arXiv:1811.08405v1 [math-ph] 20 Nov 2018

LPENSL-TH-11-18

Complete spectrum of quantum integrable lattice models associated to $\mathcal{U}_q(\widehat{gl_n})$ by separation of variables

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Abstract. In this paper we apply our new separation of variables approach to completely characterize the transfer matrix spectrum for quantum integrable lattice models associated to fundamental evaluation representations of $\mathcal{U}_q(\widehat{gl_n})$ with general quasi-periodic boundary conditions. We consider here the case of generic deformations associated to a parameter q which is not a root of unity. The Separation of Variables (SoV) basis for the transfer matrix spectral problem is generated by using the action of the transfer matrix itself on a generic co-vector of the Hilbert space, following the general procedure described in our paper [1]. Such a SoV construction allows to prove that for general values of the parameters defining the model the transfer matrix is diagonalizable and with simple spectrum for any twist matrix which is also diagonalizable with simple spectrum. Then, using together the knowledge of such a SoV basis and of the fusion relations satisfied by the hierarchy of transfer matrices, we derive a complete characterization of the transfer matrix eigenvalues and eigenvectors as solutions of a system of polynomial equations of order $n + 1$. Moreover, we show that such a SoV discrete spectrum characterization is equivalently reformulated in terms of a finite difference functional equation, the quantum spectral curve equation, under a proper choice of the set of its solutions. A construction of the associated Q-operator induced by our SoV approach is also presented.

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1 Introduction

In this paper we make use of our new approach to generate the SoV basis [1–3] to characterize the complete spectrum of quantum integrable lattice models with general quasi-periodic boundary conditions associated to the fundamental evaluation representations of higher rank trigonometric Yang-Baxter algebras. More in detail, in the framework of the quantum inverse scattering method [4–12], these models are constructed by using the rank n principal gradation R-matrix [13, 14] solution of the Yang-Baxter equation associated to the quantum group $\mathcal{U}_q(\widehat{gl_{n+1}})$ [15–19]. This R-matrix admits a nontrivial set of scalar solutions to the Yang-Baxter equation. Such symmetries of the R-matrix, here called twist K-matrices, allow for the definition of integrable quasi-periodic boundary conditions for the corresponding integrable quantum models. Our SoV basis is generated for these general twists, under the assumption that the twist matrix K has simple spectrum. As previously shown in our work [1], the existence of such a SoV basis implies the simplicity of the transfer matrix spectrum. Moreover if the twist matrix K is diagonalizable with simple spectrum it implies that the transfer matrix is also diagonalizable with simple spectrum for almost any choice of the parameters of the representations.

The transfer matrix spectrum of these integrable quantum models has been analyzed also by other approaches, in particular, for diagonal twists, in the framework of the fusion relations [20, 21] and analytic Bethe ansatz [22–26], the nested Bethe ansatz [21, 27–29], with also first interesting results toward the computation of correlations functions [30–39]. Let us also note that for anti-periodic boundary conditions an eigenvalue analysis by a functional approach has been developed in [40].

The quantum version of the separation of variables method has been pioneered by Sklyanin in a series of works [41–46] in particular to tackle models for which the standard algebraic Bethe ansatz cannot be applied. Since the Sklyanin's original papers this approach has been successfully implemented and partially generalized to several classes of integrable quantum models mainly associated to different representations of quantum algebras related of rank one type, e.g. for the 6-vertex and 8-vertex Yang-Baxter algebras and reflection algebras as well as to their dynamical deformations [41–80]. The interest in developing the separation of variables method is mainly due to some important built-in features as the ability to provide a direct proof of the completeness of the spectrum characterization as well as some first elements towards the dynamics like scalar products and form factors.

Important analysis toward the SoV description of higher rank cases have been presented in [46, 49, 78, 81, 82]. Here, we solve the long-standing problem to systematically introduce a quantum separation of variable approach capable to completely characterize the transfer matrix spectrum associated to the higher rank representations of the trigonometric Yang-Baxter algebra. While our approach bypass the construction of the so-called Sklyanin's commuting B-operator family [41–46], the results on the rank one representations as well as some evidence from the short lattices for the higher rank representations [1], plus some recent analysis developed in [83] for the rational higher rank situation, confirm that our SoV basis construction can nevertheless reproduce the Sklyanin's SoV basis (i.e. the B-operator eigenbasis). This is the case under some special choice of the generating covector, i.e. the covector from which our SoV basis is constructed by the action of a chosen set of commuting conserved charges. This type of connection for the trigonometric representations, in particular, in relation to the SoV results obtained in [49] would be interesting to study further.

However, as anticipated above, our strategy following [1], is instead to make a direct use of our

SoV basis construction to obtain the complete characterization of the spectrum (eigenvalues and eigenvectors) using in particular the hierarchy of fusion relations for the transfer matrices. Here we consider the transfer matrices associated to general twist matrices K diagonalizable and with simple spectrum for the trigonometric gl_n ($n \geq 2$) Yang-Baxter algebras in the fundamental evaluation representations. We will first obtain a complete characterization of the spectrum in terms of the set of solutions to a given system of N polynomial equations of degree $n+1$ in N unknowns, where N is the number of lattice sites. Second, we introduce the so-called quantum spectral curve functional equation and we provide the exact characterization of the set of its solutions which generates the complete transfer matrix spectrum associating to any eigenvalue solution exactly one nonzero eigenvector. These results allow also to point out, as already done for other quantum integrable models [1–3], how the SoV basis in our construction can be equivalently obtained by the action of the commuting family of Q-operator. This connection is important as it allows to bring in our SoV approach results of the Baxter's Q-operator method [84–100] ; of special interests are then the results presented in [101] for the higher rank case. In our approach we show that this Q-operator satisfies with the transfer matrices the quantum spectral curve equation and that it can be reconstructed, making use of our Sov basis, in terms of the monodromy matrix entries.

In order to make easier the reading of our results we have decided to present them first in the rank 2 case, namely for the fundamental representations generated by the principal gradation R-matrix associated to $\mathcal{U}_q(\widehat{gl_3})$. Then these results are extended to the generic higher rank cases associated to the fundamental evaluation representations of $\mathcal{U}_q(\widehat{gl_n})$.

This article is organized as follows. In section 2, we introduce the fundamental evaluation representation of the rank two trigonometric Yang-Baxter algebra, the corresponding quantum spectral invariants of the model and then we list some general analytic properties they satisfy. Then in subsection 2.2 we construct our SoV covector basis and we state the first consequences on the transfer matrix spectrum. The section 3 is then dedicated to the presentation of our results on the transfer matrix spectrum characterization. In subsection 3.1, we derive the SoV discrete characterization of the transfer matrix spectrum in terms of solutions to a system of polynomial equations of degree 3. In subsection 3.2, we give an equivalent characterization in terms of the solutions to a functional equation of third order type, the so-called quantum spectral curve equation. In subsection 3.3 we also show that our SoV characterization of the transfer matrix eigenvectors allows for their rewriting in an algebraic Bethe ansatz form. In section 4, we define the framework of the general higher rank n case by introducing the corresponding fundamental evaluation representations of the $\mathcal{U}_q(\widehat{gl_n})$ Yang-Baxter algebra, the associated quantum spectral invariants and some of their general analytic properties. In subsection 4.2 we construct our SoV covector basis for this general rank n case. The section 5 presents the complete transfer matrix spectrum characterization. We first derive the SoV discrete spectrum characterization in subsection 5.1 while in subsection 5.2 we show its equivalence to the quantum spectral curve, a functional equation of difference type of order $n+1$ for the Q-operator that we construct using the knowledge of our SoV basis. Some important technical proofs are gathered in two appendices. In appendix A, for the rank two case, we provide a proof of the complete characterization of the spectrum which is based on the explicit calculation of the transfer matrix action on our SoV covector basis. While this proof can be generalized to the general rank n along a similar path described in [2] for the rational case, we provide in appendix B a proof of the SoV discrete characterization of the transfer matrix spectrum which bypass the computation of the action of the transfer matrix on the SoV covector basis.

2 Transfer matrices for fundamental evaluation representations of $\mathcal{U}_q(\widehat{gl_3})$

We consider here the trigonometric Yang-Baxter algebra generated by the principal gradation R-matrix [13, 14] associated to the quantum group $\mathcal{U}_q(\widehat{gl_3})$[1]:

$$R_{a,b}^{(P)}(\lambda)=\left(\begin{array}{ccc} a_1(\lambda) & \lambda^{1/3}b_1 & \lambda^{-1/3}b_2 \\ \lambda^{-1/3}c_1 & a_2(\lambda) & \lambda^{1/3}b_3 \\ \lambda^{1/3}c_2 & \lambda^{-1/3}c_3 & a_3(\lambda)\end{array}\right)\in \mathrm{End}(V_a\otimes V_b), \tag{2.1}$$

where $V_a \cong V_b \cong \mathbb{C}^3$ and we have defined:

$$a_j(\lambda)=\left(\begin{array}{ccc} \lambda q^{\delta_{j,1}}-1/(\lambda q^{\delta_{j,1}}) & 0 & 0 \\ 0 & \lambda q^{\delta_{j,2}}-1/(\lambda q^{\delta_{j,2}}) & 0 \\ 0 & 0 & \lambda q^{\delta_{j,3}}-1/(\lambda q^{\delta_{j,3}})\end{array}\right), \quad \forall j\in\{1,2,3\},$$
$$b_1=\left(\begin{array}{ccc} 0 & 0 & 0 \\ q-1/q & 0 & 0 \\ 0 & 0 & 0\end{array}\right), \quad b_2=\left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q-1/q & 0 & 0\end{array}\right), \quad b_3=\left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q-1/q & 0\end{array}\right),$$
$$c_1=\left(\begin{array}{ccc} 0 & q-1/q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right), \quad c_2=\left(\begin{array}{ccc} 0 & 0 & q-1/q \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right), \quad c_3=\left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & q-1/q \\ 0 & 0 & 0\end{array}\right). \tag{2.2}$$

Note that here we have chosen to present the R-matrix in a Laurent polynomial form but clearly it can be rewritten as well in a trigonometric form. This R-matrix is a solution of the Yang-Baxter equation:

$$R_{12}(\lambda/\mu)R_{13}(\lambda)R_{23}(\mu)=R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda/\mu)\in \mathrm{End}(V_1\otimes V_2\otimes V_3), \tag{2.3}$$

and it is related by a similarity transformation to the so-called homogeneous gradation R-matrix for $U_q(\widehat{gl_3})$:

$$R_{a,b}^{(H)}(\lambda)=\left(\begin{array}{ccc} a_1(\lambda) & \lambda b_1 & \lambda b_2 \\ c_1/\lambda & a_2(\lambda) & \lambda b_3 \\ c_2/\lambda & c_3/\lambda & a_3(\lambda)\end{array}\right)\in \mathrm{End}(V_a\otimes V_b), \tag{2.4}$$

which is also a solution of the Yang-Baxter equation. More in detail, it holds:

$$R_{a,b}^{(P)}(\lambda)=S_a(\lambda)R_{a,b}^{(H)}(\lambda)S_a^{-1}(\lambda), \tag{2.5}$$

where

$$S(\lambda)=\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \lambda^{2/3} & 0 \\ 0 & 0 & \lambda^{4/3}\end{array}\right), \tag{2.6}$$

such connection has been first remarked in [14]. Let us comment that this two solutions of the Yang-Baxter equation generate the same quantum integrable models with diagonal quasi-periodic boundary conditions. Indeed, any diagonal 3×3 matrix $K\in$End(V) is a symmetry for both these R-matrices:

$$R_{12}^{(P/H)}(\lambda)K_1K_2=K_2K_1R_{12}^{(P/H)}(\lambda)\in \mathrm{End}(V_1\otimes V_2), \tag{2.7}$$

[1] As already stressed, in all this article we assume that the deformation parameter $q=e^{\eta}$ is not a root of unity

and defined the monodromy matrices:

$$M_a^{(P/H|K)}(\lambda|\{\xi\}) = \begin{pmatrix} A_1^{(K)}(\lambda) & B_1^{(K)}(\lambda) & B_2^{(K)}(\lambda) \\ C_1^{(K)}(\lambda) & A_2^{(K)}(\lambda) & B_3^{(K)}(\lambda) \\ C_2^{(K)}(\lambda) & C_3^{(K)}(\lambda) & A_3^{(K)}(\lambda) \end{pmatrix}_a \tag{2.8}$$

$$\equiv K_a R_{a,\mathsf{N}}^{(P/H)}(\lambda/\xi_{\mathsf{N}}) \cdots R_{a,1}^{(P/H)}(\lambda/\xi_1) \in \mathrm{End}(V_a \otimes \mathcal{H}), \tag{2.9}$$

where $\mathcal{H} = \bigotimes_{n=1}^{\mathsf{N}} V_n$, and the transfer matrices

$$T_1^{(P/H|K)}(\lambda|\{\xi\}) \equiv \mathrm{tr}_a M_a^{(P/H|K)}(\lambda|\{\xi\}) \in \mathrm{End}\mathcal{H} \tag{2.10}$$

then the following identity holds for the homogeneous chains:

$$T_1^{(P|K)}(\lambda|\{\xi\})\Big|_{\xi_i=0} = T_1^{(H|K)}(\lambda|\{\xi\})\Big|_{\xi_i=0}, \tag{2.11}$$

which implies our statement on the equality of the Hamiltonians with diagonal quasi-periodic boundary conditions generated by these two R-matrices. However, one has to remark that for the inhomogeneous models the above identity does not hold.

One motivation to consider the fundamental models associated with the principal gradation R-matrix is the set of symmetry K-matrices enjoyed by this Yang-Baxter solution together with the simpler properties under co-product action, and hence under fusion. Indeed, the set of solutions of the scalar Yang-Baxter equation for the principal gradation R-matrix reads:

$$K^{(a)} = \delta_{a,1} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} + \delta_{a,2} \begin{pmatrix} 0 & 0 & \alpha \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix} + \delta_{a,3} \begin{pmatrix} 0 & \beta & 0 \\ 0 & 0 & \gamma \\ \alpha & 0 & 0 \end{pmatrix}, \tag{2.12}$$

for any complex value of α, β and γ, while for the homogeneous gradation R-matrix the matrices $K^{(2)}$ and $K^{(3)}$ are symmetries if and only if $\alpha = 0$. Note that the matrices $K^{(2)}$ and $K^{(3)}$ are diagonalizable and their eigenvalues reads:

$$\mathsf{k}_0 = \sqrt[3]{\alpha\beta\gamma}, \mathsf{k}_1 = -(-1)^{1/3}\mathsf{k}_0, \mathsf{k}_2 = (-1)^{2/3}\mathsf{k}_0 \tag{2.13}$$

so that for $\alpha\beta\gamma \neq 0$ these matrices have simple spectrum and it holds:

$$\det K^{(a)} = \alpha\beta\gamma \quad \forall a \in \{1, 2, 3\}.$$

It is also interesting to remark that the principal gradation R-matrix directly generates, the q-deformed antisymmetric projector $P_{a,b}^{-} \in$End$(V_a \otimes V_b)$ which is used in the fusion representations for the $\mathcal{U}_q(\widehat{gl_3})$ case. In particular, it holds:

$$\frac{R_{12}^{(P)}(1/q)}{2(1/q-q)} = P_{a,b}^{-}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & -q^{1/3}/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & -1/(2q^{1/3}) & 0 & 0 \\ 0 & -1/(2q^{1/3}) & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & -q^{1/3}/2 & 0 \\ 0 & 0 & -q^{1/3}/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/(2q^{1/3}) & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.14}$$

2.1 Fundamental properties of the hierarchy of transfer matrices

Here, we collect some relevant known properties [20, 21, 26] of the fused transfer matrices for the representations under consideration.

