

THE MEMORY EFFECT IN IMPULSIVE PLANE WAVES: COMMENTS, CORRECTIONS, CLARIFICATIONS⁰

ROLAND STEINBAUER

*University of Vienna, Faculty of Mathematics
Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria*

ABSTRACT. Recently the “memory effect” has been studied in plane gravitational waves and, in particular, in *impulsive* plane waves. Based on an analysis of the particle motion (mainly in Baldwin-Jeffery-Rosen coordinates) a “velocity memory effect” is claimed to be found in [35]. Here we point out a conceptual mistake in this account and employ earlier works to explain how to correctly derive the particle motion and how to correctly deal with the notorious distributional Brinkmann form of the metric and its relation to the continuous Rosen form.

Keywords: impulsive gravitational waves, memory effect

MSC2010: 83C15, 83C35, 46F10, 34A36

1. INTRODUCTION

The “wave memory effect”, see e.g. [32, 3, 4, 5] has recently attracted much interest due to its possible experimental detection and a growing number of publications addresses this topic, see e.g. the introduction of [35] and the literature cited therein. In a recent series of papers [34, 33, 36] the “memory effect” has been studied in plane gravitational waves and in [35] these studies have been extended to *impulsive* plane waves.

Impulsive plane waves and, more generally, impulsive pp-waves have been introduced by Roger Penrose in the late 1960-ies, see [17, 16], and [18] for a more extensive treatment. They are spacetimes of low regularity, described alternatively by a (locally Lipschitz) continuous metric in (Baldwin-Jeffery-)¹Rosen form, or by a distributional metric in Brinkmann form. Over the years they have attracted the attention of researchers in exact spacetimes (who have widely generalized the original class of solutions), of mathematicians (who used them as relevant key-models in low regularity Lorentzian geometry), and of particle physicists (who have considered quantum scattering in these geometries).

In their work “Memory effect for impulsive gravitational waves” the authors of [35] derive the geodesics in impulsive plane waves using both forms of the metric. They find that in both coordinate systems particles initially at rest suffer a jump in their transversal velocities when crossing the impulse and start to move apart with constant speed. From this they conclude the occurrence of a “velocity memory effect”. While this behavior of the geodesics *in Brinkmann coordinates* is certainly correct and in accordance with the literature, the corresponding claim for the geodesics *in Rosen coordinates is incorrect*. In this note we analyse in detail the approach of [35] to explain why this derivation of the geodesics in Rosen coordinates is flawed and leads to inconsistent results. In particular, we address the conceptual intricacies that originate from the low regularity nature of the spacetimes involved. Moreover—based on earlier works—we show how

E-mail address: roland.steinbauer@univie.ac.at.

⁰This note was published in the present form in CQG as a ‘comment’ with the title changed to *Comment on ‘Memory effect for impulsive gravitational waves’*. The precise reference is Class. Quant. Grav. **36**(9) 098001, 14, (2019).

¹On historical grounds the name “Baldwin-Jeffery-Rosen coordinates” seems only to be accurate in the context of plane waves.

to calculate the geodesics in Rosen coordinates and how to use their C^1 -regularity to employ a “ C^1 -matching procedure” that leads to transparent “jump formulas” in Brinkmann coordinates. Finally, we explain how one can handle the subtle interrelations between the two forms of the metric in a mathematically meaningful way.

Throughout this note we have strived for maximum clarity at the expense of brevity. We aim at completely resolving the situation and we express our hope that in this way we may prevent further confusion in the literature².

In the remainder of this section we introduce our notation. Generally we follow the notations and conventions of [35] to make comparisons simple. The *metric* of plane gravitational waves in Brinkmann coordinates³ $X^\mu = (\mathbf{X}, U, V)$ with $\mathbf{X} = (X^i) = (X, Y)$ is written as⁴

$$ds^2 = \delta_{ij}dX^i dX^j + 2dUdV + K_{ij}(U)X^i X^j dU^2, \quad (2.1)$$

with the profile fixed to the + polarisation, hence given by

$$K = (K_{ij}) = \frac{1}{2} \mathcal{A}(U) \text{diag}(1, -1) = \frac{1}{2} \mathcal{A}(U) \mathbb{J}, \quad (2.2)$$

where the dependence of the function \mathcal{A} on retarded time is arbitrary but smooth and we use the abbreviation $\mathbb{J} = \text{diag}(1, -1)$. On the other hand, in Baldwin-Jeffery-Rosen coordinates⁵ $x^\mu = (\mathbf{x}, u, v)$ with $\mathbf{x} = (x^i) = (x, y)$ the metric is written as

$$ds^2 = a_{ij}(u)dx^i dx^j + 2dudv. \quad (2.4)$$

Here the profile $a = (a_{ij})$ is a positive definite (2×2) -matrix which depends again arbitrary but smoothly on retarded time u .

The *transformation* between the (R) and (B)-coordinates is written as

$$\mathbf{X} = P(u)\mathbf{x}, \quad U = u, \quad V = v - \frac{1}{4}\mathbf{x} \cdot \dot{a}(u)\mathbf{x}, \quad (2.6)$$

where the (2×2) -matrix $P(u)$ is a square root of a , i.e.,

$$a(u) = P(u)^T P(u) \quad (2.5)$$

when going from (R) to (B)-coordinates, and a solution of the Sturm-Liouville problem

$$\ddot{P} = KP, \quad P^T \dot{P} - \dot{P}^T P = 0, \quad (2.7)$$

if one goes from (B) to (R)-coordinates. The profiles are related via

$$K = \frac{1}{2}P\left(\dot{b} + \frac{1}{2}b^2\right), \quad \text{with } b = a^{-1}\dot{a}, \quad (2.8)$$

and so the Ricci-flat condition becomes $\text{tr} K = 0 \Leftrightarrow \text{tr}(\dot{b} + \frac{1}{2}b^2) = 0$.

