Bilinear forms of Kählerian twistor spinors

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Generalization of twistor spinors to Kähler manifolds which are called Kählerian twistor spinors are considered. We find the differential equation satisfied by the bilinear forms of Kählerian twistor spinors. We show that the bilinear form equation reduces to Kählerian conformal Killing-Yano equation under special conditions. We also investigate the special cases of holomorphic and anti-holomorphic Kählerian twistor spinors and the bilinear forms of alternative definitions for Kählerian twistor spinors.

I. INTRODUCTION

On an n-dimensional Riemannian spin manifold M^n, a special class of spinors which are in the kernel of the twistor operator can exist and these are called twistor spinors [1–6]. They are solutions of the twistor equation

$$\nabla_X\psi = \frac{1}{n}\widetilde{X}.\not{D}\psi$$

for any vector field X and its metric dual $\widetilde{X}$. Here, ψ is the spinor, $\not{D}$ is the Dirac operator and . denotes the Clifford multiplication. Because of the conformal covariance of the twistor equation, twistor spinors are related to the conformal symmetries of the manifold [2, 6–8]. On the other hand, Kähler spin manifolds do not admit nontrivial twistor spinors [9, 10]. However, one can define a generalization of twistor spinors to Kähler spin manifolds. By considering the decomposition of the spinor bundle into the eigenbundles of the Kähler form on a Kähler manifold, one can define a Kählerian twistor operator and the spinors in the kernel of this Kählerian twistor operator are called Kählerian twistor spinors [11]. Kählerian twistor equation for a spinor ψ of type r on a Kähler manifold M^{2m} is written as

$$\nabla_X\psi = \frac{m+2}{8(r+1)(m-r+1)}(\widetilde{X}.\not{D}\psi + J\widetilde{X}.\not{D}^c\psi) + \frac{m-2r}{8(r+1)(m-r+1)}i(J\widetilde{X}.\not{D}\psi - \widetilde{X}.\not{D}^c\psi).$$

where J is the complex structure and $\not{D}^c$ is the conjugate Dirac operator defined in (20). The manifolds admitting nontrivial Kählerian twistor spinors are investigated in [11]. Special types of Kählerian twistor spinors are also defined previously in [12, 13]. Moreover, Kählerian Killing spinors as the limiting cases of the eigenvalues of the Dirac operator on compact Kähler manifolds can also be defined as the special cases of Kählerian twistor spinors [14, 15]

From a spin invariant inner product defined on the space of spinors, one can define a squaring map which is used in the construction of differential forms from spinors. These different degree differential forms are called bilinear forms of spinors. For a twistor spinor on a Riemannian manifold, the bilinear forms are related to the conformal hidden symmetries of the manifold. Bilinear forms of twistor spinors correspond to conformal Killing-Yano (CKY) forms which are antisymmetric generalizations of conformal Killing vector fields to higher degree differential forms [6, 16, 17]. So, the bilinear p-forms ω of a twistor spinor satisfy the following CKY equation on M^n

$$\nabla_X\omega = \frac{1}{p+1}i_Xd\omega - \frac{1}{n-p+1}\widetilde{X}\wedge\delta\omega$$

where d and δ are exterior derivative and co-derivative operators, i_X is the interior derivative or contraction operator with respect to X and $\wedge$ denotes the wedge product. In this paper, we investigate the construction of bilinear forms for Kählerian twistor spinors. We show that the bilinear forms of Kählerian twistor spinors satisfy a more general differential equation given in Theorem 1 which can reduce to the Kählerian CKY equation in some special cases. Properties of Kählerian CKY forms are studied in [18]. Moreover, we also consider the bilinear forms of holomorphic

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and anti-holomorphic Kählerian twistor spinors and also the special Kählerian twistor spinor definitions of [12] and [13].

The paper is organized as follows. In Section II, we give the basic definitions and operators on Kähler manifolds. One can find more details on Kähler manifolds in [19]. Section III includes the spin geometry of Kähler manifolds and the definitions of Kählerian twistor spinors. In Section IV, we construct the bilinear forms of Riemannian and Kählerian twistor spinors and prove the theorem for the bilinear form equation of Kählerian twistor spinors. Section V includes the special cases of holomorphic and anti-holomorphic Kählerian twistor spinors and the special definitions of [12] and [13].

II. OPERATORS ON KÄHLER MANIFOLDS

Let us consider a $n = 2m$ dimensional Kähler manifold (M^{2m}, g, J) with a Riemannian metric g and a complex structure J satisfying $J^2 = -I$ where I is the identity. Hence, J is integrable and satisfies $\nabla_X J = 0$ for any vector field X, namely the Nijenhuis tensor N^J of J vanishes. g and J have the relation $g(JX, JY) = g(X, Y)$ for any vector fields X and Y.

The complex structure J induces a splitting on the complexified tangent bundle $TM^{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$;

$$TM^{\mathbb{C}} = TM^{+} \oplus TM^{-}.$$

Here, TM^{+} corresponds to $+i$ eigenspace of J and TM^{-} corresponds to $-i$ eigenspace of J. For any $X \in TM^{\mathbb{C}}$, we have $X = X^{+} + X^{-}$ and $X^{+} \in TM^{+}$ and $X^{-} \in TM^{-}$ are defined as follows

$$X^{+} = \frac{1}{2}(X - iJX) \qquad , \qquad X^{-} = \frac{1}{2}(X + iJX).$$

Then, we can define an orthonormal basis $\{X_a, JX_a\}$ for $TM^{\mathbb{C}}$ with $a = 1, 2, ..., m$. Similarly, the above splitting also induce a splitting on the cotangent bundle $T^*M^{\mathbb{C}} = T^*M \otimes_{\mathbb{R}} \mathbb{C}$ and we can define an orthonormal basis $\{e^a, Je^a\}$ on the cotangent bundle with $a = 1, 2, ..., m$. We have the following relations between the frame and coframe basis;

$$\begin{aligned} e^a(X_b) &= g(X^a, X_b) = \delta^a_b \\ Je^a(JX_b) &= g(JX^a, JX_b) = g(X^a, X_b) = \delta^a_b \\ e^a(JX_b) &= g(X^a, JX_b) = -g(JX^a, X_b) \\ Je^a(X_b) &= g(JX^a, X_b) = -g(X^a, JX_b) \end{aligned}$$

where δ^a_b denotes the Kronecker delta.

