

The Loop Equation for Special Cubic Hodge Integrals

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Abstract

As the first step of proving the Hodge-FVH correspondence recently proposed in [19], we derive the Virasoro constraints and the Dubrovin–Zhang loop equation for special cubic Hodge integrals. We show that this loop equation has a unique solution, and provide a new algorithm for the computation of these Hodge integrals. We also observe the gap phenomenon for certain special cubic Hodge free energies.

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1 Introduction

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable algebraic curves of genus g with n distinct marked points, where g and n are non-negative integers satisfying the stability condition $2g - 2 + n > 0$. For $1 \leq k \leq n$ and $0 \leq j \leq g$, denote by ψ_k the first Chern class of the k -th tautological line bundle \mathbb{L}_k on $\overline{\mathcal{M}}_{g,n}$, and by λ_j the j -th Chern class of the Hodge bundle $\mathbb{E}_{g,n}$ on $\overline{\mathcal{M}}_{g,n}$. The rational numbers defined by the formula

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_{j_1} \cdots \lambda_{j_m}$$

are called the Hodge integrals. These numbers take zero value unless the degree-dimension counting matches, i.e.

$$i_1 + \cdots + i_n + j_1 + \cdots + j_m = 3g - 3 + n. \quad (1.1)$$

Denote by $\mathcal{C}_g(z) := \sum_{j=0}^g \lambda_j z^j$ the Chern polynomial of $\mathbb{E}_{g,n}$. We will be particularly interested in the following class of Hodge integrals defined via the cubic products of Chern polynomials,

called the cubic Hodge integrals:

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \mathcal{C}_g(-p) \mathcal{C}_g(-q) \mathcal{C}_g(-r), \quad (1.2)$$

where p, q, r are complex parameters. These integrals are called *special* if p, q, r satisfy the following *local Calabi–Yau condition*:

$$pq + qr + rp = 0. \quad (1.3)$$

These Hodge integrals are important in the localization technique of computing Gromov–Witten invariants for toric three-folds [14, 17, 22]. Their significance was also manifested by the Gopakumar–Mariño–Vafa conjecture regarding the Chern–Simons/string duality [13, 20].

Let $\mathcal{H} = \mathcal{H}(\mathbf{t}; p, q, r; \epsilon)$ be the cubic Hodge free energy defined by

$$\begin{aligned} \mathcal{H}(\mathbf{t}; p, q, r; \epsilon) &:= \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{H}_g(\mathbf{t}; p, q, r), \\ \mathcal{H}_g(\mathbf{t}; p, q, r) &:= \sum_{n \geq 0} \sum_{i_1, \dots, i_n \geq 0} \frac{t_{i_1} \cdots t_{i_n}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \mathcal{C}_g(-p) \mathcal{C}_g(-q) \mathcal{C}_g(-r). \end{aligned}$$

Here $\mathcal{H}_g(\mathbf{t}; p, q, r)$ is called the genus g part of the free energy \mathcal{H} . Then the exponential

$$e^{\mathcal{H}(\mathbf{t}; p, q, r; \epsilon)} =: Z_{\text{cubic}}(\mathbf{t}; p, q, r; \epsilon) =: Z_{\text{cubic}}$$

is called the cubic Hodge partition function. Clearly, $\mathcal{H}_g(\mathbf{t}; p, q, r) \in \mathbb{C}[p, q, r][[\mathbf{t}]]$. The genus zero free energy $\mathcal{H}_0(\mathbf{t})$ is actually independent of p, q, r and has the explicit expression

$$\mathcal{H}_0(\mathbf{t}) = \sum_{n \geq 3} \frac{1}{n(n-1)(n-2)} \sum_{i_1 + \cdots + i_n = n-3} \frac{t_{i_1}}{i_1!} \cdots \frac{t_{i_n}}{i_n!}.$$

Define

$$v(\mathbf{t}) := \partial_{t_0}^2 \mathcal{H}_0(\mathbf{t}) = \sum_{n \geq 1} \frac{1}{n} \sum_{i_1 + \cdots + i_n = n-1} \frac{t_{i_1}}{i_1!} \cdots \frac{t_{i_n}}{i_n!}. \quad (1.4)$$

It satisfies the following Riemann hierarchy:

$$\frac{\partial v}{\partial t_i} = \frac{v^i}{i!} \frac{\partial v}{\partial t_0}, \quad i \geq 0. \quad (1.5)$$

More generally, if one defines $w = \epsilon^2 \partial_{t_0}^2 \mathcal{H}(\mathbf{t}; p, q, r; \epsilon)$, then w satisfies an integrable hierarchy of Hamiltonian evolutionary PDEs [2, 3, 7], called the Hodge hierarchy for the special cubic Hodge integrals, which is a deformation of the Riemann hierarchy. The first member of this integrable hierarchy reads

$$w_{t_1} = w w_{t_0} + \frac{\epsilon^2}{12} (w_{t_0 t_0 t_0} - (p + q + r) w_{t_0} w_{t_0 t_0}) + O(\epsilon^4).$$

The Hodge-FVH correspondence is given by the following conjecture [19].

Conjecture 1.1 *The Hodge hierarchy for the special cubic Hodge integrals is equivalent, under a certain Miura type transformation, to the fractional Volterra hierarchy (FVH). Furthermore, the special cubic Hodge partition function gives a tau function of the FVH.*

For the case with $p = q$, the validity of the Hodge-FVH correspondence is implied by the Hodge-GUE correspondence established in [8, 9]. The goal of this and the subsequent paper [18] is to prove the Hodge-FVH correspondence. In the present paper, we derive the Dubrovin–Zhang loop equation by studying the Virasoro constraints for the special cubic Hodge partition function. We show that this loop equation together with the genus zero free energy uniquely determines the partition function. In the subsequent paper, we will show that the fractional Volterra hierarchy admits the same Virasoro constraints and that there exists a particular tau function of this integrable hierarchy which is uniquely determined by the same loop equation, and we prove in this way the Hodge-FVH correspondence.

From now on, we assume that p, q, r satisfy the local Calabi–Yau condition (1.3). The case with $p, q, r \in \mathbb{Q}$ is called *rational*. Note that the Virasoro constraints for the special cubic Hodge partition function in the general case are quite complicated. However, in the rational case, we find explicit expressions of the Virasoro constraints which lead to the Dubrovin–Zhang loop equation. It turns out that the loop equation for the general case can be deduced from the one for the rational case.

Let us first consider the rational case. Due to the symmetry property of the cubic Hodge integrals with respect to p, q, r , and the homogeneity property (deduced from (1.1))

$$\mathcal{H}_g(\mathbf{t}; \lambda p, \lambda q, \lambda r)|_{t_i \mapsto t_i \lambda^{i-1}} = \lambda^{3g-3} \mathcal{H}_g(\mathbf{t}; p, q, r),$$

we can assume that

$$p = \frac{1}{K_1}, \quad q = \frac{1}{K_2}, \quad r = -\frac{1}{h}, \quad (1.6)$$

where $K_1, K_2 \in \mathbb{N}$, $(K_1, K_2) = 1$ and $h := K_1 + K_2$. We denote, for $\ell \geq 0$,

$$b_{\alpha+h\ell} := \frac{\alpha}{K_1} + \ell, \quad c_{\alpha+h\ell} := \begin{pmatrix} b_{\alpha+h\ell} h \\ b_{\alpha+h\ell} K_1 \end{pmatrix}, \quad \alpha = 0, \dots, K_1 - 1, \quad (1.7)$$

$$b_{\alpha+h\ell} := \frac{-\alpha}{K_2} + \ell, \quad c_{\alpha+h\ell} := \begin{pmatrix} b_{\alpha+h\ell} h \\ b_{\alpha+h\ell} K_2 \end{pmatrix}, \quad \alpha = -(K_2 - 1), \dots, -1, \quad (1.8)$$

and

$$\mathbb{N}_* = (\mathbb{N} - K_2) \setminus (\{0\} \cup (h\mathbb{N} - K_2)), \quad \text{where } a\mathbb{N} - K_2 := \{ak - K_2 | k \in \mathbb{N}\}.$$

Define

$$Z(x, \mathbf{s}; \epsilon) := \exp\left(\frac{A(x, \tilde{\mathbf{s}})}{\epsilon^2}\right) Z_{\text{cubic}}\left(\mathbf{t}(x, \mathbf{s}); \frac{1}{K_1}, \frac{1}{K_2}, -\frac{1}{h}; \epsilon\right), \quad (1.9)$$

where $\mathbf{s} := (s_k)_{k \in \mathbb{N}_*}$ is an infinite vector of indeterminates, $\tilde{s}_k = s_k - c_h^{-1} \delta_{k,h}$ ($k \in \mathbb{N}_*$),

$$t_i = t_i(x, \mathbf{s}) = \sum_{k \in \mathbb{N}_*} b_k^{i+1} c_k \tilde{s}_k + \delta_{i,1} + x \delta_{i,0}, \quad i \geq 0, \quad (1.10)$$

and A is the quadratic series

$$A := A(x, \mathbf{s}) = \frac{1}{2} \sum_{k_1, k_2 \in \mathbb{N}_*} \frac{b_{k_1} b_{k_2}}{b_{k_1} + b_{k_2}} c_{k_1} c_{k_2} s_{k_1} s_{k_2} + x \sum_{k \in \mathbb{N}_*} c_k s_k. \quad (1.11)$$

Note that for $g \geq 0$, $\mathcal{H}_g(\mathbf{t}(x, \mathbf{s}); \frac{1}{K_1}, \frac{1}{K_2}, -\frac{1}{h})$ is a well-defined formal power series in $\mathbb{C}[[x-1]][[\mathbf{s}]]$. Indeed, the coefficient of each monomial $(x-1)^{k_0} s_{k_1} \cdots s_{k_m}$ in $\mathcal{H}_g(\mathbf{t}(x, \mathbf{s}); \frac{1}{K_1}, \frac{1}{K_2}, -\frac{1}{h})$ is a finite sum because of the dimension reason (1.1).