The respective *impulsive limits* are written with profiles

$$a(u) = P^2(u) = (\mathbb{I} + u_+ c_0)^2 = (\mathbb{I} + u_+ k \mathbb{J})^2, \quad (2.16)$$

$$\mathcal{A}(u) = 2k\delta, \quad (2.17)$$

where k is the positive eigenvalue of the symmetric (2×2) -matrix c_0 , which arises by solving for the flatness condition in the after-zone of the wave, cf. [35, p. 4]. Sometimes we will set k to $k = 1/2$ or to $k = 1$ and as usual we have put $u_+ = 0$, ($u \leq 0$) and $u_+ = u$ ($u \geq 0$). Finally, δ is the Dirac-measure.

²As of Mar. 14, 2019, Inspire counts already 15 citations for the paper [35].

³Abbreviated as (B)-coordinates from now on.

⁴Again, to simplify comparison, we have taken equation numbers to coincide with [35].

⁵Abbreviated as (R)-coordinates below.

2. THE PECULIARITIES OF IMPULSIVE WAVE SPACETIMES

Here we pause for a moment and recall an essential issue in the construction of impulsive wave spacetimes, for an extensive review see [11, Ch. 20] and [19] or the more (astro-)physically oriented monograph [2]. These spacetimes were introduced by Roger Penrose in [17, p. 189ff], [16, p. 82ff] using what he called a “scissors and paste” method: two Minkowski half-spaces M^\pm are glued along a null hyperplane with a “warp”, i.e., a shift along the generators of the hyperplane when one passes from M^- to M^+ . Despite the fact that the Brinkmann form of the impulsive pp-wave contains a distributional component, the spacetime is actually (locally Lipschitz⁶) continuous but not C^1 . This is seen from the continuous Rosen form of the metric first given in (the plane wave case in) [18, p. 103] which possesses a locally bounded connection and a distributional curvature component Ψ_4 .

Let us emphasise the fact that this construction does *not provide a global background Minkowski space* on which the wave impulse can be thought to travel. This fact is somewhat obscured by the use of the so-called *Souriau-coordinates* [27], $\hat{\mathbf{x}}^\mu = (\hat{\mathbf{x}}, \hat{u}, \hat{v})$ with $\hat{\mathbf{x}} = (\hat{x}^i) = (\hat{x}, \hat{y})$ which are defined to be such that the metric is manifestly Minkowskian in both halves. Consequently, in the impulsive case, the transformation between (S) and (R)-coordinates is the identity for $u = \hat{u} < 0$, and for $u = \hat{u} > 0$ it is given by (cf. (2.6))

$$\hat{\mathbf{x}} = P(u)\mathbf{x}, \quad \hat{u} = u, \quad \hat{v} = v - \frac{1}{4}\mathbf{x} \cdot \dot{a}(u)\mathbf{x} = v - \frac{1}{2}\mathbf{x} \cdot c_0 P(u)\mathbf{x} = v - \frac{1}{2}\mathbf{x} \cdot k\mathbb{J}P(u)\mathbf{x}, \quad (2.25)$$

where we have used that in the impulsive case we have $\dot{a} = 2P\dot{P} = 2c_0P$ (cf. (2.5), (2.16)).

A useful way of thinking about the (S)-coordinates and the whole construction of impulsive spacetimes is the line of argument given in [11, Sec. 20.2], for the present setting cf. [15, eq. (9)]: Starting from Minkowski space with global coordinates now denoted by⁷ $\tilde{x}^\mu = (\tilde{\mathbf{x}}, \tilde{u}, \tilde{v})$ with $\tilde{\mathbf{x}} = \tilde{x}^i = (\tilde{x}, \tilde{y})$, i.e.⁸,

$$ds^2 = d\tilde{\mathbf{x}}^2 + 2d\tilde{u}d\tilde{v} \quad (1)$$

one uses the identity for $\tilde{u} < 0$ and the transformation

$$\begin{aligned} \tilde{u} &= u, \\ \tilde{\mathbf{x}} &= \mathbf{x} + u\partial H = (\mathbb{I} + u_+k\mathbb{J})\mathbf{x} = P(u)\mathbf{x}, \\ \tilde{v} &= v - H - \frac{1}{2}u\partial H \cdot \partial H = v - \frac{k}{2}\mathbf{x} \cdot \mathbf{x} - \frac{1}{2}uk\mathbf{x} \cdot \mathbf{x} = v - \frac{1}{2}\mathbf{x} \cdot k\mathbb{J}(\mathbb{I} + uk\mathbb{J})\mathbf{x} \\ &= v - \frac{k}{2}\mathbf{x} \cdot \mathbb{J}P(u)\mathbf{x} \end{aligned} \quad (2)$$

for positive \tilde{u} . Here we have set the profile H in [15, eq. (9)] to $H = (k/2)(x^2 - y^2) = (k/2)\mathbf{x} \cdot \mathbb{J}\mathbf{x}$ so that $\partial H = k\mathbb{J}\mathbf{x}$ and $\partial H \cdot \partial H = k^2\mathbf{x} \cdot \mathbf{x}$ and we see that we obtain exactly (2.25).

However, let us now combine the transformation (2) for $\tilde{u} > 0$ with the identity for $\tilde{u} < 0$ formally to the *discontinuous* transformation valid for all values of \tilde{u} resp. U (which was also first given by Penrose explicitly in [18])

$$\begin{aligned} \tilde{u} &= U, \\ \tilde{\mathbf{x}} &= \mathbf{x} + U_+\partial H = P(U)\mathbf{x}, \\ \tilde{v} &= V - \theta(U)H - \frac{1}{2}U_+\partial H \cdot \partial H = V - \frac{k}{2}\theta(U)\mathbb{J}\mathbf{x} \cdot (\mathbb{I} + U_+k\mathbb{J})\mathbf{x} = V - \frac{k}{2}\theta(U)\mathbf{x} \cdot \mathbb{J}P(U)\mathbf{x}. \end{aligned} \quad (3)$$

Here we have used the identity of L^∞ -functions $\theta(U)U_+ = U_+$.

⁶The Lipschitz property is decisive, since it guarantees the connection to be locally bounded. Moreover, it has recently been noted that Lipschitz-regularity of the metric is a threshold, below which even the most basic facts of causality theory fail to hold [6, 10].

⁷The coordinates \tilde{x}^μ correspond to $(x, y, \mathcal{U}, \mathcal{V})$ in [15] with a change in sign in \tilde{v} w.r.t. \mathcal{V} , which is due to the choice of $-2d\mathcal{U}d\mathcal{V}$ in the metric [15, eq. (1)]. Moreover, the (R)-coordinates x^μ correspond to (X, Y, U, V) in [15].