The above splitting of tangent and cotangent bundles can also be generalized to complex tensor bundles on M. In particular, for the exterior bundle $\Lambda M^{\mathbb{C}} = \Lambda M \otimes_{\mathbb{R}} \mathbb{C}$, we have the following splitting;

$$\Lambda^r M^{\mathbb{C}} = \bigoplus_{p+q=r} \Lambda^p M^{+} \otimes \Lambda^q M^{-}.$$

The elements of $\Lambda^{p,q} M^{\mathbb{C}} := \Lambda^p M^{+} \otimes \Lambda^q M^{-}$ are called as the forms of type (p, q). So, a complex r-form is a sum of (p, q)-forms with $p + q = r$.

A closed (1,1)-form Ω which is written in terms of coframe basis as

$$\Omega = \frac{1}{2} e^a \wedge Je_a \tag{1}$$

is called the Kähler 2-form of M and satisfies $d\Omega = 0$ where d is the exterior derivative operator. It can also be written in terms of the metric as $\Omega(X, Y) := g(JX, Y)$ for any vector fields X and Y. The exterior derivative of a (p, q)-form is written as a sum of $(p + 1, q)$ and $(p, q + 1)$-forms. So, the exterior derivative operator d acts as

$$d : \Lambda^{p,q} M^{\mathbb{C}} \longrightarrow \Lambda^{p+1,q} M^{\mathbb{C}} \oplus \Lambda^{p,q+1} M^{\mathbb{C}}.$$

Similarly, the coderivative operator δ acts as

$$\delta : \Lambda^{p,q} M^{\mathbb{C}} \longrightarrow \Lambda^{p-1,q} M^{\mathbb{C}} \oplus \Lambda^{p,q-1} M^{\mathbb{C}}.$$

On the other hand, the interior derivative operator i_X with respect to a vector field X and i_{JX} with respect to a vector field JX correspond to contractions of (p,q)-forms with respect to these vector fields and act as follows

$$i_X : \Lambda^{p,q}M^{\mathbb{C}} \longrightarrow \Lambda^{p-1,q}M^{\mathbb{C}}$$
$$i_{JX} : \Lambda^{p,q}M^{\mathbb{C}} \longrightarrow \Lambda^{p,q-1}M^{\mathbb{C}}$$

and they satisfy the following relations;

$$i_{JX} = Ji_X \qquad , \qquad Ji_{JX} = -i_X. \tag{2}$$

For zero torsion, the exterior derivative and coderivative operators can be written in terms of the covariant derivative ∇_X with respect to a vector field X as follows;

$$d = e^a \wedge \nabla_{X_a} \tag{3}$$
$$\delta = -i_{X^a}\nabla_{X_a}. \tag{4}$$

By using the Kähler form Ω given in (1), we can also define two operators acting on (p,q)-forms on M. These are the wedge product and contraction of a (p,q)-form α with Ω and defined as follows;

$$L\alpha := \Omega \wedge \alpha = \frac{1}{2}e^a \wedge Je_a \wedge \alpha \tag{5}$$
$$\Lambda\alpha := i_\Omega \alpha = \frac{1}{2}i_{JX^a}i_{X_a}\alpha \tag{6}$$

and they act as

$$L : \Lambda^{p,q}M^{\mathbb{C}} \longrightarrow \Lambda^{p+1,q+1}M^{\mathbb{C}}$$
$$\Lambda : \Lambda^{p,q}M^{\mathbb{C}} \longrightarrow \Lambda^{p-1,q-1}M^{\mathbb{C}}.$$

If a (p,q)-form α is in the kernel of the operator Λ, that is $\Lambda\alpha = 0$, then α is called as a primitive form.

The complex structure J can also be stated as a derivation on the space of (p,q)-forms acting as

$$J : \Lambda^{p,q}M^{\mathbb{C}} \longrightarrow \Lambda^{p-1,q+1}M^{\mathbb{C}}.$$

For any $\alpha \in \Lambda^{p,q}M^{\mathbb{C}}$, it can be written as

$$J\alpha := Je^a \wedge i_{X_a}\alpha \tag{7}$$

and it has the property

$$J(\alpha \wedge \beta) = J\alpha \wedge \beta + \alpha \wedge J\beta \tag{8}$$

where $\beta \in \Lambda^{p,q}M^{\mathbb{C}}$. Moreover, J commutes with both the operators L and Λ;

$$[J, \Lambda] = 0 = [J, L]. \tag{9}$$

The operator Λ commutes with the interior derivative operations i_X and i_{JX};

$$[i_X, \Lambda] = 0 \qquad , \qquad [i_{JX}, \Lambda] = 0 \tag{10}$$

and the commutation relations between the operator L and the interior derivative operations are given by

$$[i_X, L] = J\widetilde{X}\wedge \qquad , \qquad [i_{JX}, L] = -\widetilde{X}\wedge \tag{11}$$

where the 1-form $\widetilde{X}$ is the metric dual of the vector field X.

The generalizations of the operators d and δ given in (3) and (4) to include the complex structure J can also be constructed in the following way;

$$d^c = Je^a \wedge \nabla_{X_a} \tag{12}$$
$$\delta^c = -i_{JX^a}\nabla_{X_a} \tag{13}$$

and the commutation relations between the operators d, d^c, δ and δ^c and the operators L and Λ are given as follows;

$$[L, d] = 0 = [L, d^c] \qquad , \qquad [\Lambda, \delta] = 0 = [\Lambda, \delta^c] \tag{14}$$

$$[L, \delta] = d^c \quad , \quad [L, \delta^c] = -d \quad , \quad [\Lambda, d] = -\delta^c \quad , \quad [\Lambda, d^c] = \delta. \tag{15}$$

III. KÄHLERIAN TWISTOR SPINORS

Let the Kähler manifold M^{2m} admits a spin structure and ΣM denotes the spinor bundle on M. The complex volume form on M^{2m} is defined by

$$z^{\mathbb{C}} = i^m \prod_{i=1}^{m} e_i.Je_i \tag{16}$$

where . denotes the Clifford multiplication. Since the dimension of M is even, $z^{\mathbb{C}}$ has +1 and -1 eigenvalues and the spinor bundle ΣM splits into a direct sum of spaces containing different eigenvalue eigenspinors of $z^{\mathbb{C}}$;

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M.$$

The Kähler form Ω given in (1) can be written in terms of Clifford multiplication as

$$\Omega = \frac{1}{2} e^a.Je_a \tag{17}$$

and under the action of Ω, the spinor bundle ΣM splits into different eigenbundles of Ω;

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M \tag{18}$$

where each $\Sigma_r M$ is the eigenbundle of Ω corresponding to the eigenvalue $i(2r-m)$. The rank of each eigenbundle is given by $\text{rank}_{\mathbb{C}}(\Sigma_r M) = \binom{m}{r}$. The elements of $\Sigma_r M$ are called Kählerian spinors of type r. The eigenbundles of the complex volume form $z^{\mathbb{C}}$ also splits into the eigenbundles of Ω and we have

$$\begin{aligned} \Sigma^+ M &= \bigoplus_{0\leq r\leq 2m} \Sigma_r M \quad , \quad \text{for } r \text{ even} \\ \Sigma^- M &= \bigoplus_{0\leq r\leq 2m} \Sigma_r M \quad , \quad \text{for } r \text{ odd} \end{aligned}$$

