theory of graph function representations for Feynman diagrams and large Feynman diagrams which contribute to solutions of Dyson–Schwinger equations. These new analytic generalizations of Feynman diagrams can be organized into a new graded topological Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$  which can lead us to formulate a new modification of the Connes–Kreimer BPHZ renormalization program for the computation of non-perturbative counterterms and non-perturbative renormalized values. The complex Lie group associated to the Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$  is the key tools for the formulation of a non-perturbative Renormalization Group which encodes the required  $\beta$ -functions to govern the dynamics of strongly coupled Dyson–Schwinger equations.

## 2.1 Quantum Field Theory

Suppose we have a quantized field theory with the general Lagrangian  $\mathcal{L} = \mathcal{L}(\phi, \partial_{\mu}\phi)$  which can contain the free part and the interaction part. The typical free Lagrangian density is given by

$$\mathcal{L}_{\text{free}} := \frac{1}{2}((\partial_{\mu}\phi)(\partial^{\mu}\phi) - m^2\phi^2) \quad (2.1)$$

which leads us to the free Klein–Gordon equation of motion

$$(\partial_{\mu}\partial^{\mu} + m^2)\phi = 0. \quad (2.2)$$

The interaction part  $\mathcal{L}_{\text{int}}$  encodes interactions of elementary particles in the physical theory. The transition amplitudes from initial states to all finite states can be formulated under the S-Matrix setting. It is possible to calculate these matrix elements in terms of a class of correlation functions with the general form

$$G_n(x_1, \dots, x_n) := \langle 0|T\phi(x_1)\dots\phi(x_n)|0 \rangle \quad (2.3)$$

such that  $|0\rangle$  is the vacuum ground state. These equations, known as Green's functions, allow us to formulate perturbative Quantum Field Theory in terms of formal expansions with the general form

$$G_n(x_1, \dots, x_n) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int d^4y_1 \dots d^4y_j \langle 0 | T \phi_{\text{in}}(x_1) \dots \phi_{\text{in}}(x_n) \mathcal{L}_{\text{int}}(y_1) \dots \mathcal{L}_{\text{int}}(y_j) | 0 \rangle \quad (2.4)$$

such that  $\phi_{\text{in}}$  is the initial state of  $\phi$  in the infinite past. If we apply the Wick's Theorem and normal ordering, then the vacuum expectation value can be described as the integrals of propagators that typically depend on differences of space-time vectors. The rigorous challenge is the existence of divergencies in these integrals with respect to the domains of integrations where applying regularization machineries (such as Dimensional Regularization) help us to study these integrals in the context of Laurent series with finite pole parts. [32, 40]

Feynman diagrams in Quantum Field Theory are useful combinatorial tools to encapsulate the summation over probability amplitudes corresponding to all possible exchanges of virtual particles compatible with a process at a given (loop) order. These decorated diagrams, as a set of edges and a set of vertices, can simplify the description of interactions of elementary particles in terms of the time parameter in a quantum system. Their decorations are determined by fundamental parameters of the physical theory. Momentum and position are actually Fourier transforms of each other where we can translate diagrams with respect to momentum space to their corresponding iterated integrals via Feynman rules. Each closed loop associates to an integrate over the corresponding momentum. The whole diagram obeys the conservation of momenta which tells us that the amount of momenta of input particles in an interaction procedure is the same as the amount of momenta of output particles.

For example, thanks to the Schwinger parameter  $t$ , we can consider

$$\frac{1}{p^2 + m^2} = \int_0^{\infty} dt \exp(-t(p^2 + m^2)) \quad (2.5)$$

as the propagator for each edge and

$$\int d^4x \exp(i \sum_j p_j x) \quad (2.6)$$

as the propagator for each vertex. In this setting, each edge has a factor

$$G(x, y; t) = \int \frac{d^4p}{(2\pi)^4} \exp(ip \cdot (x - y) - t(p^2 + m^2)). \quad (2.7)$$

We can combinatorially reformulate Green's functions in terms of formal expansions of Feynman diagrams. It clarifies the self-similar property of these fundamental functions in Quantum Field Theory. We have

$$\begin{aligned} \mathcal{G} &:= 1 + \int I_\gamma + \int \int I_\gamma I_\gamma + \int \int \int I_\gamma I_\gamma I_\gamma + \dots \\ &= 1 + \int I_\gamma (1 + \int I_\gamma + \int \int I_\gamma I_\gamma + \dots) \end{aligned} \quad (2.8)$$

such that  $I_\gamma$  is the Feynman integral corresponding to the primitive (1PI) Feynman diagram  $\gamma$ . This can be encapsulated via equations such as

$$\mathcal{G} = 1 + \int I_\gamma \mathcal{G} \quad (2.9)$$

such that its fixed point equations determine Dyson–Schwinger equations. This formulation of Quantum Field Theory is the result of the path integral approach to Lagrangian formalism where we study the behavior of an elementary particle in a system with infinite degrees of freedom in terms of the sum over all possible situations (such as trajectories, interactions) which could be selected by the particle. In terms of some conditions dictated by physical theory, each possible situation has a particular weight which should be considered in computational processes of Feynman integrals. [32, 40, 147, 193]

For example, in QED we deal with interactions of electron and positron (as matter) with photons (as electromagnetic waves with different quantized sizes of energies). There exist six different fundamental interactions namely, the emission of photon from electron or positron, absorbing a photon via electron or positron, the creation of a photon via annihilation of the pair (electron, positron), creation of a pair (electron, positron) via the annihilation of a photon. All Feynman diagrams and Dyson–Schwinger equations in QED, which might contain complicated off-shell interactions of virtual particles, are built on the basis of those six fundamental interactions. There exists a class of elementary graphs which play the role of building blocks to make all possible Feynman diagrams in a physical theory. These graphs, which are called one particle irreducible Feynman diagrams, remain connected after removing one internal edge. By induction we can define  $n$ -particle irreducible Feynman diagrams which remain connected after removing  $n$  internal edges. It is easy to see that each  $n$ -particle irreducible graph is a  $(n - 1)$ -particle irreducible graph.

The coupling constants in Quantum Field Theory show the strength of the interactions among elementary particles. The regularization of UV divergent integrals and the renormalization procedure generate a scale dependence. The UV cut-off dependence of the couplings can be eliminated by allowing the couplings and masses (which appear in the Lagrangian) to

acquire a scale dependence. Then we normalize them to a measured value at a given scale. Generally speaking, there are two classes of couplings namely, the bare coupling constant as the original strength of a fundamental force and the running coupling constants or effective couplings as the result of renormalization procedures. Quantum Chromodynamics (QCD) is known as the most successful fundamental gauge theory of strong interactions. It studies the hadronic interactions involving quarks and gluons at both long and short distances. Its symmetry group is  $SU(3)$  where it includes  $N_f$  family of quarks  $\psi_f^i$  and gluons  $A_\mu^i$ . Some experimental evidences inform us that at a critical temperature around  $T_c \approx 170$  MeV, the QCD matter undergoes a deconfining phase transition into quark-gluon plasma. Perturbative QCD is a method based on expanding different physical quantities with respect to the gauge coupling constant  $g$  which is applied in the region  $T \gg T_c$  where  $g$  is small. The phenomenology of the (bare and running) coupling constants have been discussed in terms of the uncertainties in their values at short distances which leads us to a total theoretical uncertainty in Physics at large hadron collider such as Higgs production via gluon fusion. In this situation we can still have hope to apply asymptotic freedom and perturbative calculations of Renormalization Group equations. However at high perturbative orders it becomes necessary to evaluate large numbers of multi-loop Feynman diagrams in the effective theory. [9, 51, 137]

However the behavior of running coupling constants of the physical systems at long distances such as the scale of the proton mass in order to understand hadronic structure, quark confinement and hadronization processes should be analyzed under non-perturbative platforms. In these cases we can study the phenomenology of strong bare or running couplings in terms of the mathematical foundations of quantum motions of physical systems namely, Dyson–Schwinger equations. We can address recent theoretical progress for the computation of non-perturbative parameters derived from solutions of Dyson–Schwinger equations in the context of Combinatorics, Geometry and Category Theory. [19, 34, 110, 136, 138, 157, 162, 163, 164, 190, 195]

The running of a coupling constant originates from the renormalization procedure while predictions for observables should be determined independent of the choice of renormalization map and regularization scheme. This invariance with respect to the choice of renormalization program is encoded via a symmetry group. The running coupling is an expansion parameter in the perturbative series describing an observable and there exists the Landau pole as the point where the perturbative expression of the running coupling diverges. It means that the full perturbative expression is actually a non-observable quantity. The observable is independent of the renormalization scheme while the series's coefficients and the running coupling are related to the renormalization scheme. Under asymptotic freedom behavior at short distances, we can get the first coefficient series as an independent parameter while at very large distances dependency can not be neglected. This dis-

cussion tells us that the running couplings are not observables because they are strongly depended on the renormalization scheme at large distances. In short, the running couplings have weak scale dependence at distances smaller than  $10^{-16}$  m such that this controllable weak behavior tends to a strong scale dependence larger than a tenth of a Fermi. This dependency on the scale is restored at larger distances due to the confinement of quarks and gluons. [9, 51, 136]

The Ward–Takahashi identities on Feynman diagrams tell us that the photon propagator is the only propagator in QED which contributes to the running of the coupling constant. The Slavnov–Taylor identities on Feynman diagrams tell us that intermediate gauge-dependent quantities in non-abelian gauge theories provide final gauge-independent results for observables [117, 178]. Thanks to these facts, it is possible to rewrite Dyson–Schwinger equations in terms of some running couplings to achieve some intermediate quantities which are useful to simplify the original complicated non-perturbative type of equations by some approximations. In a general configuration, Dyson–Schwinger equations are polynomials with respect to bare or running coupling constants which means that any change in the amount of running couplings will make direct influence on the behavior of these equations. Therefore these non-perturbative type of equations are good tools for the study of the phenomenology of strong couplings.

Dimensional Regularization was introduced by 't Hooft, Veltman, Bollini and Gambiagi as a method to regularize ultraviolet divergencies in a gauge invariant way to complete the proof of renormalizability. The method works in  $D = 4 - 2\epsilon$  space-time dimensions where divergencies for  $D \rightarrow 4$  appears as poles in  $1/\epsilon$ . This method also regulates infrared singularities where if we remove the auxiliary IR regulator, the IR divergencies appear as poles in  $1/\epsilon$ . For  $\epsilon > 0$ , we can obtain a well-defined result which we can be analytically extended to the whole complex D-plane. The only essential change in the structure of Feynman rules is to replace the couplings in the Lagrangian via the transformation  $g \mapsto g\mu^\epsilon$  such that  $\mu$  is an arbitrary mass scale. Dimensional Regularization together with Minimal Subtraction can provide a practical renormalization program for Feynman integrals with nested subdivergencies. These ill-defined parts can be eliminated step by step under a forest formula setting where the Bogoliubov–Parasiuk–Hepp preparation allows us to generate some finite values. This particular renormalization program was reconsidered by Connes and Kreimer under a modern Hopf algebraic setting to generate counterterms and renormalized values in terms of the Riemann–Hilbert problem and the Birkhoff factorization. In this context, Dimensional Regularization is encapsulated by the space of loops with the domain of a punctured infinitesimal disk around zero and with values in a pro-unipotent complex Lie group associated to the renormalization Hopf algebra of Feynman diagrams. [43, 44, 69, 78, 116, 120]

While working on the applications of the BPHZ procedure to the level of

many-loop graphs is one of the interesting topics in Quantum Field Theory, the main information of a physical theory are encoded in (infinite) formal expansions of Feynman diagrams. Thanks to the Connes–Kreimer–Marcolli theory, the required mathematical tools for an extension of the BPHZ procedure to the level of Dyson–Schwinger equations has already been considered under geometric and algebraic settings. This platform enables us to study any Dyson–Schwinger equation DSE in terms of its corresponding complex Lie group  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$  where the existence of the Hopf–Birkhoff factorization on this Lie group has led us to determine counterterms (which contribute to the unique solution of the equation DSE) in the language of differential systems together with irregular singularities. These differential systems, which are on the basis of equi-singular flat  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$ -connections, determine a new class of systems of Picard–Fuchs equations with regular singularities. [164, 167]

## 2.2 Hochschild cohomology of the renormalization Hopf algebra

The basic elements of the path integral method in Quantum Field Theory are (divergent) iterated Feynman integrals over the momentum space such that the integrands are determined from a definite collection of rules originated from the physical theory. We can encode these integrals in terms of a class of combinatorial decorated finite diagrams which are known as Feynman diagrams where sub-divergencies in the original integral are presented in terms of the existence of nested or overlapping loops in the main diagram. The Kreimer’s coproduct, which highlights the combinatorics of removing sub-divergencies from integrals, enables us to factorize the original complicated Feynman diagram into its basic sub-divergencies (as sub-graphs). This factorization reduces several layers of complications in the computational processes of perturbative renormalization in terms of a certain graded commutative non-cocommutative Hopf algebra denoted by  $H_{\text{FG}}(\Phi)$ . It is a graded Hopf algebra with respect to the first Betti number of Feynman diagrams which means that  $H_{\text{FG}}(\Phi) = \bigoplus_{n \geq 0} \mathcal{H}_n$  such that  $\mathcal{H}_0 = \{\mathbb{I}\}$  and for each  $n$ ,  $\mathcal{H}_n$  is the vector space of divergent 1PI  $n$ -loop Feynman diagrams and products of Feynman diagrams with overall loop number  $n$ . There is also another graduation parameter to build a graded Hopf algebra. We can show that  $H_{\text{FG}}(\Phi)$  is a graded Hopf algebra with respect to the number of internal edges of Feynman graphs such that the components of this grading have finite dimensions as the vector spaces. [15, 46, 105, 113]

The original version of the Kreimer’s coproduct was defined in the language of parenthesized words to characterize nested, independent or overlapped sub-divergencies via sequences of letters and their linear combinations. It encapsulates the Bogoliubov–Zimmermann forest formula based on

the formal expansion

$$\Delta_{\text{FG}}(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma \quad (2.10)$$

for each Feynman diagram  $\Gamma$  such that the sum is over all disjoint unions of 1PI divergent proper Feynman subgraphs. [42, 77, 78, 105]

Generally speaking, for a unital algebra  $(A, m, e)$  and a counital coalgebra  $(C, \Delta, \varepsilon)$  over a field  $\mathbb{K}$  of characteristic zero, let  $\text{Hom}(C, A)$  be the vector space of all  $\mathbb{K}$ -linear maps from  $C$  to  $A$ . Equip this space with a convolution product defined in terms of the following composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f * g} A \otimes A \xrightarrow{m} A \quad (2.11)$$

to achieve an algebra with the unit  $e \circ \varepsilon$ . A bialgebra  $(H, m, e, \Delta, \varepsilon)$  in which the identity map  $\text{id}_H$  is invertible under the convolution product is a Hopf algebra. This particular inverse  $S$  which obeys the following property

$$\text{id}_H * S = S * \text{id}_H = e \circ \varepsilon \quad (2.12)$$

is called the antipode such that it is a unital algebra counital coalgebra antihomomorphism. [36, 78]

There are natural graduation parameters on Feynman diagrams such as number of internal edges or number of independent loops. The graduation parameter and the coproduct (2.10) determine the required antipode for the construction of a free commutative connected graded finite type Hopf algebra on Feynman diagrams of a given physical theory  $\Phi$ . The antipode deforms Feynman rules characters to obtain renormalized values. The Hopf algebra  $H_{\text{FG}}(\Phi)$  has a Lie algebraic origin and in addition, it can be simplified via decorated rooted trees to provide a universal model for perturbative renormalization. The rooted tree version of the renormalization coproduct (2.10) can be defined in terms of the notion of "admissible cuts" on trees. [33, 65, 67, 68, 77, 143]

The factorization of a Feynman diagram into its primitive components can be reversed under some conditions via the insertion operator which enables us to glue subdiagrams. It is important to note that in gauge field theories we should work on a quotient of the renormalization Hopf algebra with respect to Ward identities and Slavnov–Taylor identities to achieve a unique factorization for each Feynman diagram with respect to the insertion operator. The insertion operator provides a Lie algebraic structure on Feynman diagrams such that the graded dual of its universal enveloping algebra will be equivalent to the renormalization Hopf algebra. [78, 109, 143, 179]

Theory of Hochschild cohomology for bialgebras is useful for us to formulate the Hochschild equation on Feynman diagrams which results a recursive formulation for the renormalization coproduct. This class of equations can

provide the key tools for the reformulation of Dyson–Schwinger equations. In this part we review the foundations of the Hochschild equation with respect to the renormalization coproduct and for further details we address [64, 77, 78, 108, 109, 200] as the major sources.

For a given bialgebra  $H$  such as  $H_{\text{FG}}(\Phi)$ , the dual of the coalgebra  $(H, \Delta, \varepsilon)$  is an algebra  $H^*$  such that the unit map  $\mathbb{I}$  of  $H$  transposes to a character  $\mathbb{I}^t$  of  $H^*$ . Therefore we can build Hochschild cohomology groups  $H^n(H, H^*)$  such that  $n$ -cochains are linear maps such as  $T : H \rightarrow H^{\otimes n}$ . We can transpose them to  $n$ -linear maps such as  $\rho_T : (H^*)^n \rightarrow H^*$  where we have

$$\rho_T(\Gamma_1, \dots, \Gamma_n) := T^t(\Gamma_1 \otimes \dots \otimes \Gamma_n). \quad (2.13)$$

In this setting, the Hochschild coboundary operator  $\mathbf{b}$  can be determined by the relation

$$\langle \Gamma_1 \otimes \dots \otimes \Gamma_{n+1}, \mathbf{b}T(\Gamma) \rangle := \langle \mathbf{b}\rho_T(\Gamma_1, \dots, \Gamma_{n+1}), \Gamma \rangle \quad (2.14)$$

for each  $\Gamma \in H$ . Define  $\Delta_j : H^{\otimes n} \rightarrow H^{\otimes(n+1)}$  as the homomorphism which applies the coproduct  $\Delta$  only on the  $j$ th factor. Now we can show that

$$\langle \rho_T(\Gamma_1, \dots, \Gamma_j \Gamma_{j+1}, \dots, \Gamma_{n+1}), \Gamma \rangle = \langle \Gamma_1 \otimes \dots \otimes \Gamma_{n+1}, \Delta_j(T(\Gamma)) \rangle. \quad (2.15)$$

It leads us to rewrite the Hochschild coboundary operator as the following way

$$\mathbf{b}T(\Gamma) := (\text{id} \otimes T)\Delta(\Gamma) + \sum_{j=1}^n (-1)^j \Delta_j(T(\Gamma)) + (-1)^{n+1} T(\Gamma) \otimes \mathbb{I}. \quad (2.16)$$

The resulting cohomology groups  $H^n(H^*, H^*_{\mathbb{I}^t})$  are indeed the Hochschild cohomology theory of the bialgebra  $H$ . It is easy to check that linear forms on  $H$  are 0-cochains and one cocycles are linear maps such as  $l : H \rightarrow H$  which obeys the following relation

$$\Delta(l) = l \otimes \mathbb{I} + (\text{id} \otimes l)\Delta. \quad (2.17)$$

The Hochschild cohomology with values in a  $H$ -bimodule  $A$  (such as the regularization algebra) is defined by working on  $n$ -cochains via the vector space  $C^n := C^n(H, A)$  consisting of  $n$ -linear maps  $\psi : H^n \rightarrow A$  with the  $H$ -bimodule structure

$$(\gamma_1 \cdot \psi \cdot \gamma_2)(\Gamma_1, \dots, \Gamma_n) := \gamma_1 \cdot \psi(\Gamma_1, \dots, \Gamma_n) \cdot \gamma_2. \quad (2.18)$$

The coboundary map  $\mathbf{b} : C^n \rightarrow C^{n+1}$  is given by

$$\begin{aligned} \mathbf{b}(\psi)(\Gamma_1, \dots, \Gamma_{n+1}) &= \Gamma_1 \cdot \psi(\Gamma_2, \dots, \Gamma_{n+1}) \\ &+ \sum_{j=1}^n (-1)^j \psi(\Gamma_1, \dots, \Gamma_j \Gamma_{j+1}, \dots, \Gamma_{n+1}) + (-1)^{n+1} \psi(\Gamma_1, \dots, \Gamma_n) \cdot \Gamma_{n+1}. \end{aligned} \quad (2.19)$$

The resulting cohomology groups are denoted by  $\mathcal{H}^n(H, A)$ .

Let us consider the Hochschild equation for the algebra  $\mathbb{K}[X]$  which is also equipped with a cocommutative coalgebra structure by considering the indeterminate  $X$  as the primitive object where  $\varepsilon(X) = 0$ . For all  $k \geq 2$ , by induction, we can show that

$$\Delta(X^k) = (\Delta X)^k = \sum_{j=0}^k \binom{k}{j} X^{k-j} \otimes X^j. \quad (2.20)$$

For any linear form  $\varrho$  on  $\mathbb{K}[X]$ , we have

$$\mathbf{b}\varrho(X^k) = (\text{id} \otimes \varrho)\Delta(X^k) - \varrho(X^k) \otimes \mathbb{I} = \sum_{j=1}^k \binom{k}{j} \varrho(X^{k-j})X^j. \quad (2.21)$$

$\mathbf{b}\varrho$  is a linear transformation of polynomials which does not increase the degree. It shows that the integration map  $T(X^k) := X^{k+1}/(k+1)$  is not a 1-coboundary but it is an one cocycle.

Thanks to this Hochschild cohomology theory, it is possible to define the renormalization coproduct under a recursive setting.

A graded bialgebra  $H$  over a field  $\mathbb{K}$  is graded as an algebra and as a coalgebra. It is called connected if the degree zero component of the graduation structure consists of scalars (i.e. elements of the field  $\mathbb{K}$ ). We have

$$H = \bigoplus_{n \geq 0} H_n, \quad H_m H_n \subseteq H_{m+n}, \quad \Delta(H_n) \subseteq \bigoplus_{p+q=n} H_p \otimes H_q. \quad (2.22)$$

The coproduct in the connected graded bialgebra of Feynman diagrams can be presented in terms of the Sweedler notation such that for  $\Gamma \in \mathcal{H}_n$ , we have

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum \Gamma'_1 \otimes \Gamma'_2 \quad (2.23)$$

where terms  $\Gamma'_1$  and  $\Gamma'_2$  all have degrees between 1 and  $n-1$ . The counit equations

$$\sum \varepsilon(\Gamma'_1)\Gamma'_2 = \sum \Gamma'_1\varepsilon(\Gamma'_2) = \Gamma, \quad \sum S(\Gamma'_1)\Gamma'_2 = \sum \Gamma'_1 S(\Gamma'_2) = \varepsilon(\Gamma)\mathbb{I} \quad (2.24)$$

tell us that  $\Delta(\Gamma)$  must contain terms  $\Gamma \otimes \mathbb{I} \in \mathcal{H}_n \otimes \mathcal{H}_0$  and  $\mathbb{I} \otimes \Gamma \in \mathcal{H}_0 \otimes \mathcal{H}_n$  and the remaining terms which have intermediate bidegrees. They address the equation

$$\Gamma = (\varepsilon \otimes \text{id})(\Delta(\Gamma)) = \varepsilon(\Gamma)\mathbb{I} + \Gamma + \sum \varepsilon(\Gamma'_1)\Gamma'_2 \quad (2.25)$$

which leads us to the relation  $\varepsilon(\Gamma) = 0$  for every non-trivial Feynman diagram.

In general, the augmentation ideal in a graded bialgebra is given by

$$\text{Ker}\varepsilon = \bigoplus_{n=1}^{\infty} H_n. \quad (2.26)$$

For  $P := \text{id} - \mathbb{I}\varepsilon$  as the projector onto the augmentation ideal, define  $\text{Aug}^m := (P \otimes \dots \otimes P) \Delta^{m-1}$  and then set

$$H^{(m)} := \text{Aug}^{m+1} / \text{Aug}^m \quad (2.27)$$

for all  $m \geq 1$ . It determines the bigraded structure on our bialgebra given by

$$H = \bigoplus_{n \geq 0} H_n = \bigoplus_{m \geq 0} H^{(m)} \quad (2.28)$$

such that for all  $k \geq 1$

$$H_k \subset \bigoplus_{j=1}^k H^{(j)}, \quad H_0 \simeq H^{(0)} \simeq \mathbb{K}. \quad (2.29)$$

In addition, number of internal edges or number of independent loops can be applied as the graduation parameters on the renormalization bialgebra of Feynman diagrams to formulate its corresponding antipode inductively. We have the recursive formulation

$$S(\Gamma) = -\Gamma - \sum S(\Gamma'_1)\Gamma'_2 \quad (2.30)$$

for the antipode of each Feynman diagram  $\Gamma$  with respect to its renormalization coproduct. This formula can be applied to show the existence of a convolution inverse for the identity map.

### 2.3 Renormalization Hopf algebra of Feynman graphons and filtration of large Feynman diagrams

The combinatorial interpretation of Feynman diagrams in terms of decorated rooted trees and the recursive nature of the renormalization coproduct are the key tools to build a new Hopf algebra  $H_{\text{CK}}$  of non-planar rooted trees. This combinatorial Hopf algebra together with the grafting operator  $B^+$  has the universal property with respect to the Hochschild cohomology theory of commutative Hopf algebras. Each Feynman diagram with nested loops can be represented by a labeled rooted tree where the root is the symbol for the original graph and other vertices are symbols of nested loops. Edges among vertices determine the positions of nested loops with respect to each other. In addition, it is possible to represent Feynman diagrams with overlapping divergencies with rooted trees where we should deal with linear

combinations of decorated rooted trees [77, 78, 106, 109]. In this section we present a new interrelationship between the theory of infinite graphs and the fundamental structure of Green's functions in Quantum Field Theory on the basis of rooted trees. We provide a new analytic generalization of Feynman diagrams organized in to a new topological enriched version of the Connes–Kreimer renormalization Hopf algebra. For this purpose, we use the representation of Feynman diagrams via rooted trees to define a new graph function representation formalism in dealing with expansions of Feynman diagrams. We show the importance of analytic graphs for the study of solutions of strongly coupled Dyson–Schwinger equations in terms of sequences of partial sums and the notion of convergence with respect to the topology of graphons. We then work on the combination of the renormalization Hopf algebra and graphon models of Feynman diagrams to build a new Hopf algebra of graphons which contribute to the analytic representation of Feynman diagrams and solutions of Dyson–Schwinger equations. This Hopf algebra of graphons is useful for us to formulate a new topological generalization of the Connes–Kreimer BPHZ renormalization to generate non-perturbative counterterms and related renormalized values. We also use this graphon model approach to Dyson–Schwinger equations to explain the structure of a filtration treatment on the space of solutions of these non-perturbative equations.

On the one hand, Dyson–Schwinger equations are reformulated in terms of the renormalization Hopf algebra and the grafting operator which act on Feynman diagrams. The unique solution of each equation DSE with the general form (1.10) is an infinite formal expansion of Feynman diagrams together with powers of running couplings. In physical theories with weak coupling constants, we can expect to study the renormalization of these expansions by applying many-loop computation techniques under the perturbative setting. In physical theories with strong couplings, these expansions contain infinite number of terms such that their renormalization generate infinite number of counterterms. In this situation we need to deal with non-renormalizable gauge field theories. However we can encapsulate all these formal expansions in terms of fixed point equations of Green's functions of the given physical theory [110]. On the other hand, graph limits, as a modern branch in infinite combinatorics, study limits of finite combinatorial objects such as weighted or directed graphs, multi or hyper graphs, bipartite graphs and posets. Theory of graphons and random graphs is one of the recent progress in infinite combinatorics where we deal with (symmetric) measurable functions such as  $W$  defined on the probability space  $\Omega$ . Actually, a graph limit, as the convergent limit of an infinite sequence of graphs, can be represented by a graphon which does not have necessarily the unique representation. The key tool which allows us to concern convergence and equivalence of graphons is the concept of cut-metric [125]. It is reasonable to think about any relation between solutions of combinatorial Dyson–

Schwinger equations namely, formal expansions such as  $\sum_{n \geq 0} (\lambda g)^n X_n$ , such that  $g$  is the bare coupling constant while  $\lambda g$  is any running coupling or any rescaled version of the bare coupling, and graphons generated by infinite sequences of sparse graphs. This idea has already been discussed in [168, 169] where we have shown that the unique solution of each equation DSE can be obtained as the cut-distance convergent limit of a sequence of random graphs generated in terms of the combinatorial information of partial sums  $Y_m = \sum_{i=1}^m (\lambda g)^i X_i$ . In this work we plan to explain the structure of new Hopf algebra of graphons which can encode solutions of all Dyson–Schwinger equations in a given (strongly coupled) gauge field theory. This Hopf algebra can be equipped with the cut-distance topology which enables us to describe infinite expansions such as  $X_{\text{DSE}(\lambda g)} = \sum_{n \geq 0} (\lambda g)^n X_n$  as the unique solution of the equation  $\text{DSE}(\lambda g) \in \mathcal{S}^\Phi(\lambda g)$  in terms of objects in the boundary region of the compact topological space  $\mathcal{S}_{\text{graphon}}^\Phi$ . In other words, this new platform enables us to study  $\mathcal{S}^{\Phi, g} = \bigcup_\lambda \mathcal{S}^\Phi(\lambda g)$ , as the collection of all Dyson–Schwinger equations under different running coupling constants in the physical theory  $\Phi$ , in the context of a subspace of the topological Hopf algebra  $\mathcal{S}_{\text{graphon}}^\Phi$ .

### 2.3.1 Graphons

Generally speaking, the theory of graph limits aims to assign a limit to a sequence of finite graphs such as  $\{G_n\}_{n \geq 0}$  when number of vertices of graphs in the sequence tends to infinity. There are some different approaches to define the concept of convergence at this level but the one approach which is based on random graphs and cut-distance topology is very useful. We can say that a sequence  $\{G_n\}_{n \geq 0}$  of finite graphs is convergent when  $|G_n|$  tends to infinity, if for each fixed value  $k$ , the distribution of the random graphs  $G_n[k]$  is convergent when  $n$  tends to infinity. In this setting,  $G_n[k]$  is a labeled subgraph of  $G_n$  with vertices  $1, \dots, k$  obtained by selecting  $k$  distinct vertices  $v_1, \dots, v_k \in G_n$  under a uniformly random process. Graph limits can be generated by infinite sequences of dense or sparse graphs. We can generalize graph limits to graph functions (i.e. graphons) in an arbitrary probability space where we can study them in terms of equivalence classes of convergent sequences of finite graphs with respect to (invertible) measure preserving transformations of the ground probability space. [94, 125]

For a given probability space  $\Omega$ , graphons are bounded measurable symmetric functions such as  $W : \Omega \times \Omega \rightarrow [0, 1]$ . The symmetric condition can be removed when we work on another class of graphons known as bigraphons. Bigraphons are useful for us if we want to rebuild Feynman diagrams. We associate graph functions to Feynman diagrams via decorated rooted trees where we need to fix an orientation on trees to identify the positions of nested loops with respect to each other in Feynman diagrams. The choice of the orientation allows us to work on the upper triangular or lower trian-

gular versions of the adjacency matrix of rooted trees which are not clearly symmetric.

The graphon representation of any given graph limit is not unique. In other words, we can produce different graphon representations in terms of changing the ground probability space or changing presentation parameters. However for a fixed probability space, we can associate different labeled graphons to a given graph limit by applying invertible measure preserving transformations of the ground probability space.

For a given finite graph, the pixel pictures derived from the adjacency matrix are the simplest examples of labeled graphons. For a given sequence of finite simple graphs, the corresponding sequence of pixel pictures derived from the adjacency matrices can provide labeled graphon representations for the graph limit of the initial sequence.

A map  $\rho : \Omega_1 \rightarrow \Omega_2$  between probability spaces  $(\Omega_1, \mathcal{F}_1, \mu_1)$  and  $(\Omega_2, \mathcal{F}_2, \mu_2)$  is called measure preserving if it is measurable and  $\mu_1(\rho^{-1}(A)) = \mu_2(A)$  for each measurable set  $A \in \mathcal{F}_2$ .  $\rho$  is called measure preserving bijection if it is a bijection map and  $\rho, \rho^{-1}$  are measure preserving. It is easy to check that for a given measure preserving map  $\rho$ , the map  $\rho \otimes \rho : \Omega_1^2 \rightarrow \Omega_2^2$  defined by  $\rho \otimes \rho(x, y) := (\rho(x), \rho(y))$  is also a measure preserving map. If  $\rho$  is a bijection, then  $f^\rho, W^\rho$  are called rearrangements of  $f$  (as a function on  $\Omega_2$ ) and  $W$  (as a function on  $\Omega_2^2$ ). Actually, relabeling of labeled graphons can be understood as a kind of rearrangement. In other words, for a given measure preserving map  $\rho$ , the pull backs of  $f$  and  $W$  are defined by

$$f^\rho(x) := f(\rho(x)), \quad W^\rho(x, y) := W(\rho(x), \rho(y)). \quad (2.31)$$

If  $f \in L^1(\Omega_2)$  and  $W \in L^1(\Omega_2^2)$ , then  $\|f^\rho\|_1 = \|f\|_1$  and  $\|W^\rho\|_1 = \|W\|_1$ .

**Definition 2.3.1.** (i) For the probability space  $\Omega := [0, 1]$  together with the Lebesgue measure, an unlabeled graphon is a graphon up to relabeling such that a relabeling is defined by an invertible measure preserving transformation of the closed unit interval.

(ii) Graphons  $W_1, W_2$  are called weakly isomorphic (or weakly equivalent i.e.  $W_1 \approx W_2$ ) if there exists a graphon  $W$  and measure preserving transformations  $\rho_1, \rho_2$  on the probability space  $\Omega$  such that  $W^{\rho_1} = W_1$  and  $W^{\rho_2} = W_2$  almost everywhere.

Relabeled graphons can be organized into an equivalence class, namely an unlabeled graphon. For a given labeled graphon  $W$ , its corresponding unlabeled graphon class  $[W]$  is given by

$$[W] := \{W^\rho : (x, y) \mapsto W(\rho(x), \rho(y)) : \rho \text{ is an arbitrary rearrangement}\}. \quad (2.32)$$

Set  $\mathcal{W}(\Omega)$  as the set of all labeled graphons on a given probability space  $\Omega$ . It is not difficult to see that if  $\Omega := [0, 1]$  is the probability space, then

$\mathcal{W}(\Omega)$  is the subspace of symmetric functions in  $L_\infty([0, 1]^2)$ . By defining a suitable equivalence relation on labeled graphons, which encodes exchanging decorations (i.e. (2.32)), it is possible to associate a unique graphon class to each graph limit. This graphon class is called unlabeled graphon. Set  $[\mathcal{W}](\Omega)$  as the family of all unlabeled graphons on a given probability space  $\Omega$ .

Graphons, as edge weighted graphs on the vertex set  $[0, 1]$ , can provide a generalization of discrete graphs. Each finite simple graph  $G$  defines naturally an unlabeled graphon class  $[W_G]$  in terms of its adjacency matrix. First we build a labeled graphon  $W_G$  with respect to the information of the adjacency matrix. Consider  $V(G)$  as a probability space such that each vertex has probability  $\frac{1}{|G|}$ . Define the map  $W_G^1 : V(G) \times V(G) \rightarrow [0, 1]$  as follows

$$W_G^1(u, v) := \begin{cases} 1, & \text{if } u \text{ and } v \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}. \quad (2.33)$$

It is easy to see that  $W_G^1$  is a symmetric measurable function. In an alternative setting, we can also consider  $\Omega = (0, 1]$  as the probability space which is equipped with a partition  $\{I_i^n\}_{i=1, \dots, n}$  such that  $I_i^n := (\frac{i-1}{n}, \frac{i}{n}]$ . If the vertices of  $G$  is labeled by  $1, \dots, n$ , then the corresponding graph function  $W_G^2$  is given by

$$W_G^2(x, y) = W_G^1(i, j) = 1, \quad x \in I_i^n, y \in I_j^n \quad (2.34)$$

whenever there exists an edge between  $i$  and  $j$ .

The homomorphism density is an useful tool in dealing with the probability of the existence of a subgraph in an extremely large graph. For a given finite graph  $G$ , the homomorphism density of each subgraph  $H$  in  $G$  is given by

$$t(H, G) := \frac{\text{hom}(H, G)}{|V(G)|^{|V(H)|}} \quad (2.35)$$

such that  $\text{hom}(H, G)$  is the number of graph homomorphisms from  $H$  to  $G$ . It is possible to generalize this idea for the level of graph limits where this parameter informs the density of  $H$  as a subgraph in  $G$  asymptotically when the number of vertices of  $G$  tends to infinity. For a given graphon  $W : \Omega^2 \rightarrow [0, 1]$  and a simple finite graph  $H$ , the homomorphism density is defined by

$$t(H, W) := \int_{\Omega^{|H|}} \prod_{ij \in E(H)} W(x_i, x_j) d\mu(x_1) \dots d\mu(x_{|H|}). \quad (2.36)$$

If the graphon  $W_G$  is a labeled graphon with respect to a given graph  $G$ , then we have  $t(H, G) := t(H, W_G)$ . Homomorphism densities provide an alternative way of defining convergence of sequences of dense graphs. It is shown that the sequence  $\{G_n\}_{n \geq 0}$  converges to the labeled graphon  $W$  iff the sequence  $\{t(H, G_n)\}_{n \geq 0}$  of homomorphism densities converges to the

homomorphism density  $t(H, W)$  for each simple subgraph  $H$ . Therefore any convergent sequence of finite sparse graphs should be convergent to the graphon with zero density which is almost everywhere the 0-graphon. However the theory of graphons of sparse graphs has been developed to obtain non-zero graphons for sequences of sparse graphs in terms of the rescaled versions of the canonical graphons or other normalization methods applied on the ground probability space. [29, 94, 125, 169]

The space of graphons provides the completion of the space of finite graphs with respect to a topology generated by a particular metric namely, cut-distance. The cut-distance between labeled graphons  $W, U$  is defined by

$$\delta_{\text{cut}}(W, U) := \inf_{\rho, \tau} \sup_{S, T} \left| \int_{S \times T} W(\rho(x), \rho(y)) - U(\tau(x), \tau(y)) dx dy \right| \quad (2.37)$$

such that the infimum is taken over all relabeling  $\rho$  on  $W$  and  $\tau$  on  $U$  and the supremum is taken over all measurable subsets  $S, T$  of the closed interval. The infimum over relabeling allows us to define the cut-distance on the space of unlabeled graphons.

Weakly isomorphic graphons and relabeled graphons with respect to a given graphon have the same corresponding symmetric measurable functions almost everywhere. The distance (2.37) does not distinguish between weakly isomorphic graphons. We can see that graphons  $W, U$  are weakly isomorphic whenever for any finite subgraph  $H$ , we have  $t(H, W) = t(H, U)$ .

**Theorem 2.3.2.** *Each graphon is the cut-distance convergent limit of a sequence of finite graphs. In addition, the cut-distance  $\delta_{\text{cut}}$  determines a compact topological structure on the quotient space  $\mathcal{W}(\Omega)/\approx$  of labeled graphons with respect to the weakly isomorphic relation. [94, 125]*

### 2.3.2 Feynman graphons

It is the place to deal with a new application of graph limits to Quantum Field Theory where we aim to achieve a new interpretation of Feynman diagrams and their corresponding formal expansions. In general, any arbitrary Feynman diagram, as a weighted graph decorated by some physical parameters, might have many nested or overlapping loops. In higher order perturbation expansions in (strongly) coupled gauge field theories, we can investigate the appearance of many loops Feynman diagrams. We can encode these loops as vertices of the decorated rooted tree representation of the original Feynman diagram and then consider this sparse graph to generate its corresponding graphon model. Decorated rooted trees, as simple graphs with low densities, are useful to simplify Feynman diagrams and their complicated formal expansions which have more densities than sparse graphs. Generating non-zero graphon models for the infinite sequence of rooted trees requires to apply renormalization methods on the canonical graphons and the ground probability space. [29, 169, 170, 174]

**Lemma 2.3.3.** *For a fixed probability space  $(\Omega, \mu_\Omega)$ , the algebraic combinatorics of each Feynman diagram can be encoded by a unique unlabeled graphon class.*

*Proof.* A rooted tree  $t$  is a finite, connected oriented graph without loops in which every vertex has exactly one incoming edge, except one namely, the root which has no incoming but only outgoing edges. We can put two classes of decorations on each tree namely, vertex-labeled and edge-labeled. The rooted tree representations of Feynman diagrams can be defined via the grafting operator. The free commutative algebra generated by isomorphism classes of non-planar rooted trees is actually the polynomial algebra generated by symbols  $t$  where each symbol represents one isomorphism class. The concatenation is the product and the empty tree is the unit for this polynomial algebra. In addition, this polynomial algebra can be equipped by a modified version of the renormalization coproduct given by

$$\Delta_{\text{CK}}(t) = \mathbb{I} \otimes t + t \otimes \mathbb{I} + \sum_c R_c(t) \otimes P_c(t) \quad (2.38)$$

such that the sum is taken over all admissible cuts  $c$  on  $t$  which divides the tree into two parts. The part  $R_c(t)$  contains the original root of  $t$  and the part  $P_c(t)$  is a forest of subtrees. The resulting Hopf algebra  $H_{\text{CK}}$  of non-planar rooted trees is connected graded free commutative non-cocommutative finite type. Decorations enable us to adapt this combinatorial Hopf algebra with respect to physical theories. Each Feynman diagram  $\Gamma$ , which might contain divergent Feynman subgraphs, is encoded by a decorated non-planar rooted tree  $t_\Gamma$  such that the root represents the full graph and each leaf is a divergent subgraph which has no further subdivergencies. If the original graph has overlapping subdivergencies, then we can replace the single rooted tree by a sum of decorated rooted trees after disentangling the overlaps. Thanks to these rules, we can embed the Hopf algebra  $H_{\text{FG}}(\Phi)$  of Feynman diagrams of  $\Phi$  into the decorated Connes–Kreimer Hopf algebra  $H_{\text{CK}}(\Phi)$  as a closed Hopf subalgebra. This embedding is encapsulated by the injective Hopf algebra homomorphism

$$\Gamma \longmapsto \Xi(\Gamma) := \sum_{j=1}^r B_{\Gamma_j, G_{j,i}}^+ \left( \prod_{i=1}^{k_j} \Xi(\gamma_{j,i}) \right) \quad (2.39)$$

such that  $\Gamma = \prod_{i=1}^{k_j} \Gamma_j \star_{j,i} \gamma_{j,i}$  and each  $G_{j,i}$  is the gluing information. For a given decorated non-planar rooted tree  $t$ , if the longest path from the root to a leaf contains  $k$  edges, then the renormalization coproduct  $\Delta_{\text{CK}}(t)$  is a sum of at least  $k + 1$  terms. In other words, the decorated non-planar rooted tree  $t$  represents an iterated integral with  $k$  nested sub-divergencies while each vertex corresponds to a sub-integral without any sub-divergencies. [65, 67, 68, 77, 88]

For any Feynman diagram  $\Gamma$  without overlapping sub-divergencies, the decorated tree  $t_\Gamma := \Xi(\Gamma)$  is a simple finite weighted graph where thanks to its corresponding adjacency matrix we can determine the labeled graphons  $W_{t_\Gamma}$  of the form (2.33) or (2.34). Set  $[W_{t_\Gamma}]$  as the unlabeled graphon class associated to  $t_\Gamma$ . Up to the rearrangement, weakly isomorphic relation and the embedding (2.39), we can show that the unlabeled graphon  $[W_{t_\Gamma}]$ .

For any Feynman diagram  $\Gamma$  which has some overlapping sub-divergencies,  $u_\Gamma := \Xi(\Gamma)$  is a linear combination of decorated non-planar rooted trees such as  $u_\Gamma = \alpha_1 t_1 + \dots + \alpha_n t_n$ . In this situation, the labeled graphons  $W_{u_\Gamma}$  can be determined by normalizing or rescaling methods used on  $W_{t_i}$ s. For each  $1 \leq i \leq n$ , the labeled graphon  $W_{t_i}$  is projected or embedded into the subinterval  $I_i$  of  $[0, 1]$  and then we have

$$W_{u_\Gamma}(x, y) := \frac{W_{t_1} + \dots + W_{t_n}}{|W_{t_1} + \dots + W_{t_n}|}. \quad (2.40)$$

The subintervals  $I_i$  are determined in terms of the grading value of rooted trees and cut-distance topology.  $\square$

*Remark 2.3.4.* For each Feynman diagram  $\Gamma$  and each natural number  $n \geq 2$ , the graphon  $W_{n\Gamma}$  is actually  $n$  copies of  $W_\Gamma$  inside the closed unit interval. We can generate this graphon by measure preserving transformation  $\rho_n : x \mapsto nx$  on  $[0, 1]$  which is not invertible with respect to the Lebesgue measure. Therefore  $W_\Gamma$  and  $W_{n\Gamma}$  are weakly isomorphic.

We use the phrase "Feynman graphons" for this class of graphons which provide an analytic generalizations for Feynman diagrams.

**Definition 2.3.5.** A sequence  $\{\Gamma_n\}_{n \geq 0}$  of non-trivial Feynman diagrams is called convergent when  $n$  tends to infinity, if the corresponding sequence  $\{[W_{t_{\Gamma_n}}]\}_{n \geq 0}$  of non-zero unlabeled graphon classes is convergent to a unique non-zero Feynman graphon class with respect to the cut-distance topology when  $n$  tends to infinity.

Non-zero graphons, as the cut-distance convergent limit of the sequences of sparse graphs, can be built in terms of applying rescaling or renormalization methods to the ground probability space or canonical graphons. [29, 174]

Suppose the unlabeled graphon class  $[W]$  is the convergent limit for the sequence  $\{[W_{t_{\Gamma_n}}]\}_{n \geq 0}$ . If we consider the pixel picture representation of the graphon  $[W]$ , then we can associate an infinite tree or forest  $t$  such that  $W_t \in [W]$  and  $W \in [W_t]$ . Therefore  $[W] = [W_t]$ . Thanks to the homomorphism (2.39), it is possible to build an extremely large Feynman diagram  $\Gamma_t$  with respect to the infinite tree or forest  $t$ . This  $\Gamma_t$  can be described as the convergent limit of the sequence  $\{\Gamma_n\}_{n \geq 0}$  with respect to the cut-distance topology. We can also show that this limit is unique up to

the rearrangement, weakly isomorphic relation and the embedding (2.39). [169]

Graphon models for Feynman diagrams in a given gauge field theory can lead us to study formal expansions of Feynman diagrams in the context of the theory of random graphs.

The study of random graphs was begun by Erdos, Renyi and Gilbert when they were working on a probabilistic construction of a graph with large girth and large chromatic number. After a short period of time, work on random graphs  $G_{n,m}$  has been concerned by many mathematicians in Combinatorics and Discrete Mathematics. Nowadays it is not difficult to observe various applications of these combinatorial objects in many fields in Mathematics and other applied sciences. Generally speaking, the theory of random graphs aims to provide some results such as "a combinatorial property A almost always implies another combinatorial property B". For any integer value  $n$  and  $0 \leq p \leq 1$ , a random graph  $G(n, p)$  is defined by taking  $n$  nodes and connecting any two of them with the probability  $p$ , making an independent decision about each pair. There are alternative ways to build random graphs. As an example, consider  $\mathcal{L}_{n,m}$  as the collection of all labeled graphs with the vertex set  $V = [n] = \{1, 2, \dots, n\}$  and  $m$  edges such that  $0 \leq m \leq \binom{n}{2}$ . To each  $G \in \mathcal{L}_{n,m}$ , assign the probability

$$\mathbb{P}(G) = \frac{1}{\binom{\binom{n}{2}}{m}}. \quad (2.41)$$

In other words, start with an empty graph on the set  $[n]$  and insert  $m$  edges in such a way that all possible  $\binom{\binom{n}{2}}{m}$  choices are equally likely. The resulting graph  $G_{n,m} := ([n], E_{n,m})$  is known as the uniform random graph. As other example, fix  $0 \leq p \leq 1$  and for each graph  $G$  with the vertex set  $[n]$  and  $0 \leq m \leq \binom{n}{2}$  edges, assign the probability

$$\mathbb{P}(G) = p^m (1-p)^{\binom{n}{2}-m}. \quad (2.42)$$

In other words, start with an empty graph with the vertex set  $[n]$  and consider  $\binom{n}{2}$  to insert edges independently with the probability  $p$ . The resulting graph  $G_{n,p} := ([n], E_{n,p})$  is known as the binomial random graph.