⁸Equations *not* appearing in [35] are numbered consecutively.

Now if one *formally* transforms the metric (2.4) with the impulsive profile (2.16) according to (3) *keeping the distributional terms* then one obtains precisely the (B)-form (2.1) of the metric with the distributional profile (2.17). Here we say “formally” since in addition to the standard distributional identities

$$(U_+)' = \theta, \quad \text{and} \quad \theta' = \delta \quad (4)$$

one has to use the “multiplication rules”

$$\theta u_+ = u_+, \quad \theta^2 = \theta, \quad \text{and} \quad \theta \delta = \frac{1}{2} \delta \quad (5)$$

which come from the grey areas of distribution theory: a careless combination of (4) and (5) easily leads to contradictions as in

$$\theta^2 = \theta \Rightarrow 2\theta\theta' = 2\theta\delta = \delta \Rightarrow \theta\delta = \frac{1}{2}\delta, \quad \text{but} \quad \theta^3 = \theta \Rightarrow 3\theta^2\theta' = 3\theta\delta = \delta \Rightarrow \theta\delta = \frac{1}{3}\delta. \quad (6)$$

Structurally speaking the problems arise when one combines rules like $\theta^2 = \theta$, which perfectly hold for L^∞ -functions with taking derivatives—which then has to be carried out in the sense of distributions.

The above procedure, however, has been made mathematically rigorous (using nonlinear distributional geometry in the sense of the geometric theory of generalized functions [12]) even in the pp -wave case in [13], see also Section 3.7, below.

We summarize with the following *observation and warning*: The (S)-system does not cover the whole manifold given by the (B)-coordinates (or the (R)-coordinates near the null surface $\{U = 0\}$) but is only valid for all $U = u = \hat{u} \neq 0$ and actually consists of two patches which do not overlap and can only be joined via the (B)- or (R)-coordinates.

3. PARTICLE MOTION

We now turn to the heart of the matter, i.e., the geodesics in impulsive plane waves. We will intensively comment on the approach taken in [35, Section 5]. Since in this approach the symmetries of the spacetime are used in an essential way, we start with a general comment on this strategy.

3.1. A general remark on the use of symmetries. The symmetries of *extended* (i.e., non-impulsive) plane waves are identified in [7] to be a subgroup of the Carroll group. Using the symmetries, the geodesics (again in the *extended* case) can be calculated efficiently, especially in (R)-coordinates. This has been done in [7, Sec. 3.2], see also [34, eqs. (2.11)] and [33, Sec. IV B]. The constants of motion are given by

$$\mathbf{p} = a(u)\dot{\mathbf{x}}, \quad \mathbf{k} = \mathbf{x}(u) - H(u)\mathbf{p}, \quad (7)$$

where already the 5th conserved quantity $\mu = \dot{u}$ was used to parametrise the geodesics by the coordinate u . Here the matrix-valued function H is given by⁹

$$H(u) = \int_0^u a^{-1}(w) dw. \quad (4.2)$$

Additionally, the kinetic energy

$$e = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} \dot{\mathbf{x}} \cdot a(u) \dot{\mathbf{x}} + \dot{v} = \frac{1}{2} \mathbf{p} \cdot a^{-1}(u) \mathbf{p} + \dot{v} \quad (8)$$

is conserved and is set to $e = -1, 0, 1$ in the timelike, null and spacelike case respectively. This finally leads to the explicit expression for the geodesics, cf. [7, eqs. (3.11)], [34, eqs. (2.11)]

$$\mathbf{x}(u) = H(u)\mathbf{p} + \mathbf{k}, \quad v(u) = -\frac{1}{2} \mathbf{p} \cdot H(u)\mathbf{p} + eu + d, \quad (9)$$

where d is a constant of integration.

⁹To be consistent with [35] we here keep the letter H —this is, however, *not* to be confused with the H of Section 2.

However, while in the extended case the symmetries allow one to nicely express the geodesics, it seems less obvious that this is also an efficient approach in the impulsive case. There the particle motion off the impulsive hypersurface is trivial and if one wants to derive its form in (R)-coordinates this can be done using the transformation (2.6) or by several other simple means, see Section 3.6. This raises the question:

(Q1) Why use symmetries to calculate the geodesics in the *impulsive* case at all?

As a follow-up, a more subtle question arises from the further procedure applied in [35, Sec. 5.1] (discussed in the next section)

(Q2) How to relate the values of the constants of motion in the before- and after-zones of the impulsive wave and how to argue for that?

We will come back to this questions later on in Section 3.6 but first have a closer look on the just mentioned procedure.

3.2. The approach of section 5.1 in [35]. In the main section 5 of [35] the geodesics are computed. In particular, in section 5.1 this is done in (R)-coordinates with a substantial use of (S)-coordinates. To start with, the form of the geodesics is derived using the symmetries, where the constants of motion are allowed to be different in the before- and the after-zones of the wave, denoted by \mathbf{p}_\pm , \mathbf{k}_\pm , and e_\pm , cf. [35, eqs. (4.12), (4.13)]. This readily leads to the form of the geodesics in the before- and the after-zones (again using the \pm -notation)

$$\mathbf{x}_\pm(u) = \mathbf{k}_\pm + H(u)\mathbf{p}_\pm, \quad v_\pm = -\frac{1}{2}\mathbf{p}_\pm \cdot H(u)\mathbf{p}_\pm + e_\pm u + d_\pm, \quad (5.2)$$

where now d_\pm are constants of integration.