The Levi-Civita connection ∇ defined on $\Lambda M^{\mathbb{C}}$ can be induced on the spinor bundle ΣM and it preserves the above splitting of the spinor bundle. The Dirac operator on ΣM is defined as follows;

$$\displaystyle{\not}D := e^a.\nabla_{X_a} \tag{19}$$

and the conjugate Dirac operator is given by

$$\displaystyle{\not}D^c := Je^a.\nabla_{X_a}. \tag{20}$$

Both are square roots of the Laplacian and have the property $\displaystyle{\not}D^2 = (\displaystyle{\not}D^c)^2$. Moreover, one can also define the operators

$$\displaystyle{\not}D^+ = \frac{1}{2}(\displaystyle{\not}D - i\displaystyle{\not}D^c) \quad , \quad \displaystyle{\not}D^- = \frac{1}{2}(\displaystyle{\not}D + i\displaystyle{\not}D^c) \tag{21}$$

and we have $\displaystyle{\not}D = \displaystyle{\not}D^+ + \displaystyle{\not}D^-$ and $\displaystyle{\not}D^c = i(\displaystyle{\not}D^+ - \displaystyle{\not}D^-)$. The Dirac operator $\displaystyle{\not}D$ acts on the subbundle $\Sigma_r M$ as

$$\displaystyle{\not}D : \Sigma_r M \longrightarrow \Sigma_{r-1} M \oplus \Sigma_{r+1} M$$

The metric duals of the vector fields X^+ and X^- defined in Section II also act on the subbundle $\Sigma_r M$ and their action change the subbundles as follows;

$$\widetilde{X}^+.\Sigma_r M \subseteq \Sigma_{r+1} M \quad , \quad \widetilde{X}^-.\Sigma_r M \subseteq \Sigma_{r-1} M.$$

In general, the Clifford multiplication of a 1-form with an element of $\Sigma_r M$ gives an element of $\Sigma_{r-1} M \oplus \Sigma_{r+1} M$.

We define special types of spinors satisfying some differential equations in terms of ∇, $\displaystyle{\not}D$ and $\displaystyle{\not}D^c$.

Definition 1. *Let M be a n-dimensional Riemannian spin manifold and $\psi \in \Sigma M$ is a spinor field on M. If ψ satisfies the following equation for all vector fields $X \in TM$ and their metric duals $\widetilde{X} \in T^*M$*

$$\nabla_X \psi = \frac{1}{n} \widetilde{X}.\displaystyle{\not}D\psi, \tag{22}$$

then ψ is called as a Riemannian twistor spinor.

In the case of Kähler manifolds, the definition of twistor spinors can be generalized as follows;

Definition 2. *Let (M^{2m}, g, J) be a 2m-dimensional Kähler manifold. If $\psi \in \Sigma_r M$ is a spinor of type r and satisfies the following equalities*

$$\nabla_{X^+}\psi = \frac{1}{2(m-r+1)}\widetilde{X}^+.\not{D}^-\psi \tag{23}$$

$$\nabla_{X^-}\psi = \frac{1}{2(r+1)}\widetilde{X}^-.\not{D}^+\psi \tag{24}$$

for all $X = X^+ + X^- \in TM^{\mathbb{C}}$, then ψ is called as a Kählerian twistor spinor. The defining equations above can be combined into a single equation for Kählerian twistor spinors as in the following form

$$\nabla_X\psi = \frac{m+2}{8(r+1)(m-r+1)}(\widetilde{X}.\not{D}\psi + J\widetilde{X}.\not{D}^c\psi) + \frac{m-2r}{8(r+1)(m-r+1)}i(J\widetilde{X}.\not{D}\psi - \widetilde{X}.\not{D}^c\psi). \tag{25}$$

From the equations (23) and (24), some special types of Kählerian twistor spinors can also be defined;

Definition 3. *If a Kählerian twistor spinor ψ is in the kernel of the operator $\not{D}^+$, that is $\not{D}^+\psi = 0$, then it is called as a holomorphic Kählerian twistor spinor and is parallel with respect to all $X^- \in TM^-$. Similarly, if a Kählerian twistor spinor ψ is in the kernel of the operator $\not{D}^-$, that is $\not{D}^-\psi = 0$, then it is called as an anti-holomorphic Kählerian twistor spinor and is parallel with respect to all $X^+ \in TM^+$.*

Special cases of Definition 2 are considered as the definition of Kählerian twistor spinors in the literature. By considering any real numbers a and b and taking the first term on the right hand side of (25), one can write the equality

$$\nabla_X\psi = a\widetilde{X}.\not{D}\psi + bJ\widetilde{X}.\not{D}^c\psi \tag{26}$$

which is the definition of Kählerian twistor spinors given by Hijazi [13]. If we choose the real numbers $a = b = \frac{1}{4(r+1)}$ for $0 \leq r \leq m-2$, we have

$$\nabla_X\psi = \frac{1}{4r}(\widetilde{X}.\not{D}\psi + J\widetilde{X}.\not{D}^c\psi) \tag{27}$$

and this is the definition of Kählerian twistor spinors given by Kirchberg [12]. If we choose $a = \frac{1}{n}$ and $b = 0$ in (26), then we obtain the Riemannian twistor equation given in (22).

For a Kähler manifold M^{2m}, if m is even and the type of the Kählerian twistor spinor is $r = \frac{m}{2}$, then the Kählerian twistor equations (23) and (24) reduce to

$$\nabla_X\psi = \frac{1}{m+2}(\widetilde{X}^+.\not{D}^-\psi + \widetilde{X}^-.\not{D}^+\psi). \tag{28}$$

If M^{2m} is a compact Kähler spin manifold of positive constant scalar curvature, then (28) does not have a non-trivial solution and there are no Kählerian twistor spinors of middle dimension type [11]. For a connected Kähler spin manifold M^{2m}, the dimension of the space of Kählerian twistor spinors of type r, which is denoted by $KT(r)$, is bounded by [11]

$$\dim(KT(r)) \leq \binom{m}{r} + \binom{m}{r+1} + \binom{m}{r-1}.$$

The classification of manifolds admitting Kählerian twistor spinors are investigated in [11]. For a compact Kählerian spin manifold M^{2m} of positive constant scalar curvature, a Kählerian twistor spinor $\psi \in \Sigma_r M$ of type r is an anti-holomorphic Kählerian twistor spinor if $r < \frac{m}{2}$ or a holomorphic Kählerian twistor spinor if $r > \frac{m}{2}$. As a special case, weakly Bochner flat manifolds admit Kählerian twistor spinors [11].

