

Classification of first order sesquilinear forms

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Abstract

A natural way to obtain a system of partial differential equations on a manifold is to vary a suitably defined sesquilinear form. The sesquilinear forms we study are Hermitian forms acting on sections of the trivial \mathbb{C}^n -bundle over a smooth m -dimensional manifold without boundary. More specifically, we are concerned with first order sesquilinear forms, namely, those generating first order systems. Our goal is to classify such forms up to $GL(n, \mathbb{C})$ gauge equivalence. We achieve this classification in the special case of $m = 4$ and $n = 2$ by means of geometric and topological invariants (e.g. Lorentzian metric, spin/spin^c structure, electromagnetic covector potential) naturally contained within the sesquilinear form – a purely analytic object. Essential to our approach is the interplay of techniques from analysis, geometry, and topology.

Keywords: sesquilinear forms, first order systems, gauge transformations, spin^c structure.

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1 Introduction

In this paper we study sesquilinear forms of a particular type, namely, those that generate first order systems of partial differential equations on manifolds.

In order to provide motivation for our analysis, let us first discuss some basic facts from linear algebra in finite dimension.

Working in a k -dimensional complex vector space V , consider an Hermitian form

$$S : V \times V \rightarrow \mathbb{C}, \quad (u, v) \mapsto S(u, v).$$

Here S is assumed to be antilinear in the first argument and linear in the second. Variation of the real-valued action $S(v, v)$ produces the following linear field equation for v :

$$S(u, v) = 0, \quad \forall u \in V. \tag{1.1}$$

Suppose now that our vector space V is equipped with an additional structure, an inner product $\langle \cdot, \cdot \rangle$. Then the sesquilinear form S and inner product $\langle \cdot, \cdot \rangle$ uniquely define a self-adjoint linear operator $L : V \rightarrow V$ via the formula

$$S(u, v) = \langle u, Lv \rangle, \quad \forall u, v \in V. \tag{1.2}$$

The argument also works the other way round: a self-adjoint linear operator uniquely defines an Hermitian sesquilinear form via formula (1.2). Thus, in an inner product space the concepts of Hermitian sesquilinear form and self-adjoint linear operator are equivalent.

Given a linear operator L , we can consider the linear equation

$$Lv = 0. \tag{1.3}$$

If S and L are related as in (1.2), then equations (1.1) and (1.3) are equivalent.

It may seem that there is no point in working with Hermitian sesquilinear forms and that one can work with self-adjoint linear operators instead, which would be easier for practical purposes. However, there is a point because the statement regarding the equivalence of linear equations (1.1) and (1.3) is based on the use of an inner product. The concept of an Hermitian sesquilinear form is more fundamental than the concept of a self-adjoint linear operator in that it does not require an inner product for its definition. One can formulate and study the linear equation (1.1) without introducing an inner product.

In the class of problems we are interested in, the above toy model translates into the study of partial differential equations on manifolds in a setting when there is no natural definition of an inner product invariant under relevant gauge transformations. Such a situation arises, for instance, when dealing with physically meaningful problems in 4-dimensional Lorentzian spacetime, see Sections 9 and 10. Fully relativistic equations of mathematical physics are not always associated with a natural inner product, not even an indefinite non-degenerate one.

This is why studying sesquilinear forms and their classification is an interesting mathematical problem with relevant applications. More precisely, the goal of our paper is to study and classify sesquilinear forms acting on compactly supported smooth sections of the trivial \mathbb{C}^n -bundle over a smooth manifold M , whose coordinate representation involves the sections themselves and their first derivatives but no products of first derivatives. Adopting a non-canonical approach, we ask the question: when do two sesquilinear forms written in their coordinate representation correspond to the same abstract sesquilinear form? In other words, we are interested in establishing when two sesquilinear forms can be obtained one from the other by a pointwise change of basis in the fibre depending smoothly on the base point. As it turns out, this problem can be solved thanks to the interplay of techniques from algebraic topology, geometry and analysis of partial differential equations.

Our paper is structured as follows.

In Section 2 we provide a precise definition of the class of sesquilinear forms we work with using the language of analysis of partial differential equations.

In Section 3 we formulate the mathematical problem we want to address, namely, the classification of first order sesquilinear forms, distinguishing the two different types of classification we will be looking at.

Section 4 contains a brief description of the main result of the paper: our classification theorems in dimension four.

Sections 5 and 6 comprise preparatory work towards the proof of the main theorems. In Section 5 we analyse properties of sesquilinear forms, identifying geometric and topological objects naturally encoded in their analytic definition. In Section 6 we recast our analytic definition of equivalence of sesquilinear forms in a purely algebraic topological fashion, proving the equivalence of the two formulations.

Our main theorems are proved in Section 7.

Section 8 is concerned with a similar analysis in dimension three, under suitable additional conditions. We also examine two explicit examples.

In Section 9 we revisit the sesquilinear forms vs linear operators issue in the context of our main results.

In conclusion, in Section 10 we briefly mention some physically meaningful applications of our results.

The paper is complemented by Appendix A where we explain the relation between the traditional definitions of symbols of (pseudo)differential operators and our definitions for sesquilinear forms.

2 First order sesquilinear forms

Let M be a real connected smooth m -manifold without boundary, not necessarily compact. Local coordinates on M will be denoted by x^α , $\alpha = 1, \dots, m$.

We will be working with compactly supported smooth functions $u : M \rightarrow \mathbb{C}^n$. Such functions can be thought of as sections of the trivial \mathbb{C}^n -bundle over M or as n -columns of smooth complex-valued scalar fields. They form an (infinite-dimensional) vector space $C_0^\infty(M, \mathbb{C}^n)$.

Definition 2.1. A first order sesquilinear form is a functional

$$S(u, v) := \int_M [u^* \mathbf{A}^\alpha v_{x^\alpha} + u_{x^\alpha}^* \mathbf{B}^\alpha v + u^* \mathbf{C} v] dx, \quad u, v \in C_0^\infty(M, \mathbb{C}^n), \quad (2.1)$$

where $\mathbf{A}^\alpha(x)$, $\mathbf{B}^\alpha(x)$ and $\mathbf{C}(x)$ are some prescribed smooth complex $n \times n$ matrix-functions, the subscript x^α indicates partial differentiation, the star stands for Hermitian conjugation (transposition and complex conjugation) and $dx = dx^1 \dots dx^m$. We adopt the summation convention over repeated indices.

In formula (2.1) the elements of the matrix-function \mathbf{C} are densities, whereas the elements of the matrix-functions \mathbf{A} and \mathbf{B} are vector densities. Here and further on we use bold script for density-valued quantities.

Performing integration by parts, one can rewrite the sesquilinear form (2.1) in many different ways. We define the canonical representation of a first order sesquilinear form as

$$S(u, v) = \int_M \left[-\frac{i}{2} u^* \mathbf{E}^\alpha v_{x^\alpha} + \frac{i}{2} u_{x^\alpha}^* \mathbf{E}^\alpha v + u^* \mathbf{F} v \right] dx. \quad (2.2)$$

The matrix-functions in (2.1) and (2.2) are related by formulae

$$\mathbf{E}^\alpha = i(\mathbf{A}^\alpha - \mathbf{B}^\alpha), \quad \mathbf{F} = \mathbf{C} - \frac{1}{2} \frac{\partial(\mathbf{A}^\alpha + \mathbf{B}^\alpha)}{\partial x^\alpha}.$$

Recall the well-known fact that if \mathbf{w}^α is a vector density then $\partial\mathbf{w}^\alpha/\partial x^\alpha$ is a density, so elements of the matrix-function $\mathbf{F}(x)$ are densities.

We define the principal, subprincipal and full symbols of the sesquilinear form (2.2) as

$$\mathbf{S}_{\text{prin}}(x, p) := \mathbf{E}^\alpha(x) p_\alpha, \quad (2.3)$$

$$\mathbf{S}_{\text{sub}}(x) := \mathbf{F}(x), \quad (2.4)$$

$$\mathbf{S}_{\text{full}}(x, p) := \mathbf{S}_{\text{prin}}(x, p) + \mathbf{S}_{\text{sub}}(x) \quad (2.5)$$

respectively. Here p_α , $\alpha = 1, \dots, m$, is the dual variable (momentum) and all the above symbols are well defined on the cotangent bundle T^*M . It is easy to see that the full symbol uniquely determines our first order sesquilinear form and that our sesquilinear form is Hermitian (that is, $S(u, v) = \overline{S(v, u)}$) if and only if its full symbol is Hermitian.

Establishing a correspondence between a sesquilinear form or a (pseudo)differential operator on the one hand and a (full) symbol on the other hand is often referred to as *quantisation*. The argument in the above paragraph shows that first order sesquilinear forms admit a particularly convenient and natural quantisation.

Further on we work with Hermitian first order sesquilinear forms.

An Hermitian first order sesquilinear form $S(u, v)$ defines a real-valued action $S(v, v)$. Variation of this action produces field equations for v . This is a system of n linear scalar first order partial differential equations for n unknown complex-valued scalar fields. Our interest in such systems is the motivation for the current paper.

Note that, according to our Definition 2.1, a first order sesquilinear form does not contain the term $u_{x^\alpha}^* \mathbf{D}^{\alpha\beta} v_{x^\beta}$. The presence of such a term would fundamentally change the corresponding field equations, making them second order, whereas we are interested in first order systems.