Then on physical grounds the geodesics are assumed to be continuous across $u = 0$ such that

$$\mathbf{k}_\pm = \mathbf{x}(0) =: \mathbf{x}_0, \quad d_\pm = v_\pm(0) =: v_0, \quad (5.3)$$

where the choice that $H(0) = 0$, cf. equation (4.2) was used as well as the fact that $a(0) = 1$, cf. equation (2.16). Again using the latter equation one moreover obtains from (5.2) that

$$\mathbf{p}_\pm = \dot{\mathbf{x}}_\pm(0), \quad \dot{v}_\pm(0) = e_\pm - \frac{1}{2}|\mathbf{p}_\pm|^2, \quad (5.4)$$

where these remaining constants of motion are still allowed to be different in the two zones. Finally one may explicitly calculate H from equations (4.2) and (2.16) to obtain

$$H(u) = u_- \mathbb{I} + u_+ P^{-1}(u). \quad (5.4)$$

This gives the following form of the geodesics

$$\begin{aligned} \mathbf{x}(u) &= \mathbf{x}_0 + u_- \dot{\mathbf{x}}_-(0) + u_+ P^{-1}(u) \dot{\mathbf{x}}_+(0), \\ v(u) &= v_0 + u_- \dot{v}_-(0) + u_+ \dot{v}_+(0) + \frac{1}{2} \dot{\mathbf{x}}_+(0) \cdot u_+ (\mathbb{I} - P^{-1}(u)) \dot{\mathbf{x}}_+(0). \end{aligned} \quad (5.7)$$

These geodesics are determined by the following set of 9 real constants \mathbf{x}_0 , v_0 , $\dot{\mathbf{x}}_\pm(0)$, $\dot{v}_\pm(0)$, while there should only be 6 since we have to subtract the two constants $u(0) = 0$ and $\dot{u}(0) = 1$, which we have used to write the geodesics in the above form, from the usual 8 initial positions and speeds in a 4-dimensional spacetime.

To determine at least some of the “spurious constants” the authors of [35] now employ the (S)-coordinates and limit their considerations to the case of particles being at rest in the before-zone, a condition that can be expressed in the (S)-system of coordinates to be

$$\hat{\mathbf{x}}(\hat{u}) = \hat{\mathbf{x}}_0, \quad \hat{v}(\hat{u}) = \hat{v}_0 + e\hat{u}. \quad (5.9)$$

Their decisive argument now is (cf. [35, p. 10, bottom]):

“Now, the after-zone being also flat and indeed Minkowskian when the hatted (S) coordinates are used, we argue that the latter have the same parametric form in the after-zone: (5.9) holds for all u .” (*)

Using this argument the authors of [35] go on to rewrite the geodesics in (R)-form. To this end they apply the inverse of the transformation (2.25) for $\hat{u} = u > 0$ with the impulsive profile (2.16) inserted explicitly, i.e.,

$$\mathbf{x} = (\mathbb{I} + uk\mathbb{J})^{-1}\hat{\mathbf{x}}, \quad \hat{u} = u, \quad v = \hat{v} + \frac{1}{2}\hat{\mathbf{x}} \cdot k\mathbb{J}(\mathbb{I} + uk\mathbb{J})^{-1}\hat{\mathbf{x}}, \quad (5.8)$$

which implies the relations of the constants (just set $u = 0$)

$$\hat{\mathbf{x}}_0 = \mathbf{x}_0, \quad \hat{v}_0 = v_0 - \frac{1}{2}\mathbf{x}_0 \cdot k\mathbb{J}\mathbf{x}_0 \quad (10)$$

and obtain the following geodesics

$$\begin{aligned} \mathbf{x}(u) &= (\mathbb{I} + u_+k\mathbb{J})^{-1}\mathbf{x}_0, \\ v(u) &= v_0 + eu - \frac{1}{2}\mathbf{x}_0 \cdot k\mathbb{J}\left(\mathbb{I} - (\mathbb{I} + u_+k\mathbb{J})^{-1}\right)\mathbf{x}_0. \end{aligned} \quad (5.10)$$

3.3. Interpretation of the results and comments. After deriving the above result (5.10) the authors of [35] say:

“We see that the geodesic equation (5.10) is a special case of (5.7) where the after-zone initial velocity has been fixed by the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0-)[= \dot{\mathbf{x}}_0^-] = 0$, namely

$$\boxed{\dot{\mathbf{x}}(0+)[= \dot{\mathbf{x}}_0^+] = -c_0\mathbf{x}_0.} \quad (5.11)$$

The impulsive GW induces a (sort of) “percussion” [27], since

$$\Delta\dot{\mathbf{x}} = \dot{\mathbf{x}}(0+) - \dot{\mathbf{x}}(0-) = -c_0\mathbf{x}_0, \quad (5.12a)$$

$$\Delta v = v(0+) - v(0-) = -\frac{1}{2}|c_0\mathbf{x}_0|^2. \quad (5.12b)$$

First we note that the geodesics (5.10) are continuous not because “[...] this follows from (5.10).”, see [35, Sec. 5.1, last paragraph] but because this was assumed during the procedure explicitly in [35, p. 10, first lines] and mentioned here prior to (5.3).

The main issue is, however, that the geodesics (5.10) are *not* continuously differentiable and, moreover, that they are actually *not the correct geodesics in the first place*, as we are going to argue in the following:

First, it was shown in [15, Thm. 1] that the geodesics even in all impulsive (non-plane) *pp*-waves are C^1 -curves. There, Carathéodory’s solution concept is used, which is the most natural extension of classical ODE-theory using Lebesgue theory of integration. In fact, instead of solving the ODE with an L^∞ -right hand side one solves the classically equivalent integral equation, see e.g. [31, Ch. 3 §10, Suppl. 2]. This is actually a special case of the fact that the geodesics¹⁰ of any locally Lipschitz continuous metric are C^1 -curves, see [30].

Second¹¹, we explicitly demonstrate that the geodesics of (5.10) *do not* satisfy the geodesic equation. Let us demonstrate this just for the $x^1 = x$ -component, which reads

$$x(u) = \frac{x_0}{1 + ku_+}. \quad (11)$$

We find (using (5), as well as formally applying the product rule)

$$\dot{x}(u) = -\frac{kx_0\theta(u)}{(1 + ku_+)^2}, \quad \ddot{x}(u) = -\frac{kx_0\delta(u)}{(1 + ku_+)^2} + \frac{2k^2x_0\theta(u)}{(1 + ku_+)^3}. \quad (12)$$

¹⁰In the sense of Filippov [9], which is the appropriate solution concept there.

¹¹If one remains sceptical about the use of appropriate solution concepts for ODEs with L^∞ -right hand side, like Carathéodory’s, the following argument might be even more convincing.