It is shown that the random graph  $G_{n,p}$  with  $0 \leq m \leq \binom{n}{2}$  edges is the same as one of the  $\binom{\binom{n}{2}}{m}$  graphs that have  $m$  edges. For enough large  $n$ , random graphs  $G_{n,m}$  and  $G_{n,p}$  have the same behavior whenever the number of edges  $m$  in  $G_{n,m}$  is very close to the expected number of edges of  $G_{n,p}$ , namely,

$$m = \binom{n}{2} p \approx \frac{n^2 p}{2}. \quad (2.43)$$

It is equivalent to say that the edge probability in  $G_{n,p}$  should be  $p \approx \frac{2m}{n^2}$ . [79]

**Lemma 2.3.6.** *Each labeled graphon determines a class of random graphs.*

*Proof.* If we have a simple weighted graph  $G$ , then we can build a random simple graph  $R(G)$  by including the edge with probability equal to its weight. Thanks to this idea, suppose we have a labeled graphon  $W$  and finite subset  $S := \{s_1, \dots, s_n\}$  in  $[0, 1]$ . We can make a weighted graph  $G(S, W)$  with  $|S| = n$  nodes such that the edge  $s_i s_j$  has the weight  $W(s_i, s_j)$ . In general, the random graph  $R(n, W) := R(G(S, W))$  with respect to the weighted graph  $G(S, W)$  is our promising graph such that  $S$  is a set of  $n$  points which are selected independently from the closed interval.  $\square$

The random graphs  $R(n, W)$  are useful to approximate graphons  $W$  associated to large numbers of points in the closed interval. It is shown that with the probability 1, the sequence  $\{R(n, W)\}_{n \geq 0}$  is convergent to the graphon  $W$  with respect to the cut-distance topology when  $n$  tends to infinity. [125]

Thanks to the discussed topics, it is time to observe some new applications of the graph function representation theory of Feynman diagrams in dealing with expansions of these physical graphs in Quantum Field Theory. At the first application, we address a new interpretation of solutions of Dyson–Schwinger equations in the context of sequences of random graphs derived from partial sums of solutions.

**Theorem 2.3.7.** *The unique solution of each combinatorial Dyson–Schwinger equation can be described as the cut-distance convergent limit of a sequence of finite Feynman diagrams.*

*Proof.* The full proof is given in [169] and here we only address the main idea. Suppose DSE be a combinatorial Dyson–Schwinger equation with the general form (1.10) such that its unique solution is given by

$$X_{\text{DSE}} = \sum_{n \geq 0} (\lambda g)^n X_n \quad (2.44)$$

while  $g$  is the bare coupling constant and the generators  $X_n$  are determined by the recursive relations (1.11). Make the new sequence  $\{Y_m\}_{m \geq 1}$  of partial sums of the expansion  $\sum_{n \geq 0} (\lambda g)^n X_n$  such that we have

$$Y_m := (\lambda g)^1 X_1 + \dots + (\lambda g)^m X_m. \quad (2.45)$$

It is shown in [19, 73] that the large Feynman diagram  $X_{\text{DSE}}$  belongs to a completion of  $H_{\text{FG}}[[g]]$  with respect to the  $n$ -adic topology. We claim that the sequence  $\{Y_m\}_{m \geq 1}$  of finite graphs converges to the large Feynman diagram  $X_{\text{DSE}}$  with respect to the cut-distance topology. For this purpose, we can apply the  $n$ -adic metric and the graphon representations of the components  $X_n$  to build a random graph with respect to each graph  $Y_m$ . It leads us to associate a sequence  $\{R(Y_m)\}_{m \geq 1}$  of random graphs with respect to the sequence  $\{Y_m\}_{m \geq 1}$  which is cut-distance convergent to  $X_{\text{DSE}}$ .  $\square$

The structure of a modification of the Connes–Kreimer BPHZ renormalization for large Feynman diagrams has been formulated in [169] where we worked on a topological completion of the renormalization Hopf algebra of Feynman diagrams with respect to the cut-distance topology. As the second application, we work on Feynman graphon formalism to build a renormalization program on the collection  $\mathcal{S}^{\Phi,g}$  under a topological Hopf algebraic setting. This new approach enables us to proceed our knowledge about non-perturbative versions of Feynman rules which act on large Feynman diagrams. For this purpose we explain the structure of a new Hopf algebra derived from the renormalization coproduct on graphons.

**Theorem 2.3.8.** *Thanks to the renormalization coproduct, there exists a topological Hopf algebraic structure on the collection  $\mathcal{S}_{\text{graphon}}^{\Phi}$  of all unlabeled graphons which contribute to represent (large) Feynman diagrams of a physical theory  $\Phi$ .*

*Proof.* We plan to equip  $\mathcal{S}_{\text{graphon}}^{\Phi}$  with an enriched version of the renormalization Hopf algebra which is completed with respect to the cut-distance topology.

Thanks to Lemma 2.3.3, for each finite Feynman diagram  $\Gamma$ , we associate the unlabeled graphon class  $[W_{\Gamma}]$ . In addition, the unique solution of each combinatorial Dyson–Schwinger equation DSE in  $\mathcal{S}^{\Phi,g}$  determines a unique large Feynman diagram  $X_{\text{DSE}}$  such that thanks to Theorem 2.3.7, this infinite graph can be interpreted as the convergent limit of the sequence of partial sums with respect to the cut-distance topology. Therefore it does make sense to replace objects of  $\mathcal{S}^{\Phi,g}$  with large Feynman diagrams such as  $X_{\text{DSE}}$  as the unique solution of the equation DSE. Thanks to Lemma 2.3.3, we associate a unique unlabeled graphon class  $[W_{t_{X_{\text{DSE}}}}]$  to the large Feynman diagram  $X_{\text{DSE}}$  via its rooted tree (forest) representation. For simplicity in the presentation, from now we use the notation  $[W_{X_{\text{DSE}}}]$  for this graphon class.

It is possible to lift the renormalization coproduct (2.10) onto the level of unlabeled graphons which contribute to the description of (large) Feynman diagrams. For a given finite Feynman diagram  $\Gamma$  with the corresponding unlabeled graphon  $[W_{\Gamma}]$ , define

$$\Delta_{\text{graphon}}([W_{\Gamma}]) := \sum [W_{\gamma}] \otimes [W_{\Gamma/\gamma}] \quad (2.46)$$

such that the sum is taken over all unlabeled graphon classes such as  $[W_{\gamma}]$  associated to  $\gamma$  as the disjoint union of 1PI superficially divergent subgraphs of  $\Gamma$ .

Thanks to Theorem 2.3.7, for the unlabeled graphon class  $[W_{X_{\text{DSE}}}]$  corresponding to the large Feynman diagram  $X_{\text{DSE}}$ , define its coproduct as the convergent limit of the sequence  $\{\Delta_{\text{graphon}}([W_{Y_m}])\}_{m \geq 1}$  of the coproducts of the finite partial sums with respect to the cut-distance topology.

Now we can adapt (2.46) for the level of large Feynman diagrams and define

$$\Delta_{\text{graphon}}([W_{X_{\text{DSE}}}] := \sum [W_{\Upsilon}] \otimes [W_{X_{\text{DSE}}/\Upsilon}] \quad (2.47)$$

such that the sum is taken over all unlabeled graphon classes such as  $[W_{\Upsilon}]$  associated to  $\Upsilon$  as the disjoint union of 1PI superficially divergent subgraphs of  $X_{\text{DSE}}$ .

If we consider objects of  $\mathcal{S}_{\text{graphon}}^{\Phi}$  as generators of a free commutative algebra, then thanks to (2.46) we obtain a bialgebra structure on Feynman graphons which is graded in terms of the number of independent loops of the corresponding Feynman diagrams. The unlabeled graphon class  $[W_{\mathbb{I}}]$  corresponding to the empty graph is the unit for this bialgebra. The counit is also defined by

$$\tilde{\varepsilon}([W_{\Gamma}]) = \begin{cases} 1, & [W_{\Gamma}] = [W_{\mathbb{I}}] \\ 0, & \text{else} \end{cases}. \quad (2.48)$$

The existence of the graduation parameter is the key tool to define an antipode map. For each finite Feynman diagram  $\Gamma$ , we have

$$S_{\text{graphon}}([W_{\Gamma}]) = -[W_{\Gamma}] - \sum S_{\text{graphon}}([W_{\gamma_{(1)}}])[W_{\gamma_{(2)}}] \quad (2.49)$$

such that  $\Delta_{\text{graphon}}([W_{\Gamma}]) = \sum [W_{\gamma_{(1)}}] \otimes [W_{\gamma_{(2)}}]$ .

Thanks to Theorem 2.3.7, for the unlabeled graphon class  $[W_{X_{\text{DSE}}}]$  corresponding to the large Feynman diagram  $X_{\text{DSE}}$ , define its antipode as the convergent limit of the sequence  $\{S_{\text{graphon}}([W_{Y_m}])\}_{m \geq 1}$  of unlabeled graphons of finite partial sums  $Y_m$  with respect to the cut-distance topology. Since partial sums are finite graphs, their corresponding graphon type antipodes  $S_{\text{graphon}}([W_{Y_m}])$  can be obtained inductively by the coproduct  $\Delta_{\text{graphon}}$  where we have

$$S_{\text{graphon}}([W_{Y_m}]) = -[W_{Y_m}] - \sum S_{\text{graphon}}([W_{\Gamma_{(1)}}])[W_{\Gamma_{(2)}}] \quad (2.50)$$

such that  $\Delta_{\text{graphon}}([W_{Y_m}]) = \sum [W_{\Gamma_{(1)}}] \otimes [W_{\Gamma_{(2)}}]$ .

Now we can adapt the antipode (2.49) for the level of large Feynman diagrams and define

$$S_{\text{graphon}}([W_{\text{DSE}}]) = -[W_{\text{DSE}}] - \sum S_{\text{graphon}}([W_{\Upsilon_{(1)}}])[W_{\Upsilon_{(2)}}] \quad (2.51)$$

such that  $\Delta_{\text{graphon}}([W_{\text{DSE}}]) = \sum [W_{\Upsilon_{(1)}}] \otimes [W_{\Upsilon_{(2)}}]$ .

Therefore  $\mathcal{S}_{\text{graphon}}^{\Phi}$  becomes a connected graded free commutative non-cocommutative (not necessarily finite type) Hopf algebra. In addition, the recursive structure of the coproduct (2.47) and the antipode (2.51) together with their linear property allow us to show the compatibility of this Hopf algebraic structure with the cut-distance topology.  $\square$

Feynman graphon models of Feynman diagrams allow us to show that the space  $\mathcal{S}_{\text{graphon}}^{\Phi}$  (given by Theorem 2.3.8) is completed with respect to the cut-distance topology.

**Corollary 2.3.9.** *Let  $V$  be a complex vector space with a basis labeled by coupling constants of a given Quantum Field Theory  $\Phi$ , and suppose  $\text{Diff}(V)$  be the group of formal diffeomorphisms of  $V$  tangent to the identity at  $0 \in V$  and  $H_{\text{diff}}(V)$  be its corresponding Hopf algebra. The complex Lie group  $\mathbb{G}_{\text{graphon}}^{\Phi}(\mathbb{C})$  of characters on Feynman graphons can be represented by  $\text{Diff}(V)$ .*

*Proof.* The Hopf algebra  $H_{\text{diff}}(\mathbb{C})$  of formal diffeomorphisms of  $\mathbb{C}$  tangent to the identity has generators such as  $a_n$  which play the role of coordinates of

$$\phi(x) = x + \sum_{n \geq 2} a_n(\phi)x^n \quad (2.52)$$

such that  $\phi$  is a formal diffeomorphism satisfying  $\phi(0) = 0$ ,  $\phi'(0) = \text{id}$ . Its coproduct is given by

$$\Delta(a_n)(\phi_1 \otimes \phi_2) = a_n(\phi_2 \circ \phi_1). \quad (2.53)$$

We can define a Hopf algebra homomorphism  $\Psi : H_{\text{diff}}(V) \rightarrow H_{\text{FG}}(\Phi)$  with the corresponding dual group homomorphism  $\hat{\Psi} : \mathbb{G}_{\Phi}(\mathbb{C}) \rightarrow \text{Diff}(V)$ . The map  $\Psi$  maps the coefficients of the expansion of formal diffeomorphisms to the coefficients in the renormalization Hopf algebra of the expansions of the effective (or running) coupling constants of the physical theory as formal power series in the bare coupling constants. As the consequence, for each Dyson–Schwinger equation DSE with the corresponding Hopf subalgebra  $H_{\text{DSE}}$  and Lie (sub)group  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$ , we can define a group homomorphism  $\hat{\Psi}_{\text{DSE}}$  from  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$  to  $\text{Diff}(V)$ . [46, 164]

Thanks to Lemma 2.3.3, we can embed  $H_{\text{FG}}(\Phi)$  into the renormalization Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$  of Feynman graphons. This allows us to lift the map  $\Psi$  onto the level of Feynman graphons and build a new Hopf algebra homomorphism  $\bar{\Psi} : H_{\text{diff}}(V) \rightarrow \mathcal{S}_{\text{graphon}}^{\Phi}$  with the corresponding dual group homomorphism  $\hat{\bar{\Psi}} : \mathbb{G}_{\text{graphon}}^{\Phi}(\mathbb{C}) \rightarrow \text{Diff}(V)$ .  $\square$

The construction of a canonical filtration on terms  $X_n$ s of the unique solution of a given Dyson–Schwinger equation has been explained in [121] where each filtered term maps to a certain power of  $L$  in the log-expansion. The original idea is to filter images of Feynman diagrams in a particular universal enveloping algebra which generates a quasi-shuffle type Hopf algebra. Thanks to Theorem 2.3.7 and Theorem 2.3.8, we aim to adapt this filtration for large Feynman diagrams.

**Theorem 2.3.10.** *Renormalized Feynman rules characters of the Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$  filtrate large Feynman diagrams.*

*Proof.* Set  $H_{\text{word}}$  as the vector space of words which contains  $H_{\text{letter}}$  as the subspace of letters. Set a commutative associative map  $\Theta : H_{\text{letter}} \times H_{\text{letter}} \rightarrow$

$H_{\text{letter}}$  as the Hoffman pairing which sends two generators  $a, b$  to another generator  $\Theta(a, b)$  and adds degrees. Define the generalized quasi-shuffle product  $\ominus_{\Theta}$  on  $H_{\text{word}}$  as follows

$$au \ominus_{\Theta} bv := a(u \ominus_{\Theta} bv) + b(au \ominus_{\Theta} v) + \Theta(a, b)(u \ominus_{\Theta} v) \quad (2.54)$$

which builds a commutative associative algebra with empty word  $\mathbb{I}$  as the unit. We can equip this algebra with the following coproduct structure

$$\Delta_{\text{word}}(w) = \sum_{vu=w} u \otimes v \quad (2.55)$$

which gives us a bialgebra structure on  $H_{\text{word}}$  with the counit  $\hat{\mathbb{I}}_{\text{word}}$ . The length of each word determines a natural graduation parameter on this bialgebra which leads us to define an antipode recursively. As the consequence,  $(H_{\text{word}}, \ominus_{\Theta}, \mathbb{I}, \Delta_{\text{word}}, \hat{\mathbb{I}}_{\text{word}}, S_{\text{word}})$  is a graded connected commutative unital non-cocommutative counital Hopf algebra [66, 87]. We have

$$\ominus_{\Theta} \circ (S_{\text{word}} \otimes \text{id}) \circ \Delta_{\text{word}} = \ominus_{\Theta} \circ (\text{id} \otimes S_{\text{word}}) \circ \Delta_{\text{word}} = \mathbb{I}_{\text{word}} \circ \hat{\mathbb{I}}_{\text{word}}. \quad (2.56)$$

In this setting, the grafting operator on words allows us to add a letter to the first place

$$B_a^+(u) := au. \quad (2.57)$$

We can check that for each  $a$ , the grafting operators are Hochschild one-cocycles. It is possible to embed the renormalization Hopf algebra of Feynman diagrams into the Hopf algebra of words. This embedding is defined in terms of the homomorphism  $\nu : H_{\text{FG}}(\Phi) \rightarrow H_{\text{word}}$  determined by the relations

$$\nu(\mathbb{I}) = \mathbb{I}_{\text{word}}, \quad \ominus_{\Theta} \circ (\nu \otimes \nu) = \nu \circ m, \quad \hat{\mathbb{I}}_{\text{word}} \circ \nu = \nu \circ \hat{\mathbb{I}},$$

$$\Delta_{\text{word}} \circ \nu = (\nu \otimes \nu) \circ \Delta_{\text{FG}}, \quad S_{\text{word}} \circ \nu = \nu \circ S, \quad B_{a_n}^+ \circ \nu = \nu \circ B_{\gamma_n}^+. \quad (2.58)$$

The morphism  $\nu$  sends each primitive Feynman graph  $\gamma_n$  to a letter  $a_n$ . Thanks to Theorem 2.3.8, it is possible to lift the embedding  $\nu$  onto a new homomorphism  $\bar{\nu}$  which embeds the renormalization Hopf algebra of graphons  $\mathcal{S}_{\text{graphon}}^{\Phi}$  into the Hopf algebra of words. It is enough to replace each Feynman diagram  $\Gamma$  with its corresponding unlabeled graphon class  $[W_{\Gamma}]$ .

Consider Dyson–Schwinger equations for 1PI Green’s functions with the general form

$$\Gamma^{\bar{n}} = 1 + \sum_{\gamma, \text{res}(\gamma)=\bar{n}} \frac{g^{|\gamma|}}{\text{Sym}(\gamma)} B_{\gamma}^+(X_{\mathcal{R}}^{\gamma}) \quad (2.59)$$

such that  $B_{\gamma}^+$  are Hochschild closed one-cocycles of the Hopf algebra of Feynman diagrams indexed by Hopf algebra primitives  $\gamma$  with external legs  $\bar{n}$ ,

$X_{\mathcal{R}}^{\gamma}$  is a monomial in superficially divergent Green's functions which dress the internal vertices and edges of  $\gamma$ . If we apply the renormalized Feynman rules character  $\phi_r$  to a Feynman graph which contributes to this class of equations, then we can obtain a polynomial in a suitable external scale parameter  $L = \log S/S_0$  such that  $S_0$  fixes a reference scale for the renormalization process. At the end of the day, we can get a renormalized version  $G_r(g, L, \theta)$  of Green's functions. Lemma 2.3.3 and Theorem 2.3.8 are useful to reformulate the equation (2.59) in the language of graphons as an equation in the Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$ . The embedding  $\mathcal{V}$  enables us to lift this graphon model Dyson–Schwinger equations onto their corresponding equations in the Hopf algebra of words. We have

$$X_{\text{DSE,word}} = \mathcal{V}([W_{X_{\text{DSE}}}] ) = \mathbb{I}_{\text{word}} + \sum_{n \geq 1} g^n B_{l_n}^+(X_{\text{DSE,word}}^{\ominus \Theta(n+1)}) \quad (2.60)$$

such that  $X_{\text{DSE,word}}$  is the word representation of the unlabeled graphon class  $[W_{X_{\text{DSE}}}]$  with respect to the large graph  $X_{\text{DSE}}$ . We have  $X_{\text{DSE,word}} = \sum_{n \geq 0} g^n z_n$  such that each  $z_n = \nu(X_n)$  is determined recursively by the relations

$$z_n = \sum_{m=1}^n B_{l_m}^+ \left( \sum_{k_1 + \dots + k_{m+1} = n-m, k_i \geq 0} z_{k_1} \ominus \Theta \dots \ominus \Theta z_{k_{m+1}} \right). \quad (2.61)$$

We plan to explain the filtration structure on words and then by applying the inverse of the embedding  $\nu$ , we can adapt it for the level of Feynman graphon representations of large Feynman diagrams which contribute to solutions of Dyson–Schwinger equations.

The canonical candidate for the filtration on words is built in terms of the lower central series at the Lie algebra level where we need to apply the theory of Hall sets and Hall basis. The Milnor–Moore theorem ([144]) allows us to build the graded dual Hopf algebra to  $H_{\text{word}}$  in terms of the universal algebra of a particular Lie algebra.

A bilinear anti-symmetric map  $[\cdot, \cdot]$  on a vector space  $\mathcal{L}$  over the field  $\mathbb{K}$  with characteristic zero defines a Lie algebra structure if it obeys the conditions

$$\begin{aligned} \forall x \in \mathcal{L} : [x, x] &= 0, \\ \forall x, y, z \in \mathcal{L} : [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0. \end{aligned} \quad (2.62)$$

The lexicographical ordering enables us to build the Hall basis for the Lie algebra  $\mathcal{L}$  [83, 84]. For a given ordering  $x_1 < x_2 < \dots < [x_1, x_2] < \dots$  on  $\mathcal{L}$ , define  $[x, x']$  as an element of a Hall basis for  $\mathcal{L}$  iff

- $x, x' \in \mathcal{L}$  are Hall basis elements with  $x < x'$ ,
- if  $x' = [x_1, x_2]$ , then  $x' \geq x_2$ .

The unique universal enveloping algebra associated to  $\mathcal{L}$  is defined in terms of the tensor algebra  $(T(\mathcal{L}), \otimes, 1)$  such that

$$T(\mathcal{L}) := \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}. \quad (2.63)$$

Set

$$I := \{s \otimes (x \otimes y - y \otimes x - [x, y]) \otimes t : x, y \in \mathcal{L}; s, t \in T(\mathcal{L})\} \quad (2.64)$$

as a two sided ideal and then define the equivalent classes of the form

$$[t] := \{s \in T(\mathcal{L}) : s - t \in I\}. \quad (2.65)$$

$T(\mathcal{L})/I$  is actually the unique universal enveloping algebra  $\mathcal{U}(\mathcal{L})$  generated by  $\mathcal{L}$  where the product of this algebra is given by

$$m_{\mathcal{U}(\mathcal{L})}([s] \otimes [t]) := [s \otimes t] \quad (2.66)$$

and  $[1]$  is the unit of this algebra. In addition, we can equip  $\mathcal{U}(\mathcal{L})$  by a graded Hopf algebra structure with the coproduct

$$\Delta_{\mathcal{U}(\mathcal{L})}([x]) = [x] \otimes \mathbb{I} + \mathbb{I} \otimes [x] \quad (2.67)$$

Set  $\mathcal{L}_{\text{word}}$  as the Lie algebra corresponding to the Hopf algebra  $H_{\text{word}}$ . Define the decreasing sequence

$$\mathcal{L}_{\text{word}} = \mathcal{L}_1 \geq \mathcal{L}_2 \geq \mathcal{L}_3 \geq \dots \quad (2.68)$$

such that  $\mathcal{L}_{n+1}$  is generated by all objects  $[x, y]$  with  $x \in \mathcal{L}$  and  $y \in \mathcal{L}_n$ . For letters  $a_1, a_2, \dots, \Theta(a_1, a_2), \dots \in H_{\text{letter}}$ , set  $x_1, x_2, \dots, \Theta(x_1, x_2), \dots \in \mathcal{L}/\mathcal{L}_2$ . The duality between  $H_{\text{word}}$  and  $\mathcal{U}(\mathcal{L}_{\text{word}})$  can be determined by the unique linear invertible map  $\nu$  such that

$$\nu(a_i) = [x_i], \quad \nu(\Theta(a_i, a_j)) = [\Theta(x_i, x_j)], \quad \nu(a_i a_j) = [x_i \otimes x_j], \dots \quad (2.69)$$

The universal enveloping algebra  $\mathcal{U}(\mathcal{L}_{\text{word}})$  is a filtered bialgebra.

Thanks to the structure of the quasi-shuffle product, we can build a filtration algorithm where it requires to consider all words with length  $k$  into the lexicographical order in terms of the concatenation commutator with respect to the Hall basis which generates words with the length  $k - 1$ . This procedure, which starts with the maximal length of words, should be repeated for the full quasi-shuffle products of the  $k$  corresponding letters and then insert them into the expression [121]. Now if we apply the inverse of the embedding  $\bar{\nu}$ , then this filtration can be defined on Feynman graphons and large Feynman diagrams which live in  $\mathcal{S}_{\text{graphon}}^{\Phi}$ .

Let us now apply renormalized Feynman rules characters on large Feynman diagrams. The Hopf algebra homomorphism  $\nu$  sends the renormalized Feynman rules character  $\phi_r$  to

$$\psi_r = \phi_r \circ \nu^{-1}. \quad (2.70)$$

In [121], it is shown that for each word  $w \in H_{\text{word}}$  with the corresponding  $[x] \in \mathcal{U}(\mathcal{L}_{\text{word}})$ , if  $x \in T(\mathcal{L}_{\text{word}})$  is also an element of the Lie algebra  $\mathcal{L}_{\text{word}}$ , then  $\psi_r(u)$  maps to the  $L$ -linear part of the log-expansion of the renormalized Green's functions. In addition, we have

$$\psi_r(u \ominus_{\Theta} v) = \psi_r(u) \cdot \psi_r(v). \quad (2.71)$$

For a given Feynman diagram  $\Gamma$  with the coradical degree  $r_{\Gamma}$ , we have

$$\phi_r(\Gamma) = \sum_{j=1}^{r_{\Gamma}} c_j^{\Gamma}(\theta) L^j \quad (2.72)$$

such that

$$c_j^{\Gamma} = c_1^{\otimes j} \tilde{\Delta}_{\text{FG}}^{j-1}(\Gamma) \quad (2.73)$$

while  $c_1^{\otimes j} : H_{\text{FG}}(\Phi) \otimes \dots \otimes H_{\text{FG}}(\Phi) \rightarrow \mathbb{C}$  is a symmetric function. Thanks to the Hopf algebra homomorphism  $\nu$  which preserves the co-radical degree, for any word  $u \in H_{\text{word}}$ , we have

$$\psi_r(u) = \sum_{j=1}^{r_u} d_j^u L^j \quad (2.74)$$

such that  $d_j^u = c_j^{\nu^{-1}(u)}$ .

Thanks to the graphon representations of Dyson–Schwinger equations (Theorem 2.3.7 and Theorem 2.3.8), we want to lift the Feynman rules character (2.72) onto the level of large Feynman diagrams.

In [121], it is shown that  $\psi_r$  maps the shuffle product  $u_1 \ominus_{\Theta} \dots \ominus_{\Theta} u_n$  to the  $L^n$ -term in the log expansion such that as the result, this process filtrates coefficients  $X_n$  in the unique solution of each Dyson–Schwinger equation. We lift this story onto the level of Feynman graphons where we have the correspondences  $Y_m \mapsto [W_{Y_m}]$  for each  $m \geq 1$  and  $X_{\text{DSE}} \mapsto [W_{X_{\text{DSE}}}]$ . Now the renormalized character  $\psi_r := \tilde{\phi}_r \circ \bar{\nu}^{-1}$  can help us to map the formal expansions  $\sum_1^m u_{i_1} \ominus_{\Theta} \dots \ominus_{\Theta} u_{i_k}$  of shuffle products of words corresponding to the partial sums  $Y_m$  of  $X_{\text{DSE}}$  to a certain term in the expansion

$$\tilde{\phi}_r([W_{Y_m}]) = \sum_{j=1}^{r_{Y_m}} c_j^{Y_m}(\theta) L^j \quad (2.75)$$

such that

$$c_j^{Y_m} = c_1^{\otimes j} \tilde{\Delta}_{\text{graphon}}^{j-1}([W_{Y_m}]). \quad (2.76)$$

When  $m$  tends to infinity the sequence  $\{c_j^{Y_m}\}_{m \geq 1}$  of coefficients converges to  $c_j^{X_{\text{DSE}}}$  (for each  $j$ ) with respect to the cut-distance topology. In addition, Feynman rules characters are linear homomorphisms which means that when  $m$  tends to infinity, the sequence  $\{\tilde{\phi}_r([W_{Y_m}])\}_{m \geq 1}$  is cut-distance convergent to  $\tilde{\phi}_r([W_{X_{\text{DSE}}}]$ .

Therefore for the infinite graph  $X_{\text{DSE}}$ , we can obtain the following formal expansion as the result of the application of the renormalized Feynman rules character  $\tilde{\phi}_r$ .

$$\tilde{\phi}_r([W_{X_{\text{DSE}}}]) = \sum_{j=1}^{r_{X_{\text{DSE}}}} c_j^{X_{\text{DSE}}}(\theta) L^j \quad (2.77)$$

such that

$$c_j^{X_{\text{DSE}}} = c_1^{\otimes j} \tilde{\Delta}_{\text{graphon}}^{j-1}([W_{X_{\text{DSE}}}] ). \quad (2.78)$$

Suppose  $\mathcal{S}_{\text{graphon},(i)}^\Phi$  is the vector space generated by those Feynman graphons derived from solutions of Dyson–Schwinger equations. The considered Feynman graphons are filtered in terms of the canonical filtration on their corresponding words. Namely, the filtration  $(i)$  can be defined by applying  $\bar{\nu}$  and  $\tilde{\psi}_r$  while the associated words map to a similar term  $i$  in the log-expansion (2.77). Set

$$\mathcal{S}_{\text{graphon},(0)}^\Phi \preceq \mathcal{S}_{\text{graphon},(1)}^\Phi \preceq \dots \preceq \mathcal{S}_{\text{graphon},(i)}^\Phi \preceq \dots \preceq \mathcal{S}_{\text{graphon}}^{\Phi,g} \quad (2.79)$$

as the resulting filtration on all Feynman graphons which contribute to solutions of Dyson–Schwinger equations such that  $\mathcal{S}_{\text{graphon}}^{\Phi,g} := \bigcup_{i \geq 0} \mathcal{S}_{\text{graphon},(i)}^\Phi \subset \mathcal{S}_{\text{graphon}}^\Phi$ . It defines the graded vector space  $\mathcal{G}^\Phi$  given by

$$\begin{aligned} \mathcal{G}_{[0]}^\Phi &= \mathcal{S}_{\text{graphon},(0)}^\Phi \\ \mathcal{G}_{[i]}^\Phi &:= \mathcal{S}_{\text{graphon},(i)}^\Phi / \mathcal{S}_{\text{graphon},(i-1)}^\Phi, \quad \forall i > 0 \end{aligned} \quad (2.80)$$

where we have

$$\mathcal{G}^\Phi = \bigoplus_{i \geq 0} \mathcal{G}_{[i]}^\Phi. \quad (2.81)$$

We can show that  $\mathcal{G}^\Phi$  and  $\mathcal{S}_{\text{graphon}}^{\Phi,g}$  are isomorphic as vector spaces.  $\square$

Theory of words and quasi-shuffle products were studied in [87] where the existence of Hopf algebra structures on words have been addressed. The applications of shuffle type of products to Hopf algebraic renormalization have been addressed in different settings [66, 109, 160, 165]. There is also another alternative machinery ([160]) to lift Dyson–Schwinger equations in  $\mathcal{S}^{\Phi,g}$  onto their corresponding equations in the Hopf algebra of words. According to this approach, we apply the rooted tree representation of the Connes–Marcolli shuffle type renormalization Hopf algebra  $H_{\mathbb{U}}$  and then embed  $H_{\mathbb{U}}$  into an adapted version of the Hopf algebra  $H_{\text{CK}}$  decorated by

a particular Hall set. In this setting, we can address the question about the existence of another filtration on Dyson–Schwinger equations originated from Hopf algebra of words.

Thanks to the explained Hopf algebraic formalism we are ready to formulate a generalization of the BPHZ renormalization machinery for non-perturbative QFT in the context of Feynman graphons which will be discussed in the next part.

## 2.4 *A generalization of the BPHZ renormalization machinery for large Feynman diagrams via Feynman graphons*

In gauge field theories with strong couplings such as QCD, the size of the coupling constant even at rather large values of the exchanged momentum is in the range that the convergence of the perturbative expansion is slow. Although in higher energy levels, the theory enjoys the asymptotic freedom property, several orders of perturbation theory should be concerned to provide a greater accuracy where we need to deal with the evaluation of a large class of higher order Feynman diagrams. We can address the corrections to the quark self-energy as a complicated example in this setting. The situation goes stranger when we deal with QCD in relatively lower energy levels where non-perturbative aspects do appear. This is the level that we need to build a powerful theoretical model for the study of interactions of elementary particles. Thanks to the Hopf algebra structure  $\mathcal{S}_{\text{graphon}}^{\Phi}$ , which encodes Dyson–Schwinger equations of a given physical theory  $\Phi$ , in this part we plan to build a topological Hopf algebraic renormalization program for large Feynman diagrams in the context of the Riemann–Hilbert problem. We describe the construction of the Connes–Kreimer renormalization group at the level of Feynman graphons.

The renormalization Hopf algebra  $H_{\text{FG}}(\Phi)$  of Feynman diagrams of the physical theory  $\Phi$  encodes enough mathematical tools to explain the step by step removal of sub-divergencies. The graded dual of this Hopf algebra identifies an infinite dimensional complex pro-unipotent Lie group denoted by  $\mathbb{G}_{\Phi}(\mathbb{C})$ . Feynman rules, which allow us to replace Feynman diagrams with their corresponding Feynman integrals, are encoded by some elements of  $\mathbb{G}_{\Phi}(\mathbb{C})$ . This Lie group has been applied by Connes and Kreimer to describe perturbative renormalization as a special instance of a general mathematical procedure of multiplicative extraction of finite values in the context of the Riemann–Hilbert problem. According to this approach, the BPHZ perturbative renormalization can be described as the existence of the Birkhoff factorization for loops such as  $\gamma_{\mu}$  defined on an analytic curve  $C \subset \mathbb{C}P^1$  (as

the domain) which has values in  $\mathbb{G}_\Phi(\mathbb{C})$ . It is shown that

$$\gamma_\mu(z) = \gamma_-(z)^{-1} \gamma_{\mu,+}(z) \quad (2.82)$$

such that  $\gamma_{\mu,+}(z)$  is the boundary value of a holomorphic map from the inner domain of  $C$  to the group  $\mathbb{G}_\Phi(\mathbb{C})$  and  $\gamma_-(z)$  is the boundary value of the outer domain of  $C$  to the group  $\mathbb{G}_\Phi(\mathbb{C})$ . In addition,  $\gamma_-$  is normalized by  $\gamma_-(\infty) = 1$ . The renormalized theory is the evaluation of the holomorphic part  $\gamma_{\mu,+}$  of  $\gamma_\mu$  as a product of two holomorphic maps  $\gamma_\pm$  from the connected components  $C_\pm$  of the complement of the circle  $C$  in the Riemann sphere  $\mathbb{C}P^1$ . For Dimensional Regularization, we are interested in an infinitesimal disk around  $z = 0$  while the curve  $C$  is the boundary of this disk. Then we have  $0 \in C_+$  and  $\infty \in C_-$ . [43, 44]

Each regularized Feynman integral  $U^z(\Gamma(p_1, \dots, p_N))$  defines a loop  $\gamma_\mu(z)$  which allows us to lift the analytic formulation of the Birkhoff factorization onto the level of affine group schemes. Set

$$K = \mathbb{C}\{z\}[z^{-1}], \quad O_1 = \mathbb{C}\{z\}, \quad O_2 = \mathbb{C}[z^{-1}]. \quad (2.83)$$

It is shown that each character  $\phi \in \mathbb{G}_\Phi(K)$  has a unique Birkhoff factorization  $\phi = (\phi_- \circ S) * \phi_+$  such that  $\phi_+ \in \mathbb{G}_\Phi(O_1)$ ,  $\phi_- \in \mathbb{G}_\Phi(O_2)$  and  $\varepsilon_- \circ \phi_- = \varepsilon$ . The BPHZ renormalization procedure has been encapsulated in terms of the deformed version of the antipode with respect to the Minimal Subtraction scheme. We have

$$\Gamma \longmapsto S_{R_{ms}}^\phi * \phi(\Gamma) \quad (2.84)$$

such that

$$S_{R_{ms}}^\phi(\Gamma) = -R_{ms}(\phi(\Gamma)) - R_{ms}\left(\sum_{\gamma \subset \Gamma} S_{R_{ms}}^\phi(\gamma) \phi(\Gamma/\gamma)\right). \quad (2.85)$$

Therefore

$$S_{R_{ms}}^\phi * \phi(\Gamma) = \bar{R}(\Gamma) + S_{R_{ms}}^\phi(\Gamma) \quad (2.86)$$

such that

$$\bar{R}(\Gamma) = U_\mu^z(\Gamma) + \sum_{\gamma \subset \Gamma} c(\gamma) U_\mu^z(\Gamma/\gamma) = \phi(\Gamma) + \sum_{\gamma \subset \Gamma} S_{R_{ms}}^\phi(\gamma) \phi(\Gamma/\gamma) \quad (2.87)$$

is the Bogoliubov's operation. For any given Feynman integral  $U_\mu(\Gamma)$ , the mathematical term  $S_{R_{ms}}^\phi(\Gamma)$  generates the counterterm and the mathematical term  $S_{R_{ms}}^\phi * \phi(\Gamma)$  generates the corresponding renormalized value. [43, 44, 65, 78]

Each Dyson–Schwinger equation determines a commutative graded Hopf subalgebra  $H_{\text{DSE}}$  of  $H_{\text{FG}}(\Phi)$  where the morphism (1.14) allows us to have a projection of the complex Lie group  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$  inside  $\mathbb{G}_\Phi(\mathbb{C})$ . The Lie group  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$  has been applied to formulate the foundations of a new differential

Galois theory for the computation of counterterms which contribute to the renormalization of formal expansions in the solutions of Dyson–Schwinger equations. Work on this Lie group has already provided global  $\beta$ -functions with respect to solutions of Dyson–Schwinger equations to relate renormalized values generated via different regularization schemes. The study of Dyson–Schwinger equations in the context of this class of Lie groups have also been lifted onto a categorical setting where the renormalization of any given equation DSE can be encoded by objects of its corresponding neutral Tannakian category  $\text{Rep}_{\mathbb{G}_{\text{DSE}}^*}$ . In this setting, we can compute non-perturbative parameters derived from the equation DSE in terms of a class of equi-singular differential equations organized as a subcategory of the universal Connes–Marcolli category  $\mathcal{E}^{\text{CM}}$  of flat equi-singular vector bundles. [160, 162, 163, 164]

Thanks to Feynman graphon models, it is shown in [169] that the unique solution of any given Dyson–Schwinger equation in  $\mathcal{S}^{\Phi, g}$  is the cut-distance convergent limit of the sequence of its corresponding partial sums. It allows us to embed  $\mathcal{S}^{\Phi, g}$  as a topological sub-vector space in  $\mathcal{S}_{\text{graphon}}^{\Phi}$ . In this part we plan to provide a new interpretation of the renormalization of (large) Feynman diagrams in the context of the Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$  of Feynman graphons. We determine a new class of singular differential equations which contribute in the computation of non-perturbative counterterms corresponding to renormalized Dyson–Schwinger equations.

**Theorem 2.4.1.** *The Hopf–Birkhoff factorization process provides a renormalization program for each large Feynman diagram in  $\mathcal{S}^{\Phi, g}$ .*

*Proof.* We build a renormalization program for Feynman graphons in the Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$  and then we pull back the results to the level of Feynman diagrams and solutions of Dyson–Schwinger equations.

Thanks to Milnor–Moore Theorem ([144]), the commutative graded Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$  (given by Theorem 2.3.8) determines the complex infinite dimensional pro-unipotent Lie group  $\mathbb{G}_{\text{graphon}}^{\Phi}(\mathbb{C})$ . Choose Dimensional Regularization and Minimal Subtraction as the renormalization program where the commutative algebra  $A_{\text{dr}}$  of Laurent series with finite pole parts encodes the regularization scheme and the linear map  $R_{\text{ms}}$ , which projects series onto their pole parts, encodes the renormalization scheme. Set  $\text{Loop}(\mathbb{G}_{\text{graphon}}^{\Phi}(\mathbb{C}), \mu)$  as the space of loops  $\gamma_{\mu}$  on the infinitesimal punctured disk  $\Delta^*$ , around the origin in the complex plane, with values in  $\mathbb{G}_{\text{graphon}}^{\Phi}(\mathbb{C})$ . These loops can represent unrenormalized regularized Feynman rules characters in  $\mathbb{G}_{\text{graphon}}^{\Phi}(\mathbb{C})$  which act on Feynman graphons. The Rota–Baxter property of  $(A_{\text{dr}}, R_{\text{ms}})$  supports the existence of a unique Birkhoff factorization  $(\gamma_-, \gamma_+)$  which can be lifted onto the level of Feynman rules characters of Feynman graphons to achieve the factorization  $(\tilde{\phi}_-, \tilde{\phi}_+)$  for Feynman rules character  $\phi$ . We have

$$\tilde{\phi}^z([W_{\Gamma}]) := \phi^z(\Gamma) \quad (2.88)$$

as the modified version of the regularized Feynman rules character  $\phi^z$  which acts on Feynman graphons.

For a given Feynman graphon  $[W_X]$  corresponding to the unique solution of an equation DSE, set  $[W_{Y_m}]$  (for each  $m$ ) as the unlabeled graphon classes with respect to the partial sums  $Y_m$  of the infinite graph  $X$ . Since formal expansions  $Y_m$ s and their corresponding rooted tree representations are sparse graphs, then we need to apply rescaling methods to obtain non-zero Feynman graphon model for the solution  $X$ . Details about this subject has been discussed in [169, 174]. Now if we apply the renormalization coproduct formulas (2.46), (2.47) on Feynman graphons, the renormalization antipode formulas (2.50), (2.51) on Feynman graphons and also the Hopf algebraic BPHZ process given by (2.85), (2.86), (2.87), then we can build the sequence  $\{S_{R_{ms}}^{\tilde{\phi}}([W_{Y_m}])\}_{m \geq 1}$  of Feynman graphons which is convergent with respect to the cut-distance topology. We have

$$\begin{aligned} S_{R_{ms}}^{\tilde{\phi}}([W_X]) &= \lim_{m \rightarrow \infty} S_{R_{ms}}^{\tilde{\phi}}([W_{Y_m}]) = \lim_{m \rightarrow \infty} \sum_{i=1}^m S_{R_{ms}}^{\tilde{\phi}}([W_{X_i}]) \\ &= \lim_{m \rightarrow \infty} \sum_{i=1}^m -R_{ms}(\tilde{\phi}([W_{X_i}])) - R_{ms}(\sum S_{R_{ms}}^{\tilde{\phi}}([W_{\gamma}])\tilde{\phi}([W_{X_i/\gamma}])). \end{aligned} \quad (2.89)$$

The functional  $S_{R_{ms}}^{\tilde{\phi}}$  is the negative component of the Birkhoff factorization of  $\tilde{\phi}$ . The term  $S_{R_{ms}}^{\tilde{\phi}}([W_X])$  addresses the counterterm with respect to the Feynman graphon  $[W_X]$ . We can also build the sequence  $\{S_{R_{ms}}^{\tilde{\phi}} * \tilde{\phi}([W_{Y_m}])\}_{m \geq 1}$  of Feynman graphons which is convergent with respect to the cut-distance topology. We have

$$S_{R_{ms}}^{\tilde{\phi}} * \tilde{\phi}([W_X]) = \lim_{m \rightarrow \infty} S_{R_{ms}}^{\tilde{\phi}} * \tilde{\phi}([W_{Y_m}]) = \lim_{m \rightarrow \infty} \sum_{i=1}^m S_{R_{ms}}^{\tilde{\phi}} * \tilde{\phi}([W_{X_i}]) \quad (2.90)$$

such that the convolution product  $*$  is defined with respect to the coproduct  $\Delta_{\text{graphon}}$ . The functional  $S_{R_{ms}}^{\tilde{\phi}} * \tilde{\phi}$  is the positive component of the Birkhoff factorization of  $\tilde{\phi}$ . The term  $S_{R_{ms}}^{\tilde{\phi}} * \tilde{\phi}([W_X])$  addresses the renormalized value with respect to the Feynman graphon  $[W_X]$ .  $\square$

Thanks to the filtration parameter on Feynman graphons on the basis of the Hopf algebra of words given by Theorem 2.3.10, it is possible to define a new one-parameter group  $\{\theta_t\}_t$  of automorphisms on  $\mathbb{G}_{\text{graphon}}^{\tilde{\phi}}(\mathbb{C})$  with the infinitesimal generator

$$\left. \frac{d}{dt} \right|_{t=0} \theta_t = Y \quad (2.91)$$

such that  $Y$  sends each Feynman graphon  $[W_\Gamma]$  to its corresponding filtration rank  $n_{[W_\Gamma]}$ . In other words, for each character  $\tilde{\phi}$ ,

$$\langle \theta_t(\tilde{\phi}), [W_\Gamma] \rangle := \langle \tilde{\phi}, \theta_t([W_\Gamma]) \rangle. \quad (2.92)$$

**Lemma 2.4.2.** *Suppose the loop  $\gamma_\mu \in \text{Loop}(\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C}), \mu)$  encodes the regularized unrenormalized Feynman rules character  $\tilde{\phi}$  on Feynman graphons. Then we have*

$$\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)).$$

In addition, the limit

$$F_t = \lim_{z \rightarrow 0} \gamma_-(z) \theta_{tz}(\gamma_-(z)^{-1})$$

defines a new one-parameter subgroup of  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$  such that for each  $t \in \mathbb{R}$ ,

$$\gamma_{e^t\mu^+}(0) = F_t \gamma_{\mu^+}(0).$$

*Proof.* Thanks to the construction of the renormalization Hopf algebra of Feynman graphons (given by Theorem 2.3.8) and Proposition 1.47 in [46], we have the proof.  $\square$

More details about non-perturbative parameters generated by Feynman graphon representations of solutions of Dyson–Schwinger equations have been discussed in [169, 170, 174].

It is possible to lift the Connes–Marcolli geometric approach onto the level of the renormalization Hopf algebra of Feynman graphons. For this purpose we need to adapt the regularization bundle and then classify equi-singular flat connections, which encode counterterms, in terms of the renormalization of Feynman graphons.

**Proposition 2.4.3.** *There exists a bijective correspondence (independent of the choice of a local regular section  $\sigma : \Delta \rightarrow B$ ) between equivalence classes of flat equi-singular  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$ -connections on the regularization bundle and elements of the Lie algebra  $\mathfrak{g}_{\text{graphon}}^\Phi(\mathbb{C})$ .*

*Proof.* The regularization parameter in Dimensional Regularization can be encoded by the punctured version of an infinitesimal disk  $\Delta$  around the origin  $z = 0$ . Set  $P_{\text{graphon}} := (\Delta \times \mathbb{C}^*) \times \mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$  as the trivial principal bundle over the base space  $\Delta \times \mathbb{C}^*$ . Remove the fiber over  $z = 0$  to obtain the bundle  $P_{\text{graphon}}^0 = \Delta \times \mathbb{C}^* - \pi^{-1}(\{0\}) \times \mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$  as the regularization bundle modified with respect to the renormalization Hopf algebra of Feynman graphons.

A flat  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$ -connection  $\varpi$  on  $P_{\text{graphon}}^0$  is a  $\mathfrak{g}_{\text{graphon}}^\Phi(\mathbb{C})$ -valued one form such that  $\mathfrak{g}_{\text{graphon}}^\Phi(\mathbb{C})$  is the Lie algebra of the Lie group  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$  which contains all linear maps  $l : \mathcal{S}_{\text{graphon}}^\Phi \rightarrow \mathbb{C}$  with the property

$$l([W_{\Gamma_1}][W_{\Gamma_2}]) = l([W_{\Gamma_1}])\tilde{\varepsilon}([W_{\Gamma_2}]) + \tilde{\varepsilon}([W_{\Gamma_1}])l([W_{\Gamma_2}]). \quad (2.93)$$

The Lie bracket is given by the formula

$$[l_1, l_2]([W_\Gamma]) = \langle l_1 \otimes l_2 - l_2 \otimes l_1, \Delta_{\text{graphon}}([W_\Gamma]) \rangle. \quad (2.94)$$

The one-parameter group  $\{\theta_t\}_{t \in \mathbb{C}}$  of automorphisms of  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$  can be lifted onto the level of this Lie algebra.