Remark 1.2 *The change of the variables (1.10) is not invertible, but we do not lose any information by making the substitution (1.10) as we will see from Theorem 1.4 (cf. Proposition 3.10) that the special cubic Hodge integrals are uniquely determined by the loop equation (1.18).*

Denote $I = \{-(K_2 - 1), \dots, K_1 - 1\}$ and $I_* = I \setminus \{0\}$, and define a family of linear operators $L_m = L_m(\epsilon^{-1}x, \epsilon^{-1}\mathbf{s}, \epsilon\partial/\partial\mathbf{s})$, $m \geq 0$ by

$$L_0 = \sum_{k \in \mathbb{N}_*} b_k s_k \frac{\partial}{\partial s_k} + \frac{x^2}{2\epsilon^2} + \frac{1}{24} \left(\frac{1}{h} - \frac{1}{K_1} - \frac{1}{K_2} \right), \quad (1.12)$$

$$\begin{aligned} L_m &= \sum_{k \in \mathbb{N}_*} b_k s_k \frac{\partial}{\partial s_{k+hm}} + x \frac{\partial}{\partial s_{hm}} \\ &+ \frac{\epsilon^2}{2} \sum_{\ell=1}^{m-1} \frac{\partial^2}{\partial s_{h\ell} \partial s_{h(m-\ell)}} + \frac{\epsilon^2}{2} \sum_{\alpha, \beta \in I_*} \sum_{\ell=0}^{m-1} G^{\alpha\beta} \frac{\partial^2}{\partial s_{\alpha+h\ell} \partial s_{\beta+h(m-1-\ell)}}, \end{aligned} \quad (1.13)$$

where $(G^{\alpha\beta})_{\alpha, \beta \in I}$ is a symmetric nondegenerate constant matrix defined by

$$G^{\alpha\beta} = \begin{cases} \frac{K_1}{h} \delta^{\alpha+\beta, -K_2}, & \alpha, \beta < 0; \\ 1, & \alpha = \beta = 0; \\ \frac{K_2}{h} \delta^{\alpha+\beta, K_1}, & \alpha, \beta > 0; \\ 0, & \text{elsewhere.} \end{cases} \quad (1.14)$$

It is easy to check that the operators L_m satisfy the following Virasoro commutation relations:

$$[L_m, L_n] = (m - n) L_{m+n}, \quad \forall m, n \geq 0.$$

Theorem 1.3 *For the rational numbers p, q, r given by (1.6), the series $Z(x, \mathbf{s}; \epsilon)$ defined by (1.9) satisfies the following Virasoro constraints:*

$$L_m(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\partial/\partial\mathbf{s}) Z(x, \mathbf{s}; \epsilon) = 0, \quad m \geq 0. \quad (1.15)$$

Using Theorem 1.3 and the technique developed in [10], we derive in Section 3 the Dubrovin–Zhang loop equation for the special cubic Hodge free energies in the rational case; see Theorem 3.9.

We proceed to the general case. Denote

$$\sigma_1 = -(p + q + r), \quad \sigma_3 = -2(p^3 + q^3 + r^3). \quad (1.16)$$

From the local Calabi–Yau condition (1.3) and the fact that the integral in (1.2) is symmetric in p, q, r , we know that

$$\mathcal{H}_g := \mathcal{H}_g(\mathbf{t}; p, q, r) \in \mathbb{C}[\sigma_1, \sigma_3][[\mathbf{t}]], \quad g \geq 0. \quad (1.17)$$

The following theorem is the main result of the present paper.

Theorem 1.4 *The equation*

$$\begin{aligned}
& \sum_{i \geq 0} \left(\partial^i \Theta + \sum_{j=1}^i \binom{i}{j} P_{j-1, i-j+1} \right) \frac{\partial \Delta H}{\partial z_i} \\
&= \frac{\Theta^2}{16} - \left(\frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta + \epsilon^2 \sum_{i \geq 0} \partial^{i+2} \left(\frac{\Theta^2}{16} - \left(\frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta \right) \frac{\partial \Delta H}{\partial z_i} \\
&+ \frac{\epsilon^2}{2} \sum_{i, j \geq 0} P_{i+1, j+1} \left(\frac{\partial^2 \Delta H}{\partial z_i \partial z_j} + \frac{\partial \Delta H}{\partial z_i} \frac{\partial \Delta H}{\partial z_j} \right), \tag{1.18}
\end{aligned}$$

which is called the Dubrovin–Zhang loop equation for the special cubic Hodge integrals, has a unique solution of the form

$$\Delta H := \sum_{g \geq 1} \epsilon^{2g-2} H_g, \quad H_g := H_g(z_0, \dots, z_{3g-2}; \sigma_1, \sigma_3)$$

up to the addition of a constant to each H_g , $g \geq 1$. These constants can be uniquely determined by the following conditions:

$$H_1 = \frac{1}{24} \log z_1 + \frac{\sigma_1}{24} z_0, \quad \sum_{j=1}^{3g-2} j z_j \frac{\partial H_g}{\partial z_j} = (2g-2) H_g, \quad g \geq 2. \tag{1.19}$$

Here

$$\partial := \sum_{k \geq 0} z_{k+1} \frac{\partial}{\partial z_k}, \quad \Theta := \frac{1}{1 - e^{z_0/\mu}},$$

the coefficients $P_{i,j}$ are certain polynomials in $\Theta, \sigma_1, \sigma_3, z_1, z_2, \dots$ whose explicit expressions are given in Section 4, and μ is an arbitrary parameter. Moreover, let $v(\mathbf{t})$ be defined in (1.4), then the genus g ($g \geq 1$) special cubic Hodge free energy has the expression

$$\mathcal{H}_g = H_g \left(v(\mathbf{t}), \frac{\partial v(\mathbf{t})}{\partial t_0}, \dots, \frac{\partial^{3g-2} v(\mathbf{t})}{\partial t_0^{3g-2}}; \sigma_1, \sigma_3 \right). \tag{1.20}$$

One can recursively solve the loop equation to obtain the free energies H_g , $g \geq 1$. For example, the first two H_g are given by

$$H_1 = \frac{1}{24} \log z_1 + \frac{\sigma_1}{24} z_0, \tag{1.21}$$

$$\begin{aligned}
H_2 = & \frac{1}{1152} \frac{z_4}{z_1^2} - \frac{7}{1920} \frac{z_2 z_3}{z_1^3} + \frac{1}{360} \frac{z_2^3}{z_1^4} + \frac{\sigma_1}{480} \frac{z_3}{z_1} - \frac{11\sigma_1}{5760} \frac{z_2^2}{z_1^2} + \frac{7\sigma_1^2}{5760} z_2 \\
& + \left(\frac{\sigma_1^3}{17280} - \frac{\sigma_2}{34560} \right) z_1^2. \tag{1.22}
\end{aligned}$$

These expressions agree with the results of [7].

Besides the above theorems, we present in the next proposition some properties of the Taylor coefficients of $\mathcal{H}_g(\mathbf{t}(x, \mathbf{s}); \frac{1}{K_1}, \frac{1}{K_2}, -\frac{1}{h})$, $g \geq 1$ in the rational case, and we give in Theorem 1.6 (see below) the *gap phenomenon* for the special cubic Hodge integrals in the rational case with the additional requirement that one of p, q is equal to 1. This type of results was used by M.-X. Huang, A. Klemm and S. Quackenbush to compute the Gromov–Witten invariants of the quintic Calabi–Yau three-fold up to genus 51 [15].

Proposition 1.5 *The free energies $\mathcal{F}_g(x, \mathbf{s}) := \mathcal{H}_g(\mathbf{t}(x, \mathbf{s}); \frac{1}{K_1}, \frac{1}{K_2}, -\frac{1}{h})$, $g \geq 1$ with $\mathbf{t}(x, \mathbf{s})$ defined in (1.10) have the following property:*

$$\begin{aligned} & \mathcal{F}_1(x, \mathbf{s}) - \frac{\sigma_1 - 1}{24} \log x \\ &= \sum_{m \geq 1} \sum_{k_1, \dots, k_m \in \mathbb{N}_*} \frac{c_{1; k_1, \dots, k_m}(\sigma_1, \sigma_3)}{m!} x^{b_{k_1} + \dots + b_{k_m} - m} s_{k_1} \cdots s_{k_m}, \end{aligned} \quad (1.23)$$

$$\begin{aligned} & \mathcal{F}_g(x, \mathbf{s}) - \frac{R_g(\sigma_1, \sigma_3)}{x^{2g-2}} \\ &= \sum_{m \geq 1} \sum_{k_1, \dots, k_m \in \mathbb{N}_*} \frac{c_{g; k_1, \dots, k_m}(\sigma_1, \sigma_3)}{m!} x^{b_{k_1} + \dots + b_{k_m} - m - (2g-2)} s_{k_1} \cdots s_{k_m}, \quad g \geq 2. \end{aligned} \quad (1.24)$$

Here $c_{g; k_1, \dots, k_m}(\sigma_1, \sigma_3)$, $g \geq 1$ are certain functions of σ_1, σ_3 , and $R_g(\sigma_1, \sigma_3)$, $g \geq 2$ are certain polynomials of σ_1, σ_3 satisfying the condition

$$\deg R_g \leq 3g - 3, \quad \text{with } \deg \sigma_1 = 1, \deg \sigma_3 = 3, \quad (1.25)$$

and the degree $(3g - 3)$ -part of R_g is given by

$$\frac{(-1)^g}{2(2g-2)!} \frac{|B_{2g}| |B_{2g-2}|}{2g(2g-2)} \left(\frac{\sigma_1^3}{3} - \frac{\sigma_3}{6} \right)^{g-1}, \quad (1.26)$$

where B_j denote the Bernoulli numbers.

Note that in (1.23) and (1.24) the indeterminates of $R_g(\sigma_1, \sigma_3)$ take the values

$$\sigma_1 = \frac{1}{K_1 + K_2} - \frac{1}{K_1} - \frac{1}{K_2}, \quad \sigma_3 = \frac{2}{(K_1 + K_2)^3} - \frac{2}{K_1^3} - \frac{2}{K_2^3}.$$

The first two R_g , $g \geq 2$ have the expressions

$$\begin{aligned} R_2 &= -\frac{1}{1440} + \frac{13\sigma_1}{5760} - \frac{7\sigma_1^2}{5760} + \frac{\sigma_1^3}{17280} - \frac{\sigma_3}{34560}, \\ R_3 &= \frac{1}{181440} - \frac{107\sigma_1}{362880} + \frac{145\sigma_1^2}{290304} - \frac{961\sigma_1^3}{4354560} + \frac{31\sigma_3}{2177280} + \frac{113\sigma_1^4}{4354560} - \frac{113\sigma_1\sigma_3}{8709120} \\ &\quad - \frac{\sigma_1^6}{13063680} + \frac{\sigma_1^3\sigma_3}{13063680} - \frac{\sigma_3^2}{52254720}. \end{aligned}$$

When one of p, q is equal to 1, we have the following theorem.

Theorem 1.6 *For the rational case satisfying the condition that one of p, q is equal to 1, the free energies $\mathcal{F}_g(x, \mathbf{s})$, $g \geq 1$ satisfy the following gap condition:*

$$\mathcal{F}_1(x, \mathbf{s})|_{\mathbf{s}_{\text{II}}=0} - \frac{\sigma_1 - 1}{24} \log x \in \mathbb{C}[[x, \mathbf{s}_{\text{I}}]], \quad (1.27)$$

$$\mathcal{F}_g(x, \mathbf{s})|_{\mathbf{s}_{\text{II}}=0} - \frac{R_g(\sigma_1, \sigma_3)}{x^{2g-2}} \in \mathbb{C}[[x, \mathbf{s}_{\text{I}}]], \quad g \geq 2. \quad (1.28)$$

Here $\mathbf{s}_{\text{I}} = (s_h, s_{2h}, s_{3h}, \dots)$ and $\mathbf{s}_{\text{II}} = (s_k)_{k \in \mathbb{N}_* \setminus h\mathbb{N}}$.