Plugging this into the x -component of the geodesic equation for the metric (2.4) with the impulsive profile (2.16) (derived under the same “rules” as above, see also (21) below), i.e.,

$$\ddot{x}(u) + \frac{2k\theta(u)}{1+ku_+} \dot{x}(u) \quad (13)$$

one does *not* obtain a vanishing right hand side, but instead we have

$$\ddot{x}(u) + \frac{2k\theta(u)}{1+ku_+} \dot{x}(u) = -\frac{kx_0\delta(u)}{(1+ku_+)^2} = -kx_0\delta(u), \quad (14)$$

where in the last step we have used that for any function continuous in a neighbourhood of $u = 0$ we have $f(u)\delta(u) = f(0)\delta(u)$.

The result (14) may be explained in rough terms also as follows: Since the x -component of the geodesics (5.10) has a finite jump in its velocity at $u = 0$ its second derivative will contain a term proportional to $\delta(u)$. On the other hand such a term cannot be present on the right hand side of the geodesic equation in (R)-coordinates. Indeed the metric is Lipschitz continuous and hence possesses an L^∞ -connection: the Christoffel symbols in the geodesic equation will be proportional to the step function $\theta(u)$. Also the velocity \dot{x} will only involve a step function, so that the δ -term arising from \ddot{x} cannot be cancelled.

Third, one can resort to regularisation, which leads to different (and correct) geodesics, see item (A2) below, which again confirms that the geodesics of (5.10) are not the right ones.

But where is the error in the arguments of [35, Sec. 5.1]? It is included in the statement (*). In fact, it simply makes no sense to relate the form of the geodesic equations in the before- and the after-zones in (S) coordinates. These are actually given on two non-overlapping patches and there is no way of relating quantities on either side but using the transformation to (R)-coordinates or alternatively to (B)-coordinates *but* in this case keeping the distributional terms.

In conclusion, the geodesics (5.10) cannot be seen as actually being geodesics of the spacetime (2.4) with the impulsive profile (2.16). They do not satisfy the geodesic equations in any meaningful way across the impulse, neither in the sense of an appropriate solution concept, nor via a regularisation approach, nor formally. Finally the physical argument (*) put forward in deriving (5.10) is flawed.

3.4. Correctly deriving the geodesics using (R) & (S)-coordinates. Here we specialize the so-called C^1 -matching procedure of [15, Sec. 3] to the case of plane waves. This is a method to derive “jump conditions” for the geodesics in impulsive waves as seen with respect to “background coordinates” in the before- and after-zones. We suspect that this was also the idea underlying the (flawed) approach of [35, Sec. 5.2].

In the particular case at hand we have a Minkowskian background in the before- and the after-zone and hence we can trivially derive the geodesics there in manifestly Minkowskian coordinates, i.e., in the (S)-coordinates¹². We denote these geodesics in the usual \pm -notation as

$$\hat{\mathbf{x}}^\pm(u), \quad \text{and} \quad \hat{v}^\pm(u). \quad (15)$$

They are clearly just straight lines and are entirely determined by the following set of 2×6 constants

$$\begin{aligned} \hat{\mathbf{x}}_i^\pm &:= \lim_{u \rightarrow 0^\pm} \hat{\mathbf{x}}^\pm(u), & \hat{v}_i^\pm &:= \lim_{u \rightarrow 0^\pm} \hat{v}^\pm(u) \\ \hat{\dot{\mathbf{x}}}_i^\pm &:= \lim_{u \rightarrow 0^\pm} \dot{\hat{\mathbf{x}}}^\pm(u), & \hat{\dot{v}}_i^\pm &:= \lim_{u \rightarrow 0^\pm} \dot{\hat{v}}^\pm(u), \end{aligned} \quad (16)$$

where the subscript i stands for “interaction time”, i.e., for the instance when the geodesics cross the impulse. We can now relate the \pm -versions of these constants to one another using the fact that the geodesics in (R)-coordinates are C^1 -curves. More precisely, we transform the geodesics (15) to (R)-coordinates in which we will denote them by

$$\mathbf{x}(u), \quad \text{and} \quad v(u) \quad (17)$$

¹²This actually suggests a negative answer to question (Q1) above.

and “match” the respective constants (16). To do so most explicitly we invoke the inverse transformation of (3) to relate the (S)- to the (R)-coordinates¹³, which reads

$$\mathbf{x}(u) = P^{-1}(u)\hat{\mathbf{x}}(u), \quad v(u) = \hat{v}(u) + \frac{k}{2}\theta(u)P^{-1}(u)\hat{\mathbf{x}} \cdot \mathbb{J}\hat{\mathbf{x}}, \quad (18)$$

where again $P(u) = (\mathbb{I} + u_+ k \mathbb{J})$.

Now we can relate the \pm -versions of the constants (16) as follows

$$\begin{aligned} \hat{\mathbf{x}}_i^- &= \lim_{u \rightarrow 0^-} \hat{\mathbf{x}}^-(u) = \lim_{u \rightarrow 0^-} \mathbf{x}(u) = \lim_{u \rightarrow 0^+} \mathbf{x}(u) \\ &= \lim_{u \rightarrow 0^+} P^{-1}(u)\hat{\mathbf{x}}^+(u) = \lim_{u \rightarrow 0^+} (P^{-1}(u))\hat{\mathbf{x}}^+ = \hat{\mathbf{x}}_i^+. \end{aligned} \quad (19)$$

Here we have used the definition of $\hat{\mathbf{x}}_i^-$ in the first equality, the transformation (18) for $u < 0$ in the second, the continuity of the geodesics (17) in the third, and then again the transformation (18), now for $u > 0$. Finally, we have used the definition of $\hat{\mathbf{x}}_i^+$ and the explicit form of P^{-1} to calculate the limit.