Definition 2.2. We say that the sesquilinear form S is *non-degenerate* if

$$\mathbf{S}_{\text{prin}}(x, p) \neq 0, \quad \forall (x, p) \in T^*M \setminus \{0\}. \quad (2.6)$$

Condition (2.6) means that \mathbf{S}_{prin} does not vanish as a matrix, i.e. for any $(x, p) \in T^*M \setminus \{0\}$ the matrix $\mathbf{S}_{\text{prin}}(x, p)$ has at least one nonzero element. This is the weakest possible non-degeneracy condition.

Further on we work with non-degenerate Hermitian first order sesquilinear forms.

3 Statement of the problem

3.1 General linear classification

Consider a smooth matrix-function

$$R : M \rightarrow GL(n, \mathbb{C}). \quad (3.1)$$

Given a sesquilinear form (2.2) we can now define another sesquilinear form

$$\tilde{S}(u, v) := S(Ru, Rv). \quad (3.2)$$

We interpret this new sesquilinear form as a different representation of our original sesquilinear form. What we did is we changed, fibrewise, the basis in our \mathbb{C}^n -bundle over M using the gauge transformation R .

The explicit formula for $\tilde{S}(u, v)$ reads

$$\tilde{S}(u, v) = \int_M \left[-\frac{i}{2} u^* \tilde{\mathbf{E}}^\alpha v_{x^\alpha} + \frac{i}{2} u_{x^\alpha}^* \tilde{\mathbf{E}}^\alpha v + u^* \tilde{\mathbf{F}} v \right] dx,$$

where

$$\tilde{\mathbf{E}}^\alpha = R^* \mathbf{E}^\alpha R, \quad \tilde{\mathbf{F}} = R^* \mathbf{F} R + \frac{i}{2} [R_{x^\alpha}^* \mathbf{E}^\alpha R - R^* \mathbf{E}^\alpha R_{x^\alpha}].$$

The corresponding full symbol is

$$\tilde{\mathbf{S}}_{\text{full}} = R^* \mathbf{S}_{\text{full}} R + \frac{i}{2} [R_{x^\alpha}^* (\mathbf{S}_{\text{full}})_{p_\alpha} R - R^* (\mathbf{S}_{\text{full}})_{p_\alpha} R_{x^\alpha}]. \quad (3.3)$$

Our goal is to perform the above argument the other way round, solving, effectively, an ‘inverse problem’. Namely, suppose we are given two full symbols, $\mathbf{S}_{\text{full}}(x, p)$ and $\tilde{\mathbf{S}}_{\text{full}}(x, p)$. Do they describe the same sesquilinear form? In order to deal with this question rigorously we introduce the following definition.

Definition 3.1. We say that two full symbols $\mathbf{S}_{\text{full}}(x, p)$ and $\tilde{\mathbf{S}}_{\text{full}}(x, p)$ are *GL-equivalent* if there exists a smooth matrix-function (3.1) such that (3.3) is satisfied.

3.2 Special linear classification

We will also deal with the problem of equivalence of symbols in a more restrictive, special linear setting.

Definition 3.2. We say that two full symbols $\mathbf{S}_{\text{full}}(x, p)$ and $\tilde{\mathbf{S}}_{\text{full}}(x, p)$ are *SL-equivalent* if there exists a smooth matrix-function

$$R : M \rightarrow SL(n, \mathbb{C}) \quad (3.4)$$

such that (3.3) is satisfied.

We now explain the motivation for Definition 3.2.

Suppose that we have an additional structure in our mathematical model, a complex-valued volume form, namely, a non-vanishing map

$$\text{vol} : M \rightarrow \wedge^{n,0}(\mathbb{C}^n), \quad \text{vol}(x) = c(x) dz^1 \wedge \dots \wedge dz^n,$$

where $c(x)$ is some prescribed smooth non-vanishing complex scalar field.

The transformation $u \rightarrow Ru$, where R is a matrix-function (3.1), turns vol into the complex-valued volume form $\tilde{\text{vol}}(x) = \tilde{c}(x) dz^1 \wedge \dots \wedge dz^n$ with $\tilde{c}(x) = c(x) \det R(x)$.

As in the previous subsection, we consider the ‘inverse’ problem which now involves both the sesquilinear form and the complex-valued volume form. Namely, consider two symbols $\mathbf{S}_{\text{full}}(x, p)$ and $\tilde{\mathbf{S}}_{\text{full}}(x, p)$ and two non-vanishing scalar fields $c(x)$ and $\tilde{c}(x)$. Does there exist a smooth matrix-function (3.1) which turns $(\mathbf{S}_{\text{full}}, c)$ into $(\tilde{\mathbf{S}}_{\text{full}}, \tilde{c})$?

One way of addressing the above question is as follows. Choose an arbitrary smooth matrix-function $Q : M \rightarrow GL(n, \mathbb{C})$ such that $\det Q(x) = c(x)/\tilde{c}(x)$ (for example, one can take $Q(x) = \text{diag}(c(x)/\tilde{c}(x), 1, \dots, 1)$) and view the sesquilinear form $\tilde{\mathbf{S}}(Qu, Qv)$ as the ‘new’ sesquilinear form $\tilde{\mathbf{S}}$. The two complex-valued volume forms now have the same representation. After this we can only apply $SL(n, \mathbb{C})$ -transformations (3.4) to establish whether the two sesquilinear forms $\mathbf{S}_{\text{full}}(x, p)$ and $\tilde{\mathbf{S}}_{\text{full}}(x, p)$ are equivalent, because we do not want to change the complex-valued volume form. This reduces the problem to checking whether the symbols are *SL-equivalent* in the sense of Definition 3.2.

Alternatively, we can do the argument the other way round. Take an arbitrary smooth matrix-function $Q : M \rightarrow GL(n, \mathbb{C})$ such that $\det Q(x) = \tilde{c}(x)/c(x)$ and view the sesquilinear form $\mathbf{S}(Qu, Qv)$ as the ‘new’ sesquilinear form \mathbf{S} etc.

It is easy to see that the outcome of this exercise does not depend on which way we proceed or which Q we choose. In group-theoretic language, this corresponds to the fact that the group of matrix-functions (3.4) is a normal subgroup of the group of matrix-functions (3.1). The matrix-function Q picks a particular element in each of the left cosets (or, equivalently, right cosets) of $C^\infty(M, GL(n, \mathbb{C}))/C^\infty(M, SL(n, \mathbb{C}))$.

3.3 Gauge transformations

The transformations of symbols (and corresponding sesquilinear forms) described in this section can be interpreted as gauge transformations. Our gauge transformations come in two versions, general linear (subsection 3.1) and special linear (subsection 3.2). In the current paper we do not dwell on the physical meaning of our gauge transformations and pursue our analysis from a purely mathematical standpoint.

4 Main results

The main problem addressed in the current paper is to give necessary and sufficient conditions for a pair of full symbols to be GL -equivalent or SL -equivalent. Our explicit non-canonical approach will eventually produce a full classification of equivalence classes of sesquilinear forms for the special case

$$m = 4, \quad n = 2, \quad (4.1)$$

i.e. the case when we are dealing with a pair of complex-valued scalar fields over a 4-manifold.

Under the assumption (4.1) we have the following two theorems, which are our main results.

Theorem 4.1. *Let M be a connected 4-manifold without boundary and let S and \tilde{S} be non-degenerate sesquilinear forms acting on sections of the trivial \mathbb{C}^2 -bundle over M , see Definition 2.2. Then the corresponding full symbols $\mathbf{S}_{\text{full}}(x, p)$ and $\tilde{\mathbf{S}}_{\text{full}}(x, p)$ are GL -equivalent if and only if*

- (i) *the metrics encoded within these symbols belong to the same conformal class,*
- (ii) *the electromagnetic covector potentials encoded within these symbols belong to the same cohomology class in $H_{\text{dR}}^1(M)$,*
- (iii) *their topological charges are the same,*
- (iv) *their temporal charges are the same and*
- (v) *they have the same 2-torsion spin^c structure.*

Theorem 4.2. *Let M be a connected 4-manifold without boundary and let S and \tilde{S} be non-degenerate sesquilinear forms acting on sections of the trivial \mathbb{C}^2 -bundle over M , see Definition 2.2. Then the corresponding full symbols $\mathbf{S}_{\text{full}}(x, p)$ and $\tilde{\mathbf{S}}_{\text{full}}(x, p)$ are SL -equivalent if and only if*

- (i) *the metrics encoded within these symbols are the same,*
- (ii) *the electromagnetic covector potentials encoded within these symbols are the same,*
- (iii) *their topological charges are the same,*
- (iv) *their temporal charges are the same and*
- (v) *they have the same spin structure.*

The geometric and topological objects appearing in (i)–(v) in Theorems 4.1 and 4.2 will be introduced in Section 5 and examined further in Section 6. The proof of the above theorems will be given in Section 7.

The construction presented in Sections 5–7 is not straightforward and comes in several steps which combine techniques from differential geometry, algebraic topology and analysis of partial differential equations.

Remark 4.3. The assumption (4.1) may look quite restrictive at first glance. However, there are reasons for restricting the dimension of the manifold and number of scalar fields — reasons to do with the existence or non-existence of an inner product compatible with gauge transformations. See Section 9 and last paragraph of Appendix A for further details.