A flat  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$ -connection  $\varpi$  on  $P_{\text{graphon}}^0$  is called equi-singular if it is  $\mathbb{G}_m$ -invariant and for any solution  $f$  of the differential equation  $\mathbf{D}f = \varpi$  with respect to the logarithmic derivative, the restrictions of  $f$  to sections  $\sigma : \Delta \rightarrow \Delta \times \mathbb{C}^*$  have the same type of singularity.

Thanks to [46], the negative component of the Birkhoff factorization of each  $\gamma_\mu \in \text{Loop}(\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C}), \mu)$  determines a unique element  $\beta$  in  $\mathfrak{g}_{\text{graphon}}^\Phi(\mathbb{C})$  where we have

$$\gamma_-(z) = T \exp\left(\frac{-1}{z} \int_0^\infty \theta_{-t}(\beta) dt\right). \quad (2.95)$$

We can show that for each  $\varpi \in \mathfrak{g}_{\text{graphon}}^\Phi(K)$  with the trivial monodromy, there exists a solution  $\tilde{\psi} \in \mathbb{G}_{\text{graphon}}^\Phi(K)$  for the differential equation  $\mathbf{D}\tilde{\psi} = \varpi$ .

Two connections  $\varpi_1, \varpi_2$  with the trivial monodromy are called equivalent if they are gauge conjugate by an element regular at  $z = 0$ . It leads us to show that for equivalent connections  $\varpi_1, \varpi_2$ , the solutions of the differential equations  $\mathbf{D}\tilde{\psi}^{\varpi_1} = \varpi_1$  and  $\mathbf{D}\tilde{\psi}^{\varpi_2} = \varpi_2$  have the same negative components of the Birkhoff factorization, namely,

$$\tilde{\psi}_-^{\varpi_1} = \tilde{\psi}_-^{\varpi_2}. \quad (2.96)$$

Thanks to (2.95), (2.96) and the Connes–Marcolli Classification Theorem (given by Theorem 1.67 in [46]), each element  $\beta \in \mathfrak{g}_{\text{graphon}}^\Phi(\mathbb{C})$  determines a unique class  $\varpi$  of flat equi-singular  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$ -connections on  $P_{\text{graphon}}^0$  in terms of a differential equation with the general form  $\mathbf{D}\gamma_\mu = \varpi$  such that

$$\gamma_\mu(z, v) = T \exp\left(\frac{-1}{z} \int_0^v u^Y(\beta) \frac{du}{u}\right) \quad (2.97)$$

where  $u = tv$ ,  $t \in [0, 1]$  and  $u^Y$  is the action of  $\mathbb{G}_m$  on  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$ .  $\square$

**Proposition 2.4.4.** *The category  $\mathcal{E}^{\text{CM}}$  encodes the renormalization group corresponding to the BPHZ renormalization of large Feynman diagrams.*

*Proof.* Thanks to Proposition 2.4.3, for the Hopf algebra  $\mathcal{S}_{\text{graphon}}^\Phi$ , we can determine a family of flat equi-singular  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$ -connections which encode counterterms on the basis of the  $\beta$ -functions. Thanks to [46], these geometric objects form a new category  $\mathcal{E}_{\text{graphon}}^\Phi$  which is recovered by the category  $\text{Rep}_{\mathbb{G}_{\text{graphon}}^{\Phi,*}}$  of finite dimensional representations of the affine group scheme  $\mathbb{G}_{\text{graphon}}^{\Phi,*}$ . In addition, the renormalization Hopf algebra  $H_{\text{FG}}(\Phi)$  of Feynman diagrams of the physical theory  $\Phi$  determines the category  $\mathcal{E}^\Phi$

of geometric objects recovered by the category  $\text{Rep}_{\mathbb{G}_{\Phi}^*}$  of finite dimensional representations of the affine group scheme  $\mathbb{G}_{\Phi}^*$ . Thanks to the explained categorical formalism in [46], we can embed  $\text{Rep}_{\mathbb{G}_{\Phi}^*}$  as a sub-category into  $\mathcal{E}^{\text{CM}}$ . It is shown that  $\mathcal{E}^{\text{CM}}$  is isomorphic to the category  $\text{Rep}_{\mathbb{U}^*}$  such that the complex Lie group  $\mathbb{U}(\mathbb{C})$  can be described in terms of the Lie algebra  $L_{\mathbb{U}}$  generated by elements  $e_{-n}$  of degree  $-n$  for each  $n > 0$  such that the sum  $e = \sum e_{-n}$  is an element of this Lie algebra. We can lift  $e$  onto the morphism  $\text{rg} : \mathbb{G}_a \rightarrow \mathbb{U}$ . The universality of  $\mathcal{E}^{\text{CM}}$  supports the existence of a new class of graded representations such as

$$\vartheta : \mathbb{U}(\mathbb{C}) \rightarrow \mathbb{G}_{\text{graphon}}^{\Phi}(\mathbb{C}). \quad (2.98)$$

Now the composition  $\vartheta \circ \text{rg}$  determines the renormalization group  $\{F_t\}_{t \in \mathbb{C}}$  at the level of Feynman graphons (i.e. Lemma 2.4.2).  $\square$

Lemma 2.3.3 and Theorem 2.3.8 enable us to embed  $H_{\text{FG}}(\Phi)$  into  $\mathcal{S}_{\text{graphon}}^{\Phi}$  which leads us to define an epimorphism of affine group schemes from  $\mathbb{G}_{\text{graphon}}^{\Phi,*}$  to  $\mathbb{G}_{\Phi}^*$ . In addition, the renormalization Hopf algebra of Feynman graphons includes solutions of all Dyson–Schwinger equations in the physical theory  $\Phi$ . Therefore the category  $\mathcal{E}_{\text{graphon}}^{\Phi}$  is rich enough to recover the category  $\mathcal{E}^{\Phi}$  and also, non-perturbative information of the physical theory.

As the summary, we have shown that the renormalization topological Hopf algebra of Feynman graphons is capable to encode the renormalization of Feynman diagrams and solutions of Dyson–Schwinger equations. We have also embedded this graphon model of renormalization into the universal Connes–Marcolli categorical setting where as the result, we can study Feynman graphons under the differential Galois theory. In final, these achievements suggest the existence of a new unexplored interconnection between motivic renormalization and Dyson–Schwinger equations in the context of the theory of graphons.

## Chapter 3

# Non-perturbative computational complexity

- *A parametric representation for large Feynman diagrams*
- *The optimization of non-perturbative complexity via a multi-scale Renormalization Group*
  - *A Renormalization Group program on  $S^{\Phi,g}$*
  - *Kolmogorov complexity of Dyson–Schwinger equations*

The original motivation for the introduction of Feynman graphons is to clarify and study infinities originated from non-perturbative aspects in Quantum Field Theory with strong couplings via a new topological Hopf algebraic renormalization program. From a physicist's perspective, these infinities do not acceptable and applying some approximation methods are useful for the production of some intermediate values such as running couplings, N large methods, etc. Then the Physics of elementary particles and its phenomenology shall be described in terms of those approximations. From a mathematician's perspective, we have a different story where it is possible to deal with infinities under different settings instead of only removing them. The Cartier's cosmic Galois group as a universal group of symmetries is useful for the analysis of divergencies in perturbative gauge field theories to generate some new data about the behavior of these physical theories in higher order perturbation terms. The motivic Galois group associated to the renormalization Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$  of Feynman graphons is useful to analyze divergencies originated from solutions of Dyson–Schwinger equations in the language of singular differential equations on cut-distance topological regions of Feynman diagrams. This motivic Galois group could be a practical candidate to encode non-perturbative gauge field theories. In this chapter we plan to show some new applications of Feynman graphon models in dealing with the non-perturbative computations of physical parameters derived from Dyson–Schwinger equations. Here we focus on finding a new parametric representation in the context of Tutte polynomials and other combinatorial polynomials for the description of solutions of Dyson–Schwinger equations. This new combinatorial setting allows us to understand the graph complexity of large Feynman diagrams in terms of the graph complexities of partial sums. Then we explain the elementary foundations of the concept of Kolmogorov complexity on the space of all Dyson–Schwinger equations of a given strongly coupled gauge field theory. We define this complexity in terms of a new multi-scale non-commutative non-perturbative Renormalization Group on  $\mathcal{S}^{\Phi, g}$  which governs the changing the scales of the bare coupling constant  $g$  and running coupling constants. This platform is useful to provide a new algorithm for the study of Dyson–Schwinger equations under strong couplings in terms of sequences of Dyson–Schwinger equations with lesser complexities. We try to show that this Renormalization Group machinery can optimize the complexity of non-perturbative computations. We show how this notion of complexity is related to the Halting problem in the Theory of Computation. This study suggests a new contextualization for the description of non-perturbative situations and their complexity.

### 3.1 *A parametric representation for large Feynman diagrams*

The original task in Quantum Field Theory is to compute correlation functions (i.e. Green's functions) in a (non-)perturbative expansion setting whose terms are decorated by Feynman diagrams. Each term in this class of expansions consists of a multiple ill-defined integral such that the integrand is codified by the combinatorial information of its corresponding Feynman diagram. Generally speaking, we can work in momentum space of  $D$  dimensions such that a preliminary count of the powers of the momenta in the integrands can report the possible superficially divergence in the integral. In this situation, the renormalization program associates a counterterm to each superficially divergent subgraph to finally produce a finite result by subtraction treatment. All superficially divergent Feynman subgraphs should be considered under a recursive setting to assign a final finite value to the full Feynman diagram. Studying Feynman diagrams via tree representations enables us to formulate perturbative renormalization theory under a simplified universal setting. Furthermore, it provides also a combinatorial reformulation of Dyson–Schwinger equations where we can study solutions of these non-perturbative type of equations in the context of partial sums of decorated non-planar rooted trees. [19, 73, 74, 110, 115, 187, 200]

In the previous sections we have shown that the unique solution of each Dyson–Schwinger equation is described as the convergent limit of a sequence of Feynman graphons with respect to the cut-distance topology. This formalism has been applied to lift the BPHZ renormalization program onto the level of large Feynman graphs to generate some new expressions for the description of counterterms and renormalized values associated to fixed point equations of Green's functions. Using graph polynomials for the study of Feynman integrals has played an important role in the computational processes where this class of combinatorial polynomials can bring some powerful algorithms for the analysis of the behavior of these divergent integrals ([8, 21, 101, 102, 119, 135, 148, 185, 193]). In this section we show another application of this graphon representation of non-perturbative parameters where we deal with the concept of parametric representation of large Feynman diagrams. We study solutions of Dyson–Schwinger equations in the language of Tutte polynomial and Kirchhoff–Symanzik polynomials.

The Tutte polynomial, as a two variables graph polynomial, enjoys a universal property which enables us to evaluate any multiplicative graph invariant with a deletion/contraction reduction machinery [181, 182, 183, 184, 191]. This fundamental property provides the opportunity to demonstrate how graph polynomials can be specialized or generalized. The Aluffi–Marcolli approach has clarified the practical importance of Tutte polynomials in dealing with Feynman rules characters and Feynman integrals under

an algebro-geometric setting where a motivic perspective on perturbative renormalization program has been formulated very nicely. [7, 8, 134, 135]

We first review the basic structure of Tutte polynomials on finite graphs, its different reformulations and its universal property ([135, 182, 183, 184, 191]) and then we explain graph polynomials which can contribute to the combinatorial representations of Feynman graphons of large Feynman diagrams.

A given (finite) graph  $G$  has a set  $V(G)$  of vertices and a set  $E(G)$  of edges. Graphs  $G_1, G_2$  are isomorphic when there exists a bijection such as  $\rho$  between the sets  $V(G_1)$  and  $V(G_2)$  such that for each edge  $uv$  in  $G_1$ ,  $\rho(uv)$  is an edge in  $G_2$  and vice versa. Subsets of the set of vertices or the set of edges can give us subgraphs. For any subset  $A \subset E(G)$  of edges, the rank  $r(A)$  and the nullity  $n(A)$  are defined by the relations

$$r(A) := |V(G)| - \kappa(A), \quad n(A) := |A| - r(A) \quad (3.1)$$

such that  $\kappa(A)$  is the number of connected components of the graph. In general, finite graphs can be classified in terms of their numbers of non-trivial connected components. A graph is called  $n$ -connected, if we should remove at least  $n$  edges from the graph to obtain a disconnected graph. Rooted trees, as fundamental tools for us, are connected graphs which have no cycles or loops. They are non-trivial connected components of forests as more complicated graphs. Sometimes working on subgraphs of a given complicated (finite) graph enables us to clarify some fundamental properties of the original graph. Spanning subgraphs are applied as one important class of subgraphs for the construction of graph polynomials. A spanning subgraph covers all vertices of the original graph with the optimum number of edges.

The notion of "dual" in Graph Theory enables us to build the algebraic combinatorics of graphs. If we can embed a graph into the plane without any crossing in edges, then the graph is called planar. Each planar graph can separate the plane into regions known as faces. Faces are key tools for the construction of the dual of a graph. For a given planar graph  $G$ , its corresponding connected dual graph is built by assigning a vertex to each face where there exist  $m$  edges between two vertices in the dual graph if the corresponding faces of the original graph have  $m$  edges in their boundaries. We denote  $G^*$  as the dual of the connected planar graph  $G$  and it can be seen that

$$(G^*)^* = G. \quad (3.2)$$

There are two fundamental commutative operations on graphs namely, deletion and contraction which enable us to build the algebraic combinatorics of graphs. For a given finite graph  $G$ , we can build a new graph  $G \setminus e$  as the result of deleting an edge  $e \in E(G)$ . This new graph has the same set of vertices  $V(G)$  and the set of edges  $E(G) - \{e\}$ . We can also build

another new graph  $G/e$  as the result of contracting an edge  $e$  in terms of identifying the endpoints of the edge  $e$  by shrinking this edge. It is easy to check that the deletion and the insertion on a self-loop edge determine the same resulting graph.

**Lemma 3.1.1.** (i) For any given different edges  $e_1, e_2$  of a given planar graph  $G$ , the graph  $(G \setminus e_1)/e_2$  is isomorphic to the graph  $(G/e_2) \setminus e_1$ .

(ii) A planar graph and its dual have the same numbers of spanning trees.

(iii) The rank of a dual graph is well-defined. [182, 191]

The deletion or contraction of an edge determines a minor of a graph. In more general setting, if a graph  $H$  is isomorphic to  $G \setminus A_1/A_2$  for some choice of disjoint subsets  $A_1, A_2$  of  $E(G)$ , then it is called a minor graph. In this setting, a class of graphs is called minor closed if whenever the graph  $G$  is in the class, then any minor of  $G$  is also in the class.

Graph invariants are useful tools for the characterization of graphs in terms of some particular properties. A graph invariant is a function on the class of all graphs such that it has the same output on isomorphic graphs. Graph polynomials (such as Tutte polynomials) are indeed some graph invariants such that their images belong to some polynomial rings.

There are several different (but equivalent) (re)formulations for Tutte polynomials in terms of rank–nullity generating function method, linear recursion machinery and spanning tree expansion method which was originally applied by Tutte. The linear recursion form can be described as a collection of reduction rules to rewrite a graph as a weighted formal sum of graphs that are less complicated than the original graph. This method is useful to identify a collection of simplest or irreducible graphs. [135, 182, 183, 184, 191]

**Definition 3.1.2.** The Tutte polynomial  $T(G; x, y)$  of a given (finite) graph  $G$  is a two variables polynomial with respect to the independent variables  $x, y$  which is defined in terms of the following recursive machinery:

- If  $G$  has no edge, then  $T(G; x, y) = 1$ ; otherwise, for any edge  $e \in E(G)$ ,
- $T(G; x, y) = T(G \setminus e; x, y) + T(G/e; x, y)$ ,
- $T(G; x, y) = xT(G/e; x, y)$ , if  $e$  is a coloop,
- $T(G; x, y) = yT(G \setminus e; x, y)$ , if  $e$  is a loop.

We can see that if  $G$  has  $i$  bridges and  $j$  loops, then its corresponding Tutte polynomial is given by

$$T(G; x, y) = x^i y^j. \quad (3.3)$$

In addition, Definition 3.1.2 shows us that the Tutte polynomial of the disjoint union of finite number of graphs can be defined in terms of the Tutte polynomials of each graph in the union. In other words,

$$T(G_1 \sqcup G_2 \sqcup \dots \sqcup G_n; x, y) = T(G_1; x, y) \dots T(G_n; x, y). \quad (3.4)$$

We can redefine Tutte polynomials in the language of the rank–nullity generating functions. They are (infinite) polynomials with coefficients which can count structures which are encoded by the exponents of variables. In this setting we have

$$T(G; x, y) = \sum_{A \subset E(G)} (x-1)^{r(E(G))-r(A)} (y-1)^{n(A)}. \quad (3.5)$$

*Remark 3.1.3.* The Tutte polynomials of a planar graph  $G$  and its dual graph  $G^*$  can determine each other. This means that

$$T(G; x, y) = T(G^*; y, x). \quad (3.6)$$

We can also redefine Tutte polynomials in terms of spanning trees. In this setting, we need to define a total order  $\prec$  on the set of edges  $E(G) = \{v_1, \dots, v_n\}$  of a given graph  $G$  such as

$$v_i \prec v_j \leftrightarrow i > j. \quad (3.7)$$

For a given tree  $t$ , an edge  $e$  is called internally active if  $e$  is an edge of  $t$  and it is the smallest edge in the cut defined by  $e$ . We can lift this concept onto the dual level where an edge  $u$  is called externally active if  $u \notin t$  and it is the smallest edge in the cycle defined by  $u$ . Now the Tutte polynomial of the totally ordered graph  $G$  can be defined (independent of the chosen total order) by the formal expansion

$$T(G; x, y) = \sum_{i,j} t_{ij} x^i y^j \quad (3.8)$$

such that  $t_{ij}$  counts spanning trees with internal activity  $i$  and external activity  $j$ . [8, 101, 135, 181, 182, 191]

The most fundamental property of the Tutte polynomial is its universality under a graph invariant setting. This means that any multiplicative graph invariant on disjoint unions and one-point joins of graphs which is formulated via a deletion/contraction reduction can be described as an evaluation of the Tutte polynomial. There are different notions for the generalization of the Tutte polynomials and here we address the one which is useful for us. [8, 135]

**Definition 3.1.4.** Let  $\mathfrak{G}$  be the set of isomorphism classes of finite graphs. A graph invariant  $F$  from  $\mathfrak{G}$  to a commutative ring such as the polynomial ring  $\mathbb{C}[\alpha, \beta, \eta, x, y]$  is called Tutte–Grothendieck invariant of graphs, if it has the following properties:

- $F(G) = \eta^{\# V(G)}$  if the set of edges is empty,
- $F(G) = xF(G/e)$  if the edge  $e \in E(G)$  is a bridge,

- $F(G) = yF(G \setminus e)$  if the edge  $e \in E(G)$  is a looping edge,
- For any ordinary edge, which is not a bridge nor a looping edge,

$$F(G) = \alpha F(G/e) + \beta F(G \setminus e), \quad (3.9)$$

- For every  $G, H \in \mathfrak{G}$ , if  $G \cup H \in \mathfrak{G}$  or  $G \bullet H \in \mathfrak{G}$ , then  $F(G \cup H) = F(G)F(H)$  and  $F(G \bullet H) = F(G)F(H)$  such that the one-point join  $G \bullet H$  is defined by identifying a vertex of  $G$  and a vertex of  $H$  into a new single vertex of  $G \bullet H$ .

The induction machinery can show that

$$T(G \bullet H) = T(G)T(H), \quad (3.10)$$

which means that the Tutte polynomial does not distinguish between the one-point join of two graphs and their disjoint union.

The Tutte polynomial is a special version of the Tutte–Grothendieck invariant which is independent of the choice of any ordering of edges of the graph. We can show that for any given map  $f : \mathfrak{G} \rightarrow R$ , if there exist  $a, b \in R$  such that  $f$  is another Tutte–Grothendieck invariant, then  $f$  can be presented in terms of the Tutte polynomial such that we have

$$f(G) = a^{|E(G)|-r(E(G))} b^{r(E(G))} T(G; \frac{x_0}{b}, \frac{y_0}{a}). \quad (3.11)$$

Here we want to apply Feynman graphon representations of Feynman diagrams in a given gauge field theory to build a new class of Tutte polynomials which contribute to the combinatorial presentations of solutions of Dyson–Schwinger equations. Our idea is to implement an efficient algorithm for the computation of the Tutte polynomial  $T(X_{\text{DSE}}; x, y)$  of the unique solution of a given Dyson–Schwinger equation DSE in terms of handling intermediate graphs (i.e. partial sums) and their corresponding Tutte polynomials to avoid unnecessary recomputations.

**Theorem 3.1.5.** *There exists a new class of Tutte polynomials with respect to large Feynman diagrams.*

*Proof.* We work on the unique solution  $X_{\text{DSE}} = \sum_{n \geq 0} X_n$  of a given Dyson–Schwinger equation DSE under the coupling constant  $\lambda g = 1$ . Thanks to Theorem 2.3.7, the sequence  $\{Y_m\}_{m \geq 1}$  of partial sums is convergent to  $X_{\text{DSE}}$  with respect to the cut-distance topology.

Thanks to Definition 3.1.2 and the formula (3.8), for each  $m \geq 1$ , the Tutte polynomial  $T(Y_m; x, y)$  of the finite disjoint union graph  $Y_m := \mathbb{I} + X_1 + \dots + X_m$  is defined in terms of the Tutte polynomials of the components  $X_k$ s. We have

$$T(Y_m; x, y) = \prod_{k=1}^m T(X_k; x, y) = \prod_{k=1}^m \sum_{i_k, j_k} t_{i_k j_k} x^{i_k} y^{j_k} \quad (3.12)$$

such that  $t_{i_k j_k}$  is the number of spanning trees in  $X_k$  with internal activity  $i_k$  and external activity  $j_k$ .

We know that  $\lim_{m \rightarrow \infty} Y_m = X_{\text{DSE}}$  with respect to the cut-distance topology. It means that for each  $\epsilon > 0$ , there exists  $N_\epsilon$  such that for each  $m_1, m_2 \geq N_\epsilon$ , we have

$$d(Y_{m_1}, Y_{m_2}) = d_{\text{cut}}([W_{Y_{m_1}}], [W_{Y_{m_2}}]) < \epsilon. \quad (3.13)$$

Therefore

$$d_{\text{cut}}([W_{Y_{m_1}}], [W_{Y_{m_2}}]) = 0 \Leftrightarrow [W_{Y_{m_1}}] \approx [W_{Y_{m_2}}]. \quad (3.14)$$

For each  $m$ , the Feynman graphon class  $[W_{Y_m}]$  is determined in terms of the rooted tree representations of Feynman diagrams  $X_1, \dots, X_m$  where decorated rooted trees (or forests)  $t_{X_1}, \dots, t_{X_m}$  are the only spanning trees (or forests) in themselves. Thanks to the relation (3.14), for enough large orders, unlabeled graphon classes corresponding to partial sums are going to be weakly isomorphic while they converge to the unique Feynman graphon class  $[W_{X_{\text{DSE}}}]$ . It means that spanning forests of partial sums for enough large orders tend to the spanning forest  $t_{X_{\text{DSE}}}$  of the unique graph limit  $X_{\text{DSE}}$ .

In addition, the Tutte polynomial for each arbitrary rooted tree  $t$  is given by

$$T(t; x, y) = \sum_{s \in R(t)} x^{|E(s)|} (y+1)^{|E(s)| - |L(s)|} \quad (3.15)$$

such that  $R(t)$  is the set of all subtrees of  $t$ ,  $|E(s)|$  is the number of edges of a subtree  $s$  and  $|L(s)|$  is the number of leaves of a subtree  $s$ .

Now for the collection  $\{\prod_{k=1}^m T(t_{X_k}; x, y)\}_{m \geq 1}$  of Tutte polynomials, we can define a collection  $\{p_m : \prod_{k=1}^\infty T(t_{X_k}; x, y) \rightarrow \prod_{k=1}^m T(t_{X_k}; x, y)\}_{m \geq 1}$  of projections. Thanks to the universal property of the Tutte polynomial, for any graph invariant  $T$  (which enjoys the properties in Definition 3.1.2) together with the collection  $\{f_m : T \rightarrow \prod_{k=1}^m T(t_{X_k}; x, y)\}_{m \geq 1}$ , we can define the unique map

$$F : T \rightarrow \prod_{k=1}^{\infty} T(t_{X_k}; x, y) \quad (3.16)$$

such that  $f_m = p_m \circ F$ . As the consequence, we can consider the direct product  $\prod_{k=1}^{\infty} T(t_{X_k}; x, y)$  as the Tutte polynomial for the infinite tree (or forest)  $t_{X_{\text{DSE}}}$ .

If we replace rooted tree representations with the original Feynman diagrams, then we can build the Tutte polynomial for the large Feynman diagram  $X_{\text{DSE}}$  in terms of the direct product over the Tutte polynomials for simpler finite graphs (i.e. partial sums)  $\{T(X_k; x, y)\}_{k \geq 1}$ . We have

$$T(X_{\text{DSE}}; x, y) = \prod_{k=1}^{\infty} T(X_k; x, y). \quad (3.17)$$

□

We can address here some interesting applications of this class of Tutte polynomials in dealing with the complexity of non-perturbative parameters.

As the first application, it is possible to describe the complexity of a large Feynman diagram  $X_{\text{DSE}}$  in terms of the complexity of finite Feynman diagrams which live in partial sums  $Y_m$ , ( $m \geq 1$ ). The complexity of  $Y_m$  is interpreted in terms of the number of different spanning trees (or forests) which live in the graph. We can compute the complexity of  $Y_m$  under a recursive algorithm where at each stage of the algorithm, only an edge belonging to the proper cycle is chosen. The algorithm starts with a given graph and produces two graphs at the end of the first stage. By applying the elementary contraction to a multiple edge, the resulting graph can have a loop and therefore the procedure can be still continued. At each subsequent stage one proper cyclic edge from each graph is chosen (if it exists) for applying the recurrence. On termination of the algorithm, we get a set of graphs (or general graphs) none of which have a proper cycle. The complexity of  $Y_m$  is the sum of the number of these graphs. If we perform this recursive algorithm for each  $Y_m$  when  $m$  tends to infinity, then we can get a sequence which presents the behavior of complexities when the partial sums converge to  $X_{\text{DSE}}$ .

As the second application, we can interpret Feynman rules characters of the renormalization Hopf algebra of Feynman graphons in the context of deletion and contraction operators. This approach leads us to formulate a universal motivic Feynman rule character on large Feynman diagrams.

**Corollary 3.1.6.** *The Tutte polynomial invariant defines an abstract version of Feynman rules characters on the renormalization Hopf algebra of Feynman graphons.*

*Proof.* Feynman graphon classes in  $\mathcal{S}_{\text{graphon}}^\Phi$  can recover finite Feynman diagrams and their finite or infinite formal expansions which contribute to Dyson–Schwinger equations in the physical theory  $\Phi$ .

For each unlabeled graphon class  $[W_\Gamma]$  corresponding to a finite Feynman diagram  $\Gamma$ , we can define its corresponding Tutte polynomial  $T([W_\Gamma]; x, y)$  via

$$T([W_\Gamma]; x, y) := T(\Gamma; x, y). \quad (3.18)$$

Thanks to Proposition 2.2 in [8], the Tutte polynomial is multiplicative over disjoint unions of finite (Feynman) diagrams. To see this property requires to describe each connected Feynman diagram  $\Gamma$  as a tree  $t_\Gamma$  with 1PI graphs inserted at the vertices of that tree. Then we can compute the Tutte polynomials of the resulting trees (i.e. formula (3.15)) to show that the Tutte polynomial of the disjoint union of Feynman graphons  $[W_{\Gamma_1}], [W_{\Gamma_2}]$  corresponding to finite Feynman diagrams  $\Gamma_1, \Gamma_2$  can be determined by the

Tutte polynomial of the disjoint union of decorated trees  $t_{\Gamma_1}$  and  $t_{\Gamma_2}$  which is multiplicative. So we have

$$\begin{aligned} T([W_{\Gamma_1}] \sqcup [W_{\Gamma_2}]; x, y) &= T(\Gamma_1 \sqcup \Gamma_2; x, y) \quad (3.19) \\ &= \sum_{s=(s_1, s_2)} (x-1)^{b_0(s_1)+b_0(s_2)-b_0(\Gamma_1 \sqcup \Gamma_2)} (y-1)^{b_1(s_1)+b_1(s_2)} \\ &= T(\Gamma_1; x, y) T(\Gamma_2; x, y) = T([W_{\Gamma_1}]; x, y) T([W_{\Gamma_2}]; x, y) \end{aligned}$$

such that the sum is taken over all pairs  $s = (s_1, s_2)$  of subgraphs of  $\Gamma_1$  and  $\Gamma_2$ , respectively where  $V(s_i) \subseteq V(\Gamma_i)$ ,  $E(s_i) \subset E(\Gamma_i)$ ,  $b_0(s) = b_0(s_1) + b_0(s_2)$ .

Furthermore, for a finite connected Feynman diagram  $\Gamma$ , we have

$$\Gamma = \bigcup_{v \in V(t_\Gamma)} \Gamma_v \quad (3.20)$$

such that  $\Gamma_v$ s are 1PI Feynman diagrams inserted at the vertices of the tree  $t_\Gamma$ . The internal edges of the tree  $t_\Gamma$  are all bridges in the resulting graph and thus

$$T(\Gamma; x, y) = x^{|E_{\text{int}}(t_\Gamma)|} T(\Gamma / \cup_{e \in E_{\text{int}}(t_\Gamma)} e; x, y). \quad (3.21)$$

It is possible to lift this property onto the level of Feynman graphons where the decomposition (3.20) can be described by the disjoint unions of Feynman graphons. In other words, for each  $v_1, \dots, v_r \in V(t_\Gamma)$ , set  $[W_{\Gamma_v}]$  as the unlabeled Feynman graphon class with respect to the graph  $\Gamma_v$  such that

$$[W_\Gamma] = [W_{\Gamma_{v_1}}] \sqcup \dots \sqcup [W_{\Gamma_{v_r}}]. \quad (3.22)$$

It means that the Feynman graphon  $[W_\Gamma]$  can be defined in terms of a rescaled version of the linear combination of Feynman graphons  $[W_{\Gamma_{v_j}}]$ . In other words,

$$W_{\Gamma_{v_1} \sqcup \dots \sqcup \Gamma_{v_r}} = \frac{\sum_{j=1}^r W_{\Gamma_{v_j}}}{|\sum_{j=1}^r W_{\Gamma_{v_j}}|}. \quad (3.23)$$

Thanks to (3.18), the Tutte polynomial of each  $[W_{\Gamma_{v_j}}]$  is defined in terms of the Tutte polynomial of the graph  $\Gamma_{v_j}$ . Then we have

$$T([W_\Gamma]; x, y) = \prod_{j=1}^r T([W_{\Gamma_{v_j}}]; x, y) \quad (3.24)$$

which leads us to a Feynman graphon version of the relation (3.21).

Theorem 3.1.5 describes the Tutte polynomial of a large Feynman diagram  $X_{\text{DSE}}$  on the basis of the Tutte polynomials of the partial sums  $\{Y_m\}_{m \geq 1}$ . The cut-distance convergence of the sequence of partial sums to  $X_{\text{DSE}}$  and the universality of the Tutte polynomial enable us to lift the properties (3.19) and (3.21) onto the Feynman graphon  $[W_{X_{\text{DSE}}}]$ . It allows us

to define the abstract Feynman rules characters on large Feynman diagrams in terms of the Tutte polynomial where we have

$$\tilde{U}([W_{X_{\text{DSE}}}] := T([W_{X_{\text{DSE}}}] ; x, y) = T(X_{\text{DSE}} ; x, y). \quad (3.25)$$

□

Now we explain the construction of another important class of combinatorial polynomials namely, the first Kirchhoff–Symanzik polynomials for large Feynman diagrams.

The Feynman parametric representation of a given Feynman integral  $U(\Gamma)$  can be described by the integration theory over a topological simplex such as  $\sigma_n$  with respect to Feynman parameters  $w = (w_1, \dots, w_n) \in \sigma_n$  such that  $n$  is the number of internal edges of the corresponding Feynman diagram  $\Gamma$ . If  $l = b_1(\Gamma)$  be the first Betti number of  $\Gamma$  (as the maximum number of independent loops in the graph) and an orientation were fixed on the graph, then we can define the circuit matrix  $\hat{\eta} = (\eta_{ik})_{ik}$  such that  $e_i \in E(\Gamma)$  and  $k$  ranges over the chosen basis of loops. If an edge  $e_i$  belongs to a loop  $l_k$  with the same/reverse orientations, then  $\eta_{ik} = 1$  and  $\eta_{ik} = -1$ , respectively. If the edge  $e_i$  does not belong to a loop  $l_k$ , then  $\eta_{ik} = 0$ . The arrays of the corresponding  $l \times l$  Kirchhoff–Symanzik matrix  $M_\Gamma(w)$  are given by

$$(M_\Gamma(w))_{kr} = \sum_{i=1}^n w_i \eta_{ik} \eta_{ir} \quad (3.26)$$

which defines a function  $M_\Gamma : \mathbb{A}^n \longrightarrow \mathbb{A}^{l^2}$ ,  $w = (w_1, \dots, w_n) \longmapsto M_\Gamma(w)$  over higher dimensional affine spaces. The first Kirchhoff–Symanzik polynomial of the graph  $\Gamma$  is then defined by the equation

$$\Psi_\Gamma(w) = \det(M_\Gamma(w)) \quad (3.27)$$

which is independent of the choice of an orientation on the graph and the basis of loops. This function on  $\mathbb{A}^n$ , which is a homogeneous polynomial of degree  $l$ , can be formulated in the language of spanning trees. We have

$$\Psi_\Gamma(w) = \sum_{T \subset \Gamma} \prod_{e \notin E(T)} w_e \quad (3.28)$$

such that the sum is over all spanning trees  $T$  of the graph  $\Gamma$  and for each spanning tree, the product is over all edges of  $\Gamma$  that are not in the selected spanning tree. We can show that this product is multiplicative over connected components.

Now consider a large Feynman diagram  $X$  with the corresponding sequence  $\{Y_m\}_{m \geq 1}$  of partial sums. For each  $m$ , we know that the first

Kirchhoff–Symanzik polynomial of  $Y_m$  is the product of the polynomials of each of its components which means that

$$\Psi_{Y_m}(w) = \prod_{j=1}^m \Psi_{X_j} \quad (3.29)$$

where

$$\Psi_{X_j}(w) = \sum_{T_j \subset X_j} \prod_{e \notin E(T_j)} w_e \quad (3.30)$$

such that the sum is taken over all the spanning forests  $T_j$  of  $X_j$  and for each spanning forest, the product is taken over all edges of  $X_j$  that are not in that spanning forest.

Thanks to the cut-distance convergence of the sequence  $\{Y_m\}_{m \geq 1}$  to  $X$ , for each  $\epsilon > 0$ , there exists  $N_\epsilon$  such that for each  $m_1, m_2 \geq N_\epsilon$ , we have

$$d(Y_{m_1}, Y_{m_2}) = d_{\text{cut}}([W_{Y_{m_1}}], [W_{Y_{m_2}}]) < \epsilon. \quad (3.31)$$

Therefore

$$d_{\text{cut}}([W_{Y_{m_1}}], [W_{Y_{m_2}}]) = 0 \Leftrightarrow [W_{Y_{m_1}}] \approx [W_{Y_{m_2}}], \quad (3.32)$$

which means that for enough large  $m$ , spanning forests of  $Y_m$  tend to the spanning forests of the unique graph limit  $X$ .

**Definition 3.1.7.** The first Kirchhoff–Symanzik polynomial  $\Psi_X(w)$  of the large Feynman diagram  $X$  is defined as the convergent limit of the sequence  $\{\Psi_{Y_m}(w)\}_{m \geq 1}$  of the first Kirchhoff–Symanzik polynomials of finite graphs  $Y_m = X_1 \sqcup \dots \sqcup X_m = X_1 + \dots + X_m$  with respect to the cut-distance topology.

We can present this polynomial by the expansion

$$\Psi_X(w) = \prod_{j=1}^{\infty} \Psi_{X_j} = \sum_{T \subset X} \prod_{e \notin E(T)} w_e \quad (3.33)$$

such that the sum is taken over all the spanning forests  $T$  of  $X$  and for each spanning forest, the product is taken over all edges of  $X$  that are not in that spanning forest.

**Lemma 3.1.8.** *The first Kirchhoff–Symanzik polynomial  $\Psi_X(w)$  of the large Feynman diagram  $X$  can be defined recursively in terms of the deletion and the contraction operators.*

*Proof.* Set

$$F := \frac{\partial \Psi_X}{\partial w_n} = \Psi_X \setminus e \quad (3.34)$$

as the deletion operator which is the result of deleting the edge  $e = e_n$  from the original graph. In addition, set

$$G := \Psi_X|_{w_n=0} = \Psi_X / e \quad (3.35)$$

as the contraction operator which is the result of contracting the edge  $e = e_n$  to a point in the original graph.

For each edge  $e$  which is not a bridge or self-loop in the large Feynman diagram  $X$ , we can show that

$$\Psi_X = w_e F + G \tag{3.36}$$

such that  $w_e F$  collects the monomials corresponding to spanning forests that do not include  $e$ .  $\square$

At the end of this section, we address a new application of the first Kirchhoff–Symanzik polynomial for the study of polynomial invariants of large Feynman diagrams and Feynman rules characters which act on Feynman graphons.

For a given large Feynman diagram  $X$  with the corresponding first Kirchhoff–Symanzik polynomial  $\Psi_X(w)$ , define

$$\hat{V}_X = \{w \in \mathbb{A}^\infty := \prod_{i=1}^\infty \mathbb{A}^{n_i} : \Psi_X(w) = 0\} \tag{3.37}$$

such that the affine hypersurface complement  $\mathbb{A}^\infty \setminus \hat{V}_X$  enjoys the multiplicative property. We have

$$\mathbb{A}^\infty \setminus \hat{V}_X = \prod_{i=1}^\infty \mathbb{A}^{n_i} \setminus \hat{V}_{X_i} \tag{3.38}$$

such that  $n_i$  is the number of internal edges of the Feynman diagram  $X_i$ .

Consider the Grothendieck ring  $\mathcal{F}$  of immersed conical varieties generated by the equivalence classes  $[\hat{V}]$  up to linear changes of coordinates of varieties  $\hat{V} \subset \mathbb{A}^\infty$  embedded into some affine space. These varieties are defined in terms of homogeneous ideals with the usual inclusion–exclusion relation

$$[\hat{V}] = [\hat{R}] + [\hat{V} \setminus \hat{R}] \tag{3.39}$$

for the closed embedding. Now we can define algebro–geometric Feynman rules characters on the renormalization Hopf algebra of Feynman graphons. It is an abstract Feynman rules character  $\hat{U} : \mathcal{S}_{\text{graphon}}^\Phi \rightarrow A_{\text{dr}}$  with the general form

$$\hat{U}([W_X]) = I([\mathbb{A}^\infty \setminus \hat{V}_X]) \tag{3.40}$$

such that  $[\mathbb{A}^\infty \setminus \hat{V}_X]$  is the class in  $\mathcal{F}$  and  $I : \mathcal{F} \rightarrow A_{\text{dr}}$  is a ring homomorphism.

We can also define a new invariant of infinite Feynman diagrams in terms of a generalization of the Chern–Schwartz–MacPherson (CSM) characteristic classes of singular varieties. The algebro–geometric Feynman rules have been constructed in terms of a polynomial invariant originated from the CSM characteristic classes ([7, 8, 135]) and here we plan to lift that study onto the level of Feynman graphons.

**Corollary 3.1.9.** *There exists an extension of the CSM homomorphism for the level of large Feynman diagrams generated by Dyson–Schwinger equations.*

*Proof.* The existence of the CSM-homomorphism  $I_{\text{CSM}}^\infty$  is another consequence of the cut-distance topology and graphon representations of Feynman diagrams.

For a given large Feynman diagram  $X_{\text{DSE}}$  as the unique solution of an equation DSE, suppose  $\Psi_{X_{\text{DSE}}}(w)$  is the first Kirchhoff–Symanzik polynomial and  $\hat{V}_{X_{\text{DSE}}}$  is its associated hypersurface. In addition, suppose  $1_{\hat{V}_{X_{\text{DSE}}}}$  is the function for  $\hat{V}_{X_{\text{DSE}}} \subset \mathbb{A}^\infty$  and  $A(\mathbb{P}^\infty)$  is the associated Chow group. The natural transformation

$$1_{\hat{V}_{X_{\text{DSE}}}} \longmapsto a_0[\mathbb{P}^0] + a_1[\mathbb{P}^1] + a_2[\mathbb{P}^2] + \dots \in A(\mathbb{P}^\infty) \quad (3.41)$$

allows us to define

$$G_{\hat{V}_{X_{\text{DSE}}}}(T) := a_0 + a_1T + a_2T^2 + \dots + a_NT^N + \dots \quad (3.42)$$

Now define

$$I_{\text{CSM}}^\infty : \mathcal{F} \longrightarrow \mathbb{Z}[T], \quad [\hat{V}_{X_{\text{DSE}}}] \longmapsto G_{\hat{V}_{X_{\text{DSE}}}}(T) \quad (3.43)$$

and extend it by linearity to achieve a group homomorphism.  $\square$

### 3.2 The optimization of non-perturbative complexity via a multi-scale Renormalization Group

In Complexity Theory, the efficiency of an algorithm against a problem is judged in terms of the algorithm’s capability in dealing with computational demands about quantities originated from the intrinsic complexity of that problem. An algorithm is known as feasible if it has a polynomial-time asymptotic scaling and it is known as infeasible if it has a super-polynomial (typically, exponential) scaling. The calculations of quantum field-theoretical scattering amplitudes at high precision or strong couplings are infeasible on classical computers but recently, there are some research efforts which aim to show that these calculations can be feasible on quantum computers. Traditional calculations of scattering amplitudes in Quantum Field Theory is on the basis of a series expansion in powers of the coupling constant (i.e. the coefficients of the interaction terms) such that the running coupling constant is taken to be small. Feynman diagrams provide an intuitive way to organize this class of perturbative expansions where the loop number is associated with the power of the (running) coupling constant. The number of this class of combinatorial diagrams gives us a reasonable measure

to evaluate the computational complexity of perturbative calculations. This measure increases factorially in terms of the number of loops and the number of external particles. Furthermore, if the amount of the coupling constant is insufficiently small, then the perturbative machinery can not provide correct results while the series expansions are divergent or asymptotic even at weak coupling constants. Indeed, if we include higher-order terms beyond a certain point, then the approximations can be inappropriate. In fact, by increasing the coupling constant, one eventually reaches a quantum phase transition at some critical couplings such that in the parameter space near this phase transition perturbative methods become unreliable. This region can be studied under strong-coupling regimes.

Generally speaking, limits of computations and the efficiently computing of things are the most important topics in Theory of Computation and Information Theory where people deal with the Halting problem as an undecidable type of problem which determines whether the program will finish running or continue to run forever. Thanks to rooted trees decorated by primitive recursive functions, Manin discovered a new reinterpretation of the Halting problem in the context of the BPHZ perturbative renormalization. He encapsulated the amount of (non-)computability in terms of the existence of the Birkhoff factorization at the level of the renormalization Hopf algebra of the Halting problem [63, 131, 132, 133].

Algorithms belong to the intermediate steps between programs and functions which means that they are classified as substructures in the context of Galois theory. This fundamental fact has already been applied to describe the foundations of a new categorical–geometric setting for the study of (systems) of Dyson–Schwinger equations (as the generators of intermediate steps) in the renormalization Hopf algebra of the Halting problem under Dimensional Regularization and the global  $\beta$ -functions. As the consequence of this treatment, we already have the construction of a new class of neutral Tannakian subcategories of the universal Connes–Marcolli category  $\mathcal{E}^{\text{CM}}$  which encode intermediate algorithms in the context of systems of differential equations together with singularities. In addition, these subcategories can address the existence of a new interrelationship between mixed Tate motives and the Halting problem in Theory of Computation. Furthermore, thanks to the combinatorial reformulation of the universal counterterm, some new computational techniques for the study of the amount of non-computability in the language of the theory of Hall words have been obtained. It is now possible to analyze infinities or non-computability in the Theory of Computation in terms of a renormalization theory on (systems) of Dyson–Schwinger equations and vice versa. [63, 131, 167]

It is so difficult to have an optimal solution when we want to consider a complex problem under a limited period of time. In this situation we work on the construction of anytime algorithms by computing an initial potentially highly suboptimal solution and then we improve the computed

suboptimal solution as time allows.

The Kolmogorov complexity, as an uncomputable concept, aims to determine the length of the shortest algorithm which produces an object as the output of a procedure. For a given set  $\Sigma$  of alphabets or letters, let  $f$  be a computable function on the set of all possible strings generated by elements in  $\Sigma$ . A description of a string  $\sigma$  is some string  $\tau$  with  $f(\tau) = \sigma$ . The Kolmogorov complexity  $K_f$  is defined by

$$K_f(\sigma) := \begin{cases} \min\{|\tau|:f(\tau)=\sigma\} \\ \infty, & \text{otherwise} \end{cases}. \quad (3.44)$$

It is possible to modify this definition independent of choosing  $f$  where we need to apply a universal Turing machine. In fact, there exists a Turing machine  $U$  such that for all partial computable functions  $f$ , there exists a program  $p$  such that for all  $y$ , we have  $U(p, y) = f(y)$ . It enables us to define  $K(\sigma)$  as the Kolmogorov complexity of  $\sigma$ . It is shown that for all  $n$ , there exists some  $\sigma$  with  $|\sigma| = n$  such that  $K(\sigma) \geq n$ . Such  $\sigma$  is called Kolmogorov random.

In this section, we plan to apply the graphon representation theory of solutions of Dyson–Schwinger equations to build a new concept of complexity of non-perturbative parameters in terms of the Kolmogorov complexity. We will show that optimal algorithms in dealing with non-perturbative parameters can be achieved by working on a multi-scale non-perturbative Wilsonian renormalization group defined on the space  $\mathcal{S}^{\Phi, g}$ .

### 3.2.1 A Renormalization Group program on $\mathcal{S}^{\Phi, g}$

One important method for the study of the dynamics of quantum systems is changing the scales of fundamental parameters of the physical theory such as momentum, energy and mass. Theory of Renormalization Group aims to describe the behavior of quantum systems under this class of re-scalings where the possibility of exchanging information from scale to scale is considered under the fundamental principles of Quantum Mechanics. The interpretation of the concept of mass in the context of time and distance by using the Planck constant and the interpretation of the concept of time in the context of distance by using the speed of light enable us to study the dynamics of relativistic quantum systems under the re-scaling of the distance parameter. In this situation, small distances and times are equivalent to large momenta, energies and masses which produce divergencies in Quantum Field Theory.

There is another important parameter in Quantum Field Theory which encodes the strength of the fundamental forces. This parameter, which is known as the coupling constant, appears in the interaction part of the Lagrangian where we encode information of physical theory in the language of Green’s functions and Feynman integrals. The amount of the coupling

constant has direct influence on the complexity of Green's functions. As the basic fact, in QED we deal with couplings smaller than 1 while in low energy QCD we deal with couplings at the size of 1 or larger than 1. In theoretical and experimental studies we study coupling constants under two settings namely, the bare couplings and the running couplings. Running coupling constants are the outputs of (dimensional) regularization and renormalization schemes and they have been applied in high energy levels to generate some intermediate quantities which are useful for the approximation of non-perturbative parameters. Running couplings guide us to deal with changing the scale of the momentum parameter where the Wilsonian model of the Renormalization Group has been formulated.

Generally speaking, there are two different well-known approaches for the formulation of non-perturbative Renormalization Group in Theoretical and Mathematical Physics namely, Wilson–Polchinski framework and effective average action. We can address following references [48, 103, 136, 145, 146, 149, 150, 154, 192] for the study of these methods and here we only review their general facts.