The proofs of Proposition 1.5 and Theorem 1.6 are given in [18]. We hope that the results will be useful in the study of Gromov–Witten invariants for toric Calabi–Yau varieties.

Organization of the paper In Section 2, we derive the explicit expressions of Virasoro constraints for $Z(x, \mathbf{s}; \epsilon)$. In Section 3, we derive the Dubrovin–Zhang loop equation for the special cubic Hodge free energies in the rational case. In Section 4, we prove Theorem 1.4.

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2 Virasoro constraints

In this section we first give two versions of Virasoro constraints for Z_{cubic} , and then we prove Theorem 1.3. Denote by Z_{WK} the Witten–Kontsevich partition function

$$Z_{\text{WK}}(\mathbf{t}; \epsilon) = \exp \left(\sum_{g \geq 0} \epsilon^{2g-2} \sum_{n \geq 0} \frac{t_{i_1} \cdots t_{i_n}}{n!} \int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \right).$$

It is well known that $Z_{\text{WK}}(\mathbf{t}; \epsilon)$ satisfies the following Virasoro constraints [4, 16, 24]:

$$L_m^{\text{KdV}} (\epsilon^{-1} \tilde{\mathbf{t}}, \epsilon \partial / \partial \tilde{\mathbf{t}}) Z_{\text{WK}}(\mathbf{t}; \epsilon) = 0, \quad (2.1)$$

where $\tilde{t}_i = t_i - \delta_{i,1}$, and $L_m^{\text{KdV}} := L_m^{\text{KdV}} (\epsilon^{-1} \mathbf{t}, \epsilon \partial / \partial \mathbf{t})$ are linear operators given by

$$\begin{aligned} L_{-1}^{\text{KdV}} &:= \sum_{i \geq 1} t_i \frac{\partial}{\partial t_{i-1}} + \frac{t_0^2}{2\epsilon^2}, \\ L_0^{\text{KdV}} &:= \sum_{i \geq 0} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16}, \\ L_m^{\text{KdV}} &:= \sum_{i \geq 0} \frac{(2i+2m+1)!!}{2^{m+1}(2i-1)!!} t_i \frac{\partial}{\partial t_{i+m}} + \frac{\epsilon^2}{2} \sum_{i+j=m-1} \frac{(2i+1)!!(2j+1)!!}{2^{m+1}} \frac{\partial^2}{\partial t_i \partial t_j}, \quad m \geq 1. \end{aligned}$$

These operators satisfy the Virasoro commutation relations

$$\left[L_m^{\text{KdV}}, L_n^{\text{KdV}} \right] = (m-n) L_{m+n}^{\text{KdV}}, \quad \forall m, n \geq -1.$$

Lemma 2.1 ([11]) *The partition function $Z_{\text{cubic}}(\mathbf{t}; p, q, r; \epsilon)$ has the expression*

$$Z_{\text{cubic}}(\mathbf{t}; p, q, r; \epsilon) = e^{G(\epsilon^{-1} \tilde{\mathbf{t}}, \epsilon \partial / \partial \tilde{\mathbf{t}})} Z_{\text{WK}}(\mathbf{t}; \epsilon), \quad (2.2)$$

where the operator $G = G(\epsilon^{-1} \mathbf{t}, \epsilon \partial / \partial \mathbf{t})$ is defined by

$$G(\epsilon^{-1} \mathbf{t}, \epsilon \partial / \partial \mathbf{t}) = - \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} (p^{2i-1} + q^{2i-1} + r^{2i-1}) D_i,$$

B_{2i} are Bernoulli numbers, and $D_i = D_i(\epsilon^{-1} \mathbf{t}, \epsilon \partial / \partial \mathbf{t})$ are given by

$$D_i := - \sum_{j \geq 0} t_j \frac{\partial}{\partial t_{j+2i-1}} + \frac{\epsilon^2}{2} \sum_{j=0}^{2i-2} \frac{\partial^2}{\partial t_j \partial t_{2i-2-j}}, \quad i \geq 1.$$

The lemma below follows from equation (2.1) and Lemma 2.1.

Lemma 2.2 ([25]) *Define a sequence of operators $L_m^{\text{cubic}} = L_m^{\text{cubic}}(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t})$ by*

$$L_m^{\text{cubic}}(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t}) := e^G \circ L_m^{\text{KdV}}(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t}) \circ e^{-G}, \quad m \geq -1.$$

Then L_m^{cubic} satisfy the Virasoro commutation relations

$$\left[L_m^{\text{cubic}}, L_n^{\text{cubic}} \right] = (m - n)L_{m+n}^{\text{cubic}}, \quad m, n \geq -1.$$

Moreover, $Z_{\text{cubic}}(\mathbf{t}; p, q, r; \epsilon)$ satisfies the equations

$$L_m^{\text{cubic}}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\partial/\partial\tilde{\mathbf{t}}) Z_{\text{cubic}}(\mathbf{t}; p, q, r; \epsilon) = 0, \quad m \geq -1. \quad (2.3)$$

We call (2.3) the Virasoro constraints for $Z_{\text{cubic}}(\mathbf{t}; p, q, r; \epsilon)$. Note that these Virasoro operators L_m^{cubic} are in general quite complicated, and it is difficult to use (2.3) directly for the computation of the cubic Hodge integrals. For example, the operator L_0^{cubic} has the explicit expression

$$L_0^{\text{cubic}} = L_0^{\text{KdV}} - \sum_{k \geq 1} \frac{B_{2k}}{2k} \frac{\sigma_{2k-1}}{(2k-2)!} D_k,$$

where $\sigma_{2k-1} = -(2k-2)!(p^{2k-1} + q^{2k-1} + r^{2k-1})$, and the corresponding constraint given in (2.3) has the expression

$$\begin{aligned} & \left(\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{k \geq 1} \frac{B_{2k}}{2k} \frac{\sigma_{2k-1}}{(2k-2)!} \frac{\partial}{\partial t_{2k}} \right) Z_{\text{cubic}} \\ &= \left(\sum_{i \geq 0} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16} + \sum_{k \geq 1} \frac{B_{2k}}{2k} \frac{\sigma_{2k-1}}{(2k-2)!} \left(\sum_{j \geq 0} t_j \frac{\partial}{\partial t_{j+2k-1}} - \sum_{j=0}^{2k-2} \frac{\partial^2}{\partial t_j \partial t_{2k-2-j}} \right) \right) Z_{\text{cubic}}. \end{aligned}$$

Following [8] we consider the following linear combinations of Virasoro operators:

$$\tilde{L}_m^{\text{cubic}} = \tilde{L}_m^{\text{cubic}}(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t}) := \sum_{k \geq -1} \frac{m^{k+1}}{(k+1)!} L_k^{\text{cubic}}, \quad m \geq 0. \quad (2.4)$$

As it is shown in [8], the operators $\tilde{L}_m^{\text{cubic}}$ also satisfy the Virasoro commutation relations

$$\left[\tilde{L}_m^{\text{cubic}}, \tilde{L}_n^{\text{cubic}} \right] = (m - n)\tilde{L}_{m+n}^{\text{cubic}}, \quad \forall m, n \geq 0.$$

From (2.3) it follows that

$$\tilde{L}_m^{\text{cubic}}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\partial/\partial\tilde{\mathbf{t}}) Z_{\text{cubic}}(\mathbf{t}; p, q, r; \epsilon) = 0, \quad m \geq 0. \quad (2.5)$$

We call (2.5) the second version of the Virasoro constraints for $Z_{\text{cubic}}(\mathbf{t}; p, q, r; \epsilon)$.

Let us proceed to derive the explicit expressions of $\tilde{L}_m^{\text{cubic}}$. To this end, we are to use the Givental quantization technique [12] to simplify the computation. Note that we will use a slightly different convention of notations from Givental's, and we refer to Appendix A of [8] for

the details. Let (\mathcal{V}, ω) denote the Givental symplectic space, where \mathcal{V} is the space of Laurent series over \mathbb{C} , and ω is the bilinear form on \mathcal{V} defined by

$$\omega(f, g) := -\operatorname{res}_{z=\infty} f(-z)g(z)\frac{dz}{z^2}, \quad \forall f, g \in \mathcal{V}. \quad (2.6)$$

Denote by q_i, p_i ($i \geq 0$) the Darboux coordinates associated to ω , and define a family of infinitesimal symplectic transformations l_k , $k \geq -1$ on \mathcal{V} by

$$l_k := (-1)^{k+1} z^{3/2} \partial_z^{k+1} z^{-1/2}, \quad k \geq -1. \quad (2.7)$$

Then we have the following formulae [12]:

$$L_k^{\text{KdV}} = \widehat{l}_k \Big|_{q_i \mapsto t_i, \partial_{q_i} \mapsto \partial_{t_i}, i \geq 0} + \frac{\delta_{k,0}}{16}. \quad (2.8)$$

Similarly, define a family of infinitesimal symplectic transformations $d_i := z^{1-2i}$, $i \geq 1$ on \mathcal{V} . Then we have

$$D_i(\epsilon^{-1} \mathbf{t}, \epsilon \partial / \partial \mathbf{t}) = \widehat{d}_i \Big|_{q_j \mapsto t_j, \partial_{q_j} \mapsto \partial_{t_j}, j \geq 0}. \quad (2.9)$$

Note that the above results of this section hold true for general p, q, r . The local Calabi–Yau condition (1.3) for p, q, r will be needed in what follows.

Following [8] we have the following lemma.