5 Invariant objects encoded within sesquilinear forms

5.1 Geometric objects

Let us first explain why the case (4.1) is special.

We start by observing that having the weaker constraint

$$m = n^2 \tag{5.1}$$

already brings about important geometric consequences. Namely, under the condition (5.1) a manifold M admits a non-degenerate Hermitian first order sesquilinear form if and only if it is parallelizable. The proof of this statement retraces that of [1, Lemma 1.2].

Further on we assume that our manifold M is parallelizable. Without this assumption we would not have any non-degenerate Hermitian first order sesquilinear forms to work with.

Setting $n = 2$ and $m = 4$ has even more profound geometric consequences. Namely, observe that the determinant of the principal symbol is a quadratic form in momentum p :

$$\det \mathbf{S}_{\text{prin}}(x, p) = -\mathbf{g}^{\alpha\beta}(x) p_\alpha p_\beta, \tag{5.2}$$

where $\mathbf{g}^{\alpha\beta}(x)$ is a real symmetric 4×4 matrix-function with values in 2-densities. More precisely, \mathbf{g} is a rank two symmetric tensor density of weight two.

The quadratic form $\mathbf{g}^{\alpha\beta}$ has Lorentzian signature, i.e. it has three positive eigenvalues and one negative eigenvalue, see [4, Lemma 2.1]. This implies, in particular, that

$$\det \mathbf{g}^{\alpha\beta}(x) < 0, \quad \forall x \in M.$$

Put

$$\rho(x) := (-\det \mathbf{g}^{\mu\nu}(x))^{1/6}. \tag{5.3}$$

The quantity (5.3) is a density. This observation allows us to define the Lorentzian metric

$$g^{\alpha\beta}(x) := (\rho(x))^{-2} \mathbf{g}^{\alpha\beta}(x). \tag{5.4}$$

Of course, formula (5.3) can now be rewritten in more familiar form as

$$\rho(x) = (-\det g_{\mu\nu}(x))^{1/2}.$$

We see that the case (4.1) is special in that there is a Lorentzian metric encoded within our sesquilinear form. This Lorentzian metric g is defined by the explicit formulae (5.4), (5.3), (5.2).

Let $g^{\alpha\beta}$ be the contravariant metric tensor encoded within the sesquilinear for S . Then the contravariant metric tensor $\tilde{g}^{\alpha\beta}$ encoded within the sesquilinear form \tilde{S} defined by (3.2) is

$$\tilde{g}^{\alpha\beta} = |\det R|^{-2/3} g^{\alpha\beta}. \tag{5.5}$$

We see that the metric transforms conformally under the action of R as in (3.1). In particular, it is invariant under (3.4).

The second geometric object encoded within our sesquilinear form is the electromagnetic covector potential. In order to single it out we first introduce the concept of covariant sub-principal symbol

$$\mathbf{S}_{\text{csub}} := \mathbf{S}_{\text{sub}} + \frac{i}{16} \mathbf{g}_{\alpha\beta} \{ \mathbf{S}_{\text{prin}}, \text{adj } \mathbf{S}_{\text{prin}}, \mathbf{S}_{\text{prin}} \}_{p_\alpha p_\beta}, \tag{5.6}$$

where $\mathbf{g}_{\alpha\beta}$ is the inverse of $\mathbf{g}^{\alpha\beta}$,

$$\{F, G, H\} := F_{x^\alpha} G H_{p_\alpha} - F_{p_\alpha} G H_{x^\alpha}$$

is the generalised Poisson bracket on matrix-functions and adj is the operator of matrix adjugation

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \text{adj } F.$$

We define the electromagnetic covector potential A as the — unique, due to (2.6) — real-valued solution of

$$\mathbf{S}_{\text{csub}}(x) = \mathbf{S}_{\text{prin}}(x, A(x)). \quad (5.7)$$

Note that (5.7) is a system of four linear algebraic equations for the four components of A .

Lemma 5.1. *The electromagnetic covector potential is given explicitly by the following formula*

$$A_\alpha = -\frac{1}{2} \mathbf{g}_{\alpha\beta} \text{tr}((\mathbf{S}_{\text{csub}}) (\text{adj}(\mathbf{S}_{\text{prin}})_{p_\beta})). \quad (5.8)$$

Proof. In view of (5.2), multiplication of both sides of (5.7) by $\text{adj}(\mathbf{S}_{\text{prin}}(x, p))$ gives

$$\begin{aligned} (\mathbf{S}_{\text{csub}}(x)) (\text{adj}(\mathbf{S}_{\text{prin}}(x, p))) &= (\mathbf{S}_{\text{prin}}(x, A(x))) (\text{adj}(\mathbf{S}_{\text{prin}}(x, p))) \\ &= (-\mathbf{g}^{\mu\nu}(x) A_\mu(x) p_\nu) \text{Id}. \end{aligned} \quad (5.9)$$

Differentiating both sides of (5.9) with respect to p_β , taking the matrix trace and lowering the index with the ((-2)-density valued) metric yields (5.8). \square

Formulae (5.2), (5.6) and (5.7) tell us that the full symbol is completely determined by principal symbol and electromagnetic covector potential.

Lemma 5.2. *Let A be the electromagnetic covector potential encoded within the sesquilinear form S . Then the electromagnetic covector potential \tilde{A} encoded within the sesquilinear form \tilde{S} defined by (3.2) is*

$$\tilde{A} = A + \frac{1}{2} \text{grad}(\arg \det R). \quad (5.10)$$

Proof. From [4, formulae (5.1), (5.2), (D.4)–(D.6)] it follows that

$$\tilde{\mathbf{S}}_{\text{csub}} = R^* \mathbf{S}_{\text{csub}} R - \mathbf{Q} - \mathbf{Q}^*, \quad (5.11)$$

with

$$\mathbf{Q} = -\frac{i}{8} \mathbf{g}_{\alpha\beta} R^* (\mathbf{S}_{\text{prin}})_{p_\alpha} R_{x^\gamma} R^{-1} (\text{adj } \mathbf{S}_{\text{prin}})_{p_\beta} (\mathbf{S}_{\text{prin}})_{p^\gamma} R. \quad (5.12)$$

The matrix-function R can be written locally as

$$R(x) = r(x) e^{i\varphi(x)} R_1(x), \quad (5.13)$$

where $r, \varphi : M \rightarrow \mathbb{R}$ and $R_1 : M \rightarrow SL(2, \mathbb{C})$ are smooth real and matrix-valued functions respectively. In particular, $\varphi(x) = \frac{1}{2} \arg \det R(x)$. From (5.13) we obtain

$$R_{x^\gamma} R^{-1} = (R_1)_{x^\gamma} R_1^{-1} + \frac{r_{x^\gamma}}{r} \text{Id} + i \varphi_{x^\gamma} \text{Id}. \quad (5.14)$$

The first term on the RHS of (5.14) is trace-free and hence, by [4, formula (C.1)], it does not contribute to (5.12). The second term is real, and, when multiplied by i , it does not contribute to $\mathbf{Q} + \mathbf{Q}^*$. Therefore, by substituting (5.14) into (5.12) and, in turn, (5.12) into (5.11), we obtain

$$\tilde{\mathbf{S}}_{\text{csub}} = R^* \mathbf{S}_{\text{csub}} R + R^* (\mathbf{S}_{\text{prin}})_{p^\gamma} \varphi_{x^\gamma} R, \quad (5.15)$$

from which (5.10) ensues. \square

Remark 5.3. The use of the term ‘electromagnetic covector potential’ for the covector field A is motivated by the fact that this A is, in our context, a counterpart of what in gauge theory is a $U(1)$ -connection, see formula (5.10).

5.2 Topological objects

As explained in the beginning of the previous subsection, our manifold M is *a priori* parallelizable, hence orientable. We specify an orientation on our manifold and define the topological charge of our sesquilinear form as

$$c_{\text{top}} := -\frac{i}{2} \sqrt{-\det \mathbf{g}_{\alpha\beta}} \operatorname{tr}((\mathbf{S}_{\text{prin}})_{p_1} (\mathbf{S}_{\text{prin}})_{p_2} (\mathbf{S}_{\text{prin}})_{p_3} (\mathbf{S}_{\text{prin}})_{p_4}), \quad (5.16)$$

where tr stands for the matrix trace. Straightforward calculations show that the number c_{top} can take only two values, $+1$ or -1 . It describes the orientation of the principal symbol relative to our chosen orientation of local coordinates $x = (x^1, x^2, x^3, x^4)$.

Our Lorentzian 4-manifold (M, g) does, in fact, possess an additional property: it is automatically time-orientable, i.e. it admits a timelike (co)vector field. Indeed, consider the quantity

$$f_x(p) := \frac{1}{\rho(x)} \operatorname{tr} \mathbf{S}_{\text{prin}}(x, p).$$

We are looking at a linear map

$$f_x : T_x^* M \rightarrow \mathbb{R}, \quad p \mapsto f_x(p),$$

depending smoothly on $x \in M$. Non-degeneracy of our principal symbol implies that

$$\operatorname{range} f_x \neq \{0\}, \quad \forall x \in M.$$

By duality the linear map f_x can be represented in terms of a nonvanishing vector field t ,

$$f_x(p) = t(p) = t^\alpha(x) p_\alpha,$$

which can be shown to be timelike.