IV. BILINEAR FORMS

For a spinor $\psi \in \Sigma M$, a dual spinor $\overline{\psi} \in \Sigma^* M$ can be defined by using the spin invariant inner product (,) on ΣM. For $\psi, \phi \in \Sigma M$

$$\begin{aligned} \Sigma^* M \times \Sigma M &\longrightarrow \mathbb{F} \\ \overline{\psi}, \phi &\longmapsto \overline{\psi}(\phi) = (\psi, \phi) \end{aligned}$$

where $\mathbb{F}$ is the division algebra on which the Clifford algebra is defined. The tensor product of spinors and dual spinors $\Sigma M \otimes \Sigma^* M$ acts on ΣM by the Clifford multiplication. For $\psi, \phi, \kappa \in \Sigma M$ and $\overline{\phi} \in \Sigma^* M$, we have

$$(\psi\overline{\phi}).\kappa = (\phi, \kappa)\psi \tag{29}$$

where we denote $\psi \otimes \overline{\phi} = \psi\overline{\phi}$. So, the elements of $\Sigma M \otimes \Sigma^* M$ can be regarded as linear transformations on ΣM and hence they can be identified with the elements of the Clifford bundle $Cl(M)$. Then, for an orthonormal basis $\{e^a\}$, we can write $\psi\overline{\phi}$ as a sum of different degree differential forms as follows

$$\begin{aligned}\psi\overline{\phi} &= (\phi, \psi) + (\phi, e_a.\psi)e^a + (\phi, e_{ba}.\psi)e^{ab} + ... \\ &\quad +(\phi, e_{a_p...a_2a_1}.\psi)e^{a_1a_2...a_p} + ... + (-1)^{\lfloor n/2 \rfloor}(\phi, z.\psi)z\end{aligned} \tag{30}$$

where $e^{a_1a_2...a_p} = e^{a_1} \wedge e^{a_2} \wedge ... \wedge e^{a_p}$, n is the dimension of the manifold and z is the volume form. If we choose $\psi = \phi$, each term on the right hand side of (30) are called spinor bilinears of ψ and the p-form bilinear is defined as

$$(\psi\overline{\psi})_p = (\psi, e_{a_p...a_2a_1}.\psi)e^{a_1a_2...a_p}. \tag{31}$$

The inner product (,) is said to have $\mathcal{J}$ involution with the following equality

$$(\phi, \omega.\psi) = (\omega^{\mathcal{J}}.\phi, \psi) \tag{32}$$

for any inhomogeneous Clifford form ω and $\mathcal{J}$ can be ξ, ξ^*, $\xi\eta$ and $\xi\eta^*$ with ξ and η acts on a p-form α as $\alpha^\xi = (-1)^{\lfloor p/2 \rfloor}\alpha$ and $\alpha^\eta = (-1)^p\alpha$ where $\lfloor\,\rfloor$ is the floor function taking the integral part of the argument and * is the complex conjugation. Moreover, for any Clifford forms α and β, we have $(\alpha.\beta)^\xi = \beta^\xi.\alpha^\xi$.

Proposition 1. *For a Riemannian twistor spinor $\psi \in \Sigma M$ on a n-dimensional spin manifold M which satisfies (22), all the p-form bilinears $(\psi\overline{\psi})_p$ satisfies the following conformal Killing-Yano (CKY) equation*

$$\nabla_X(\psi\overline{\psi})_p = \frac{1}{p+1}i_X d(\psi\overline{\psi})_p - \frac{1}{n-p+1}\widetilde{X} \wedge \delta(\psi\overline{\psi})_p \tag{33}$$

for all vector fields $X \in TM$ and their metric duals $\widetilde{X} \in T^*M$.

The proof of the proposition can be found in [16]. CKY forms correspond to antisymmetric generalizations of conformal Killing vector fields to higher degree differential forms.

On a Kähler manifold M^{2m}, the tensor product of spinors and dual spinors can also be written as a sum of different degree differential forms. Because of the decomposition of the exterior bundle given in Section II, the sum is written in terms of (p,q)-forms. For a spinor $\psi \in \Sigma M$ and its dual $\overline{\psi} \in \Sigma^* M$, we have the following decomposition

$$\begin{aligned}\psi\overline{\psi} &= (\psi, \psi) + (\psi, e_a.\psi)e^a + (\psi, Je_a.\psi)Je^a + (\psi, (e_b \wedge e_a).\psi)e^a \wedge e^b \\ &\quad +(\psi, (e_b \wedge Je_a).\psi)Je^a \wedge e^b + (\psi, (Je_b \wedge Je_a).\psi)Je^a \wedge Je^b + ... \\ &\quad +(\psi, (e_{a_p} \wedge ... \wedge e_{a_1} \wedge Je_{b_q} \wedge ... \wedge Je_{b_1}).\psi)Je^{b_1} \wedge ... \wedge Je^{b_q} \wedge e^{a_1} \wedge ... \wedge e^{a_p} + ... \\ &\quad +(\psi, (e_{a_m} \wedge ... \wedge e_{a_1} \wedge Je_{b_m} \wedge ... \wedge Je_{b_1}).\psi)Je^{b_1} \wedge ... \wedge Je^{b_m} \wedge e^{a_1} \wedge ... \wedge e^{a_m}.\end{aligned} \tag{34}$$

So, we can define the (p,q)-form bilinears of a spinor ψ as

$$(\psi\overline{\psi})_{(p,q)} = (\psi, (e_{a_p} \wedge ... \wedge e_{a_1} \wedge Je_{b_q} \wedge ... \wedge Je_{b_1}).\psi)Je^{b_1} \wedge ... \wedge Je^{b_q} \wedge e^{a_1} \wedge ... \wedge e^{a_p}. \tag{35}$$

Theorem 1. *For a Kählerian twistor spinor $\psi \in \Sigma M$ of type r on a $2m$-dimensional Kählerian spin manifold M^{2m} which satisfies (25), the (p,q)-form bilinears $(\psi\overline{\psi})_{(p,q)}$ satisfies the following equation*

$$\begin{aligned}\nabla_{X_a}(\psi\overline{\psi})_{(p,q)} &= \frac{1}{p+1}i_{X_a}\left(d(\psi\overline{\psi})_{(p,q)} - 2L\alpha_{(p,q-1)} - 2J\beta_{(p,q+1)}\right) \\ &\quad -\frac{1}{m-p+1}e_a \wedge \left(\delta(\psi\overline{\psi})_{(p,q)} - J\alpha_{(p,q-1)} - 2\Lambda\beta_{(p,q+1)}\right) \\ &\quad +\frac{1}{q+1}i_{JX_a}\left(d^c(\psi\overline{\psi})_{(p,q)} + 2L\gamma_{(p-1,q)} - 2J\mu_{(p+1,q)}\right) \\ &\quad -\frac{1}{m-q+1}Je_a \wedge \left(\delta^c(\psi\overline{\psi})_{(p,q)} - J\gamma_{(p-1,q)} - 2\Lambda\mu_{(p+1,q)}\right).\end{aligned} \tag{36}$$