Lemma 2.3 *Assume that p, q, r satisfy the local Calabi–Yau condition (1.3). Define*

$$\Psi(z) = \left(p^{1/p} q^{1/q} r^{1/r} \right)^{-z} \sqrt{\frac{\Gamma(1-z/p)\Gamma(1-z/q)\Gamma(1-z/r)}{\Gamma(1+z/p)\Gamma(1+z/q)\Gamma(1+z/r)}}. \quad (2.10)$$

Then the asymptotic expansion of $\log \Psi(z)$ as $z \rightarrow \infty$ within a properly chosen sector is given by

$$\log \Psi(z) \sim \pm \frac{\pi i}{4} - \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)} (p^{2i-1} + q^{2i-1} + r^{2i-1}) z^{1-2i}. \quad (2.11)$$

Proof One can obtain formula (2.11) by using Stirling’s formula [23] and the definition (2.10) after a careful calculation. \square

We note that the constants $\pm \pi i/4$ in the above asymptotic formula (2.11) do not affect the results of Givental’s quantization. For simplicity, denote by $\log \Phi(z)$ the following formal power series of z^{-1} :

$$\log \Phi(z) := - \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)} (p^{2i-1} + q^{2i-1} + r^{2i-1}) z^{1-2i}. \quad (2.12)$$

Lemma 2.4 *The operators $\widetilde{L}_m^{\text{cubic}}$, $m \geq 0$ have the expressions*

$$\widetilde{L}_m^{\text{cubic}}(\epsilon^{-1} \mathbf{t}, \epsilon \partial / \partial \mathbf{t}) = \left(z V_m(z) e^{-m \partial_z} \right)^\wedge \Big|_{q_i \mapsto t_i, \partial_{q_i} \mapsto \partial_{t_i}, i \geq 0} + \frac{m}{16} - \frac{p+q+r}{24}, \quad (2.13)$$

where $V_m(z)$, $m \geq 0$ are given by

$$V_m(z) = \frac{\Phi(z)}{\Phi(z-m)} \sqrt{\frac{1}{1-m/z}}. \quad (2.14)$$

Proof Denote

$$\widehat{\Phi} := \exp \left((\log \Phi(z))^\wedge \Big|_{q_i \mapsto t_i, \partial_{q_i} \mapsto \partial_{t_i}, i \geq 0} \right).$$

It follows from formula (2.9) and Lemma 2.2 that

$$L_m^{\text{cubic}}(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t}) = \widehat{\Phi} \circ L_m^{\text{KdV}}(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t}) \circ \widehat{\Phi}^{-1}, \quad m \geq 0.$$

Then by using (2.8) we obtain

$$L_m^{\text{cubic}}(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t}) = (\Phi \circ l_m \circ \Phi^{-1})^\wedge \Big|_{q_i \mapsto t_i, \partial_{q_i} \mapsto \partial_{t_i}} + \frac{1}{16}\delta_{m,0} - \frac{p+q+r}{24}\delta_{m,-1}.$$

Note that the term $-\frac{p+q+r}{24}\delta_{m,-1}$ comes from the cocycle [12] in the quantization of $\Phi \circ l_m \circ \Phi^{-1}$. The quantization formula (2.13) then follows from (2.4) as well as the identity

$$z^{3/2}\Phi(z) \circ e^{-m\partial_z} \circ \frac{1}{\sqrt{z}\Phi(z)} = z \frac{\Phi(z)}{\Phi(z-m)} \sqrt{\frac{1}{1-m/z}} e^{-m\partial_z}.$$

The lemma is proved. \square

We are now ready to prove Theorem 1.3, where we recall that p, q, r are assumed to be rational numbers given by (1.6).

Proof of Theorem 1.3 In order to prove the validity of (1.15), it suffices to show that the following identities hold true:

$$L_m(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\partial/\partial\mathbf{s}) = K^m e^{\frac{A(x, \tilde{\mathbf{s}})}{\epsilon^2}} \circ \tilde{L}_m^{\text{cubic}}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\partial/\partial\mathbf{t}) \circ e^{-\frac{A(x, \tilde{\mathbf{s}})}{\epsilon^2}}, \quad m \geq 0, \quad (2.15)$$

where

$$K = h^h K_1^{-K_1} K_2^{-K_2}, \quad (2.16)$$

and

$$\tilde{t}_i = \tilde{t}_i(x, \mathbf{s}) = \sum_{k \in \mathbb{N}_*} b_k^{i+1} c_k \tilde{s}_k + x \delta_{i,0}, \quad i \geq 0. \quad (2.17)$$

For simplicity we denote

$$\bar{s}_k := c_k s_k, \quad k \in \mathbb{N}_*.$$

Then the above substitution of the independent variables (2.17) reads

$$\tilde{t}_i(x, \mathbf{s}) = \sum_{k \in \mathbb{N}_*} b_k^{i+1} \tilde{s}_k + x \delta_{i,0}, \quad \tilde{s}_k = \bar{s}_k - \delta_{k,h}. \quad (2.18)$$

Since both sides of (2.15) satisfy the Virasoro commutation relations, we only need to prove (2.15) for $m = 0, 1, 2$.

Lemma 2.5 *The operator $\tilde{L}_0^{\text{cubic}}$ satisfies the following relation:*

$$e^{\frac{A(x, \tilde{\mathbf{s}})}{\epsilon^2}} \circ \tilde{L}_0^{\text{cubic}}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\partial/\partial\mathbf{t}) \circ e^{-\frac{A(x, \tilde{\mathbf{s}})}{\epsilon^2}} = \sum_{k \in \mathbb{N}_*} b_k \tilde{s}_k \frac{\partial}{\partial \bar{s}_k} + \frac{x^2}{2\epsilon^2} + \frac{\sigma_1}{24}. \quad (2.19)$$

Proof By using Lemma 2.4 we have

$$\tilde{L}_0^{\text{cubic}}(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t}) = \widehat{z} |_{q_i \mapsto t_i, \partial_{q_i} \mapsto \partial_{t_i}} + \frac{\sigma_1}{24} = \sum_{i \geq 1} t_i \frac{\partial}{\partial t_{i-1}} + \frac{t_0^2}{2\epsilon^2} + \frac{\sigma_1}{24}.$$

Under the substitution (2.18), we arrive at

$$\begin{aligned} & \tilde{L}_0^{\text{cubic}}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\partial/\partial\tilde{\mathbf{t}}) \\ &= \sum_{i \geq 1} \left(\sum_{k \in \mathbb{N}_*} b_k^{i+1} \tilde{s}_k \right) \frac{\partial}{\partial t_{i-1}} + \frac{1}{2\epsilon^2} \left(\sum_{k \in \mathbb{N}_*} b_k \tilde{s}_k + x \right)^2 + \frac{\sigma_1}{24} \\ &= \sum_{k \in \mathbb{N}_*} b_k \tilde{s}_k \frac{\partial}{\partial \tilde{s}_k} + \frac{1}{2\epsilon^2} \left(\sum_{k \in \mathbb{N}_*} b_k \tilde{s}_k + x \right)^2 + \frac{\sigma_1}{24}. \end{aligned}$$

The lemma is proved by applying the conjugation by $e^{\frac{A(x, \tilde{s})}{\epsilon^2}}$ to the above equality. \square

Lemma 2.6 *The operator $\tilde{L}_1^{\text{cubic}}$ satisfies the following relation:*

$$\begin{aligned} & e^{\frac{A(x, \tilde{s})}{\epsilon^2}} \circ \tilde{L}_1^{\text{cubic}}(\epsilon^{-1}\tilde{\mathbf{t}}, \epsilon\partial/\partial\tilde{\mathbf{t}}) \circ e^{-\frac{A(x, \tilde{s})}{\epsilon^2}} \\ &= \sum_{k \in \mathbb{N}_*} b_k V_1(-b_k) \tilde{s}_k \frac{\partial}{\partial \tilde{s}_{k+h}} + x V_1(0) \frac{\partial}{\partial \tilde{s}_h} + \frac{\epsilon^2}{2} \sum_{\alpha=1}^{K_1-1} \frac{\text{res}_{z=b_\alpha} V_1(z)}{b_\alpha} \frac{\partial^2}{\partial \tilde{s}_\alpha \partial \tilde{s}_{K_1-\alpha}} \\ & \quad + \frac{\epsilon^2}{2} \sum_{\alpha=-(K_2-1)}^{-1} \frac{\text{res}_{z=b_\alpha} V_1(z)}{b_\alpha} \frac{\partial^2}{\partial \tilde{s}_\alpha \partial \tilde{s}_{-\alpha-K_2}}, \end{aligned} \tag{2.20}$$

where $V_1(z)$ is defined in (2.14).

Proof Since $p = 1/K_1, q = 1/K_2, r = -1/h$, we find by using the definition (2.14) and Lemma 2.3 that $V_1(z)$ can be given by the asymptotic series (as $z \rightarrow \infty$) of a rational function, namely,

$$\frac{\prod_{i=1}^{K_1+K_2} \left(z - \frac{i}{K_1+K_2} \right)}{\prod_{i=1}^{K_1} \left(z - \frac{i}{K_1} \right) \prod_{i=1}^{K_2} \left(z - \frac{i}{K_2} \right)} \sim V_1(z).$$

For simplicity the above rational function will still be denoted by $V_1(z)$ as this will not introduce ambiguity. (Without further mentioning, similar notations will also be used for $V_m(z)$ as they are also asymptotic series of rational functions.) Note that the rational function $V_1(z)$ has simple poles at $b_{-(K_2-1)}, \dots, b_{-1}, b_1, \dots, b_{K_1-1}$, and 1, therefore,

$$V_1(z) = 1 + \frac{\text{res}_{y=1} V_1(y)}{z-1} + \sum_{\alpha \in I_*} \frac{\text{res}_{y=b_\alpha} V_1(y)}{z-b_\alpha}.$$

Hence we obtain

$$V_1(z) = 1 + \sum_{n \geq 1} z^{-n} \left(\text{res}_{y=1} V_1(y) + \sum_{\alpha \in I_*} b_\alpha^{n-1} \text{res}_{y=b_\alpha} V_1(y) \right).$$

Denote $a_\alpha := b_\alpha^{-1} \operatorname{res}_{z=b_\alpha} V_1(z)$, $\alpha \in I_*$. By taking $m = 1$ in (2.13) we obtain

$$\begin{aligned} \tilde{L}_1^{\text{cubic}} &= \left(z V_1(z) e^{-\partial z} \right)^\wedge \Big|_{q_i \mapsto t_i, \partial_{q_i} \mapsto \partial_{t_i}} + \frac{1}{16} + \frac{\sigma_1}{24} \\ &= \sum_{i \geq 0} \left(\sum_{j=0}^{i+1} t_j \binom{i+1}{j} + \sum_{j=0}^i t_j \sum_{n=1}^{i+1-j} (-1)^n \binom{i+1}{j+n} \left(\operatorname{res}_{y=1} V_1(y) + \sum_{\alpha \in I_*} a_\alpha b_\alpha^n \right) \right) \frac{\partial}{\partial t_i} \\ &\quad + \frac{\epsilon^2}{2} \sum_{i,j \geq 0} \left(\sum_{n=0}^{i+1} (-1)^n \binom{i+1}{n} \left(\operatorname{res}_{y=1} V_1(y) + \sum_{\alpha \in I_*} a_\alpha b_\alpha^{n+j+1} \right) \right) \frac{\partial^2}{\partial t_i \partial t_j} + \frac{t_0^2}{2\epsilon^2} + \frac{1}{16} + \frac{\sigma_1}{24}. \end{aligned}$$

Now by performing the variables substitution (2.18) we arrive at

$$\begin{aligned} &\tilde{L}_1^{\text{cubic}} (\epsilon^{-1} \tilde{\mathbf{t}}, \epsilon \partial / \partial \mathbf{t}) \\ &= \sum_{k \in \mathbb{N}_*} b_k V_1(-b_k) \tilde{s}_k \frac{\partial}{\partial \tilde{s}_{k+h}} + x V_1(0) \frac{\partial}{\partial \tilde{s}_h} + \frac{1}{2\epsilon^2} \left(\sum_{k \in \mathbb{N}_*} b_k \tilde{s}_k + x \right)^2 + \frac{1}{16} + \frac{\sigma_1}{24} \\ &\quad + \frac{\epsilon^2}{2} \sum_{\alpha=1}^{K_1-1} a_\alpha \frac{\partial^2}{\partial \tilde{s}_\alpha \partial \tilde{s}_{K_1-\alpha}} + \frac{\epsilon^2}{2} \sum_{\alpha=-(K_2-1)}^{-1} a_\alpha \frac{\partial^2}{\partial \tilde{s}_\alpha \partial \tilde{s}_{-\alpha-K_2}} \\ &\quad + \sum_{\alpha=1}^{K_1-1} a_\alpha \frac{\partial A(x, \tilde{\mathbf{s}})}{\partial \tilde{s}_\alpha} \frac{\partial}{\partial \tilde{s}_{K_1-\alpha}} + \sum_{\alpha=-(K_2-1)}^{-1} a_\alpha \frac{\partial A(x, \tilde{\mathbf{s}})}{\partial \tilde{s}_\alpha} \frac{\partial}{\partial \tilde{s}_{-\alpha-K_2}}. \end{aligned}$$

Here we used the identity

$$\sum_{j=0}^i \sum_{k=1}^{i+1-j} \binom{i+1}{j+k} \lambda^j \mu^k = \mu \frac{(\lambda+1)^{i+1} - (\mu+1)^{i+1}}{\lambda - \mu}.$$

Then dressing the operator $\tilde{L}_1^{\text{cubic}} (\epsilon^{-1} \tilde{\mathbf{t}}, \epsilon \partial / \partial \mathbf{t})$ by $e^{\frac{A(x, \tilde{\mathbf{s}})}{\epsilon^2}}$ we obtain (2.20) after a long but straightforward computation. The lemma is proved. \square

We can prove in a similar way the following lemma.