Let us specify a time orientation by choosing a reference timelike covector field q . We define the temporal charge of our sesquilinear form as

$$c_{\text{tem}} := \operatorname{sgn} t(q). \quad (5.17)$$

It describes the orientation of the principal symbol relative to our chosen time orientation.

Definition 5.4. Consider symbols corresponding to metrics from a given conformal class and with the same topological and temporal charges. We define *2-torsion spin^c structure* to be the equivalence class of symbols

$$[\mathbf{S}] := \{\tilde{\mathbf{S}} \mid \tilde{\mathbf{S}}_{\text{prin}} = R^* \mathbf{S}_{\text{prin}} R, R \in C^\infty(M, GL(n, \mathbb{C}))\}. \quad (5.18)$$

Definition 5.5. Consider symbols corresponding to a given metric and with the same topological and temporal charges. We define *spin structure* to be the equivalence class of symbols

$$[\mathbf{S}] := \{\tilde{\mathbf{S}} \mid \tilde{\mathbf{S}}_{\text{prin}} = R^* \mathbf{S}_{\text{prin}} R, R \in C^\infty(M, SL(n, \mathbb{C}))\}. \quad (5.19)$$

In the above definitions we use topological terminology, even though the definitions themselves are stated in a purely analytic fashion. A rigorous justification for this is provided in the next section.

6 Transition from analysis to topology

The aim of this section is to perform an analysis of Definitions 5.4 and 5.5, so as to show that these analytic definitions are equivalent to standard topological ones. We will establish this equivalence by rewriting the principal symbol of a sesquilinear form in a way that is better suited for revealing topological content, see formula (6.1) below.

6.1 Framings and their equivalence

Let M be an oriented time-oriented Lorentzian 4-manifold. By a *frame* at a point $x \in M$ we mean a positively oriented and positively time-oriented orthonormal, in the Lorentzian sense, frame e_j , $j = 1, 2, 3, 4$, in the tangent space $T_x M$:

$$\det e_j^\alpha > 0, \quad q(e^4) > 0,$$

$$g_{\alpha\beta} e_j^\alpha e_k^\beta = \begin{cases} 0 & \text{if } j \neq k, \\ +1 & \text{if } j = k \neq 4, \\ -1 & \text{if } j = k = 4. \end{cases}$$

Here each vector e_j has coordinate components e_j^α , $\alpha = 1, 2, 3, 4$. By a *framing* of M we mean a choice of frame at every point $x \in M$ depending smoothly on the point. Of course, the contravariant metric tensor is expressed via the framing as

$$g^{\alpha\beta} = e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta + e_3^\alpha e_3^\beta - e_4^\alpha e_4^\beta$$

and the Lorentzian density is expressed via the framing as $\rho = (\det e_j^\alpha)^{-1}$.

Let

$$s^1 = s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s^2 = s_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s^3 = s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s^4 = -s_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be the standard basis in the real vector space of 2×2 Hermitian matrices. Then the principal symbols of sesquilinear forms with $c_{\text{top}} = c_{\text{temp}} = +1$ are in one-to-one correspondence with framings. This correspondence is realised explicitly by the formula

$$\mathbf{S}_{\text{prin}}(x, p) = \rho(x) s^j e_j^\alpha(x) p_\alpha. \quad (6.1)$$

The nondegeneracy condition (2.6) implies that the vector fields e_j , $j = 1, 2, 3, 4$, are linearly independent. Moreover, they are automatically Lorentz-orthogonal with respect to the metric encoded within \mathbf{S}_{prin} , see [1, Sections 1 and 2]. Thus, the e_j , $j = 1, 2, 3, 4$, provide a framing. Observe that one can also argue the other way around: in view of (6.1) a framing completely determines the principal symbol.

The point of the above argument is that instead of working with an analytic object, a principal symbol, we can work with an equivalent geometric object, a framing.

In what follows $SO^+(3, 1)$ denotes the identity component of the Lorentz group and $CSO^+(3, 1)$ denotes its conformal extension. Here ‘conformal extension’ refers to multiplication of matrices from $SO^+(3, 1)$ by arbitrary positive factors. The Lie group $SO^+(3, 1)$ is 6-dimensional, so $CSO^+(3, 1)$ is 7-dimensional. The conformal extension of the Lorentz group is needed because gauge transformations (3.2), (3.1) result in the scaling of the Lorentzian metric encoded within the principal symbol, see formula (5.5).

Let us now fix a conformal class of Lorentzian metrics and within this class choose a pair of principal symbols \mathbf{S}_{prin} and $\tilde{\mathbf{S}}_{\text{prin}}$. Let e_j and \tilde{e}_j be the corresponding framings. Then

$$\tilde{e}_j = O_j^k e_k \quad (6.2)$$

for some uniquely defined smooth matrix-function $O : M \rightarrow CSO^+(3, 1)$.

Suppose now that there exists a matrix-function $R : M \rightarrow GL(2, \mathbb{C})$ such that $\tilde{\mathbf{S}}_{\text{prin}} = R^* \mathbf{S}_{\text{prin}} R$. A straightforward calculation shows that the matrix-function O appearing in (6.2) is expressed via R as

$$O_j^k = \frac{1}{2} |\det R|^{-4/3} \text{tr}(s_j R^* s^k R). \quad (6.3)$$

It is convenient to define

$$\mathcal{R} := |\det R|^{2/3} R. \quad (6.4)$$

Of course, the above formula can be inverted:

$$R = |\det \mathcal{R}|^{-2/7} \mathcal{R}. \quad (6.5)$$

The advantage of working with the matrix-function

$$\mathcal{R} : M \rightarrow GL(2, \mathbb{C})$$

rather than the original matrix-function (3.1) is that formula (6.3) simplifies and reads now

$$O_j^k = \frac{1}{2} \operatorname{tr}(s_j \mathcal{R}^* s^k \mathcal{R}). \quad (6.6)$$

The switch from R to \mathcal{R} does not affect the topological issues we are addressing, it just makes formulae simpler.

Observe that when $\mathcal{R} \in SL(2, \mathbb{C})$, (6.6) is the standard spin homomorphism formula which provides a map

$$\Pi : SL(2, \mathbb{C}) \longrightarrow SO^+(3, 1), \quad \Pi(\mathcal{R}) = O. \quad (6.7)$$

When we allow \mathcal{R} to take values in $GL(2, \mathbb{C})$, formula (6.6) gives us a map

$$\Pi : GL(2, \mathbb{C}) \longrightarrow CSO^+(3, 1), \quad \Pi(\mathcal{R}) = O. \quad (6.8)$$

We are now in a position to rephrase Definitions 5.4 and 5.5 as follows.

Consider symbols corresponding to metrics from a given conformal class and with the same topological and temporal charges. We define *2-torsion spin^c structure* to be the equivalence class of symbols, where two symbols are called equivalent if the matrix-function O relating them, see (6.2), can be written in the form (6.6) for some $\mathcal{R} : M \rightarrow GL(2, \mathbb{C})$. In other words, the matrix-function $O : M \rightarrow CSO^+(3, 1)$ admits a factorization

$$O : M \xrightarrow{\mathcal{R}} GL(2, \mathbb{C}) \xrightarrow{\Pi} CSO^+(3, 1). \quad (6.9)$$

If the metric is the same, we define *spin structure* to be the equivalence class of symbols, where two symbols are called equivalent if the matrix-function O relating them can be written in the form (6.6) for some $\mathcal{R} : M \rightarrow SL(2, \mathbb{C})$.

Remark 6.1. It is easy to see that in the GL case the matrix-function \mathcal{R} , if it exists, is defined uniquely modulo multiplication by $e^{i\varphi}$, where φ is an arbitrary smooth real-valued scalar function. In the SL case the matrix-function \mathcal{R} , if it exists, is defined uniquely modulo multiplication by ± 1 .

It follows from [1] that our definition of spin structure agrees with the accepted topological one¹. In the remainder of this section we establish a similar result for 2-torsion spin^c structure.

Let us remind the reader that it follows from our assumptions (2.6) and (4.1) that M is a Lorentzian manifold which is parallelizable and time-orientable. In particular, it is spin. A choice of reference framing on M provides a trivialization of the tangent bundle TM so that any other framing is related to this reference framing by a smooth function $O : M \rightarrow CSO^+(3, 1)$. Two framings corresponding to functions O_1 and O_2 are equivalent in the above sense if and only if there exists a smooth function $\mathcal{R} : M \rightarrow GL(2, \mathbb{C})$ such that $O_2 \cdot (\Pi \circ \mathcal{R}) = O_1$ as functions $M \rightarrow CSO^+(3, 1)$.

¹The map called $\operatorname{Ad} : SL(2, \mathbb{C}) \rightarrow SO^+(3, 1)$ in [1] should in fact be understood as the spin homomorphism Π .