Proposition 2.1. *The transfer matrices:*

$$T_1^{(K)}(\lambda) \equiv tr_a M_a^{(K)}(\lambda), \qquad T_2^{(K)}(\lambda) \equiv tr_{a,b} P_{a,b}^{-} M_b^{(K)}(\lambda) M_a^{(K)}(\lambda/q), \tag{2.15}$$

satisfy the following commutation relations:

$$\left[T_1^{(K)}(\lambda), T_1^{(K)}(\mu)\right] = \left[T_1^{(K)}(\lambda), T_2^{(K)}(\mu)\right] = \left[T_2^{(K)}(\lambda), T_2^{(K)}(\mu)\right] = 0. \tag{2.16}$$

The quantum determinant:

$$q\text{-}detM^{(K)}(\lambda) \equiv tr_{abc} P_{abc}^{-} M_c^{(K)}(\lambda) M_b^{(K)}(\lambda/q) M_a^{(K)}(\lambda/q^2) \tag{2.17}$$

is a central element of the algebra, i.e.

$$[q\text{-}detM^{(K)}(\lambda), M_a^{(K)}(\mu)] = 0. \tag{2.18}$$

Furthermore, let us define the operators $\mathsf{N}_i \in \mathrm{End}(\mathcal{H})$ by the following action:

$$\mathsf{N}_i \otimes_{n=1}^{\mathsf{N}} |n, a_n\rangle = \otimes_{n=1}^{\mathsf{N}} |n, a_n\rangle \sum_{n=1}^{\mathsf{N}} \delta_{i,a_n}, \ \forall \ i \in \{1,2,3\}, \tag{2.19}$$

in the basis

$$|n, a_n\rangle = \begin{pmatrix} \delta_{1,a_n} \\ \delta_{2,a_n} \\ \delta_{3,a_n} \end{pmatrix}_n \in V_n \quad \forall n \in \{1, ..., \mathsf{N}\} \text{ and } a_n \in \{1,2,3\}, \tag{2.20}$$

for which we have:

$$\mathsf{N}_1 + \mathsf{N}_2 + \mathsf{N}_3 = \mathsf{N}\ . \tag{2.21}$$

These operators generalize to higher rank the symmetry of the R-matrix given in the rank one case by the third component of spin. Then it holds:

Proposition 2.2. *The transfer matrices $T_1^{(K)}(\lambda)$ and $T_2^{(K)}(\lambda)$ satisfy the following properties:*

i) For any K-matrix defining a symmetry of the R-matrix, $T_2^{(K)}(\lambda)$ has the following $2\mathsf{N}$ central zeroes:

$$T_2^{(K)}(\pm q\xi_b)=0 \quad \forall b\in\{1,...,\mathsf{N}\}, \tag{2.22}$$

the quantum determinant reads:

$$q\text{-}detM^{(K)}(\lambda)=detK\prod_{b=1}^{\mathsf{N}}(\lambda q/\xi_b-\xi_b/(q\lambda))(\lambda/(q\xi_b)-(q\xi_b)/\lambda)(\lambda/(q^2\xi_b)-(q^2\xi_b)/\lambda), \tag{2.23}$$

and the following fusion identities hold for any $b\in\{1,...,\mathsf{N}\}$:

$$\begin{aligned} T_1^{(K)}(\xi_b)T_1^{(K)}(\xi_b/q) &= T_2^{(K)}(\xi_b), &(2.24)\\ T_1^{(K)}(\xi_b)T_2^{(K)}(\xi_b/q) &= q\text{-}detM^{(K)}(\xi_b). &(2.25)\end{aligned}$$

Moreover, in the case of a diagonal twist $K^{(1)}$:

ii) $\lambda^{\mathsf{N}}T_1^{(K)}(\lambda)$ is a degree N polynomial in λ^2 with the following asymptotics:

$$T_1^{(\pm\infty|K)}\equiv\lim_{\log\lambda\to\pm\infty}\lambda^{\mp\mathsf{N}}T_1^{(K)}(\lambda)=(\pm1)^{\mathsf{N}}\frac{\alpha q^{\pm\mathsf{N}_1}+\beta q^{\pm\mathsf{N}_2}+\gamma q^{\pm\mathsf{N}_3}}{\prod_{n=1}^{\mathsf{N}}\xi_n^{\pm1}}. \tag{2.26}$$

iii) $\lambda^{2\mathsf{N}}T_2^{(K)}(\lambda)$ is a degree $2\mathsf{N}$ polynomial in λ^2 with the following asymptotics:

$$T_2^{(\pm\infty|K)}\equiv\lim_{\log\lambda\to\pm\infty}\lambda^{\mp2\mathsf{N}}T_2^{(K)}(\lambda)=\frac{\alpha\beta q^{\pm(\mathsf{N}_1+\mathsf{N}_2)}+\alpha\gamma q^{\pm(\mathsf{N}_1+\mathsf{N}_3)}+\beta\gamma q^{\pm(\mathsf{N}_2+\mathsf{N}_3)}}{q^{\pm\mathsf{N}}\prod_{n=1}^{\mathsf{N}}\xi_n^{\pm2}}. \tag{2.27}$$

iv) The operators N_i commute with the transfer matrices.

In the case of non-diagonal twists $K^{(2)}$ or $K^{(3)}$:

v) $\lambda^{(\mathsf{N}-1/3)}T_1^{(K)}(\lambda)$ is a degree $\mathsf{N}-1$ polynomial in λ^2.

vi) $\lambda^{(2\mathsf{N}-2/3)}T_2^{(K)}(\lambda)$ is a degree $2\mathsf{N}-1$ polynomial in λ^2.

Let us introduce the functions

$$\begin{aligned} f_{l,\mathbf{h}}^{(a,m)}(\lambda) &= \left(\prod_{b=1}^{\mathsf{N}}\frac{\lambda/\xi_b^{(-1)}-\xi_b^{(-1)}/\lambda}{\xi_l^{(h_l)}/\xi_b^{(-1)}-\xi_b^{(-1)}/\xi_l^{(h_l)}}\right)^{\delta_{m,2}}\left(\frac{t_{h_1,...,h_{\mathsf{N}}}\lambda/\xi_n+\xi_n/(t_{h_1,...,h_{\mathsf{N}}}\lambda)}{t_{h_1,...,h_{\mathsf{N}}}+1/t_{h_1,...,h_{\mathsf{N}}}}\right)^{\delta_{1,a}} \\ &\times\prod_{b\neq n,b=1}^{\mathsf{N}}\frac{\lambda/\xi_b^{(h_b)}-\xi_b^{(h_b)}/\lambda}{\xi_n^{(h_n)}/\xi_b^{(h_b)}-\xi_b^{(h_b)}/\xi_n^{(h_n)}},\qquad \xi_b^{(h)}=\xi_b/q^h,\ t_{h_1,...,h_{\mathsf{N}}}=q^{-\sum_{a=1}^{\mathsf{N}}h_a}, \end{aligned} \tag{2.28}$$

that are well defined under the assumption that q is not a root of unity and $h_n\in\{0,...,n-1\}$ for any $n\in\{1,...,\mathsf{N}\}$. We also define the following functions of the operators N_i:

$$T_{1,\mathbf{h}}^{(K^{(a)},\infty)}(\lambda)=\delta_{1,a}\frac{\alpha\cosh\eta\mathsf{N}_1+\beta\cosh\eta\mathsf{N}_2+\gamma\cosh\eta\mathsf{N}_3}{\cosh(\eta\sum_{b=1}^{\mathsf{N}}h_b)}\prod_{b=1}^{\mathsf{N}}(\lambda/\xi_b^{(h_b)}-\xi_b^{(h_b)}/\lambda), \tag{2.29}$$

and

$$T_{2,\mathbf{h}}^{(K^{(a)},\infty)}(\lambda)=\delta_{1,a}\frac{\alpha\beta\cosh\eta(\mathsf{N}_1+\mathsf{N}_2)+\alpha\gamma\cosh\eta(\mathsf{N}_1+\mathsf{N}_3)+\beta\gamma\sinh\eta(\mathsf{N}_2+\mathsf{N}_3)}{\cosh(\eta\sum_{b=1}^{\mathsf{N}}h_b)}$$
$$\times\prod_{b=1}^{\mathsf{N}}(\lambda/\xi_b^{(h_b)}-\xi_b^{(h_b)}/\lambda)(\lambda/\xi_b^{(-1)}-\xi_b^{(-1)}/\lambda)\ . \tag{2.30}$$

Then the next corollary holds:

Corollary 2.1. *The transfer matrix* $T_1^{(K^{(a)})}(\lambda)$ *and* $T_2^{(K^{(a)})}(\lambda)$ *admit the following interpolation formulae:*

$$T_1^{(K^{(a)})}(\lambda)=T_{1,\mathbf{h}}^{(K^{(a)},\infty)}(\lambda)+\sum_{n=1}^{\mathsf{N}}f_{n,\mathbf{h}}^{(a,1)}(\lambda)T_1^{(K^{(a)})}(\xi_n^{(h_n)}), \tag{2.31}$$

and

$$T_2^{(K^{(a)})}(\lambda)=T_{2,\mathbf{h}}^{(K^{(a)},\infty)}(\lambda)+\sum_{n=1}^{\mathsf{N}}f_{n,\mathbf{h}}^{(a,2)}(\lambda)T_2^{(K^{(a)})}(\xi_n^{(h_n)}), \tag{2.32}$$

under the assumption that q *is not a root of unity and* $h_n\in\{0,1,2\}$ *for any* $n\in\{1,...,\mathsf{N}\}$*. Moreover, the following sum rules are satisfied:*

$$\delta_{1,a}\left(\alpha\sinh\eta(\mathsf{N}_1-\sum_{b=1}^{\mathsf{N}}h_b)+\beta\sinh\eta(\mathsf{N}_2-\sum_{b=1}^{\mathsf{N}}h_b)+\gamma\sinh\eta(\mathsf{N}_3-\sum_{b=1}^{\mathsf{N}}h_b)\right)$$
$$=\sum_{n=1}^{\mathsf{N}}\frac{T_1^{(K^{(a)})}(\xi_n^{(h_n)})}{2\prod_{b\neq n,b=1}^{\mathsf{N}}\xi_n^{(h_n)}/\xi_b^{(h_b)}-\xi_b^{(h_b)}/\xi_n^{(h_n)}}, \tag{2.33}$$

and

$$\delta_{1,a}\left(\alpha\beta\sinh\eta(\mathsf{N}_1+\mathsf{N}_2-\sum_{b=1}^{\mathsf{N}}h_b)+\alpha\gamma\sinh\eta(\mathsf{N}_1+\mathsf{N}_3-\sum_{b=1}^{\mathsf{N}}h_b)+\beta\gamma\sinh\eta(\mathsf{N}_2+\mathsf{N}_3-\sum_{b=1}^{\mathsf{N}}h_b)\right)$$
$$=\sum_{n=1}^{\mathsf{N}}\frac{T_2^{(K^{(a)})}(\xi_n^{(h_n)})}{2\prod_{b\neq n,b=1}^{\mathsf{N}}(\xi_n^{(h_n)}/\xi_b^{(h_b)}-\xi_b^{(h_b)}/\xi_n^{(h_n)})\prod_{b=1}^{\mathsf{N}}(\xi_n^{(h_n)}/\xi_b^{(-1)}-\xi_b^{(-1)}/\xi_n^{(h_n)})}. \tag{2.34}$$

$T_1^{(K^{(a)})}(\lambda)$ *then completely characterizes* $T_2^{(K^{(a)})}(\lambda)$ *in terms of the fusion equations by:*

$$T_2^{(K^{(a)})}(\lambda)=T_{2,\mathbf{h=0}}^{(K^{(a)},\infty)}(\lambda)+\sum_{n=1}^{\mathsf{N}}f_{n,\mathbf{h=0}}^{(a,2)}(\lambda)T_1^{(K^{(a)})}(\xi_n/q)T_1^{(K^{(a)})}(\xi_n). \tag{2.35}$$

Proof. We have to use just the known central zeroes and asymptotic behavior to prove the above interpolation formula once $T_2^{(K^{(a)})}(\xi_n)$ is given by the fusion equations. The sum rules just follow from the fact that in this trigonometric case we know the asymptotic behavior of the transfer matrices in two points (at λ going to zero and to infinity) while we still reconstruct these degree N polynomials in N points which leads to the sum rule. □

2.2 SoV covector basis generated by transfer matrix action

The general Proposition 2.6 of our article [1] for the construction of the SoV covector basis applies in particular to the fundamental representations of the trigonometric Yang-Baxter algebra.

The twist $K^{(a)}$ are diagonalizable and with simple spectrum 3×3 matrices, as soon as α, β and γ are all different in the case $a=1$ and $\alpha\beta\gamma\neq 0$ in the case $a=2$ and $a=3$. Let us denote by K_J the diagonal form of the matrix K and W_K the invertible matrix defining the change of basis:

$$K=W_K K_J W_K^{-1} \quad \text{with} \quad K_J=\left(\begin{array}{ccc} \mathsf{k}_1 & 0 & 0 \\ 0 & \mathsf{k}_2 & 0 \\ 0 & 0 & \mathsf{k}_3 \end{array}\right). \tag{2.36}$$

Then the following theorem holds:

Theorem 2.1. *For any diagonalizable* 3×3 *twist matrix* K *having simple spectrum, the following set:*

$$\langle h_1,...,h_{\mathsf{N}}|\equiv\langle S|\prod_{n=1}^{\mathsf{N}}(T_1^{(K)}(\xi_n))^{h_n} \quad \text{for any } \{h_1,...,h_{\mathsf{N}}\}\in\{0,1,2\}^{\otimes\mathsf{N}}, \tag{2.37}$$

forms a covector basis of $\mathcal{H}$*, for almost any choice of* $\langle S|$ *and of the inhomogeneities under the condition*

$$\xi_a\neq\xi_b q^h \quad \forall a\neq b\in\{1,...,\mathsf{N}\} \text{ and } h\in\{-2,-1,0,1,2\}. \tag{2.38}$$

Moreover, a proper choice of the state $\langle S|$ *has the following tensor product form:*

$$\langle S|=\bigotimes_{a=1}^{\mathsf{N}}(x,y,z)_a\Gamma_W^{-1}, \quad \Gamma_W=\bigotimes_{a=1}^{\mathsf{N}}W_{K,a} \tag{2.39}$$

under the condition $x\,y\,z\neq 0$*.*

Proof. As a corollary of the general Proposition 2.6 of [1], this set of covectors is a covector basis of $\mathcal{H}$ as soon as we can show that the covectors:

$$(x,y,z)_a W^{-1},(x,y,z)_a W^{-1}K,(x,y,z)_a W^{-1}K^2, \tag{2.40}$$

or equivalently:

$$(x,y,z)_a,(x,y,z)_a K_J,(x,y,z)_a K_J^2, \tag{2.41}$$

form a basis in $V_a\cong\mathbb{C}^3$, that is that the following determinant is non-zero:

$$\det||\left((x,y,z)K_J^{i-1}e_j(a)\right)_{i,j\in\{1,2,3\}}||=-xyzV(\mathsf{k}_0,\mathsf{k}_1,\mathsf{k}_2), \tag{2.42}$$

which leads to the given requirements on the components $x,y,z\in\mathbb{C}$ of the three dimensional covector. □

3 Transfer matrix spectrum in our SoV approach: the $\mathcal{U}_q(\widehat{gl_3})$ case

3.1 Discrete spectrum characterization by SoV

For any given twist matrix K diagonalizable and with simple spectrum, the following characterization of the transfer matrix spectrum holds:

Theorem 3.1. *The spectrum of $T_1^{(K)}(\lambda)$ is characterized by:*

$$\Sigma_{T^{(K)}} = \bigcup_{0\leq l+m\leq \mathsf{N}} \Sigma_{T^{(K)}}^{(l,m)}, \tag{3.1}$$

where l, m are positive integers and

$$\Sigma_{T^{(K)}}^{(l,m)} = \left\{ t_1(\lambda) = t_1(l,m,\alpha,\beta,\gamma) \prod_{b=1}^{\mathsf{N}} (\lambda/\xi_b - \xi_b/\lambda) + \sum_{a=1}^{\mathsf{N}} f_{n,\mathbf{h=0}}^{(a,1)}(\lambda) x_a \right\}, \tag{3.2}$$

for any set $\{x_1, \ldots, x_{\mathsf{N}}\}$ belonging to $S_{T^{(K)}}^{(l,m)}$ where we have defined:

$$t_1(l,m,\alpha,\beta,\gamma) \equiv \delta_{1,a} \left(\alpha \cosh \eta l + \beta \cosh \eta m + \gamma \cosh \eta (l+m-\mathsf{N})\right) \tag{3.3}$$

under the assumption that the 3×3 twist matrix K is simple and diagonalizable and the inhomogeneities satisfy (2.38). Here, $S_{T^{(K)}}^{(l,m)}$ stand for the set of solutions $\{x_1, \ldots, x_{\mathsf{N}}\}$ to the following system of N cubic equations:

$$\begin{aligned} &x_n[\delta_{1,a} \left(\alpha\beta \cosh \eta(l+m) + \alpha\gamma \cosh \eta(\mathsf{N}-m) + \beta\gamma \sinh \eta(\mathsf{N}-l)\right) \prod_{b=1}^{\mathsf{N}} (\xi_n^{(1)}/\xi_b - \xi_b/\xi_n^{(1)}) \\ &\times (\xi_n^{(1)}/\xi_b^{(-1)} - \xi_b^{(-1)}/\xi_n^{(1)}) + \sum_{m=1}^{\mathsf{N}} f_{m,\mathbf{h=0}}^{(a,2)}(\xi_m^{(1)}) t_1(\xi_m^{(1)}) x_m] = q\text{-}det M^{(K)}(\xi_a), \end{aligned} \tag{3.4}$$

in N unknown $\{x_1, ..., x_{\mathsf{N}}\}$. Moreover, $T_1^{(K)}(\lambda)$ is diagonalizable with simple spectrum and for any $t_1(\lambda) \in \Sigma_{T^{(K)}}$ the associated unique eigenvector $|t\rangle$ has the following wave function in the covector SoV basis:

$$\langle h_1, ..., h_{\mathsf{N}} | t\rangle = \prod_{n=1}^{\mathsf{N}} t_1^{h_n}(\xi_n), \tag{3.5}$$

where the overall normalization has been fixed by imposing $\langle S|t\rangle = 1$.

Proof. The transfer matrix fusion equations:

$$T_1^{(K)}(\xi_b) T_2^{(K)}(\xi_b/q) = q\text{-det} M^{(K)}(\xi_b), \ \forall b \in \{1, ..., \mathsf{N}\}, \tag{3.6}$$

when rewritten for the eigenvalues take exactly the form of the given system of N cubic equations in N unknowns $\{x_1, ..., x_{\mathsf{N}}\}$, once we use the known analyticity and central zeroes. Consequently, this system has to be satisfied and the given characterization of any eigenvector $|t\rangle$ is implied.

The reverse statement has to be shown now. In particular, we have to prove that given a polynomial of the above form satisfying this system then it is an eigenvalue of the transfer matrix $T_1^{(K)}(\lambda)$. This is done by proving that the vector $|t\rangle$ defined in (3.5) is a transfer matrix eigenvector using our SoV basis:

$$\langle h_1, ..., h_{\mathsf{N}} | T_1^{(K)}(\lambda) | t\rangle = t_1(\lambda) \langle h_1, ..., h_{\mathsf{N}} | t\rangle, \ \forall \{h_1, ..., h_{\mathsf{N}}\} \in \{0,1,2\}^{\otimes \mathsf{N}}. \tag{3.7}$$

Let us point out that as a consequence of the Corollary 2.1, in order to prove this identity it is enough to prove it in a generic N-upla of points $\xi_a^{(k_a)}$ for any $a \in \{1, ..., \mathsf{N}\}$. Indeed, let us assume that it holds:

$$\langle h_1, ..., h_{\mathsf{N}} | T_1^{(K)}(\xi_a^{(k_a)}) | t\rangle = t_1(\xi_a^{(k_a)}) \langle h_1, ..., h_{\mathsf{N}} | t\rangle, \ \forall \{h_1, ..., h_{\mathsf{N}}\} \in \{0,1,2\}^{\otimes \mathsf{N}}, \tag{3.8}$$

then we have that:

$$
\begin{aligned}
&\delta_{1,a}\langle h_1,...,h_{\mathsf{N}}|\left(\alpha\sinh\eta(\mathsf{N}_1-\sum_{b=1}^{\mathsf{N}}k_b)+\beta\sinh\eta(\mathsf{N}_2-\sum_{b=1}^{\mathsf{N}}k_b)+\gamma\sinh\eta(\mathsf{N}_3-\sum_{b=1}^{\mathsf{N}}k_b)\right)|t\rangle \\
&=\langle h_1,...,h_{\mathsf{N}}|\left(\sum_{n=1}^{\mathsf{N}}\frac{T_1^{(K^{(a)})}(\xi_n^{(k_n)})}{2\prod_{b\neq n,b=1}^{\mathsf{N}}\xi_n^{(k_n)}/\xi_b^{(k_b)}-\xi_b^{(k_b)}/\xi_n^{(k_n)}}\right)|t\rangle \qquad (3.9)\\
&=\left(\sum_{n=1}^{\mathsf{N}}\frac{t_1(\xi_n^{(k_n)})}{2\prod_{b\neq n,b=1}^{\mathsf{N}}\xi_n^{(k_n)}/\xi_b^{(k_b)}-\xi_b^{(k_b)}/\xi_n^{(k_n)}}\right)\langle h_1,...,h_{\mathsf{N}}|t\rangle \qquad (3.10)\\
&=\delta_{1,a}(\alpha\sinh\eta(l-\sum_{b=1}^{\mathsf{N}}k_b)+\beta\sinh\eta(m-\sum_{b=1}^{\mathsf{N}}k_b)+\gamma\sinh\eta(\mathsf{N}-(l+m)-\sum_{b=1}^{\mathsf{N}}k_b))\langle h_1,...,h_{\mathsf{N}}|t\rangle, \qquad (3.11)
\end{aligned}
$$

which implies that for $a=1$, $|t\rangle$ is an eigenvector of the charges N_1, N_2 and N_3 with eigenvalues, respectively, l, m and $\mathsf{N}-(l+m)$ which in turn fix the asymptotics of the transfer matrices in that case. That is for $a=1$, any $t_1(\lambda)$ in (3.2), $t_1(\lambda)$ and $|t\rangle$ can be transfer matrix eigenvalue and eigenvector only associated to the common eigenspace of N_1 and N_2 corresponding to the nonnegative integer eigenvalues l and m, respectively. Notice that if $a\neq 1$ the asymptotic term is zero and the N_i are no longer symmetries of the transfer matrices and so we don't need to distinguish those values in the discussion.

Let $h_a=0,1$ and $h_b\in\{0,1,2\}$ for any $b\in\{1,...,\mathsf{N}\}\backslash a$, then we have the following identities:

$$
\begin{aligned}
\langle h_1,...,h_{\mathsf{N}}|T_1^{(K^{(a)})}(\xi_a)|t\rangle &= \langle h_1,...,h_a+1,...,h_{\mathsf{N}}|t\rangle \\
&= t_1(\xi_a)\langle h_1,...,h_a,...,h_{\mathsf{N}}|t\rangle, \qquad (3.12)
\end{aligned}
$$

as a direct consequence of the definition of the covector SoV basis and of the state $|t\rangle$. So that we are left with the proof of the statement in the case $h_a=2$. In this case we want to prove that it holds:

$$
\langle h_1,...,h_{\mathsf{N}}|T_1^{(K^{(a)})}(\xi_a/q)|t\rangle = t_1(\xi_a/q)\langle h_1,...,h_a,...,h_{\mathsf{N}}|t\rangle, \qquad (3.13)
$$

the proof is done by induction on the number of zeros contained in $\{h_1,...,h_{\mathsf{N}}\}\in\{0,1,2\}^{\otimes\mathsf{N}}$. It is developed just following the same steps we have developed in the case of the fundamental representation of the $Y(gl_3)$ rational Yang-Baxter algebra in our paper [1]. In fact we have only to take into account the different functional form of the transfer matrix, i.e. they are Laurent polynomials and not simple polynomials, and the fact that the asymptotic behavior of the transfer matrices are not central for $a=1$ but take fixed values in any common eigenspace of N_1 and N_2 that is stable by the action of the transfer matrices since for $a=1$ they commute with each N_i. For completeness we dedicate Appendix A to make explicit these steps of the proof. □

3.2 Spectrum characterization by quantum spectral curve

The discrete characterization of the transfer matrix spectrum derived in the previous section in our SoV basis allows us to introduce the following quantum spectral curve functional reformulation.

Let us first introduce the functions:

$$\delta_3(\lambda) = \delta_1(\lambda)\delta_1(\lambda/q)\delta_1(\lambda/q^2), \tag{3.14}$$

$$\delta_2(\lambda) = \delta_1(\lambda)\delta_1(\lambda/q), \tag{3.15}$$

$$\delta_1(\lambda) = \delta_0 \prod_{a=1}^{\mathsf{N}} (\lambda q/\xi_a - \xi_a/(\lambda q)), \tag{3.16}$$

Theorem 3.2. *Here we consider the case of a twist K which is diagonal and with simple spectrum, then the entire functions $t_1(\lambda)$ is a $T_1^{(K)}(\lambda)$ transfer matrix eigenvalue belonging to $\Sigma_{T^{(K)}}^{(\nu_1,\nu_2)}$, with ν_1, ν_2 two nonnegative integers satisfying:*

$$\nu_1 + \nu_2 \leq \mathsf{N}, \tag{3.17}$$

iff there exists an unique Laurent polynomial of the form:

$$\varphi_t(\lambda) = \prod_{a=1}^{\mathsf{M}} (\lambda/\lambda_a - \lambda_a/\lambda) \quad \text{with } \mathsf{M} \leq \mathsf{N} \text{ and } \lambda_a \neq \xi_n \ \forall (a,n) \in \{1, ..., \mathsf{M}\} \times \{1, ..., \mathsf{N}\}, \tag{3.18}$$

solution of the following quantum spectral curve functional equation:

$$\delta_3(\lambda)\varphi_t(\lambda/q^3) - \delta_2(\lambda)t_1(\lambda/q^2)\varphi_t(\lambda/q^2) + \delta_1(\lambda)t_2(\lambda/q)\varphi_t(\lambda/q) - q\text{-}det M_a^{(K)}(\lambda)\varphi_t(\lambda) = 0 \tag{3.19}$$

where we have defined

$$\begin{aligned} t_2(\lambda) &= (\alpha\beta\cosh\eta(\nu_1+\nu_2) + \alpha\gamma\cosh\eta(\nu_1+\nu_3) + \beta\gamma\sinh\eta(\nu_2+\nu_3)) \\ &\times \prod_{b=1}^{\mathsf{N}} (\lambda/\xi_b - \xi_b/\lambda)(\lambda/\xi_b^{(-1)} - \xi_b^{(-1)}/\lambda) + \sum_{n=1}^{\mathsf{N}} f_{n,\mathbf{h}=\mathbf{0}}^{(a)}(\xi_n^{(1)}) t_1(\xi_n^{(1)}) t_1(\xi_n), \end{aligned} \tag{3.20}$$

with $\nu_3 = \mathsf{N} - \nu_1 - \nu_2$ and fixed[2]*:*

$$\delta_0 = \mathsf{k}_i \text{ for one fixed } i \in \{1, 2, 3\}, \tag{3.21}$$

with

$$\mathsf{M} = \mathsf{M} - \nu_i. \tag{3.22}$$

Moreover, up to an overall normalization the common transfer matrix eigenvector $|t\rangle$ admits the following separate representation:

$$\langle h_1, ..., h_{\mathsf{N}}|t\rangle = \prod_{a=1}^{\mathsf{N}} \delta_1^{h_a}(\xi_a)\varphi_t^{h_a}(\xi_a/q)\varphi_t^{2-h_a}(\xi_a). \tag{3.23}$$

Proof. This is a special case of the proof that will be given in the general $U_q(\widehat{gl_n})$ case. □

[2]That is we have to fix δ_0 to be one of the three distinct eigenvalues of the matrix K and then the degree of the Laurent polynomial $\varphi_t(\lambda)$ is fixed by:

$$\mathsf{M} = \mathsf{N} - \nu_1, \text{ for } \delta_0 = \alpha, \ \mathsf{M} = \mathsf{N} - \nu_2, \text{ for } \delta_0 = \beta, \ \mathsf{M} = \mathsf{N} - \nu_3, \text{ for } \delta_0 = \gamma, \text{ with } \nu_3 = \mathsf{N} - (\nu_1 + \nu_2).$$

Let us comment that the Theorem 3.1 applies for any integrable boundary conditions while the previous Theorem 3.2 applies only to the case of diagonal twists. The non-diagonal case is not presented explicitly but we can similarly derive a functional equation of third order type which however is of inhomogeneous type, if we ask that the $\varphi_t(\lambda)$ has the same Laurent polynomial form as indicated in (3.18). While at this stage this is a simple exercise we think that the main interesting question to investigate about the non-diagonal twists is if, with a different (periodicity) definition of the $\varphi_t(\lambda)$ function, we can reestablish an homogeneous equation as it has been proven for the $U_q(\widehat{gl_2})$ case in [74].

3.3 ABA rewriting of transfer matrix eigenvectors

An equivalent rewriting of algebraic Bethe ansatz type for the transfer matrix eigenvectors can be derived on the basis of their SoV representation. Let us first define one common eigenvector of the transfer matrix $T_1^{(K)}(\lambda)$ and $T_2^{(K)}(\lambda)$:

Lemma 3.1. *Let K be a diagonal 3×3 matrix having three distinct eigenvalues k_i:*

$$K = \begin{pmatrix} \mathsf{k}_1 & 0 & 0 \\ 0 & \mathsf{k}_2 & 0 \\ 0 & 0 & \mathsf{k}_3 \end{pmatrix}, \tag{3.24}$$

then:

$$|t_0\rangle = \bigotimes_{a=1}^{\mathsf{N}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_a, \tag{3.25}$$

is a common eigenvector of the transfer matrices $T_1^{(K)}(\lambda)$ and $T_2^{(K)}(\lambda)$:

$$T_1^{(K)}(\lambda)|t_0\rangle = |t_0\rangle t_{1,0}(\lambda) \quad \textit{with } t_{1,0}(\lambda) = \mathsf{k}_1 \prod_{a=1}^{\mathsf{N}} (\lambda q/\xi_a - \xi_a/(\lambda q)) + (\mathsf{k}_2 + \mathsf{k}_3) \prod_{a=1}^{\mathsf{N}} (\lambda/\xi_a - \xi_a/\lambda), \tag{3.26}$$

$$T_2^{(K)}(\lambda)|t_0\rangle = |t_0\rangle t_{2,0}(\lambda) \quad \textit{with } t_{2,0}(\lambda) = \left\{ \begin{array}{l} \prod_{a=1}^{\mathsf{N}} (\lambda/(q\xi_a) - (q\xi_a)/\lambda)(\mathsf{k}_3\mathsf{k}_2 \prod_{a=1}^{\mathsf{N}} (\lambda/\xi_a - \xi_a/\lambda) \\ +(\mathsf{k}_2\mathsf{k}_1 + \mathsf{k}_3\mathsf{k}_1) \prod_{a=1}^{\mathsf{N}} (\lambda q/\xi_a - \xi_a/(\lambda q)), \end{array} \right. \tag{3.27}$$

and the quantum spectral curve

$$\delta_3(\lambda) - \delta_2(\lambda) t_1(\lambda/q^2) + \delta_1(\lambda) t_2(\lambda/q) - q\text{-}detM_a^{(K)}(\lambda) = 0 \tag{3.28}$$

with constant $\varphi_t(\lambda)$ is satisfied by the couple of eigenvalues $t_{1,0}(\lambda)$ and $t_{2,0}(\lambda)$ for $\delta_0 = \mathsf{k}_1$.