In Wilson–Polchinski framework, Physics at very small scale corresponds to a scale  $\Lambda$  in momentum space which is actually the inverse of a microscopic length where the partition function is given by

$$Z[B] = \int d\mu_{C_\Lambda}(\phi) \exp\left(-\int V(\phi) + \int B\phi\right) \quad (3.45)$$

such that  $d\mu_{C_\Lambda}$  is a functional Gaussian measure with a cut-off at scale  $\Lambda$ . Now if we separate the field  $\phi_p = \phi(p)$  into rapid and slow modes  $\phi_{p,<}, \phi_{p,>}$  with respect to a scale  $k$ , then we can rewrite the partition function in terms of these components which lead us to define a running potential  $V_k$  at scale  $k$  via performing the integration on  $\phi_{>}$ . So we can have

$$Z = \int d\mu_{C_k}(\phi_{<}) \exp\left(-\int V_k(\phi_{<})\right) \quad (3.46)$$

such that when  $k \leq \Lambda$ ,  $V_k$  involves derivative terms with any power of the derivatives of  $\phi_{<}$ . The Wilson–Polchinski equation is indeed a differential equation for the evolution of  $V_k$  with  $k$  such that the flow of potentials  $V_k(\phi_{<})$  do not contain all information on the initial theory and in addition,  $V_k(\phi)$  involves infinitely many couplings contrarily to perturbation theory that involves only the renormalizable ones. In this method of non-perturbative Renormalization Group there is no general achievement about the convergence of the series of approximations that are used. In addition, the anomalous dimension is depended on the choice of cut-off parameters that separate the rapid and the slow modes whereas it should be independent of it.

In effective average action method, the basic idea is to build the 1-parameter family of models for which a momentum depended mass term is

added to the original Hamiltonian where we have

$$Z_k[B] = \int \mathcal{D}\phi(x) \exp(-H[\phi] - \Delta H_k[\phi] + \int B\phi) \quad (3.47)$$

$$\Delta H_k[\phi] = \frac{1}{2} \int_q R_k(q) \phi_q \phi_{-q}. \quad (3.48)$$

For  $0 < k < \Lambda$ , the rapid modes are almost unaffected by the cut-off function  $R_k(q)$  (as a homogeneous to a mass square) which means that  $R_k(|q| > k) \simeq 0$ . Set

$$W_k[B] := \log Z_k[B] \quad (3.49)$$

with the corresponding Legendre transformation  $\Gamma_k[M(x)] = \Gamma_k[\frac{\delta W_k}{\delta B(x)}]$ . The Renormalization Group equation on  $\Gamma_k$  is the differential equation of the type

$$\partial_k \Gamma_k = f(\Gamma_k). \quad (3.50)$$

It is shown that by working on dimensionless and renormalized quantities, the resulting non-perturbative Renormalization Group can be written independently of the scales  $k$  and  $\Lambda$ . The geometry of the resulting Renormalization Group flow from this framework supports the universality of self-similarity and decoupling of massive modes.

In this part we address an alternative Renormalization Group platform for the study of non-perturbative parameters in terms of changing the scales of Dyson–Schwinger equations in terms of the rescaling of bare and running couplings. Our study can provide a mathematical setting to exchange information among non-perturbative aspects under different scales. For this purpose, we plan to build a new multi-scale Renormalization Group machinery on the space  $\mathcal{S}^{\Phi, g}$  of all Dyson–Schwinger equations in the physical theory  $\Phi$  under different scales  $\lambda g$  of the bare coupling constant  $g$ .

The bare couplings are independent of any regularization and renormalization schemes and therefore their rescaling can be helpful for us to approximate non-perturbative parameters originated from Dyson–Schwinger equations under a universal setting. Using running coupling constants makes some fundamental issues. On the one hand, running couplings are unobservable. On the other hand, the most important mission of the Renormalization Group is to show that the predictions for the observables do not depend on theoretical conventions such as renormalization or regularization schemes, the initial state, the choice of effective charge or the choice of running coupling constants. Therefore different choices of these couplings should be related to each other which means that search for an optimal choice is very important. Our promising non-perturbative Renormalization Group enables us to study Dyson–Schwinger equations under changing the scales of bare couplings and running couplings not simultaneously but related to each other. We expect that this alternative machinery is helpful to

provide a theoretical algorithm for the determination of effective couplings where the complexity of the corresponding Dyson–Schwinger equations will be controlled in terms of changing the scale of the bare coupling constant.

Let  $\mathcal{X}^g$  be the collection of all interacting Lagrangians with coefficients in the ring  $\mathbb{R}[[g]]$ , invariant under the change  $\phi \rightarrow -\phi$ , and interaction parts with the general form

$$I(\phi) := \sum_{k \geq 2} I_k(\phi) \quad (3.51)$$

such that for all  $k$ ,  $I_k = O(g)$  with respect to the bare coupling constant  $g$ . Changing the scale of  $g$  allows us to obtain an effective Lagrangian at the scale  $\tau \leq \lambda$  of a Lagrangian  $L$  at the initial scale  $\lambda$ . The interaction part is the original source of Dyson–Schwinger equations and therefore the re-scaling of the coupling constants lead us to rescale these non-perturbative type of equations. The Renormalization Group with respect to this class of momentum type rescaling enables us to discuss about the possibility of exchanging information among rescaled Dyson–Schwinger equations.

For each  $k$ , set  $F_k$  as the set of all smooth functions on the hyperplane  $\sum_{i=1}^k v_i = 0$  in  $(V^*)^{\oplus k}$  such that  $V$  is the Euclidean 4-dimensional space-time. Define  $F := \prod_{i=1}^{\infty} F_{2i}$  to formulate Green’s functions  $\mathcal{G}$  given by

$$\mathcal{G} : \mathcal{X}^g \times M_m \longrightarrow F, \quad \mathcal{G} := (\mathcal{G}_2, \mathcal{G}_4, \dots) \quad (3.52)$$

such that

- $M_m$  is the set of scales of the momentum parameter,
- The value  $\mathcal{G}_k$  at  $(L, \lambda)$  is called the  $k$ -point correlation function of the Lagrangian  $L$  at the scale  $\lambda$ ,
- For each  $k$ ,  $\mathcal{G}_k$  is the formal expansion of amplitudes of all Feynman diagrams with  $k$  external edges.

Dyson–Schwinger equations are actually formulated as the fixed point equations of  $\mathcal{G}$  with the general form

$$\mathcal{G} = 1 + \int I_\gamma \mathcal{G} \quad (3.53)$$

such that  $I_\gamma$  is the integral kernel with respect to the (IPI) primitive Feynman diagram  $\gamma$ .

**Definition 3.2.1.** An equation DSE’ in  $\mathcal{S}^{\Phi, g}$  is called effective at the scale  $\tau$  of the original Dyson–Schwinger equation DSE at the initial scale  $\lambda$ , if the fixed point equation of the Green’s function  $\mathcal{G}(L^\Phi, \lambda)$  corresponding to the equation DSE coincides with the fixed point equation of the Green’s function  $\mathcal{G}(L^\Phi, \tau)$  corresponding to the equation DSE’.

It is shown in [171] that we can build a unique effective equation at the scale  $\tau$  for any equation DSE in  $\mathcal{S}^{\Phi, g}$  at the original scale  $\lambda$  of the momentum

parameter. It enables us to change the scale of the momenta of internal edges of each term in the formal expansion of the solution of DSE.

In higher orders in perturbation theory we should deal with a large number of Feynman diagrams which cost us exponentially growing of the momentum scale. Therefore all orders in perturbation theory do not accessible for any scale of the momentum parameter. The asymptotic freedom behavior of QCD at very high energies enables us to study Physics of hadrons under perturbative setting. At a relatively low energy scale, coupling constants become too large where non-perturbative situations of the physical system can be observed. Running coupling constants, as the functions of the momentum parameter, describe the strength of the interactions among quarks and gluons. The determination of this class of couplings has very uncertainty nature which makes so many computational and phenomenological difficulties. Dimensional Regularization allows us to replace the bare coupling constant with a class of scale depended couplings. The ultraviolet divergencies are eliminated by normalizing the coupling at a specific momentum scale. In addition, the ultraviolet cut-off dependency is removed by allowing the couplings and masses in the Lagrangian to have a scale dependency. Therefore we can produce running couplings in terms of normalizing them to a measured value at a given scale. This normalization of the coupling to a measured value makes the running coupling to not have sensitivity to the ultraviolet cut-off. The scale dependency of the strong coupling can be controlled by the  $\beta$ -function as the infinitesimal generator of the Renormalization Group. [51, 137, 138]

However we plan to apply effective Dyson–Schwinger equations under different rescaling of the bare coupling constant and running coupling constants to define a new multi-scale Renormalization Group on the space  $\mathcal{S}^{\Phi,g}$  to analyze the behavior of rescaled Dyson–Schwinger equations. This new Renormalization Group is also useful to rescale running couplings in terms of changing the scale of the bare coupling constant independent of any regularization method.

**Theorem 3.2.2.** *There exists a Renormalization Group machinery on  $\mathcal{S}^{\Phi,g}$  which encodes the dynamics of Dyson–Schwinger equations under changing the scales of the bare and running coupling constants.*

*Proof.* Set  $M_{\text{running}}$  as the set of scales of the running couplings. For scales  $\Lambda_1, \Lambda_2, \Lambda_3 \in M_{\text{running}}$  such that  $\Lambda_1 < \Lambda_2 < \Lambda_3$ , define the scale map  $R_{--}^{\text{running}}$  on  $\mathcal{S}^{\Phi,g}$  which satisfies the property

$$R_{\Lambda_1\Lambda_2}^{\text{running}} R_{\Lambda_2\Lambda_3}^{\text{running}} = R_{\Lambda_1\Lambda_3}^{\text{running}}. \quad (3.54)$$

For each equation DSE,  $R_{\Lambda_1\Lambda_2}^{\text{running}}$  DSE is the effective Dyson–Schwinger equation at the scale  $\Lambda_2$  of the equation DSE at the original scale  $\Lambda_1$ . Now define

an action of the semigroup  $\mathbb{R}_{\leq 1}^+$  on the space  $\mathcal{S}^{\Phi,g} \times M_{\text{running}}$  given by

$$r \circ (\text{DSE}, \Lambda) := (R_{\Lambda, r\Lambda}^{\text{running}} \text{DSE}, r\Lambda). \quad (3.55)$$

The equation  $R_{\Lambda, r\Lambda}^{\text{running}} \text{DSE}$  is the rescaled version of the Dyson–Schwinger equation DSE under the running coupling  $r\Lambda$  while the equation

$$R_{r\Lambda \ \Lambda}^{\text{running}} \text{DSE} := (R_{\Lambda, r\Lambda}^{\text{running}} \text{DSE}, r\Lambda) \quad (3.56)$$

is the corresponding unique effective Dyson–Schwinger equation in the effective Lagrangian  $L_{r\Lambda}^{\Phi}(g)$ . The resulting Renormalization Group allows us to study the dynamics of Dyson–Schwinger equations under the rescaling of the running couplings.

Set  $M_{\text{bare}}$  as the set of scales of the bare coupling constant  $g$ . For scales  $\tau_1, \tau_2, \tau_3 \in M_{\text{bare}}$  such that  $\tau_1 < \tau_2 < \tau_3$ , define the scale map  $R_{\tau_1 \tau_2}^{\text{bare}}$  on  $\mathcal{S}^{\Phi,g}$  which satisfies the property

$$R_{\tau_1 \tau_2}^{\text{bare}} R_{\tau_2 \tau_3}^{\text{bare}} = R_{\tau_1 \tau_3}^{\text{bare}}. \quad (3.57)$$

For each equation DSE in  $\mathcal{S}^{\Phi,g}$  define a new equation  $R_{\tau_1 \tau_2}^{\text{bare}} \text{DSE}$  which is the effective Dyson–Schwinger equation at the scale  $\tau_2$  of the equation DSE at the initial scale  $\tau_1$ . Now define an action of the semigroup  $\mathbb{R}_{\leq 1}^+$  on the space  $\mathcal{S}^{\Phi,g} \times M_{\text{bare}}$  given by

$$r \circ (\text{DSE}, \tau) := (R_{\tau, r\tau}^{\text{bare}} \text{DSE}, r\tau). \quad (3.58)$$

The equation  $R_{\tau, r\tau}^{\text{bare}} \text{DSE}$  is the rescaled version of the Dyson–Schwinger equation DSE under the rescaled bare coupling constant  $r\tau$  while the equation

$$R_{r\tau \ \tau}^{\text{bare}} \text{DSE} := (R_{\tau, r\tau}^{\text{bare}} \text{DSE}, r\tau) \quad (3.59)$$

is the corresponding unique effective Dyson–Schwinger equation in the effective Lagrangian  $L^{\Phi}(r\tau g)$ . The resulting Renormalization Group allows us to study the dynamics of Dyson–Schwinger equations under the rescaling of the bare coupling constant  $g$ .

Thanks to (3.55) and (3.58), we can define a new multi-scale renormalization group on  $\mathcal{S}^{\Phi,g}$  where it is possible to rescale the bare coupling constant  $g \mapsto \tau g$  before the application of regularization schemes.

Each triple  $(\text{DSE}, \tau g, \Lambda_{\tau})$  in  $\mathcal{S}^{\Phi,g} \times M_{\text{bare}} \times M_{\text{running}}$  presents an effective Dyson–Schwinger equation such that its unique solution is a polynomial with respect to the rescaled bare and running coupling constants. This polynomial is an infinite formal expansion of Feynman integrals together with the powers of the rescaled bare coupling constant bare coupling constant  $\tau g$  (as the initial scale) such that each Feynman integral in the expansion is defined in terms of the momentum parameter at the initial scale  $\Lambda_{\tau}$ . Now

we can define a new action of the semi-group  $\mathbb{R}_{\leq 1}^+$  on  $\mathcal{S}^\Phi \times M_{\text{bare}} \times M_{\text{running}}$  as the following way

$$\lambda \circ (\text{DSE}, \tau g, \Lambda_\tau) := (R_{(\tau g, \Lambda_\tau), (\lambda \tau g, \lambda \Lambda_\tau)}^{\text{multi}} \text{DSE}, (\lambda \tau g, \lambda \Lambda_\tau)). \quad (3.60)$$

The equation  $R_{(\tau g, \Lambda_\tau), (\lambda \tau g, \lambda \Lambda_\tau)}^{\text{multi}} \text{DSE}$  is the multi-rescaled version of the Dyson–Schwinger equation DSE obtained by changing scales  $\tau g \mapsto \lambda \tau g$  and  $\Lambda_\tau \mapsto \lambda \Lambda_\tau$  of the bare and running coupling constants. The equation

$$R_{(\lambda \tau g, \lambda \Lambda_\tau)}^{\text{multi}} (\tau g, \Lambda_\tau) \text{DSE} := (R_{(\tau g, \Lambda_\tau), (\lambda \tau g, \lambda \Lambda_\tau)}^{\text{multi}} \text{DSE}, (\lambda \tau g, \lambda \Lambda_\tau)) \quad (3.61)$$

in  $\mathcal{S}^{\Phi, g}$  is the corresponding unique effective Dyson–Schwinger equation in the effective Lagrangian  $L_{\lambda \Lambda_\tau}^\Phi(\lambda \tau g)$ .  $\square$

In Theorem 3.2.2, the equation  $R_{(\lambda \tau g, \lambda \Lambda_\tau)}^{\text{multi}} (\tau g, \Lambda_\tau) \text{DSE}$  can be seen as the unique effective Dyson–Schwinger equation at the multi-scale  $(\tau g, \Lambda_\tau)$  of the equation DSE at the original multi-scale  $(\lambda \tau g, \lambda \Lambda_\tau)$ . This means that this new multi-scale Renormalization Group can generate observable running coupling constants which are independent of any regularization or renormalization schemes. This fundamental property clarifies the universality of this non-perturbative Renormalization Group with respect to generating observable intermediate values for a given strong bare coupling constant.

Roughly speaking, the renormalization machinery enables us to redefine the unrenormalized constants which exist in the Lagrangian in such a way that the observable quantities remain finite when the ultraviolet cut-off is removed. This machinery requires a new quantity  $\mu$  with the dimension of a mass where all intermediate quantities are depended on  $\mu$ . The confinement in QCD does not allow us to determine a natural scale for  $\mu$ . The  $\mu$  dependence of the coupling constant and various quark masses in QCD force us to define running coupling constants and running masses where the Renormalization Group equations can control the  $\mu$  dependence of the resulting renormalized quantities. The running coupling constant  $g(\mu^2)$  can be studied in terms of the equation

$$\mu^2 \frac{dg(\mu^2)}{d\mu^2} = \beta(g(\mu^2)) \quad (3.62)$$

which leads us to

$$g(\mu^2) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)} \quad (3.63)$$

such that the dimensional scale  $\Lambda$  is the scale at which the coupling diverges and perturbation theory becomes meaningless. When the cut-off parameter tends to infinity,  $\beta(g(\mu^2))$  remains finite such that in perturbation theory we have

$$\beta(g(\mu^2)) = -g(\mu^2)^2(\beta_0 + \beta_1 g(\mu^2) + \beta_2 g(\mu^2)^2 + \dots). \quad (3.64)$$

The Renormalization Group machinery defined by Theorem 3.2.2 is non-commutative because the scale of the momentum parameter is completely depended on the chosen rescaling of the bare coupling. It encodes the dynamics of non-perturbative aspects of quantum systems by Dyson–Schwinger equations under different running coupling constants derived from changing the scales of couplings.

Dimensional Regularization or other regularization schemes can change the nature of the bare couplings to describe QFT under a perturbative setting but it fails to be functional in higher orders. Theorem 3.2.2 enables us to generate a new class of running couplings in terms of the rescaled bare coupling which are independent of any regularization process. Therefore the resulting running couplings preserve the nature of the bare coupling which means that they have physical meanings.

The one important application of this multi-scale Renormalization Group together with Feynman graphon models is a new way for the description of any strongly coupled Dyson–Schwinger equation  $DSE(g)$  in terms of a cut-distance convergent sequence of Dyson–Schwinger equations under weaker couplings  $\lambda g$  in the space  $\mathcal{S}^{\Phi,g}$ . This means that we can approximate the complexity of non-perturbative parameters under the strong coupling constant  $g$  in terms of a sequence of complexities of solutions of Dyson–Schwinger equations with lesser rate of complexities. In this setting, we can compute the unique solution  $X(\lambda g)$  of the equation  $DSE(\lambda g)$  for couplings  $\lambda g < 1$  as intermediate values for the approximation of the large Feynman diagram  $X(g)$ .

**Corollary 3.2.3.** *For each large Feynman diagram  $X(g) = \sum_{m=0}^{\infty} g^m X_m$  derived from an equation DSE in  $\mathcal{S}^{\Phi,g}$  at the strong bare coupling constant  $g \geq 1$ , there exists a sequence of large Feynman diagrams at weaker effective couplings which converges to  $X(g)$  with respect to the cut-distance topology.*

*Proof.* Thanks to Theorem 3.2.2, we can build the sequence  $\{R_{g, \frac{n}{n+1}}^{\text{bare}} DSE\}_{n \geq 1}$  of Dyson–Schwinger equations with respect to the rescaled bare coupling constants  $\frac{n}{n+1}g$  for each  $n \geq 1$  where the initial scale of the equation DSE is at least 1.

For each  $n$ ,  $R_{g, \frac{n}{n+1}}^{\text{bare}} DSE$  is an equation in  $\mathcal{S}^{\Phi,g}$  which has the unique solution

$$Y\left(\frac{n}{n+1}g\right) = \sum_{m=0}^{\infty} \left(\frac{n}{n+1}g\right)^m X_m. \quad (3.65)$$

The scales  $\frac{n}{n+1}$  for each  $n \geq 1$  provide an increasing sequence of effective couplings derived from the bare coupling constant  $g$  where  $\frac{n}{n+1}g < g$ . Therefore for each  $n$ , the solution  $Y\left(\frac{n}{n+1}g\right)$  of the equation  $R_{g, \frac{n}{n+1}}^{\text{bare}} DSE$  is actually a disjoint union of multi-loop Feynman diagrams which can be handled by higher order perturbation methods. It remains to show that the sequence

$\{Y(\frac{n}{n+1}g)\}_{n \geq 1}$  is convergent to  $X(g)$  with respect to the cut-distance topology. Thanks to Lemma 2.3.3, for each  $n \geq 1$ , we can associate a unique unlabeled graphon class  $[W_{Y(\frac{n}{n+1}g)}]$  with respect to each large Feynman diagram  $Y(\frac{n}{n+1}g)$ . Thanks to Definition 2.3.5, it is enough to show that the sequence  $\{[W_{Y(\frac{n}{n+1}g)}]\}_{n \geq 1}$  is convergent to the unlabeled graphon class  $[W_{X(g)}]$ .

On the one hand, we can show that when  $n$  goes to infinity, the labeled Feynman graphons  $W_{(\frac{n}{n+1}g)^m X_m}$  and  $W_{g^m X_m}$  are weakly isomorphic for each  $m \geq 0$ . On the other hand, if we replace the Lebesgue measure with the Gaussian measure in the ground probability measure of our Feynman graphon model, then we can show that for a fixed  $n \geq 1$ , the labeled Feynman graphons  $W_{(\frac{n}{n+1}g)^m X_m}$  and  $W_{g^m X_m}$  are weakly isomorphic (or equivalent) for each  $m \geq 0$ .

Therefore when  $m$  tends to infinity, labeled Feynman graphons  $W_{Y(\frac{n}{n+1}g)}$  and  $W_{X(g)}$  are also weakly isomorphic (or equivalent).  $\square$

**Corollary 3.2.4.** *For any given strongly coupled Dyson–Schwinger DSE in  $\mathcal{S}^{\Phi,g}$ , there exists a sequence of many-loop Feynman diagrams such that their corresponding BPHZ counterterms and renormalized values converge to the counterterm and the renormalized value generated by the renormalization of the equation DSE.*

*Proof.* Thanks to Corollary 3.2.3, there exists a sequence of Dyson–Schwinger equations under weaker rescaled bare coupling constants  $\lambda_n g$  and running couplings  $\Lambda_{\lambda_n}$  in  $\mathcal{S}^{\Phi,g}$  with the corresponding sequence  $\{\Gamma_n\}_{n \geq 1}$  of their solutions which is cut-distance convergent to the unique solution  $X_{\text{DSE}}(g)$ . Now apply Theorem 2.4.1 to each large Feynman diagram  $\Gamma_n$  to build sequences

$$\{S_{R_{\text{ms}}}^{\tilde{\phi}}([W_{\Gamma_n}])\}_{n \geq 1} \quad (3.66)$$

and

$$\{S_{R_{\text{ms}}}^{\tilde{\phi}} * \tilde{\phi}([W_{\Gamma_n}])\}_{n \geq 1} \quad (3.67)$$

which are cut-distance convergent to  $S_{R_{\text{ms}}}^{\tilde{\phi}}([W_{X_{\text{DSE}}(g)}])$  and  $S_{R_{\text{ms}}}^{\tilde{\phi}} * \tilde{\phi}([W_{X_{\text{DSE}}(g)}])$ , respectively  $\square$

Thanks to these investigations, the multi-scale Renormalization Group defined by Theorem 3.2.2 is capable to optimize the computational procedures in dealing with non-perturbative parameters. We consider this topic in the next section where our main effort is to determine an order of complexity on Dyson–Schwinger equations in a given strongly coupled gauge field theory.

### 3.2.2 Kolmogorov complexity of Dyson–Schwinger equations

In this part, we plan to build a new concept of complexity on the space  $\mathcal{S}^{\Phi,g}$  of Dyson–Schwinger equations of a given strongly coupled gauge field theory  $\Phi$  in terms of our new multi-scale Renormalization Group (i.e. Theorem 3.2.2). Then we use the Manin renormalization Hopf algebra of the Halting problem for non-perturbative Feynman rules characters on Feynman graphons to formulate a new way of computing non-perturbative parameters in the context of the Halting problem for partial recursive functions on the new constructive world  $\mathcal{S}^{\Phi,g}$ . The required structural numbering for this new constructive world can be determined via our multi-scale Renormalization Group (i.e. Theorem 3.2.2) and the density of rational numbers in real numbers.

We define the Kolmogorov complexity of each Dyson–Schwinger equation DSE in  $\mathcal{S}^{\Phi,g}$  in terms of changing the scale of the bare coupling constant where exchanging information among equations at different scales have been encoded by our new non-perturbative multi-scale Renormalization Group. We can generate partial recursive (or semi-computable) functions in terms of increasing sequences of rational numbers which provide different rescaling of the bare coupling constant and the momentum parameter. For example, thanks to Corollary 3.2.3, define

$$u^g : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}, \quad (n, \text{DSE}(g)) \longmapsto \text{DSE}\left(\frac{n}{n+1}g\right) \quad (3.68)$$

as a semi-computable function. It means that there exists an algorithm which encodes the application of  $u^g$  on Dyson–Schwinger equations to generate effective equations under different running couplings. The equation  $\text{DSE}\left(\frac{n}{n+1}g\right)$  is the effective Dyson–Schwinger equation  $R_{\frac{n}{n+1}g}^{\text{bare}} g$  generated by changing the scale of the bare coupling constant in terms of the sequence  $\left\{\frac{n}{n+1}\right\}_{n \geq 1}$ .

**Definition 3.2.5.** The Kolmogorov complexity of an equation  $\text{DSE}(\lambda g)$  at the scale  $\lambda g$  with respect to the function  $u^g$  (3.68) is determined by the relation

$$K_{u^g}(\text{DSE}(\lambda g)) := \text{Min}\{n \in \mathbb{Z}_+ : u^g(n, \text{DSE}'(g)) \subseteq \text{DSE}(\lambda g)\} \quad (3.69)$$

such that the inclusion means that, up to the weakly isomorphic relation, the large Feynman diagram  $X_{\text{DSE}'\left(\frac{n}{n+1}g\right)}$  can be embedded as a subgraph into the large Feynman diagram  $X_{\text{DSE}(\lambda g)}$ .

**Lemma 3.2.6.** *There exists the Kolmogorov total order on  $\mathcal{S}^{\Phi,g}$ .*

*Proof.* The Kolmogorov order of  $\mathcal{S}^{\Phi,g}$  is defined as a bijection  $\mathbf{K}_{u^g} : \mathcal{S}^{\Phi,g} \rightarrow \mathbb{Z}_+$  which arranges all Dyson–Schwinger equations of the physical theory  $\Phi$

in the increasing order of their complexities  $\mathbf{K}_{ug}(\text{DSE}(\lambda g))$ . Define

$$\text{DSE}_1(\lambda_1 g) \preceq \text{DSE}_2(\lambda_2 g) \iff K_{ug}(\text{DSE}_1(\lambda_1 g)) < K_{ug}(\text{DSE}_2(\lambda_2 g)). \quad (3.70)$$

It is possible to determine some constants  $c_0 > 0$  such that for all Dyson–Schwinger equations such as  $\text{DSE}(\lambda g)$ ,

$$c_0 K_{ug}(\text{DSE}(\lambda g)) \leq \mathbf{K}_{ug}(\text{DSE}(\lambda g)) \leq K_{ug}(\text{DSE}(\lambda g)). \quad (3.71)$$

□

This is actually the most simple example of this class of semi-computable functions and we can generate other examples in terms of any arbitrary increasing sequence of rational numbers, which provides some rescaling for the bare coupling constant  $g$  and the running couplings, to define more general semi-computable functions in terms of our new multi-scale Renormalization Group. We have addressed a more general setting for the structure of the Kolmogorov complexity in other research work and here we only focus on the partial recursive (or semi-computable) function (3.68).

Thanks to Definition 3.2.5 and Definition 3.2.6, it is now possible to consider  $\mathcal{S}^{\Phi, g}$  as a poset such that for any given partial recursive map  $\sigma : \mathcal{S}^{\Phi, g} \rightarrow \mathcal{S}^{\Phi, g}$  which generates a permutation, we can define a new map

$$\sigma_{\mathbf{K}_{ug}} := \mathbf{K}_{ug} \circ \sigma \circ \mathbf{K}_{ug}^{-1} \quad (3.72)$$

where it provides a permutation of the subset

$$D(\sigma_{\mathbf{K}_{ug}}) := \mathbf{K}_{ug}(\text{Dom}(\sigma)) \subseteq \mathbb{Z}_+. \quad (3.73)$$

Consider the equation  $\text{DSE}(\lambda g) \in \text{Dom}(\sigma)$  such that its corresponding orbit  $\sigma^{\mathbb{Z}}(\text{DSE}(\lambda g))$  is infinite. Set

$$\mathbf{K}_{ug}(\text{DSE}(\lambda g)) := k_{\text{DSE}}^\lambda \quad (3.74)$$

such that for each  $n > 0$ , we have

$$\sigma_{\mathbf{K}_{ug}}^n(k_{\text{DSE}}^\lambda) = \mathbf{K}_{ug}(\sigma^n(\text{DSE}(\lambda g))) \leq c \mathbf{K}_{ug}(n). \quad (3.75)$$

In [130] it is discussed that for any partial recursive function  $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  and  $x \in \text{Dom}(f)$  we have

$$K(f(x)) \leq c_f K(x) \leq c'_f x. \quad (3.76)$$

We want to apply the inequality (3.76) for the Kolmogorov complexity of Dyson–Schwinger equations defined by Definition 3.2.5 and bijection  $\mathbf{K}_{ug}$ . Define

$$Y := \{\sigma^n(\text{DSE}(\lambda g)) : n \in \mathbb{Z}_+\} \quad (3.77)$$

as a recursively enumerable subset of  $\mathcal{S}^{\Phi,g}$  which plays the role of the domain for a partial recursive function  $A : \mathcal{S}^{\Phi,g} \rightarrow \mathbb{Z}_+$  given by

$$A(\text{DSE}(\tau g)) = n, \quad \text{if } \sigma^n(\text{DSE}(\lambda g)) \cong \text{DSE}(\tau g). \quad (3.78)$$

We then have

$$\mathbf{K}_{u^g}^{-1}(n) = \mathbf{K}_{u^g}^{-1}(A(\text{DSE}(\tau g))) \leq c' \mathbf{K}_{u^g}(\text{DSE}(\tau g)) = c' \mathbf{K}_{u^g}(\sigma^n(\text{DSE}(\lambda g))) \quad (3.79)$$

such that as the consequence, it is possible to determine some upper and lower boundaries for the permutation  $\sigma_{\mathbf{K}_{u^g}}$  such as

$$c_1 \mathbf{K}_{u^g}^{-1}(n) \leq \sigma_{\mathbf{K}_{u^g}}^n(k_{\text{DSE}}^\lambda) \leq c_2 \mathbf{K}_{u^g}^{-1}(n). \quad (3.80)$$

**Lemma 3.2.7.** *Consider  $\mathcal{S}^{\Phi,g}$  as the constructive world and equip this collection with a total recursive structure of additive group without torsion with the zero element  $\mathbf{0}$ . The Halting problem for any partial recursive function  $f : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \rightarrow \mathcal{S}^{\Phi,g}$  can be described in the language of fixed points of some permutations on  $\mathbb{Z}_+ \times \mathcal{S}^{\Phi,g}$  derived from  $f$ .*

*Proof.* Feynman graphon models of solutions of Dyson–Schwinger equations enable us to define a total recursive structure of the additive group without torsion on the constructive world  $\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}$  such that  $\mathbf{0}$  is the zero element. Now extend  $f$  to a new function

$$g_f : \mathbb{Z}_+ \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \longrightarrow (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \quad (3.81)$$

such that

$$g_f((n, X)) := \mathbf{0}, \quad \text{if } (n, X) \notin \text{Dom}(f). \quad (3.82)$$

Now define a new permutation

$$\begin{aligned} \tau_f : \mathbb{Z}_+ \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) &\longrightarrow \mathbb{Z}_+ \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}), \\ \tau_f(n, (X, Y)) &:= (g_f(n, \mathbf{0}), X + g_f((n, Y)), Y). \end{aligned} \quad (3.83)$$

We can check that finite orbits of  $\tau_f$  are fixed points of the permutation. It leads us to build a new partial recursive permutation  $\sigma_f$  with the domain

$$\text{Dom}(\sigma_f) := (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \times \text{Dom}(f). \quad (3.84)$$

Thanks to [130, 132] and the definition of  $g_f$ , we can show that the complement to  $\text{Dom}(\sigma_f)$  in the constructive world  $(\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\}) \times (\mathcal{S}^{\Phi,g} \sqcup \{\mathbf{0}\})$  covers the fixed points of  $\tau_f$ . This process reduces the Halting problem for  $f$  to the determination of the fixed points of  $\tau_f$ .  $\square$

For the constructive world  $\mathcal{S}^{\Phi,g}$ , the map  $u^g$  given by (3.68), the map  $\sigma_{\mathbf{K}_{u^g}}$  given by (3.72) and (3.73), the integer value  $k_{\text{DSE}}^\lambda$  given by (3.74), define

$$\Psi(k_{\text{DSE}}^\lambda, \sigma, u^g, z) := \frac{1}{(k_{\text{DSE}}^\lambda)^2} + \sum_{n \geq 1} \frac{z^{\mathbf{K}_{u^g}(\text{DSE}(\lambda \frac{n}{n+1}g))}}{(\sigma_{\mathbf{K}_{u^g}}^n(k_{\text{DSE}}^\lambda))^2}. \quad (3.85)$$

**Corollary 3.2.8.** - *If the  $\sigma$ -orbit of the equation  $\text{DSE}(\lambda g) \in \text{Dom}(\sigma)$  is finite, then  $\Psi(k_{\text{DSE}}^\lambda, \sigma, u^g, z)$  is a rational function in the complex variable  $z$ . All poles of this formal series, which are of the first order, live at the roots of unity.*

- *If the  $\sigma$ -orbit of the equation  $\text{DSE}(\lambda g) \in \text{Dom}(\sigma)$  is infinite, then  $\Psi(k_{\text{DSE}}^\lambda, \sigma, u^g, z)$  is the Taylor series of an analytic function on the region  $|z| < 1$  which is continuous at the boundary of this region.*

*Proof.* It is a direct result of the discussions in [130, 132] where we need to replace the constructive world  $\mathbb{Z}_+$  with  $\mathcal{S}^{\Phi,g}$ .  $\square$

Thanks to the Manin's reconstruction of the Halting problem in the language of the BPHZ renormalization program ([131, 132, 133]) and the explained machinery with respect to the constructive world  $\mathcal{S}^{\Phi,g}$ , now it is possible to relate the Halting problem for a given partial recursive map  $f : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \rightarrow \mathcal{S}^{\Phi,g}$  to our new version of the Kolmogorov complexity of Dyson–Schwinger equations. For this purpose we reduce  $f$  to a partial recursive permutation

$$\sigma_f : \text{Dom}(\sigma_f) \subset \mathcal{S}^{\Phi,g} \longrightarrow \text{Dom}(\sigma_f) \subset \mathcal{S}^{\Phi,g} \quad (3.86)$$

to interpret the problem of recognizing whether a positive integer number  $k$  belongs to the domain  $\text{Dom}(\sigma_f)$  or not to the problem of whether the corresponding analytic function  $\Psi(k, \sigma_f, u^g, z)$  of a complex parameter  $z$  has a pole at  $z = 1$  or not.

**Theorem 3.2.9.** *The BPHZ renormalization of Feynman graphons encodes the Halting problem for a given partial recursive map  $f : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \longrightarrow \mathcal{S}^{\Phi,g}$ .*

*Proof.* Thanks to the construction of the renormalization Hopf algebra of Feynman graphons  $\mathcal{S}_{\text{graphon}}^\Phi$  and the BPHZ renormalization of large Feynman diagrams, consider the character

$$\varphi_k : \mathcal{S}_{\text{graphon}}^\Phi \longrightarrow A_{\text{dr}}, \quad \varphi_k([W_{X_{\text{DSE}}}] ) := \Psi(k_{\text{DSE}}^\lambda, \sigma_f, u^g, z). \quad (3.87)$$

Thanks to the Birkhoff factorization on the regularization algebra  $A_{\text{dr}}$ , we have  $A_{\text{dr}} = \mathcal{A}_+ \oplus \mathcal{A}_-$  such that  $\mathcal{A}_+$  is the unital algebra of analytic functions in the region  $|z| < 1$  which are continuous on the boundary  $|z| = 1$  and  $\mathcal{A}_- := (1-z)^{-1} \mathbb{C}[(1-z)^{-1}]$ .

If we apply Lemma 3.2.7, Corollary 3.2.8 and discussion about the existence of a pole at  $z = 1$  for the analytic function  $\Psi(k_{\text{DSE}}^\lambda, \sigma_f, u^g, z)$ , then we can determine whether  $k_{\text{DSE}}^\lambda$  lives in  $D(\sigma_f)$  or not.  $\square$

The main reason of this important result is the existence of a class of semi-computable maps such as  $u^g$  (for a given strong coupling  $g$ ) which has led us to define a modified version of the Kolmogorov complexity for Dyson–Schwinger equations (i.e. Definition 3.2.5). The dynamics of the well-defined map  $u^g$  (3.68) can be studied by the multi-scale Renormalization Group machinery which is defined on  $\mathcal{S}^{\Phi,g}$ . We can define the Kolmogorov complexity  $K_w$  on Dyson–Schwinger equations with respect to other arbitrary elements  $w$  of the set of Kolmogorov optimal functions. In this setting, the optimality means that for any partial recursive  $v : \mathbb{Z}_+ \times \mathcal{S}^{\Phi,g} \rightarrow \mathcal{S}^{\Phi,g}$  there exists a constant  $c_{v,w} > 0$  such that for each Dyson–Schwinger equation  $\text{DSE}(\lambda g)$ ,

$$K_w((n, \text{DSE}(\lambda g))) \leq c_{v,w} K_v((n, \text{DSE}(\lambda g))). \quad (3.88)$$

Thanks to Corollary 3.2.3, relations (3.71) and (3.80) and Theorem 3.2.9, which determines the amount of non-computability via the Halting problem at the level of Feynman graphons, those semi-computable maps defined in terms of the map  $R_{--}^{\text{multi}}$  can be considered as the truth candidate to search for the optimal option.

## Chapter 4

# The dynamics of non-perturbative QFT in the language of noncommutative geometry

- *A spectral triple model for quantum motions*
- *A noncommutative symplectic geometry model for  $S_{\text{graphon}}^{\Phi}$*

Noncommutative Geometry studies geometric properties of singular spaces on the basis of suitable coordinate algebras where point spaces are replaced by (noncommutative) function algebras. The standard differential and integral calculi have been adapted to a more general setting in the way compatible with the interpretation of variable quantities in Quantum Mechanics as operators on the Hilbert space of states and spectral analysis. The interplay between Algebra and Topology has been studied conceptually and contextually under two general settings on the basis of the theory of Hopf algebras (or quantum groups) and the theory of  $C^*$ -algebras. In the resulting dictionary, noncommutative  $C^*$ -algebras, which are interpreted as the algebras of continuous functions on some virtual noncommutative spaces, are the dual arena for noncommutative topology. As the important consequence of this interrelationship, the theory of spectral triples and the theory of noncommutative differential graded algebras enable us to build the foundations of differential and integral calculi in Noncommutative Geometry. [45, 54]

The idea of applying Noncommutative Geometry to Quantum Field Theory has already been considered and developed by different groups of mathematicians and mathematical/theoretical physicists where we can address new models of gauge field theories or the mathematical foundations of Standard Model and its modified versions in dealing with elementary particles [39, 46, 52, 55, 56, 127, 140]. Furthermore, thanks to the renormalization Hopf algebra, some new applications of noncommutative geometric tools in dealing with Dyson–Schwinger equations were found where two classes of differential graded algebras had been formulated to describe the geometry of quantum motions. The first class of differential graded algebras was built in the way to determine a family of connections which encode quantum motions independent of the chosen regularization scheme [161]. The second class of differential graded algebras was built in the way to encode regularization and renormalization processes of Feynman diagrams which contribute to solutions of Dyson–Schwinger equations in the language of noncommutative differential forms. This setting, which applies shuffle products and Rota–Baxter algebras ([80]), has provided a new geometric interpretation of the Connes–Kreimer renormalization group in the context of integrable systems under a non-perturbative setting [166].

In this chapter, we plan to continue our search for some new applications of Noncommutative Geometry to non-perturbative aspects ([171]). At the first step, we explain the construction of a new class of spectral triples which encodes the geometry of Dyson–Schwinger equations under an operator theoretic setting. This study provides the foundations of a theory of spectral geometry for the description of large Feynman diagrams. At the second step, we search for a new class of differential graded algebras on Feynman graphons to build a noncommutative differential geometry machinery for the description of physical parameters generated by large Feynman diagrams.

## 4.1 A spectral triple model for quantum motions

Geometric objects associated to any  $n$ -dimensional  $C^\infty$  manifold  $M$  such as vector fields, differential forms, general tensor fields, vector bundles, Riemannian metric, connections, curvature tensor, etc are encoded via the commutative algebra  $C^\infty(M)$  (i.e. infinite times differentiable functions on  $M$ ) and some extra operators on this algebra. If we replace the algebra  $C^\infty(M)$  with a noncommutative algebra  $A$  (such as the algebra generated by some deformation procedures on  $C^\infty(M)$ ), then we can achieve the basic elements of Noncommutative Geometry as a generalization of the standard commutative geometry of manifolds. The basic pedagogical example of a noncommutative space is given via Gelfand–Naimark Theorem where studying commutative  $C^*$ -algebras is translated to studying compact topological (Hausdorff) spaces and vice versa. It leads us to a general idea that studying noncommutative  $C^*$ -algebras becomes to studying "noncommutative" compact topological spaces. [45]

Classical Mechanics can be interpreted as the fundamental example of a commutative geometry where the phase space of a system of  $N$  non-relativistic particles is a  $6N$  dimensional symplectic manifold  $M$  and the physical observables, energy, angular momentum, etc are functions in  $C^\infty(M)$ . Quantum Mechanics can be interpreted as the fundamental example of a model in Noncommutative Geometry where we should deal with a noncommutative algebra of quantum observables consisting of operators on the Hilbert space of states. The position operator  $Q$  and the momentum operator  $P$  (as unbounded self-adjoint operators) satisfy the canonical Heisenberg's commutation relation

$$PQ - QP = -i\hbar I. \quad (4.1)$$

The physical observables are represented by hermitian operators. If we apply one-parameter unitary groups  $U_s = e^{isP}$ ,  $V_t = e^{itQ}$ , then we have the Weyl form of the commutation relation namely,

$$U_s V_t = e^{-i\hbar st} V_t U_s. \quad (4.2)$$

Set  $s = t = 1$  and  $\lambda = -2\pi\hbar$  to obtain the unitary bounded operators  $U, V$  on the same Hilbert space which enjoy the property  $UV = e^{2\pi i\lambda} VU$ . The noncommutative polynomial algebra  $A_\lambda$  generated by  $U, V$  together with their corresponding adjoint operators where equipped with the operator norm is actually a noncommutative  $C^*$ -algebra derived from Quantum Mechanics.

Deformation quantization focuses on the construction of a noncommutative algebra of quantum observables in terms of defining some new noncommutative type of products on the vector space  $C^\infty(M)$ . The deformation of the coordinates of space-time with respect to relations such as  $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$

is another machinery in this setting to build a Noncommutative Geometry model.

In an alternative approach, Connes developed a formulation of differential geometry in terms of commutative algebras to build a noncommutative generalization where we can consider a compact manifold of arbitrary dimension with a well-defined Riemannian structure which gives rise to a first order differential operator known as the Dirac operator. It is shown that the manifold, including the metric tensor, can be completely reconstructed from the discrete eigenvalues of the Dirac operator such that the properties of the spectrum can be encoded by a spectral triple which contains some algebraic information. In summary, an ordinary compact Riemannian manifold  $M$  is reinterpreted in terms of the spectral triple  $(A = C^\infty(M), \mathbb{H} = L^2(\cdot), D = i\gamma_\mu \partial x^\mu)$  which is called a commutative spectral triple. Thanks to this setting, Connes achieved a new modified version of the Gelfand–Naimark Theorem for compact Riemannian manifolds and spectral triples. The generalization of this approach has led us to the concept of noncommutative spectral triples where some new applications of Noncommutative Geometry to the description of relativistic quantum theory, elementary particles and space-time at the micro-scale Physics have been discovered by mathematicians and mathematical/theoretical physicists. As an example we can address the mathematical foundations of Standard Model, its modified versions and perturbative renormalization program on these physical theories in terms of noncommutative geometric tools. [45, 46, 178, 179]

Here we plan to explain the structure of a new class of spectral triples originated from solutions of Dyson–Schwinger equations. The resulting spectral triples encode the geometry of those parts of Quantum Field Theories under strong running coupling constants where quantum motions have complicated non-perturbative behaviors.

In general, a spectral triple is a collection  $(A, \mathbb{H}, D)$  of related mathematical structures such that  $A$  is a (unital) involutive algebra which is faithfully represented on a given Hilbert space  $\mathbb{H}$  via a representation  $\pi$ . The operator  $D$  is a self-adjoint operator acting on  $\mathbb{H}$  with the compact resolvent. For any  $a \in A$ ,  $\pi(a)$  maps  $\text{dom}(D)$  into itself. The operator  $[D, \pi(a)]$  extends to a bounded operator on  $\mathbb{H}$ .

Theory of Clifford algebras and spin structures have provided the foundations of the algebraic reconstruction of the geometry of smooth (compact) Riemannian manifolds in the context of the theory of spectral triples. For a given  $n$ -dimensional (locally) compact  $C^\infty$ -Riemannian manifold  $M$  without boundary, set  $A^1(M) := \Gamma(M, T_{\mathbb{C}}^*M)$  as the space of sections of the complex cotangent bundle, which are differentiable 1-forms on  $M$ , with the corresponding dual space  $\mathfrak{N}(M) := \Gamma(M, T_{\mathbb{C}}M)$ , as the space of sections of the tangent bundle, which are differentiable vector fields on  $M$ . The metric  $g$  is therefore a  $C^\infty(M)$ -valued symmetric bilinear positive definite form

on  $A^1(M)$  (or  $\mathfrak{K}(M)$ ). The Cech cohomology theory of the algebra of Clifford sections enables us to define  $\text{spin}^c$  structures. Then we determine the corresponding spin structures under Morita equivalent relation. A  $\text{spin}^c$  connection on a spinor module  $\Gamma(M, S)$  is defined (compatible with the action of the algebra of Clifford sections) as a Hermitian connection

$$\nabla^S : \Gamma(M, S) \longrightarrow A^1(M) \otimes_{C^\infty(M)} \Gamma(M, S). \quad (4.3)$$

It is called a spin connection, if it commutes with the anti-linear charge conjugation  $c$  for each real vector field. The Riemannian distance on the manifold  $M$  is determined in terms of the Dirac operator as a complex linear operator such as  $D : \Gamma(M, S) \longrightarrow \Gamma(M, S)$  defined by the composition  $-i\widehat{c} \circ \nabla^S$  such that

$$\widehat{c} \in \text{Hom}_{C^\infty(M)}(B \otimes \Gamma(M, S), \Gamma(M, S)) \quad (4.4)$$

is given by  $\widehat{c}(\rho_1, \rho_2) := c(\rho_1)\rho_2$  while  $B$  is the Clifford algebra bundle. It is also possible to present this operator under a local setting in terms of the spaces of vector fields and 1-forms. This explains the Dirac operator as an essentially self-adjoint operator on its original domain, where we can see that  $[D, f] = -ic(df)$  for any smooth function  $f$ . Thanks to this treatment the relation

$$d(x, y) = \sup\{|f(y) - f(x)| : f \in C^\infty(M), \|[D, f]\| \leq 1\} \quad (4.5)$$

describes the geodesic distance in terms of an unbounded Fredholm module over the  $C^*$ -algebra  $C^\infty(M)$  [45, 46]. Therefore all geometric information of the manifold  $M$  can be encapsulated by the spectral triple

$$(C^\infty(M), L^2(M, S), D). \quad (4.6)$$

**Theorem 4.1.1.** *Consider  $\{(A_m, \mathbb{H}_m, D_m)\}_{m \geq 1}$  as a countable family of spectral triples with the corresponding family of representations  $\{\pi_m\}_{m \geq 1}$ . For each  $m$ , let  $\|\cdot\|_m$  be the norm on  $\mathbb{H}_m$  and then choose  $\{\alpha_m\}_{m \geq 1}$  as a sequence of non-zero real numbers such that the sequence  $\{\|(1 + \alpha_m^2 D_m^2)^{-\frac{1}{2}}\|_m\}_{m \geq 1}$  converges to zero when  $m$  goes to infinity. There exists a spectral triple*

$$(A^\oplus, \mathbb{H}^\oplus, \overline{D}^\oplus) \quad (4.7)$$

such that  $\mathbb{H}^\oplus := \bigoplus_{m \geq 1} \mathbb{H}_m$ ,  $D^\oplus := \bigoplus_{m \geq 1} \alpha_m D_m$  with the corresponding self-adjoint extension  $\overline{D}^\oplus$ . In addition,

$$A^\oplus := \{(a_m)_{m \geq 1} \in \prod_m A_m :$$

$$\sup_{m \geq 1} \|\pi_m(a_m)\|_m < +\infty, \quad \sup_{m \geq 1} \|[\alpha_m D_m, \pi_m(a_m)]\|_m < +\infty\}$$

such that for each  $a^\oplus \in A^\oplus$ ,  $\pi^\oplus(a^\oplus) := \bigoplus_{m \geq 1} \pi_m(a_m)$ . [71]

The graduation parameter on the renormalization Hopf algebra and Hopf subalgebras generated by Dyson–Schwinger equations enable us to describe the corresponding complex Lie groups  $\mathbb{G}_\Phi(\mathbb{C})$  and  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$  under projective limits of Lie subgroups.