6.2 Topological characterization

In this section, we will characterize the equivalence relation we used to define the 2-torsion spin^c structures in purely topological terms. We begin by recalling that the compact subgroups $U(2) \subset GL(2, \mathbb{C})$ and $SO(3) \subset CSO^+(3, 1)$ are deformation retracts of the respective non-compact Lie groups compatible with the map (6.8) in the sense that the following diagram commutes

$$\begin{array}{ccc}
 U(2) & \longrightarrow & GL(2, \mathbb{C}) \\
 \downarrow \text{Ad} & & \downarrow \Pi \\
 SO(3) & \longrightarrow & CSO^+(3, 1)
 \end{array} \tag{6.10}$$

Here we used the fact that the restriction of the map (6.8) to the subgroup $U(2)$ coincides with the adjoint map $\text{Ad} : U(2) \rightarrow SO(3)$. The two vertical arrows in this diagram are principal $U(1)$ -bundles, the action being multiplication by a diagonal matrix; see Remark 6.1.

Lemma 6.2. *The principal $U(1)$ -bundles $U(2) \rightarrow SO(3)$ and $GL(2, \mathbb{C}) \rightarrow CSO^+(3, 1)$ are non-trivial.*

Proof. A principal bundle is known to be trivial if and only if it admits a section. Assuming that the bundle $U(2) \rightarrow SO(3)$ admits a section $s : SO(3) \rightarrow U(2)$, we immediately obtain a contradiction because the composition

$$H^2(SO(3); \mathbb{Z}) \xrightarrow{\text{Ad}^*} H^2(U(2); \mathbb{Z}) \xrightarrow{s^*} H^2(SO(3); \mathbb{Z})$$

must be identity while $H^2(U(2); \mathbb{Z}) = \mathbb{Z}$ and $H^2(SO(3); \mathbb{Z}) = \mathbb{Z}/2$. The argument for the other bundle is similar. \square

We will be mostly interested in the bundle $GL(2, \mathbb{C}) \rightarrow CSO^+(3, 1)$. Given a map $f : M \rightarrow CSO^+(3, 1)$, associate with it the cohomology class $\mathcal{O}(f) = f^*(1) \in H^2(M; \mathbb{Z})$, where $1 \in H^2(CSO^+(3, 1); \mathbb{Z}) = \mathbb{Z}/2$ is the generator. Note that $\mathcal{O}(f)$ is an element of order at most two in $H^2(M; \mathbb{Z})$; in particular, it automatically vanishes whenever the group $H^2(M; \mathbb{Z})$ has no 2-torsion.

Proposition 6.3. *A map $f : M \rightarrow CSO^+(3, 1)$ admits a factorization (6.9) if and only if $\mathcal{O}(f) = 0$.*

Proof. We begin by constructing, for a given map $f : M \rightarrow CSO^+(3, 1)$, the pull back principal bundle

$$\begin{array}{ccc}
 E(f) & \longrightarrow & GL(2, \mathbb{C}) \\
 \downarrow \pi & & \downarrow \Pi \\
 M & \xrightarrow{f} & CSO^+(3, 1)
 \end{array}$$

where $E(f) = \{(x, p) \mid f(x) = \Pi(p)\} \subset M \times GL(2, \mathbb{C})$ and the maps $\pi : E(f) \rightarrow M$ and $E(f) \rightarrow GL(2, \mathbb{C})$ are projections onto the respective factors. It is well known (and can be checked by comparing the definitions) that $f : M \rightarrow CSO^+(3, 1)$ admits a factorization (6.9)

if and only if the bundle $\pi : E(f) \rightarrow M$ admits a section. Since $\pi : E(f) \rightarrow M$ is a principal bundle it admits a section if and only if it is trivial. The latter happens if and only if the first Chern class $c_1(E(f)) \in H^2(M; \mathbb{Z})$ vanishes. Since c_1 is natural with respect to pull backs, $c_1(E(f))$ is the pull back via $f^* : H^2(CSO^+(3, 1); \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z})$ of the first Chern class of the bundle $GL(2, \mathbb{C}) \rightarrow CSO^+(3, 1)$. According to Lemma 6.2, the latter bundle is non-trivial, hence its first Chern class must be the generator $1 \in H^2(CSO^+(3, 1); \mathbb{Z}) = \mathbb{Z}/2$ and $c_1(E(f)) = f^*(1) = \mathcal{O}(f)$. \square

Proposition 6.4. *Every element of $H^2(M; \mathbb{Z})$ of order two can be realized as $\mathcal{O}(f)$ for some map $f : M \rightarrow CSO^+(3, 1)$.*

Proof. Let us consider the short exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$. The associated long exact sequence

$$\dots \longrightarrow H^1(M; \mathbb{Z}/2) \xrightarrow{\partial} H^2(M; \mathbb{Z}) \xrightarrow{\cdot 2} H^2(M; \mathbb{Z}) \longrightarrow H^2(M; \mathbb{Z}/2) \longrightarrow \dots$$

implies that every element $b \in H^2(M; \mathbb{Z})$ of order two belongs to the image of the Bockstein homomorphism $\partial : H^1(M; \mathbb{Z}/2) \rightarrow H^2(M; \mathbb{Z})$. We will show that every cohomology class $a \in H^1(M; \mathbb{Z}/2)$ is of the form $a = f^*(1)$ for some $f : M \rightarrow SO^+(3, 1)$ and $1 \in H^1(SO^+(3, 1); \mathbb{Z}/2) = \mathbb{Z}/2$. The result will then follow from the commutative diagram

$$\begin{array}{ccc} H^1(SO^+(3, 1); \mathbb{Z}/2) & \xrightarrow{\partial} & H^2(SO^+(3, 1); \mathbb{Z}) \\ \downarrow f^* & & \downarrow f^* \\ H^1(M; \mathbb{Z}/2) & \xrightarrow{\partial} & H^2(M; \mathbb{Z}) \end{array} \quad (6.11)$$

whose upper row is an isomorphism, and the fact that $SO^+(3, 1) \subset CSO^+(3, 1)$ is a deformation retract.

Let us consider the double covering $SL(2, \mathbb{C}) \rightarrow SO^+(3, 1)$ given by the spin homomorphism (6.7) and its associated fibration sequence (see, for instance, [2, Lemma 8.23])

$$\mathbb{Z}/2 \longrightarrow SL(2, \mathbb{C}) \longrightarrow SO^+(3, 1) \longrightarrow K(\mathbb{Z}/2, 1) \longrightarrow BSL(2, \mathbb{C}),$$

where $K(\mathbb{Z}/2, 1)$ is the Eilenberg–MacLane space and $BSL(2, \mathbb{C})$ the classifying space of the Lie group $SL(2, \mathbb{C})$. It gives rise to the exact sequence of homotopy sets (see [2, Theorem 6.29])

$$[M, SO^+(3, 1)] \longrightarrow H^1(M; \mathbb{Z}/2) \longrightarrow [M, BSL(2, \mathbb{C})]$$

using the fact that $H^1(M; \mathbb{Z}/2) = [M, K(\mathbb{Z}/2, 1)]$. We wish to show that the first map in this sequence is surjective or, equivalently, that the second map is zero. Write $H^1(M; \mathbb{Z}/2) = [M, \mathbb{RP}^\infty]$ using the homotopy equivalence between $K(\mathbb{Z}/2, 1)$ and the real projective space \mathbb{RP}^∞ . Also observe that, up to homotopy equivalence, $BSL(2, \mathbb{C}) = BSU(2) = \mathbb{HP}^\infty$, the quaternionic projective space. Then the question becomes whether, for any continuous map $M \rightarrow \mathbb{RP}^\infty$, the composition $M \rightarrow \mathbb{RP}^\infty \rightarrow \mathbb{HP}^\infty$ with the natural inclusion $\mathbb{RP}^\infty \rightarrow \mathbb{HP}^\infty$ is homotopic to zero. Since $\dim M = 4$ and the 5–skeleton of the CW–complex \mathbb{HP}^∞ is $\mathbb{HP}^1 = S^4$, the cellular approximation theorem reduces this question to an identical question about the composition $M \rightarrow \mathbb{RP}^4 \rightarrow S^4$. By the Hopf theorem, a map $M \rightarrow S^4$ is homotopic to zero if and only if the induced map $H^4(S^4; \mathbb{Z}) \rightarrow H^4(M; \mathbb{Z})$ is zero. In our case, this last map splits as the composition

$$H^4(S^4; \mathbb{Z}) \longrightarrow H^4(\mathbb{RP}^4; \mathbb{Z}) \longrightarrow H^4(M; \mathbb{Z}),$$

with $H^4(\mathbb{RP}^4; \mathbb{Z}) = \mathbb{Z}/2$. Since M is orientable, $H^4(M; \mathbb{Z})$ is a free abelian group, hence the second map in this composition must vanish.