Similarly we may use the C^1 -property of the geodesics (17) to relate the respective *velocities* on either side of the impulse and we obtain the following set of “jump conditions”:

$$\begin{aligned} \hat{\mathbf{x}}_i^- &= \hat{\mathbf{x}}_i^+, & \hat{v}_i^- &= \hat{v}_i^+ + \frac{k}{2}\hat{\mathbf{x}}_i^+ \cdot \mathbb{J}\hat{\mathbf{x}}_i^+, \\ \dot{\hat{\mathbf{x}}}_i^- &= \dot{\hat{\mathbf{x}}}_i^+ - k\mathbb{J}\hat{\mathbf{x}}_i^+, & \dot{\hat{v}}_i^- &= \dot{\hat{v}}_i^+ + k\hat{\mathbf{x}}_i^+ \cdot \mathbb{J}\dot{\hat{\mathbf{x}}}_i^+ - \frac{k^2}{2}\hat{\mathbf{x}}_i^+ \cdot \mathbb{J}\dot{\hat{\mathbf{x}}}_i^+. \end{aligned} \quad (20)$$

Observe that these relations are just a special case of the relations derived in [15, Sec. 3] with the same identifications as explained below (2).

We conclude with a remark on the “philosophy” of the C^1 -matching, cf. [20, Rem. 4.1]. The matching *presupposes* the following knowledge of the geodesics on the *entire* spacetime: the geodesics heading towards the impulse have to *cross* it, have to be *unique* and of C^1 -*regularity*. All these properties have been established for the situation at hand in [15]. Also the C^1 -matching procedure has been generalised to the case of non-expanding impulsive waves in any constant curvature background in [20] and to expanding impulsive waves, again in all constant curvature backgrounds in [21].

3.5. The geodesics in (B)-coordinates. Here we very briefly comment on the derivation of the geodesics in impulsive waves in (B)-coordinates. Indeed in [35, Sec. 5.2] the \mathbf{X} -components of the geodesics in impulsive plane waves are derived by basically integrating the geodesic equations and the use of the “multiplication rules” (5) to yield

$$\mathbf{X}(U) = P(U)\mathbf{X}_0 = (\mathbb{I} + u_+ k \mathbb{J})\mathbf{X}_0, \quad (5.16)$$

where $\mathbf{X}_0 = \mathbf{X}(0)$. Observe that (5.16) is in perfect agreement with the left equations in (20). The authors of [35] correctly remark in footnote 11 on p. 12 that the derivation of the V -component is more involved from the distribution theoretic point of view.

However, an ad-hoc procedure has been employed in the *pp*-wave case to derive the geodesics in [8], which—to the author’s best knowledge—is the earliest account explicitly calculating the geodesics in the distributional form of impulsive gravitational waves. A more reliable account has been put forward in [1], again in the *pp*-wave case. Here some nonlinear theory of distributions was applied but still an ad-hoc assumption (preservation of the geodesic’s tangent across the impulse) was needed to derive the result. The full solution was finally given in [28, 14]. We remark that in these approaches, which essentially are based on regularisation of the impulsive profile by a sequence of general sandwich waves, it becomes a nontrivial task to show that the solutions of the now nonlinear geodesic equations live long enough to cross the regularised (and hence extended) wave zone, i.e., the impulse *at all*. This is done using a fixed point argument which has been subsequently refined to allow a generalisation of the procedure to ever wider classes of impulsive waves, cf. [22, 23, 26, 25, 24].

¹³Note that we hence have to identify $\tilde{x}^\mu = (\tilde{\mathbf{x}}, \tilde{u}, \tilde{v})$ with $\hat{x}^\mu = (\hat{\mathbf{x}}, \hat{u}, \hat{v})$ in (3).

3.6. Returning to (R)-coordinates. In this section we finally comment on [35, Sec. 7], where the geodesics (5.7) are compared to the ones we have given in [29, Sec. 4]¹⁴.

In fact [29] uses quite different conventions and there is a lapse in the geodesics presented in eq. (14) there—in fact the X - the Y - components should be interchanged. However, these geodesics have been correctly transferred to the present setting in [35, Sec. 7] to read (using (R)-coordinates $\mathbf{x} = (x, y)$)

$$x(u) = x_0 + \dot{x}_0 \left(\frac{u_+}{1 + u_+} + u_- \right), \quad y(u) = y_0 + \dot{y}_0 \left(\frac{u_+}{1 - u_+} + u_- \right), \quad (7.1)$$

where for brevity we again restrict attention to the spatial components leaving aside the more complicated v -equation. Also we set $k = 1$ and use the usual definition $u_- = u$ if $u \leq 0$ and $u_- = 0$ for $u \geq 0$.

Actually we are aware of three ways to directly derive the explicit form of the geodesics in (R)-form in the plane wave case, i.e. (7.1), all without the use of the symmetries of the spacetime and all leading to the same result:

(A1) Solving the geodesic equations, which are given e.g. in [35, eq. (7.3)] and explicitly read

$$\ddot{x} + 2 \frac{k\theta}{1 + u_+} \dot{x} = 0, \quad \ddot{y} - 2 \frac{k\theta}{1 - u_+} \dot{y} = 0, \quad (21)$$

separately in the before- and the after-zones and matching the integration constant to obtain a global C^1 -curve.

(A2) Regularising the step function in the metric e.g. by setting $\theta_\varepsilon(u) = \int_{-\infty}^u \rho_\varepsilon(t) dt$ (with $\rho_\varepsilon \rightarrow \delta$ in distributions), then integrating the regularised equations, and finally performing the limit $\varepsilon \rightarrow 0$.

(A3) Making an ad-hoc ansatz (essentially guessing the solutions) and checking that the equations do hold again using the “multiplication rules” (5).

In [29, Sec. 4] we actually only mention approaches (A1) and (A2). However, the fact that the C^1 -property is used in the matching is explicitly stated above eq. (14)—contrary to the claim made below [35, eq. (7.1)]. Anyhow, it has meanwhile been proven that the C^1 -property holds, cf. Section 3.3, above.

Moreover, approach (A2) and hence also indirectly the C^1 -property is confirmed in [35, Caption of Fig. 7], where the authors acknowledge the fact that a regularisation by Gaussians leads to geodesics converging to (7.1).