Lemma 2.7 *The operator $\tilde{L}_2^{\text{cubic}}$ satisfies the relation*

$$\begin{aligned} &e^{\frac{A(x, \tilde{\mathbf{s}})}{\epsilon^2}} \circ \tilde{L}_2^{\text{cubic}} (\epsilon^{-1} \tilde{\mathbf{t}}, \epsilon \partial / \partial \mathbf{t}) \circ e^{-\frac{A(x, \tilde{\mathbf{s}})}{\epsilon^2}} \\ &= \sum_{k \in \mathbb{N}_*} b_k V_2(-b_k) \tilde{s}_k \frac{\partial}{\partial \tilde{s}_{k+2h}} + x V_2(0) \frac{\partial}{\partial \tilde{s}_{2h}} + \frac{\epsilon^2}{2} b_h^{-1} \operatorname{res}_{z=b_h} V_2(z) \frac{\partial^2}{\partial \tilde{s}_h \partial \tilde{s}_h} \\ &\quad + \frac{\epsilon^2}{2} \sum_{\alpha=1}^{K_1-1} \sum_{\ell=0}^1 b_\alpha^{-1} \operatorname{res}_{z=b_\alpha} V_2(z) \frac{\partial^2}{\partial \tilde{s}_{\alpha+h\ell} \partial \tilde{s}_{K_1-\alpha+h(1-\ell)}} \\ &\quad + \frac{\epsilon^2}{2} \sum_{\alpha=-(K_2-1)}^{-1} \sum_{\ell=0}^1 b_\alpha^{-1} \operatorname{res}_{z=b_\alpha} V_2(z) \frac{\partial^2}{\partial \tilde{s}_{\alpha+h\ell} \partial \tilde{s}_{-\alpha-K_2+h(1-\ell)}}. \end{aligned} \tag{2.21}$$

By using Lemma 2.3 we can also obtain the following lemma.

Lemma 2.8 *The numbers c_k , $k \in \mathbb{N}_*$ defined in (1.7)–(1.8) have the following properties:*

- (i) *For $k \in \mathbb{N}_*$ and $\ell \geq 1$, $c_{k+h\ell}/c_k = K^\ell V_\ell(-b_k)$.*
- (ii) *For $\ell \geq 1$, $c_{h\ell} = K^\ell V_\ell(0)$.*
- (iii) *For $m, n \geq 0$,*

$$\begin{aligned} c_{\alpha+hm} c_{K_1-\alpha+hn} &= \frac{h}{K_2} \frac{K^{m+n+1}}{b_{\alpha+hm}} \operatorname{res}_{z=b_{\alpha+hm}} V_{m+n+1}(z), \quad \alpha = 1, \dots, K_1 - 1, \\ c_{\alpha+hm} c_{-\alpha-K_2+hn} &= \frac{h}{K_1} \frac{K^{m+n+1}}{b_{\alpha+hm}} \operatorname{res}_{z=b_{\alpha+hm}} V_{m+n+1}(z), \quad \alpha = -(K_2 - 1), \dots, -1, \\ c_{h(m+1)} c_{h(n+1)} &= \frac{K^{m+n+2}}{b_{h(m+1)}} \operatorname{res}_{z=h(m+1)} V_{m+n+2}(z). \end{aligned}$$

By comparing (2.20)–(2.21) with the definition of the left-hand side of (2.15) and by using Lemma 2.8 we find that (2.15) is true for $m = 1, 2$. Thus Theorem 1.3 is proved. \square

3 Loop equation: the rational case

In this section, we derive the Dubrovin–Zhang loop equation for the special cubic Hodge integrals in the rational case, namely, we take

$$p = 1/K_1, \quad q = 1/K_2, \quad r = -1/h,$$

where K_1 and K_2 are coprime positive integers, and $h = K_1 + K_2$.

Introduce a generating series for the operators L_m , $m \geq 0$ defined in (1.12)–(1.13) as follows:

$$L(\lambda) = \sum_{m \geq 0} \frac{L_m(\epsilon^{-1}x, \epsilon^{-1}\tilde{\mathbf{s}}, \epsilon\partial/\partial\mathbf{s})}{\lambda^{m+1}}. \quad (3.1)$$

Lemma 3.1 *The generating series $L(\lambda)$ can be represented as*

$$L(\lambda) = \left[J_1(\lambda)J_2(\lambda) + \frac{1}{2}J_2(\lambda)^2 \right]_{\operatorname{reg},-} + \frac{1}{\lambda} \left(\frac{x}{2\epsilon^2} + \frac{\sigma_1}{24} \right), \quad (3.2)$$

where the operators $J_1(\lambda)$ and $J_2(\lambda)$ are defined by

$$\begin{aligned} J_1(\lambda) &:= \epsilon^{-1} \frac{x}{\sqrt{\lambda}} + \epsilon^{-1} \sum_{k \in \mathbb{N}_*} b_k \lambda^{b_k - \frac{1}{2}} \tilde{s}_k, \\ J_2(\lambda) &:= \sum_{\ell \geq 1} \frac{\epsilon}{\lambda^{b_{h\ell} + 1/2}} \frac{\partial}{\partial s_{h\ell}} + \sqrt{\frac{K_2}{h}} \sum_{\alpha=1}^{K_1-1} \sum_{\ell \geq 0} \frac{\epsilon}{\lambda^{b_{\alpha+h\ell} + \frac{1}{2}}} \frac{\partial}{\partial s_{\alpha+h\ell}} \\ &\quad + \sqrt{\frac{K_1}{h}} \sum_{\alpha=-(K_2-1)}^{-1} \sum_{\ell \geq 0} \frac{\epsilon}{\lambda^{b_{\alpha+h\ell} + \frac{1}{2}}} \frac{\partial}{\partial s_{\alpha+h\ell}}, \end{aligned}$$

and $[\cdot]_{\operatorname{reg},-}$ means to take terms of the series with negative integer powers of λ .

Proof By comparing the coefficients of $\lambda^{-(m+1)}$ for $m \geq 0$ on both sides of (3.2) with L_m given by (1.12)–(1.13) we obtain the validity of the identity (3.2). The lemma is proved. \square

Theorem 1.3 implies that $L(\lambda)Z(x, \mathbf{s}; \epsilon) = 0$ holds true *identically* in λ . From the definition of $Z(x, \mathbf{s}; \epsilon)$ given by (1.9) we know that its logarithm admits the genus expansion

$$\log Z(x, \mathbf{s}; \epsilon) = \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g(x, \mathbf{s}).$$

Here we denote, as we do in Proposition 1.5,

$$\mathcal{F}_g(x, \mathbf{s}) := \mathcal{H}_g \left(\mathbf{t}(x, \mathbf{s}); \frac{1}{K_1}, \frac{1}{K_2}, -\frac{1}{h} \right) + A(x, \tilde{\mathbf{s}}) \delta_{g,0}, \quad \text{with } \tilde{s}_k = s_k - c_h^{-1} \delta_{k,h}. \quad (3.3)$$

In what follows we will also use the notations

$$u(x, \mathbf{s}) := \partial_x^2 \mathcal{F}_0(x, \mathbf{s}) \quad (3.4)$$

and

$$\Delta \mathcal{F} = \sum_{g \geq 1} \epsilon^{2g-2} \mathcal{F}_g(x, \mathbf{s}). \quad (3.5)$$

Lemma 3.2 *The function $u(x, \mathbf{s})$ satisfies the following equations*

$$\partial_x^i u(x, \mathbf{s}) = \partial_{t_0}^i v(\mathbf{t}(x, \mathbf{s})), \quad i \geq 0. \quad (3.6)$$

$$\frac{\partial u}{\partial s_k} = c_k \partial_x e^{b_k u}, \quad \frac{\partial^2 \mathcal{F}_0}{\partial x \partial s_k} = c_k e^{b_k u}, \quad k \in \mathbb{N}_*. \quad (3.7)$$

Proof By using (1.10) we obtain

$$\partial_x = \partial_{t_0}, \quad \frac{\partial}{\partial s_k} = c_k \sum_{i \geq 0} b_k^{i+1} \frac{\partial}{\partial t_i}, \quad k \in \mathbb{N}_*.$$

The lemma then follows from the Riemann hierarchy (1.5). \square

Lemma 3.3 ([7]) *For any given $g \geq 1$, there exists a polynomial $H_g(z_0, \dots, z_{3g-2}; p, q, r)$ of p, q, r, z_2, z_3, \dots with coefficients depending smoothly on z_0 and rationally on z_1 , such that*

$$\mathcal{H}_g(\mathbf{t}; p, q, r) = H_g \left(v(\mathbf{t}), \frac{\partial v(\mathbf{t})}{\partial t_0}, \dots, \frac{\partial^{3g-2} v(\mathbf{t})}{\partial t_0^{3g-2}}; p, q, r \right). \quad (3.8)$$

It should be noted that Lemma 3.3 does not require p, q, r satisfy the local Calabi–Yau condition.

Let us denote

$$u^{(i)} = \partial_x^i u(x, \mathbf{s}), \quad i \geq 0.$$

Then from Lemmas 3.2–3.3 we know that

$$\mathcal{F}_g(x, \mathbf{s}) = H_g \left(u^{(0)}, u^{(1)}, \dots, u^{(3g-2)}; p, q, r \right), \quad g \geq 1, \quad (3.9)$$

where $p = 1/K_1$, $q = 1/K_2$, $r = -1/(K_1 + K_2)$. Introduce a derivation $D(\lambda)$ on $\mathbb{C}[[x-1, \mathbf{s}]]$ by

$$D(\lambda) = -[J_1(\lambda)J_2(\lambda)]_{\text{reg},-} - \epsilon^{-2} [J_2(\lambda)(\mathcal{F}_0)J_2(\lambda)]_{\text{reg},-},$$

where $J_1(\lambda), J_2(\lambda)$ are defined in Lemma 3.1.