It is worth mentioning that the orientability of M in this argument is essential: in general, realizability of cohomology classes in $H^2(M; \mathbb{Z})$ can be obstructed by the non-trivial quadruple cup-product on $H^1(M; \mathbb{Z}/2)$. \square

Corollary 6.5. *The set of 2-torsion spin^c structures on M is in a bijective correspondence with the 2-torsion subgroup of $H^2(M; \mathbb{Z})$.*

6.3 Differential geometric characterization

Our goal in this subsection is to identify the equivalence classes of framings with the 2-torsion spin^c structures on M , whose definition is modelled after that in Riemannian geometry [7]; see Remark 6.7 below. In the special case at hand, when the tangent bundle TM is trivialized via the reference frame, it reads as follows. A 2-torsion spin^c structure on M is an equivalence class of commutative diagrams

$$\begin{array}{ccc} M \times GL(2, \mathbb{C}) & & \\ \downarrow \Phi & \searrow \pi & \\ & & M \\ & \nearrow \pi & \\ M \times CSO^+(3, 1) & & \end{array}$$

where π stands for the projection onto the first factor, and the map Φ is equivariant in that $\Phi(x, g) = \Phi(x, 1) \cdot \Pi(g)$ for all $x \in M$ and $g \in GL(2, \mathbb{C})$. Two diagrams as above with the vertical maps Φ_1 and Φ_2 are called equivalent if there is a commutative diagram

$$\begin{array}{ccc} M \times GL(2, \mathbb{C}) & \xrightarrow{A} & M \times GL(2, \mathbb{C}) \\ \searrow \Phi_1 & & \nearrow \Phi_2 \\ & & M \times CSO^+(3, 1) \end{array}$$

such that $\pi \circ A = \pi$ and the map A is equivariant in that $A(x, g) = A(x, 1) \cdot g$ for all $x \in M$ and $g \in GL(2, \mathbb{C})$.

Theorem 6.6. *For parallelizable time-orientable Lorentzian 4-manifolds, the equivalence classes of framings as above are in bijective correspondence with the 2-torsion spin^c structures.*

Proof. Using the commutativity of the first diagram, write $\Phi(x, g) = (x, \phi(x, g))$ for some function $\phi : M \times GL(2, \mathbb{C}) \rightarrow CSO^+(3, 1)$ and observe that the equivariance condition on Φ translates into the equation $\phi(x, g) = \phi(x, 1) \cdot \Pi(g)$. Therefore, the map Φ is uniquely determined by the map $\psi : M \rightarrow CSO^+(3, 1)$ given by $\psi(x) = \phi(x, 1)$.

Similarly, write $A(x, g) = (x, \alpha(x, g))$ and observe that the equivariance condition on A translates into the equation $\alpha(x, g) = \alpha(x, 1) \cdot g$. Therefore, the map A is uniquely determined by the map $\beta : M \rightarrow GL(2, \mathbb{C})$ given by $\beta(x) = \alpha(x, 1)$. One can easily check that the second commutative diagram then simply means that $\psi_2 \cdot \Pi(\beta) = \psi_1$ as functions $M \rightarrow CSO^+(3, 1)$. \square

Theorem 6.6 rigorously shows, in view of (6.1), the equivalence of two definitions of 2-torsion spin^c structure, the standard topological one and Definition 5.4. Note that the equivalence we established is not canonical in that it depends on the choice of reference frame.

Remark 6.7. It may be worth explaining the origin of the term ‘2-torsion spin^c structure’. Following the analogy with Riemannian geometry, one can define a spin^c structure on M as the equivalence class of lifts of the principal frame bundle of M to a $GL(2, \mathbb{C})$ bundle; even though

the frame bundle of M is trivial, it may lift to a non-trivial $GL(2, \mathbb{C})$ bundle. Among these lifts is the lift to an $SL(2, \mathbb{C})$ bundle P associated with the spin structure on M . The bundle P must be trivial for topological reasons: it is classified by its second Chern class $c_2(P)$, and we know that $4c_2(P) = -p_1(TM) = 0 \in H^4(M; \mathbb{Z})$. As in the Riemannian case, one can use P to establish a bijective correspondence between spin^c structures on M and the group $H^2(M; \mathbb{Z})$. Under this correspondence, the spin^c structure corresponding to a cohomology class $a \in H^2(M; \mathbb{Z})$ lives in a Hermitian rank-two bundle with the first Chern class $c_1(P) + 2a = 2a \in H^2(M; \mathbb{Z})$. Since we restrict ourselves to trivial bundles, the class $2a$ must vanish. This means that $a \in H^2(M; \mathbb{Z})$ is a 2-torsion, hence the name of the corresponding spin^c structure.

7 Proofs of main theorems

7.1 Proof of Theorem 4.1

Necessity

Let us first show that conditions (i)–(v) of Theorem 4.1 are necessary.

(i) Formula (5.5) tells us that the conformal class of metrics is preserved under GL transformations, so condition (i) is necessary.

(ii) Lemma 5.2 tells us that condition (ii) is necessary.

(iii)–(iv) In order to deal with conditions (iii) and (iv) we observe that the two charges, topological (5.16) and temporal (5.17), can be expressed via the framing as

$$c_{\text{top}} = \text{sgn} \det e_j^\alpha, \quad c_{\text{tem}} = \text{sgn} q(e^4).$$

We showed in Section 6 that under GL transformations the framing stays within the original connected component of the conformally extended Lorentz group, hence conditions (iii) and (iv) are necessary.

(v) As to the necessity of condition (v), it follows immediately from Definition 5.4.

Sufficiency

Let us now show that conditions (i)–(v) of Theorem 4.1 are sufficient.

We need to find a GL transformation which turns one full symbol into the other. As explained in subsection 5.1, a full symbol is completely determined by principal symbol and electromagnetic covector potential. Thus, we need to find a GL transformation which turns one principal symbol into the other and one electromagnetic covector potential into the other.

Conditions (i) and (iii)–(v) ensure that we can find a matrix-function (3.1) which turns one principal symbol into the other, see formula (5.18). Remark 6.1 and formulae (6.4), (6.5) tell us that this matrix-function (3.1) is defined uniquely modulo multiplication by $e^{i\varphi}$, where φ is an arbitrary smooth real-valued scalar function. In view of condition (ii) this function φ can be chosen so as to turn one electromagnetic covector potential into the other.

All in all, we obtain a matrix-function R defined uniquely modulo multiplication by a constant $c \in \mathbb{C}$, $|c| = 1$. \square

7.2 Proof of Theorem 4.2

The proof of Theorem 4.2 is similar to that of Theorem 4.1, with only two modifications.

- SL transformations preserve the metric, so the requirement is that the two metrics are the same as opposed to the two metrics being in the same conformal class.

- SL transformations preserve the electromagnetic covector potential, so the requirement is that the two electromagnetic covector potentials are the same as opposed to the two electromagnetic covector potentials being in the same cohomology class in $H_{\text{dR}}^1(M)$.

All in all, we obtain a matrix-function R defined uniquely modulo multiplication by ± 1 . \square

8 The 3-dimensional Riemannian case

Let us consider first order sesquilinear forms satisfying the additional assumption

$$\text{tr } \mathbf{S}_{\text{prin}}(x, p) = 0, \quad \forall (x, p) \in T^*M. \quad (8.1)$$

In this setting it is natural to look at transformations of symbols generated by matrix-functions

$$R : M \rightarrow U(n) \quad (8.2)$$

or

$$R : M \rightarrow SU(n). \quad (8.3)$$

Of course, $U(n) \subset GL(n, \mathbb{C})$ and $SU(n) \subset SL(n, \mathbb{C})$, so (8.2) and (8.3) are special cases of (3.1) and (3.4) respectively. We are now more restrictive in our choice of matrix-functions R because we want to preserve condition (8.1).

It turns out that for sesquilinear forms with trace-free principal symbol one can perform a classification similar to that described in previous sections. We list the main results below, skipping detailed proofs as these are modifications of arguments presented earlier in the paper.

Condition (5.1) is now replaced by

$$m = n^2 - 1. \quad (8.4)$$

Under the condition (8.4) a manifold M admits a non-degenerate Hermitian first order sesquilinear form with trace-free principal symbol if and only if it is parallelizable. So further on we assume that our manifold is parallelizable.

In this section we deal with the special case

$$m = 3, \quad n = 2, \quad (8.5)$$

compare with (4.1). It is known [8, 6] that a 3-manifold is parallelizable if and only if it is orientable. Therefore, orientability is our only topological restriction on M .

It is easy to see that under the assumption (8.1) the non-degeneracy condition (2.6) is equivalent to the condition

$$\det \mathbf{S}_{\text{prin}}(x, p) \neq 0, \quad \forall (x, p) \in T^*M \setminus \{0\}. \quad (8.6)$$

But (8.6) is the standard ellipticity condition. Thus, in this section we work with formally self-adjoint elliptic first order sesquilinear forms S with trace-free principal symbols which act on sections of the trivial \mathbb{C}^2 -bundle over a connected smooth oriented 3-manifold M without boundary.

We define $\mathbf{g}^{\alpha\beta}(x)$ via (5.2). It is easy to see that the quadratic form $\mathbf{g}^{\alpha\beta}$ is positive definite. This implies, in particular, that

$$\det \mathbf{g}^{\alpha\beta}(x) > 0, \quad \forall x \in M.$$

Put

$$\rho(x) := (\det \mathbf{g}^{\mu\nu}(x))^{1/4}. \quad (8.7)$$

The quantity (8.7) is a density. This observation allows us to define the Riemannian metric $g^{\alpha\beta}(x) := (\rho(x))^{-2} \mathbf{g}^{\alpha\beta}(x)$. Of course, formula (8.7) can now be rewritten in more familiar form as $\rho(x) = (\det g_{\mu\nu}(x))^{1/2}$. And it is easy to see that our metric tensor is invariant under transformations (3.2), (8.2).

