

Manifold-Topology from K-Causal Order

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Abstract. To a significant extent, the metrical and topological properties of spacetime can be described purely order-theoretically. The K^+ relation has proven to be useful for this purpose, and one could wonder whether it could serve as the primary causal order from which everything else would follow. In that direction, we prove, by defining a suitable order-theoretic boundary of $K^+(p)$, that in a K -causal spacetime, the manifold-topology can be recovered from K^+ . We also state a conjecture on how the chronological relation I^+ could be defined directly in terms of K^+ .

Keywords: Causal structure, Causal order, K-causality, Manifold topology

1. Introduction

There is much information in the causal structure of spacetime, including information about the topology, differentiable structure, and metric (indeed the full metric up to a conformal factor) [1, 2, 3]. Accordingly, a field of study that could be called global causal analysis has grown up to utilize the causal structure directly, beginning with the well known singularity theorems [4, 5]. In [6] and [7] a positive energy theorem was proven using arguments similar to those that feature in the proofs of the singularity theorems. In [8] the authors show that the causality relation J^+ turns a globally hyperbolic spacetime into a bicontinuous poset, which in turn allows one to define an intrinsic topology called the interval topology that turns out to be the manifold topology. Moreover, several approaches to understanding relativity theory, some of them as old as relativity itself, make primary use of causal structure ([9, 10]). Applications of causal structure are far-reaching and extend beyond the description of spacetime at the classical level. Causal set theory is an approach to quantum gravity for which a type of causal structure is fundamental. The causal set itself is a discrete set of elements structured as a partial order, and its defining order-relation corresponds macroscopically to the causal order of spacetime. (See [11, 12, 13] for more details on causal set theory.)

A natural question is how far can one get with describing manifold properties using nothing more than order-theoretic concepts. This is of interest generally, but also in the context of theories such as causal set theory. Workers in other fields, like computer science, have been interested in this question as well [8]. But if one wishes to conceive of order-theoretic relationships as the foundation of spacetime structure, it seems simplest to have in mind a single order-relation, rather than the two or more that one sees in most presentations.‡ The relation K^+ , defined in [7], was originally conceived for this purpose. A closed and transitive generalization of I^+ and J^+ , it has properties that make it suitable to serve as the primary causal order of a spacetime. Because it is transitive (and acyclic in a K -causal spacetime) it lends itself naturally to order-theoretic reasoning. And because it is topologically closed it avoids the problems with supremums that would arise if one took a relation like I^+ as basic.

The relation K^+ was also designed to be a tool of use when one attempts to generalize causal analysis to C^0 metrics that may fail to be everywhere smooth and invertible. In [7] Sorkin and Woolgar used K^+ to extend certain results like the compactness of the space of causal curves from C^2 Lorentzian metrics to C^0 Lorentzian metrics, as needed for the positive-energy proof in [6]. In [14] Dowker, Garcia, and Surya showed that K^+ is robust against the addition and subtraction of isolated points or metric degeneracies. This allows such degeneracies to be present in metrics that contribute to the gravitational path-integral, thus enabling one to include in the space of histories topology-changing spacetimes (Lorentzian cobordisms) which contain such degeneracies. This is of course a very interesting physical consideration. For some recent articles on K^+ see [15] and [16].

Another line of thought, which comes from causal sets, also points to the desirability of a sole order-relation (and to K^+ as a reasonable choice thereof). In a fundamentally discrete context the fine topological differences that distinguish I , J , and K from each other lose their meaning. On the other hand, the discrete-continuum correspondence is normally introduced (at the kinematic level) in terms of so-called *sprinklings* which place points at random in a Lorentzian manifold \mathcal{M} via a Poisson process with a constant density of $\rho = 1$ in natural units. By definition, the probability of sprinkling N points into a region with spacetime volume V is $P(N) = \frac{(\rho V)^N}{N!} e^{-\rho V}$. This produces a causal set whose elements are the sprinkled points and whose partial order relation is most simply taken to be that of the manifold's causal order restricted to the sprinkled points. Since with a Poisson process, the probability for two sprinkled points to be lightlike related vanishes, it makes no difference which continuum order one uses to induce an order on the sprinkled points. However, it is sometimes convenient to consider, instead of a random sprinkling, something like a “diamond lattice” in two-dimensional Minkowski space, in which case either J^+ or K^+ would be the most useful choice. In all such cases