Proof. This is a standard result which follows by proving that it holds:

$$A_i^{(I)}(\lambda)|t_0\rangle = |t_0\rangle \prod_{a=1}^{\mathsf{N}} (\lambda q^{\delta_{i,1}}/\xi_a - \xi_a/(\lambda q^{\delta_{i,1}})), \quad C_i^{(I)}(\lambda)|t_0\rangle = 0, \quad i \in \{1, 2, 3\}, \tag{3.29}$$

where the upper index I stands for the identity twist matrix, from which it is simple to verify by direct computation that the $t_{1,0}(\lambda)$ and $t_{2,0}(\lambda)$ satisfies the fusion equations (3.6) and that it holds:

$$t_{1,0} \equiv \lim_{\lambda\to\pm\infty} \lambda^{\mp\mathsf{N}} t_{1,0}(\lambda) = (-1)^{\frac{1\pm1}{2}\mathsf{N}} \left(\mathsf{k}_1 q^{\pm\mathsf{N}} + \mathsf{k}_2 + \mathsf{k}_3\right) \left(\prod_{a=1}^{\mathsf{N}} \xi_a\right)^{\mp 1}, \tag{3.30}$$

$$t_{2,0} \equiv \lim_{\lambda\to\pm\infty} \lambda^{\mp2\mathsf{N}} t_{2,0}(\lambda) = \left(\prod_{a=1}^{\mathsf{N}} \xi_a\right)^{\mp 2} q^{\mp\mathsf{N}}(\mathsf{k}_3\mathsf{k}_2 + (\mathsf{k}_2\mathsf{k}_1 + \mathsf{k}_3\mathsf{k}_1)q^{\pm\mathsf{N}}), \tag{3.31}$$

so that $t_{1,0}(\lambda)$ satisfies the SoV characterization of the eigenvalues of $T_1^{(K)}(\lambda)$. Observing now that it holds:

$$t_{1,0}(\xi_a) = \delta_1(\xi_a) \quad \text{for any } a \in \{1, ..., \mathsf{N}\} \tag{3.32}$$

it follows that the associated $\varphi_t(\lambda)$ satisfies the equations:

$$\varphi_t(\xi_a) = \varphi_t(\xi_a/q) \quad \text{for any } a \in \{1, ..., \mathsf{N}\} \tag{3.33}$$

and so $\varphi_t(\lambda)$ is a constant. □

In our SoV basis we can now define the operator $\mathbb{B}^{(K)}(\lambda)$ as the one parameter family of commuting operators through the following characterization:

$$\langle h_1, ..., h_{\mathsf{N}}|\mathbb{B}^{(K)}(\lambda) = b_{h_1,...,h_{\mathsf{N}}}(\lambda)\langle h_1, ..., h_{\mathsf{N}}|, \tag{3.34}$$

where we have defined

$$b_{h_1,...,h_{\mathsf{N}}}(\lambda) = \prod_{a=1}^{\mathsf{N}} (\lambda/\xi_a - \xi_a/\lambda)^{2-h_a} (\lambda q/\xi_a - \xi_a/(q\lambda))^{h_a} . \tag{3.35}$$

Then the following corollary holds:

Lemma 3.2. *The following algebraic Bethe ansatz type representation*

$$|t\rangle = \prod_{a=1}^{\mathsf{M}} \mathbb{B}^{(K)}(\lambda_a)|t_0\rangle \quad \text{with } \mathsf{M} \leq \mathsf{N} \text{ and } \lambda_a \neq \xi_n \ \forall (a,n) \in \{1, ..., \mathsf{M}\} \times \{1, ..., \mathsf{N}\}, \tag{3.36}$$

holds for the unique (up to trivial scalar multiplication) eigenvector $|t\rangle$ associated to any given $t_1(\lambda) \in \Sigma_{T^{(K)}} \equiv \bigcup_{\forall \nu_i \geq 0\, :\, \nu_1+\nu_2 \leq \mathsf{N}} \Sigma_{T^{(K)}}^{(\nu_1,\nu_2)}$. Here the λ_a are the roots of the unique Laurent polynomial $\varphi_t(\lambda)$ associated to $t_1(\lambda) \in \Sigma_{T^{(K)}}$.

Proof. The proof is standard, the following chain of identities holds

$$\begin{aligned}
\langle h_1, ..., h_{\mathsf{N}}| \prod_{a=1}^{\mathsf{M}} \mathbb{B}^{(K)}(\lambda_a)|t_0\rangle &= \prod_{j=1}^{\mathsf{M}} b_{h_1,...,h_{\mathsf{N}}}(\lambda_j) \ \langle h_1, ..., h_{\mathsf{N}}|t_0\rangle \\
&= \prod_{j=1}^{\mathsf{M}} \prod_{a=1}^{\mathsf{N}} (\lambda_j/\xi_a - \xi_a/\lambda_j)^{2-h_a} (\lambda_j q/\xi_a - \xi_a/(q\lambda_j))^{h_a} \prod_{a=1}^{\mathsf{N}} \delta_1^{h_a}(\xi_a) \\
&= \prod_{a=1}^{\mathsf{N}} \delta_1^{h_a}(\xi_a) \varphi_t(\xi_a)^{2-h_a} \varphi_t(\xi_a/q)^{h_a},
\end{aligned} \tag{3.37}$$

which coincides with the SoV characterization of the transfer matrix eigenvector once we recall that:

$$\langle h_1, ..., h_{\mathsf{N}}|t_0\rangle = \prod_{a=1}^{\mathsf{N}} \delta_1^{h_a}(\xi_a). \tag{3.38}$$

□

4 Transfer matrices for fundamental evaluation representations of $\mathcal{U}_q(\widehat{gl_n})$

Let us consider now the general higher rank $n-1$, with $n \geq 3$ case. In particular, here we take the following R-matrix:

$$R_{a,b}(\lambda) = \left(\frac{\lambda}{q} - \frac{q}{\lambda}\right)\sum_{k=1}^{n} E_{kk}^{(a)} \otimes E_{kk}^{(b)} + \left(\lambda - \frac{1}{\lambda}\right)\sum_{p=1}^{n}\sum_{k=1,k\neq p}^{n} E_{kk}^{(a)} \otimes E_{pp}^{(b)}$$
$$+ \left(q - \frac{1}{q}\right)\sum_{1\leq k<p\leq n}\left(\lambda^{(n-2(p-k))/n} E_{kp}^{(a)} \otimes E_{pk}^{(b)} + \lambda^{-(n-2(p-k))/n} E_{pk}^{(a)} \otimes E_{kp}^{(b)}\right) \in \text{End}(V_a \otimes V_b) \tag{4.1}$$

which is the *trigonometric* principal gradation solution[3] [13, 14] of the Yang-Baxter equation:

$$R_{12}(\lambda/\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda/\mu) \in \text{End}(V_1 \otimes V_2 \otimes V_3), \tag{4.2}$$

where $V_i \simeq \mathbb{C}^n$ for any $i \in \{1, 2, 3\}$, and it is associated to the fundamental evaluation representations of $\mathcal{U}_q(\widehat{gl_n})$ [15–19]. Above, we have used the standard notation for the elementary matrices E_{lm} ∈End$(V \simeq \mathbb{C}^n)$, $(l, m) \in \{1, ..., n\} \times \{1, ..., n\}$:

$$(E_{lm})_{\alpha\beta} = \delta_{\alpha l}\delta_{m\beta} \quad \forall(\alpha, \beta) \in \{1, ..., n\} \times \{1, ..., n\}. \tag{4.3}$$

In this paper, we analyze the fundamental representations of these rank $n-1$ trigonometric Yang-Baxter algebras, associated to the following monodromy matrices:

$$M_a^{(K)}(\lambda) \equiv K_a R_{a,\mathsf{N}}(\lambda/\xi_{\mathsf{N}}) \cdots R_{a,1}(\lambda/\xi_1) \in \text{End}(V_a \otimes \mathcal{H}), \tag{4.4}$$

where $\mathcal{H} = \bigotimes_{n=1}^{\mathsf{N}} V_n$ and K ∈End(V) is a symmetry (twist matrix):

$$R_{12}(\lambda)K_1K_2 = K_2K_1R_{12}(\lambda) \in \text{End}(V_1 \otimes V_2). \tag{4.5}$$

Then the one parameter family of operators

$$T_1^{(K)}(\lambda) \equiv \text{tr}_a M_a^{(K)}(\lambda) \in \text{End}\mathcal{H}, \tag{4.6}$$

are the associated commuting transfer matrices. Here, we focus our attention on the case of diagonal quasi-periodic boundary conditions which are associated to generic diagonal $n \times n$ twist matrices

[3]These R-matrices are associated to general values of q, the root of unity case has been also analyzed see for example [102] for a review. However in all the present article we assume that q is not a root of unity.

having pairwise distinct eigenvalues k_i:

$$K=\begin{pmatrix} \mathsf{k}_1 & 0 & \cdots & 0 \\ 0 & \mathsf{k}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathsf{k}_n \end{pmatrix}. \tag{4.7}$$

For this class of representations we will prove the complete spectrum characterization of the transfer matrix in terms of a specific class of polynomial solutions to the quantum spectral curve equation, an homogeneous functional equation to the finite difference of order n. Let us comment that, as for the case $n=3$, the symmetry of the principal gradation R-matrix extends also to non-diagonal twist matrices[4] and that our construction of the SoV basis applies for these cases, as well as the derivation of the complete SoV characterization of the transfer matrix spectrum. However, for the non-diagonal twist matrices a natural reformulation of the transfer matrix spectrum leads to an inhomogeneous functional equation. We have decided to develop the case of non-diagonal twist matrix in some future analysis where, as already discussed at the end of section 3.2, the main interesting question is if under an apropiate choice of the Q-functions one can derive an homogeneous quantum spectral curve characterization. Such a statement indeed holds for both the fundamental and higher spin representations of the rank one trigonometric Yang-Baxter algebra, as proven in [74].

4.1 Fundamental properties of the hierarchy of transfer matrices

Let us introduce the following operators, $\mathsf{N}_i \in \mathrm{End}(\mathcal{H})$

$$\mathsf{N}_i \otimes_{l=1}^{\mathsf{N}} |l,a_l\rangle = \otimes_{l=1}^{\mathsf{N}} |l,a_l\rangle \sum_{l=1}^{\mathsf{N}} \delta_{i,a_l} \quad \forall i\in\{1,...,n\}, \tag{4.8}$$

in the basis

$$|l,a_l\rangle = \begin{pmatrix} \delta_{1,a_l} \\ \delta_{2,a_l} \\ \vdots \\ \delta_{n,a_l} \end{pmatrix}_l \in V_l \quad \forall l\in\{1,...,\mathsf{N}\} \text{ and } a_l\in\{1,...,n\}, \tag{4.9}$$

so that it holds:

$$\sum_{i=1}^{\mathsf{N}} \mathsf{N}_i = \mathsf{N}. \tag{4.10}$$

We will denote by ν_i the eigenvalues of the operators N_i. Then we can recall the following relevant and known properties of the fused transfer matrices:

Proposition 4.1. *The higher transfer matrices:*

$$T_m^{(K)}(\lambda) \equiv tr_{a_1,\dots,a_m} P^{-}_{a_1,\dots,a_m} M^{(K)}_{a_m}(\lambda)\cdots M^{(K)}_{a_1}(\lambda/q^{m-1}), \quad m\in\{1,...,n\} \tag{4.11}$$

[4] In the special case associated to the so-called anti-periodic boundary conditions a first eigenvalue analysis has been developed in [40].

defines one parameter families of mutually commuting operators:

$$\left[T_i^{(K)}(\lambda), T_j^{(K)}(\mu)\right] = 0, \quad i, j \in \{1, ..., n\}. \tag{4.12}$$

The last quantum spectral invariant

$$q\text{-}detM^{(K)}(\lambda) \equiv T_n^{(K)}(\lambda), \tag{4.13}$$

the so-called quantum determinant, is a central element of the algebra with the following explicit form:

$$q\text{-}detM^{(K)}(\lambda) = detK \prod_{b=1}^{\mathsf{N}}(\lambda q/\xi_b - \xi_b/(q\lambda)) \prod_{k=1}^{n-1}(\lambda/(q^k\xi_b) - (q^k\xi_b)/\lambda). \tag{4.14}$$

Moreover, the quantum spectral invariants have the following analyticity properties:

a) The following fusion identities holds for any $a \in \{1, \ldots, \mathsf{N}\}$:

$$T_1^{(K)}(\xi_a)T_{m-1}^{(K)}(\xi_a/q) = T_m^{(K)}(\xi_a), \quad \forall m \in \{2, ..., n\}, \tag{4.15}$$

and the following $(m-1)\mathsf{N}$ central zero conditions:

$$T_m^{(K)}(\pm q^r \xi_a) = 0 \quad \forall r \in \{1, ..., m-1\}, a \in \{1, ..., \mathsf{N}\}, \tag{4.16}$$

holds for any $m \in \{1, ..., n-1\}$ and for any above diagonal K-matrix.

b) $\lambda^{m\mathsf{N}}T_m^{(K)}(\lambda)$ is a degree $m\mathsf{N}$ polynomial in λ^2 with the following asymptotics:

$$T_m^{(\pm\infty|K)} \equiv \lim_{\log\lambda\to\pm\infty} \lambda^{\mp\mathsf{N}}T_m^{(K)}(\lambda) = (\pm 1)^{m\mathsf{N}} \frac{\sigma_m^{(n)}(\mathsf{k}_1 q^{\pm\mathsf{N}_1}, ..., \mathsf{k}_n q^{\pm\mathsf{N}_n})}{q^{\pm m(m-1)\mathsf{N}/2}\prod_{n=1}^{\mathsf{N}}\xi_n^{\pm m}}, \tag{4.17}$$

in the case of the diagonal twist K, where $\sigma_m^{(n)}$ is the standard symmetric polynomial of degree m in n variables.