We define the covariant subprincipal symbol in accordance with formula (5.6). The magnetic covector potential $A = (A_1, A_2, A_3)$ and electric potential A_4 are defined as the solution of

$$\mathbf{S}_{\text{csub}}(x) = \mathbf{S}_{\text{prin}}(x, A(x)) + A_4 \text{Id},$$

compare with (5.7). For the magnetic potential we still have the explicit formula (5.8) and for the electric potential we have

$$A_4 = \frac{1}{2} \text{tr } \mathbf{S}_{\text{csub}}.$$

The full symbol is completely determined by principal symbol, magnetic covector potential and electric potential. The electric potential is invariant under transformations (3.2), (8.2), whereas the magnetic covector potential transforms in accordance with formula (5.10).

We specify an orientation on our manifold and define the topological charge of our sesquilinear form as

$$c_{\text{top}} := -\frac{i}{2} \sqrt{\det \mathbf{g}_{\alpha\beta}} \text{tr}((\mathbf{S}_{\text{prin}})_{p_1} (\mathbf{S}_{\text{prin}})_{p_2} (\mathbf{S}_{\text{prin}})_{p_3}) = \text{sgn } \det e_j^\alpha, \quad (8.8)$$

compare with (5.16).

Definition 8.1. Consider symbols corresponding to a given metric and with the same topological charge. We define *2-torsion spin^c structure* to be the equivalence class of symbols

$$[\mathbf{S}] = \{ \tilde{\mathbf{S}} \mid \tilde{\mathbf{S}}_{\text{prin}} = R^* \mathbf{S}_{\text{prin}} R, R \in C^\infty(M, U(2)) \}. \quad (8.9)$$

Definition 8.2. Consider symbols corresponding to a given metric and with the same topological charge. We define *spin structure* to be the equivalence class of symbols

$$[\mathbf{S}] = \{ \tilde{\mathbf{S}} \mid \tilde{\mathbf{S}}_{\text{prin}} = R^* \mathbf{S}_{\text{prin}} R, R \in C^\infty(M, SU(2)) \}. \quad (8.10)$$

Our analytic definition of 2-torsion spin^c structure in dimension three, Definition 8.1, is equivalent to the standard topological one. This follows by the argument of Section 6.2 and Section 6.3 once the map $GL(2, \mathbb{C}) \rightarrow CSO^+(3, 1)$ is replaced by the map $U(2) \rightarrow SO(3)$. Our analytic definition of spin structure in dimension three, Definition 8.2, is also equivalent to the standard topological one, which follows from [1] with the help of Diagram 6.10.

We define U -equivalence and SU -equivalence of symbols as in Definition 3.1, replacing (3.1) by (8.2) and (8.3) respectively.

We have the following analogues of Theorems 4.1 and 4.2.

Theorem 8.3. *Two full symbols $\mathbf{S}_{\text{full}}(x, p)$ and $\tilde{\mathbf{S}}_{\text{full}}(x, p)$ are U -equivalent if and only if*

- (i) *the metrics encoded within these symbols are the same,*
- (ii) *the electric potentials encoded within these symbols are the same,*
- (iii) *the magnetic covector potentials encoded within these symbols belong to the same cohomology class in $H_{\text{dR}}^1(M)$,*
- (iv) *their topological charges are the same and*
- (v) *they have the same 2-torsion spin^c structure.*

Theorem 8.4. *Two full symbols $\mathbf{S}_{\text{full}}(x, p)$ and $\tilde{\mathbf{S}}_{\text{full}}(x, p)$ are SU -equivalent if and only if*

- (i) *the metrics encoded within these symbols are the same,*
- (ii) *the electric potentials encoded within these symbols are the same,*
- (iii) *the magnetic covector potentials encoded within these symbols are the same,*
- (iv) *their topological charges are the same and*
- (v) *they have the same spin structure.*

8.1 Explicit examples

Concluding this section, we examine two explicit examples. The first one illustrates how topological obstructions may arise when classifying symbols in accordance with (8.9). The second demonstrates the difference between spin and spin^c .

8.1.1 The Lie group $SO(3)$

Let $M = SO(3)$. We claim that $SO(3)$ has more than one 2-torsion spin^c structure. This follows from Corollary 6.5 and the non-vanishing of the group $H^2(SO(3); \mathbb{Z})$ but can also be seen directly as follows. With reference to Section 6.2, consider the identity map

$$\text{Id} : M \rightarrow SO(3).$$

The map Id does not lift to a map $SO(3) \rightarrow U(2)$, namely, there does not exist a map $s : SO(3) \rightarrow U(2)$ such that the diagram

$$\begin{array}{ccc} & & U(2) \\ & \nearrow s & \downarrow \pi \\ SO(3) & \xrightarrow{\text{Id}} & SO(3) \end{array}$$

commutes. A cohomological argument can be found in the proof of Lemma 6.2. Another way to see this is as follows. Let us restrict ourselves to $SU(2)$ matrices with zero trace. These matrices form a sphere $S^2 \subset SU(2)$, which can also be viewed as the conjugacy class of $\text{diag}(i, -i) \in SU(2)$. Explicitly, the matrices in S^2 are of the form

$$A = \begin{pmatrix} ia & b + ic \\ -b + ic & -ia \end{pmatrix},$$

where a, b , and c are real numbers such that $a^2 + b^2 + c^2 = 1$. The adjoint representation sends matrices A and $-A \in S^2$ to the same matrix, giving rise to the double covering $S^2 \rightarrow \mathbb{RP}^2$ of the real projective plane. We shall show that the bundle $U(2) \rightarrow SO(3)$ does not admit a section even over the subset $\mathbb{RP}^2 \subset SO(3)$. The issue one encounters with finding such a section is adjusting for the signs of $SU(2)$ matrices in S^2 mapping to the same matrix in \mathbb{RP}^2 . To make this adjustment, we need to find a continuous function $h : S^2 \rightarrow U(1)$ such that $h(-x) = -h(x)$, where $-x$ stands for the antipodal map on the sphere. If such a function h existed, its composition with the standard inclusion $U(1) \rightarrow \mathbb{R}^2$ would give rise to a function $f : S^2 \rightarrow \mathbb{R}^2$ with the property that $f(-x) = -f(x)$. However, such a function does not exist by the Borsuk–Ulam theorem [5, Theorem 1.10]: the Borsuk–Ulam theorem states that, for any continuous function $f : S^2 \rightarrow \mathbb{R}^2$, there exists $x \in S^2$ such that $f(-x) = f(x)$. Combined with $f(-x) = -f(x)$, this means that $f(x) = 0$ for some x , which contradicts the fact that the image of f belongs to the unit circle.

In fact, one can show that $SO(3)$ has precisely two distinct 2-torsion spin^c structures and precisely two distinct spin structures because $H^2(SO(3); \mathbb{Z}) = \mathbb{Z}/2$ and $H^1(SO(3); \mathbb{Z}/2) = \mathbb{Z}/2$. In this particular case, spin^c and spin structures are matched via the Bockstein isomorphism $H^1(SO(3); \mathbb{Z}/2) \rightarrow H^2(SO(3); \mathbb{Z})$, cf. Diagram (6.11).

8.1.2 The 3-torus

Let $M = \mathbb{T}^3$ be the 3-dimensional torus parameterised by mod 2π coordinates x^α , $\alpha = 1, 2, 3$. Put

$$\mathbf{S}(x, p) = \mathbf{S}_{\text{prin}}(x, p) := \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix},$$

$$\tilde{\mathbf{S}}(x, p) = \tilde{\mathbf{S}}_{\text{prin}}(x, p) := \begin{pmatrix} p_3 & e^{ix^3}(p_1 - ip_2) \\ e^{-ix^3}(p_1 + ip_2) & -p_3 \end{pmatrix}.$$

We have

$$\det \mathbf{S}_{\text{prin}}(x, p) = \det \tilde{\mathbf{S}}_{\text{prin}}(x, p) = -(p_1^2 + p_2^2 + p_3^2),$$

which means that the metric encoded within the symbols \mathbf{S} and $\tilde{\mathbf{S}}$ is the same, namely, the Euclidean metric. Furthermore, the topological charge (8.8) encoded within the symbols \mathbf{S} and $\tilde{\mathbf{S}}$ is the same, $+1$. Do these symbols have the same spin^c structure? The answer is yes, because if we take

$$R(x) = \begin{pmatrix} e^{-ix^3} & 0 \\ 0 & 1 \end{pmatrix} \in C^\infty(M, U(2))$$

we get

$$\tilde{\mathbf{S}}_{\text{prin}} = R^* \mathbf{S}_{\text{prin}} R. \quad (8.11)$$

However, it is easy to see that there does not exist a matrix-function $R \in C^\infty(M, SU(2))$ which would give (8.11), so our two symbols, \mathbf{S} and $\tilde{\mathbf{S}}$, have different spin structure.

In fact, it follows from Corollary 6.5 that the 3-torus has a unique 2-torsion spin^c structure because the cohomology group $H^2(\mathbb{T}^3; \mathbb{Z}) = \mathbb{Z}^3$ has no 2-torsion, but it has eight distinct spin structures because the cohomology group $H^1(\mathbb{T}^3; \mathbb{Z}/2) = (\mathbb{Z}/2)^3$ has eight elements.

9 Sesquilinear forms vs linear operators

Having developed our theory, we are now in a position to connect the motivational ideas outlined in the Introduction with the theory of partial differential equations.