Lemma 3.4 *The series $D(\lambda)(\Delta\mathcal{F})$ has the following expression:*

$$\begin{aligned} & D(\lambda)(\Delta\mathcal{F}) \\ &= \frac{\sigma_1}{24\lambda} + \left[\frac{1}{2\epsilon^2} J_2(\lambda)^2(\mathcal{F}_0) + \frac{1}{2} \sum_{i \geq 0} J_2(\lambda)^2(u^{(i)}) \frac{\partial \Delta\mathcal{F}}{\partial u^{(i)}} \right. \\ & \quad \left. + \frac{1}{2} \sum_{i,j \geq 0} J_2(\lambda)(u^{(i)}) J_2(\lambda)(u^{(j)}) \left(\frac{\partial^2 \Delta\mathcal{F}}{\partial u^{(i)} \partial u^{(j)}} + \frac{\partial \Delta\mathcal{F}}{\partial u^{(i)}} \frac{\partial \Delta\mathcal{F}}{\partial u^{(j)}} \right) \right]_{\text{reg,-}}. \end{aligned} \quad (3.10)$$

Proof From Lemma 3.1 we know that the Virasoro constraints $L(\lambda)Z(x, \mathbf{s}; \epsilon) = 0$ can be represented as

$$\left[J_1(\lambda) J_2(\lambda)(\mathcal{F}_0) + \frac{1}{2\epsilon^2} (J_2(\lambda)(\mathcal{F}_0))^2 \right]_{\text{reg,-}} + \frac{x^2}{2\lambda} = 0, \quad (3.11)$$

$$\begin{aligned} & \left[J_1(\lambda) J_2(\lambda)(\Delta\mathcal{F}) + \frac{1}{\epsilon^2} J_2(\lambda)(\mathcal{F}_0) J_2(\lambda)(\Delta\mathcal{F}) \right. \\ & \quad \left. + \frac{1}{2} (J_2(\lambda)^2(\Delta\mathcal{F}) + (J_2(\lambda)(\Delta\mathcal{F}))^2) + \frac{1}{2\epsilon^2} J_2(\lambda)^2(\mathcal{F}_0) \right]_{\text{reg,-}} + \frac{\sigma_1}{24\lambda} = 0. \end{aligned} \quad (3.12)$$

It is easy to see that (3.12) can be rewritten in the form

$$D(\lambda)(\Delta\mathcal{F}) = \frac{\sigma_1}{24\lambda} + \left[\frac{1}{2} (J_2(\lambda)^2(\Delta\mathcal{F}) + (J_2(\lambda)(\Delta\mathcal{F}))^2) + \frac{1}{2\epsilon^2} J_2(\lambda)^2(\mathcal{F}_0) \right]_{\text{reg,-}}. \quad (3.13)$$

Since $J_2(\lambda)$ is a derivation, we obtain (3.10). The lemma is proved. \square

Let us proceed to simplify the equation (3.10). Introduce the following Puiseux series:

$$E_\alpha = \begin{cases} \sum_{n \geq 0} c_{\alpha+hn} \zeta^{n + \frac{\alpha}{K_1}}, & \alpha = 0, \dots, K_1 - 1, \\ \sum_{n \geq 0} c_{\alpha+hn} \zeta^{n - \frac{\alpha}{K_2}}, & \alpha = -(K_2 - 1), \dots, -1, \end{cases} \quad (3.14)$$

where $\zeta := e^{u(x, \mathbf{s})}/\lambda$. Define

$$B_{i,j} := \sum_{\alpha, \beta \in I} \partial_x^i (E_\alpha) G^{\alpha\beta} \partial_x^j (E_\beta), \quad (3.15)$$

where $G^{\alpha\beta}$ are constants given by (1.14). Note that $B_{i,j} \in \mathbb{C}[u^{(1)}, u^{(2)}, \dots][[\zeta]]$ for all $i, j \geq 0$.

Lemma 3.5 *The following formulae hold true:*

$$D(\lambda)(u^{(i)}) = \frac{1}{\lambda} \left(\partial_x^i B_{0,0} + \sum_{j=1}^i \binom{i}{j} B_{j-1, i-j+1} \right), \quad \forall i \geq 0, \quad (3.16)$$

$$\epsilon^{-2} \left[J_2(\lambda)(u^{(i)}) J_2(\lambda)(u^{(j)}) \right]_{\text{reg,-}} = \frac{1}{\lambda} B_{i+1, j+1}, \quad \forall i, j \geq 0. \quad (3.17)$$

Proof Using Lemma 3.2 and the definition of $J_2(\lambda)$ given in Lemma 3.1, we have

$$\epsilon^{-1}J_2(\lambda)(\partial_x\mathcal{F}_0) + \frac{1}{\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}} \left(E_0 + \sqrt{\frac{K_2}{h}} \sum_{\alpha=1}^{K_1-1} E_\alpha + \sqrt{\frac{K_1}{h}} \sum_{\alpha=-(K_2-1)}^{-1} E_\alpha \right). \quad (3.18)$$

Acting ∂_x^2 on both sides of the equation (3.11) yields

$$D(\lambda)(u) = \left[\left(\epsilon^{-1}J_2(\lambda)(\partial_x\mathcal{F}_0) + \frac{1}{\sqrt{\lambda}} \right)^2 \right]_{\text{reg},-} = \frac{1}{\lambda} B_{0,0},$$

which gives the validity of (3.16) for $i = 0$. Assume that the formula (3.16) is true for $i = k$; then for $i = k + 1$ we have

$$\begin{aligned} D(\lambda)(u^{(k+1)}) &= \partial_x D(\lambda)(u^{(k)}) + [D(\lambda), \partial_x](u^{(k)}) \\ &= \frac{1}{\lambda} \left(\partial_x^{k+1} B_{0,0} + \sum_{j=1}^k \binom{k}{j} (B_{j,k-j+1} + B_{j-1,k-j+2}) \right) \\ &\quad + \epsilon^{-1} \left[J_2(\lambda)(\partial_x\mathcal{F}_0) + \frac{1}{\sqrt{\lambda}} J_2(\lambda)(u^{(k)}) \right]_{\text{reg},-} \\ &= \frac{1}{\lambda} \left(\partial_x^{k+1} B_{0,0} + \sum_{j=1}^{k+1} \binom{k+1}{j} B_{j-1,k-j+2} \right). \end{aligned}$$

Hence by induction we arrive at the formula (3.16). The formula (3.17) can be proved by using (3.15) and (3.18). The lemma is proved. \square

Let us now introduce a family of polynomials $f_{i,j} \in \mathbb{C}[u^{(1)}, u^{(2)}, \dots]$, $i, j \geq 0$ by requiring the validity of the identity

$$\partial_x^i y(\zeta) = \sum_{j=0}^i f_{i,j} (\zeta \partial_\zeta)^j y(\zeta) \quad (3.19)$$

for any smooth function $y(\zeta)$ of $\zeta = e^{u(x,\mathbf{s})}/\lambda$. Clearly, $f_{i,j}$ vanishes if $i < j$. For $i \geq j$, the polynomials $f_{i,j}$ can be uniquely determined by the following recursive relations:

$$\begin{aligned} f_{i,0} &= \delta_{i,0}, \\ f_{i+1,j+1} &= \partial_x f_{i,j+1} + u^{(1)} f_{i,j}. \end{aligned}$$

Explicit expressions for $f_{i,j}$ will be given in Section 4. The functions $B_{i,j}$ defined in (3.15) can now be written as

$$B_{i,j} = \sum_{k=0}^i \sum_{l=0}^j f_{i,k} f_{j,l} \tilde{B}_{k,l}, \quad (3.20)$$

where

$$\tilde{B}_{k,l} := \sum_{\alpha, \beta \in I} (\zeta \partial_\zeta)^k (E_\alpha) G^{\alpha\beta} (\zeta \partial_\zeta)^l (E_\beta), \quad (3.21)$$

and E_α are defined by (3.14).

Lemma 3.6 *The functions $\tilde{B}_{i,j}$, $i, j \geq 0$ defined in (3.21) satisfy the relations*

$$\sum_{k \geq 0} z^{-k} \tilde{B}_{0,k} = \sqrt{z} \Phi(z) \frac{1}{1 - K\zeta e^{-\partial_z}} \left(\frac{1}{\sqrt{z} \Phi(z)} \right), \quad (3.22)$$

$$\tilde{B}_{i,j} = \tilde{B}_{j,i}, \quad \zeta \partial_\zeta \tilde{B}_{i,j} = \tilde{B}_{i+1,j} + \tilde{B}_{i,j+1}. \quad (3.23)$$

Moreover, $\tilde{B}_{i,j}$ are polynomials of $(1 - K\zeta)^{-1}$ with degrees less than or equal to $i + j + 1$.

Proof By substituting (3.14) into (3.21) we obtain

$$\tilde{B}_{0,k} = \sum_{n \geq 0} A_{k,n} \zeta^n, \quad (3.24)$$

where

$$\begin{aligned} A_{k,n} := & c_{hn} \delta_{k,0} + \sum_{\ell=1}^{n-1} b_{h\ell}^k c_{h\ell} c_{h(n-\ell)} + \frac{K_2}{h} \sum_{\alpha=1}^{K_1-1} \sum_{\ell=0}^{n-1} b_{\alpha+h\ell}^k c_{\alpha+h\ell} c_{K_1-\alpha+h(n-1-\ell)} \\ & + \frac{K_1}{h} \sum_{\alpha=-(K_2-1)}^{-1} \sum_{\ell=0}^{n-1} b_{\alpha+h\ell}^k c_{\alpha+h\ell} c_{-\alpha-K_2+h(n-1-\ell)}. \end{aligned}$$

Using Lemma 2.8 we find that

$$K^{-n} A_{k,n} = \sum_{\alpha \in I} \sum_{\ell=0}^{n-1} b_{\alpha+h\ell}^{k-1} \operatorname{res}_{z=b_{\alpha+h\ell}} V_n(z) + \frac{\operatorname{res}_{z=n} V_n(z)}{n} \delta_{k,0}. \quad (3.25)$$

In particular, we have

$$A_{0,n} = K^n \left(1 - V_n(0) + \frac{1}{n} \operatorname{res}_{z=n} V_n(z) \right) = K^n.$$

Hence

$$\begin{aligned} \sum_{k \geq 0} z^{-k} \tilde{B}_{0,k} &= \sum_{k,n \geq 0} z^{-k} A_{k,n} \zeta^n = \sum_{n \geq 0} V_n(z) (K\zeta)^n \\ &= \sum_{n \geq 0} \frac{\Phi(z)}{\Phi(z-n)} \sqrt{\frac{1}{1-n/z}} (K\zeta)^n = \sqrt{z} \Phi(z) \frac{1}{1 - K\zeta e^{-\partial_z}} \left(\frac{1}{\sqrt{z} \Phi(z)} \right). \end{aligned} \quad (3.26)$$