‡ The two most common are I^+ and J^+ . The “chronological future” of a point-event p , $I^+(p)$, is the set of events accessible from p by future-directed timelike curves starting from p . The “causal future” $J^+(p)$ is the set of points accessible from p by future-directed timelike or null curves starting from p . The past sets, $I^-(p)$ and $J^-(p)$, are defined analogously, with future replaced by past.

one loses nothing by thinking of K^+ as the basic continuum-order. One sees again how it is more natural to work with only one causal relation, i.e. one does not distinguish between lightlike and timelike related pairs of elements, only between causally related and unrelated pairs.

If K^+ is to be taken as primitive, then it must be possible in particular to recover the manifold-topology from it, something which was not addressed in [7]. We could perhaps obtain the manifold-topology indirectly by first defining I^+ in terms of K^+ , but we will not do this herein (although we will provide a conjecture suggesting how it might be done). Instead we proceed directly from K^+ to the topology by defining an *order-theoretic boundary* of $K^+(p)$ and demonstrating that it coincides with the topological boundary. Removing it, we obtain a family of open sets $A(p, q)$ from which the topology can be reconstructed.

We begin in Section 2 by reviewing the definition and properties of K^\pm . We then introduce the derived sets $A^\pm(p)$, which we use throughout this paper. In Section 3 we show that $A^\pm(p)$ is open and locally equivalent to $I^\pm(p)$. From this it follows that the order-interval $A(p, q)$ is locally equal to $I(p, q)$, which completes the proof that the K -causal order yields the manifold topology. In Section 4 we state a conjecture for how to obtain $I^+(p)$ more directly from $K^+(p)$. The appendix collects the lemmas from [7] that we use in this paper.

Our results do not depend on the spacetime-dimension of the manifold \mathcal{M} that we work in.

2. K-Causality

2.1. Some Definitions

Definition 2.1. $K^+ := \prec$ (respectively K^-) is the smallest relation that contains I^+ (respectively I^-) and is both transitive§ and topologically closed [7].

Regarded as a subset of $\mathcal{M} \times \mathcal{M}$, the relation K^+ can be obtained by intersecting all the closed and transitive sets R_i that include I^+ [7]:

$$K^+ = \bigcap_i R_i, \quad \text{where} \quad \forall i, I^+ \subset R_i. \quad (1)$$

By $K^+(p, \mathcal{M})$ or $K^+(p)$ we denote all the points q such that $p \prec q$, where $p, q \in \mathcal{M}$. Let \mathcal{O} be an open subset of \mathcal{M} . For $q \in K^+(p, \mathcal{O})$ we write $p \prec_{\mathcal{O}} q$. Figure 1 shows an example of $K^+(p)$ and how it differs from $I^+(p)$ and $J^+(p)$. In the figure, $q, r, s \in K^+(p)$ while $q \in J^+(p)$, $r, s \notin J^+(p)$ (and also $q \in I^+(p)$, $r, s \notin I^+(p)$).

An open set is *K-causal* if and only if \prec induces a (reflexive) partial ordering on it, in other words iff \prec restricted to the open set is asymmetric.¶

§ $p \prec q \prec r \Rightarrow p \prec r$

¶ Minguzzi has proven [17] that K -causality coincides with stable causality, and that in consequence, K^+ coincides with the the Seifert relation J_S^+ provided that K -causality is in force.

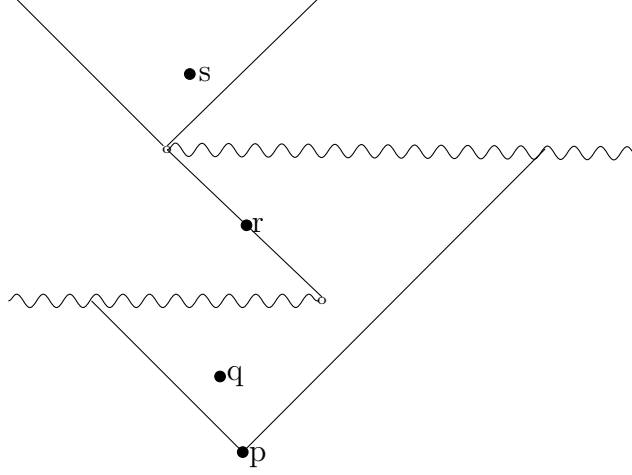


Figure 1. An example of $K^+(p)$. The wiggly lines are removed from the manifold. $q, r, s \in K^+(p)$ while $q \in J^+(p)$, $r, s \notin J^+$.