Let us introduce the functions

$$f_{l,\mathbf{h}}^{(m)}(\lambda) = \frac{t_{h_1,\ldots,h_{\mathsf{N}}}\lambda/\xi_l + \xi_l/(t_{h_1,\ldots,h_{\mathsf{N}}}\lambda)}{t_{h_1,\ldots,h_{\mathsf{N}}} + 1/t_{h_1,\ldots,h_{\mathsf{N}}}} \prod_{b=1}^{\mathsf{N}}\prod_{r=1}^{m-1} \frac{\lambda/\xi_b^{(-r)} - \xi_b^{(-r)}/\lambda}{\xi_l^{(h_l)}/\xi_b^{(-r)} - \xi_b^{(-r)}/\xi_l^{(h_l)}}$$

$$\times \prod_{b\neq l,b=1}^{\mathsf{N}} \frac{\lambda/\xi_b^{(h_b)} - \xi_b^{(h_b)}/\lambda}{\xi_n^{(h_n)}/\xi_b^{(h_b)} - \xi_b^{(h_b)}/\xi_n^{(h_n)}}, \tag{4.18}$$

$$\tag{4.19}$$

well defined under the assumption that q is not a root of unity and $h_l \in \{0, \ldots, n-1\}$ for any $l \in \{1, ..., \mathsf{N}\}$, and

$$T_{m,\mathbf{h}}^{(K,\infty)}(\lambda|\mathsf{N}_1, ..., \mathsf{N}_n) = \frac{\sum_{1\leq i_1<i_2<\cdots<i_{m-1}<i_m\leq n}\prod_{k=1}^m \mathsf{k}_{i_k}\cosh(\eta\sum_{k=1}^m \mathsf{N}_{i_k})}{\cosh(\eta\sum_{a=1}^{\mathsf{N}} h_a)}$$

$$\times \prod_{b=1}^{\mathsf{N}}(\lambda/\xi_b^{(h_b)} - \xi_b^{(h_b)}/\lambda) \prod_{r=1}^{m-1}(\lambda/\xi_b^{(-r)} - \xi_b^{(-r)}/\lambda), \tag{4.20}$$

then the following corollary holds:

Corollary 4.1. *Under the assumption that q is not a root of unity and $h_l \in \{0, \dots, n-1\}$ for any $l \in \{1, ..., \mathsf{N}\}$, the following interpolation formulae:*

$$T_m^{(K)}(\lambda) = T_{m,\mathbf{h}}^{(K,\infty)}(\lambda|\mathsf{N}_1, ..., \mathsf{N}_n) + \sum_{l=1}^{\mathsf{N}} f_{l,\mathbf{h}}^{(m)}(\lambda) T_m^{(K)}(\xi_l^{(h_l)}) \tag{4.21}$$

holds for the transfer matrix $T_m^{(K)}(\lambda)$ with $m \in \{1, ..., n-1\}$ together with the following sum rules:

$$\begin{aligned} &\sum_{1 \le i_1 < i_2 < \cdots < i_{m-1} < i_m \le n} \prod_{k=1}^{m} \mathsf{k}_{i_k} \cosh(\eta \sum_{k=1}^{m} \mathsf{N}_{i_k} - \sum_{l=1}^{\mathsf{N}} h_l) \\ &= \sum_{l=1}^{\mathsf{N}} \frac{T_m^{(K)}(\xi_l^{(h_l)})}{2 \prod_{b \neq l, b=1}^{\mathsf{N}} (\xi_l^{(h_l)}/\xi_b^{(h_b)} - \xi_b^{(h_b)}/\xi_l^{(h_l)}) \prod_{b=1}^{\mathsf{N}} \prod_{r=1}^{m-1} (\xi_l^{(h_l)}/\xi_b^{(-r)} - \xi_b^{(-r)}/\xi_l^{(h_l)})}. \end{aligned} \tag{4.22}$$

The fusion equations allow to completely characterize all the $T_m^{(K)}(\lambda)$ in terms of $T_1^{(K)}(\lambda)$ by the following interpolation formulae:

$$T_m^{(K)}(\lambda) = T_{m,\mathbf{h}=\mathbf{0}}^{(K,\infty)}(\lambda) + \sum_{l=1}^{\mathsf{N}} f_{n,\mathbf{h}=\mathbf{0}}^{(m)}(\lambda) T_{m-1}^{(K)}(\xi_l/q) T_1^{(K)}(\xi_l). \tag{4.23}$$

4.2 SoV covector basis generated by transfer matrix action

The following theorem holds as a corollary of Proposition 2.6 of [1]:

Theorem 4.1. *Let K be a $n \times n$ simple and diagonalizable symmetry of the R-matrix, then the following set:*

$$\langle h_1, ..., h_{\mathsf{N}}| \equiv \langle S| \prod_{n=1}^{\mathsf{N}} (T_1^{(K)}(\xi_n))^{h_n} \quad \textit{for any } \{h_1, ..., h_{\mathsf{N}}\} \in \{0, ..., n-1\}^{\otimes \mathsf{N}}, \tag{4.24}$$

forms a covector basis of $\mathcal{H}$, for almost any choice of $\langle S|$ and of the inhomogeneities satisfying (2.38)*. A proper choice of the state $\langle S|$ has the following tensor product form:*

$$\langle S| = \bigotimes_{l=1}^{\mathsf{N}} (x_1, ..., x_n)_l \Gamma_W^{-1}, \quad \Gamma_W = \bigotimes_{a=1}^{\mathsf{N}} W_{K,a} \tag{4.25}$$

under the condition $\prod_{l=1}^{n} x_l \neq 0$, where W is the invertible matrix defining the similarity transformation to the diagonal matrix K_J by $K = W_K K_J W_K^{-1}$.

5 Transfer matrix spectrum in our SoV approach: the $\mathcal{U}_q(\widehat{gl_n})$ case

5.1 Discrete spectrum characterization by SoV

In the following we need $n-1$ Laurent polynomials in λ:

$$t_1^{(K,\{x\},\{\nu_i\})}(\lambda) = T_{m+1,\mathbf{h}=\mathbf{0}}^{(K,\infty)}(\lambda|\nu_1, ..., \nu_n) + \sum_{l=1}^{\mathsf{N}} f_{l,\mathbf{h}=\mathbf{0}}^{(1)}(\lambda) x_l, \tag{5.1}$$

where we have used the notation $T_{m+1,\mathbf{h=0}}^{(K,\infty)}(\lambda|\nu_1,...,\nu_n)$ for the eigenvalue of the already defined asymptotic operator $T_{m+1,\mathbf{h=0}}^{(K,\infty)}(\lambda|\mathsf{N}_1,...,\mathsf{N}_n)$ on the common eigenspaces of the operators N_i and

$$t_{m+1}^{(K,\{x\},\{\nu_i\})}(\lambda) = T_{m+1,\mathbf{h=0}}^{(K,\infty)}(\lambda|\nu_1,...,\nu_n) + \sum_{l=1}^{\mathsf{N}} f_{l,\mathbf{h=0}}^{(m+1)}(\lambda) t_m^{(K^{(a)},\{x\},\{\nu_i\})}(\xi_l/q) x_l, \tag{5.2}$$

for any $m \in \{1,...,n-2\}$, which are as well functions of a $n \times n$ twist matrix K, of a point $\{x_1,...,x_\mathsf{N}\} \in \mathbb{C}^\mathsf{N}$ and of an n-upla $\{\nu_1,...,\nu_n\}$ of nonnegative integers (the eigenvalues of the operators N_i) satisfying:

$$\sum_{l=1}^{n} \nu_l = \mathsf{N}. \tag{5.3}$$

Then, the following characterization of the transfer matrix spectrum holds:

Theorem 5.1. *We consider a twist K matrix symmetry which is diagonal and with simple spectrum and inhomogeneity parameters in generic position. Then the spectrum of $T_1^{(K)}(\lambda)$ is characterized by:*

$$\Sigma_{T^{(K)}} = \bigcup_{\forall \nu_i \geq 0 \,:\, \sum_{l=1}^n \nu_l = \mathsf{N}} \Sigma_{T^{(K)}}^{(\{\nu_i\})}, \tag{5.4}$$

where

$$\Sigma_{T^{(K)}}^{(\{\nu_i\})} = \left\{ t_1(\lambda) : t_1(\lambda) = t_1^{(K,\{x\},\{\nu_i\})}(\lambda), \quad \forall \{x_1,...,x_\mathsf{N}\} \in S_{T^{(K)}}^{(\{\nu_i\})} \right\}, \tag{5.5}$$

and $S_{T^{(K)}}^{(\{\nu_i\})}$ is defined as the set of solutions to the next system of N polynomial equations of order n:

$$x_a t_{n-1}^{(K,\{x\},\{\nu_i\})}(\xi_a/q) = q\text{-}detM^{(K)}(\xi_a), \tag{5.6}$$

in N unknown $\{x_1,...,x_\mathsf{N}\}$. Moreover, $T_1^{(K)}(\lambda, \{\xi\})$ is diagonalizable with simple spectrum and

$$\langle h_1,...,h_\mathsf{N}|t\rangle = \prod_{n=1}^{\mathsf{N}} t_1^{h_n}(\xi_n) \tag{5.7}$$

uniquely characterizes the eigenvector $|t\rangle$ associated to any fixed $t_1(\lambda) \in \Sigma_{T^{(K)}}$ in our SoV basis.

Proof. The proof works by some simple modifications of the case of the Yangian $Y(gl_n)$ fundamental representations developed in our second paper [2]. We have just to handle the fact that the asymptotic behavior of the transfer matrices is now not central in the full representation space but only in each common eigenspaces of all the operators N_i. Since those commute with all transfer matrices, the proof can be achieved in each of these subspaces and hence in the full Hilbert space.

In appendix we present an alternative proof of our statement which is a corollary of the diagonalizabilty and simplicity of the transfer matrix spectrum which follows from the Proposition 2.7 of our paper [1]. □

5.2 Spectrum characterization by quantum spectral curve

Let us first introduce the functions:

$$\delta_1(\lambda) = \delta_0 \prod_{a=1}^{\mathsf{N}} (\lambda q/\xi_a - \xi_a/(\lambda q)), \quad \delta_m(\lambda) = \prod_{i=0}^{m-1} \delta_1(\lambda/q^i), \tag{5.8}$$

then the discrete SoV characterization of the transfer matrix spectrum implies:

Theorem 5.2. *Under the same conditions of the previous theorem, the entire functions* $t_1(\lambda)$ *is a* $T_1^{(K)}(\lambda)$ *transfer matrix eigenvalue belonging to*[5] $\Sigma_{T^{(K)}}^{(\{\nu_i\})}$, *with* ν_i *nonnegative integers satisfying:*

$$\sum_{l=1}^{n} \nu_l = \mathsf{N}, \tag{5.9}$$

iff there exists the unique Laurent polynomial:

$$\varphi_t(\lambda) = \prod_{a=1}^{\mathsf{M}} (\lambda/\lambda_a - \lambda_a/\lambda) \quad \textit{with } \mathsf{M} \leq \mathsf{N} \textit{ and } \lambda_a \neq \xi_m \; \forall (a,m) \in \{1,...,\mathsf{M}\} \times \{1,...,\mathsf{N}\}, \tag{5.10}$$

such that $t_1(\lambda)$, $t_m(\lambda) \equiv t_m^{(K^{(a)},\{t_1(\xi_1),...,t_1(\xi_\mathsf{N})\},\{\nu_i\})}(\lambda)$ *and* $\varphi_t(\lambda)$ *are solutions of the following quantum spectral curve functional equation:*

$$\sum_{l=0}^{n} (-1)^l \, \delta_l(\lambda)\varphi_t(\lambda/q^l)t_{n-l}(\lambda/q^l) = 0 \tag{5.11}$$

where $t_0(\lambda) = 1$ *and we have to fix*[6]*:*

$$\delta_0 = \mathsf{k}_i \quad \textit{for one fixed } i \in \{1,...,n\}, \tag{5.12}$$

and

$$\mathsf{M} = \mathsf{N} - \nu_i. \tag{5.13}$$

Moreover, up to a normalization, the corresponding common transfer matrix eigenvector $|t\rangle$ *admits the following separate representation:*

$$\langle h_1, ..., h_\mathsf{N}|t\rangle = \prod_{a=1}^{\mathsf{N}} \delta_1^{h_a}(\xi_a)\varphi_t^{h_a}(\xi_a/q)\varphi_t^{n-h_a}(\xi_a). \tag{5.14}$$

Proof. Let us start proving that the asymptotics of the functional equation are indeed compatibles with those of the transfer matrix eigenvalues. That is, if we assume that $t_1(\lambda) \in \Sigma_{T^{(K)}}^{(\{n_i\})}$, then we have to show that the leading asymptotics associated to the degree $\mathsf{M} + n\mathsf{N}$ of the l.h.s. of the equation must be zero and vice versa.

Let us remark that from the known asymptotics $T_m^{(\pm\infty|K)}$ of the transfer matrices, the following identities hold:

$$\lim_{\log\lambda \to \pm\infty} \lambda^{\mp(n-a)\mathsf{N}} T_{n-a}^{(K,\infty)}(\lambda/q^a|\mathsf{N}_1, ..., \mathsf{N}_n) = \frac{(\pm 1)^{(n-a)\mathsf{N}} \, \sigma_{n-a}^{(n)}(\mathsf{k}_1 q^{\pm \mathsf{N}_1}, ..., \mathsf{k}_n q^{\pm \mathsf{N}_n})}{q^{\pm(n-a)(a+(n-a-1)/2)\mathsf{N}} \prod_{n=1}^{\mathsf{N}} \xi_n^{\pm(n-a)}}, \tag{5.15}$$

while it is easy to verify that it holds:

$$\lim_{\log\lambda \to \pm\infty} \lambda^{\mp a\mathsf{N}} \delta_a(\lambda) = \frac{(\pm 1)^{a\mathsf{N}} \, \mathsf{k}_i^a}{q^{\pm((a-1)(a-2)/2-1)\mathsf{N}} \prod_{n=1}^{\mathsf{N}} \xi_n^{\pm a}}, \tag{5.16}$$

[5] i.e. the transfer matrix eigenvalues associated to the common eigenspace of the $\mathsf{N}_1, ..., \mathsf{N}_n$ with eigenvalues $\nu_1, ..., \nu_n$.

[6] That is we have to fix δ_0 to be one of the n distinct eigenvalue k_i of the twist matrix K and then the degree of the Laurent polynomial $\varphi_t(\lambda)$ is fixed by $\mathsf{M} = \mathsf{N} - \nu_i$.

where we have imposed the choice $\delta_0 = \mathsf{k}_i$ and

$$\lim_{\log\lambda\to\pm\infty} \lambda^{\mp\mathsf{M}}\varphi_t(\lambda/q^a) = \frac{(\pm 1)^{\mathsf{M}}}{q^{\pm a\mathsf{M}}}, \tag{5.17}$$

from which it follows:

$$\lim_{\log\lambda\to\pm\infty} \lambda^{\mp(\mathsf{M}+n\mathsf{N})}(l.h.s.)_{(5.11)} = \frac{(\pm 1)^{(\mathsf{M}+n\mathsf{N})}\sum_{l=0}^{n}(-\mathsf{k}_i)^l\, q^{\pm l(\mathsf{N}-\mathsf{M})}\sigma_{n-l}^{(n)}(\mathsf{k}_1 q^{\pm\nu_1},...,\mathsf{k}_n q^{\pm\nu_n})}{q^{\pm n(n-1)\mathsf{N}/2}\prod_{n=1}^{\mathsf{N}}\xi_n^{\pm n}} \tag{5.18}$$

$$= \frac{(\pm 1)^{(\mathsf{M}+n\mathsf{N})}\sum_{l=0}^{n}(-\mathsf{k}_i q^{\pm\nu_i})^l\, \sigma_{n-l}^{(n)}(\mathsf{k}_1 q^{\pm\nu_1},...,\mathsf{k}_n q^{\pm\nu_n})}{q^{\pm n(n-1)\mathsf{N}/2}\prod_{n=1}^{\mathsf{N}}\xi_n^{\pm n}} \tag{5.19}$$

$$= 0, \tag{5.20}$$

where according to our choice $\delta_0 = \mathsf{k}_i$ we also fix $\mathsf{N}-\mathsf{M} = \nu_i$. The last identity to zero, holding for any choice of $i \in \{1,..,n\}$, as a trivial consequence of the defining identity of the symmetric polynomials:

$$\sum_{l=0}^{n}(-\lambda)^l\,\sigma_{n-l}^{(n)}(x_1,...,x_n) = (-1)^n\prod_{a=1}^{n}(\lambda - x_a), \tag{5.21}$$

which is zero if and only if $\lambda = x_i$ for any fixed $i \in \{1,..,n\}$. Vice versa, if $t_1(\lambda)$ satisfies with the polynomial $t_m(\lambda)$ and $\varphi_t(\lambda)$ the functional equation then it is a degree N Laurent polynomial in λ with leading coefficients forced to be:

$$t_1^{(\pm)} \equiv \lim_{\log\lambda\to\pm\infty}\lambda^{\mp\mathsf{N}}t_1(\lambda) = (\pm 1)^{\mathsf{N}}\frac{\sigma_1^{(n)}(\mathsf{k}_1 q^{\pm n_1},...,\mathsf{k}_n q^{\pm n_n})}{\prod_{n=1}^{\mathsf{N}}\xi_n^{\pm 1}}, \tag{5.22}$$

as a consequence of the asymptotic of the satisfied functional equation for any $\delta_0 = \mathsf{k}_i$ we also fix $\mathsf{N}-\mathsf{M} = \nu_i$.