Finally, we extend the calculations of eqs. (11)–(14) by formally showing that also the geodesics (5.7) (with arbitrary initial speeds) do *not* satisfy the geodesic equations [35, eq. (7.3)], i.e., (21). Indeed the spatial components of (5.7) take the explicit form

$$x(u) = x_0 + \dot{x}_0^+ \frac{u_+}{1 + u_+} + \dot{x}_0^- u_-, \quad y(u) = y_0 + \dot{y}_0^+ \frac{u_+}{1 - u_+} + \dot{y}_0^- u_-, \quad (7.2)$$

and using again the usual set of “multiplication rules” (5) one obtains e.g. for the x -component in $-\infty < u < 1$ that

$$\ddot{x} + 2 \frac{k\theta}{1 + u_+} \dot{x} = \delta(u) (\dot{x}_0^+ - \dot{x}_0^-). \quad (22)$$

This equation again tells us that in order to satisfy the geodesic equation we need to have $\Delta \dot{\mathbf{x}} = \dot{\mathbf{x}}_0^+ - \dot{\mathbf{x}}_0^- = 0$, i.e., no jumps in the velocities of the geodesics in (R)-coordinates.

3.7. Comparing the geodesics in (B)- and (R)-coordinates. In the final section of this chapter we comment on [35, Sec. 5.3] and the interrelations between the geodesics in (R)-form and in (B)-form. The authors of [35] say on this matter:

¹⁴We remark that this note was prepared for the “Proceedings of the 8-th National Romanian Conference on GRG, Bistritza, June 1998”. However, it was never peer-reviewed and to the best of my knowledge the said volume never appeared, cf. the comment on ArXiv.

“The naive expectation might be that this [interrelation] could be achieved by using the transformation formula between the coordinates, (2.6), i.e.,

$$\mathbf{X}(U) = P(U) \mathbf{x}(u), \quad (5.20)$$

which is indeed correct in the case of continuous wave profiles for particles initially at rest, [34, 33], for which $\mathbf{x}(u) = \mathbf{x}_0 = \text{const}$ for all u . However, identifying the initial positions, $\mathbf{x}_0 = \mathbf{X}_0$ and combining (5.16) and (5.10) yields instead,

$$\mathbf{X}(U) = (P^T P)(u) \mathbf{x}(u) = a(u) \mathbf{x}(u). \quad (5.21)$$

Where does the extra P -factor come from? The clue is that the delta-function $\delta(u)$ makes the velocity jump both in B [(B)] and BJR [(R)] coordinates — and does it in the opposite way, see in (5.15)¹⁵ and (5.12b) [that actually should read (5.12a)], respectively. The extra P factor takes precisely care of these jumps: the first P in (5.21) straightens the trajectory (5.10) to the trivial one, $P(u)\mathbf{x}(u) = \mathbf{x}_0$, which has zero initial BJR velocity as in the smooth case [34, 33]; then the second $P(u)$ factor curls it up according to (5.20), yielding $\mathbf{X}(u)$ in (5.16).”

This explanation remains dubious. While it is true that the combination of (5.16) and (5.10) yields (5.21), this just confirms that the geodesics in (R)-form (5.10) are *not* the correct ones. In fact, replacing the incorrect geodesics (5.10) by the correct ones, i.e., (7.1), which in the case at hand, i.e., vanishing speeds, simply read $\mathbf{x}(u) = \mathbf{x}_0$, we correctly obtain (5.20) (as is also acknowledged in the above quotation):

$$\mathbf{X}(U) = P(U)\mathbf{X}_0 = P(u)\mathbf{x}_0 = P(u)\mathbf{x}(u). \quad (23)$$

The fact that formally transforming the geodesics in (R)-form (7.1) with the “discontinuous change of coordinates” (3) yields exactly the geodesics in (B)-form (5.16) has already been noted in [29, Sec. 4] just below eq. (14)¹⁶. In fact, it *does nothing else but* transforming the (B)-geodesics with vanishing initial speeds into the (R)-geodesics. In other words, the broken and jumping (B)-geodesics become the new coordinate lines in the (R)-system. And this is the ultimate reason why the regularity of the metric improves from distributional in the (B)-coordinates to continuous in (R)-coordinates.

The formal calculation establishing these ideas has been turned into a solid piece of mathematics even for the pp -wave case in [13], using nonlinear distributional geometry. A good way to describe the situation in physical terms is given there in Sec. 5: The “discontinuous change” of coordinates is the distributional limit of a family of smooth transformations which can be obtained by a general regularisation procedure, which is adapted to the spacetime geometry. From this regularisation point of view, the (B) and (R)-forms of the impulsive metric arise as the distributional limits of the *same* sandwich wave in different coordinate systems. In such a scenario, in general, different spacetimes may result and the fact that in this case the geometries are “physically equivalent” is reflected by the fact that the resulting transformation is merely discontinuous rather than unbounded. Nevertheless, it introduces finite jumps of the geodesics and their velocities.

4. SUMMARY AND CONCLUSIONS

We have clarified the intricacies of the particle motion in impulsive plane—and effectively in pp -waves. In (B)-coordinates the geodesics possess a discontinuous v -component and the v -velocity as well as the transverse velocities exhibit a finite jump across the impulse. Then again in (R)-coordinates the geodesics are continuously differentiable curves and hence there is *no* jump in the velocities. This seemingly odd behaviour is due to the fact that the transformation between the (B)- and (R)-coordinates is discontinuous. It nevertheless allows one to correctly and consistently

¹⁵Equation (5.15) in [35] reads $\dot{\mathbf{X}}(0+) = c_0 \mathbf{X}_0$.

¹⁶Note that in the last line on p. 7 there is a typo: the equation number (11) should actually be (10).

transform the geodesics (formally) from one form to the other. Moreover, this procedure has been handled in a mathematically meaningful way using nonlinear distributional geometry.

All this is in perfect agreement with the geometric picture: The (R)-coordinates are *comoving* and hence discontinuous as seen from the two Minkowski halves to either “side” of the impulse. This is why the regularity of the metric improves from distributional in (B)-coordinates to continuous in (R)-coordinates. But it is also the reason why one does not see any particle motion in (R)-coordinates: Particles initially at rest remain so after the impulse until they eventually reach the coordinate singularity of the (R)-coordinates.

Finally, this author remains agnostic regarding the question whether or not these results mean that impulsive plane waves exhibit a “velocity memory effect”. The reason simply is that we are not aware of an invariant definition of the “memory effect” for the spacetimes at hand, which are *not* asymptotically flat.