Note that

$$\frac{1}{1 - K\zeta e^{-\partial_z}} = \sum_{n \geq 0} \sum_{k=1}^{n+1} \frac{Q(n,k)}{(1 - K\zeta)^k} \frac{\partial_z^n}{n!}, \quad Q(n,k) = \frac{1}{k} \sum_{i=1}^k (-1)^{i-1} \binom{k}{i} i^{n+1}. \quad (3.27)$$

So from (3.23) and (3.26) it follows that $\tilde{B}_{i,j}$ belong to $\mathbb{C}[(1 - K\zeta)^{-1}]$ with degrees less than or equal to $i + j + 1$. The lemma is proved. \square

Remark 3.7 *Introduce the trinomial curve*

$$XY^{K_1+K_2} - Y^{K_2} + 1 = 0. \quad (3.28)$$

Near $X = 0$ we have two Puiseux series solutions:

$$Y_- = \sum_{m \geq 0} c_{-,m} X^m, \quad c_{-,0} = 1,$$

$$Y_+ = X^{-\frac{1}{K_1}} \sum_{m \geq 0} c_{+,m} X^{m \frac{K_2}{K_1}}, \quad c_{+,0} = 1.$$

By using the Lagrange inversion formula we obtain

$$(Y_-)^{K_2} = \sum_{m \geq 0} \binom{m \frac{K_1+K_2}{K_2}}{m} \frac{X^m}{1 + m \frac{K_1}{K_2}}, \quad (Y_+)^{-K_1} = \sum_{m \geq 0} \binom{m \frac{K_1+K_2}{K_1}}{m} \frac{X^{1+m \frac{K_2}{K_1}}}{1 + m \frac{K_2}{K_1}}.$$

We also observe the validity of the following relations:

$$\sum_{\alpha=0}^{K_1-1} E_\alpha = \frac{d}{dX} ((Y_+)^{-K_1}), \quad \sum_{\alpha=-(K_2-1)}^0 E_\alpha = (Y_-)^{K_2} + \frac{K_1}{K_2} X \frac{d}{dX} ((Y_-)^{K_2}) \quad (3.29)$$

with $X = \zeta^{1/K_2}$. We conjecture that the curve (3.28) can be used to give an alternative way of computing the special cubic Hodge integrals by using the topological recursion of Chekhov–Eynard–Orantin type. We note that the $K_2 = 1$ case was considered by Bouchard–Klemm–Mariño–Pasquetti [1].

Corollary 3.8 *The following formula holds true:*

$$\epsilon^{-2} [J_2(\lambda)^2(\mathcal{F}_0)]_{\text{reg},-} = \frac{1}{\lambda} \left(\frac{1}{8(1-K\zeta)^2} - \left(\frac{1}{8} - \frac{\sigma_1}{12} \right) \frac{1}{1-K\zeta} - \frac{\sigma_1}{12} \right). \quad (3.30)$$

Proof By using the definition of $J_2(\lambda)$ given in Lemma 3.1 and by using Lemma 3.2 we obtain

$$\epsilon^{-2} [J_2(\lambda)^2(\mathcal{F}_0)]_{\text{reg},-} = \int \frac{1}{\lambda} \frac{B_{1,1}}{u^{(1)}} dx = \frac{1}{\lambda} \int \frac{\tilde{B}_{1,1}}{K\zeta} d(K\zeta). \quad (3.31)$$

Here we recall that $B_{1,1}$ is defined in (3.15), and the integration constant is chosen such that the left hand side tends to zero as $\lambda \rightarrow \infty$. From Lemma 3.6 it follows that

$$\tilde{B}_{1,1} = \frac{1}{4} \frac{1}{(1-K\zeta)^3} - \left(\frac{3}{8} - \frac{\sigma_1}{12} \right) \frac{1}{(1-K\zeta)^2} + \left(\frac{1}{8} - \frac{\sigma_1}{12} \right) \frac{1}{1-K\zeta},$$

which, together with (3.31), leads to the formula (3.30). The corollary is proved. \square

By using Lemmas 3.4, 3.5 and Corollary 3.8 we arrive at the following theorem.

Theorem 3.9 *The series $\Delta\mathcal{F}$ given in (3.5), (3.9) satisfies the Dubrovin–Zhang loop equation*

$$\sum_{i \geq 0} \frac{\partial \Delta\mathcal{F}}{\partial u^{(i)}} \partial_x^i \Theta + \sum_{i \geq 1} \sum_{j=1}^i \sum_{\alpha, \beta \in I} \frac{\partial \Delta\mathcal{F}}{\partial u^{(i)}} \binom{i}{j} \partial_x^{j-1}(E_\alpha) G^{\alpha\beta} \partial_x^{i-j+1}(E_\beta)$$

$$\begin{aligned}
&= \frac{\Theta^2}{16} - \left(\frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta + \epsilon^2 \sum_{i \geq 0} \frac{\partial \Delta \mathcal{F}}{\partial u^{(i)}} \partial_x^{i+2} \left(\frac{\Theta^2}{16} - \left(\frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta \right) \\
&\quad + \frac{\epsilon^2}{2} \sum_{i,j \geq 0} \sum_{\alpha, \beta \in I} \left(\frac{\partial^2 \Delta \mathcal{F}}{\partial u^{(i)} \partial u^{(j)}} + \frac{\partial \Delta \mathcal{F}}{\partial u^{(i)}} \frac{\partial \Delta \mathcal{F}}{\partial u^{(j)}} \right) \partial_x^{i+1}(E_\alpha) G^{\alpha\beta} \partial_x^{j+1}(E_\beta), \tag{3.32}
\end{aligned}$$

where $\Theta = \frac{1}{1-K\zeta}$ with $K = h^h K_1^{-K_1} K_2^{-K_2}$, and $\sigma_1 = -(p+q+r) = \frac{1}{h} - \frac{1}{K_1} - \frac{1}{K_2}$.

Note that each side of the loop equation (3.32) is a power series of Θ . It is understood that this equation for $\Delta \mathcal{F}$ holds true identically in Θ . It is easy to check that \mathcal{F}_g also satisfy the equations

$$\mathcal{F}_1 = \frac{1}{24} \log u^{(1)} + \frac{\sigma_1}{24} u^{(0)}, \quad \sum_{j=1}^{3g-2} j u^{(j)} \frac{\partial \mathcal{F}_g}{\partial u^{(j)}} = (2g-2) \mathcal{F}_g, \quad g \geq 2. \tag{3.33}$$

Proposition 3.10 *The solution to equations (3.32)–(3.33) is unique.*

Proof Expanding (3.32) with respect to ϵ and comparing the coefficients of powers of ϵ we find that (3.32) is equivalent to the following equations:

$$\begin{aligned}
&\sum_{i \geq 0} \left(\partial_x^i B_{0,0} + \sum_{j=1}^i \binom{i}{j} B_{j-1, i-j+1} \right) \frac{\partial \mathcal{F}_1}{\partial u^{(i)}} = \frac{\Theta^2}{16} - \left(\frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta, \\
&\sum_{i \geq 0} \left(\partial_x^i B_{0,0} + \sum_{j=1}^i \binom{i}{j} B_{j-1, i-j+1} \right) \frac{\partial \mathcal{F}_g}{\partial u^{(i)}} = \sum_{i \geq 0} \partial_x^{i+2} \left(\frac{\Theta^2}{16} - \left(\frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta \right) \frac{\partial \mathcal{F}_{g-1}}{\partial u^{(i)}} \\
&\quad + \frac{1}{2} \sum_{i,j \geq 0} B_{i+1, j+1} \left(\frac{\partial^2 \mathcal{F}_{g-1}}{\partial u^{(i)} \partial u^{(j)}} + \sum_{k=1}^{g-1} \frac{\partial \mathcal{F}_k}{\partial u^{(i)}} \frac{\partial \mathcal{F}_{g-k}}{\partial u^{(j)}} \right), \quad g \geq 2.
\end{aligned}$$

Here we used the formulae (3.15). By using the fact that $\partial \mathcal{F}_g / \partial u^{(i)} = 0$ for $i \geq 3g-1$, and that $B_{i,j}$ are polynomials in Θ of degrees $i+j+1$, we arrive at the following system of equations:

$$(\Theta, \dots, \Theta^{3g-1}) M_g \left(\frac{\partial \mathcal{F}_g}{\partial u^{(0)}}, \dots, \frac{\partial \mathcal{F}_g}{\partial u^{(3g-2)}} \right)^T = (\Theta, \dots, \Theta^{3g-1}) N_g, \quad g \geq 1,$$

where M_g is an invertible upper triangular $(3g-1)$ by $(3g-1)$ matrix, and N_g is a column vector. All the elements of the matrix and the vector are polynomials of $1/u^{(1)}, u^{(0)}, u^{(1)}, \dots$. So the gradient of \mathcal{F}_g is uniquely determined. The proposition then follows from (3.33). \square

4 Loop equation: the general case

In this section we drop the rational condition (1.6) and consider the general case when p, q, r are arbitrary complex numbers satisfying the *local Calabi–Yau condition* (1.3). We will give and prove the general version of the loop equation for the corresponding Hodge free energies.

As it is pointed out in Lemma 3.3, there are polynomials $H_g(z_0, z_1, \dots, z_{3g-2}; p, q, r)$ in p, q, r satisfying the relations (3.8) and the relations (1.19) (see [7]), where $g \geq 1$. It is then

clear that $H_g(z_0, \dots, z_{3g-2}; p, q, r)$, $g \geq 1$ are also polynomials of σ_1, σ_3 . For simplicity, we will denote them by $H_g(z_0, \dots, z_{3g-2}; \sigma_1, \sigma_3)$.

We will need the following lemma.

Lemma 4.1 *Let $\mathcal{P}(\sigma_1, \sigma_3)$ be a polynomial in $\mathbb{C}[\sigma_1, \sigma_3]$. If \mathcal{P} vanishes for the values*

$$(\sigma_1, \sigma_3) = \left(\frac{1}{K_1 + K_2} - \frac{1}{K_1} - \frac{1}{K_2}, \frac{2}{(K_1 + K_2)^3} - \frac{2}{K_1^3} - \frac{2}{K_2^3} \right),$$

where $K_1, K_2 \in \mathbb{N}$, $(K_1, K_2) = 1$, then $\mathcal{P} \equiv 0$.