Generally speaking, for a given commutative (graded) Hopf algebra  $H$ , let  $\text{Spec}(H)$  be the set of all prime ideals of  $H$  equipped with the Zariski topology and the structure sheaf. This topological space accepts a group structure generated by the coproduct of  $H$ . Under a categorical setting, the functional  $\text{Spec}$  is a contravariant functor from the category of commutative algebras to the category of topological spaces which leads us to define another functional  $\mathbb{G}_H = \text{Spec}(H)$  as a covariant representable functor from the category of commutative algebras to the category of groups. For each commutative algebra  $A$ , the Lie group  $\mathbb{G}_H(A) = \text{Spec}(H)(A)$  is the set of morphisms with the general form

$$\varphi : H \longrightarrow A, \quad \varphi(h_1 h_2) = \varphi(h_1)\varphi(h_2), \quad \varphi(1_H) = 1_A, \quad (4.8)$$

which is equipped with the convolution product

$$\varphi_1 * \varphi_2(h) := m \circ (\varphi_1 \otimes \varphi_2) \circ \Delta_H(h). \quad (4.9)$$

Thanks to Milnor–Moore Theorem ([144]), the finite dimensional complex Lie group  $\text{GL}_n$  of  $n \times n$  matrices with non-zero determinants corresponds to the Hopf algebra

$$H_{\text{GL}_n} = k[x_{i,j}, t]_{i,j=1,\dots,n} / \det(x_{i,j})t - 1 \quad (4.10)$$

with the coproduct

$$\Delta(x_{i,j}) = \sum_s x_{i,s} \otimes x_{s,j}. \quad (4.11)$$

It is shown that if the Hopf algebra  $H$  is finitely generated as an algebra, then its corresponding affine group scheme is a linear algebraic group which can be embedded as a Zariski closed subset of some  $\text{GL}_n$ . [139]

If we have a graduation parameter on the commutative Hopf algebra  $H$ , then there exists a family  $\{H_n\}_{n \geq 0}$  of commutative Hopf subalgebras such that  $H = \bigcup_{n \geq 0} H_n$  and for all  $n$  and  $m$ , we can find some  $k$  where  $H_n \cup H_m \subset H_k$ . It is called a finite type graded Hopf algebra if each component of the grading structure is finitely generated as an algebra which means that for each  $n$ , there exists the corresponding linear algebraic group of the form

$$\mathbb{G}_n(\mathbb{C}) = \text{Spec}(H_n)(\mathbb{C}) < \text{GL}_{m_n}(\mathbb{C}) \quad (4.12)$$

for some  $m_n$ . These algebraic groups can generate the affine group scheme  $\mathbb{G}_H$  corresponding to the Hopf algebra  $H$  via the projective limit

$$\mathbb{G}_H = \lim_{\leftarrow n} \mathbb{G}_n. \quad (4.13)$$

**Theorem 4.1.2.** *There exists a class of infinite dimensional spectral triples which describes the geometry of quantum motions in physical theories with strong coupling constants.*

*Proof.* We are going to build a spectral triple with respect to each Dyson–Schwinger equation in  $\mathcal{S}^{\Phi, g}$  such that the bare coupling constant  $g$  is strong enough to produce non-perturbative situations. For simplicity in notation we set  $g = 1$  and suppose the large Feynman diagram  $X_{\text{DSE}} = \sum_{n \geq 0} X_n$  is the unique solution of an equation DSE. It is discussed that terms  $X_n$  are generators of the free graded connected commutative finite type Hopf subalgebra  $H_{\text{DSE}}(\Phi)$  of the Connes–Kreimer renormalization Hopf algebra  $H_{\text{FG}}(\Phi)$  of Feynman diagrams graded in terms of the number of internal edges. Present  $H_{\text{DSE}}(\Phi)$  in terms of its graded components as follows

$$H_{\text{DSE}}(\Phi) = \bigcup_{n \geq 0} H_{\text{DSE}}^{(n)}(\Phi). \quad (4.14)$$

For each  $n$ , the finite dimensional Hopf subalgebra  $H_{\text{DSE}}^{(n)}(\Phi)$  determines the finite dimensional complex Lie subgroup  $\mathbb{G}_{\text{DSE}}^{(n)}(\mathbb{C})$  which is embedded as a closed subset of the linear algebraic group  $\text{GL}_{m_n}(\mathbb{C})$  for some  $m_n$  with respect to the Zariski topology. Thanks to (4.13), the complex pro-unipotent graded Lie group  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$  is the projective limit of  $\mathbb{G}_{\text{DSE}}^{(n)}(\mathbb{C})$ s as closed subsets of  $\text{GL}_{m_n}(\mathbb{C})$ s.

For each  $m_n$ ,  $\text{GL}_{m_n}(\mathbb{C})$  is a finite dimensional Riemannian manifold with the corresponding spectral triple

$$\mathcal{S}^{(m_n)} := (C^\infty(\text{GL}_{m_n}(\mathbb{C})), L^2(\text{GL}_{m_n}(\mathbb{C}), \mathcal{S}), D_{\text{GL}_{m_n}(\mathbb{C})}). \quad (4.15)$$

A restriction of this spectral triple enables us to build the spectral triple corresponding to the complex Lie group  $\mathbb{G}_{\text{DSE}}^{(n)}(\mathbb{C})$ . We present it by

$$\mathcal{S}_{\text{DSE}}^{(n)} = (A_{\text{DSE}}^{(n)}, \mathbb{H}_{\text{DSE}}^{(n)}, D_{\text{DSE}}^{(n)}). \quad (4.16)$$

Now consider the family  $\{\mathcal{S}_{\text{DSE}}^{(n)}\}_{n \geq 0}$  of countable number of spectral triples derived from components of the graduation structure of the Hopf subalgebra  $H_{\text{DSE}}(\Phi)$  generated by the equation DSE. Let  $\{\alpha_n\}_{n \geq 1}$  be a sequence of non-zero real numbers such that

$$\{ \|(1 + \alpha_n^2 (D_{\text{DSE}}^{(n)})^2)^{-\frac{1}{2}}\|_n \}_{n \geq 1} \quad (4.17)$$

converges to zero when  $n$  tends to infinity where  $\|\cdot\|_n$  is the norm on  $\mathbb{H}_{\text{DSE}}^{(n)}$ . Apply Theorem 4.1.1 to achieve the infinite dimensional spectral triple

$$\mathcal{S}_{\text{DSE}}^\oplus := (A_{\text{DSE}}^\oplus, \mathbb{H}_{\text{DSE}}^\oplus, \overline{D_{\text{DSE}}^\oplus}) \quad (4.18)$$

originated from the five-tuples  $(A_{\text{DSE}}^{(n)}, \mathbb{H}_{\text{DSE}}^{(n)}, D_{\text{DSE}}^{(n)}, \pi_{\text{DSE}}^{(n)}, \alpha_n)$  for each  $n$ . The norm of the Hilbert space  $\mathbb{H}_{\text{DSE}}^{\oplus}$  is given by

$$\|\cdot\|^{\oplus} := \sup_n \|\cdot\|_n. \tag{4.19}$$

In addition, we can check that the representation  $\pi_{\text{DSE}}^{\oplus}$  and the commutator  $[D_{\text{DSE}}^{\oplus}, \pi_{\text{DSE}}^{\oplus}(A_{\text{DSE}}^{\oplus})]$  are bounded where the sequence  $\{\alpha_n\}_{n \geq 1}$  controls the behavior of the sequence  $\{D_{\text{DSE}}^{(n)}\}_{n \geq 1}$ . It means that

$$\sum_n \dim(\text{Ker} D_{\text{DSE}}^{(n)}) < \infty. \tag{4.20}$$

□

It is reasonable to name  $\mathcal{S}_{\text{DSE}}^{\oplus}$  as the non-perturbative spectral triple with respect to the Dyson–Schwinger equation DSE.

*Remark 4.1.3.* If the coupling constant of a physical theory is weak enough where higher order perturbation methods can handle solutions of Dyson–Schwinger equations, then we can describe the geometry of this class of quantum motions in terms of summing a finite number of finite dimensional spectral triples.

**Corollary 4.1.4.** *Each non-perturbative spectral triple has a graphon representation.*

*Proof.* For a given spectral triple  $\mathcal{S}_{\text{DSE}}^{\oplus}$  with respect to the equation DSE, we can associate the unlabeled Feynman graphon class  $[W_{t_{X_{\text{DSE}}}}]$  determined by the labeled graph functions of the infinite tree (or forest)  $t_{X_{\text{DSE}}}$  corresponding to the unique solution of DSE. □

The geometry of the underlying manifold determines the spectrum but the main challenge is the possibility of recovering geometrical information from the spectrum to determine completely the metric or the shape of the boundary. While the answer to this challenge is negative but Noncommutative Geometry can provide an operator theoretic setting to deal with the theory of spectral geometry. The fundamental integral in Noncommutative Geometry is described as the Dixmier trace which extends the Wodzicki residue from pseudodifferential operators on a manifold to a general framework which concerns spectral triples [45]. In other words, for a given spectral triple, we have

$$\overline{\int} T := \text{Res}_{s=0} \text{Tr}(T|D|^{-s}). \tag{4.21}$$

It is possible to adapt this integral to deal with the geometry of Dyson–Schwinger equations. The construction of the non-perturbative spectral triple  $\mathcal{S}_{\text{DSE}}^{\oplus}$  (i.e. Theorem 4.1.2) shows that for each  $n$ ,  $\mathcal{S}_{\text{DSE}}^{(n)}$  is a finite

dimensional spectral triple. Actually, for each  $n \geq 1$ ,  $\mathcal{S}_{\text{DSE}}^{(n)}$  is the result of the restriction of the spectral triple associated to the complex Lie group  $\text{Gl}_{m_n}(\mathbb{C})$  for some  $m_n$ . Therefore for each  $n \geq 1$ , the functional

$$a \longmapsto \text{Tr}^+(a|D_{\text{DSE}}^{(n)}|^{-m_n}) \quad (4.22)$$

determines a differential calculus theory and spectral geometry with respect to the Riemannian volume form for  $\mathcal{S}_{\text{DSE}}^{(n)}$ . This differential calculus is describing the geometric behavior of a quantum motion in terms of its approximation with respect to partial sums of the unique solution  $X_{\text{DSE}}$  of the corresponding equation DSE. Thanks to this interpretation, we may have chance to search for the existence of a noncommutative integral with the general form

$$a^\oplus \longmapsto \text{Tr}_\omega(a^\oplus|D_{\text{DSE}}^\oplus|^{-p}) \quad (4.23)$$

for some  $p \geq 1$  and state  $\omega$ . This noncommutative integral, which is on the basis of the Connes–Dixmier traces, can lead us to build a theory of spectral geometry for large Feynman diagrams.

## 4.2 A noncommutative symplectic geometry model for $\mathcal{S}_{\text{graphon}}^\Phi$

We have discussed that for a given smooth manifold  $M$  with the corresponding complex commutative unital  $*$ -algebra  $C^\infty(M)$ , it is possible to reconstruct  $M$  together with its smooth structure and the objects attached to the manifold (such as smooth vector fields) in terms of the spaces of characters and derivations of the algebra  $C^\infty(M)$ . The choice of the generalization method for the notion of module over a commutative algebra when this algebra is replaced by a noncommutative algebra is related to the choice of the noncommutative generalization of the classical commutative case. There are some approaches to build the algebraic generalizations of differential geometry such as Koszul framework. This framework is on the basis of the space  $\text{Der}(A)$  of all derivations of a commutative associative algebra  $A$ . A graded differential algebra (as the generalization of the algebra of differential forms) determines another graded differential algebra  $C_\wedge(\text{Der}(A), A)$  of  $A$ -valued Chevalley–Eilenberg cochains of the Lie algebra  $\text{Der}(A)$ . The Koszul framework admits a generalization to the noncommutative setting via differential calculus with respect to derivations. It is actually the suitable differential calculus for Quantum Mechanics. In this setting, an algebraic version of differential geometry in terms of a commutative associative algebra  $A$ ,  $A$ -modules and connections on these modules have been designed. If we replace the commutativity of the algebra with non-commutativity, then different classes of generalizations of the notion of

a module over a noncommutative algebra can be resulted such as the notions of left or right  $A$ -modules and left or right  $Z(A)$ -modules. [52, 53, 54, 56, 127]

The algebraic interpretation of Classical Geometry requires a commutative setting where we have two options to fix the algebra. The first one is the real commutative algebra  $A_{\mathbb{R}}$  of smooth real valued functions where its complexified extension is canonically a complex commutative  $*$ -algebra. The second one is the complex commutative  $*$ -algebra  $A_{\mathbb{C}}$  of smooth complex valued functions where the set  $A^{\text{hermitian}}$  of its hermitian elements is a real commutative algebra and thus  $A_{\mathbb{C}}$  will be the complexification of  $A^{\text{hermitian}}$ .

The algebraic interpretation of Quantum Physics requires a noncommutative setting where we already have two classes of generalizations of the algebra of real valued functions. The first one is the real Jordan algebra  $A^{\text{hermitian}}$  of all hermitian elements of a complex noncommutative associative  $*$ -algebra  $A$ . The second one is a real associative noncommutative algebra. The most important challenge at this level is the choice of the mathematical machinery to build a differential calculus theory. One generalization approach has been formulated by Connes in terms of theory of cyclic cohomology of an algebra where the generalization of the cohomology of a manifold in Noncommutative Geometry is actually the reduced cyclic homology of an algebra which replaces the standard algebra of smooth functions. As we know the computation of cohomology theory of classical manifolds is not a unique way and furthermore, we can expect the construction of noncommutative generalizations of differential geometry for which the generalization of de Rham theorem fails to be true. These facts show that any cochain complex, which has the reduced cyclic homology as cohomology, can not be an acceptable generalization of differential forms. Thanks to these efforts, the best candidate for the construction of a noncommutative differential calculus is on the basis of the space of derivations as generalizations of vector fields. This platform, which had been initiated and developed by Kozul and Dubois-Violette, has already provided the foundations of a noncommutative symplectic geometry for the study of quantum theories. [52, 56, 57, 99]

In a different story, the Connes–Kreimer Hopf algebraic renormalization is the direct result of the existence of the Hopf–Birkhoff factorization on a class of Lie groups. The original source of this particular factorization is the multiplicativity of perturbative renormalization which is encoded by the theory of Rota–Baxter algebras. The determination of a class of Hopf subalgebras via Dyson–Schwinger equations together with the renormalization of these equations under Dimensional Regularization had been applied to build a class of Dubois–Violette’s differential graded algebras which encode the geometric information of these equations in the context of noncommutative differential forms. The basic idea in this approach is to associate a noncommutative algebra to each equation DSE and then build a theory of noncommutative (symplectic) geometry to encode the behavior of infinitesimal characters corresponding to Feynman diagrams which contribute to

the solution of DSE under the renormalization process. This platform is also useful to formulate a new interpretation of the Connes–Kreimer non-perturbative renormalization group in the context of quantum integrable systems. [166]

Our main task in this part is to develop this new Hopf algebraic approach and explain the construction of a noncommutative differential calculus theory for the topological Hopf algebra  $\mathcal{S}_{\text{graphon}}^\Phi$  of Feynman graphons which is originated from the BPHZ non-perturbative renormalization (i.i. Theorem 2.4.1) and the theory of Rota–Baxter algebras ([80]). The basic step is to associate a (noncommutative) algebra to  $\mathcal{S}_{\text{graphon}}^\Phi$  and then build a theory of noncommutative differential forms on this algebra. Our study can determine a new class of non-perturbative quantum integrable systems generated by solutions of Dyson–Schwinger equations.

The BPHZ renormalization program is on the basis of Dimensional Regularization and Minimal Subtraction map  $R_{\text{ms}}$  which is an idempotent Rota–Baxter map on the regularization algebra of Laurent series with finite pole parts. We want to show that the application of each step of the BPHZ renormalization program to Feynman graphons can determine a theory of noncommutative differential calculus. These differential calculi are useful to formulate some new geometric tools for the evaluation of solutions of Dyson–Schwinger equations under the renormalization procedure.

**Theorem 4.2.1.** *The Minimal Subtraction map  $R_{\text{ms}}$  in the BPHZ renormalization of Feynman graphons (i.e. Theorem 2.3.8 and Theorem 2.4.1) determines a noncommutative symplectic geometry model for the Hopf algebra  $\mathcal{S}_{\text{graphon}}^\Phi$ .*

*Proof.* Consider  $A_{\text{dr}} := \mathcal{A}_+ \oplus \mathcal{A}_-$  as the algebra of Laurent series with finite pole parts which encodes Dimensional Regularization (i.e. regularization scheme) and  $R_{\text{ms}}$  as the linear map on  $A_{\text{dr}}$  which projects a series onto its corresponding pole parts. The pair  $(A_{\text{dr}}, R_{\text{ms}})$  satisfies the conditions of a Rota–Baxter algebra which enables us to define a new family of convolution products on the space  $L(\mathcal{S}_{\text{graphon}}^\Phi, A_{\text{dr}})$  of linear maps in terms of the following steps.

- Lift the map  $R_{\text{ms}}$  onto a new map  $\mathcal{R}$  on  $L(\mathcal{S}_{\text{graphon}}^\Phi, A_{\text{dr}})$  defined by

$$\mathcal{R}(\phi) := R_{\text{ms}} \circ \phi. \quad (4.24)$$

The pair  $(L(\mathcal{S}_{\text{graphon}}^\Phi, A_{\text{dr}}), \mathcal{R})$  is a new Rota–Baxter algebra.

- Set  $\hat{\mathcal{R}} := Id - \mathcal{R}$  and for each  $\lambda \in \mathbb{R}$ , define a new class of Nijenhuis maps  $\mathcal{R}_\lambda := \mathcal{R} - \lambda \hat{\mathcal{R}}$ .

- Define a new family of products on  $L(\mathcal{S}_{\text{graphon}}^\Phi, A_{\text{dr}})$  with the general form

$$\phi_1 \circ_\lambda \phi_2 := \mathcal{R}_\lambda(\phi_1) *_{\text{gr}} \phi_2 + \phi_1 *_{\text{gr}} \mathcal{R}_\lambda(\phi_2) - \mathcal{R}_\lambda(\phi_1 *_{\text{gr}} \phi_2) \quad (4.25)$$

such that  $*_{\text{gr}}$  is the convolution product with respect to the coproduct  $\Delta_{\text{graphon}}$  on Feynman graphons (2.46) where we have

$$\psi_1 *_{\text{gr}} \psi_2([W_\Gamma]) = \sum \psi_1([W_{\Gamma'}]) \psi_2([W_{\Gamma''}]), \quad \Delta_{\text{graphon}}([W_\Gamma]) = \sum [W_{\Gamma'}] \otimes [W_{\Gamma''}]. \quad (4.26)$$

The non-cocommutativity of the renormalization Hopf algebra of Feynman graphons shows that the convolution product  $*_{\text{gr}}$  and new products  $\circ_\lambda$  are noncommutative.

The Nijenhuis property of  $\mathcal{R}_\lambda$  shows that

$$\mathcal{R}_\lambda(\phi_1 \circ_\lambda \phi_2) = \mathcal{R}_\lambda(\phi_1) *_{\text{gr}} \mathcal{R}_\lambda(\phi_2) \quad (4.27)$$

which supports the associativity of these new products.

Now set

$$C_\lambda^{\text{graphon}} := (L(\mathcal{S}_{\text{graphon}}^\Phi, A_{\text{dr}}), \circ_\lambda) \quad (4.28)$$

as the unital associative noncommutative algebra generated by the Minimal Subtraction map. For each  $\lambda$ , the commutator with respect to  $\circ_\lambda$  determines a new Lie bracket  $[\cdot, \cdot]_\lambda$  on the space  $L(\mathcal{S}_{\text{graphon}}^\Phi, A_{\text{dr}})$  given by

$$[\phi_1, \phi_2]_\lambda = [\mathcal{R}_\lambda(\phi_1), \phi_2] + [\phi_1, \mathcal{R}_\lambda(\phi_2)] - \mathcal{R}_\lambda[\phi_1, \phi_2]. \quad (4.29)$$

This class of Lie brackets is the key tool for us to build a new noncommutative differential calculus on  $C_\lambda^{\text{graphon}}$  in terms of the following steps.

- Set  $\text{Der}_{\text{graphon}}^\lambda$  as the space of all derivations on  $C_\lambda^{\text{graphon}}$ . It has all linear maps such as  $\theta : C_\lambda^{\text{graphon}} \rightarrow C_\lambda^{\text{graphon}}$  which enjoys the Leibniz rule.

- The Lie bracket  $[\cdot, \cdot]_\lambda$  determines naturally the Poisson bracket  $\{\cdot, \cdot\}_\lambda$  on  $C_\lambda^{\text{graphon}}$ . For each  $\phi \in C_\lambda^{\text{graphon}}$ , its corresponding Hamiltonian derivation is defined by

$$\text{ham}(\phi) : \psi \mapsto \{\phi, \psi\}_\lambda. \quad (4.30)$$

Set  $\text{Ham}_{\text{graphon}}^\lambda$  as the  $Z(C_\lambda^{\text{graphon}})$ -module generated by all Hamiltonian derivations on  $C_\lambda^{\text{graphon}}$ .

- Define

$$\Omega_{\lambda, \text{graphon}}^\bullet(C_\lambda^{\text{graphon}}) := \left( \bigoplus_{n \geq 0} \Omega_{\lambda, \text{graphon}}^n(C_\lambda^{\text{graphon}}), d_\lambda \right) \quad (4.31)$$

as the differential graded algebra on  $C_\lambda^{\text{graphon}}$ . For each  $n \geq 1$ ,  $\Omega_{\lambda, \text{graphon}}^n(C_\lambda^{\text{graphon}})$  is the space of all  $Z(C_\lambda^{\text{graphon}})$ -multilinear antisymmetric mappings from  $\text{Ham}_{\text{graphon}}^\lambda \times \dots \times \text{Ham}_{\text{graphon}}^\lambda$  into  $C_\lambda^{\text{graphon}}$ . The zero component of this differential graded algebra is the initial algebra  $C_\lambda^{\text{graphon}}$ . In addition, for each  $\omega \in \Omega_{\lambda, \text{graphon}}^n(C_\lambda^{\text{graphon}})$  and  $\theta_i \in \text{Ham}_{\text{graphon}}^\lambda$ , the anti-derivative degree one differential operator  $d_\lambda$  is defined by

$$d_\lambda \omega(\theta_0, \dots, \theta_n) := \sum_{k=0}^n (-1)^k \theta_k \omega(\theta_0, \dots, \hat{\theta}_k, \dots, \theta_n) +$$

$$\sum_{0 \leq r < s \leq n} (-1)^{r+s} \omega([\theta_r, \theta_s]_\lambda, \theta_0, \dots, \hat{\theta}_r, \dots, \hat{\theta}_s, \dots, \theta_n) \quad (4.32)$$

such that we have  $d_\lambda^2 = 0$ .

Thanks to this differential graded (Lie) algebraic machinery, we can determine a new class of symplectic structures generated by the Lie bracket  $[\cdot, \cdot]_\lambda$ . Define

$$\begin{aligned} \omega_\lambda : \text{Ham}_{\text{graphon}}^\lambda \times \text{Ham}_{\text{graphon}}^\lambda &\longrightarrow C_\lambda^{\text{graphon}} \\ \omega_\lambda(\theta, \theta') &:= \sum_{i,j} u_i \circ_\lambda v_j \circ_\lambda [f_i, h_j]_\lambda \end{aligned} \quad (4.33)$$

such that  $\{f_1, \dots, f_m, h_1, \dots, h_n\} \subset C_\lambda^{\text{graphon}}$ ,  $\{u_1, \dots, u_m, v_1, \dots, v_n\} \subset Z(C_\lambda^{\text{graphon}})$  and

$$\theta = \sum_i u_i \circ_\lambda \text{ham}(f_i), \quad \theta' = \sum_j v_j \circ_\lambda \text{ham}(h_j). \quad (4.34)$$

The differential form  $\omega_\lambda$  is a  $Z(C_\lambda^{\text{graphon}})$ -bilinear anti-symmetric non-degenerate closed 2-form in  $\Omega_{\lambda, \text{graphon}}^2(C_\lambda^{\text{graphon}})$ . For a given  $f \in C_\lambda^{\text{graphon}}$  with the corresponding symplectic vector field  $\theta_f^\lambda$ , we have

$$\{f, g\}_\lambda := i_{\theta_f^\lambda}(d_\lambda g) \quad (4.35)$$

such that

$$i_\theta(\omega_0 d_\lambda \omega_1 \dots d_\lambda \omega_n) = \sum_{j=1}^n (-1)^{j-1} \omega_0 d_\lambda \omega_1 \dots \theta(\omega_j) \dots d_\lambda \omega_n \quad (4.36)$$

is the super-derivation of degree -1. We can check that

$$\{f, g\}_\lambda = i_{\theta_f^\lambda} i_{\theta_g^\lambda} \omega_\lambda. \quad (4.37)$$

□

**Theorem 4.2.2.** *The Dimensional Regularization in the BPHZ renormalization of Feynman graphons (i.e. Theorem 2.3.8 and Theorem 2.4.1) determines a noncommutative symplectic geometry model for the Hopf algebra  $\mathcal{S}_{\text{graphon}}^\Phi$ .*

*Proof.* There exists a universal setting for the construction of a new Nijenhuis algebra from a given commutative unital algebra  $A_{\text{dr}}$ . We present the product of formal series by  $m(f, g) = [fg]$  and consider the graded tensor module  $T(A_{\text{dr}}) := \bigoplus_{n \geq 0} A_{\text{dr}}^{\otimes n}$  generated by expressions such as  $f_1 \otimes f_2 \otimes \dots \otimes f_n$ . From now we name each series in  $A_{\text{dr}}$  as a letter and each sequence  $U := f_1 f_2 \dots f_n$  of letters as a word with the length  $n$ . The empty word  $e$  which has the length zero is the unit object in  $T(A_{\text{dr}})$ .

By induction we can define a new shuffle product on  $T(A_{\text{dr}})$  given by

$$fU \odot gV := f(U \odot gV) + g(fU \odot V) - e[fg](U \odot V) \quad (4.38)$$

which is unital and associative. The product (4.38) defines another new quasi-shuffle product on  $\overline{T}(A_{\text{dr}}) := \bigoplus_{n \geq 1} A_{\text{dr}}^{\otimes n}$  given by

$$fU \odot gV := [fg](U \odot V) \quad (4.39)$$

which is also unital and associative.

The linear map  $B_e^+$  on  $\overline{T}(A_{\text{dr}})$  sends each word  $U$  of length  $n$  to the new word  $eU$  of length  $n + 1$ . Thanks to investigations discussed in [66], the triple  $(\overline{T}(A_{\text{dr}}), \odot, B_e^+)$  is the universal Nijenhuis algebra in a category of Nijenhuis algebras generated by the initial algebra  $A_{\text{dr}}$ .

Now we can lift the linear map  $B_e^+$  onto  $L(\mathcal{S}_{\text{graphon}}^\Phi, \overline{T}(A_{\text{dr}}))$  to define the new Nijenhuis map

$$\mathcal{N}_{\text{graphon}}(\psi) := B_e^+ \circ \psi. \quad (4.40)$$

The resulting Nijenhuis algebra is the key tool for us to build a new product  $\circ_u$  on  $L(\mathcal{S}_{\text{graphon}}^\Phi, \overline{T}(A_{\text{dr}}))$  defined by

$$\psi_1 \circ_u \psi_2 := \mathcal{N}_{\text{graphon}}(\psi_1) *_{\odot} \psi_2 + \psi_1 *_{\odot} \mathcal{N}_{\text{graphon}}(\psi_2) - \mathcal{N}_{\text{graphon}}(\psi_1 *_{\odot} \psi_2) \quad (4.41)$$

such that  $*_{\odot}$  is the convolution product with respect to the coproduct  $\Delta_{\text{graphon}}$  on Feynman graphons (2.46) and the product  $\odot$ . We have

$$\psi_1 *_{\odot} \psi_2([W_\Gamma]) = \sum \psi_1([W_{\Gamma'}]) \odot \psi_2([W_{\Gamma''}]), \quad \Delta_{\text{graphon}}([W_\Gamma]) = \sum [W_{\Gamma'}] \otimes [W_{\Gamma''}]. \quad (4.42)$$

The non-cocommutativity of the renormalization Hopf algebra of Feynman graphons shows that the convolution product  $*_{\odot}$  and the new product  $\circ_u$  are noncommutative. In addition, the Nijenhuis property shows that

$$\mathcal{N}_{\text{graphon}}(\psi_1 \circ_u \psi_2) = \mathcal{N}_{\text{graphon}}(\psi_1) *_{\odot} \mathcal{N}_{\text{graphon}}(\psi_2) \quad (4.43)$$

which supports the associativity of this new product.

Set

$$C_u^{\text{graphon}} := (L(\mathcal{S}_{\text{graphon}}^\Phi, \overline{T}(A_{\text{dr}})), \circ_u) \quad (4.44)$$

as the unital associative noncommutative algebra generated by Dimensional Regularization. In addition, the commutator with respect to the product  $\circ_u$  determines a new Lie bracket  $[\cdot, \cdot]_u$  on the space  $L(\mathcal{S}_{\text{graphon}}^\Phi, \overline{T}(A_{\text{dr}}))$  given by

$$[\psi_1, \psi_2]_u := [\mathcal{N}_{\text{graphon}}(\psi_1), \psi_2] + [\psi_1, \mathcal{N}_{\text{graphon}}(\psi_2)] - \mathcal{N}_{\text{graphon}}[\psi_1, \psi_2]. \quad (4.45)$$

This class of Lie brackets enables us to build a new noncommutative differential calculus on  $C_u^{\text{graphon}}$  in terms of the following steps.

- Set  $\text{Der}_{\text{graphon}}^u$  as the space of all derivations on  $C_u^{\text{graphon}}$ . It has all linear maps such as  $\theta : C_u^{\text{graphon}} \longrightarrow C_u^{\text{graphon}}$  which enjoys the Leibniz rule.

- The Lie bracket  $[\cdot, \cdot]_u$  naturally determines the Poisson bracket  $\{\cdot, \cdot\}_u$  on  $C_u^{\text{graphon}}$ . For each  $\phi \in C_u^{\text{graphon}}$ , its corresponding Hamiltonian derivation is defined by

$$\text{ham}(\phi) : \psi \longmapsto \{\phi, \psi\}_u. \quad (4.46)$$

Set  $\text{Ham}_{\text{graphon}}^u$  as the  $Z(C_u^{\text{graphon}})$ -module generated by all Hamiltonian derivations on  $C_u^{\text{graphon}}$ .

- Define

$$\Omega_{u, \text{graphon}}^\bullet(C_u^{\text{graphon}}) := \left( \bigoplus_{n \geq 0} \Omega_{u, \text{graphon}}^n(C_u^{\text{graphon}}), d_u \right) \quad (4.47)$$

as the differential graded algebra on  $C_u^{\text{graphon}}$ . For each  $n \geq 1$ ,  $\Omega_{u, \text{graphon}}^n(C_u^{\text{graphon}})$  is the space of all  $Z(C_u^{\text{graphon}})$ -multilinear antisymmetric mappings from  $\text{Ham}_{\text{graphon}}^u \times \dots \times \text{Ham}_{\text{graphon}}^u$  into  $C_u^{\text{graphon}}$ . The zero component of this differential graded algebra is the initial algebra  $C_u^{\text{graphon}}$ . In addition, for each  $\omega \in \Omega_{u, \text{graphon}}^n(C_u^{\text{graphon}})$  and  $\theta_i \in \text{Ham}_{\text{graphon}}^u$ , the anti-derivative degree one differential operator  $d_u$  is defined by

$$\begin{aligned} d_u \omega(\theta_0, \dots, \theta_n) &:= \sum_{k=0}^n (-1)^k \theta_k \omega(\theta_0, \dots, \hat{\theta}_k, \dots, \theta_n) + \\ &\sum_{0 \leq r < s \leq n} (-1)^{r+s} \omega([\theta_r, \theta_s]_u, \theta_0, \dots, \hat{\theta}_r, \dots, \hat{\theta}_s, \dots, \theta_n) \end{aligned} \quad (4.48)$$

such that we have  $d_u^2 = 0$ .

Thanks to this differential graded (Lie) algebraic machinery, we can determine a new class of symplectic structures generated by the Lie bracket  $[\cdot, \cdot]_u$ . Define

$$\begin{aligned} \omega_u : \text{Ham}_{\text{graphon}}^u \times \text{Ham}_{\text{graphon}}^u &\longrightarrow C_u^{\text{graphon}} \\ \omega_u(\theta, \theta') &:= \sum_{i,j} u_i \circ_u v_j \circ_u [f_i, h_j]_u \end{aligned} \quad (4.49)$$

such that  $\{f_1, \dots, f_m, h_1, \dots, h_n\} \subset C_u^{\text{graphon}}$ ,  $\{u_1, \dots, u_m, v_1, \dots, v_n\} \subset Z(C_u^{\text{graphon}})$ ,

$$\theta = \sum_i u_i \circ_u \text{ham}(f_i), \quad \theta' = \sum_j v_j \circ_u \text{ham}(h_j). \quad (4.50)$$

The differential form  $\omega_u$  is a  $Z(C_u^{\text{graphon}})$ -bilinear anti-symmetric non-degenerate closed 2-form in  $\Omega_{u, \text{graphon}}^2(C_u^{\text{graphon}})$ . For a given  $f \in C_u^{\text{graphon}}$  with the corresponding symplectic vector field  $\theta_f^u$ , we have

$$\{f, g\}_u := i_{\theta_f^u}(d_u g) \quad (4.51)$$

such that

$$i_\theta(\omega_0 d_u \omega_1 \dots d_u \omega_n) = \sum_{j=1}^n (-1)^{j-1} \omega_0 d_u \omega_1 \dots \theta(\omega_j) \dots d_u \omega_n \quad (4.52)$$

is the super-derivation of degree -1. We can check that

$$\{f, g\}_u = i_{\theta_f^u} i_{\theta_g^u} \omega_u. \quad (4.53)$$

□

The modified version of the Connes–Kreimer Renormalization Group for Feynman graphons is defined by Lemma 2.4.2 where we should apply the filtration parameter on Feynman graphons given by Theorem 2.3.10. Thanks to the built noncommutative differential geometry on  $\mathcal{S}_{\text{graphon}}^\Phi$ , we can provide a new geometric interpretation for the behavior of the Connes–Kreimer Renormalization Group whenever it acts on large Feynman diagrams.

**Lemma 4.2.3.** *Let  $\{F_t\}_t$  be the Renormalization Group on Feynman graphons defined by Lemma 2.4.2. For each  $t$  and any large Feynman diagram  $X$ ,  $F_t(X)$  is the convergent limit of the sequence  $\{F_t(X_n)\}_{n \geq 1}$  with respect to the cut-distance topology.*

*Proof.* Consider the loop  $\gamma_\mu \in \text{Loop}(\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C}), \mu)$  which encodes the Feynman rules characters in the renormalization Hopf algebra of Feynman graphons with respect to a given physical theory  $\Phi$ . For a given Dyson–Schwinger equation DSE with the unique solution  $X = \sum_{n \geq 0} X_n$ , we have

$$\gamma_\mu(z)([W_X]) := U_\mu^z(X) \quad (4.54)$$

such that  $U_\mu^z(X)$  is a Laurent series as the regularized large Feynman integral with respect to  $X$ . The one-parameter group  $\{\theta_t\}_{t \in \mathbb{C}}$  sends the unlabeled graphon class  $[W_X]$  to the filtration rank of the equation DSE (i.e. Theorem 2.3.10). The resulting Renormalization Group  $\{F_t\}_t$  (i.e. Lemma 2.4.2) is a subgroup of  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$  which means that for each  $t$ ,  $F_t$  is a linear homomorphism. On the other hand, thanks to Theorem 2.3.7, we know that the large Feynman diagram  $X$  is the convergent limit of the sequence of its partial sums with respect to the cut-distance topology. Therefore we have

$$\begin{aligned} F_t(X) &= F_t(\lim_{m \rightarrow \infty} Y_m) = F_t(\lim_{m \rightarrow \infty} \sum_{n=1}^m X_n) = \\ \lim_{m \rightarrow \infty} \sum_{n=1}^m F_t(X_n) &= \lim_{m \rightarrow \infty} \sum_{n=1}^m \lim_{z \rightarrow 0} \gamma_-(z)(X_n) \theta_{tz}(\gamma_-^{-1}(z)(X_n)) \\ &= \lim_{m \rightarrow \infty} \lim_{z \rightarrow 0} \sum_{n=1}^m \gamma_-(z)(X_n) \theta_{tz}(\gamma_-^{-1}(z)(X_n)) \end{aligned} \quad (4.55)$$

such that according to Proposition 1.47 in [46], for each  $t$ ,  $F_t(X_n)$  is a polynomial in  $t$ .  $\square$

**Corollary 4.2.4.** *The non-perturbative Connes–Kreimer Renormalization Group on Feynman graphons can determine an infinite dimensional integrable system.*

*Proof.* We work on the unital associative noncommutative algebra  $C_0^{\text{graphon}} := (L(\mathcal{S}_{\text{graphon}}^\Phi, A_{\text{dr}}), \circ_0)$  generated by the Minimal Subtraction map for  $\lambda = 0$ . Thanks to Theorem 4.2.1, consider the differential graded algebra

$$\Omega_{0,\text{graphon}}^\bullet(C_0^{\text{graphon}}) := \left( \bigoplus_{n \geq 0} \Omega_{0,\text{graphon}}^n(C_0^{\text{graphon}}), d_0 \right) \quad (4.56)$$

with respect to the Lie bracket  $[\cdot, \cdot]_0$ . Each character  $F_t$  of the Renormalization Group  $\{F_t\}_t$  given by Lemma 4.2.3 is an object in the algebra  $C_0^{\text{graphon}}$ . Therefore the motion integral equation with respect to the character  $F_{t_0}$  is given by the equation

$$\{f, F_{t_0}\}_0 = 0 \quad (4.57)$$

such that  $f \in C_0^{\text{graphon}}$ . Thanks to the existence of a noncommutative symplectic form  $\omega_0$  on  $C_0^{\text{graphon}}$  with respect to the Lie bracket  $[\cdot, \cdot]_0$  (i.e. Theorem 4.2.1), the motion integral can be determined by the equation

$$\{f, F_{t_0}\}_0 = i_{\theta_{F_{t_0}}^0} i_{\theta_f^0} \omega_0 = w_0(\theta_{F_{t_0}}^0, \theta_f^0) = [f, F_{t_0}] = 0. \quad (4.58)$$

On the one hand, from the definition of the deformed Lie bracket  $[\cdot, \cdot]_0$  and the idempotent Rota–Baxter property of  $(A_{\text{dr}}, R_{\text{ms}})$ , we have

$$\{F_t, F_s\}_0 = [R_{\text{ms}}(F_t), F_s] + [F_t, R_{\text{ms}}(F_s)] - R_{\text{ms}}([F_t, F_s]). \quad (4.59)$$

On the other hand, for each  $t$ ,  $F_t([W_\Gamma])$  is a polynomial in  $t$  which means that  $R_{\text{ms}}(F_t([W_\Gamma])) = 0$  and in addition, for each  $s, t$ ,  $F_t * F_s = F_{t+s}$ . Thanks to these facts, we can observe that for each  $s, t$ ,

$$\{F_t, F_s\}_0 = 0. \quad (4.60)$$

$\square$

## Chapter 5

# A theory of functional analysis for large Feynman diagrams

- *The Haar integration on  $S^{\Phi, g}$  and its application*
- *The Gâteaux differential calculus on  $S^{\Phi, g}$  and its application*
  - *Feynman random graphs via homomorphism densities*
  - *Differentiability*

This chapter aims to provide the foundations of a functional analysis machinery for the study of large Feynman diagrams which contribute to solutions of Dyson–Schwinger equations. We build an integration theory and a differentiation theory for the functionals on the space  $\mathcal{S}^{\Phi,g}$  with respect to a given strongly coupled gauge field theory  $\Phi$ . The space  $\mathcal{S}^{\Phi,g}$  can be embedded into the Hopf algebra  $H_{\text{FG}}^{\text{cut}}(\Phi)$  of (large) Feynman diagrams topologically completed with the cut-distance topology. The space  $H_{\text{FG}}^{\text{cut}}(\Phi)$  consists of all Feynman diagrams and their corresponding finite or infinite formal expansions where solutions of all non-perturbative Dyson–Schwinger equations belong to the boundary region of this compact topological Hopf algebra. As we have shown in the previous parts, this enriched Hopf algebra of Feynman diagrams can be encoded in terms of the renormalization Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$  of Feynman graphons.

At the first step, we build a measure theory on  $\mathcal{S}^{\Phi,g}$  where we equip this space with a new topological group structure which leads us to a new Haar measure integration theory for functionals on large Feynman diagrams. Then we deal with some applications of the resulting measure space where a new generalization of the classical Johnson–Lapidus Dyson series for large Feynman diagrams will be obtained. In addition, we work on the construction of a new Fourier transformation machinery on the Banach algebra  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  which enables us to describe the evolution of large Feynman diagrams on the basis of their corresponding partial sums under a functional setting. At the second step, we concern the Gâteaux differentiability of real valued functionals on  $\mathcal{S}_{\text{graphon}}^{\Phi}$  where we obtain Taylor expansion representations for these functionals under some conditions.

Achievements of these steps can be adapted for the level of the topological Hopf algebra  $H_{\text{FG}}^{\text{cut}}(\Phi)$ . In other words, the promising differential calculus and integration theory enable us to describe the dynamics of topological regions of Feynman diagrams on the basis of the behavior of functionals with respect to the built Haar integration theory and Gâteaux differentiation theory on Feynman graphons.

## 5.1 *The Haar integration on $\mathcal{S}^{\Phi,g}$ and its application*

For a given physical theory  $\Phi$  with strong coupling constant  $g \geq 1$ , set  $V(\Phi)$  as the set of all vertices which appear in Feynman diagrams and their corresponding formal expansions as interactions among elementary particles. This infinite countable set allows us to count interactions independent of their physical types. Set  $K_{V(\Phi)}$  as the complete graph with  $V(\Phi)$  as the set of vertices and all possible edges among these vertices except self-loops. The collection  $\{0, 1\}^{K_{V(\Phi)}}$ , as the family of all functions from  $K_{V(\Phi)}$  to  $\{0, 1\}$ , allows us to characterize Feynman diagrams which contribute to physical

theory  $\Phi$ . Therefore, each (large) Feynman diagram  $\Gamma$  can be determined by its corresponding characteristic function  $\chi_\Gamma$  which sends vertices  $v \in K_{V(\Phi)}$  to 1 if  $v \in \Gamma$  and sends other vertices to 0. For each edge  $e \in K_{V(\Phi)}$ , if  $V_e$  be the set of vertices in  $V(\Phi)$  which are attached to the edge  $e$ , then the infinite Cartesian product  $\times_{e \in K_{V(\Phi)}} \mathcal{P}(K_{V_e})$  has enough vertices and edges to contain all Feynman graphs in  $H_{\text{FG}}^{\text{cut}}(\Phi)$ .

**Lemma 5.1.1.**  $\mathcal{S}^{\Phi,g}$  can be equipped with an abelian compact Hausdorff topological group structure.

*Proof.* For a given Dyson–Schwinger DSE in  $\mathcal{S}^{\Phi,g}$  with the unique solution  $X_{\text{DSE}}$ , we can associate the characteristic function  $\chi_{X_{\text{DSE}}} \in \{0,1\}^{K_{V(\Phi)}}$  to DSE. The function  $\chi_{X_{\text{DSE}}}$  allows us to identify vertices and edges which contribute to the large Feynman diagram  $X_{\text{DSE}}$ . It means that we can embed the collection  $\mathcal{S}^{\Phi,g}$  into  $\{0,1\}^{K_{V(\Phi)}}$  which is useful to define new addition and multiplication operators on Dyson–Schwinger equations in terms of the pointwise addition and multiplication of their corresponding characteristic functions. These operators provide a vector space structure generated by  $X_{\text{DSE}}$  for each DSE where as the result, we have a commutative  $\mathbb{Z}_2$ -algebra structure on  $\mathcal{S}^{\Phi,g}$ .

In addition, we can also define a new binary operation on  $\mathcal{S}^{\Phi,g}$  in terms of the symmetric difference operator

$$(\text{DSE}_1, \text{DSE}_2) \mapsto X_{\text{DSE}_1} \Delta X_{\text{DSE}_2}. \quad (5.1)$$

By adding the empty graph  $\mathbb{I}$  to  $\mathcal{S}^{\Phi,g}$  as the zero element, the pair  $(\mathcal{S}^{\Phi,g}, \Delta)$  is an abelian group which can be equipped with a compatible topology to obtain a compact topological group. For this purpose, suppose  $\alpha$  be a bijection between  $K_{V(\Phi)}$  and the set of natural numbers  $\mathbb{N}$ . For the fixed coupling constant  $g \geq 1$  and each  $\epsilon > 0$ , define a new map  $d_{g,\alpha,\epsilon} : \mathcal{S}^{\Phi,g} \times \mathcal{S}^{\Phi,g} \rightarrow [0, \infty)$  given by

$$d_{g,\alpha,\epsilon}(\text{DSE}_1, \text{DSE}_2) := \sum_{e \in X_{\text{DSE}_1} \Delta X_{\text{DSE}_2}} (g + \epsilon)^{-\alpha(e)} \quad (5.2)$$

such that the sum is taken over vertices such as  $e$  which belongs to only one of the large Feynman diagrams  $X_{\text{DSE}_1}$  or  $X_{\text{DSE}_2}$ . The map  $d_{g,\alpha,\epsilon}$  is a translation invariant metric such that  $d_{g,\alpha,\epsilon_1}$  and  $d_{g,\alpha,\epsilon_2}$  have the equivalent topology.

The space  $\mathcal{S}^{\Phi,g}$  together with the symmetric difference operator and the topology generated by the metric  $d_{g,\alpha,\epsilon}$  is a compact Hausdorff abelian topological group.  $\square$

Thanks to the translation-invariant metric  $d_{g,\alpha,\epsilon}$  defined by Lemma 5.1.1, for each equation DSE in  $\mathcal{S}^{\Phi,g}$  with the corresponding large Feynman diagram  $X_{\text{DSE}}$  define

$$\| X_{\text{DSE}} \|_{g,\alpha,\epsilon} := d_{g,\alpha,\epsilon}(\mathbb{I}, X_{\text{DSE}}). \quad (5.3)$$

In this setting, a sequence  $\{\Gamma_n\}_{n \geq 1}$  of large Feynman diagrams in  $\mathcal{S}^{\Phi, g}$  is convergent to a unique large Feynman diagram  $\Gamma$ , if each indicator sequence  $\{\mathbf{1}_{e \in \Gamma_n}\}_{n \geq 1}$  converges to the indicator  $\mathbf{1}_{e \in \Gamma}$  for any  $e \in K_{V(\Phi)}$ .

**Theorem 5.1.2.** *The topological group  $\mathcal{S}^{\Phi, g}$  can be equipped with the Haar measure  $\mu_{\text{Haar}}$ .*

*Proof.* Lemma 5.1.1 supports the existence of the unique Haar measure  $\mu_{\text{Haar}}$  on  $\mathcal{S}^{\Phi, g}$  originated from the compact topological structure. We build this measure which is actually of the type Bernoulli probability.