Now that the asymptotic behavior is verified the proof of the theorem follows mainly the same steps used for the Yangian $Y(gl_n)$ case. For completeness let us reproduce them here. We complete first the proof of the fact that given a $t_1(\lambda)$ entire function satisfying with the polynomials $t_m(\lambda)$ and $\varphi_t(\lambda)$ the functional equation implies that it is a transfer matrix eigenvalue. Let us observe now that, for $\lambda = \xi_a$ it holds:

$$\delta_{1+j}(\xi_a) = 0, \text{ for } 1 \leq j \leq n-1,\ \delta_1(\xi_a) \neq 0,\ \det_q M_a^{(K)}(\xi_a) \neq 0, \tag{5.23}$$

and the quantum spectral curve in these points reads:

$$\frac{\delta_1(\xi_a)\varphi_t(\xi_a/q)}{\varphi_t(\xi_a)} = \frac{\det_q M_a^{(K)}(\xi_a)}{t_{n-1}(\xi_a/q)}. \tag{5.24}$$

Consider instead $1 \leq s \leq n-1$, then for $\lambda = \xi_a q^s$ it holds:

$$\delta_{r\geq s+2}(\xi_a q^s) = 0,\ t_{n-b}(\xi_a q^{s-b}) = 0, \text{ for any } 0 \leq b \leq s-1 \tag{5.25}$$

$$\delta_{r\leq s+1}(\xi_a q^s) \neq 0, \tag{5.26}$$

and the quantum spectral curve in these points reads:

$$\frac{\delta_{s+1}(\xi_a q^s)\varphi_t(\xi_a/q)}{\delta_s(\xi_a q^s)\varphi_t(\xi_a)} = \frac{t_{n-s}(\xi_a)}{t_{n-s-1}(\xi_a/q)}. \tag{5.27}$$

Then the chain of identities:

$$\frac{\delta_{s+1}(\xi_a q^s)}{\delta_s(\xi_a q^s)} = \delta_1(\xi_a) \text{ for any } 1 \leq s \leq n-1, \tag{5.28}$$

imply the following ones:

$$t_{m+1}(\xi_a) = t_m(\xi_a/q)t_1(\xi_a), \ \forall m \in \{1, ..., n-1\}, a \in \{1, ..., \mathsf{N}\}. \tag{5.29}$$

So that $t_m(\lambda)$ are eigenvalues of the transfer matrices $T_m^{(K)}(\lambda)$, for the same eigenvector $|t\rangle$, thanks to the SoV characterization given in our previous theorem.

Let us now prove the reverse statement. Let $t_1(\lambda)$ be eigenvalue of the transfer matrix $T_1^{(K)}(\lambda)$ then we have to prove the existence of $\varphi_t(\lambda)$ a Laurent polynomial which satisfies the quantum spectral curve with the $t_m(\lambda)$. By imposing the following set of conditions:

$$\delta_1(\xi_a)\frac{\varphi_t(\xi_a/q)}{\varphi_t(\xi_a)} = t_1(\xi_a), \tag{5.30}$$

we characterize uniquely a Laurent polynomial $\varphi_t(\lambda)$ of the form (3.18). Indeed, following the same steps given in the proof of the Theorem 4.1 of our second paper [2] for the Yangian $Y(gl_n)$ case, we have that there exists a unique Laurent polynomial $\varphi_t(\lambda)$ of the form (3.18) with some degree $\mathsf{M} \leq \mathsf{N}$ so that one is left with the proof of the identity $\mathsf{M} = \mathsf{N} - \nu_i$. This is done just generalizing to the present case the argument based on the sum rules as presented in the proof of the Theorem 4.3 of our first paper [1], see equations (4.69) and (4.70) there.

Here, we recall that this characterization of $\varphi_t(\lambda)$ indeed implies that the functional equation is satisfied. The l.h.s. of the quantum spectral curve is a Laurent polynomial in λ of maximal degree $n\mathsf{N} + \mathsf{M} \leq (n+1)\mathsf{N}$ then to prove that it is identically zero it is enough to show that it is zero in $(n+1)\mathsf{N}$ distinct points. Indeed, when this is the case the above argument on the sum rules shows that the leading coefficients of the quantum spectral curve are indeed zero. The chosen distinct points are the following $(n+1)\mathsf{N}$ ones $\xi_a q^{k_a}$, for any $a \in \{1, ..., \mathsf{N}\}$ and $k_a \in \{-1, 0, ..., n-1\}$. For $\lambda = \xi_a/q$ we have:

$$\delta_r(\xi_a/q) = 0 \text{ for any } 1 \leq r \leq n, \text{ as well as } \det M_a^{(K)}(\xi_a/q) = 0, \tag{5.31}$$

so that for any $a \in \{1, ..., \mathsf{N}\}$ the quantum spectral curve equation is satisfied while in the remaining $n\mathsf{N}$ points this equation coincides with the $n\mathsf{N}$ equations (5.24) and (5.27) which are satisfied by the transfer matrix eigenvalues as they are all equivalent to the discrete characterization (5.30) thanks to the fusion equations.

Let us verify now the equivalence of the SoV characterization of the transfer matrix eigenvector with the one presented in this theorem. It is enough to multiply by the non-zero product of the $\varphi_t^{n-1}(\xi_a)$ over all the $a \in \{1, ..., \mathsf{N}\}$ the eigenvector $|t\rangle$ getting our result:

$$\prod_{a=1}^{\mathsf{N}} \varphi_t^{n-1}(\xi_a) \prod_{a=1}^{\mathsf{N}} t_1^{h_a}(\xi_a) \overset{(5.30)}{=} \prod_{a=1}^{\mathsf{N}} \delta_1^{h_a}(\xi_a)\varphi_t^{h_a}(\xi_a/q)\varphi_t^{n-1-h_a}(\xi_a). \tag{5.32}$$

□

5.3 Q-operator reconstruction by SoV

The Q-operator commuting family, satisfying the quantum spectral curve equation with the transfer matrices at the operator level, can be constructed in terms of the fundamental transfer matrix thanks to the above SoV characterization of the transfer matrix spectrum. Let us denote by $\delta_b^{(i)}(\lambda)$ the polynomials defined in the previous section just making explicit that we have fixed $\delta_0 = \mathsf{k}_i$ for some fixed $i \in \{1, ..., n\}$. Moreover, let us define the following $\mathsf{N} \times \mathsf{N}$ operator matrix of elements:

$$[C_{i,\xi_{\mathsf{N}+1}}^{(T_1^{(K)})}]_{ab} = -\delta_{ab} \frac{T_1^{(K)}(\xi_a)}{\delta_1^{(i)}(\xi_a)} + \prod_{\substack{c=1 \\ c\neq b}}^{\mathsf{N}+1} \frac{\xi_a/(q\xi_c) - (q\xi_c)/\xi_a}{\xi_b/\xi_c - \xi_c/\xi_b} \qquad \forall a, b \in \{1, \dots, \mathsf{N}\}, \tag{5.33}$$

and the rank one central matrix:

$$[\Delta_{\xi_{\mathsf{N}+1}}(\lambda)]_{ab} = \frac{\lambda/\xi_{\mathsf{N}+1} - \xi_{\mathsf{N}+1}/\lambda}{\lambda/\xi_b - \xi_b/\lambda} \frac{\prod_{c=1}^{\mathsf{N}}(\xi_a/(q\xi_c) - (q\xi_c)/\xi_a)}{\prod_{c=1,c\neq b}^{\mathsf{N}+1}(\xi_b/\xi_c - \xi_c/\xi_b)} \qquad \forall a, b \in \{1, \dots, \mathsf{N}\}, \tag{5.34}$$

then it holds:

Corollary 5.1. *Given*[7] $\xi_{\mathsf{N}+1} \neq \xi_{i\leq N}$*, then for almost any values of the parameters* $\{\xi_{i\leq N}\}$ *and of the nonzero eigenvalues* $\{\mathsf{k}_{j\leq n}\}$ *of the diagonal and simple spectrum twist matrix* K*, the following Laurent polynomial family of* Q*-operators:*

$$Q_i(\lambda) = \frac{det_N[C_{i,\xi_{\mathsf{N}+1}}^{(T_1^{(K)})} + \Delta_{\xi_{\mathsf{N}+1}}(\lambda)]}{det_N[C_{i,\xi_{\mathsf{N}+1}}^{(T_1^{(K)})}]} \prod_{c=1}^{\mathsf{N}} \frac{\lambda/\xi_c - \xi_c/\lambda}{\xi_{\mathsf{N}+1}/\xi_c - \xi_{\mathsf{N}+1}/\xi_c}, \tag{5.35}$$

satisfies the operator quantum spectral curve equation

$$\sum_{b=0}^{n} \delta_b^{(i)}(\lambda) Q_i(\lambda - b\eta) T_{n-b}^{(K)}(\lambda - b\eta) = 0, \tag{5.36}$$

where we have defined $T_0^{(K)}(\lambda) \equiv 1$*, and moreover* $Q_i(\xi_a)$ *are invertible operators for any* $a \in \{1, \dots, \mathsf{N}\}$*.*

Proof. The SoV characterization of the transfer matrix spectrum and the proof of its reformulation in terms of the quantum spectral curve functional equation imply this corollary. Indeed, following the same proof given in the case of the fundamental representations of the Yangian $Y(gl_n)$, see appendix B of our second paper [2], one can prove that the Laurent polynomial $\varphi_t(\lambda)$ of the form (3.18) solution of the quantum spectral curve equation has the following determinant representation:

$$\varphi_t^{(i)}(\lambda) = \frac{\det_N[C_{i,\xi_{\mathsf{N}+1}}^{(t_1)} + \Delta_{\xi_{\mathsf{N}+1}}(\lambda)]}{\det_N[C_{i,\xi_{\mathsf{N}+1}}^{(t_1)}]} \prod_{c=1}^{\mathsf{N}} \frac{\lambda/\xi_c - \xi_c/\lambda}{\xi_{\mathsf{N}+1}/\xi_c - \xi_{\mathsf{N}+1}/\xi_c}, \tag{5.37}$$

obtained by substituting to the transfer matrix $T_1^{(K)}(\xi_a)$ the corresponding eigenvalue $t_1(\xi_a)$. As a consequence of the Proposition 2.7 of [1], the transfer matrix $T_1^{(K)}(\lambda)$ is diagonalizable and

[7] Note that we can fix for example $\xi_{N+1} = \xi_h - \eta$ for any fixed $h \in \{1, \dots, N\}$.

with simple spectrum in our current setting. The Laurent polynomial family Q_i-operator is then completely characterized by its action on the eigenbasis of the transfer matrix:

$$Q_i(\lambda)|t\rangle = |t\rangle \varphi_t^{(i)}(\lambda), \tag{5.38}$$

for any eigenvalue $t_1(\lambda)$ and uniquely associated eigenvector $|t\rangle$ of the transfer matrix $T_1^{(K)}(\lambda)$, which is equivalent to the characterization given in the corollary. This also imply that this operator family satisfies the quantum spectral curve equation with the transfer matrices. □

Acknowledgements

J. M. M. and G. N. are supported by CNRS and ENS de Lyon.

A Appendix A

In this appendix we complete the proof of the Theorem 3.1 by computing the direct computation of the action of transfer matrices on the SoV covector basis. Once again let us comment that these computations are obtained by adapting those of the fundamental representations of the $Y(gl_3)$ rational Yang-Baxter algebra in our first paper [1], taking into account the fact that the transfer matrices commute with the operators N_i that define their (non-central) asymptotic behavior. hence, all computations can be done in each common eigenspaces of these operators that give a complete decomposition of the full Hilbert space.

Complement to proof of Theorem 3.1. In order to complete the proof of Theorem 3.1, we have to prove that in the case $h_j = 2$ the following identities hold

$$\langle h_1, ..., h_{\mathsf{N}}| T_1^{(K^{(a)})}(\xi_j/q)|t\rangle = t_1(\xi_j/q)\langle h_1, ..., h_a, ..., h_{\mathsf{N}}|t\rangle, \tag{A.1}$$

and this is done by making an induction on the number R of zeros contained in $\{h_1, ..., h_{\mathsf{N}}\} \in \{0, 1, 2\}^{\otimes \mathsf{N}}$. Let us start proving this identity for $R = 0$, the fusion identities imply:

$$\langle h_1, ..., h_a = 2, ..., h_{\mathsf{N}}| T_1^{(K^{(a)})}(\xi_j/q)|t\rangle = \langle h_1, ..., h_j = 1, ..., h_{\mathsf{N}}| T_2^{(K^{(a)})}(\xi_j)|t\rangle, \tag{A.2}$$

so that:

$$\langle h_1, ..., h'_j = 1, ..., h_{\mathsf{N}}| T_2^{(K^{(a)})}(\xi_j)|t\rangle = T_{2,\mathbf{h=1}}^{(K^{(a)},\infty)}(\xi_j)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m}\langle h_1, ..., h'_j, ..., h_{\mathsf{N}}|t\rangle \tag{A.3}$$

$$+\sum_{n=1}^{\mathsf{N}} f_{n,\mathbf{h=1}}^{(a,2)}(\xi_j)\langle h_1, ..., h'_j, ..., h_{\mathsf{N}}| T_2^{(K)}(\xi_n/q)|t\rangle, \tag{A.4}$$

thanks to the following interpolation formula:

$$T_2^{(K)}(\xi_j) = T_{2,\mathbf{h=1}}^{(K^{(a)},\infty)}(\xi_j)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} + \sum_{n=1}^{\mathsf{N}} f_{n,\mathbf{h=1}}^{(a,2)}(\xi_j) T_2^{(K)}(\xi_n/q). \tag{A.5}$$