Definition 2.2. The order theoretic or causal boundary of $K^+(p)$, denoted $\partial K^+(p)$, is the set indicated by

$$\partial K^+(p) = \{x \in K^+(p) | x = \sup\{y_i\} \text{ for some } y_i \notin K^+(p)\}. \quad (2)$$

More precisely, a point x of $K^+(p)$ belongs to $\partial K^+(p)$ iff it is the supremum with respect to \prec of an increasing sequence[¶] of points y_i all belonging to the complement of $K^+(p)$. (Intuitively this says that one can approach x arbitrarily closely from outside of $K^+(p)$, which is a precise order-theoretic counterpart of the topological definition of boundary.)

We next define the open sets $A^+(p)$ which we will use in the next section to recover the manifold topology.

Definition 2.3. $A^+(p)$ is the set $K^+(p)$ without its causal boundary defined in (2):

$$A^+(p) = K^+(p) \setminus \partial K^+(p). \quad (3)$$

By definition, $A^+(p)$ is open in the order-theoretic sense. We will see in the next section that the order-theoretic boundary coincides with the topological one, therefore making $A^+(p)$ open in the topological sense as well.

Definition 2.4. The set $A(p, q)$ is

$$A(p, q) = A^+(p) \cap A^-(q). \quad (4)$$

The set $A(p, q)$ is a kind of “open order-interval”. Figure 2 shows an example.

3. Manifold Topology from the sets $A(p, q)$

Let us demonstrate that the sets $A(p, q)$ are locally the same as the order-intervals $I(p, q)$. To that end, we first prove that the future- and past- sets $A^\pm(p)$ are topologically open and locally equivalent to $I^\pm(p)$.

[¶] Instead of “increasing sequence”, one could say “directed set”

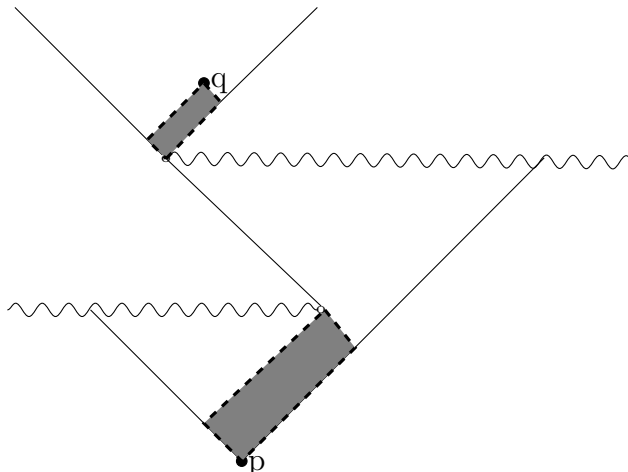


Figure 2. The region in gray is an example of a K-causal open interval $A(p, q)$.

To show that $A^+(p)$ is open, it suffices to show that the causally defined boundary $\partial K^+(p)$ that we removed from $K^+(p)$ to yield $A^+(p)$ is also a topological boundary.

We will assume henceforth that \mathcal{M} is K-causal and without boundary.

Lemma 3.1. $x = \sup y_i \Rightarrow x = \lim_{i \rightarrow \infty} \{y_i\}$ in the topological sense. Hence if x is (with respect to K^+) the supremum of an increasing sequence (or directed set) of points y_i then x is also the limit of $\{y_i\}$ with respect to the manifold topology.

Proof. Let the increasing sequence $y_1 \prec y_2 \prec y_3 \dots$ have x as its supremum, and let U be an arbitrarily small open neighborhood of x . From Lemma A.4 we can assume without loss of generality that U is K-convex.⁺ It suffices to prove that the y_i eventually enter U (after which they will necessarily remain in U because of the latter's convexity, and because $y_i \prec x$).

Now there are two possibilities. Either the y_i enter U or they don't. In the former case we are done, so suppose the latter case, and let $B = \text{Fr}(U)$ be the topological (not causal) boundary of U . The symbol 'Fr' stands for "Frontier", and we follow the convention of [18] in referring to the topological boundary this way. Since U is arbitrarily small and B is closed, we can assume without loss of generality that it is compact. By Lemma A.3 there then exist points z_i in B such that $(\forall i) y_i \prec z_i \prec x$.