Consider an Hermitian first order sesquilinear form of the type (2.2) on the infinite-dimensional vector space $C_0^\infty(M, \mathbb{C}^2)$.

9.1 Four-dimensional case

In dimension $m = 4$, introduce an inner product

$$\langle u, v \rangle := \int_M u^* G v \rho dx, \quad (9.1)$$

where G is some positive definite Hermitian 2×2 matrix-function and ρ is the Lorentzian density defined as in subsection 5.1. Our first order Hermitian sesquilinear form S and inner product (9.1) define a formally self-adjoint first order linear differential operator L . The problem here is that it is impossible to choose G so as to have

$$R^* G R = G, \quad \forall R \in GL(2, \mathbb{C}),$$

or even

$$R^* G R = G, \quad \forall R \in SL(2, \mathbb{C}),$$

i.e. one cannot introduce an inner product compatible with our gauge transformations. Hence, in the 4-dimensional case the construction presented in our paper defines a linear field equation but not a linear operator.

9.2 Three-dimensional case

Working in dimension $m = 3$ and within the framework of Section 8 (see, in particular, formulae (8.1)–(8.3)), introduce the inner product

$$\langle u, v \rangle := \int_M u^* v \rho dx, \quad (9.2)$$

where ρ is the Riemannian density encoded within our sesquilinear form in accordance with formulae (5.2) and (8.7). Now (9.2) is compatible with our gauge transformations. Hence, in the 3-dimensional case our construction defines a formally self-adjoint elliptic first order linear differential operator.

10 Applications

In dimension $m = 4$ a distinguished physically meaningful sesquilinear form is the so-called *Weyl form*. It is defined by the condition that the electromagnetic covector potential is zero. The gauge group is $SL(2, \mathbb{C})$. One cannot use here the gauge group $GL(2, \mathbb{C})$ because the electromagnetic covector potential is not invariant under the action of this group, see Lemma 5.2. The corresponding linear field equation is called *Weyl's equation*, the accepted mathematical model for the massless neutrino in curved spacetime. The condition $A = 0$ translates, in physical terms, into the neutrino having no electric charge and, therefore, not interacting with the electromagnetic field.

In dimension $m = 3$ and under the assumption (8.1) a distinguished physically meaningful sesquilinear form is the so-called *massless Dirac form*. It is defined by the condition that the electric potential and magnetic covector potential are both zero. By analogy with the previous paragraph, the gauge group here is $SU(2)$ and one cannot use $U(2)$ because the magnetic covector potential is not invariant under the action of the latter. The corresponding linear differential operator is called *massless Dirac operator*. Its spectrum describes the energy levels of a massless neutrino in curved space.

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Appendix A The concepts of principal and subprincipal symbol

The concepts of principal and subprincipal symbol are widely used in modern analysis, however they are traditionally employed for the description of (pseudo)differential *operators*. In the main text of our paper we use these concepts for the description of *sesquilinear forms*. We explain below the relation between the two seemingly different versions of, essentially, the same objects.

A half-density is a spatially varying complex-valued quantity on M which under changes of local coordinates transforms as the square root of a density. Analysts, especially those working in the spectral theory of partial differential operators, often prefer working with half-densities rather than with scalar functions.

Let $L^{(1/2)}$ be a first order linear differential operator acting on n -columns of half-densities. In local coordinates this operator reads

$$L^{(1/2)} = -iE^\alpha(x) \frac{\partial}{\partial x^\alpha} + F(x), \quad (A.1)$$

where $E^\alpha(x)$ and $F(x)$ are some $n \times n$ matrix-functions, compare with (2.2). Here the superscript $(1/2)$ indicates that we are dealing with an operator acting on half-densities.

We define the principal, subprincipal and full symbols of the operator (A.1) as

$$L_{\text{prin}}^{(1/2)}(x, p) := E^\alpha(x) p_\alpha, \quad (\text{A.2})$$

$$L_{\text{sub}}^{(1/2)}(x) := F(x) + \frac{i}{2} (L_{\text{prin}}^{(1/2)})_{x^\alpha p_\alpha}(x) = F(x) + \frac{i}{2} (E^\alpha)_{x^\alpha}(x), \quad (\text{A.3})$$

$$L_{\text{full}}^{(1/2)}(x, p) := L_{\text{prin}}^{(1/2)}(x, p) + L_{\text{sub}}^{(1/2)}(x) \quad (\text{A.4})$$

respectively. It is easy to see that the full symbol $L_{\text{full}}^{(1/2)}$ uniquely determines our first order linear differential operator $L^{(1/2)}$.

The definition of the subprincipal symbol (A.3) originates from the classical paper [3] of J.J. Duistermaat and L. Hörmander: see formula (5.2.8) in that paper. Unlike [3], we work with matrix-valued symbols, but this does not affect the formal definition of the subprincipal symbol. The correction term $\frac{i}{2} (L_{\text{prin}}^{(1/2)})_{x^\alpha p_\alpha}$ plays a crucial role in formula (A.3): its presence ensures that the subprincipal symbol is invariant under changes of local coordinates.

Our formulae (2.3)–(2.5) are analogues of the standard formulae (A.2)–(A.4). The bold script in the former indicates that we are dealing with density-valued quantities.

In order to establish the relation between symbols of sesquilinear forms and symbols of operators, let us fix a particular positive density μ and introduce the inner product

$$\langle u, v \rangle := \int_M u^* v \mu dx \quad (\text{A.5})$$

on n -columns of scalar fields. Formulae (1.2), (2.2) and (A.5) define a linear operator L .

The main result of this appendix is the following lemma.

Lemma A.1. *Conditions*

$$L = \mu^{-1/2} L^{(1/2)} \mu^{1/2} \quad (\text{A.6})$$

and

$$\mathbf{S}_{\text{full}} = \mu L_{\text{full}}^{(1/2)} \quad (\text{A.7})$$

are equivalent.

Proof. Formula (A.1) implies

$$\mu^{-1/2} L^{(1/2)} \mu^{1/2} = -i E^\alpha \frac{\partial}{\partial x^\alpha} + F - \frac{i}{2} E^\alpha (\ln \mu)_{x^\alpha}. \quad (\text{A.8})$$

Performing integration by parts, we rewrite formula (2.2) as

$$S(u, v) = \int_M u^* \left[\frac{1}{\mu} \left(-i \mathbf{E}^\alpha \frac{\partial}{\partial x^\alpha} + \mathbf{F} - \frac{i}{2} (\mathbf{E}^\alpha)_{x^\alpha} \right) v \right] \mu dx,$$

which gives us the following explicit local representation of the operator L :

$$L = \frac{1}{\mu} \left(-i \mathbf{E}^\alpha \frac{\partial}{\partial x^\alpha} + \mathbf{F} - \frac{i}{2} (\mathbf{E}^\alpha)_{x^\alpha} \right). \quad (\text{A.9})$$

Substituting (A.9) and (A.8) into (A.6), we see that the latter reduces to the pair of equations

$$\mathbf{E}^\alpha = \mu E^\alpha, \quad (\text{A.10})$$

$$\mathbf{F} - \frac{i}{2} (\mathbf{E}^\alpha)_{x^\alpha} = \mu \left(F - \frac{i}{2} E^\alpha (\ln \mu)_{x^\alpha} \right). \quad (\text{A.11})$$

Substituting (A.10) into (A.11) we rewrite the latter in equivalent form

$$\mathbf{F} = \mu \left(F + \frac{i}{2} (E^\alpha)_{x^\alpha} \right). \quad (\text{A.12})$$

In view of (2.3)–(2.5) and (A.2)–(A.4) conditions (A.10) and (A.12) are equivalent to (A.7). \square

As already pointed out in Section 9, in the most general setting of arbitrary m (dimension of the manifold), arbitrary n (number of scalar fields) and arbitrary sesquilinear form the introduction of an inner product of the form (A.5) does not make much sense because this inner product is incompatible with general linear and special linear gauge transformations. However, it makes sense in the special case (8.5), (8.1) because the inner product (A.5) is compatible with unitary and special unitary gauge transformations. And in this special case it is natural to take $\mu = \rho$, where ρ is the Riemannian density encoded within our sesquilinear form in accordance with formulae (5.2) and (8.7).

References

- [1] Z. Avetisyan, Y.-L. Fang, N. Saveliev and D. Vassiliev, Analytic definition of spin structure. *J. Math. Phys.* **58** (2017) 082301.
- [2] J. Davis and P. Kirk, *Lecture Notes in Algebraic Topology* (American Mathematical Society, Providence, 2001).
- [3] J.J. Duistermaat and L. Hörmander, Fourier integral operators II. *Acta Math.* **128** (1972) 183–269.
- [4] Y.-L. Fang and D. Vassiliev, Analysis as a source of geometry: a non-geometric representation of the Dirac equation. *J. Phys. A* **48** (2015) 165203.
- [5] A. Hatcher, *Algebraic topology* (Cambridge University Press, Cambridge, 2002).
- [6] R. C. Kirby, *The topology of 4-manifolds* (Lecture Notes in Mathematics, 1374, Springer-Verlag, Berlin, 1989).
- [7] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry* (Princeton University Press, 1989).
- [8] E. Stiefel, Richtungsfelder und Fernparallelismus in n -dimensionalen Mannigfaltigkeiten. *Comment. Math. Helv.* **8** (1935–1936) 305–353.