Consider the product  $\sigma$ -algebra  $\sum_{\text{prod}}$  on  $\mathcal{S}^{\Phi, g}$  generated by cylinder sets

$$S_{\Gamma_0} := \times_{\Gamma \neq \Gamma_0} \{\mathbb{I}, \{\Gamma\}\} \times \{\Gamma_0\} \quad (5.4)$$

for each large Feynman diagram  $\Gamma_0$  corresponding to an equation DSE<sub>0</sub> in  $\mathcal{S}^{\Phi, g}$ . For each large Feynman diagram  $\Gamma$ , the characteristic function  $\chi_{\Gamma}$  is useful to see  $\Gamma$  as an infinite countable subset of vertices in  $K_{V(\Phi)}$  which contribute to the unique solution of the equation DSE <sub>$\Gamma$</sub> . Therefore each function  $P \in \{0, 1\}^{K_{V(\Phi)}}$  can identify a new function  $\tilde{P} : \mathcal{S}^{\Phi, g} \rightarrow [0, 1]$  which can be applied to define the measure  $\mu_{\tilde{P}}$  on the  $\sigma$ -algebra  $\sum_{\text{prod}}$  in terms of the following steps.

- For finite intersections of cylinder sets  $S_{\Gamma_1}, \dots, S_{\Gamma_n}$ , we have

$$\mu_{\tilde{P}}(S_{\Gamma_1} \cap S_{\Gamma_2} \cap \dots \cap S_{\Gamma_n}) = \prod_{i=1}^n \tilde{P}(\Gamma_i) \quad (5.5)$$

for large Feynman diagrams  $\Gamma_1, \dots, \Gamma_n$  as solutions of Dyson–Schwinger equations DSE<sub>1</sub>, ..., DSE<sub>n</sub> in  $\mathcal{S}^{\Phi, g}$ .

- The function  $\mu_{\tilde{P}}$  is a probability measure on  $\mathcal{S}^{\Phi, g}$  which can be presented with the general form

$$\mu_{\tilde{P}} := \prod_{X \in \mathcal{S}^{\Phi, g}} \mu_{\tilde{P}, X}. \quad (5.6)$$

Now we need to show that the measure  $\mu_{\tilde{P}}$  is the Haar measure. In other words, we claim that the Haar measure is equal with  $\mu_{\tilde{P}}$  where  $P(X) = 1/2$  for each large Feynman diagram  $X \in \mathcal{S}^{\Phi, g}$ .

For subsets  $Z_1, Z_2$  of  $K_{V(\Phi)}$ , define

$$I(Z_1, Z_2) := \{\text{DSE} \in \mathcal{S}^{\Phi, g} : Z_1 \subset X_{\text{DSE}}, Z_2 \subset K_{V(\Phi)} \setminus X_{\text{DSE}}\} \quad (5.7)$$

and then consider the  $\sigma$ -algebra  $\sum_{\mathcal{I}}$  generated by all sets  $I(Z_1, Z_2)$ . The  $\sigma$ -algebra  $\sum_{\mathcal{I}}$  is the same as the  $\sigma$ -algebra generated by all sets  $I(Z_1, Z_2)$  for disjoint sets  $Z_1, Z_2$ . In addition, we have  $S_{\Gamma} = I(\mathbb{I}, \{\Gamma\})$  which leads us to show that  $\sum_{\text{prod}} = \sum_{\mathcal{I}}$ .

Thanks to some standard methods in Analysis [158], we can determine the unique translation-invariant probability measure  $\mu_{\text{Haar}}$  on the compact

topological group  $\mathcal{S}^{\Phi,g}$ . For a given large Feynman diagram  $X$  and a subset  $Z$  of  $K_{V(\Phi)}$ , define

$$Z + X := \{\gamma \sqcup X : \gamma \in Z\}. \quad (5.8)$$

Then we can show that

$$I(Z_1, Z_2) = I(\mathbb{I}, Z_1 \sqcup Z_2) + Z_2. \quad (5.9)$$

Thanks to this fact, for given large Feynman diagrams  $\Gamma_1, \Gamma_2$ , set  $\Gamma = \Gamma_1 \sqcup \Gamma_2$ . Then we have

$$\mu_{\text{Haar}}(I(\Gamma_1, \Gamma_2)) = \mu_{\text{Haar}}(I(\Gamma_1, \Gamma_2) + \Gamma_2) = \mu_{\text{Haar}}(I(\mathbb{I}, \Gamma)) \quad (5.10)$$

which informs the translation-invariance.

In general, if  $(\Omega, A)$  be a  $\sigma$ -algebra generated by a subset  $C \subset A$  which is closed under finite intersections, then two probability measures on  $A$  are equal if and only if they agree on  $C$ . If  $C$  has an algebraic structure which is equipped by a probability measure  $\mu$ , then we can extend  $\mu$  to a unique measure on  $A$  [158]. Now let the subset  $Z$  of  $K_{V(\Phi)}$  can determine a finite number of large Feynman diagrams  $\Gamma_1, \dots, \Gamma_n$ . Then as the set we have

$$\mathcal{S}^{\Phi,g} = \bigsqcup_{Z_0 \subset Z} I(Z_0, Z \setminus Z_0) \quad (5.11)$$

which can be used to show that  $\mu_{\text{Haar}}$  agrees with  $\mu_{1/2}$  on  $\sigma_{\mathcal{I}}$ . In other words,

$$\mu_{\text{Haar}}(I(Z_1, Z_2)) = 2^{-n} = \mu_{1/2}(I(Z_1, Z_2)). \quad (5.12)$$

It is also possible to check that  $\sigma_{\mathcal{I}}$  is equal with the Borel  $\sigma$ -algebra generated by all open sets in  $\mathcal{S}^{\Phi,g}$  with respect to the metric  $d_{g,\alpha,\epsilon}$ . For a given bijection  $\alpha : K_{V(\Phi)} \rightarrow \mathbb{N}$ , set

$$E_n(\alpha) := \{e \in K_{V(\Phi)} : \alpha(e) \leq n\}. \quad (5.13)$$

For each large Feynman diagram  $X$  in  $\mathcal{S}^{\Phi,g}$ , set  $B(X, r, \|\cdot\|_{g,\alpha,\epsilon})$  as the open ball in  $(\mathcal{S}^{\Phi,g}, d_{g,\alpha,\epsilon})$  with the center  $X$  and the radius  $r$ . In addition, for given disjoint finite subsets  $N_1, N_2 \subset \mathbb{N}$  such that  $N_1 \cup N_2 = \{1, 2, \dots, n\}$ , define

$$I(N_1, N_2) := I(\alpha^{-1}(N_1), \alpha^{-1}(N_2)). \quad (5.14)$$

Then we have

$$I(N_1, N_2) = B(\alpha^{-1}(N_2), 2^{-n}, \|\cdot\|_{g,\alpha,2}) \bigcup B(K_{V(\Phi)} \setminus \alpha^{-1}(N_1), 2^{-n}, \|\cdot\|_{g,\alpha,2}) \quad (5.15)$$

such that  $B(\alpha^{-1}(N_i), 2^{-n}, \|\cdot\|_{g,\alpha,2})$  is the open ball in  $(\mathcal{S}^{\Phi,g}, d_{g,\alpha,g+\epsilon=2})$  with the center  $\alpha^{-1}(N_i)$  and the radius  $2^{-n}$ .

Now a large Feynman diagram  $X \in \mathcal{S}^{\Phi,g}$  can be described as the convergent limit of the sequence  $\{\Gamma_n\}_{n \geq 1}$  such that  $\Gamma_n := X \cap E_n(\alpha)$ .

Furthermore, we know that  $\mathcal{S}^{\Phi,g}$  is a compact Hausdorff topological group (i.e. Lemma 5.1.1). Therefore  $K \subset \mathcal{S}^{\Phi,g}$  is compact iff  $\mathcal{S}^{\Phi,g} \setminus K$  is open. It shows that the  $\sigma$ -algebra generated by all compact sets is the same as the Borel  $\sigma$ -algebra generated by all open sets. As the result,  $\mu_{\text{Haar}}$  on  $\sigma_{\text{prod}}$  determines uniquely the Haar measure. Therefore  $\mu_{\text{Haar}} = \mu_{1/2}$ .  $\square$

**Theorem 5.1.3.** *For a given bijection  $\alpha$ , the Haar measure of any ball of the radius  $0 \leq r \leq 1$  in the normed vector space  $(\mathcal{S}^{\Phi,g}, \|\cdot\|_{g,\alpha,g+\epsilon=2})$  is  $r$ .*

*Proof.* The proof is a direct result of Theorem 5.1.2 and the proof of Theorem A in [98].  $\square$

The resulting measure space enables us to initiate an integration theory on the family of (large) Feynman diagrams which contribute to solutions of Dyson–Schwinger equations of a given strongly coupled gauge field theory. This integration theory can be interpreted in the context of the Riemann–Lebesgue integration theory on the measure space  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ .

**Theorem 5.1.4.** *The integration theory on the measure space  $(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  can be formulated in terms of the Riemann–Lebesgue integration theory on real numbers with respect to the Borel  $\sigma$ -algebra generated by all open sets.*

*Proof.* Thanks to the structure of the topological group  $\mathcal{S}^{\Phi,g}$  (i.e. Lemma 5.1.1) where the norm  $\|\cdot\|_{g,\alpha,2}$  (given by (5.3)) and the Haar measure  $\mu_{\text{Haar}}$  (given by Theorem 5.1.2) are defined on large Feynman diagrams, we can adapt the proofs of Lemma 3.22 and Proposition 3.23 in [168] for large Feynman diagrams to obtain the following results.

(i) We can show that the norm  $\|\cdot\|_{g,\alpha,2}$  (as a real valued function on  $\mathcal{S}^{\Phi,g}$ ) is the Haar measure-preserving map.

(ii) We can show that for any Lebesgue integrable real valued function  $f$  on  $[0, 1]$ ,

$$\mathbb{E}_{\mu_{\text{Haar}}}[f(\|\cdot\|_{g,\alpha,2})] = \int_0^1 f(x)dx. \quad (5.16)$$

(iii) We can show that for any integrable real valued function  $h$  on  $\mathcal{S}^{\Phi,g}$ ,

$$\mathbb{E}_{\mu_{\text{Haar}}}[h] = \int_0^1 h((\|\cdot\|_{g,\alpha,2})^{-1}(x))dx. \quad (5.17)$$

Therefore the Haar integration theory on  $(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  can be described by transferring the Riemann–Lebesgue integration theory from the unit interval to the space of large Feynman diagrams.  $\square$

The rest of this section provides some applications of this integration theory for functionals on the measure space  $(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ .

Consider a single quantum particle which moves in a given potential such that its behavior can be studied by a class of functionals on  $C[0, t]$  given by

$$Z(y) := \exp\left\{\int_{(0,t)} \theta(s, y(s)) ds\right\} \quad (5.18)$$

where the complex valued function  $\theta$  on  $[0, t] \times \mathbb{R}^n$  is the given potential. This formulation is on the basis of the standard Lebesgue–Stieltjes measure while under some conditions it is possible to formulate these functionals with respect to other complex Borel measures. It has been shown that for each complex number with positive real part  $\lambda$ , the operators  $K_\lambda(Z_n)$  exist for each  $n$  such that  $Z_n(y) := (\int_{(0,t)} \theta(s, y(s)) d\eta)^n$  and  $K_\lambda(Z) = \sum_{n \geq 0} a_n K_\lambda(Z_n)$ . The central motivation of this formulation was to deal with Feynman’s operational calculus in QED and other quantum theories. [96]

A modification of the Johnson–Lapidus Dyson series for a measure space of graphs which contribute to the topological Hopf algebra  $H_{\text{FG}}^{\text{cut}}(\Phi)$  has been obtained in [168]. Now we want to apply the Haar integration theory on the topological group  $\mathcal{S}^{\Phi, g}$  to formulate the Johnson–Lapidus Dyson series on  $\mathcal{S}^{\Phi, g}$ . This class of series allows us to explain the evolution of (strongly coupled) Dyson–Schwinger equations in terms of sequences of partial sums or sequences of large Feynman diagrams corresponding to weakly coupled Dyson–Schwinger equations.

**Theorem 5.1.5.** *Let  $\theta$  be a complex valued function on  $\mathcal{S}^{\Phi, g} \times \mathbb{R}^2$  and  $v(z) = \sum_{n \geq 0} a_n z^n$  with the radius of convergence strictly greater than  $\|\theta\|_{\infty; \mu_{\text{Borel}}}$ . For a functional  $Z$  on the measure space  $L^1(\mathcal{S}^{\Phi, g}, \mu_{\text{Haar}})$  of all complex valued  $\mu_{\text{Haar}}$ -integrable functions on  $\mathcal{S}^{\Phi, g}$  given by*

$$Z(F) := v\left(\int_{\mathcal{S}^{\Phi, g}} \theta(X, F(X)) d\mu_{\text{Borel}}\right) \quad (5.19)$$

, there exists a family of operators  $\{K_\lambda(Z_n)\}_{n \in \mathbb{N}}$  such that

- parameters  $\lambda$  are complex numbers with positive real parts,
- $Z_n(F) := (\int_{\mathcal{S}^{\Phi, g}} \theta(X, F(X)) d\mu_{\text{Borel}})^n$ ,
- $K_\lambda(Z) = \sum_{n \geq 0} a_n K_\lambda(Z_n)$ .

*Proof.* Theorem 5.1.2 and Theorem 5.1.4 enable us to understand the Haar integration theory on  $\mathcal{S}^{\Phi, g}$  in terms of the Riemann–Lebesgue integration theory for real valued functions on the closed interval. In addition, we have discussed the equivalence between the product  $\sigma$ -algebra  $\sum_{\text{prod}}$  on cylinders determined by large Feynman diagrams and the Borel  $\sigma$ -algebra of open balls with respect to the norm  $\|\cdot\|_{g, \alpha, 2}$ . It is useful to determine uniquely the Borel measure  $\mu_{\text{Borel}}$  on  $\mathcal{S}^{\Phi, g}$  corresponding to the Haar measure  $\mu_{\text{Haar}}$  (i.e. Theorem 5.1.2). Therefore we can now extend the classical Johnson-Lapidus Dyson series to the level of the Haar measure  $\mu_{\text{Haar}}$  on  $\mathcal{S}^{\Phi, g}$ .

In addition, we have shown the existence of a compact Hausdorff topological group structure on  $\mathcal{S}^{\Phi,g}$ . Thanks to standard methods in Analysis ([158]), it is easy to show that the topological space  $C_c(\mathcal{S}^{\Phi,g})$  consisting of continuous functions on  $\mathcal{S}^{\Phi,g}$  with compact support is dense in  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ . Apply (5.17) to transfer the Haar measure integral

$$\mathbb{E}_{\mu_{\text{Haar}}}[h] = \int_{\mathcal{S}^{\Phi,g}} h(X) d\mu_{\text{Haar}} \tag{5.20}$$

of each  $h \in C_c(\mathcal{S}^{\Phi,g})$  to its corresponding Riemann–Lebesgue integral. It remains only to lift the proof of the classical Johnson–Lapidus generalized Dyson series given in [96] onto  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ .  $\square$

**Corollary 5.1.6.** (i) *The Johnson–Lapidus Dyson series can describe the behavior of a combinatorial Dyson–Schwinger equation in a given potential.*

(ii) *The functionals  $K_\lambda(Z)$  (determined by Theorem 5.1.5) enable us to describe the evolution of each large Feynman diagram  $X$  in terms of a sequence of Dyson–Schwinger equations in  $\mathcal{S}^{\Phi,g}$ .*

*Proof.* (i) Suppose  $X_{\text{DSE}}(g) = \sum_{n \geq 0} g^n X_n$  is a large Feynman diagram as the unique solution of an equation DSE in the normed vector space  $(\mathcal{S}^{\Phi,g}, \|\cdot\|_{g,\alpha,2})$ . Thanks to the Hahn–Banach Theorem ([158]), there exists a continuous linear map  $\psi_{\text{DSE}} : \mathcal{S}^{\Phi,g} \rightarrow \mathbb{R}$  such that

$$\psi_{\text{DSE}}(X_{\text{DSE}}) = \|X_{\text{DSE}}\|_{g,\alpha,2}, \quad \|\psi_{\text{DSE}}\| \leq 1 \tag{5.21}$$

where the operator norm  $\|\psi_{\text{DSE}}\|$  is defined by

$$\|\psi_{\text{DSE}}\| := \inf\{c \geq 0 : |\psi_{\text{DSE}}(X)| \leq c \|X\|_{g,\alpha,2}, \forall X \in \mathcal{S}^{\Phi,g}\}. \tag{5.22}$$

Now apply Theorem 5.1.5 for  $\psi_{\text{DSE}} \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ .

(ii) For any given Dyson–Schwinger equation DSE, if we apply the multi-scale Renormalization Group given by Theorem 3.2.2, then we can build a sequence  $\{\text{DSE}_n(\frac{n}{n+1}g)\}_{n \geq 1}$  of Dyson–Schwinger equations under rescaled values of the bare coupling constant where we have

$$X_{\text{DSE}_n}(\frac{n}{n+1}g) \subset X_{\text{DSE}_{n+1}}(\frac{n+1}{n+2}g). \tag{5.23}$$

For each  $n$ , set  $\chi_{\text{DSE}}^n$  as the characteristic function with respect to the large Feynman diagram  $X_{\text{DSE}_n}(\frac{n}{n+1}g)$  on  $\mathcal{S}^{\Phi,g}$  such that  $\chi_{\text{DSE}}^n \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ . Now apply Theorem 5.1.5 to the sequence  $\{\chi_{\text{DSE}}^n\}_{n \geq 1}$  to obtain a description for the evolution of  $X_{\text{DSE}}(g)$  in terms of large sub-graphs. A free evolution from  $X_{\text{DSE}}(0) = \mathbb{I}$  (i.e. the empty graph) to  $X_{\text{DSE}_1}(\frac{1}{2}g)$ , interactions of particles in  $X_{\text{DSE}_1}(\frac{1}{2}g)$  with the potential  $\theta$ , free evolution from  $X_{\text{DSE}_1}(\frac{1}{2}g)$  to  $X_{\text{DSE}_2}(\frac{2}{3}g)$ , and so on up to  $n^{\text{th}}$  integration with  $\theta$  at the level  $X_{\text{DSE}_n}(\frac{n}{n+1}g)$  followed by a free evolution from  $X_{\text{DSE}_n}(\frac{n}{n+1}g)$  to  $X_{\text{DSE}}(g)$  when  $n$  tends to infinity.  $\square$

**Lemma 5.1.7.** *Thanks to the symmetric difference as a binary operation on large Feynman diagrams, there exists a complex commutative Banach algebra structure on  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ .*

*Proof.* We have seen that the binary operation  $\Delta$  determines an abelian compact Hausdorff topological group structure on  $\mathcal{S}^{\Phi,g}$  such that the empty graph  $\mathbb{I}$  is the zero element of this group. Now define the following convolution product on  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$

$$\begin{aligned} & F_1 *_{\Delta} F_2(\Gamma_1) \\ &= \int_{\mathcal{S}^{\Phi,g}} F_1(\Gamma_2) F_2(\Gamma_2^{-1} \Delta \Gamma_1) d\mu_{\text{Haar}}(\Gamma_2), \quad F_1, F_2 \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}) \end{aligned} \quad (5.24)$$

such that  $\Gamma_2^{-1}$  is the inverse of the graph with respect to the group structure  $\Delta$ . The compatibility between the product (5.24) and the  $L^1$ -norm

$$\|F\|_1 := \int_{\mathcal{S}^{\Phi,g}} |F(X)| d\mu_{\text{Haar}}(X) \quad (5.25)$$

provides our promising Banach algebra. The abelian property of the group  $(\mathcal{S}^{\Phi,g}, \Delta)$  guarantees the commutativity of this Banach algebra. In addition, we add the infinitesimal delta function  $\delta$  as the multiplicative unit for this Banach algebra. Then we have

$$\int_{\mathcal{S}^{\Phi,g}} F(X) \delta(X) d\mu_{\text{Haar}}(X) = F(\mathbb{I}) \quad (5.26)$$

for each  $F \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  and each large Feynman diagram  $X \in \mathcal{S}^{\Phi,g}$ .  $\square$

**Proposition 5.1.8.** (i) *Each functional  $F \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  has a non-empty spectrum.*

(ii) *The space  $\Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}))$  of all characters of the complex Banach algebra  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  is a compact Hausdorff topological space.*

*Proof.* We need only to adapt the standard procedures in Functional Analysis ([158]) to achieve the results.

(i) We show that

$$\text{sp}(F) := \{\lambda \in \mathbb{C} : F - \lambda\delta \text{ not invertible}\} \quad (5.27)$$

is non-empty. If  $F = 0$ , then thanks to the definition of the infinitesimal delta function, we have the result. If  $F$  be a non-zero functional, suppose its spectrum is empty which means that the new function

$$R : \mathbb{C} \longrightarrow L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}), \quad \lambda \longmapsto (F - \lambda\delta)^{-1} \quad (5.28)$$

is well-defined, holomorphic, non-constant and bounded. For any bounded linear functional  $\Upsilon$  on  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ , define a new function  $\tilde{\Upsilon}$  on  $\mathbb{R}^2$  given by

$$\tilde{\Upsilon}(x, y) := \Upsilon(R(xe^{iy})). \quad (5.29)$$

We can show that  $\tilde{\Upsilon}$  is continuously differentiable with respect to variables  $x$  and  $y$ . Now by differentiation under the integral sign from the holomorphic bounded function  $K(x) := \int_0^{2\pi} \tilde{\Upsilon}(x, y) dy$ , we have  $K'(x) = 0$ . Therefore  $K$  is a constant function which is a contradiction with the initial assumption.

(ii) The ideal generated by kernel of any character provides a natural correspondence between the set of maximal ideals of the Banach algebra  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  and the set of characters on the space  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ .

Each character  $\psi \in \Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}))$  is actually an algebra homomorphism from  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  to  $\mathbb{C}$  such that  $\psi(\delta) = 1$ . We can show that  $\psi$  is continuous of norm 1, otherwise there exists a function  $F \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  such that  $\|F\| < 1$  and  $\psi(F) = 1$ . Apply the convolution product to define  $G := \sum_{n \geq 1} G^n$ . From the equation  $G = F + FG$  we have

$$\psi(G) = \psi(F) + \psi(F)\psi(G) = 1 + \psi(G) \quad (5.30)$$

which shows a contradiction. So the norm of  $\psi$  is less than or equal to 1 and  $\psi(\delta) = 1$  which implies that  $\|\psi\| = 1$ . Thanks to this fact, we can see that  $\Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}))$  is a closed subset of the unit ball of the dual space  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})^*$  which is a compact Hausdorff space with respect to the weak- $\star$  topology. As the consequence,  $\Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}))$  is a compact Hausdorff topological space.  $\square$

Thanks to the Gelfand transform, define

$$L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}) \longrightarrow C_0(\Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}))) \quad (5.31)$$

$$F \longmapsto \tilde{F}, \quad \tilde{F}(\psi) := \psi(F).$$

It is a norm decreasing algebraic homomorphism such that its image separates  $\mu_{\text{Haar}}$ -integrable functions on  $\mathcal{S}^{\Phi,g}$ . It can be seen that

$$\|\tilde{F}\|_{\infty} = \max\{|\lambda| : \lambda \in \text{sp}(F)\}. \quad (5.32)$$

Thanks to the Pontryagin duality Theorem [158], we can obtain a correspondence between elements of the topological space  $\Omega(L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}}))$  and elements of the Pontryagin dual. In this situation, the canonical isomorphism

$$\text{ev}_{L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})}(X)(\rho) = \rho(X) \in S^1 \subset \mathbb{C} \quad (5.33)$$

can be applied to define a modification of the Fourier transformation on  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ .

**Definition 5.1.9.** For a given Quantum Field Theory  $\Phi$  with the strong coupling constant  $g \geq 1$  and the corresponding collection  $\mathcal{S}^{\Phi,g}$  of all large Feynman diagrams generated by Dyson–Schwinger equations, the Fourier transformation  $\mathcal{F}$  on the complex commutative unital Banach algebra  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  is well-defined. For  $F \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ , we have

$$\widehat{F}(\rho) = \int_{\mathcal{S}^{\Phi,g}} F(X) \overline{\rho(X)} d\mu_{\text{Haar}}(X). \quad (5.34)$$

for any character  $\rho$ .

For functionals  $G, H \in L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ , we have

$$\mathcal{F}\{G *_{\Delta} H\} = \mathcal{F}\{G\} \mathcal{F}\{H\}. \quad (5.35)$$

The original motivation to formulate the Gelfand transform (5.31) is providing a way to separate functionals in  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$ . Thanks to this idea, the Fourier transformation (5.34) encodes the mathematical procedure for the decomposition of the functional  $F$  in terms of large Feynman diagrams which contribute to Dyson–Schwinger equations in  $\mathcal{S}^{\Phi,g}$ . In particular, if we restrict our discussion to a fixed Dyson–Schwinger equation DSE with the unique solution  $X_{\text{DSE}}$ , then our generalized Fourier transformation describes the evolution of the large Feynman diagram  $X_{\text{DSE}}$  with respect to  $\mu$ -integrable functions originated from large subdiagrams (or partial sums) which converge to  $X_{\text{DSE}}$ .

As the final note, we have developed a theory of Haar integration on the space of all Dyson–Schwinger equations of a given (strongly coupled) gauge field theory in the language of the classical Riemann–Lebesgue integral. This new measure theoretic approach is useful to build an analogous version of the classical Newton–Leibniz differentiation theory with respect to metrics  $d_{g,\alpha,\epsilon}$  for the study of functionals on large Feynman diagrams.

## 5.2 The Gâteaux differential calculus on $\mathcal{S}^{\Phi,g}$ and its application

In the second section of the previous chapter we have explained the construction of a noncommutative differential geometry model for the Hopf algebra  $\mathcal{S}_{\text{graphon}}^{\Phi}$  which is derived from the BPHZ non-perturbative renormalization process of Feynman graphons. In [168] we have applied the analysis of linear spaces to discuss the construction of a theory of differentiation on the space of Feynman diagrams in terms of the graph function representations of these physical diagrams and Gâteaux differentiability under the cut-distance topology where we worked on admissible directions to define well-defined differentiations. It has led us to obtain Taylor series type representations for continuous functionals on Feynman graphons on the basis of

homomorphism densities. The homomorphism densities can be considered as functionals  $W_\Gamma \mapsto t(G, W_\Gamma)$  on Feynman graphons such that  $G$  is an arbitrary finite graph. If  $G$  is a simple graph (such as rooted trees), then we can show that the corresponding homomorphism density is continuous with respect to the cut-distance topology and it is also  $L^1$ -integrable. Thanks to the disjoint union operator, we can build an algebraic structure on the linear span of homomorphism densities with respect to finite simple graphs which determine a dense subset in the space  $C(\mathcal{S}_{\text{graphon}}^\Phi)$  of all continuous functions on  $\mathcal{S}_{\text{graphon}}^\Phi$  with respect to the topology of uniform convergence. In addition, we can compute the Gâteaux derivatives of homomorphism densities where their Fréchet differentiability can be achieved with respect to simple graphs (i.e. decorated non-planar rooted trees) under some conditions where we might need to remove the symmetric condition of Feynman graphons and work on Feynman bigraphons. These observations can clarify the importance of homomorphism densities to study graphons under a functional analysis setting.

In this section, we concern the question of how to endow with a differential calculus on large Feynman diagrams independent of any renormalization program. We consider the real or complex vector space  $\mathcal{S}^{\Phi, g}$  generated by all Dyson–Schwinger equations in the physical theory  $\Phi$  and equip this space with the cut-distance topology determined by

$$d(\text{DSE}_1, \text{DSE}_2) := d_{\text{cut}}([f^{X_{\text{DSE}_1}}], [f^{X_{\text{DSE}_2}}]). \quad (5.36)$$

It defines the cut-norm for each large Feynman diagram. We have

$$\|X_{\text{DSE}}\|_{\text{cut}} = \sup_{A, B \subseteq [0, 1]} \left| \int_{A \times B} f^{X_{\text{DSE}}}(x, y) dx dy \right| \quad (5.37)$$

such that the supremum is taken over Lebesgue measurable subsets  $A, B$  of the closed interval. The resulting space can be interpreted as a closed topological subspace of the compact topological space of all unlabeled graphons. It means that we can consider  $\mathcal{S}^{\Phi, g}$  as a new Banach space to build a new Gâteaux differential calculus on the space of solutions of Dyson–Schwinger equations of a given physical theory. Then we apply the functional analysis of graphons ([50]) to obtain a new Gâteaux differential calculus machinery for the study of  $C(\mathcal{S}^{\Phi, g})$ .

Total derivative and directional derivatives are the most common differentiation machineries in finite dimensions such that their analogous versions in infinite dimensions are Fréchet derivative and Gâteaux derivative.

For a given function  $F : X \rightarrow Y$  between two Banach spaces (or normed vector spaces), the Gâteaux derivative at  $x_0 \in X$  is by definition a bounded linear operator  $T_{x_0} : X \rightarrow Y \in \mathcal{B}(X, Y)$  such that for every  $u \in X$ ,

$$\lim_{t \rightarrow 0} \frac{F(x_0 + tu) - F(x_0)}{t} = T_{x_0} u. \quad (5.38)$$

If for some fixed  $u$  the limit

$$\delta_u F(x) := \left. \frac{d}{dt} F(x + tu) \right|_{t=0} = \lim_{t \rightarrow 0} \frac{F(x + tu) - F(x)}{t} \quad (5.39)$$

exists, then we call  $F$  has a directional derivative at  $x$  in the direction  $u$ . Therefore  $F$  is Gâteaux derivative at  $x_0$  if and only if all the directional derivatives  $\delta_u F(x)$  exist and form a bounded linear operator

$$DF(x) : u \longmapsto \delta_u F(x). \quad (5.40)$$

$T_{x_0}$  is called the Fréchet derivative of  $F$  at  $x_0$ , if the limit (in the sense of the Gâteaux derivative) exists uniformly in  $u$  on the unit ball in  $X$ . If we set  $y = tu$  then  $t$  tends to zero is equivalent to  $y$  tends to zero. Now  $F$  is Fréchet differentiable at  $x_0$ , if for all  $y$  we have

$$F(x_0 + y) = F(x_0) + T_{x_0}(y) + o(\|y\|), \quad (5.41)$$

which means that

$$\lim_{\|h\| \rightarrow 0} \frac{\|F(x+h) - F(x) - Th\|}{\|h\|} = 0 \quad (5.42)$$

holds. As we can see, the limit in the Fréchet derivative only depends on the norm of  $y$  where the operator  $T$  defines the natural linear approximation of  $F$  in a neighborhood of the point  $x_0$ . In this setting, we call  $T_{x_0} = DF(x_0)$  as the derivative of  $F$  at  $x_0$ . In addition, we can show that being Fréchet differentiable at a point implies being Gâteaux differentiable at a point such that in this case the Gâteaux derivative is equal to the Fréchet derivative.

If  $F$  is Gâteaux differentiable on  $X$ , then we have the mean value formula

$$\|F(y) - F(x)\| \leq \|x - y\| \sup_{0 \leq \theta \leq 1} \|DF(\theta x + (1 - \theta)y)\|. \quad (5.43)$$

This enables us to show that if  $F$  is Gâteaux differentiable on an open neighborhood  $U$  of  $x$  and  $DF(x)$  is continuous, then  $F$  is Fréchet differentiable at  $x$ . [72, 158]

We plan to study (smooth) real valued continuous functions on the Banach space  $\mathcal{S}^{\Phi, g}$  in terms of their Taylor series representation under the higher orders Gâteaux differentiations. We show that the solution space of the natural generalization of the equation  $\frac{d^n}{dx^n} F(x) \equiv 0$  to large Feynman diagrams namely, Gâteaux type differential equations with the general form

$$d^{N+1} F(X; Z_1, \dots, Z_{N+1}) = 0 \quad (5.44)$$

for all large Feynman diagrams  $X, Z_1, \dots, Z_{N+1} \in \mathcal{S}^{\Phi, g}$ , can be described by homomorphism densities. Solutions of this class of equations enjoy the property

$$F(Z) = F(0) + dF(0; Z) + \frac{d^2 F(0; Z, Z)}{2} + \dots + \frac{d^n F(0; Z, \dots, Z)}{n!}. \quad (5.45)$$

**Definition 5.2.1.** For a given function  $F : \mathcal{S}^{\Phi, g} \rightarrow \mathbb{R}$  and each large Feynman diagram  $X$ , the Gâteaux derivative exists at  $X$  in the direction  $Y \in \mathcal{S}^{\Phi, g}$ , if the limit

$$dF(X; Y) = \lim_{t \rightarrow 0} \frac{F(X + tY) - F(X)}{t} \quad (5.46)$$

exists. The higher orders of the Gâteaux differentiability can be defined by induction where for any  $n \geq 2$ ,  $F$  is called  $n$ -time Gâteaux differentiable at  $X$  in directions  $Z_1, \dots, Z_n$  if at the first, the higher mixed Gâteaux derivatives

$$d^{n-1}F(X + \lambda Z_n; Z_1, \dots, Z_{n-1}) \quad (5.47)$$

exist for each real number  $\lambda$  and at the second, the limit

$$d^n F(X; Z_1, \dots, Z_n) = \lim_{\lambda \rightarrow 0} \frac{d^{n-1}F(X + \lambda Z_n; Z_1, \dots, Z_{n-1}) - d^{n-1}F(X; Z_1, \dots, Z_{n-1})}{\lambda} \quad (5.48)$$

exists.

It is easy to check that the Gâteaux derivatives  $d^n F(X; Z_1, \dots, Z_n)$  are multilinear maps in  $Z_i$  and in addition, for any permutation  $\tau \in S_n$ , we have

$$d^n F(X; Z_1, \dots, Z_n) = d^n F(X; Z_{\tau(1)}, \dots, Z_{\tau(n)}). \quad (5.49)$$

### 5.2.1 Feynman random graphs via homomorphism densities

The description of large Feynman diagrams via Feynman graphons in  $\mathcal{S}_{\text{graphon}}^{\Phi}$  allows us to think about the concept of "density" of infinite Feynman graphs in the solution of a given Dyson–Schwinger equation. From the view point of Quantum Field Theory, any infinite Feynman graph can contain nested loops which present different types of subdivergencies derived from virtual particles and their interactions. One important task is to search for any algorithm which could estimate the appearance of a particular class of subdivergencies (or subgraphs) in an infinite expansion of Feynman diagrams. The homomorphism density, as a class function on  $\mathcal{S}_{\text{graphon}}^{\Phi}$  or  $\mathcal{S}^{\Phi, g}$ , is a useful tool to formulate a new mathematical model for this fundamental challenge.

**Proposition 5.2.2.** *The homomorphism density is a well-defined operator on large Feynman diagrams.*

*Proof.* Proposition 4.6 in [169] shows that the unique solution  $X_{\text{DSE}}$  of a given Dyson–Schwinger equation DSE can be interpreted as the convergent limit of the sequence  $\{Y_m\}_{m \geq 1}$  of its partial sums with respect to the cut-distance topology.

For each unlabeled Feynman graphon class  $[W]$ , we can build the homomorphism density  $t(X_{\text{DSE}}, W)$  as the "limit" of the sequence  $\{t(Y_m, W)\}_{m \geq 1}$  of homomorphism densities corresponding to finite expansions  $Y_m$  of finite graphs (which do not have self-loops but have loops).

For each  $m \geq 1$ , if the partial sum  $Y_m := X_1 + \dots + X_m$  has  $k_m$  vertices, then we have

$$t(Y_m, W) = \int_{[0,1]^{k_m}} \prod_{(i,j) \in E(Y_m)} W(x_i, x_j) dx_1 \dots dx_{k_m}. \quad (5.50)$$

The induction is useful to show that

$$t(Y_{m+1}, W) = t(Y_m, W)t(X_{m+1}, W). \quad (5.51)$$

The condition  $d_{\text{cut}}(W', W) = 0$  for weakly isomorphic graphons can be applied to show that  $t(Y_m, W') = t(Y_m, W)$  which leads us to  $t(X_{\text{DSE}}, W') = t(X_{\text{DSE}}, W)$ . Therefore we can define a poset on homomorphism densities where a family of injections  $\{f_{ij} : t(Y_i, -) \rightarrow t(Y_j, -)\}_{i \leq j}$  on the space of Feynman graphons can be formulated. This gives us an inverse system where its inverse limit can be identified as a subset of the direct product of the homomorphism densities  $t(Y_i, -)$ s. This inverse limit can be considered as the homomorphism density with respect to the large Feynman diagram  $X_{\text{DSE}}$ . We have

$$\begin{aligned} t(X_{\text{DSE}}, -) &= \lim_{\leftarrow m} t(Y_m, -) \\ &= \{\vec{W} \in \prod_{m=1}^{\infty} t(Y_m, -) : f_{ij}(W_j) = W_i, \forall i \leq j\} \subseteq \prod_{m=1}^{\infty} t(Y_m, -). \end{aligned} \quad (5.52)$$

□

For  $1 \leq n \leq \infty$ , set  $[n] := \{i \in \mathbb{N} : i \leq n\}$ . For a given Feynman graphon  $W$ , define a random graph  $G(n, W)$  with the vertex set  $[n]$  (chosen points  $\{x_1, \dots, x_n\}$  at random from the closed unit interval) by letting  $ij$  be an edge in  $G(n, W)$  with the probability  $W(x_i, x_j)$ . It is possible to build  $G(n, W)$  for all  $n$  by constructing  $G(\infty, W)$  as an exchangeable random graph namely, its distribution is invariant under permutations of the vertices and every exchangeable random graph is a mixture of such graphs. Then take the subgraph defined by the first  $n$  vertices. In general, for any Feynman diagram  $\Gamma$  (as a labeled graph), the homomorphism density  $t(\Gamma, W)$  equals the probability that  $\Gamma$  is a subgraph of  $G(\infty, W)$  or of  $G(n, W)$  for any  $n \geq |\Gamma|$ . In other words, the family  $\{t(\Gamma, W)\}_{\Gamma}$  and the distribution of  $G(\infty, W)$  can determine each other.

It is important to note that for given Feynman graphons  $W, W', G(\infty, W)$  and  $G(\infty, W')$  have the same distribution if and only if those two graphons are weakly isomorphic or equivalent.

**Lemma 5.2.3.** *Homomorphism densities can determine a new class of random graphs with respect to solutions of Dyson–Schwinger equations.*

*Proof.* We know that  $X_{\text{DSE}} = \lim_{m \rightarrow \infty} Y_m$  with respect to the cut-distance topology such that  $|Y_m| \rightarrow \infty$ . For each  $[k]$  there exists a random graph  $Y_m[k]$  on the vertex set  $[k]$  such that  $\{Y_m[k]\}_{m \geq 1}$  converges to  $X_{\text{DSE}}[k]$  with respect to the metric

$$d_{\text{dens}}(\Gamma, \Gamma') = \sum_i 2^{-i} |t(W_i, \Gamma) - t(W_i, \Gamma')| \quad (5.53)$$

which is equivalent to the cut-distance ([26, 94]). Therefore there exists an infinite random graph  $\overline{X_{\text{DSE}}}$  on  $[\infty]$  such that  $X_{\text{DSE}}[k] \equiv \overline{X_{\text{DSE}}}|_{[k]}$  with respect to the metric  $d_{\text{dens}}$ . This means that

$$\lim_{m \rightarrow \infty} Y_m[k] = \overline{X_{\text{DSE}}}|_{[k]}. \quad (5.54)$$

□

**Corollary 5.2.4.** *The distribution of the random graphs  $Y_m[k]$  with respect to partial sums of  $X_{\text{DSE}}$  converges when  $m$  tends to infinity.*

*Proof.* Thanks to Lemma 5.2.3, for each  $k \leq |Y_m|$ , there exists  $Y_m[k]$  as the random induced subgraph of  $Y_m$  with  $k$  vertices determined by selecting  $k$  separate vertices  $v_1, \dots, v_k$  of  $Y_m$  at uniformly random procedures. Now thanks to graphon representations of Feynman diagrams, it is enough to apply the definition of convergent sequences in the theory of graphons via random graphs ([94]) to the sequence  $\{Y_m\}_{m \geq 1}$  to show its convergence to  $X_{\text{DSE}}$ . □

**Lemma 5.2.5.** *Dyson–Schwinger equations which generate isomorphic Hopf subalgebras have the same homomorphism density.*

*Proof.* Suppose Hopf subalgebras  $H_{\text{DSE}_1}, H_{\text{DSE}_2}$  corresponding to the equations  $\text{DSE}_1$  and  $\text{DSE}_2$  are isomorphic which means that the unique solutions  $X_{\text{DSE}_1}$  and  $X_{\text{DSE}_2}$  are isomorphic infinite graphs. We can lift the weakly isomorphic relation on graphons onto the level of large Feynman diagrams. We say that two large Feynman diagrams  $X_{\text{DSE}_1}, X_{\text{DSE}_2}$  are weakly isomorphic or equivalent if their corresponding labeled graphons have the same unlabeled measurable function almost everywhere. In other words,  $X_{\text{DSE}_1}$  and  $X_{\text{DSE}_2}$  are weakly isomorphic or equivalent whenever

$$d_{\text{cut}}(f^{X_{\text{DSE}_1}}, f^{X_{\text{DSE}_2}}) = 0. \quad (5.55)$$

Thanks to Borgs–Chayes–Lovasz Theorem ([125]) and Proposition 5.2.2, we can show that two weakly equivalent large Feynman diagrams  $X_{\text{DSE}_1}, X_{\text{DSE}_2}$  have the same homomorphism density for all (finite) simple graphs. In other words,

$$t(H, f^{X_{\text{DSE}_1}}) = t(H, f^{X_{\text{DSE}_2}}). \quad (5.56)$$

□

### 5.2.2 Differentiability

In this part our task focuses on the study of the space of smooth real valued functions on  $\mathcal{S}^{\Phi, g}$  in terms of homomorphism densities of Feynman graphons which contribute to solutions of Dyson–Schwinger equations. We show that this class of homomorphism densities can play the role of a basis for the Taylor series representations of smooth functions.

Define a new graduation parameter on the collection  $\mathcal{H}(\Phi)$  of all Feynman diagrams of a physical theory  $\Phi$  in terms of the number of edges. For each  $n$ , set  $\mathcal{H}_n(\Phi)$  as the isomorphism classes of Feynman diagrams with  $n$  internal and external edges, no isolated vertices, no self-loops but possible multi-edges. Set  $\mathcal{H}_{\leq n}(\Phi) = \bigcup_{j \leq n} \mathcal{H}_j(\Phi)$ . The homomorphism density for all Feynman graphs  $\Gamma \in \mathcal{H}_j(\Phi)$  is well-defined.

**Theorem 5.2.6.** *For each  $n \geq 1$ , define*

$$F_n : \mathcal{S}_{\text{graphon}}^{\Phi} \longrightarrow \mathbb{R}, \quad F_n(W_{\Gamma}) := \sum_{\gamma \in \mathcal{H}_{\leq n}(\Phi)} a_{\gamma} t(\gamma, W_{\Gamma}) \quad (5.57)$$

for any finite Feynman diagram  $\Gamma$  and some constants  $a_{\gamma}$ . Then

(i)  $F_n$  is continuous in the  $L^1$ -topology.

(ii) It is possible to lift  $F_n$  onto the space of large Feynman diagrams and define a new real valued map  $\tilde{F}$  on  $\mathcal{S}^{\Phi, g}$  which is continuous in the  $L^1$ -topology.

*Proof.* (i) Define a new real valued multilinear functional  $\tau$  on  $(\mathcal{S}_{\text{graphon}}^{\Phi})^n$  given by

$$\tau((W_{\gamma_t})_{\gamma_t}) := \int_{[0,1]^{\sum |\gamma_t|}} \prod_{\gamma_t} W_{\gamma_t}(x_i, x_j) \prod_i dx_i. \quad (5.58)$$

We can show that

$$|\tau((W_{\gamma_t})_{\gamma_t}) - \tau((W'_{\gamma_t})_{\gamma_t})| \leq \sum_{\gamma_t} \|W_{\gamma_t} - W'_{\gamma_t}\|_1 \quad (5.59)$$

which leads us to

$$|t(\sqcup \gamma_t, W) - t(\sqcup \gamma_t, W')| \leq |E(\sqcup \gamma_t)| \|W_{\gamma_t} - W'_{\gamma_t}\|_1. \quad (5.60)$$

(ii) Thanks to Proposition 5.2.2, we plan to lift the above process onto the level of large Feynman diagram  $X_{\text{DSE}}$  with the partial sums  $Y_m$ ,  $m \geq 1$ . It is enough to extend the relation (5.60) to  $Y_{m+1} = Y_m + X_{m+1}$ . We have

$$\begin{aligned} & |t(Y_m \sqcup X_{m+1}, W) - t(Y_m \sqcup X_{m+1}, W')| = \\ & |t(Y_m \sqcup X_{m+1}, W) - t(Y_m \sqcup X_{m+1}, W') \pm t(Y_m, W')t(X_{m+1}, W)| = \\ & |t(X_{m+1}, W)(t(Y_m, W) - t(Y_m, W')) + t(Y_m, W')(t(X_{m+1}, W) - t(X_{m+1}, W'))| \end{aligned}$$

$$\begin{aligned} &\leq |t(X_{m+1}, W)| |(t(Y_m, W) - t(Y_m, W'))| + |t(Y_m, W')| |t(X_{m+1}, W) - t(X_{m+1}, W')| \\ &\hspace{20em} (5.61) \\ &\leq |t(X_{m+1}, W)| (|E(Y_m)| \|W_{Y_m} - W'_{Y_m}\|_1) + |t(Y_m, W')| (|E(X_{m+1})| \|W_{Y_m} - W'_{Y_m}\|_1) \end{aligned}$$

such that  $\|W_{Y_m}\|_\infty, \|W'_{Y_m}\|_\infty \leq 1$ .

Lift the multilinear operator  $\tau$  onto the multilinear operator  $\tilde{\tau}$  defined as a bounded operator on the Banach space  $\mathcal{S}^{\Phi, g}$ . Now define the new map  $\tilde{F}$  on  $\mathcal{S}^{\Phi, g}$  given by

$$\tilde{F}(X_{\text{DSE}}) := \prod_{m=1}^{\infty} F_m(W_{Y_m}) \tag{5.62}$$

such that each term  $F_m(W_{Y_m})$  is a  $L^1$ -continuous function. Therefore  $\tilde{F}$ , as the product of continuous functions, is also continuous with respect to the  $L^1$ - topology.  $\square$

The space  $\mathcal{W}_{[0,1]}$  of all (bi-)graphons can be embedded into the vector space  $\mathcal{W}$  of bounded (symmetric) measurable functions  $f : [0, 1]^2 \rightarrow \mathbb{R}$  which is equipped by a semi-norm. Under weakly isomorphic relation  $\approx$ , we can build a complete metric structure on the quotient space  $\mathcal{W}_{[0,1]}/\approx$ . The topological space  $\mathcal{S}_{\text{graphon}}^\Phi$  of all unlabeled graphon classes which contribute to the representations of Feynman diagrams and Dyson–Schwinger equations sits inside  $\mathcal{W}_{[0,1]}/\approx$ . As we have discussed each Feynman graphon  $[W_\Gamma] \in \mathcal{S}_{\text{graphon}}^\Phi$  is a class of bounded (symmetric) measurable functions on  $[0, 1]^2$  up to relabeling and weakly isomorphic relation. This class of graphons are generated by rooted tree representations of (large) Feynman diagrams. Rooted trees are simple graphs where their adjacency matrices can determine their corresponding graphon classes. Orientations on decorated non-planar rooted trees, which encode positions of nested loops in the main Feynman diagram, inform us that we might need only the upper part or the lower part of the adjacency matrix for the reconstruction of any Feynman diagram from its graphon representation. It means that we do not need the symmetric property of graphons and we can work only on the bounded measurable functions  $f : [0, 1]^2 \rightarrow [0, 1]$  up to the relabeling and weakly isomorphic relation. This class of objects, which are known as bi-graphons, enables us to have Fréchet differentiability of homomorphism densities of Feynman graphons.