Then, being $R=0$, it follows:

$$\langle h_1,...,h'_a,...,h_{\mathsf{N}}|T_2^{(K^{(a)})}(\xi_j)|t\rangle = T_{2,\mathbf{h=1}}^{(K^{(a)},\infty)}(\xi_j)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m}\langle h_1,...,h'_j,...,h_{\mathsf{N}}|t\rangle + \sum_{n=1}^{\mathsf{N}} q\text{-det}M^{(K^{(a)})}(\xi_n) f_{n,\mathbf{h=1}}^{(a,2)}(\xi_j)\langle h_1,...,h''_n,...,h_{\mathsf{N}}|t\rangle, \quad (A.6)$$

where $h''_n = h_n - 1$ for $n \neq j$ and $h''_j = h'_j - 1 = 0$. Now the function:

$$t_2(\lambda) = T_{2,\mathbf{h=1}}^{(K^{(a)},\infty)}(\lambda)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} + \sum_{n=1}^{\mathsf{N}} f_{n,\mathbf{h=0}}^{(a,2)}(\lambda)t_1(\xi_n/q)t_1(\xi_n), \quad (A.7)$$

satisfies by definition the equations:

$$t_2(\xi_n) = t_1(\xi_n/q)t_1(\xi_n), \ \forall n \in \{1,...,\mathsf{N}\}, \quad (A.8)$$

$$t_1(\xi_n)t_2(\xi_n/q) = q\text{-det}M^{(K^{(a)})}(\xi_n), \ \forall n \in \{1,...,\mathsf{N}\}, \quad (A.9)$$

where the the quantum determinant equation is indeed a consequence of the definition of $t_1(\lambda)$. Then we get:

$$\langle h_1,...,h'_j,...,h_{\mathsf{N}}|T_2^{(K^{(a)})}(\xi_j)|t\rangle =$$

$$= \left(T_{2,\mathbf{h=1}}^{(K^{(a)},\infty)}(\xi_j)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} + \sum_{n=1}^{\mathsf{N}} t_2(\xi_n/q) f_{n,\mathbf{h=1}}^{(a,2)}(\xi_j)\right)\langle h_1,...,h'_j,...,h_{\mathsf{N}}|t\rangle, \quad (A.10)$$

$$= t_2(\xi_n)\langle h_1,...,h'_j,...,h_{\mathsf{N}}|t\rangle \quad (A.11)$$

$$= t_1(\xi_n/q)\langle h_1,...,h_j=2,...,h_{\mathsf{N}}|t\rangle, \quad (A.12)$$

where we have used the interpolation formula:

$$t_2(\xi_j) = T_{2,\mathbf{h=1}}^{(K^{(a)},\infty)}(\xi_j)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} + \sum_{n=1}^{\mathsf{N}} t_2(\xi_n/q) f_{n,\mathbf{h=1}}^{(a,2)}(\xi_j), \quad (A.13)$$

i.e. we have shown our identity (3.13) for $R=0$. Then we can do our proof by induction; we assume that it holds for the generic $\{h_1,...,h_{\mathsf{N}}\} \in \{0,1,2\}^{\otimes \mathsf{N}}$ containing $R-1$ zeros and we prove it for the generic $\{h_1,...,h_{\mathsf{N}}\} \in \{0,1,2\}^{\otimes \mathsf{N}}$ containing R zeros. Let us fix the generic $\{h_1,...,h_{\mathsf{N}}\} \in \{0,1,2\}^{\otimes \mathsf{N}}$ with $h_j = 2$ and let us denote with π a permutation of $\{1,...,\mathsf{N}\}$ such that:

$$\begin{aligned} h_{\pi(i)} &= 0, \ \forall i \in \{1,...,R\}, \\ h_{\pi(i)} &= 1, \ \forall i \in \{R+1,...,R+S\}, \\ h_{\pi(i)} &= 2, \ \forall i \in \{R+S+1,...,\mathsf{N}\}, \end{aligned} \quad (A.14)$$

with $j = \pi(R+S+1)$. Let us use now the following interpolation formula:

$$T_2^{(K)}(\xi_j) = T_{2,\mathbf{k}}^{(K^{(a)},\infty)}(\xi_j)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} + \sum_{n=1}^{\mathsf{N}} f_{n,\mathbf{k}}^{(a,2)}(\xi_j)T_2^{(K)}(\xi_n^{(k_n)}), \quad (A.15)$$

where we have defined $\mathbf{k}$ by:

$$\begin{aligned} k_{\pi(i)} &= 1, \ \forall i \in \{1, ..., R\}, \\ h_{\pi(i)} &= 2, \ \forall i \in \{R+1, ..., \mathsf{N}\}, \end{aligned} \tag{A.16}$$

then it holds:

$$\begin{aligned} \langle h_1, ..., h'_j = 1, ..., h_{\mathsf{N}}| T_2^{(K^{(a)})}(\xi_j)|t\rangle &= T_{2,\mathbf{k}}^{(K^{(a)},\infty)}(\xi_j)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} \langle h_1, ..., h'_j, ..., h_{\mathsf{N}}|t\rangle \\ &+ \sum_{n=1}^{R} f_{\pi(n),\mathbf{k}}^{(a,2)}(\xi_j)\langle h_1, ..., h'_j, ..., h_{\mathsf{N}}|T_2^{(K)}(\xi_{\pi(n)})|t\rangle \\ &+ \sum_{n=R+1}^{\mathsf{N}} f_{\pi(n),\mathbf{k}}^{(a,2)}(\xi_j)\langle h_1, ..., h'_j, ..., h_{\mathsf{N}}|T_2^{(K)}(\xi_{\pi(n)}/q)|t\rangle. \end{aligned} \tag{A.17}$$

and which by the fusion identity reads:

$$\begin{aligned} \langle h_1, ..., h'_j, ..., h_{\mathsf{N}}| T_2^{(K^{(a)})}(\xi_j)|t\rangle &= T_{2,\mathbf{k}}^{(K^{(a)},\infty)}(\xi_j)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} \langle h_1, ..., h'_j, ..., h_{\mathsf{N}}|t\rangle \\ &+ \sum_{n=1}^{R} f_{\pi(n),\mathbf{k}}^{(a,2)}(\xi_j)\langle h_1^{(n)}, ..., h_{\mathsf{N}}^{(n)}|T_1^{(K)}(\xi_{\pi(n)}/q)|t\rangle \\ &+ \sum_{n=R+1}^{\mathsf{N}} q\text{-det}M^{(K^{(a)})}(\xi_{\pi(n)}) f_{\pi(n),\mathbf{k}}^{(a,2)}(\xi_j)\langle h_1^{(n)}, ..., h_{\mathsf{N}}^{(n)}|t\rangle, \end{aligned} \tag{A.18}$$

where we have defined:

$$h_{\pi(m)}^{(n)} = \begin{cases} h_{\pi(m)} + \theta(R-m)\delta_{m,n} & \text{for } n \leq R \\ h_{\pi(m)} - \theta(m-(R+1))\delta_{m,n} - \delta_{m,R+S+1} & \text{for } R+1 \leq n \end{cases}. \tag{A.19}$$

To compute $\langle h_1^{(n)}, ..., h_{\mathsf{N}}^{(n)}|T_1^{(K)}(\xi_{\pi(n)}/q)|t\rangle$ for $n \leq R$, we use the following interpolation formula:

$$T_1^{(K)}(\xi_{\pi(n)}/q) = T_{1,\mathbf{k}'}^{(K^{(a)},\infty)}(\xi_{\pi(n)}/q)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} + \sum_{r=1}^{\mathsf{N}} f_{r,\mathbf{k}'}^{(a,1)}(\xi_{\pi(n)}/q) T_1^{(K)}(\xi_r^{(k'_r)}), \tag{A.20}$$

where we have defined:

$$k'_{\pi(m)} = \begin{cases} 0 & \text{for } m \leq R+S+1 \\ 1 & \text{for } R+S+2 \leq m \end{cases}, \tag{A.21}$$

which gives:

$$\begin{aligned} \langle h_1^{(n)}, ..., h_{\mathsf{N}}^{(n)}|T_1^{(K)}(\xi_{\pi(n)}/q)|t\rangle &= T_{1,\mathbf{k}'}^{(K^{(a)},\infty)}(\xi_{\pi(n)}/q)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} \langle h_1^{(n)}, ..., h_{\mathsf{N}}^{(n)}|t\rangle \\ &+ \sum_{r=1}^{R+S+1} f_{\pi(r),\mathbf{k}'}^{(a,1)}(\xi_{\pi(n)}/q)\langle h_1^{(n)}, ..., h_{\mathsf{N}}^{(n)}|T_1^{(K)}(\xi_{\pi(r)})|t\rangle \\ &+ \sum_{r=R+S+2}^{\mathsf{N}} f_{\pi(r),\mathbf{k}'}^{(a,1)}(\xi_{\pi(n)}/q)\langle h_1^{(n)}, ..., h_{\mathsf{N}}^{(n)}|T_1^{(K)}(\xi_{\pi(r)}/q)|t\rangle, \end{aligned} \tag{A.22}$$

which becomes:

$$\begin{aligned}\langle h_1^{(n)},...,h_{\mathsf{N}}^{(n)}|T_1^{(K)}(\xi_{\pi(n)}/q)|t\rangle =& \left.T_{1,\mathbf{k}'}^{(K^{(a)},\infty)}(\xi_{\pi(n)}/q)\right|_{\mathsf{N}_1=l,\mathsf{N}_2=m}\langle h_1^{(n)},...,h_{\mathsf{N}}^{(n)}|t\rangle \\ &+\sum_{r=1}^{R+S+1} f_{\pi(r),\mathbf{k}'}^{(a,1)}(\xi_{\pi(n)}/q)t_1(\xi_{\pi(r)})\langle h_1^{(n)},...,h_{\mathsf{N}}^{(n)}|t\rangle \\ &+\sum_{r=R+S+2}^{\mathsf{N}} f_{\pi(r),\mathbf{k}'}^{(a,1)}(\xi_{\pi(n)}/q)t_1(\xi_{\pi(r)}/q)\langle h_1^{(n)},...,h_{\mathsf{N}}^{(n)}|t\rangle, \end{aligned} \tag{A.23}$$

where in the second line we have used the identity (3.12) while in the third line the identity (3.13), which holds by assumption being $R-1$ the number of zeros in $\{h_1^{(n)},...,h_{\mathsf{N}}^{(n)}\}$. So that we have shown for any $n \leq R$:

$$\langle h_1^{(n)},...,h_{\mathsf{N}}^{(n)}|T_1^{(K)}(\xi_{\pi(n)}/q)|t\rangle = t_1(\xi_{\pi(n)}/q)\langle h_1^{(n)},...,h_{\mathsf{N}}^{(n)}|t\rangle, \tag{A.24}$$

and substituting it in the second line of (A.18), we get:

$$\begin{aligned}\langle h_1,...,h'_j,...,h_{\mathsf{N}}|T_2^{(K^{(a)})}(\xi_j)|t\rangle =& \left.T_{2,\mathbf{k}}^{(K^{(a)},\infty)}(\xi_j)\right|_{\mathsf{N}_1=l,\mathsf{N}_2=m}\langle h_1,...,h'_j,...,h_{\mathsf{N}}|t\rangle \\ &+\sum_{n=1}^{R} t_1(\xi_{\pi(n)}/q)f_{\pi(n),\mathbf{k}}^{(a,2)}(\xi_j)\langle h_1^{(n)},...,h_{\mathsf{N}}^{(n)}|t\rangle \\ &+\sum_{n=R+1}^{\mathsf{N}} q\text{-det}M^{(K^{(a)})}(\xi_{\pi(n)})f_{\pi(n),\mathbf{k}}^{(a,2)}(\xi_j)\langle h_1^{(n)},...,h_{\mathsf{N}}^{(n)}|t\rangle, \end{aligned} \tag{A.25}$$

and so $\langle h_1,...,h'_j,...,h_{\mathsf{N}}|T_2^{(K^{(a)})}(\xi_j)|t\rangle$ reads:

$$\begin{aligned}&\left(\left.T_{2,\mathbf{k}}^{(K^{(a)},\infty)}(\xi_j)\right|_{\mathsf{N}_1=l,\mathsf{N}_2=m}+\sum_{n=1}^{R} t_1(\xi_{\pi(n)})t_1(\xi_{\pi(n)}/q)f_{\pi(n),\mathbf{k}}^{(a,2)}(\xi_j)+\sum_{n=R+1}^{\mathsf{N}} t_2(\xi_{\pi(n)}/q)f_{\pi(n),\mathbf{k}}^{(a,2)}(\xi_j)\right) \\ &\times\langle h_1,...,h'_j,...,h_{\mathsf{N}}|t\rangle \\ &= t_2(\xi_j/q)\langle h_1,...,h'_j=1,...,h_{\mathsf{N}}|t\rangle = t_1(\xi_j/q)\langle h_1,...,h_j=2,...,h_{\mathsf{N}}|t\rangle, \end{aligned} \tag{A.26}$$

i.e. we have proven our formula (3.13). Finally, taken the generic $\{h_1,...,h_{\mathsf{N}}\} \in \{0,1,2\}^{\otimes \mathsf{N}}$ with:

$$\begin{aligned} h_{\pi(i)} &= 0,\ \forall i \in \{1,...,R\}, \\ h_{\pi(i)} &= 1,\ \forall i \in \{R+1,...,R+S\}, \\ h_{\pi(i)} &= 2,\ \forall i \in \{R+S+1,...,\mathsf{N}\}, \end{aligned} \tag{A.27}$$

and by using the interpolation formula:

$$T_1^{(K)}(\lambda) = \left.T_{1,\mathbf{k}}^{(K^{(a)},\infty)}(\lambda)\right|_{\mathsf{N}_1=l,\mathsf{N}_2=m}+\sum_{n=1}^{\mathsf{N}} f_{n,\mathbf{p}}^{(a,1)}(\lambda)T_1^{(K)}(\xi_n^{(p_n)}), \tag{A.28}$$

where we have defined $\mathbf{p}$ by:

$$\begin{aligned} p_{\pi(i)} &= 0,\ \forall i \in \{1,...,R+S\}, \\ p_{\pi(i)} &= 1,\ \forall i \in \{R+S+1,...,\mathsf{N}\}, \end{aligned} \tag{A.29}$$

then it holds:

$$\langle h_1, ..., h_{\mathsf{N}}|T_1^{(K)}(\lambda)|t\rangle = T_{1,\mathbf{k}}^{(K^{(a)},\infty)}(\lambda)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} \langle h_1, ..., h_{\mathsf{N}}|t\rangle + \sum_{n=1}^{R} f_{\pi(n),\mathbf{p}}^{(a,1)}(\lambda)\langle h_1, ..., h_{\mathsf{N}}|T_1^{(K)}(\xi_{\pi(n)})|t\rangle + \sum_{n=R+1}^{\mathsf{N}} f_{\pi(n),\mathbf{p}}^{(a,1)}(\lambda)\langle h_1, ..., h_{\mathsf{N}}|T_1^{(K)}(\xi_{\pi(n)}/q)|t\rangle \tag{A.30}$$

then by using in the second line the identity (3.12) and (3.13) in the third line we get:

$$\langle h_1, ..., h_{\mathsf{N}}|T_1^{(K)}(\lambda)|t\rangle = \left(T_{1,\mathbf{k}}^{(K^{(a)},\infty)}(\lambda)\Big|_{\mathsf{N}_1=l,\mathsf{N}_2=m} + \sum_{n=1}^{\mathsf{N}} f_{\pi(n),\mathbf{p}}^{(a,1)}(\lambda)t_1(\xi_{\pi(n)}^{(p_{\pi(n)})}) \right) \langle h_1, ..., h_{\mathsf{N}}|t\rangle \tag{A.31}$$

$$= t_1(\lambda)\langle h_1, ..., h_{\mathsf{N}}|t\rangle, \tag{A.32}$$

which complete the proof of our theorem. □

B Appendix B

In this appendix, we provide a proof of the discrete SoV characterization of the transfer matrix spectrum given in Theorem 5.1 bypassing the computation of the transfer matrix action in the SoV basis. The proof is presented bellow in the case of the rational fundamental representations of $Y(\widehat{gl_n})$. Then one can either use the argument that the fundamental evaluation representations of $\mathcal{U}_q(\widehat{gl_n})$ lead under the rational limit to the rational ones, so inferring that the same result has to hold for the trigonometric case too for almost any values of the parameters. Otherwise one can just repeat the same type of proof directly in the trigonometric case, only taking into account that the asymptotic behavior for the trigonometric case are not central in the full representation space but only in the common eigenspaces of the operators N_i. Moreover, the case of non-fundamental representation can be handled similarly. In fact, the proof of the Theorem 2.3 of our third paper [3] can be seen as the first step in the proof by induction for these non-fundamental representations.