Here, the forms $\alpha_{(p,q-1)}$, $\beta_{(p,q+1)}$, $\gamma_{(p-1,q)}$ and $\mu_{(p+1,q)}$ are defined as follows

$$\alpha_{(p,q-1)} = \big(k\mathcal{A}+l\mathcal{B}\big)_{(p,q-1)} \tag{37}$$
$$\beta_{(p,q+1)} = \big(k\mathcal{C}+l\mathcal{D}\big)_{(p,q+1)} \tag{38}$$
$$\gamma_{(p-1,q)} = \big(k\mathcal{B}-l\mathcal{A}\big)_{(p-1,q)} \tag{39}$$
$$\mu_{(p+1,q)} = \big(k\mathcal{D}-l\mathcal{C}\big)_{(p+1,q)} \tag{40}$$

where

$$\mathcal{A} = \not{d}^c(\psi\overline{\psi}) - 2Je^b\wedge(\psi\overline{\nabla_{X_b}\psi}) \tag{41}$$
$$\mathcal{B} = \not{d}(\psi\overline{\psi}) - 2e^b\wedge(\psi\overline{\nabla_{X_b}\psi}) \tag{42}$$
$$\mathcal{C} = \not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi}) \tag{43}$$
$$\mathcal{D} = \not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi}) \tag{44}$$

and

$$k := \frac{m+2}{8(r+1)(m-r+1)} \quad , \quad l := \frac{m-2r}{8(r+1)(m-r+1)}. \tag{45}$$

Proof. First we calculate the covariant derivative of the (p,q)-form bilinears $(\psi\overline{\psi})_{(p,q)}$ of a Kählerian twistor spinor ψ satisfying (25). Since the covariant derivative ∇ is compatible with the spin invariant inner product $(\, , \,)$, we can write

$$\nabla_{X_a}(\psi\overline{\psi})_{(p,q)} = \big((\nabla_{X_a}\psi)\overline{\psi}\big)_{(p,q)} + \big(\psi\overline{\nabla_{X_a}\psi}\big)_{(p,q)}$$

and from (25), we have

$$\begin{aligned}\nabla_{X_a}(\psi\overline{\psi})_{(p,q)} &= k\Big[\big((e_a.\not{D}\psi)\overline{\psi}\big)_{(p,q)} + \big((Je_a.\not{D}^c\psi)\overline{\psi}\big)_{(p,q)} + \big(\psi\overline{e_a.\not{D}\psi}\big)_{(p,q)} + \big(\psi\overline{Je_a.\not{D}^c\psi}\big)_{(p,q)}\Big] \\ &\quad + l\Big[\big((Je_a.\not{D}\psi)\overline{\psi}\big)_{(p,q)} - \big((e_a.\not{D}^c\psi)\overline{\psi}\big)_{(p,q)} + \big(\psi\overline{Je_a.\not{D}\psi}\big)_{(p,q)} - \big(\psi\overline{e_a.\not{D}^c\psi}\big)_{(p,q)}\Big]\end{aligned}$$

where we have used the definitions of k and l given in (45). From the equalities (29) and (32), we can write the identites

$$\begin{aligned}\overline{e_a.e^b.\nabla_{X_b}\psi} &= \overline{\nabla_{X_b}\psi}.e^{b\mathcal{J}}.e_a{}^{\mathcal{J}} \\ &= \overline{\nabla_{X_b}\psi}.e^b.e_a\end{aligned}$$

which is true for all choices of $\mathcal{J}$. Similar identites can also be obtained for the terms including Je_a basis. By using these identites and the definitions of $\not{D}$ and $\not{D}^c$ given in (19) and (20), we obtain

$$\begin{aligned}\nabla_{X_a}(\psi\overline{\psi})_{(p,q)} &= k\Big[\big(e_a.e^b.(\nabla_{X_b}\psi)\overline{\psi}\big)_{(p,q)} + \big(Je_a.Je^b.(\nabla_{X_b}\psi)\overline{\psi}\big)_{(p,q)} + \big(\psi\overline{\nabla_{X_b}\psi}.e^b.e_a\big)_{(p,q)} + \big(\psi\overline{\nabla_{X_b}\psi}.Je^b.Je_a\big)_{(p,q)}\Big] \\ &\quad + l\Big[\big(Je_a.e^b.(\nabla_{X_b}\psi)\overline{\psi}\big)_{(p,q)} - \big(e_a.Je^b.(\nabla_{X_b}\psi)\overline{\psi}\big)_{(p,q)} + \big(\psi\overline{\nabla_{X_b}\psi}.e^b.Je_a\big)_{(p,q)} - \big(\psi\overline{\nabla_{X_b}\psi}.Je^b.e_a\big)_{(p,q)}\Big]\end{aligned}$$

and from the identity $(\nabla_{X_b}\psi)\overline{\psi} = \nabla_{X_b}(\psi\overline{\psi}) - \psi\overline{\nabla_{X_b}\psi}$, we can write

$$\begin{aligned}\nabla_{X_a}(\psi\overline{\psi})_{(p,q)} &= k\Big[\big(e_a.e^b.\nabla_{X_b}(\psi\overline{\psi})\big)_{(p,q)} - \big(e_a.e^b.\psi\overline{\nabla_{X_b}\psi}\big)_{(p,q)} + \big(Je_a.Je^b.\nabla_{X_b}(\psi\overline{\psi})\big)_{(p,q)} \\ &\quad - \big(Je_a.Je^b.\psi\overline{\nabla_{X_b}\psi}\big)_{(p,q)} + \big(\psi\overline{\nabla_{X_b}\psi}.e^b.e_a\big)_{(p,q)} + \big(\psi\overline{\nabla_{X_b}\psi}.Je^b.Je_a\big)_{(p,q)}\Big] \\ &\quad + l\Big[\big(Je_a.e^b.\nabla_{X_b}(\psi\overline{\psi})\big)_{(p,q)} - \big(Je_a.e^b.\psi\overline{\nabla_{X_b}\psi}\big)_{(p,q)} - \big(e_a.Je^b.\nabla_{X_b}(\psi\overline{\psi})\big)_{(p,q)} \\ &\quad + \big(e_a.Je^b.\psi\overline{\nabla_{X_b}\psi}\big)_{(p,q)} + \big(\psi\overline{\nabla_{X_b}\psi}.e^b.Je_a\big)_{(p,q)} - \big(\psi\overline{\nabla_{X_b}\psi}.Je^b.e_a\big)_{(p,q)}\Big].\end{aligned}$$