**Lemma 5.2.7.** (i) *The homomorphism densities on  $\mathcal{S}_{\text{graphon}}^\Phi$  are Fréchet differentiable.*

(ii) *The homomorphism densities on  $\mathcal{S}^{\Phi, g}$  are Fréchet differentiable.*

*Proof.* (i) We compute the Fréchet derivatives of the homomorphism densities on  $\mathcal{S}_{\text{graphon}}^\Phi$  for ladder trees  $l_1, l_2, l_3$  and the rooted tree  $\vee$  where vertices 2, 3 are adjacent to the root 1.

For the tree with only one vertex,  $t(l_1, -) \equiv 1$  which is obviously Fréchet differentiable.

For the oriented decorated ladder tree  $l_2$  with two vertices 1, 2 and one edge  $e_{12}$  from 1 to 2, the Gâteaux derivative can be computed by

$$d(t(H, W_\Gamma); W_{\Gamma'}) = \int_{[0,1]^k} \sum_{(i_1, j_1) \in E(H)} W_{\Gamma'}(x_{i_1}, x_{j_1}) \prod_{(i,j) \in E(H) \setminus \{(i_1, j_1)\}} W_\Gamma(x_i, x_j) dx_1 \dots dx_k \quad (5.63)$$

which leads us to compute the unique Fréchet derivative by the linear map

$$W_{\Gamma'} \mapsto d(t(l_2, W_\Gamma); W_{\Gamma'}) = \int_{[0,1]^2} W_{\Gamma'}(x_1, x_2) dx_1 dx_2. \quad (5.64)$$

We have

$$\begin{aligned} \lim_{W_{\Gamma'} \rightarrow 0} \frac{|t(H, W_\Gamma + W_{\Gamma'}) - t(H, W_\Gamma) - \int_{[0,1]^2} W_{\Gamma'}(x_1, x_2) dx_1 dx_2|}{\|W_{\Gamma'}\|_{\text{cut}}} & \quad (5.65) \\ & = \lim_{W_{\Gamma'} \rightarrow 0} \frac{0}{\|W_{\Gamma'}\|_{\text{cut}}} = 0 \end{aligned}$$

which approves the Fréchet differentiability in terms of the formula (5.42).

For the oriented decorated ladder tree  $l_3$  with three vertices 1, 2, 3 and two edges  $e_{12}, e_{23}$  which connect the vertices 1 to 2 and 2 to 3, the unique candidate for the Fréchet derivative of  $t(l_3, -)$  should be

$$\begin{aligned} W_{\Gamma'} \mapsto d(t(l_3, W_\Gamma); W_{\Gamma'}) & = \\ \int_{[0,1]^3} W_{\Gamma'}(x_1, x_2) W_\Gamma(x_2, x_3) + W_\Gamma(x_1, x_2) W_{\Gamma'}(x_2, x_3) dx_1 dx_2 dx_3 & \quad (5.66) \\ = 2 \int_{[0,1]^3} W_\Gamma(x_1, x_2) W_{\Gamma'}(x_2, x_3) dx_1 dx_2 dx_3. & \end{aligned}$$

$t(l_3, -)$  is Fréchet differentiable if the following limit exists and equals to zero,

$$\begin{aligned} \lim_{W_{\Gamma'} \rightarrow 0} \frac{|t(l_3, W_\Gamma + W_{\Gamma'}) - t(l_3, W_\Gamma) - d(t(l_3, W_\Gamma); g)|}{\|W_{\Gamma'}\|_{\text{cut}}} & \\ = \lim_{W_{\Gamma'} \rightarrow 0} \frac{|\int_{[0,1]^3} W_{\Gamma'}(x_1, x_2) W_{\Gamma'}(x_2, x_3) dx_1 dx_2 dx_3|}{\|W_{\Gamma'}\|_{\text{cut}}}. & \quad (5.67) \end{aligned}$$

If this limit is not zero, then there are some below boundaries  $c > 0$  which means that we can define a sequence  $\{W_n\}_{n \geq 1}$  of graphons which converges to zero with respect to the cut-norm but the limit (5.67) does not zero when  $n$  tends to infinity. In other words, we might have

$$0 < c \leq \frac{1}{n} = \left| \int_{[0,1]^2, x_2 \in [0,1/n]} 1 dx_1 dx_2 dx_3 \right| =$$

$$\left| \int_{[0,1]^2, x_2 \in [0,1/n]} W_n(x_1, x_2) W_n(x_2, x_3) dx_1 dx_2 dx_3 \right| \leq \quad (5.68)$$

$$\left| \int_{[0,1]^3} W_n(x_1, x_2) W_n(x_2, x_3) dx_1 dx_2 dx_3 \right|.$$

This situation supports the existence of sequences of graphons such as  $W_n = \mathbf{1}$  on  $\min(x_1, x_2) < 1/n$  which satisfy the above inequality. On the other hand, we can build the topological renormalization Hopf algebra  $\mathcal{S}_{\text{graphon}}^\Phi$  of Feynman graphons by measurable bounded functions from  $[0, 1]^2$  to  $[0, 1]$  in terms of working only on the upper parts or lower parts of the adjacency matrices of oriented decorated non-planar rooted trees. In this non-symmetric setting, the sequences such as  $\{W_n\}_{n \geq 1}$  of graphons does not belong to  $\mathcal{S}_{\text{graphon}}^\Phi$ . As the consequence, the only lower boundary for Feynman graphons is zero itself which means that the limit (5.67) is zero.

By a similar discussion, we can show the existence of Fréchet derivative for other oriented rooted trees  $H$  which contains the tree  $\vee$  where vertices 2, 3 are adjacent to the root 1. In this situation, we need to deal with

$$\lim_{n \rightarrow \infty} \frac{|t(H, W_\Gamma + W_{\Gamma'}) - t(H, W_\Gamma) - d(t(H, W_\Gamma); W_{\Gamma'})|}{\|W_{\Gamma'}\|_{\text{cut}}} \quad (5.69)$$

where if this limit is not zero then we get some lower boundaries  $c > 0$  such that

$$c^{|E(H)|-2} \int_{[0,1]^3} W_{\Gamma'}(x_1, x_2) W_{\Gamma'}(x_2, x_3) dx_1 dx_2 dx_3 \leq$$

$$\int_{[0,1]^{|V(H)|}} W_{\Gamma'}(x_1, x_2) W_{\Gamma'}(x_2, x_3) \prod_{(ij) \in E(H) \setminus \{(1,2), (2,3)\}} W_\Gamma(x_i, x_j) dx_1 \dots dx_{|V(H)|}$$

$$\leq t(H, W_\Gamma + W_{\Gamma'}) - t(H, W_\Gamma) - d(t(H, W_\Gamma); W_{\Gamma'}). \quad (5.70)$$

This situation allows us to determine sequences of graphons which can not belong to  $\mathcal{S}_{\text{graphon}}^\Phi$  whenever we work on Feynman bigraphons. Therefore the limit (5.69) should be zero.

(ii) Objects in the Banach space  $\mathcal{S}^{\Phi, g}$  are large Feynman diagrams namely, infinite formal expansions of Feynman diagrams which have nested or overlapping loops. Thanks to Theorem 5.2.2, homomorphism densities on large Feynman diagrams can be computed in terms of homomorphism densities of finite partial sums. For large Feynman diagrams  $X, Z$ , we have

$$t(X, Z) = \lim_{\leftarrow m} t(Y_m, Z) \quad (5.71)$$

such that for each  $m \geq 1$ ,

$$t(Y_m, Z) = \prod_{i=1}^m t(X_i, W_Z). \quad (5.72)$$

We have  $X(g) = \sum_{n \geq 0} g^n X_n$  and  $W_Z$  as a Feynman bigraphon which lives in  $\mathcal{S}_{\text{graphon}}^\Phi$  such that it can not have a non-zero lower boundary.

Furthermore, thanks to the formula (5.41), we know that the Fréchet differentiability depends only on the norm  $W_Z$  where by applying (i), each  $t(X_m, W_Z)$  is Fréchet differentiable. Thanks to the product rule,  $t(Y_m, Z)$  as the product of Fréchet differentiable functions is also Fréchet differentiable for each  $m$ . Since  $t(X, Z)$  can be identified by a subset of the direct product  $\prod_{m=1}^\infty t(Y_m, Z)$ ,  $t(X, Z)$  will be also Fréchet differentiable.  $\square$

**Lemma 5.2.8.** *For a given  $C^n$  ( $n > 0$ ) class function  $G : \mathcal{S}^{\Phi, g} \rightarrow \mathbb{R}$ ,  $d^n G(0; Z_1, \dots, Z_n)$  is a symmetric  $S_{[0,1]}$ -invariant multilinear functional.*

*Proof.* We can extend the functions

$$\mathcal{G}_{X,Z}(\lambda_1, \dots, \lambda_m) := G(X + \lambda_1 Z_1 + \dots + \lambda_m Z_m) \quad (5.73)$$

to  $C^n$  functions on  $\mathbb{R}^m$  where the equality of mixed partial derivatives show that for any permutation  $\sigma \in S_n$ , we have

$$d^n G(X; Z_1, \dots, Z_n) = d^n G(X; Z_{\sigma(1)}, \dots, Z_{\sigma(n)}). \quad (5.74)$$

In addition, we can extend  $d^n G(0; Z_1, \dots, Z_n)$  multilinearly to each

$$(\text{span}_{\mathbb{R}}(\Gamma_1, \dots, \Gamma_m))^n. \quad (5.75)$$

$\square$

**Corollary 5.2.9.** *Let  $\tilde{F} : \mathcal{S}^{\Phi, g} \rightarrow \mathbb{R}$  be a  $L^1$ -continuous functional (determined by Theorem 5.2.6) such that for some  $N \geq 1$ ,  $\tilde{F}$  is  $N + 1$  times Gâteaux differentiable. Then for each large Feynman diagram  $X$  and also  $Z_1, \dots, Z_{N+1}$  in the Banach space  $\mathcal{S}^{\Phi, g}$ ,*

$$d^{N+1} \tilde{F}(X; Z_1, \dots, Z_{N+1}) = 0$$

*if and only if there exists a unique family  $\{a_\gamma\}_\gamma$  of real constants such that  $\tilde{F}(X) = \sum_{\gamma \in \mathcal{H}_{\leq N}(\Phi)} a_\gamma t(\gamma, W_X)$ .*

*Proof.* Thanks to Definition 5.2.1, Proposition 5.2.2, Theorem 5.2.6, Lemma 5.2.7, Lemma 5.2.8, we can apply the main result in [50].  $\square$

**Corollary 5.2.10.** *Let  $G : \mathcal{S}^{\Phi, g} \rightarrow \mathbb{R}$  be a continuous functional with respect to the cut-distance topology and smooth with respect to the Gâteaux derivation. For each large Feynman diagram  $X$  define the following sequence of Taylor polynomials*

$$P_n(X) := \sum_{m=0}^n \frac{1}{m!} d^m G(0; X, \dots, X), \quad \forall n \geq 0 \quad (5.76)$$

which converges to  $G(X)$  when  $n$  tends to infinity. In addition, let the Taylor expansion

$$\sum_{m=0}^{\infty} \sum_{\gamma \in \mathcal{H}_m(\Phi)} a_{\gamma} t(\gamma, W_X) \quad (5.77)$$

is absolutely convergent to  $P(G)(X)$  such that

$$\sum_{\gamma \in \mathcal{H}_m(\Phi)} a_{\gamma} t(\gamma, W_X) = \frac{1}{m!} d^m G(0; X, \dots, X). \quad (5.78)$$

Then  $G(X) = P(G)(X)$ .

*Proof.* In [50], the required conditions for the existence of the convergent Taylor series of a smooth function on the space of unlabeled graphons have been provided. Now it is enough to adapt that procedure for Feynman graphons (which is already addressed in [168]) and then lift it onto the whole space  $\mathcal{S}^{\Phi, g}$  in terms of Definition 5.2.1, Proposition 5.2.2, Theorem 5.2.6, Lemma 5.2.7, Lemma 5.2.8 and Corollary 5.2.9.  $\square$

## Chapter 6

# The intrinsic foundations of QFT under a non-perturbative setting

- *Some historical remarks*
- *QFT-entanglement via lattices of Dyson–Schwinger equations*
  - *A new lattice model*
  - *Intermediate algorithms for QFT-entanglement*
  - *Tannakian subcategories as intermediate algorithms*
- *Quantum logic via non-perturbative propositional calculus*
  - *Quantum Topos*
  - *Non-perturbative Topos*

The first purpose in this chapter is to build a new mathematical model for the description of information flow among particles in (strongly coupled) interacting gauge field theories. This new platform enables us to analyze quantum entanglement via fundamental tools in Category Theory and Theoretical Computer Science. We apply combinatorial Dyson–Schwinger equations as the building blocks of information flow among distant elementary particles in a system with infinite degrees of freedom. The cut-distance topology is applied to construct topological regions around elementary particles which encode passing information. We organize these cut-distance topological regions into a new class of lattices of topological Hopf subalgebras. This setting allows us to understand quantum entanglement in the language of intermediate algorithms which contribute to transferring information among entangled particles [172]. The second purpose in this chapter is to build a new mathematical model for the description of logical propositions of non-perturbative aspects in strongly coupled gauge field theories. We explain the basic foundations of a new topos of presheaves which is capable to encode topological regions of elementary particles and the strength of coupling constants. This topos has enough physical information to evaluate logical propositions about infinite formal expansions of Feynman diagrams which contribute to quantum motions [173].

## 6.1 *Some historical remarks*

Entanglement, non-locality and indeterminism are actually the most complicated and challenging concepts in Quantum Mechanics. These concepts have been originated from pioneering efforts to clarify the complicated methodologies applied in dealing with the outputs of Quantum Mechanics. The study of the behavior of electrons via Quantum Mechanics had shown some results which were against the objections of Einstein and his followers such as the double slit experiment, the photon box experiment and the Einstein–Podolsky–Rosen paradox. The completeness of Quantum Mechanics has already been clarified in terms of successful experiments on photons polarization and formulating modern gauge field theories. However there still remain some philosophical challenges for the interpretation of predications in Quantum Mechanics. [1, 11, 14, 70, 176]

On the one hand, there are some rigorous efforts for the interpretation of Quantum Mechanics under a deterministic setting. Theory of many worlds and universal wave function, as the key tools in this setting, can provide predictions completely different from Quantum Mechanics predictions. These tools introduce a different theory for quantum world. Bohmian Mechanics, hidden variables, many-world interpretation and theory of universal wave function are well-known tools for the deterministic interpretation of Quantum Mechanics. This class of theories has tried to show that

the form of all hidden variable models is capable to reproduce Quantum Mechanics of a spin-singlet by satisfying both the assumptions of free will and no signaling, which correspond, measurement-independence and setting-independence, respectively. [5, 11, 13, 49, 188, 196]

On the other hand, Bell's Theorem ([11, 12]), Kochen–Specker Theorem ([100, 153]) and collapse theories such as von Neumann method ([189]), Ghirardi–Rimini–Weber Theory ([82]), no-collapse theories such as modal interpretations ([85]) are rigorous well-known efforts to show the inconsistency of the Einstein–Podolsky–Rosen paradox and other deterministic observations with the foundations of Quantum Mechanics. In this direction, the existence of certain class of observables which can not consistently be assigned values at all has been considered. Then it is discussed that a quantum system evolves based on the Schrodinger equation between measurements at which it collapses to the eigenstate of the measured variable. In non-measurement interactions, the evolution of states obeys a linear and unitary equation of motion such that the particle pair in the Einstein–Podolsky–Rosen experiment remains in an entangled state. This class of equations dictates that in a spin measurement, the pointers of the measurement tools are entangled with the particle pair in a non-separable state in which the indefiniteness of spins of particles is transmitted to the pointer's position. The critical challenge in this description is the lack of explicit definitions for the notions of measurement and time, duration and nature of state collapses. There are some efforts to handle this issue such as adding a nonlinear term to the Schrodinger equation, modal no-collapse interpretations. In addition, Kochen–Specker Theorem shows the impossibility of the reproduction of Quantum Mechanics predictions in terms of a hidden variable model where the hidden variables could assign a value to every projector deterministically and non-contextually. We can also address Free Will Theorem which proves that no theory, whether it extends Quantum Mechanics or not, can correctly predict the results of future spin experiments. These topics could provide strong reasons for the intrinsic non-local indeterministic nature of Quantum Mechanics. [11, 12, 18, 47, 153]

The notion of entanglement in Quantum Mechanics was derived from Schrodinger's efforts to describe quantum systems extended over physically distant parts. Then it was modified by Bohr to deal with the Einstein–Podolsky–Rosen paradox. Bohr was trying to solve the question about the spin components of a pair of particles emitted from a source to move in opposite directions where no slower than light or light signal can travel between them. Bohr addressed that since two particles have interacted, they were part of one whole phenomenon which means that the two particles are entangled. In other words, these particles are part of one whole phenomenon or one whole system that has one wave function [14, 176]. On the other hand, Bell's Theorem made the free choice of the experimenter as one of the axioms where it is proved mathematically that certain quantum correlations

violate realism, locality or freedom of choice. The Bell's inequality is on the basis of the assumption that the quantum state is not the ultimate limit and additional parameters such as hidden variables could provide a modified description. Bell discovered that no local hidden variable theory can reproduce all possible results of Quantum Mechanics. This framework shows that any local model of the Bohm's version of the Einstein–Podolsky–Rosen experiment is committed to certain inequalities about the probabilities of measurement outcomes which could be incompatible with the predications of Quantum Mechanics. Bell's Theorem has achieved the non-locality of the quantum realm in terms of an alternative interpretation of the factorization as a locality condition. This perspective postulates that for each quantum mechanical state there exists a distribution over all possible pair states which is independent of the settings of the equipments. However, experiments (such as polarization of photons) violate the Bell's inequality and moreover, they confirm the predictions of Quantum Mechanics with high accuracy. The violation of the Bell's inequality is enough to show that there is no underlying classical description for Quantum Mechanics. [10, 11, 12, 18]

Quantum concepts such as entanglement and superposition are fundamental tools for a theory of quantum computation which performs operations on information in terms of quantum bits. The state of a quantum bit lives in a superposition of two orthonormal states such as  $|\psi\rangle := \alpha|0\rangle + \beta|1\rangle$  such that  $\alpha, \beta$  are complex numbers. The measurement of one quantum bit collapses the wave function of the other quantum bit. Quantum entanglement deals with three fundamental subjects which can be studied under deterministic and indeterministic settings. The first challenge is to explain how we can detect optimally entanglement under theoretical models and experimental tests. The second challenge is to build theoretical models and experimental tests which reverse an inevitable process of degradation of entanglement. The third challenge is to design computational algorithms which enable us to characterize, control and quantify entanglement. The main objective in dealing with these challenges is to find a way to estimate optimally the amount of quantum entanglement of the compound system in an unknown state if only incomplete data in the form of average values of some operators detecting entanglement are accessible. In this direction, a notion of minimization of entanglement has been formulated under a chosen measure of entanglement with constrains in the form of incomplete set of data from experiment. In addition, theory of positive maps has been developed to provide strong tools for the detection of entanglement. [38, 89, 93, 155, 159]

Entanglement in Quantum Field Theory have also been considered recently where the measurements of the amount of entanglement in a quantum system with infinite degrees of freedom were modeled under some settings such as entropy, kinematic entanglement, particle mixing and oscillations, theory of neutrino oscillations and entangled space-time points. Entropy

is on the basis of partitioning an extended quantum system into two complementary subsystems and calculating the entanglement entropy defined as the von Neumann entropy of the reduced density matrix of one subsystem. This treatment does not provide information about the entanglement between two non-complementary parts of a larger system because of the existence of a mixed state. Negativity is one interesting tool to deal with this issue in Quantum Field Theory. Multi-mode entanglement of single-particle states has been concerned via particle mixing and flavor oscillations. It is shown that in Quantum Field Theory these phenomena exhibit a fine structure of quantum correlations as multi-mode multi-particle entanglement appears. Quantum information theory is capable to provide appropriate tools to quantify the content of multi-particle flavor entanglement in QFT systems. The multi-particle flavor-species entanglement associated with flavor oscillations of the QFT neutrino system has been studied in terms of the particle-antiparticle species as further quantum modes. Neutrino oscillations are due to neutrino mixing and neutrino mass differences. Theory of entanglement in neutrino oscillations is another progress in this direction where mode entanglement can be expressed in terms of flavor transition probabilities. Charged-current weak-interaction processes together with their associated charged leptons enable us to identify flavor neutrinos. Neutrino oscillations and CP violation concern neutrino mixing such that neutrino masses as corrections to Standard Model play their essential roles in the procedure [6, 23, 24, 25, 38, 81, 97]. In this chapter we plan to build a new mathematical model for the description of quantum entanglement in terms of the space of Dyson–Schwinger equations of a given gauge field theory. We show the importance of analytic generalization of solutions of Dyson–Schwinger equations (i.e. Feynman graphon models) for the analysis of quantum informational bridges in terms of lattices of substructures.

Passing from Classical Physics to Quantum Physics changes the logical foundations of our mathematical frameworks. If we have a rigorous formulation for the logic of quantum systems with infinite degrees of freedom, then it definitely helps us to develop our knowledge about entanglement machinery in QFTs. The logical foundations of Quantum Mechanics were firstly built in the context of propositional calculus, Hilbert space of states and the space of observations. The original aim of the propositional calculus in logic is to evaluate propositions with the general form ” *the physical quantity such as  $A$  of a given system  $S$  has a value in the subset  $\Delta$  of real numbers.*” In this context, the main task is to find what truth-values such propositions have in a given state of the system and how the truth-value changes with the state in time. In Classical Physics, there is a space of states such as points in a topological space equipped with some additional structures such as Poisson brackets, symplectic forms, .... In any given state, each physical quantity has its value and each proposition of the form  $A \in \Delta$ , which is represented by some Borel subsets of the state space, has a truth-value true

or false. The Borel subsets of the state space form a Boolean  $\sigma$ -algebra which means that the logic of classical systems can be encoded by a definite Boolean logic. This description enables us to label Classical Physics as a realist theory. Quantum Physics does not have this explicit realistic nature and according to the Kochen–Specker Theorem, there is no state space of a quantum system analogous to the classical state space. As the assumptions of this Theorem, the physical quantities are represented as real-valued functions on the hypothetical state space of a quantum system. Then it is shown that such a space does not exist and it is impossible to assign values to all physical quantities at once. Therefore it is also impossible to assign true or false values to all propositions. Birkhoff and von Neumann built the foundations of an instrumentalist approach to quantum logic where upon measurement of the physical quantity  $A$ , we could find the result belong in  $\Delta$  with a determined probability. In this approach, pure states are represented by unit vectors in one particular Hilbert space and propositions with the general form  $A \in \Delta$  are represented by projection operators on this Hilbert space. These projections form a non-distributive lattice. Set  $\hat{E}[A \in \Delta]$  as the projection which represents the proposition  $A \in \Delta$ . The probability of  $A \in \Delta$  being true in a given state  $|\psi\rangle$  is determined by

$$P(A \in \Delta | \psi) := \langle \psi | \hat{E}[A \in \Delta] | \psi \rangle \in [0, 1]. \quad (6.1)$$

Non-distributivity, dependence on measurement tools and the use of real numbers as continuum are the most fundamental and conceptual issues of this instrumentalist approach and its generalizations. [1, 4, 10, 20, 37, 41]

Category Theory had been applied to formulate a modern topos model for the logical descriptions of physical phenomena. This modern approach clarified the passing from Classical Physics to Quantum Physics in terms of their corresponding topos models. Study in this direction has provided a new contextual form of quantum logic where it is possible to reconstruct the foundations of physical theories in the context of search for a suitable representation in a topos of a certain formal language. Classical Physics is reconstructed via the category of sets while Quantum Physics is reconstructed via the category of presheaves on a particular base category. This categorical machinery has been developed to QFT models where nowadays we have some topos models for gauge field theories [17, 58, 59, 60, 61, 62, 90, 91, 92, 122, 126, 129, 141, 152]. In this chapter we plan to build a new topos model for strongly coupled gauge field theories which is capable to recognize the logical differences between perturbative and non-perturbative aspects of the physical theory.

## 6.2 *QFT-entanglement via lattices of Dyson–Schwinger equations*

The mathematical formulation of Standard Model in the context of Noncommutative Geometry can motivate to bring a new approach for the description of quantum entanglement in gauge field theories. In Standard Model we have six quarks, six leptons and gauge bosons which are responsible to carry fundamental forces. Gauge bosons describe exchanging information between elementary particles and their interactions in strong, weak and electromagnetic forces. For example, the exchanging virtual photons (as the gauge boson in quantum electrodynamics) makes transferring information as the force between two electrons which is repulsive. Gluons are involved gauge bosons in strong interactions among hadrons (i.e. six quarks) which live in the nucleus of an atom. Electrons and neutrinos do not feel strong nuclear force. Every charge particle feels the electromagnetic force.  $W^\pm, Z$  are involved gauge bosons in weak interactions where everything is effected by the weak nuclear force. Graviton is the theoretical candidate for gauge bosons of gravity which effects everything. The modified versions of Standard Model aim to describe the contribution of gravity. Heavier gauge bosons are bosons of the fundamental force with the shortest range of effect. Photons are massless which means that the electromagnetic force has infinite range.  $W^\pm, Z$  bosons are extremely heavy and they have very short range. For example, a neutron can decay into a proton and the gauge boson  $W^-$  where at the very short time, this boson quickly decays into an electron and an antielectron neutrino. A proton can decay into a neutron and the gauge boson  $W^+$  where at the very short time this boson quickly decays into a positron and a neutrino. Protons and neutrons are built by quarks.  $W^\pm$  bosons can contribute to exchanging a type of quark to another type where as the result a proton converts to a neutron and vice versa. Since  $W^\pm$  are heavy, they need to borrow energy to perform this exchange and then they should pay back the energy by converting to pairs (positron, neutrino) or (electron, antielectron neutrino) very quickly. Quarks enjoy the Pauli exclusion principle which means that quarks should be in different quantum states. This distinction is encoded by colors. Gluons govern any possible interactions among quarks which convert or exchange the colors of quarks by absorption or emission of gluons. Gluons can also produce other gluons and they glue quarks together. Force between two quarks is independent of distance between them and therefore we need infinite amount of energy to separate quarks. This fact, known as quark confinement, tells us that we can not isolate a quark. Thanks to gluons, the strong force also governs the existence of protons and neutrons together inside the nucleus but the force at this level is not independent of distance. Theoretically, the amount of energy can be converted to a pair of quark and anti-quark where some interactions could happen to

exchange colours. [46, 112, 147, 156, 157, 186]

### 6.2.1 A new lattice model

It is possible to encapsulate all possible interactions in terms of Green's functions where their self-similar nature enable us to study interactions in the context of fixed point equations of Green's functions namely, Dyson–Schwinger equations. The strength of the fundamental forces dictate the appearance of perturbative, asymptotic freedom or non-perturbative behaviors to these equations. It is mentioned that gauge bosons provide information exchange and here we plan to mathematically describe the existence of information flow among elementary particles at strong levels of the coupling constants in interacting gauge field theories via towers of Dyson–Schwinger equations, cut-distance topological regions of Feynman diagrams which contribute to solutions of these equations and the vacuum energy. The vacuum energy guarantees the existence of virtual particles in the vacuum state which will be used in our setting.

*Remark 6.2.1.* The vacuum state in free field theory can be described in terms of a tensor product of the Fock space vacuum states for each independent field mode where there is no entanglement between the field modes at different momenta. The full vacuum state in interacting Quantum Field Theory can be described in terms of a superposition of Fock basis states where the modes of different momenta are entangled.

Our promising mathematical model tries to engineer divergencies of non-perturbative aspects of strongly coupled gauge field theories in the context of lattices of topological regions of elementary particles. It shows a deep dependence of the quantum entanglement on the indeterminateness of elementary particles. This platform enables us to understand quantum entanglement as an intrinsic non-local property of non-perturbative QFT-models which approves the indeterministic nature of strongly coupled gauge field theories.

**Definition 6.2.2.** For each  $n$ , consider  $\gamma_n^p$  as a primitive (1PI) Feynman diagram which presents some interactions of any elementary particle  $p$  with other (virtual) particles in the physical theory. For each Feynman diagram  $\Gamma$ ,  $B_{\gamma_n^p}^+(\Gamma)$  builds a new disjoint union of Feynman diagrams as the result of all possibilities for the insertion of  $\Gamma$  into  $\gamma_n^p$  in terms of the types of vertices in  $\gamma_n^p$  and types of external edges in  $\Gamma$ . Each family  $\{B_{\gamma_n^p}^+\}_{n \geq 0}$  of this class of Hochschild one cocycles can determine a particular Dyson–Schwinger equation  $DSE_p$  which encodes a collection of possible interactions between  $p$  and other (virtual) particles in the physical theory.

We plan to apply lattice structures to describe quantum entanglement. A lattice is a partially ordered set such that each pair of its elements has a unique join  $\vee$  which is the least upper bound and a unique meet  $\wedge$  which

is the greatest lower bound. A lattice is called bounded if there exist the greatest element and the least element. A lattice is called distributive, if the operations meet and join obey the distributive conditions.

**Theorem 6.2.3.** *The information flow between the particle  $p$  and all unobserved intermediate states can be described in terms of a lattice of topological Hopf subalgebras.*

*Proof.* Intermediate states address virtual particles. Dyson–Schwinger equations are the best tools for us to build Hopf subalgebras of the Connes–Kreimer renormalization Hopf algebra  $H_{FG}(\Phi)$  of Feynman diagrams of a given gauge field theory  $\Phi$ . Thanks to the cut-distance topology defined on Feynman graphons (i.e. Theorem 2.3.8), we can naturally equip each Hopf subalgebra  $H_{DSE}$  with this topology such that the distance between Feynman diagrams  $\Gamma_1, \Gamma_2$  is given by

$$d(\Gamma_1, \Gamma_2) := d_{\text{cut}}([f^{\Gamma_1}], [f^{\Gamma_2}]). \tag{6.2}$$

The class  $[f^{\Gamma_i}]$  is the unique unlabeled Feynman graphon with respect to the Feynman diagram  $\Gamma_i$  and

$$d_{\text{cut}}([f^{\Gamma_1}], [f^{\Gamma_2}]) = \inf_{\phi, \psi} \sup_{A, B} \left| \int_{A \times B} f^{\Gamma_1}(\phi(x), \phi(y)) - f^{\Gamma_2}(\psi(x), \psi(y)) dx dy \right| \tag{6.3}$$

where the infimum is taken over all different relabeling  $\phi, \psi$  for the labeled graphons  $f^\phi, f^\psi$ , respectively. The supremum is taken over all Lebesgue measurable subsets  $A, B$  of the closed interval.

In addition, the coproduct of  $H_{DSE}$  is a linear bounded operator on normed space of Feynman diagrams which leads us to consider each  $H_{DSE}^{\text{cut}}$  as a topological Hopf subalgebra.

Thanks to Definition 6.2.2, choose an equation  $DSE_p$  which contains some interactions related to the particle  $p$ . For each  $j \geq 1$ , build a new collection  $\{\Gamma_n^{(j)}\}_{n \geq 1}$  of primitive graphs

$$\Gamma_n^{(j)} := \Gamma_1^{(j-1)} + \dots + \Gamma_n^{(j-1)} \tag{6.4}$$

such that  $\Gamma_n^{(0)} = \gamma_n^p$  for each  $n \geq 1$ .

Thanks to the Milnor–Moore Theorem ([144]), the Connes–Kreimer renormalization Hopf algebra  $H_{FG}(\Phi)$  is isomorphic to the graded dual of the universal enveloping algebra of the Lie algebra of primitive elements in  $H_{FG}(\Phi)^*$  where for each Feynman diagram  $\Gamma$ , we can consider its corresponding infinitesimal character  $Z_\Gamma$  in the dual space. Since the renormalization coproduct is a linear map, we can check easily that the sum of a finite number of primitive graphs is primitive. Therefore for each  $j \geq 1$  and  $n \geq 1$ , the operator  $B_{\Gamma_n^{(j)}}^+$  is the corresponding Hochschild one cocycle such that for each

Feynman diagram  $\Gamma$ , this operator concerns all possible situations for the insertion of  $\Gamma$  into the disjoint unions of Feynman diagrams  $\Gamma_1^{(j-1)}, \dots, \Gamma_n^{(j-1)}$ . In addition, by induction, we can show that for each  $j$ , the resulting graph  $B_{\Gamma_n^{(j)}}^+(\Gamma)$  covers  $B_{\Gamma_n^{(j-1)}}^+(\Gamma)$  as a subgraph.

For each  $j \geq 1$ , we can consider the Dyson–Schwinger equation

$$\text{DSE}_p^{(j)} := \langle \{B_{\Gamma_n^{(j)}}^+\}_{n \geq 1} \rangle, \tag{6.5}$$

with the corresponding Hopf subalgebra  $H_{\text{DSE}_p^{(j)}}$ . There exists a natural injective Hopf algebra homomorphism from  $H_{\text{DSE}_p^{(j)}}$  to  $H_{\text{DSE}_p^{(j+1)}}$  which leads us to build the following increasing chain of Hopf subalgebras

$$H_{\text{DSE}_p} \leq H_{\text{DSE}_p^{(1)}} \leq H_{\text{DSE}_p^{(2)}} \leq \dots \leq H_{\text{DSE}_p^{(j)}} \leq \dots \tag{6.6}$$

If  $X_{\text{DSE}_p^{(j)}} = \sum_{n_{(j)} \geq 0} g^{n_{(j)}} X_{n_{(j)}}^j$  is the unique solution of the equation  $\text{DSE}_p^{(j)}$ , then for each  $n_{(j)} \geq 1$ , set

$$H(X_1^j, \dots, X_{n_{(j)}}^j) \tag{6.7}$$

as the graded Hopf subalgebra of  $H_{\text{DSE}_p^{(j)}}$  free commutative generated algebraically by graphs  $X_1^j, \dots, X_{n_{(j)}}^j$ .

We can also equip these Hopf subalgebras with the cut-distance topology and then work on the topologically completed enrichment of these Hopf subalgebras. We have discussed that solutions of Dyson–Schwinger equations are actually graph limits of sequences of finite Feynman diagrams with respect to the cut-distance topology. The Hopf subalgebra  $H_{\text{DSE}}$  generated by a given Dyson–Schwinger equation DSE is graded with respect to the number of internal edges or the number of independent loops. We have

$$H_{\text{DSE}} = \bigoplus_{n \geq 0} H_{\text{DSE},(n)} \tag{6.8}$$

such that  $H_{\text{DSE},(n)}$  is the homogeneous component of degree  $n$  and

$$H_{\text{DSE},(p)} H_{\text{DSE},(q)} \subseteq H_{\text{DSE},(p+q)},$$

$$\Delta(H_{\text{DSE},(n)}) \subseteq \bigoplus_{p+q=n} H_{\text{DSE},(p)} \otimes H_{\text{DSE},(q)}, \quad S(H_{\text{DSE},(p)}) \subseteq H_{\text{DSE},(p)}. \tag{6.9}$$

Define a new parameter  $\text{val}$  for Feynman diagrams in  $H_{\text{DSE}}$  given by

$$\text{val}(\Gamma) := \text{Max}\{n \in \mathbb{N} : \Gamma \in \bigoplus_{k \geq n} H_{\text{DSE},(k)}\} \tag{6.10}$$

which determines the  $n$ -adic metric on  $H_{\text{DSE}}$  defined by the formula

$$d_{\text{adic}}(\Gamma_1, \Gamma_2) := 2^{-\text{val}(\Gamma_1 - \Gamma_2)}. \tag{6.11}$$

Thanks to Proposition 4.6 in [169], the  $n$ -adic metric enables us to build a sequence  $\{R_n(\text{DSE})\}_{n \geq 1}$  of random graphs which is convergent to the unique solution  $X_{\text{DSE}}$  with respect to the cut-distance topology. Now if we apply the graphon representations of Feynman diagrams, then we can embed  $H_{\text{DSE}}$  into the renormalization topological Hopf algebra of Feynman graphons  $\mathcal{S}_{\text{graphon}}^\Phi$ . We can consider the completed enrichment of  $H_{\text{DSE}}^{\text{cut}}$  with respect to the cut-distance topology to add graph-limits to this topological Hopf algebra. The coproduct  $\Delta_{H_{\text{DSE}}}$  is a linear bounded map  $H_{\text{DSE}}^{\text{cut}}$  which makes it a continuous operator. Thanks to the graduation parameter, the antipode is also a continuous operator in this setting. Denote  $H_{\text{DSE}}^{\text{cut}}$  as the resulting topological Hopf algebra.

Now for each  $j \geq 1$ , set  $H_{\text{DSE}_p^{(j)}}^{\text{cut}}$  as the resulting topological Hopf subalgebra corresponding to each equation  $\text{DSE}_p^{(j)}$ . The family  $\mathcal{C}_p$  of these topological Hopf subalgebras can be equipped by the following binary relation

$$V \preceq W \iff \tag{6.12}$$

there exists a finite sequence of topological Hopf subalgebras  $V_1, \dots, V_r \in \mathcal{C}_p$  together with injective morphisms  $V \rightarrow V_1 \rightarrow V_2 \rightarrow \dots \rightarrow V_r \rightarrow W$  which connect  $V$  to  $W$ .

We can check that  $(\mathcal{C}_p, \preceq)$  is a lattice of topological Hopf subalgebras with the greatest lower bound  $H(X_{1(0)}^{(0)})^{\text{cut}}$ . For each subset  $\{V_1, V_2\}$  of  $(\mathcal{C}_p, \preceq)$ , if  $V_1 \preceq V_2$  then define

$$V_1 \wedge V_2 := V_1, \quad V_1 \vee V_2 := V_2. \tag{6.13}$$

The lattice  $(\mathcal{C}_p, \preceq)$  enables us to relate a class of Dyson–Schwinger equations derived from the basic equation  $\text{DSE}_p$  to each other with respect to morphisms among their corresponding topological Hopf subalgebras. This means that the lattice  $(\mathcal{C}_p, \preceq)$  mathematically describes the transferring of information from one equation to another. It describes the informational bridge between  $p$  and its intermediate states.  $\square$

**Definition 6.2.4.** Set  $R^p$  as the smallest collection of all Feynman diagrams in  $\Phi$  which contribute to the equations  $\text{DSE}_p$  and  $\text{DSE}_p^{(j)}$  for all  $j \geq 1$ . Then equip it with the cut-distance topology and add graph limits to obtain a complete topological region  $\overline{R^p}$ .

Thanks to Theorem 6.2.3, the lattice  $(\mathcal{C}_p, \preceq)$  shows us the information flow processes among all (virtual) particles which contribute to the topological region  $\overline{R^p}$ .

**Theorem 6.2.5.** *There exists a lattice of topological Hopf subalgebras which describes the information flow in an entangled system of elementary particles in a given interacting Quantum Field Theory.*

*Proof.* At the first step, we are going to show the existence of a class of Dyson–Schwinger equations for the description of information flow between two space-time far distant particles which belong to an entangled system in a given interacting Quantum Field Theory.

Thanks to Theorem 6.2.3, we already have described the entanglement process in a topological region around an elementary particle on the basis of the cut-distance topology. Here we need to show the possibility of information flow between two entangled particles  $p, q$  while  $p$  does not belong to  $\overline{R^q}$ ,  $q$  does not belong to  $\overline{R^p}$  and  $\overline{R^p} \cap \overline{R^q} = \emptyset$ .

We have identified topological subspaces  $\overline{R^p}$  and  $\overline{R^q}$  of the topological Hopf algebra  $H_{\text{FG}}^{\text{cut}}(\Phi)$  in terms of their contribution to the entanglement of intermediate states (i.e. Definition 6.2.4). Now thanks to the metric (6.2), define the distance between this class of regions in terms of

$$d(\overline{R^p}, \overline{R^q}) := \inf\{d(X, Y) : X \in \overline{R^p}, Y \in \overline{R^q}\}. \quad (6.14)$$

We want to show the existence of topological regions such as  $R^{cpq}$  in  $H_{\text{FG}}^{\text{cut}}(\Phi)$  with the following conditions

$$\overline{R^p} \cap \overline{R^{cpq}} \neq \emptyset, \quad \overline{R^q} \cap \overline{R^{cpq}} \neq \emptyset. \quad (6.15)$$

For  $d(\overline{R^p}, \overline{R^q}) > 0$ , there exist  $j_1, j_2 \geq 0$  such that the corresponding equations  $\text{DSE}_p^{(j_1)}$  and  $\text{DSE}_q^{(j_2)}$  have the following conditions

$$X_{\text{DSE}_p^{(j_1)}} = \lim_{n \rightarrow \infty} \sum_{k=0}^n X_k^{(j_1)}, \quad X_{\text{DSE}_q^{(j_2)}} = \lim_{n \rightarrow \infty} \sum_{k=0}^n X_k^{(j_2)} \quad (6.16)$$

$$d(\overline{R^p}, \overline{R^q}) = d_{\text{cut}}(X_{\text{DSE}_p^{(j_1)}}, X_{\text{DSE}_q^{(j_2)}}) > 0. \quad (6.17)$$

For each  $\epsilon > 0$ , we can determine Hochschild one cocycles  $B_{\gamma_n^+, p}^+$ ,  $n \geq 1$  which fulfills the following conditions:

- Each  $\gamma_n^\epsilon$  is a finite primitive (1PI) Feynman diagram such that

$$\forall n \geq 1, \quad \gamma_n^\epsilon \notin R^p, \quad \gamma_n^\epsilon \notin R^q, \quad \gamma_n^\epsilon \in H_{\text{FG}}(\Phi). \quad (6.18)$$

- The equation  $\text{DSE}_p^{(\epsilon)}$  as the Dyson–Schwinger equation originated from the family  $\{B_{\gamma_n^+, p}^+\}_{n \geq 1}$  with the unique solution  $X_\epsilon^p = \sum_{n \geq 0} X_n^{(\epsilon)p}$  has the following property that there exists  $N_\epsilon \in \mathbb{N}$  such that for each  $n > N_\epsilon$ , we have  $d(X_n^{(j_1)}, X_n^{(\epsilon)p}) \leq \epsilon$ .

Thanks to the triangle inequality of the cut-distance metric, we can obtain

$$d(X_{\text{DSE}_p^{(j_1)}}, X_\epsilon^p) \leq \epsilon \quad (6.19)$$

In addition, for each  $\epsilon > 0$ , we can determine Hochschild one cocycles  $B_{\gamma_n^+, q}^+$ ,  $n \geq 1$  which fulfills the following conditions:

- Each  $\eta_n^\epsilon$  is a finite primitive (1PI) Feynman diagram such that

$$\forall n \geq 1, \quad \eta_n^\epsilon \notin R^p, \quad \eta_n^\epsilon \notin R^q, \quad \eta_n^\epsilon \in H_{\text{FG}}(\Phi). \quad (6.20)$$

- The equation  $\text{DSE}_q^{(\epsilon)}$  as the Dyson–Schwinger equation originated from the family  $\{B_{\eta_n^\epsilon, q}^+\}_{n \geq 1}$  with the unique solution  $X_\epsilon^q = \sum_{n \geq 0} X_n^{(\epsilon)q}$  has the following property that there exists  $N'_\epsilon \in \mathbb{N}$  such that for each  $n > N'_\epsilon$ , we have  $d(X_n^{(j_2)}, X_n^{(\epsilon)q}) \leq \epsilon$ .

Thanks to the triangle inequality of the cut-distance metric, we can obtain

$$d(X_{\text{DSE}_q^{(j_2)}}, X_\epsilon^q) \leq \epsilon. \quad (6.21)$$

The vacuum in an interacting physical theory can be described as a homogeneous system of virtual particles where its states are invariant by all transformations of the invariance group. Some particles in the vacuum have negative energies where without violating the conservation laws they can annihilate ([156]). Thanks to this fact, we can determine  $R^{cpq}$  as the smallest topological subset of  $H_{\text{FG}}^{\text{cut}}(\Phi)$  consisting of Feynman graphs which contribute to equations of the types  $\text{DSE}_p^{(\epsilon)}$  and  $\text{DSE}_q^{(\epsilon)}$ . This region contains virtual particles (created by the vacuum energy) which has separate contributive parts with topological regions  $R^p$  and  $R^q$ . The relations (6.19) and (6.21) guarantee that

$$R^p \cap R^{cpq} \neq \emptyset, \quad R^q \cap R^{cpq} \neq \emptyset. \quad (6.22)$$

Now if we apply Theorem 6.2.3, then graphs which belong to the region  $\overline{R^{cpq}}$  (as the completion of  $R^{cpq}$  with respect to the cut-distance topology) make informational bridges between entangled particles  $p$ ,  $q$  and their corresponding intermediate states (virtual particles) which live in  $\overline{R^p} \cup \overline{R^q} \cup \overline{R^{cpq}}$ .

At the second step, we want to encapsulate the above machinery in terms of lattice models. Suppose  $\text{DSE}_p$  and  $\text{DSE}_q$  are the basic Dyson–Schwinger equations corresponding to entangled particles  $p$  and  $q$ . Thanks to the built lattice by Theorem 6.2.3, let  $(\mathcal{C}_p, \preceq)$  be the lattice of topological Hopf subalgebras  $H_{\text{DSE}_p^{(j)}}^{\text{cut}}$  generated by equations of the type  $\text{DSE}_p^{(j)}$  which live in the topological region  $\overline{R^p}$  and let  $(\mathcal{C}_q, \preceq)$  be the lattice of topological Hopf subalgebras  $H_{\text{DSE}_q^{(l)}}^{\text{cut}}$  generated by equations of the type  $\text{DSE}_q^{(l)}$  which live in the topological region  $\overline{R^q}$ . Thanks to the previous part of the proof, we can show the existence of  $j_1, j_2 \geq 0$  such that  $\text{DSE}_p^{(j_1)}$  and  $\text{DSE}_q^{(j_2)}$  contribute in the description of the distance between two topological regions  $\overline{R^p}$  and  $\overline{R^q}$  (i.e. metric (6.14)). Set

$$j_1^* := \text{Min}\{j_1 : \text{DSE}_p^{(j_1)}\}, \quad j_2^* := \text{Min}\{j_2 : \text{DSE}_q^{(j_2)}\}. \quad (6.23)$$

Consider the sub-lattice  $(\mathcal{C}_p^{j_1^*}, \preceq)$  which contains only the first  $j_1^*$  columns of the original lattice  $(\mathcal{C}_p, \preceq)$  and the sub-lattice  $(\mathcal{C}_q^{j_2^*}, \preceq)$  which contains only

the first  $j_2^*$  columns of the original lattice  $(\mathcal{C}_q, \preceq)$ . These two sub-lattices have the greatest lower bound and the smallest upper bound.