Recalling that B is compact, and passing if necessary to a subsequence, we can assume without loss of generality that the z_i converge to some point z_∞ of B . Then since the relation \prec is closed, and since all of the $z_i \prec x$, we see immediately that $z_\infty \prec x$. On the other hand, if (for fixed i) $k > i$, then we have $y_i \prec y_k \prec z_k$, from which follows $y_i \prec z_k$. Therefore we see in the same way from $y_i \prec z_k$ that $y_i \prec z_\infty \forall i$. But by the definition of supremum, x is the least point of \mathcal{M} for which this holds, hence $x \prec z_\infty$, which together with $z_\infty \prec x$ implies that they are equal. This is a contradiction, since

⁺ A set or subset is said to be K-convex if and only if it contains the closed interval $K(p, q) = K^+(p) \cap K^-(q)$ between every pair of its elements.

$x \in U$ where U is an open set, whereas $z_\infty \in B$ and $B \cap U = \emptyset$. Therefore $\{y_i\}$ must enter U and in doing so, they converge (topologically) to x , as desired. \square

Lemma 3.2. *Every point p of \mathcal{M} has a neighborhood in which J^+ and K^+ agree.*

Proof. Given our standing assumption that \mathcal{M} is K -causal, Lemma A.4 tells us that p has an arbitrarily small K -convex neighborhood U , and then Lemmas A.1-A.2 (with $\mathcal{O} = \mathcal{M}$) tell us that K^+ restricted to U coincides with K^+ relativized to U (i.e. computed as if U were the whole spacetime). But we know that in a small enough U , the relation J^+ is closed*, transitive, and includes I^+ . It follows (from the definition of K^+) that $J^+(U)$ includes $K^+(U)$, and therefore coincides with it (since* $J^+(U)$ is the closure of $I^+(U)$). Notice here, that we can take J^+ to be $J^+(U)$. \square

Theorem 3.3. *The topological boundary of $K^+(p)$ equals its causal boundary $\partial K^+(p)$, $\forall p \in \mathcal{M}$.*

Proof. We use the criterion that x is in $\text{Fr}(K^+(p))$, the topological boundary of $K^+(p)$, iff every neighborhood of x contains points both inside and outside of $K^+(p)$.

First let's show that $\partial K^+(p) \subseteq \text{Fr}(K^+(p))$. Let x be any point of $\partial K^+(p)$. Since x itself is in $K^+(p)$, it suffices to show that every neighborhood of x contains points in the complement of $K^+(p)$. This follows directly from Lemma 3.1 and the definition of $\partial K^+(p)$.

Conversely let us show that $\text{Fr}(K^+(p)) \subseteq \partial K^+(p)$. Let x be any point of $\text{Fr}(K^+(p))$.

First of all, we easily check that $x \in K^+(p)$, because the topological boundary of any set lies within its closure, and because $K^+(p)$ is closed.

Now let U be a small K -convex open neighborhood of x (which exists by Lemma A.4). By Lemmas A.1-A.2, we can reason as if U were all of \mathcal{M} . And we also know from Lemma 3.2 that if U is sufficiently small then within it, K^+ and J^+ coincide.

It's also clear that $I^-(x)$ must be disjoint from $K^+(p)$. Otherwise choose any $z \in K^+(p) \cap I^-(x)$ and notice that (since $I^+ \subseteq K^+$) $I^+(z) \subseteq K^+(p)$ would be an open neighborhood of x in $K^+(p)$, hence x would be in $K^+(p)$'s topological interior and not in $\text{Fr}(K^+(p))$.

Obviously $I^-(x)$ will contain a timelike increasing sequence of points y_i converging topologically to x , and this sequence will be disjoint from $K^+(p)$ since $I^-(x)$ is. It remains to be proven that $x = \sup\{y_i\}$.

That x bounds the y_i from above is obvious. And because the relation ' \prec ' is topologically closed, we have for any other upper bound z that $z \succ y_i \rightarrow x \Rightarrow z \succ x$. Therefore x is a least upper bound, as required. \square

Lemma 3.4. *$A^+(p)$ is open. In fact it is the interior of $K^+(p)$.*

* See Proposition 4.5.1 in [4] for a proof of this.