For any Clifford form ω, one can write the Clifford product of a 1-form e^a and ω in terms of wedge product and interior product as follows

$$\begin{aligned} e^a.\omega &= e^a \wedge \omega + i_{X^a}\omega \\ \omega.e^a &= e^a \wedge \omega^\eta - i_{X^a}\omega^\eta. \end{aligned} \tag{46}$$

From this property, we can write the identites

$$\begin{aligned} \left(\psi\overline{\nabla_{X_b}\psi}.e^b.e_a\right)_{(p,q)} - \left(e_a.e^b.\psi\overline{\nabla_{X_b}\psi}\right)_{(p,q)} &= -2e_a \wedge e^b \wedge \left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p-2,q)} - 2i_{X_a}i_{X^b}\left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p+2,q)} \\ \left(\psi\overline{\nabla_{X_b}\psi}.Je^b.Je_a\right)_{(p,q)} - \left(Je_a.Je^b.\psi\overline{\nabla_{X_b}\psi}\right)_{(p,q)} &= -2Je_a \wedge Je^b \wedge \left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p,q-2)} - 2i_{JX_a}i_{JX^b}\left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p,q+2)} \\ \left(\psi\overline{\nabla_{X_b}\psi}.e^b.Je_a\right)_{(p,q)} - \left(Je_a.e^b.\psi\overline{\nabla_{X_b}\psi}\right)_{(p,q)} &= -2Je_a \wedge e^b \wedge \left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p-1,q-1)} - 2i_{JX_a}i_{X^b}\left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p+1,q+1)} \\ -\left(\psi\overline{\nabla_{X_b}\psi}.Je^b.e_a\right)_{(p,q)} - \left(e_a.Je^b.\psi\overline{\nabla_{X_b}\psi}\right)_{(p,q)} &= 2e_a \wedge Je^b \wedge \left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p-1,q-1)} + 2i_{X_a}i_{JX^b}\left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p+1,q+1)}. \end{aligned}$$

So, from these identities and the definitions of $\not{d} = e^a.\nabla_{X_a}$ and $\not{d}^c = Je^a.\nabla_{X_a}$ acting on Clifford forms, the covariant derivative of $(\psi\overline{\psi})_{(p,q)}$ is written as

$$\begin{aligned} \nabla_{X_a}(\psi\overline{\psi})_{(p,q)} &= k\Big[\left(e_a.\not{d}(\psi\overline{\psi})\right)_{(p,q)} - 2e_a \wedge e^b \wedge \left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p-2,q)} - 2i_{X_a}i_{X^b}\left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p+2,q)} \\ &\quad + \left(Je_a.\not{d}^c(\psi\overline{\psi})\right)_{(p,q)} - 2Je_a \wedge Je^b \wedge \left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p,q-2)} - 2i_{JX_a}i_{JX^b}\left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p,q+2)}\Big] \\ &\quad + l\Big[\left(Je_a.\not{d}(\psi\overline{\psi})\right)_{(p,q)} - 2Je_a \wedge e^b \wedge \left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p-1,q-1)} - 2i_{JX_a}i_{X^b}\left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p+1,q+1)} \\ &\quad - \left(e_a.\not{d}^c(\psi\overline{\psi})\right)_{(p,q)} + 2e_a \wedge Je^b \wedge \left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p-1,q-1)} + 2i_{X_a}i_{JX^b}\left(\psi\overline{\nabla_{X_b}\psi}\right)_{(p+1,q+1)}\Big] \end{aligned}$$

and by using (46) again, we finally obtain

$$\begin{aligned} \nabla_{X_a}(\psi\overline{\psi})_{(p,q)} &= k\Big[e_a \wedge \left(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\right)_{(p-1,q)} + i_{X_a}\left(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\right)_{(p+1,q)} \\ &\quad + Je_a \wedge \left(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\right)_{(p,q-1)} + i_{JX_a}\left(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\right)_{(p,q+1)}\Big] \\ &\quad + l\Big[Je_a \wedge \left(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\right)_{(p,q-1)} + i_{JX_a}\left(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\right)_{(p,q+1)} \\ &\quad - e_a \wedge \left(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\right)_{(p-1,q)} - i_{X_a}\left(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\right)_{(p+1,q)}\Big]. \end{aligned} \tag{47}$$

Now, by using the equalities (3), (4), (12) and (13), we can calculate the action of the operators d, δ, d^c and δ^c on the bilinears $(\psi\overline{\psi})_{(p,q)}$. Because of the antisymmetry of the wedge product, we have $e^a \wedge e_a = 0$ and from the property $e^a \wedge i_{X_a}\alpha_{(p,q)} = p\alpha_{(p,q)}$ for any (p,q)-form α, we obtain

$$\begin{aligned} d(\psi\overline{\psi})_{(p,q)} &= e^a \wedge \nabla_{X_a}(\psi\overline{\psi})_{(p,q)} \\ &= k\Big[(p+1)\left(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\right)_{(p+1,q)} + e^a \wedge Je_a \wedge \left(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\right)_{(p,q-1)} \\ &\quad + e^a \wedge i_{JX_a}\left(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\right)_{(p,q+1)}\Big] \\ &\quad + l\Big[e^a \wedge Je_a \wedge \left(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\right)_{(p,q-1)} + e^a \wedge i_{JX_a}\left(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\right)_{(p,q+1)} \\ &\quad - (p+1)\left(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\right)_{(p+1,q)} \\ &= (p+1)\Big(k\left(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\right) - l\left(\not{d}^c(\psi\overline{\psi} - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi}))\right)\Big)_{(p+1,q)} \\ &\quad + e^a \wedge Je_a \wedge \Big(k\left(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\right) + l\left(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\right)\Big)_{(p,q-1)} \\ &\quad + e^a \wedge i_{JX_a}\Big(k\left(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\right) + l\left(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\right)\Big)_{(p,q+1)}. \end{aligned} \tag{48}$$