Thanks to the structure of the topological region  $R^{c_{pq}}$ , we can build a new lattice  $(\mathcal{C}_{c_{pq}}^{j_1^* j_2^*}, \preceq)$  which is the result of the disjoint union of the sub-lattices  $(\mathcal{C}_p^{j_1^*}, \preceq)$  and  $(\mathcal{C}_q^{j_2^*}, \preceq)$  which are connected to each other by topological Hopf algebra homomorphisms associated to Dyson–Schwinger equations of the types  $\text{DSE}_p^{(\epsilon)}$  and  $\text{DSE}_q^{(\epsilon)}$ . We have

$$H_{\text{DSE}_p}^{\text{cut}} \leq H_{\text{DSE}_p^{(1)}}^{\text{cut}} \leq \dots \leq H_{\text{DSE}_p^{(j_1^*)}}^{\text{cut}} \quad (6.24)$$

$$H_{\text{DSE}_q}^{\text{cut}} \leq H_{\text{DSE}_q^{(1)}}^{\text{cut}} \leq \dots \leq H_{\text{DSE}_q^{(j_2^*)}}^{\text{cut}} \quad (6.25)$$

which belong to the new lattice in terms of one of the following topological Hopf algebra homomorphisms

$$H_{\text{DSE}_p^{(j_1^*)}}^{\text{cut}} \longrightarrow H_{\text{DSE}_p^{(\epsilon)}}^{\text{cut}} \longrightarrow H_{\text{DSE}_c^{(k)}}^{\text{cut}} \longrightarrow H_{\text{DSE}_q^{(\epsilon)}}^{\text{cut}} \longrightarrow H_{\text{DSE}_q^{(j_2^*)}}^{\text{cut}} \quad (6.26)$$

or

$$H_{\text{DSE}_q^{(j_2^*)}}^{\text{cut}} \longrightarrow H_{\text{DSE}_q^{(\epsilon)}}^{\text{cut}} \longrightarrow H_{\text{DSE}_c^{(k)}}^{\text{cut}} \longrightarrow H_{\text{DSE}_p^{(\epsilon)}}^{\text{cut}} \longrightarrow H_{\text{DSE}_p^{(j_1^*)}}^{\text{cut}} \quad (6.27)$$

such that  $H_{\text{DSE}_c^{(k)}}^{\text{cut}}$  is the topological Hopf subalgebra associated to the Dyson–Schwinger equation  $\text{DSE}_c^{(k)}$  which lives in the region  $R^{c_{pq}}$  and derived from the virtual particle  $c$ .  $\square$

### 6.2.2 Intermediate algorithms for QFT-entanglement

The Milnor–Moore theorem provides a correspondence between pro-unipotent Lie groups and graded commutative Hopf algebras ([36, 139]). This correspondence enables us to translate the determination of Hopf subalgebraic structures in the renormalization Hopf algebra to a problem in Lie groups. One interesting application of Galois theory is to find a fundamental interrelationship between subgroups of the group of all automorphisms and intermediate algorithmic structures which live in the middle of programs and computable functions [198, 199]. Dyson–Schwinger equations can be applied for the determination of substructures in the renormalization Hopf algebra  $H_{\text{FG}}(\Phi)$  and the determination of Lie subgroups of the complex Lie group  $\mathbb{G}_\Phi(\mathbb{C})$  of characters. We can consider each  $H_{\text{DSE}}$  as the Hopf ideal in  $H_{\text{FG}}(\Phi)$  and then consider the resulting quotient Hopf subalgebra. The complex Lie group corresponding to this quotient Hopf algebra is a Lie subgroup of  $\mathbb{G}_\Phi(\mathbb{C})$  while we have a surjective group homomorphism from  $\mathbb{G}_\Phi(\mathbb{C})$  to  $\mathbb{G}_{\text{DSE}}(\mathbb{C})$ . Therefore these non-perturbative equations play the middle bridge between Theory of Computation and Quantum Field Theory [63, 167].

Theorem 6.2.3 and Theorem 6.2.5 have explained the entanglement of elementary particles in terms of the existence of substructures originated from Dyson–Schwinger equations. This new mathematical platform can address a deep connection between the concept of information flow in Quantum Field Theory and the existence of subobjects inside an object determined by the Galois Fundamental Theorem. The immediate consequence of this investigation is to recognize a new approach to quantum entanglement in the language of intermediate algorithms in Theoretical Computer Science. We deal with this interesting challenge on the basis of the representation theory of Lie groups.

**Theorem 6.2.6.** *The intermediate algorithms corresponding to Lie subgroups of the complex Lie group  $\mathbb{G}_{\text{graphon}}^{\Phi}(\mathbb{C})$  can encode the information flow in an entangled system in a given interacting Quantum Field Theory  $\Phi$  in terms of a lattice of Lie subgroups.*

*Proof.* At the first step, we plan to show that the information flow between the particle  $p$  and all unobserved intermediate states can be encoded via a lattice of Lie subgroups of  $\mathbb{G}_{\Phi}(\mathbb{C})$ . Thanks to Theorem 6.2.3, the entanglement of the particle  $p$  and its related virtual particles is encapsulated by the lattice  $(\mathcal{C}_p, \preceq)$  of topological Hopf subalgebras. Each pair of Hopf subalgebras  $H_{\text{DSE}_p^{(k)}} \preceq H_{\text{DSE}_p^{(l)}}$  in this lattice determines the injective Hopf algebra homomorphism

$$i_{kl} : H_{\text{DSE}_p^{(k)}} \longrightarrow H_{\text{DSE}_p^{(l)}}. \quad (6.28)$$

The passing from Hopf subalgebras to Lie subgroups can be formulated by applying the contravariant functor  $\text{Spec}$  which sends a commutative algebra to a topological space. For each object  $H_{\text{DSE}_p^{(k)}} \in (\mathcal{C}_p, \preceq)$ ,  $\text{Spec}(H_{\text{DSE}_p^{(k)}})$  is the set of all prime ideals of the commutative algebra  $H_{\text{DSE}_p^{(k)}}$  equipped with the Zariski topology and the structure sheaf. The homomorphism  $i_{kl}$  can be lifted onto the surjective homomorphism

$$\tilde{i}_{kl} : \text{Spec}(H_{\text{DSE}_p^{(l)}}) \longrightarrow \text{Spec}(H_{\text{DSE}_p^{(k)}}) \quad (6.29)$$

of affine group schemes. For a fixed Hopf subalgebra  $H_{\text{DSE}_p^{(k)}}$ ,  $\text{Spec}(H_{\text{DSE}_p^{(k)}})$  is a representable covariant functor which sends a topological space to a group. Set  $G_{p^{(k)}} := \text{Spec}(H_{\text{DSE}_p^{(k)}})(\mathbb{C})$  as the complex Lie subgroup corresponding to the Hopf subalgebra  $H_{\text{DSE}_p^{(k)}}$  such that its group structure is given by the convolution product generated by the coproduct  $\Delta_{H_{\text{DSE}_p^{(k)}}}$ .

Thanks to this setting, we can build a new lattice  $(\mathcal{G}_p, \succeq)$  of complex Lie groups such that

$$G \succeq K \iff \quad (6.30)$$

There exists a finite sequence of complex Lie subgroups  $G_1, \dots, G_r \in \mathcal{G}_p$  together with surjective group homomorphisms  $G \rightarrow G_1 \rightarrow G_2 \rightarrow \dots \rightarrow G_r \rightarrow K$  which connect  $G$  to  $K$ .

In addition, for each  $n \geq 1$ , define  $G(X_1^j, \dots, X_n^j)$  as the finite dimensional complex Lie subgroup corresponding to the free commutative graded Hopf subalgebra  $H(X_1^j, \dots, X_n^j)$  of  $H_{\text{DSE}_p^{(j)}}$ . The lattice  $(\mathcal{G}_p, \succeq)$  of Lie groups encodes the information flow between  $p$  and its related virtual particles.

At the second step, we plan to show that there exists a lattice of Lie subgroups which describes the information flow in an entangled system of elementary particles in a given interacting gauge field theory. For this purpose, we need to build a lattice of Lie subgroups for the description of the quantum entanglement process between space-time far distant elementary particles which belong to an entangled system in the physical theory  $\Phi$ .

Theorem 6.2.5 determines a lattice  $(\mathcal{C}_{c_{pq}}^{j_1^* j_2^*}, \preceq)$  of Hopf subalgebras which describes the entanglement process. Thanks to the first part of the Proof, it is possible to lift the increasing chains (6.24), (6.25) onto the following decreasing chains of Lie subgroups

$$G_{p^{(j_1^*)}} \geq G_{p^{(j_1^*-1)}} \geq \dots \geq G_{p^{(1)}} \geq G_{\text{DSE}_p} \quad (6.31)$$

$$G_{q^{(j_2^*)}} \geq G_{q^{(j_2^*-1)}} \geq \dots \geq G_{q^{(1)}} \geq G_{\text{DSE}_q} \quad (6.32)$$

with the corresponding group homomorphisms

$$G_{q^{(j_2^*)}} \longrightarrow G_{q^{(\epsilon)}} \longrightarrow G_{c^{(k)}} \longrightarrow G_{p^{(\epsilon)}} \longrightarrow G_{p^{(j_1^*)}} \quad (6.33)$$

or

$$G_{p^{(j_1^*)}} \longrightarrow G_{p^{(\epsilon)}} \longrightarrow G_{c^{(k)}} \longrightarrow G_{q^{(\epsilon)}} \longrightarrow G_{q^{(j_2^*)}} \quad (6.34)$$

such that  $G_{c^{(k)}}$  is the complex Lie subgroup corresponding to the Hopf subalgebra  $H_{\text{DSE}_c^{(k)}}$  generated by the equation  $\text{DSE}_c^{(k)}$  which lives in the topological region  $\overline{R^{c_{pq}}}$ . The existence of the virtual particle  $c$ , which contributes to interactions of the particles  $p, q$ , is the consequence of the energy of the vacuum in interacting physical theory. Now we can define a new lattice

$$(\mathcal{G}_{c_{pq}}^{j_1^* j_2^*}, \succeq) \quad (6.35)$$

of Lie subgroups and Lie group epimorphisms. This lattice encodes the entanglement processes between  $p$  and  $q$ .

As we have shown in the previous parts of this work, the renormalization Hopf algebra of Feynman graphons  $\mathcal{S}_{\text{graphon}}^\Phi$  is capable to recover the renormalization Hopf algebra  $H_{\text{FG}}(\Phi)$  and Hopf subalgebras generated by all Dyson–Schwinger equations. Therefore for each Dyson–Schwinger equation DSE with the corresponding Hopf subalgebra  $H_{\text{DSE}}$ , we can cover the associated complex Lie subgroup  $G_{\text{DSE}}(\mathbb{C})$  via the complex Lie group  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$  by a natural surjection while in the quotient Hopf algebra level  $\text{Hom}(\frac{\mathcal{S}_{\text{graphon}}^\Phi}{H_{\text{DSE}}}, \mathbb{C})$  is a Lie subgroup of  $\mathbb{G}_{\text{graphon}}^\Phi(\mathbb{C})$ .  $\square$

### 6.2.3 Tannakian subcategories as intermediate algorithms

The study of Dyson–Schwinger equations had been developed to a categorical setting where we associated a category of geometric objects to each system of these equations. Then we have embedded this class of categories into the universal Connes–Marcolli category  $\mathcal{E}^{\text{CM}}$  of flat equi-singular vector bundles. Thanks to this machinery, some new geometric and combinatorial tools for the computation of non-perturbative parameters have already been obtained [160, 162, 164, 167]. Thanks to Theorem 6.2.6, now it is possible to describe quantum entanglement in the context of the representation theory of Lie groups where we can address a new application of Tannakian formalism to Quantum Field Theory.

**Theorem 6.2.7.** *There exists a lattice of Tannakian subcategories which describes quantum entanglement in a given interacting Quantum Field Theory.*

*Proof.* At the first step, we show the existence of a lattice  $(\text{Cat}_p, \succeq)$  of Tannakian subcategories which encodes the quantum entanglement between an elementary particle  $p$  and all unobserved intermediate states (as virtual particles).

Thanks to Theorem 6.2.3, the entanglement of the particle  $p$  and its related virtual particles is encapsulated by the lattice  $(\mathcal{C}_p, \preceq)$  of topological Hopf subalgebras. Each pair of objects  $H_{\text{DSE}_p^{(k)}}^{\text{cut}} \preceq H_{\text{DSE}_p^{(l)}}^{\text{cut}}$  can determine the natural injective Hopf algebra homomorphism

$$i_{kl} : H_{\text{DSE}_p^{(k)}} \longrightarrow H_{\text{DSE}_p^{(l)}} \quad (6.36)$$

which can be lifted onto the surjective group homomorphism

$$\tilde{i}_{kl} : G_{p^{(l)}} \longrightarrow G_{p^{(k)}}. \quad (6.37)$$

For each object  $G_{p^{(l)}}$  of the lattice  $(\mathcal{G}_p, \succeq)$ , let

$$G_{p^{(l)}}^* := G_{p^{(l)}} \rtimes \mathbb{G}_m \quad (6.38)$$

such that  $\mathbb{G}_m$  is the multiplicative group which acts on the original group.

Define  $\text{Rep}_{G_{p^{(l)}}^*}$  as the category of finite dimensional representations of the complex Lie group  $G_{p^{(l)}}^*$  which is a neutral Tannakian category. Thanks to the representation theory of affine group schemes ([139]), the surjective morphism  $\tilde{i}_{kl}$  allows us to send each representation  $\sigma : G_{p^{(k)}} \longrightarrow GL_V$  to a representation

$$\sigma \circ \tilde{i}_{kl} : G_{p^{(l)}} \longrightarrow GL_V \quad (6.39)$$

which leads us to achieve an exact fully faithful functor

$$\text{Rep}_{G_{p^{(k)}}^*} \longrightarrow \text{Rep}_{G_{p^{(l)}}^*}. \quad (6.40)$$

This information is enough to build a new lattice  $(\text{Cat}_p, \succeq)$  of subcategories such that

$$\text{Rep}_{H^*} \succeq \text{Rep}_{K^*} \iff \tag{6.41}$$

there exists a finite sequence of subcategories  $\text{Rep}_{H_1^*}, \dots, \text{Rep}_{H_t^*} \in \text{Cat}_p$  together with exact fully faithful functors

$$\text{Rep}_{K^*} \rightarrow \text{Rep}_{H_1^*} \rightarrow \dots \rightarrow \text{Rep}_{H_t^*} \rightarrow \text{Rep}_{H^*} \tag{6.42}$$

derived from the epimorphisms  $\tilde{i}_{kl}$ .

At the second step, we show the existence of a lattice of Tannakian subcategories for the description of the entanglement between space-time far distant elementary particles which belong to an entangled system.

Thanks to Theorem 6.2.5, the information flow between  $p$  and other distant particle  $q$  is described by the lattice  $(\mathcal{C}_{pq}^{j_1^* j_2^*}, \preceq)$  of topological Hopf subalgebras which inherits a lattice  $(\mathcal{G}_{pq}^{j_1^* j_2^*}, \succeq)$  of Lie subgroups (i.e. Theorem 6.2.6). The decreasing chains (6.31) and (6.32) can be lifted onto the categorical setting to achieve the following chains of categories and exact fully faithful functors

$$\text{Rep}_{G_{\text{DSE}_p}^*} \geq \text{Rep}_{G_{p(1)}^*} \geq \dots \geq \text{Rep}_{G_{p(j_1^*-1)}^*} \geq \text{Rep}_{G_{p(j_1^*)}^*} \tag{6.43}$$

$$\text{Rep}_{G_{\text{DSE}_q}^*} \geq \text{Rep}_{G_{q(1)}^*} \geq \dots \geq \text{Rep}_{G_{q(j_2^*-1)}^*} \geq \text{Rep}_{G_{q(j_2^*)}^*}. \tag{6.44}$$

We can connect these two chains to each other in terms of one of the following sequences of exact fully faithful functors

$$\text{Rep}_{G_{q(j_2^*)}^*} \longrightarrow \text{Rep}_{G_{q(\epsilon)}^*} \longrightarrow \text{Rep}_{G_{c(k)}^*} \longrightarrow \text{Rep}_{G_{p(\epsilon)}^*} \longrightarrow \text{Rep}_{G_{p(j_1^*)}^*} \tag{6.45}$$

or

$$\text{Rep}_{G_{p(j_1^*)}^*} \longrightarrow \text{Rep}_{G_{p(\epsilon)}^*} \longrightarrow \text{Rep}_{G_{c(k)}^*} \longrightarrow \text{Rep}_{G_{q(\epsilon)}^*} \longrightarrow \text{Rep}_{G_{q(j_2^*)}^*}. \tag{6.46}$$

This information is enough to define  $(\text{Cat}_{c_{pq}^{j_1^* j_2^*}}, \succeq)$  as a lattice of Tannakian subcategories which encodes the entanglement process between  $p$  and  $q$ .  $\square$

**Theorem 6.2.8.** *Flat equi-singular vector bundles provide a new geometric description for the information flow in interaction Quantum Field Theories on the basis of the Riemann–Hilbert correspondence.*

*Proof.* Flat equi-singular vector bundles have been applied for the construction of the Connes–Marcolli category  $\mathcal{E}^{\text{CM}}$  which is a neutral Tannakian category. It is isomorphic to the category  $\text{Rep}_{\mathbb{U}^*}$  of finite dimensional representations of the universal affine group scheme  $\mathbb{U}^*$ . [46]

This universal category is rich enough to recover all categories  $\text{Rep}_{G_p^*(j)}$  as subcategories which enable us to define a surjective functor

$$\pi_j^* : \text{Rep}_{\mathbb{U}^*} \longrightarrow \text{Rep}_{G_p^*(j)} \quad (6.47)$$

of categories. Now if we apply the chain (6.45) or (6.46), then we can determine one of the functors

$$\rho_{j_1^* j_2^*}^{pq} : \text{Rep}_{G_p^*(j_1^*)} \longrightarrow \text{Rep}_{G_q^*(j_2^*)} \quad (6.48)$$

or

$$\rho_{j_2^* j_1^*}^{pq} : \text{Rep}_{G_q^*(j_2^*)} \longrightarrow \text{Rep}_{G_p^*(j_1^*)}. \quad (6.49)$$

These functors allow us to formulate one of the following equations

$$\pi_{j_2^*} = \rho_{j_1^* j_2^*}^{pq} \circ \pi_{j_1^*}, \quad \text{or} \quad \pi_{j_1^*} = \rho_{j_2^* j_1^*}^{pq} \circ \pi_{j_2^*} \quad (6.50)$$

at the level of functors. They are the key tools for us to determine some flat equi-singular vector bundles which contribute to the information flow in entangled systems.  $\square$

**Corollary 6.2.9.** *There exists a category of mixed Tate motives which interprets the information flow in an entangled system of particles in a given interacting Quantum Field Theory.*

*Proof.* On the one hand, the category  $\mathcal{E}^{\text{CM}}$  is equivalent to the motivic category

$$\mathcal{T}\mathcal{M}_{\text{mix}}(\text{Spec } \mathcal{O}[1/N]) \quad (6.51)$$

of mixed Tate motives (i.e. Proposition 1.110, Corollary 1.111 in [46]). Thanks to Theorem 6.2.8, neutral Tannakian subcategories  $\text{Rep}_{G_p^*(j_1^*)}$  and  $\text{Rep}_{G_q^*(j_2^*)}$  can be embedded into this motivic category. Therefore the information flow is equivalent to determining subcategories of the category  $\mathcal{T}\mathcal{M}_{\text{mix}}(\text{Spec } \mathcal{O}[1/N])$  which contain those mixed Tate motives identified by motivic Galois groups  $G_p^*(j_1^*)$  and  $G_q^*(j_2^*)$ . We denote the resulting motivic subcategories with  $\text{Mot}(G_p^*(j_1^*))$  and  $\text{Mot}(G_q^*(j_2^*))$ , respectively.

On the other hand, thanks to Theorem 6.2.6, we have already determined a new class of Dyson–Schwinger equations  $\text{DSE}_c^{(k)}$  which describes the information flow in the topological region  $\overline{R^{c,pq}}$ . Now by applying Theorem 6.2.7, we have the category  $\text{Rep}_{G_c^*(k)}$  with respect to this class of Dyson–Schwinger equations which can encode the information flow. This category can be also embedded into  $\mathcal{E}^{\text{CM}}$  which leads us to characterize another class of mixed Tate motives identified with the motivic Galois group  $G_c^*(k)$ . We denote the resulting motivic subcategory with  $\text{Mot}(G_c^*(k))$ .

The disjoint union of objects of these motivic subcategories make a new subcategory

$$\text{Mot}_{pq} := \text{Mot}(G_{p^{(j_1^*)}}^*) \bigsqcup \text{Mot}(G_{c^{(k)}}^*) \bigsqcup \text{Mot}(G_{q^{(j_2^*)}}^*) \quad (6.52)$$

of  $\mathcal{TM}_{\text{mix}}(\text{Spec } \mathcal{O}[1/N])$ .  $\text{Mot}_{pq}$  is a category of mixed Tate motives which contribute to the entanglement processes between space-time far distant particles  $p, q$  via virtual particles of the vacuum energy.  $\square$

The explained mathematical machinery for the description of quantum entanglement is related to the strength of the bare or running coupling constants of physical theories where we deal with Dyson–Schwinger equations under different running couplings. It means that our machinery still works after changing the scale of the momenta (i.e. running coupling constant). As we know that theory of Renormalization Group is the key tool in Quantum Field Theory to study the changes of the dynamics of a quantum system with infinite degrees of freedom when the scales of some physical parameters such as momentum, energy and mass have been changed. It allows us to concern the possibility of exchanging information from scale to scale in the appearance of uncertainty principle. Now we can apply the Connes–Marcolli universal affine group scheme to define a suitable Renormalization Group which encodes the information flow under the rescaling of the momentum parameter.

**Corollary 6.2.10.** *The information flow between an elementary particle  $p$  and all unobserved intermediate states exists independent of changing the scale of the momenta of particles.*

*Proof.* The universal category  $\mathcal{E}^{\text{CM}}$  is isomorphic to the category  $\text{Rep}_{\mathbb{U}^*}$  with respect to the universal affine group scheme  $\mathbb{U}^*$ . The Connes–Marcolli universal shuffle type Hopf algebra of renormalization  $H_{\mathbb{U}}$  is the result of the graded dual of the universal enveloping algebra of the free graded Lie algebra  $L_{\mathbb{U}}$  which is generated by elements  $e_{-n}$  of degree  $-n$  for each  $n > 0$ . The Milnor–Moore Theorem can determine the corresponding affine group scheme  $\mathbb{U}$ . The sum  $e := \sum_n e_{-n}$  is an element of the Lie algebra  $L_{\mathbb{U}}$  where thanks to the pro-unipotent structure of  $\mathbb{U}$ , we can lift it onto the morphism  $\text{rg} : \mathbb{G}_a \rightarrow \mathbb{U}$ . [46]

Now we can apply Theorem 1.106 in [46] to determine a graded representation

$$\tau_p : \mathbb{U}^*(\mathbb{C}) \rightarrow G_{\text{DSE}_p}^* \quad (6.53)$$

such that the resulting map  $\tau_p \circ \text{rg}$  provides the Renormalization Group with respect to the equation  $\text{DSE}_p$ . By this method, we can build a Renormalization Group with respect to each Dyson–Schwinger equation  $\text{DSE}_p^{(j)}$  for each  $j \geq 1$ . These Renormalization Groups allow us to study the behavior of the information flow in the topological regions such as  $R^p$  under changing the scales of the momentum parameter.  $\square$

### 6.3 *Quantum logic via non-perturbative propositional calculus*

The fundamental purpose in this section is to address a new category model for the study of logical propositions about solutions of (strongly coupled) Dyson–Schwinger equations. We use cut-distance topological regions of Feynman diagrams to build a new topos model which can encode the role of the strength of coupling constants in the logical evaluation about non-perturbative aspects of physical theories. This new topos model can lead us to clarify a new class of computable Heyting algebras for the logical study of gauge field theories.

#### 6.3.1 *Quantum Topos*

Generally speaking, a quantum system is described by its von Neumann algebra  $\mathcal{B}(H)$  of observables which contains all bounded operators on an infinite dimensional separable Hilbert space  $H$ . Each physical quantity  $A$  has a corresponding self-adjoint operator  $\hat{A}$  in  $\mathcal{B}(H)$  and vice versa. Set  $\mathcal{V}(H)$  as the poset of all unital abelian von Neumann subalgebras of  $\mathcal{B}(H)$  which can be seen also as the context category. For objects  $V_1 \subset V_2$  in the context category, the subalgebra  $V_1$  has less number of self-adjoint operators and less number of projections than the subalgebra  $V_2$ . The restriction process from the subalgebra  $V_2$  to the subalgebra  $V_1$  or the lifting process from  $V_1$  onto  $V_2$  are fundamental issues in propositional calculus of quantum systems. These translation issues have been studied under coarse-graining process. On the one hand, it enables us to map self-adjoint operators and projections from  $V_2$  to  $V_1$ . For a proposition  $A \in \Delta$  about a given physical quantity  $A$ , suppose its corresponding projection  $\hat{P}_A^\Delta$  belongs to  $V_2$  but not belong to  $V_1$ . It means that this proposition can not be evaluated from the perspective of  $V_1$ . The daseinisation process has been designed to adapt the projection  $\hat{P}_A^\Delta$  and the proposition  $A \in \Delta$  to  $V_1$  by making them coarser. On the other hand, every self-adjoint operator and every projection in  $V_1$  belong also to  $V_2$  but the embedding of the smaller subalgebra into the larger one requires some extra structures and objects which live in  $V_2$ . To concern this issue has led people to build a topos of contravariant functors from the context category  $\mathcal{V}(H)$  to the category **Set**. This categorical setting has been developed very fast for the reconstruction of physical theories in the context of higher order logic. [4, 20, 59, 60, 61, 62, 90, 91, 92, 122, 129, 141, 175]

The original motivation for the construction of a new topos model is to provide a new analogous of this propositional calculus for the study of situations beyond perturbation theory in Quantum Field Theories with strong couplings in the context of logical conceptions. Our new topos model shows the importance of the strength of the (bare) couplings in the construction of the category of context (i.e. base category). The context category of our

new topos is built in terms of cut-distance topological Hopf subalgebras of the enriched renormalization Hopf algebra  $H_{FG}^{\text{cut}}(\Phi)$ . Its complexity is more than the complexity of the standard context category  $\mathcal{V}(H)$ . The inclusion  $H_{DSE_1}^{\text{cut}} \subset H_{DSE_2}^{\text{cut}}$  in our context category does not mean in general that the equation  $DSE_1$  should have less physical information than the equation  $DSE_2$ . We can remind the calculus of ordinals in Set Theory, where we deal with different types of infinities while sometimes a subset of a set and the set can have the same cardinal. Therefore coarse-graining process is not noticed in the foundations of our topos model and we need to concern other parameters to deal with propositions at the level of large Feynman diagrams.

Let us give a short overview about the concept of topos. The fundamental motivation for the study of topos came from the concept of abstraction in Mathematics. In fact, Category Theory, as a modern discipline, comes to the game whenever we plan to study a general theory of structures. Categories enable us to concern mathematical structures in terms of interrelationships among objects (which are formally known as morphisms) while under a set theoretic perspective, we choose to deal with properties of mathematical structures on the basis of elements and membership relations. Many basic concepts such as spaces and elements in Set Theory can be replaced by objects and arrows in the categorical setting, respectively. It is reasonable to think about Category Theory as a generalization of Set Theory where we are capable to study a mathematical structure in terms of its relations with other structures. This approach leads us to a universal fundamental language in dealing with mathematical structures where we have general powerful tools such as functors between categories and natural transformations between functors instead of the equality relation between elements of sets in Set Theory. Actually, the language of Category Theory provides a new understanding of the notion of "element" of an object in a mathematical structure which is more general than its set theoretic version. Each arrow is indeed a generalized element of its own codomain which means that each object  $X$  can be described in terms of consisting of different collections of arrows  $Y \rightarrow X$ . This interpretation is known as the varying of elements of  $X$  over the stages  $Y$  which corresponds to the notion of absolute element  $x$  of a set  $X$  in Set Theory. This story is encapsulated in terms of a map  $x : \{*\} \rightarrow X$  where we enable to address the terminal object underlying the categorical setting. Questions about the existence of a class of categories which could be regarded as a categorical-theoretic replacement and generalization of the category of sets and functions have led people to build elementary topos and Grothendieck topos such that the second class is known as a replacement for the notion of "space". The concept of topos has all tools of the set-theoretical world which are necessary for the construction of mathematical structures and their models under a categorical configuration. It provides a generalized version of the notions of space and logic where we enable to interpret it as a categorical-theoretic generalization

of the structure of a universe of sets and functions that disappears certain logical and geometric restrictions of the base mathematical structure. Generally speaking, for a given mathematical theory, we can have a treatment to evaluate and study the theory under different stages with respect to objects of a fixed base category. So there is a chance to consider possible relations among toposes in the context of a special family of functors which are called geometric morphisms. The category **Set** of all sets and functions is an example of a topos which is the basis for the construction of more complicated toposes such as the Grothendieck topos of sheaves over a given base category. Consider the category  $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$  of contravariant functors from the base category  $\mathcal{C}$  to the standard category **Set** of sets and functions. Elements of this mathematical theory corresponded to the base category  $\mathcal{C}$ , which have been already modeled as objects in the mentioned topos, become representable in terms of set-valued functors over the base category  $\mathcal{C}$ . Roughly speaking, a topos is a Cartesian closed category with equalisers and subobject classifier. In other words, a topos has terminal object, equalisers, pullbacks, all other limits, exponential objects and subobject classifier. [10, 95, 123, 124, 126, 152, 151, 177]

### 6.3.2 Non-perturbative Topos

In this part we build the context category of our new topos model. Then we show the existence of a new class of computable Heyting algebras which encode truth-values of propositions about non-perturbative aspects of gauge field theories.

**Definition 6.3.1.** Topological Hopf algebras  $H_{\text{DSE}(\lambda g)}^{\text{cut}}$  generated by solutions of Dyson–Schwinger equations under different running couplings in a given gauge field theory  $\Phi$  with the strong bare coupling constant  $g$  and their closed Hopf subalgebras can be organized into a poset structure. For each pair  $(H_1, H_2)$  of objects, we can define arrows pointing from  $H_1$  to  $H_2$  (i.e.  $H_1 \leq H_2$ ) if and only if there exists a homomorphism  $H_1 \rightarrow H_2$  of Hopf algebras which is continuous with respect to the cut-distance topology.

This poset can be seen as a category denoted by  $\mathcal{C}_{\Phi}^{\text{non},g}$ .

**Lemma 6.3.2.**  $\mathcal{C}_{\Phi}^{\text{non},g}$  is a small category.

*Proof.* The existence of the graduation parameters on the renormalization Hopf algebra  $H_{\text{FG}}(\Phi)$  allow us to represent it in terms of an infinite system of Dyson–Schwinger equations generated by fixed point equations of vertex-type and edge-type Green’s functions of the physical theory under different running couplings. Therefore  $H_{\text{FG}}(\Phi)$  can be seen as an object in the category  $\mathcal{C}_{\Phi}^{\text{non},g}$  which makes this category as a small category.  $\square$

The small category  $\mathcal{C}_{\Phi}^{\text{non},g}$  is useful to determine cut-distance topological neighborhoods of Feynman diagrams and expansions around a fixed Feynman diagram. These topological regions are Hausdorff and therefore they allow us to separate not weakly isomorphic Feynman diagrams from each other. It gives us an advantage to study the unique solution of a given Dyson–Schwinger equation in terms of Feynman diagrams which contribute to arbitrary small neighborhoods around the corresponding large Feynman diagram.

**Lemma 6.3.3.** *There exists a topos structure on the small category  $\mathcal{C}_{\Phi}^{\text{non},g}$ .*

*Proof.* The natural choice is the topos of presheaves on the category  $\mathcal{C}_{\Phi}^{\text{non},g}$  (as the category of context). We denote this new category by  $\mathbf{T}_{\Phi}^{\text{non},g}$  and call it a non-perturbative topos.

An object in  $\mathbf{T}_{\Phi}^{\text{non},g}$  is a contravariant functor from the category  $\mathcal{C}_{\Phi}^{\text{non},g}$  to the standard category **Set** of sets and functions.

A morphism between a pair  $(F_1, F_2)$  of objects is a natural transformation such as  $\eta : F_1 \rightarrow F_2$  which is actually a family of morphisms  $\{\eta_H : F_1(H) \rightarrow F_2(H)\}_{H \in \text{Obj}(\mathcal{C}_{\Phi}^{\text{non},g})}$  with respect to the contravariant property. It means that for each morphism  $f : H_1 \rightarrow H_2$  in  $\mathcal{C}_{\Phi}^{\text{non},g}$ , we have  $\eta_{H_1} \circ F_1(f) = F_2(f) \circ \eta_{H_2}$ .

A sieve on an object  $H \in \mathcal{C}_{\Phi}^{\text{non},g}$  is defined as a collection  $S$  of morphisms  $f : H \rightarrow H'$  in  $\mathcal{C}_{\Phi}^{\text{non},g}$  with the property that if  $f$  belongs to  $S$  and if  $g : H' \rightarrow H''$  is any other morphism in  $\mathcal{C}_{\Phi}^{\text{non},g}$ , then  $g \circ f : H \rightarrow H''$  also belongs to  $S$ .

The terminal object  $1 : \mathcal{C}_{\Phi}^{\text{non},g} \rightarrow \mathbf{Set}$  can be defined by  $1(H) := \{*\}$  at all stages  $H$  in  $\mathcal{C}_{\Phi}^{\text{non},g}$ . If  $f : H \rightarrow H'$  is a morphism in  $\mathcal{C}_{\Phi}^{\text{non},g}$ , then  $1(f) : \{*\} \rightarrow \{*\}$ . It is the terminal object because for any other presheaf  $F$  we can define a unique natural transformation  $\eta : F \rightarrow 1$  such that its components  $\eta_H : F(H) \rightarrow 1(H) = \{*\}$  are the constant maps  $\Gamma \mapsto *$  for all  $\Gamma \in F(H)$ .

The subobject classifier  $\Omega^{\text{non}}$  is the presheaf  $\Omega^{\text{non}} : \mathcal{C}_{\Phi}^{\text{non},g} \rightarrow \mathbf{Set}$  such that for any object  $H \in \mathcal{C}_{\Phi}^{\text{non},g}$ ,  $\Omega^{\text{non}}(H)$  is identified by the set of all sieves on  $H$ . If  $f : H' \rightarrow H''$  is a morphism in  $\mathcal{C}_{\Phi}^{\text{non},g}$ , then  $\Omega^{\text{non}}(f) : \Omega^{\text{non}}(H'') \rightarrow \Omega^{\text{non}}(H')$  is given by

$$\Omega^{\text{non}}(f)(S) := \{h : H'' \rightarrow H''', \quad h \circ f \in S\} \quad (6.54)$$

for all sieves  $S$  which lives in  $\Omega^{\text{non}}(H)$ .

The outer presheaf is the contravariant functor  $\underline{Q}$  which maps each Hopf subalgebra  $H_{\text{DSE}}$  in the base category to the set  $\text{In}(\text{DSE})$  of all infinitesimal characters corresponding to Feynman diagrams in  $H_{\text{DSE}}$ .

The spectral presheaf is the contravariant functor  $\underline{\Sigma}$  which sends each Hopf subalgebra  $H_{\text{DSE}}$  in the base category to its corresponding complex Lie group of characters.  $\square$

More details about the basic structure of the non-perturbative topos has been studied by author in other separate research work.

Heyting algebras are practical models of the intuitionistic logic where we do not have the law of excluded middle. It means that the proposition  $\phi \vee \neg \phi$  is not intuitionistically valid. The importance of this class of logical algebras in physics have been clarified when people started to describe quantum logics in terms of topos models. [10, 86, 90, 91, 92, 122]

**Definition 6.3.4.** A Heyting algebra  $A$  is a bounded distributive lattice with the largest element 1 and the smallest element 0 which obeys this condition that for each couple  $(a, b)$  of its elements there exists a greatest element  $x \in A$  such that  $a \wedge x \leq b$ . This particular element is called the relative pseudo-complement of  $a$  with respect to  $b$ .  $A$  is called a complete Heyting algebra, if it is a complete lattice.

**Theorem 6.3.5.** *The topos  $\mathbf{T}_{\Phi}^{\text{non},g}$  can encode the logical evaluation of propositions about topological regions of Feynman diagrams.*

*Proof.* Truth objects corresponding to cut-distance topological regions of Feynman diagrams can be determined by the Heyting algebraic structure defined naturally on the subobject classifier of the topos  $\mathbf{T}_{\Phi}^{\text{non},g}$ .

For a given topological Hopf algebra  $H_{\text{DSE}}^{\text{cut}}$ , consider the space  $\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$  which contains all sieves on  $H_{\text{DSE}}^{\text{cut}}$ . Now for arbitrary sieves  $S_1, S_2$  on  $H_{\text{DSE}}^{\text{cut}}$  which live in  $\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$ , the partial order relation on  $\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$  can be determined naturally by the relation

$$S_1 \leq S_2 \Leftrightarrow S_1 \subset S_2. \quad (6.55)$$

It leads us to make the following elementary logical statements

$$S_1 \wedge S_2 := S_1 \cap S_2, \quad S_1 \vee S_2 := S_1 \cup S_2,$$

$$S_1 \Rightarrow S_2 :=$$

$$\{f : H_{\text{DSE},1}^{\text{cut}} \rightarrow H_{\text{DSE},2}^{\text{cut}} \text{ s.t. } \forall g : H_{\text{DSE},2}^{\text{cut}} \rightarrow H_{\text{DSE},3}^{\text{cut}}, g \circ f \in S_1 \Rightarrow g \circ f \in S_2\}. \quad (6.56)$$

The negation of an element  $S$  is defined by the proposition  $\neg S := S \Rightarrow 0$  which means that

$$\neg S := \{f : H_{\text{DSE},1}^{\text{cut}} \rightarrow H_{\text{DSE},2}^{\text{cut}} \text{ s.t. } \forall g : H_{\text{DSE},2}^{\text{cut}} \rightarrow H_{\text{DSE},3}^{\text{cut}}, g \circ f \notin S\}. \quad (6.57)$$

Thanks to the defined partial order (6.55), for  $S_1, S_2 \in \Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$ , there exists a proposition  $S_1 \Rightarrow S_2$  of  $\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$  with the property that for all  $S \in \Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$ ,

$$S \leq (S_1 \Rightarrow S_2) \Leftrightarrow S \wedge S_1 \leq S_2. \quad (6.58)$$

In addition, the unit element in  $\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$  is the principal sieve on  $H_{\text{DSE}}^{\text{cut}}$  and the null element is the empty sieve  $\emptyset$ .

The presheaf  $\Omega^{\text{non}}$  as the subobject classifier shows that subobjects of any object  $F$  in the topos  $\mathbf{T}_{\Phi}^{\text{non},g}$  are in an one to one correspondence with morphisms such as  $\chi : F \rightarrow \Omega^{\text{non}}$ . In other words, for a subobject  $K$  of  $F$ , its associated characteristic morphism  $\chi^K$  is determined by its components  $\chi_{H_{\text{DSE}}^{\text{cut}}}^K : F(H_{\text{DSE}}^{\text{cut}}) \rightarrow \Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$  where

$$\chi_{H_{\text{DSE}}^{\text{cut}}}^K(A) := \{f : H_{\text{DSE}}^{\text{cut}} \rightarrow H_{\text{DSE},1}^{\text{cut}} : F(f)(A) \in K(H_{\text{DSE},1}^{\text{cut}})\}, \quad (6.59)$$

for all  $A \in F(H_{\text{DSE}}^{\text{cut}})$ , is actually a sieve on  $H_{\text{DSE}}^{\text{cut}}$ . Furthermore, each morphism  $\chi : F \rightarrow \Omega^{\text{non}}$ , which is a natural transformation between presheaves, defines a subobject  $K^\chi$  of  $F$  which is given by

$$K^\chi(H_{\text{DSE}}^{\text{cut}}) := \chi_{H_{\text{DSE}}^{\text{cut}}}^{-1} \{1_{\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})}\}. \quad (6.60)$$

As the conclusion, for each equation DSE,  $(\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}}), \leq, \wedge, \vee, \rightarrow)$  is our promising Heyting algebra.  $\square$

A Heyting algebra is called finitely free, if it is generated by the equivalence classes of formulas of finite number of propositional variables under provable equivalence in the intuitionistic logic.

A subset  $A$  of natural numbers is called computable if there exists an algorithm to decide whether a natural number belongs to  $A$  or not. In other words,  $A$  is computable if its corresponding characteristic function is computable. An algebraic structure is called computable if its domain can be identified with a computable set of natural numbers where the (finitely many) operations and relations on the structure are computable. If the structure is infinite, people usually identify the cardinal of its domain with the symbol  $\omega$ . The computable dimension of a computable structure is the number of classically isomorphic computable copies of the structure up to the computable isomorphism. [175]

**Definition 6.3.6.** A Heyting algebra  $(H, \leq, \wedge, \vee, \rightarrow)$  is called computable, if  $H$  and all its corresponding operations are computable.

For a given Heyting algebra with one generator, there exist infinitely nonequivalent intuitionistic formulas of one propositional variable. The connection between free Heyting algebras and the intuitionistic logic leads us to the concept of "computable dimension" for Heyting algebras in particular the ones which can encode the logics of the non-perturbative toposes  $\mathbf{T}_{\Phi}^{\text{non},g}$ .

**Theorem 6.3.7.** *There exists a computable Heyting algebra which encodes truth objects associated to topological regions of Feynman diagrams which contribute to the unique solution of the Dyson–Schwinger equation DSE in a given gauge field theory  $\Phi$ .*

*Proof.* We work on the non-perturbative topos  $\mathbf{T}_\Phi^{\text{non},g}$ . Thanks to Theorem 6.3.5, we can associate the Heyting algebra  $\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$  to each combinatorial Dyson–Schwinger equation DSE. Therefore the subobject classifier  $\Omega^{\text{non}}$  has the internal structure of our interesting Heyting algebra (as the algebraic structure appropriate for the intuitionistic logic). We want to lift this logical type of algebra onto an enriched version  $\hat{\Omega}^{\text{non}}$  which is computable at the level of dimension.

Consider the propositional intuitionistic logic over the given language  $(\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}}), \wedge, \vee, \rightarrow, \perp, \top)$  such that  $\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$  can be seen as the collection of propositional formulas in infinitely many variables modulo equivalence under the intuitionistic logic where  $\wedge, \vee, \rightarrow$  are the logical connectives,  $\perp$  is false and  $\top$  is truth. The codes for formulas such as  $\phi \wedge \psi$ ,  $\phi \vee \psi$  or  $\phi \rightarrow \psi$  are always greater than the codes for  $\phi$  and  $\psi$ .

The intuitionistic propositional logic is decidable ([37, 180]) which means that we need a finite constructive process to apply uniformly to every propositional formula to understand either it produces an intuitionistic proof of the formula or it shows no such proof can exist. Therefore we have a computable copy of the free Heyting algebra on  $\omega$  generators. Now we can consider elements of  $\Omega^{\text{non}}(H_{\text{DSE}}^{\text{cut}})$  as the equivalence classes  $[\phi]$  under provable equivalence in the intuitionistic logic which leads us to the following computational operations.

$[\phi] \leq [\psi] \Leftrightarrow \phi \rightarrow \psi$  is provable under the intuitionistic logic,

$$[\phi] \wedge [\psi] = [\phi \wedge \psi], \quad [\phi] \vee [\psi] = [\phi \vee \psi]. \quad (6.61)$$

The plan is to build  $\hat{\Omega}^{\text{non}}$  as a computable copy which is not computability isomorphic to  $\Omega^{\text{non}}$ . Let  $\alpha_s(n)$  be a label at stage  $s$  determined by the domain of  $\hat{\Omega}^{\text{non}}$  in the construction process. It is a propositional formula in the intuitionistic logic such that

- $\alpha(n) = \lim_s \alpha_s(n)$ ,
- For  $n \neq m$ , the propositional formulas  $\alpha(n)$  and  $\alpha(m)$  are not intuitionistically equivalent,
- For each intuitionistic propositional formula  $\phi$ , there exists such  $n$  such that  $\alpha(n)$  is intuitionistically equivalent to  $\phi$ ,
- Morphisms with the general form  $\phi_e : \hat{\Omega}^{\text{non}} \rightarrow \Omega^{\text{non}}$  can be applied to deal with the diagonalization against all possible computable isomorphisms ([180]).

Once we define the join, meet or relative pseudo-complement of elements, these relationships never change in future stages. Therefore, the function  $\alpha$ , which indicates an isomorphism map between  $\hat{\Omega}^{\text{non}}$  and  $\Omega^{\text{non}}$ , makes  $\hat{\Omega}^{\text{non}}$  computable.  $\square$

## Chapter 7

# Conclusion

In this part we summarize the original achievements of this monograph which focused on the mathematical foundations of non-perturbative gauge field theories in the context of combinatorial, geometric–analytic and categorical settings.

**(A)** We have studied solutions of Dyson–Schwinger equations in terms of an infinite combinatorial setting. We applied the theory of analytic graphs (for sparse graphs) to formulate non-trivial graphon models of Feynman diagrams and their formal expansions. Then we encapsulated the renormalization of these graphons in the context of a new topological Hopf algebra  $\mathcal{S}_{\text{graphon}}^\Phi$  of Feynman graphons with respect to Feynman diagrams of a given (strongly) coupled gauge field theory  $\Phi$ . The topology of this Hopf algebra is the topology of graphons, its coproduct is derived from the Kreimer’s renormalization coproduct and its graduation parameter can be determined in terms of loop numbers of the corresponding Feynman diagrams. The compactness of the topology of graphons enables us to identify Feynman graph limits as the convergent limits of sequences of finite Feynman diagrams. In addition, solutions of fixed point equations of Green’s functions have also been studied as the cut-distance convergent limits of the sequences of their corresponding partial sums. We then applied these Feynman graphon models of Dyson–Schwinger equations to formulate a new enrichment of the BPHZ renormalization theory for infinite formal expansions of Feynman diagrams. Our study completes the foundations of a differential Galois theory (in terms of the Riemann–Hilbert problem) for the study of non-perturbative parameters derived from the renormalization of Dyson–Schwinger equations under strong running or bare coupling constants.

We have also studied Dyson–Schwinger equations in the language of some combinatorial polynomials. We built a new parametric representation for solutions of these non-perturbative equations in terms of Tutte polynomials and Kirchhoff–Symanzik polynomials. We then formulated a new multi-scale Renormalization Group on the collection  $\mathcal{S}^{\Phi,g}$  of all Dyson–Schwinger equations in a given gauge field theory in terms of changing simultaneously the scales of momenta (i.e. running) and bare coupling constants. This Renormalization Group is useful to study strongly coupled Dyson–Schwinger equations in terms of cut-distance convergent limits of sequences of large Feynman diagrams under weaker running couplings. Feynman graphon models and the topology of graphons are the essential tools for this fundamental result. In addition, this Renormalization Group has been applied to formulate a new concept of complexity for Dyson–Schwinger equations under different running coupling constants. We considered  $\mathcal{S}^{\Phi,g}$  as a new constructive world to formulate a new generalization of the Kolmogorov complexity for Dyson–Schwinger equations. We showed that our generalized non-perturbative BPHZ renormalization program can encode the Halting problem of partial recursive functions on  $\mathcal{S}^{\Phi,g}$ .

**(B)** We have built a new Noncommutative Geometry model for the study

of Dyson–Schwinger equations in terms of a new class of infinite dimensional spectral triples. These spectral triples are derived from graded Hopf subalgebras generated by solutions of Dyson–Schwinger equations. The strength of running couplings has a direct influence on the structures of these spectral triples. This is a fundamental achievement for the description of the geometry of non-perturbative quantum motions via noncommutative differential geometry and the theory of Spectral Geometry.

Thanks to Feynman graphon models, we have explained the foundations of a new Functional Analysis approach for the study of solutions of Dyson–Schwinger equations in terms of the Haar integration theory on  $\mathcal{S}^{\Phi,g}$  as a modification of the classical Riemann–Lebesgue integration theory with respect to the Borel  $\sigma$ -algebra on real numbers. As we have shown this new approach is useful for the analytic description of evolution of large Feynman diagrams in terms of sequences of their corresponding partial sums where we have formulated a new generalization of . Johnson–Lapidus Dyson series for strongly coupled Dyson–Schwinger equations.

We have also worked on the Banach algebra  $L^1(\mathcal{S}^{\Phi,g}, \mu_{\text{Haar}})$  to formulate a generalized version of the Fourier transformation on the basis of the Gelfand transform. It is useful to encode the evolution of Dyson–Schwinger equations in terms of  $\mu_{\text{Haar}}$ -integrable functions on  $\mathcal{S}^{\Phi,g}$ .

In addition, we explained the basic elements of the Gâteaux differential calculus machinery on the separable Banach space  $\mathcal{S}^{\Phi,g}$  with respect to the cut-norm to study smooth functions on  $\mathcal{S}^{\Phi,g}$  in the language of Taylor series of higher order Gâteaux differentiations and homomorphism densities.

**(C)** Thanks to Feynman graphon models of Dyson–Schwinger equations, we have obtained a new mathematical model for the description of quantum entanglement in interacting (strongly coupled) gauge field theories in the language of intermediate algorithms organized in some new lattice models. These intermediate algorithms have been encoded via lattices of topological Hopf subalgebras generated by solutions of Dyson–Schwinger equations. We then lifted these lattices on to the level of lattices of Lie subgroups and Tannakian subcategories. Our study provides a new bridge between information flow in Quantum Field Theory and the Theory of Computation and Complexity.

In addition, we have organized topological Hopf subalgebras derived from solutions of Dyson–Schwinger equations in a given gauge field theory into a new topos. This new topos can provide the logical foundations of gauge field theories under strongly coupled running couplings. This new topos model has been developed by Author in his recent research works. We have shown the existence of a new class of computable Heyting algebras which encode the evaluation of logical propositions about topological regions of Feynman diagrams.

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