Proof Suppose $\mathcal{P} \neq 0$. Fix K_1 to be any positive integer. We observe that the points

$$\left(\frac{1}{K_1 + K_2} - \frac{1}{K_1} - \frac{1}{K_2}, \frac{2}{(K_1 + K_2)^3} - \frac{2}{K_1^3} - \frac{2}{K_2^3} \right), \quad K_2 \in \mathbb{N}$$

belong to the irreducible algebraic curve

$$2K_1^3\sigma_1^3 - 6K_1\sigma_1 - 6 - K_1^3\sigma_3 = 0$$

on the (σ_1, σ_3) plane. It is easy to see that there are infinitely many of such points with $(K_1, K_2) = 1$. Hence the polynomial

$$2K_1^3\sigma_1^3 - 6K_1\sigma_1 - 6 - K_1^3\sigma_3$$

must divide $\mathcal{P}(\sigma_1, \sigma_3)$. This contradicts with the fact that a polynomial has a finite degree. The lemma is proved. \square

It follows from Lemma 4.1 that if a polynomial P in $\mathbb{C}[\sigma_1, \sigma_3][z_2, z_3, \dots][z_1, z_1^{-1}]$ is equal to $H_g(z_0, \dots, z_{3g-2}; \sigma_1, \sigma_3)$ for

$$(\sigma_1, \sigma_3) = \left(\frac{1}{K_1 + K_2} - \frac{1}{K_1} - \frac{1}{K_2}, \frac{2}{(K_1 + K_2)^3} - \frac{2}{K_1^3} - \frac{2}{K_2^3} \right)$$

with K_1, K_2 being arbitrary coprime positive integers, then $P \equiv H_g(z_0, \dots, z_{3g-2}; \sigma_1, \sigma_3)$, where $g \geq 2$. For $g = 1$, we have the explicit expression

$$H_1(z_0, z_1; \sigma_1, \sigma_3) = \frac{1}{24} \log z_1 + \frac{\sigma_1}{24} z_0.$$

To prove Theorem 1.4, we further introduce some notations. Let $\mathcal{B}_{n,k}(X_1, \dots, X_{n-k+1})$ be the exponential Bell polynomials. They can be defined via the generating function

$$\sum_{n,k \geq 0} \mathcal{B}_{n,k}(X_1, \dots, X_{n-k+1}) \frac{y^n}{n!} z^k = \exp \left(z \sum_{j \geq 1} X_j \frac{y^j}{j!} \right).$$

Let \mathcal{B}_n denote the complete Bell polynomials, i.e.

$$\mathcal{B}_0 := 1, \quad \mathcal{B}_n(X_1, \dots, X_n) := \sum_{k=1}^n \mathcal{B}_{n,k}(X_1, \dots, X_{n-k+1}), \quad n \geq 1.$$

Now we define $\tilde{P}_{0,n}(\xi; \sigma_1, \sigma_3) \in \mathbb{C}[\sigma_1, \sigma_3][(1 - \xi)^{-1}]$, $n \geq 0$ by the equation

$$\sum_{n \geq 0} z^{-n} \tilde{P}_{0,n}(\xi; \sigma_1, \sigma_3) = \sqrt{z} \Phi(z; \sigma_1, \sigma_3) \frac{1}{1 - \xi e^{-\partial_z}} \left(\frac{1}{\sqrt{z} \Phi(z; \sigma_1, \sigma_3)} \right), \quad (4.1)$$

where

$$\Phi(z; \sigma_1, \sigma_3) = \exp \left(- \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} (p^{2i-1} + q^{2i-1} + r^{2i-1}) z^{1-2i} \right).$$

Denote $\Phi = \Phi(z; \sigma_1, \sigma_3)$. Then the equation (4.1) can be written more explicitly as follows:

$$\sum_{n \geq 0} z^{-n} \tilde{P}_{0,n}(\xi; \sigma_1, \sigma_3) = \sum_{n \geq 0} \sum_{k=1}^{n+1} \frac{Q(n, k)}{(1 - \xi)^k} \sum_{m=0}^n \frac{(-1)^m (2m-1)!!}{2^m m! (n-m)!} z^{-m} \Phi \partial_z^{n-m} \left(\frac{1}{\Phi} \right), \quad (4.2)$$

where $Q(n, k)$ are the numbers defined in (3.27), and

$$\Phi \partial_z^\ell \left(\frac{1}{\Phi} \right) = \mathcal{B}_\ell \left(-\partial_z \log \Phi, \dots, -\partial_z^\ell \log \Phi \right), \quad \ell \geq 0. \quad (4.3)$$

We define, for $i, j \geq 0$, $\tilde{P}_{i,j} = \tilde{P}_{i,j}(\xi; \sigma_1, \sigma_3)$ by the following recursion:

$$\begin{aligned} \tilde{P}_{i,j} &= \tilde{P}_{j,i}, \\ \xi \partial_\xi \tilde{P}_{i,j} &= \tilde{P}_{i+1,j} + \tilde{P}_{i,j+1}. \end{aligned}$$

Note that the functions $f_{i,k}$ defined in (3.19) are just the Bell polynomials $\mathcal{B}_{i,k}(z_1, \dots, z_{i-k+1})$. By using these functions we introduce the following notations:

$$P_{i,j} = \sum_{k=0}^i \sum_{\ell=0}^j \mathcal{B}_{i,k}(z_1, \dots, z_{i-k+1}) \mathcal{B}_{j,\ell}(z_1, \dots, z_{j-\ell+1}) \tilde{P}_{k,\ell}(\xi; \sigma_1, \sigma_3), \quad i, j \geq 0.$$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4 It is easy to verify that the functions $P_{i,j}$ defined above coincide with the functions $B_{i,j}$ that are given in (3.20) when $\xi = K\zeta$ and

$$\begin{aligned} (\sigma_1, \sigma_3) &= \left(\frac{1}{h} - \frac{1}{K_1} - \frac{1}{K_2}, \frac{2}{h^3} - \frac{2}{K_1^3} - \frac{2}{K_2^3} \right), \\ (z_1, z_2, \dots) &= \left(u^{(1)}, u^{(2)}, \dots \right), \end{aligned}$$

where K_1, K_2 are arbitrary coprime positive integers. It then follows from Theorem 3.9 that $\Delta H := \sum_{g \geq 1} \epsilon^{2g-2} H_g(z_0, \dots, z_{3g-2}; \sigma_1, \sigma_3)$ satisfies equation (1.18) for the particular values of (σ_1, σ_3) , where $H_g(z_0, \dots, z_{3g-2}; \sigma_1, \sigma_3)$ are introduced in the beginning of this section. By definition we also see that $P_{i,j} \in \mathbb{C}[\sigma_1, \sigma_3][(1 - \xi)^{-1}; z_1, z_2, \dots]$. Then by using Lemma 4.1 we find that ΔH is a solution to the loop equation (1.18). As we do in the proof of Proposition 3.10, we deduce from (1.18) the following equation for each $g \geq 1$:

$$(\Theta, \dots, \Theta^{3g-1}) \widetilde{M}_g(\sigma_1, \sigma_3) \left(\frac{\partial H_g}{\partial z_0}, \dots, \frac{\partial H_g}{\partial z_{3g-2}} \right)^T = (\Theta, \dots, \Theta^{3g-1}) \widetilde{N}_g(\sigma_1, \sigma_3),$$

where $\widetilde{M}_g(\sigma_1, \sigma_3)$ is an invertible $(3g - 1)$ by $(3g - 1)$ matrix, $\widetilde{N}_g(\sigma_1, \sigma_3)$ is a column vector, and their entries are polynomials of σ_1, σ_3 . Therefore we verified that the gradient of H_g is uniquely determined by (1.18). The theorem is proved. \square

Example 4.2 By taking the coefficient of ϵ^0 in (1.18), we obtain the following equation for H_1 :

$$P_{0,0} \frac{\partial H_1}{\partial z_0} + (\partial P_{0,0} + P_{0,1}) \frac{\partial H_1}{\partial z_1} = \frac{\Theta^2}{16} - \left(\frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta.$$

The coefficients read

$$P_{0,0} = \Theta, \quad P_{0,1} = \frac{z_1}{2} (\Theta^2 - \Theta).$$

So we have

$$\frac{3z_1}{2} \frac{\partial H_1}{\partial z_1} \Theta^2 + \left(\frac{\partial H_1}{\partial z_0} - \frac{3z_1}{2} \frac{\partial H_1}{\partial z_1} \right) \Theta = \frac{\Theta^2}{16} - \left(\frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta,$$

which gives

$$\frac{\partial H_1}{\partial z_0} = \frac{\sigma_1}{24}, \quad \frac{\partial H_1}{\partial z_1} = \frac{1}{24z_1}.$$

Hence we obtain (1.21). In a similar way, we obtain formula (1.22) and the following expression of H_3 :

$$\begin{aligned} H_3 = & \frac{1}{82944} \frac{z_7}{z_1^3} - \frac{7}{46080} \frac{z_2 z_6}{z_1^4} - \frac{53}{161280} \frac{z_3 z_5}{z_1^4} + \frac{353}{322560} \frac{z_2^2 z_5}{z_1^5} - \frac{103}{483840} \frac{z_4^2}{z_1^4} \\ & + \frac{1273}{322560} \frac{z_2 z_3 z_4}{z_1^5} - \frac{83}{15120} \frac{z_2^3 z_4}{z_1^6} + \frac{59}{64512} \frac{z_3^3}{z_1^5} - \frac{83}{7168} \frac{z_2^2 z_3^2}{z_1^6} + \frac{59}{3024} \frac{z_2^4 z_3}{z_1^7} \\ & - \frac{5}{648} \frac{z_2^6}{z_1^8} + \frac{7\sigma_1}{138240} \frac{z_6}{z_1^2} - \frac{383\sigma_1}{967680} \frac{z_2 z_5}{z_1^3} + \frac{41\sigma_1^2}{580608} \frac{z_5}{z_1} - \frac{689\sigma_1}{967680} \frac{z_3 z_4}{z_1^3} + \frac{185\sigma_1}{96768} \frac{z_2^2 z_4}{z_1^4} \\ & - \frac{373\sigma_1^2}{1451520} \frac{z_2 z_4}{z_1^2} + \left(\frac{23\sigma_1^3}{580608} - \frac{11\sigma_3}{2903040} \right) z_4 + \frac{869\sigma_1}{322560} \frac{z_2 z_3^2}{z_1^4} - \frac{61\sigma_1^2}{322560} \frac{z_3^2}{z_1^2} \\ & - \frac{9343}{1451520} \frac{z_2^3 z_3}{z_1^5} + \frac{151\sigma_1^2}{207360} \frac{z_2^2 z_3}{z_1^3} - \left(\frac{19\sigma_1^3}{1451520} - \frac{19\sigma_3}{2903040} \right) \frac{z_2 z_3}{z_1} + \frac{131\sigma_1}{45360} \frac{z_2^5}{z_1^6} \\ & - \frac{19\sigma_1^2}{53760} \frac{z_2^4}{z_1^4} + \left(\frac{41\sigma_1^4}{4354560} - \frac{41\sigma_1 \sigma_3}{8709120} \right) z_1 z_3 + \left(\frac{\sigma_1^3}{108864} - \frac{\sigma_3}{217728} \right) \frac{z_2^3}{z_1^2} \\ & + \left(\frac{31\sigma_1^4}{4354560} - \frac{31\sigma_1 \sigma_3}{8709120} \right) z_2^2 - \left(\frac{\sigma_1^6}{13063680} - \frac{\sigma_1^3 \sigma_3}{13063680} + \frac{\sigma_3^2}{52254720} \right) z_1^4. \end{aligned} \quad (4.4)$$

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