In a similar way, by considering the equalities $Je^a \wedge Je_a = 0$ and $Je^a \wedge i_{JX_a}\alpha_{(p,q)} = q\alpha_{(p,q)}$, we can find

$$
\begin{aligned}
d^c(\psi\overline{\psi})_{(p,q)} &= Je^a \wedge \nabla_{X_a}(\psi\overline{\psi})_{(p,q)} \\
&= (q+1)\Big(k\big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}(\psi\overline{\psi} - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi}))\big)\Big)_{(p,q+1)} \\
&\quad + Je^a \wedge e_a \wedge \Big(k\big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) - l\big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p-1,q)} \\
&\quad + Je^a \wedge i_{X_a}\Big(k\big(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\big) - l\big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p+1,q)}.
\end{aligned} \tag{49}
$$

We also have the identities $i_{X^a}i_{X_a} = 0$, $i_{X^a}(Je_a) = 0$ and $i_{X^a}(e_a \wedge \alpha_{(p,q)}) = (m-p)\alpha_{(p,q)}$. By using them, we find

$$
\begin{aligned}
\delta(\psi\overline{\psi})_{(p,q)} &= -i_{X^a}\nabla_{X_a}(\psi\overline{\psi})_{(p,q)} \\
&= (m-p+1)\Big(-k\big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}^c(\psi\overline{\psi} - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi}))\big)\Big)_{(p-1,q)} \\
&\quad + Je^a \wedge i_{X_a}\Big(k\big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p,q-1)} \\
&\quad + i_{X^a}i_{JX_a}\Big(k\big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p,q+1)}.
\end{aligned} \tag{50}
$$

Similarly, from the identities $i_{JX^a}i_{JX_a} = 0$, $i_{JX^a}(e_a) = 0$ and $i_{JX^a}(Je_a \wedge \alpha_{(p,q)}) = (m-q)\alpha_{(p,q)}$, we obtain

$$
\begin{aligned}
\delta^c(\psi\overline{\psi})_{(p,q)} &= -i_{JX^a}\nabla_{X_a}(\psi\overline{\psi})_{(p,q)} \\
&= -(m-q+1)\Big(k\big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}(\psi\overline{\psi} - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi}))\big)\Big)_{(p,q-1)} \\
&\quad + e^a \wedge i_{JX_a}\Big(k\big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) - l\big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p-1,q)} \\
&\quad + i_{JX^a}i_{X_a}\Big(-k\big(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p+1,q)}.
\end{aligned} \tag{51}
$$

The next step in the proof consists of the contractions of $d(\psi\overline{\psi})_{(p,q)}$ and $d^c(\psi\overline{\psi})_{(p,q)}$ given in (48) and (49) with the vector fields X_a and JX_a respectively and the wedge products of $\delta(\psi\overline{\psi})_{(p,q)}$ and $\delta^c(\psi\overline{\psi})_{(p,q)}$ given in (50) and (51) with the 1-forms e_a and Je_a respectively. If we contract $d(\psi\overline{\psi})_{(p,q)}$ with X_a, we find from (48)

$$
\begin{aligned}
i_{X_a}d(\psi\overline{\psi})_{(p,q)} &= (p+1)i_{X_a}\Big(k\big(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\big) - l\big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p+1,q)} \\
&\quad + 2i_{X_a}\Big[L\Big(k\big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p,q-1)}\Big] \\
&\quad + 2i_{JX_a}\Big(k\big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p,q+1)}
\end{aligned} \tag{52}
$$

where we have used the definitions of L and J given in (5) and (7), the identity $i_{JX_a} = Ji_{X_a}$ and the commutator relations in (11). Similarly, by contracting $d^c(\psi\overline{\psi})_{(p,q)}$ with JX_a, we obtain from (49)

$$
\begin{aligned}
i_{JX_a}d^c(\psi\overline{\psi})_{(p,q)} &= (q+1)i_{JX_a}\Big(k\big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p,q+1)} \\
&\quad - 2i_{JX_a}\Big[L\Big(k\big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) - l\big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p-1,q)}\Big] \\
&\quad - 2i_{X_a}\Big(k\big(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\big) - l\big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p+1,q)}.
\end{aligned} \tag{53}
$$

By taking the wedge product of e_a with $\delta(\psi\overline{\psi})_{(p,q)}$, we find from (50)

$$
\begin{aligned}
e_a \wedge \delta(\psi\overline{\psi})_{(p,q)} &= (m-p+1)\Big(-k\big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p-1,q)} \\
&\quad + e_a \wedge J\Big(k\big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p,q-1)} \\
&\quad + 2e_a \wedge \Lambda\Big(k\big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p,q+1)}
\end{aligned} \tag{54}
$$

where we have used the definitons of J and Λ given in (7) and (6). Similarly, we can also find the wedge product of Je_a with $\delta^c(\psi\overline{\psi})_{(p,q)}$ from (51) as

$$
\begin{aligned}
Je_a \wedge \delta^c(\psi\overline{\psi})_{(p,q)} &= -(m-q+1)Je_a \wedge \Big(k\big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p,q-1)} \\
&\quad + Je_a \wedge J\Big(k\big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big) - l\big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p-1,q)} \\
&\quad + 2Je_a \wedge \Lambda\Big(-k\big(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\big) + l\big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\big)\Big)_{(p+1,q)}.
\end{aligned} \tag{55}
$$

Hence, by considering the definitions $\alpha_{(p,q-1)}$, $\beta_{(p,q+1)}$, $\gamma_{(p-1,q)}$ and $\mu_{(p+1,q)}$ given in (37)-(40) and (41)-(44) and comparing the equality obtained in (47) with the equalities (52)-(55), we finally find the following equation

$$
\begin{aligned}
\nabla_{X_a}(\psi\overline{\psi})_{(p,q)} &= \frac{1}{p+1} i_{X_a}\Big(d(\psi\overline{\psi})_{(p,q)} - 2L\alpha_{(p,q-1)} - 2J\beta_{(p,q+1)}\Big) \\
&\quad - \frac{1}{m-p+1} e_a \wedge \Big(\delta(\psi\overline{\psi})_{(p,q)} - J\alpha_{(p,q-1)} - 2\Lambda\beta_{(p,q+1)}\Big) \\
&\quad + \frac{1}{q+1} i_{JX_a}\Big(d^c(\psi\overline{\psi})_{(p,q)} + 2L\gamma_{(p-1,q)} - 2J\mu_{(p+1,q)}\Big) \\
&\quad - \frac{1}{m-q+1} Je_a \wedge \Big(\delta^c(\psi\overline{\psi})_{(p,q)} - J\gamma_{(p-1,q)} - 2\Lambda\mu_{(p+1,q)}\Big)
\end{aligned} \tag{56}
$$

which proves the theorem. □

V. SPECIAL CASES

Bilinear form equation (56) for Kählerian twistor spinors reduces to a more simple form for special cases of Kirchberg's and Hijazi's Kählerian twistor spinors given in (27) and (26), respectively.