Proof of rational version of Theorem 5.1. For the fundamental representations of $Y(gl_n)$, the quantum separation of variable characterization of the first transfer matrix spectrum reads

$$\Sigma_{T^{(K)}} = \left\{ t_1(\lambda) : t_1(\lambda) = t_1^{(K,\{x\})}(\lambda), \quad \forall\{x_1, ..., x_{\mathsf{N}}\} \in S_T \right\}, \tag{B.1}$$

in terms of the solutions to the following system S_T of N polynomial equations of degree n:

$$x_a t_{n-1}^{(K,\{x\})}(\xi_a - \eta) = \det K \; q\text{-}\det M^{(I)}(\xi_a), \tag{B.2}$$

in N unknown $\{x_1, ..., x_{\mathsf{N}}\}$, where we recall the definitions used in our second paper [1]:

$$t_1^{(K,\{x\})}(\lambda) = \operatorname{tr} K \prod_{a=1}^{\mathsf{N}}(\lambda - \xi_a) + \sum_{a=1}^{\mathsf{N}} g_{a,\mathbf{h}=0}^{(1)}(\lambda)x_a, \tag{B.3}$$

and:

$$t_{m+1}^{(K,\{x\})}(\lambda)=\prod_{b=1}^{\mathsf{N}}\prod_{r=1}^{m}(\lambda-\xi_b-r\eta)\left[T_{m+1,\mathbf{h}=\mathbf{0}}^{(K,\infty)}(\lambda)+\sum_{a=1}^{\mathsf{N}}g_{a,\mathbf{h}=\mathbf{0}}^{(m+1)}(\lambda)x_a t_m^{(K,\{x\})}(\xi_a-\eta)\right], \tag{B.4}$$

for any $m\in\{1,...,n-2\}$, and

$$T_{m,\mathbf{h}}^{(K,\infty)}(\lambda) = \operatorname{tr}_{1,...,m}\left[P_{1,...,m}^{-}K_1K_2\cdots K_m\right]\prod_{b=1}^{\mathsf{N}}(\lambda-\xi_b^{(h_n)}), \tag{B.5}$$

$$g_{a,\mathbf{h}}^{(m)}(\lambda) = \prod_{b\neq a,b=1}^{\mathsf{N}}\frac{\lambda-\xi_b^{(h_b)}}{\xi_a^{(h_a)}-\xi_b^{(h_b)}}\prod_{b=1}^{\mathsf{N}}\prod_{r=1}^{m-1}\frac{1}{\xi_a^{(h_a)}-\xi_b^{(-r)}}. \tag{B.6}$$

Here we are interested in giving a proof of this characterization bypassing the computation of the action of the first transfer matrix in the SoV basis. The fact that any eigenvalue defines a solutions of this system follows from the fusion relations. So the only nontrivial thing to show is that indeed any solution of the above system defines one eigenvalue. The Theorem of Bežout[8] states that the above system of polynomial equations admits n^{N} solutions if the N polynomials, defining the system, have no common components[9]. The transfer matrix, being diagonalizable and with simple spectrum, has exactly n^{N} distinct eigenvalues and so, under the condition of no common components, there are indeed exactly n^{N} distinct solutions to the above system and each one is uniquely associated to a transfer matrix eigenvalue.

We have to show now that the condition of no common components indeed holds for almost any values of the parameters. The proof of this statement can be done by induction on $n-1$ the rank of the Yang-Baxter algebra. Let us start with the rank 1 case, i.e. $n=2$ and fundamental representations of $Y(gl_2)$. Here, we fix the eigenvalue of the twist matrix to be $\mathsf{k}_1\neq 0$ and $\mathsf{k}_2=0$, then the system of equations reads:

$$t_1^{(K,\{x\})}(\xi_a)t_1^{(K,\{x\})}(\xi_a-\eta)=x_a t_1^{(K,\{x\})}(\xi_a-\eta)=\det K\ q\text{-}\det M^{(I)}(\xi_a)=0, \tag{B.7}$$

now taking into account that by definition $t_1^{(K,\{x\})}(\lambda)$ is a degree N polynomial in λ and that it holds:

$$\xi_a^{(h)}\neq\xi_b^{(k)}\quad\forall h,k\in\{0,1\},a\neq b\in\{1,...,\mathsf{N}\}, \tag{B.8}$$

then a solution to the system can be obtained iff for any $a\in\{1,...,\mathsf{N}\}$ there exists a unique $h_a\in\{0,1\}$ such that $t_1^{(K,\{x\})}(\xi_a^{(h_a)})=0$, or equivalently:

$$t_{1,\mathbf{h}}^{(K,\{x\})}(\lambda)=\mathsf{k}_1\prod_{a=1}^{\mathsf{N}}(\lambda-\xi_a^{(h_a)}). \tag{B.9}$$

So we have that the system has exactly 2^{N} distinct solutions associated to the 2^{N} distinct N-upla $\mathbf{h}=\{h_{1\leq n\leq\mathsf{N}}\}$ in $\bigotimes_{n=1}^{\mathsf{N}}\{0,1\}$. So there are no common components for $\mathsf{k}_1\neq 0$ and $\mathsf{k}_2=0$, and being the polynomials defining the system (B.7) also polynomial in twist matrix eigenvalues we infer that this statement is true for almost any choice of k_1 and k_2. So we have proven our statement for $n=2$.

[8]See for example William Fulton (1974). Algebraic Curves. Mathematics Lecture Note Series. W.A. Benjamin.

[9]Indeed, if there are common components the system admits instead an infinite number of solutions.

Let us now prove it for $n = 3$, we fix here the twist matrix eigenvalues as it follows $\mathsf{k}_1 \neq 0$, $\mathsf{k}_2 \neq 0$, $\mathsf{k}_2 \neq \mathsf{k}_1$ and $\mathsf{k}_3 = 0$, then the system of equations reads:

$$x_a t_2^{(K,\{x\})}(\xi_a - \eta) = \det K \; q\text{-}\det M^{(I)}(\xi_a) = 0, \tag{B.10}$$

where by definition it holds

$$x_a t_1^{(K,\{x\})}(\xi_a - \eta) = t_2^{(K,\{x\})}(\xi_a), \tag{B.11}$$

so that it holds too

$$t_2^{(K,\{x\})}(\xi_a) t_2^{(K,\{x\})}(\xi_a - \eta) = 0, \tag{B.12}$$

now taking into account that by definition $t_2^{(K,\{x\})}(\lambda)$ is a degree $2\mathsf{N}$ polynomial in λ, zero in the points $\xi_a + \eta$ for any $a \in \{1, ..., \mathsf{N}\}$, it follows that a solution to the system (B.12) can be obtained iff for any $a \in \{1, ..., \mathsf{N}\}$ there exists a unique $h_a \in \{0, 1\}$ such that $t_2^{(K,\{x\})}(\xi_a^{(h_a)}) = 0$, or equivalently:

$$t_{2,\mathbf{h}}^{(K,\{x\})}(\lambda) = \mathsf{k}_1 \mathsf{k}_2 \prod_{a=1}^{\mathsf{N}} (\lambda - \xi_a - \eta)(\lambda - \xi_a^{(h_a)}). \tag{B.13}$$

So that the system (B.12) has exactly 2^{N} distinct solutions associated to the 2^{N} distinct N-upla $\mathbf{h} = \{h_{1 \leq n \leq \mathsf{N}}\}$ in $\bigotimes_{n=1}^{\mathsf{N}} \{0, 1\}$. Now for any fixed $\mathbf{h} \in \bigotimes_{n=1}^{\mathsf{N}} \{0, 1\}$ we can define a permutation $\pi_{\mathbf{h}} \in S_{\mathsf{N}}$ and a nonnegative integer $m_{\mathbf{h}} \leq \mathsf{N}$ such that:

$$h_{\pi_{\mathbf{h}}(a)} = 0 \;\; \forall a \in \{1, ..., m_{\mathbf{h}}\} \;\text{ and }\; h_{\pi_{\mathbf{h}}(a)} = 1 \;\; \forall a \in \{m_{\mathbf{h}} + 1, ..., \mathsf{N}\}. \tag{B.14}$$

It is easy to remark now that fixed $\mathbf{h} \in \bigotimes_{n=1}^{\mathsf{N}} \{0, 1\}$ then (B.11), for $a \in \{1, ..., m_{\mathbf{h}}\}$, and (B.10) are satisfied iff it holds:

$$x_{\pi_{\mathbf{h}}(a)} = t_1^{(K,\{x\})}(\xi_{\pi_{\mathbf{h}}(a)}) = 0 \;\; \forall a \in \{1, ..., m_{\mathbf{h}}\}. \tag{B.15}$$

Indeed, if this is not the case for a given $b \in \{1, ..., m_{\mathbf{h}}\}$, then the (B.10) implies $t_{2,\mathbf{h}}^{(K,\{x\})}(\xi_{h_{\pi_{\mathbf{h}}(b)}} - \eta) = 0$ which is not compatible with our choice of $t_{2,\mathbf{h}}^{(K,\{x\})}(\lambda)$. So, for any fixed $\mathbf{h} \in \bigotimes_{n=1}^{\mathsf{N}} \{0, 1\}$, we are left with the requirement to satisfy the fusion equation (B.11) for $a \in \{m_{\mathbf{h}} + 1, ..., \mathsf{N}\}$ which results in the following system of equation:

$$t_1^{(K,\{x\},\mathbf{h})}(\xi_{\pi_{\mathbf{h}}(a)}) t_1^{(K,\{x\},\mathbf{h})}(\xi_{\pi_{\mathbf{h}}(a)} - \eta) = t_{2,\mathbf{h}}^{(K,\{x\})}(\xi_{\pi_{\mathbf{h}}(a)}), \;\; \forall a \in \{m_{\mathbf{h}} + 1, ..., \mathsf{N}\}, \tag{B.16}$$

where $t_1^{(K,\{x\},\mathbf{h})}(\lambda)$ is a degree N polynomial in λ of the form (B.3) with the $m_{\mathbf{h}}$ zeros given by (B.3). Then let us define the following degree $\mathsf{N} - m_{\mathbf{h}}$ polynomial in λ:

$$\bar{t}_1^{(K,\{x\},\mathbf{h})}(\lambda) = t_1^{(K,\{x\},\mathbf{h})}(\lambda) / \prod_{a=1}^{m_{\mathbf{h}}} (\lambda - \xi_{\pi_{\mathbf{h}}(a)}), \tag{B.17}$$

and the degree $2(\mathsf{N} - m_{\mathbf{h}})$ polynomial in λ:

$$\bar{t}_{2,\mathbf{h}}^{(K,\{x\})}(\lambda) = t_{2,,\mathbf{h}}^{(K,\{x\})}(\lambda) / \prod_{a=1}^{m_{\mathbf{h}}} \left[(\lambda - \xi_{\pi_{\mathbf{h}}(a)})(\lambda - \xi_{\pi_{\mathbf{h}}(a)} - \eta)\right] \tag{B.18}$$

$$= \mathsf{k}_1 \mathsf{k}_2 \prod_{a=1+m_{\mathbf{h}}}^{\mathsf{N}} (\lambda - \xi_{\pi_{\mathbf{h}}(a)} - \eta)(\lambda - \xi_{\pi_{\mathbf{h}}(a)} + \eta), \tag{B.19}$$

the previous system of equations simplifies to:

$$\bar{t}_1^{(K,\{x\},\mathbf{h})}(\xi_{\pi_{\mathbf{h}}(a)})\bar{t}_1^{(K,\{x\},\mathbf{h})}(\xi_{\pi_{\mathbf{h}}(a)}-\eta)=\bar{t}_{2,\mathbf{h}}^{(K,\{x\})}(\xi_{\pi_{\mathbf{h}}(a)}),\ \forall a\in\{m_{\mathbf{h}}+1,...,\mathsf{N}\}. \tag{B.20}$$

Such a system coincides with the system associated to the case $n=2$ for a lattice with $\mathsf{N}-m_{\mathbf{h}}$ sites and inhomogeneities $\xi_{\pi_{\mathbf{h}}(a)}$ with $a\in\{m_{\mathbf{h}}+1,...,\mathsf{N}\}$. Indeed, $\bar{t}_{2,\mathbf{h}}^{(K,\{x\})}(\lambda)$ is just the quantum determinant for such a lattice associated to the 2×2 twist matrix with distinct non-zero eigenvalues $\mathsf{k}_1\neq 0$ and $\mathsf{k}_2\neq 0$ and $\bar{t}_1^{(K,\{x\},\mathbf{h})}(\lambda)$ has the functional form of a transfer matrix eigenvalue with asymptotic given by the trace of this twist matrix. Now, we can use our result for $n=2$ to state that this system has exactly $2^{\mathsf{N}-m_{\mathbf{h}}}$ distinct solutions, which allows to count the full set of solutions to our original system:

$$\sum_{m=0}^{\mathsf{N}}2^{\mathsf{N}-m}\begin{pmatrix}\mathsf{N}\\ m\end{pmatrix}=3^{\mathsf{N}}, \tag{B.21}$$

where we have used that for any fixed $m\in\{1,...,\mathsf{N}\}$ the number of $\mathbf{h}\in\bigotimes_{n=1}^{\mathsf{N}}\{0,1\}$ such that $m_{\mathbf{h}}=m$ is exactly given by the binomial symbol:

$$\begin{pmatrix}\mathsf{N}\\ m\end{pmatrix}=\frac{\mathsf{N}!}{(\mathsf{N}-m)!m!}. \tag{B.22}$$

So we proved that the system has exactly 3^{N} distinct solutions and no common components for $\mathsf{k}_1\neq 0$, $\mathsf{k}_2\neq 0$ and $\mathsf{k}_3=0$, and being the polynomials defining the system (B.10) also polynomial in twist matrix eigenvalues we can infer that this statement is true for almost any choice of three distinct eigenvalues k_1, k_2 and k_3. So we have proven our statement for $n=3$.

At this point it is easy to understand how to implement the proof by induction, i.e. we assume that the statement is proven for the rank $n-1$ case and we prove it for the rank n case and this is done in the case of a diagonalizable and simple spectrum $(n+1)\times(n+1)$ twist matrix with pairwise distinct eigenvalues $\mathsf{k}_a\neq 0$, for any $a\in\{1,...,n\}$, and $\mathsf{k}_{n+1}=0$. Then following similar steps to those illustrated above, we see that the function $t_n^{(K,\{x\})}(\lambda)$ is forced to take the form

$$t_{n,\mathbf{h}}^{(K,\{x\})}(\lambda)=\prod_{a=1}^{\mathsf{N}}\mathsf{k}_a(\lambda-\xi_a^{(h_a)})\prod_{r=1}^{n-1}(\lambda-\xi_a-r\eta), \tag{B.23}$$

associated to the 2^{N} distinct N-upla $\mathbf{h}=\{h_{1\leq n\leq\mathsf{N}}\}$ in $\bigotimes_{n=1}^{\mathsf{N}}\{0,1\}$ and that for any fixed $\mathbf{h}$ the system is reduced to that associated to the case of rank $n-1$ with general diagonalizable and simple spectrum $n\times n$ twist matrix with eigenvalues $\mathsf{k}_a\neq 0$, for any $a\in\{1,...,n\}$, on a number of site $\mathsf{N}-m_{\mathbf{h}}$. Then using the induction we know that this system admits $n^{\mathsf{N}-m_{\mathbf{h}}}$ distinct solutions for any such $\mathbf{h}$ and that for any fixed $m\in\{1,...,\mathsf{N}\}$ the number of $\mathbf{h}\in\bigotimes_{n=1}^{\mathsf{N}}\{0,1\}$ such that $m_{\mathbf{h}}=m$ is exactly given by the binomial symbol (B.22). So that the total counting gives:

$$\sum_{m=0}^{\mathsf{N}}n^{\mathsf{N}-m}\begin{pmatrix}\mathsf{N}\\ m\end{pmatrix}=(n+1)^{\mathsf{N}}, \tag{B.24}$$

which proves the no common component statement for the rank n case too when we repeat the polynomiality argument of the dependence w.r.t. the twist matrix eigenvalues. □

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