Proof. As defined above in (3), $A^+(p)$ is what remains of $K^+(p)$ after we remove its order-theoretic boundary $\partial K^+(p)$. But in the theorem just proven we have seen that $\partial K^+(p)$ is also the topological boundary of $K^+(p)$, and removing the topological boundary of any set whatsoever produces its interior, which by definition is open. \square

It follows immediately that the sets $A(p, q)$ are open for all p and q . We claim furthermore that $A^+(p)$ coincides locally with $I^+(p)$, whence $A(p, q)$ coincides locally with $I(p, q)$. This follows from Lemmas 3.2 and 3.4, which inform us that locally $A^+(p)$ is the interior of $K^+(p)$ and $K^+(p) = J^+(p)$. To establish our claim, then, simply notice that locally the interior of $J^+(p)$ is $I^+(p)$. \sharp

We now have all the pieces we need to derive the manifold topology from the order ‘ \prec ’, which we do by proving that the open sets $A(p, q)$ furnish a basis for the topology of \mathcal{M} in the sense that every open subset of \mathcal{M} is a union of sets of the form $A(p, q)$. For this, it suffices in turn that:

Criterion: for any $x \in \mathcal{M}$ and any open set U containing x , we can find p and q such that $x \in A(p, q) \subseteq U$.

Clearly this criterion is local in the sense that it’s enough for it to hold for U being arbitrarily small.

Theorem 3.5. *The sets $A(p, q)$ furnish a basis for the manifold-topology.*

Proof. Since K -causality is in force, strong causality also holds [17], whereby the manifold-topology is the same as the Alexandrov topology, for which by definition the sets $I(p, q)$ are a basis. (See theorem 4.24 in [19].) But because locally $I(p, q) = A(p, q)$, as we have just seen, the sets $A(p, q)$ are also a basis.

To summarize: because the $I(p, q)$ constitute a basis they satisfy the criterion stated above, and because this criterion is purely local, the sets $A(p, q)$, which locally coincide with the $I(p, q)$, also satisfy it. \square

Remark. As seen in figure 2, there will in general be pairs of points, p, q , for which the set $A(p, q)$ does not agree with $I(p, q)$. Such sets $A(p, q)$ are still included in our basis, but this does no harm, since by definition, any basis for a topology remains a basis when more sets are added to it, provided that the additional sets are also open, which of course the $A(p, q)$ are.

4. From $K^+(p)$ to $I^+(p)$ and $J^+(p)$

Once we have access to the manifold topology, it is a relatively easy matter to define continuous curve, and from there to characterize I^\pm and J^\pm . Thus, for example, a curve could be the image of a continuous function from $[0, 1] \subseteq \mathbb{R}$ into \mathcal{M} , and we might

\sharp This follows from the discussion on pages 33-34 and 103-105 of [4], which establishes that the exponential map at p induces a diffeomorphism between a neighborhood of p in \mathcal{M} and a neighborhood of 0 in Minkowski-space which preserves the sets $I^+(p)$ and $J^+(p)$.

define a causal (respectively timelike) curve as one which is linearly ordered by K^+ (respectively A^+).

Nevertheless, it might be nice to characterize I^+ and J^+ more directly in terms of K^+ . We conclude with a conjecture of that nature (concerning I^+).

Conjecture 4.1. $K^+(p) \setminus S = I^+(p)$, where $S = \{r \in K^+(p) \mid \text{every 'full chain' from } p \text{ to } r \text{ meets } \partial K^+(p)\}$

Here, by a “full chain from p to r ” we mean a subset C of \mathcal{M} containing p and r that is: *linearly ordered* by \prec (it is a chain); *order-theoretically closed* in the sense that it contains all its suprema and infima; and *dense* in the sense that it contains between any two of its points a third point $[(\forall x, y \in C)(\exists z \in C)(z \neq x, y \wedge x \prec z \prec y)]$.

Remark. We didn’t need to remove $\partial K^+(p)$ explicitly from $K^+(p)$, because it is automatically included within S .

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A. Lemmas from Sorkin-Woolgar (SW) [7]

In this appendix we collect the lemmas from [7] that we have used in this paper.