Corollary 1. *Bilinears forms of Kirchberg's Kählerian twistor spinors given in (27) satisfy the equation (56) with $k = \frac{1}{4r}$ and $l = 0$ in the definitions (37)-(40). Similarly, bilinear forms of Hijazi's Kählerian twistor spinors given in (26) satisfy the equation (56) with the following new definitions*

$$
\alpha_{p,q-1} = b\Big(\not{d}^c(\psi\overline{\psi}) - 2Je^b \wedge (\psi\overline{\nabla_{X_b}\psi})\Big)_{(p,q-1)} \tag{57}
$$

$$
\beta_{(p,q+1)} = b\Big(\not{d}^c(\psi\overline{\psi}) - 2i_{JX^b}(\psi\overline{\nabla_{X_b}\psi})\Big)_{(p,q+1)} \tag{58}
$$

$$
\gamma_{(p-1,q)} = a\Big(\not{d}(\psi\overline{\psi}) - 2e^b \wedge (\psi\overline{\nabla_{X_b}\psi})\Big)_{(p-1,q)} \tag{59}
$$

$$
\mu_{(p+1,q)} = a\Big(\not{d}(\psi\overline{\psi}) - 2i_{X^b}(\psi\overline{\nabla_{X_b}\psi})\Big)_{(p,q-1)} \tag{60}
$$

where a and b are constants given in (26).

We can also determine the bilinear form equations of holomorphic and anti-holomorphic Kählerian twistor spinors of type r.

Theorem 2. *If ψ is a holomorphic Kählerian twistor spinor of type r on a Kählerian spin manifold M^{2m}, then the bilinear forms $(\psi\overline{\psi})_p$ satisfy the equation (56) with the new definitions of*

$$k=-l=\frac{1}{16(m-r+1)}. \tag{61}$$

Similarly, if ψ is an anti-holomorphic Kählerian twistor spinor of type r on a Kählerian spin manifold M^{2m}, then the bilinear forms $(\psi\overline{\psi})_p$ satisfy the equation (56) with the new definitions of

$$k=l=\frac{1}{16(r+1)}. \tag{62}$$

Proof. If ψ is a Kählerian twistor spinor of type r, then it satisfies

$$\nabla_X\psi=\frac{m+2}{8(r+1)(m-r+1)}\big(\widetilde{X}.\not{D}\psi+J\widetilde{X}.\not{D}^c\psi\big)+\frac{m-2r}{8(r+1)(m-r+1)}i\big(J\widetilde{X}.\not{D}\psi-\widetilde{X}.\not{D}^c\psi\big). \tag{63}$$

By considering the definitions

$$X=X^++X^- \qquad , \qquad JX=i(X^+-X^-)$$

and

$$\not{D}=\not{D}^++\not{D}^- \qquad , \qquad \not{D}^c=i(\not{D}^+-\not{D}^-),$$

one can write the above equation in the following form

$$\nabla_X\psi=\frac{1}{4(m-r+1)}\widetilde{X}^+.\not{D}^+\psi-\frac{1}{4(r+1)}\widetilde{X}^-.\not{D}^+\psi. \tag{64}$$

If ψ is holomorphic, then we have $\not{D}^+\psi=0$ and (64) transforms into

$$\begin{aligned}\nabla_X\psi &= \frac{1}{4(m-r+1)}\widetilde{X}^+.\not{D}^-\psi\\ &= \frac{1}{16(m-r+1)}\big(\widetilde{X}.\not{D}\psi+i\widetilde{X}.\not{D}^c\psi-iJ\widetilde{X}.\not{D}\psi+J\widetilde{X}.\not{D}^c\psi\big)\end{aligned} \tag{65}$$

where we have used $X^+=\frac{1}{2}(X-iJX)$ and $\not{D}^-=\dfrac{1}{2}(\not{D}+i\not{D}^c)$. So, (65) is exactly in the same form with Kählerian twistor equation for $k=-l=\frac{1}{16(m-r+1)}$ and the bilinears $(\psi\overline{\psi})_p$ satisfy exactly the same equation for these new values of k and l.

Similarly, if ψ is anti-holomorphic, then we have $\not{D}^-\psi=0$ and (64) transforms into

$$\begin{aligned}\nabla_X\psi &= \frac{1}{4(r+1)}\widetilde{X}^-.\not{D}^+\psi\\ &= \frac{1}{16(r+1)}\big(\widetilde{X}.\not{D}\psi-i\widetilde{X}.\not{D}^c\psi+iJ\widetilde{X}.\not{D}\psi+J\widetilde{X}.\not{D}^c\psi\big).\end{aligned} \tag{66}$$

So, (66) is also exactly in the same form with Kählerian twistor equation for $k=l=\frac{1}{16(r+1)}$ and the bilinears $(\psi\overline{\psi})_p$ satisfy exactly the same equation for these new values of k and l. □

We obtain another special case for the bilinear forms of Kählerian twistor spinors when some constraints on $\alpha_{(p,q-1)}$, $\beta_{(p,q+1)}$, $\gamma_{(p-1,q)}$ and $\mu_{(p+1,q)}$ given in (37)-(40) are satisfied.

Proposition 2. *For a Kählerian twistor spinor ψ of type r, if the forms $\alpha_{(p,q-1)}$, $\beta_{(p,q+1)}$, $\gamma_{(p-1,q)}$ and $\mu_{(p+1,q)}$ defined in (37)-(40) satisfy the following conditions*

$$\begin{aligned}L\alpha_{(p,q-1)}=-J\beta_{(p,q+1)} \qquad &, \qquad J\alpha_{(p,q-1)}=-2\Lambda\beta_{(p,q+1)}\\ L\gamma_{(p-1,q)}=J\mu_{(p+1,q)} \qquad &, \qquad J\gamma_{(p-1,q)}=-2\Lambda\mu_{(p+1,q)},\end{aligned} \tag{67}$$

then the bilinear forms $(\psi\overline{\psi})_p$ of ψ satisfy the Kählerian CKY equation.

Proof. One can easily see from (36) that if the conditions (67) are satisfied, then the bilinear forms $(\psi\overline{\psi})_p$ satisfy the following equation

$$\nabla_{X_a}(\psi\overline{\psi})_p = \frac{1}{p+1}i_{X_a}d(\psi\overline{\psi})_p - \frac{1}{m-p+1}e_a\wedge\delta(\psi\overline{\psi})_p + \frac{1}{q+1}i_{JX_a}d^c(\psi\overline{\psi})_p - \frac{1}{m-q+1}Je_a\wedge\delta^c(\psi\overline{\psi})_p \qquad (68)$$

which is the Kählerian generalization of the CKY equation given in (33). □

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