Lemma A.1. (Lemma 12 of SW): Let \mathcal{U} and \mathcal{O} be open subsets of \mathcal{M} and $\mathcal{U} \subseteq \mathcal{O}$. For $p, q \in \mathcal{U}$, $p \prec_{\mathcal{U}} q$ implies $p \prec_{\mathcal{O}} q$.

Lemma A.2. (Lemma 13 of SW): Let \mathcal{U} and \mathcal{O} be open subsets of \mathcal{M} and $\mathcal{U} \subseteq \mathcal{O}$. For $p, q \in \mathcal{U}$, $p \prec_{\mathcal{O}} q$ implies $p \prec_{\mathcal{U}} q$, if \mathcal{U} is causally convex relative to $\prec_{\mathcal{O}}$.

Lemma A.3. (Lemma 14 of SW): Let S be a subset of \mathcal{M} with compact boundary $\text{Fr}(S)$, and let $x \prec y$ with $x \in S$, $y \notin S$ (or vice versa). Then $\exists w \in \text{Fr}(S)$ such that $x \prec w \prec y$.

Lemma A.4. (Lemma 16 of SW): If \mathcal{M} is K -causal then every element of \mathcal{M} possesses arbitrarily small K -convex open neighborhoods (\mathcal{M} is locally K -convex).

References

- [1] Malament D B 1977 *Journal of Mathematical Physics* **18** 1399–1404
- [2] Hawking S W, King A R and Mccarthy P J 1976 *J. Math. Phys.* **17** 174–181
- [3] Geroch R, Kronheimer E H and Penrose R 1972 *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* **327** 545–567 ISSN 0080-4630 URL <http://rspa.royalsocietypublishing.org/content/327/1571/545>

- [4] Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space-Time* Cambridge Monographs on Mathematical Physics (Cambridge University Press)
- [5] Penrose R 1965 *Phys. Rev. Lett.* **14**(3) 57–59 URL <https://link.aps.org/doi/10.1103/PhysRevLett.14.57>
- [6] Penrose R, Sorkin R D and Woolgar E 1993 [arXiv:gr-qc/9301015](https://arxiv.org/abs/gr-qc/9301015)
- [7] Sorkin R D and Woolgar E 1996 *Class. Quant. Grav.* **13** 1971–1994 [arXiv:gr-qc/9508018](https://arxiv.org/abs/gr-qc/9508018)
- [8] Martin K and Panangaden P 2006 *Commun. Math. Phys.* **267** 563–586 [arXiv:gr-qc/0407094](https://arxiv.org/abs/gr-qc/0407094)
- [9] Robb A A 1914 *A Theory of Time and Space* (Cambridge: University Press)
- [10] Robb A A 1936 *Geometry Of Time And Space* (At The University Press)
- [11] Bombelli L, Lee J, Meyer D and Sorkin R 1987 *Phys. Rev. Lett.* **59** 521–524
- [12] Sorkin R D 2003 Causal sets: Discrete gravity *Lectures on quantum gravity. Proceedings, School of Quantum Gravity, Valdivia, Chile, January 4-14, 2002* pp 305–327 [arXiv:gr-qc/0309009](https://arxiv.org/abs/gr-qc/0309009)
- [13] Henson J 2006 [arXiv:gr-qc/0601121](https://arxiv.org/abs/gr-qc/0601121)
- [14] Dowker H F, Garcia R S and Surya S 2000 *Class. Quant. Grav.* **17** 4377–4396 [arXiv:gr-qc/9912090](https://arxiv.org/abs/gr-qc/9912090)
- [15] Saraykar R V and Janardhan S 2014 [arXiv:1411.1836](https://arxiv.org/abs/1411.1836)
- [16] Miller T 2017 *Universe* **3** 27 [arXiv:1702.00702](https://arxiv.org/abs/1702.00702)
- [17] Minguzzi E 2009 *Commun. Math. Phys.* **290** 239–248 [arXiv:0809.1214](https://arxiv.org/abs/0809.1214)
- [18] Dieudonné J 2009 *A History of Algebraic and Differential Topology, 1900 - 1960* Modern Birkhäuser Classics (Birkhäuser Boston) ISBN 9780817649074 URL <https://www.springer.com/la/book/9780817649067>
- [19] Penrose R 1972 *Techniques of Differential Topology in Relativity* (Society for Industrial and Applied Mathematics)