ACKNOWLEDGEMENTS

I wish to thank my frequent coauthors Jiří Podolský, Robert Švarc, and Clemens Sämann for their encouragement during my writing of this note and for their ongoing support, and Michael Kunzinger for the critical reading of the manuscript. Also I acknowledge a very friendly email conversation with Peter Horvathy. This work was supported by project P28770 of the Austrian Science Fund FWF and the WTZ-grant CZ12/2018 of OeAD.

REFERENCES

- [1] H. Balasin. Geodesics for impulsive gravitational waves and the multiplication of distributions. *Classical Quantum Gravity*, 14(2):455–462, 1997. 3.5
- [2] C. Barrabès and P. A. Hogan. *Singular null hypersurfaces in general relativity*. World Scientific Publishing Co., Inc., River Edge, NJ, 2003. Light-like signals from violent astrophysical events. 2
- [3] V. B. Braginsky and L. P. Grishchuk. Kinematic resonance and the memory effect in free mass gravitational antennas. *Sov. Phys. JETP*, 62:427, 1985. 1
- [4] V. B. Braginsky and K. S. Thorne. Present status of gravitational-wave experiments. In *Proceedings of the ninth international conference on general relativity and gravitation (Jena, 1980)*, pages 239–253. Cambridge Univ. Press, Cambridge, 1983. 1
- [5] D. Christodoulou. Nonlinear nature of gravitation and gravitational-wave experiments. *Phys. Rev. Lett.*, 67(12):1486–1489, 1991. 1
- [6] P. T. Chruściel and J. D. E. Grant. On Lorentzian causality with continuous metrics. *Classical Quantum Gravity*, 29(14):145001, 32, 2012. 6
- [7] C. Duval, G. W. Gibbons, P. A. Horvathy, and P.-M. Zhang. Carroll symmetry of plane gravitational waves. *Class. Quant. Grav.*, 34(17):175003, 2017. 3.1, 3.1
- [8] V. Ferrari, P. Pendenza, and G. Veneziano. Beam-like gravitational waves and their geodesics. *Gen. Relativity Gravitation*, 20(11):1185–1191, 1988. 3.5
- [9] A. F. Filippov. *Differential equations with discontinuous righthand sides*, volume 18 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian. 10
- [10] J. D. E. Grant, M. Kunzinger, C. Sämann, and R. Steinbauer. The future is not always open. *preprint, arXiv:1901.07996 [math.DG]*, 2019. 6
- [11] J. B. Griffiths and J. Podolský. *Exact Space-Times in Einstein’s General Relativity*. Cambridge University Press, Cambridge, 2009. 2, 2
- [12] M. Grosser, M. Kunzinger, M. Oberguggenberger, and R. Steinbauer. *Geometric theory of generalized functions with applications to general relativity*, volume 537 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2001. 2
- [13] M. Kunzinger and R. Steinbauer. A note on the Penrose junction conditions. *Class. Quant. Grav.*, 16:1255–1264, 1999. 2, 3.7
- [14] M. Kunzinger and R. Steinbauer. A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves. *J. Math. Phys.*, 40(3):1479–1489, 1999. 3.5
- [15] A. Lecke, R. Steinbauer, and R. Švarc. The regularity of geodesics in impulsive pp-waves. *Gen. Rel. Grav.*, 46:1648, 2014. 2, 2, 7, 3.3, 3.4, 3.4
- [16] R. Penrose. Structure of space-time. In *Battelle Rencontres, 1967 Lectures in Mathematics and Physics*, pages 121–235. Benjamin, New York, 1968. 1, 2
- [17] R. Penrose. Twistor quantization and curved space-time. *Int. J. Theor. Phys.*, 1:61–99, 1968. 1, 2
- [18] R. Penrose. The geometry of impulsive gravitational waves. In *General relativity (papers in honour of J. L. Synge)*, pages 101–115. Clarendon Press, Oxford, 1972. 1, 2, 2

- [19] J. Podolský. Exact impulsive gravitational waves in space-times of constant curvature. In *Gravitation: Following the Prague Inspiration*, pages 205–246. Singapore: World Scientific Publishing Co., 2002. 2
- [20] J. Podolský, C. Sämann, R. Steinbauer, and R. Švarc. The global existence, uniqueness and C^1 -regularity of geodesics in nonexpanding impulsive gravitational waves. *Classical Quantum Gravity*, 32(2):025003, 23, 2015. 3.4
- [21] J. Podolský, C. Sämann, R. Steinbauer, and R. Švarc. The global uniqueness and C^1 -regularity of geodesics in expanding impulsive gravitational waves. *Classical Quantum Gravity*, 33(19):195010, 23, 2016. 3.4
- [22] C. Sämann and R. Steinbauer. On the completeness of impulsive gravitational wave spacetimes. *Classical Quantum Gravity*, 29(24):245011, 11, 2012. 3.5
- [23] C. Sämann and R. Steinbauer. Geodesic completeness of generalized space-times. In *Pseudo-differential operators and generalized functions*, volume 245 of *Oper. Theory Adv. Appl.*, pages 243–253. Birkhäuser/Springer, Cham, 2015. 3.5
- [24] C. Sämann and R. Steinbauer. Geodesics in nonexpanding impulsive gravitational waves with Λ . II. *J. Math. Phys.*, 58(11):112503, 18, 2017. 3.5
- [25] C. Sämann, R. Steinbauer, A. Lecke, and J. Podolský. Geodesics in nonexpanding impulsive gravitational waves with Λ , part I. *Classical Quantum Gravity*, 33(11):115002, 33, 2016. 3.5
- [26] C. Sämann, R. Steinbauer, and R. Švarc. Completeness of general pp -wave spacetimes and their impulsive limit. *Classical Quantum Gravity*, 33(21):215006, 27, 2016. 3.5
- [27] J.-M. Souriau. Le milieu élastique soumis aux ondes gravitationnelles. *Colloq. Internat. CNRS*, 220:243–256, 1974. Avec discussion. 2, 3.3
- [28] R. Steinbauer. Geodesics and geodesic deviation for impulsive gravitational waves. *J. Math. Phys.*, 39(4):2201–2212, 1998. 3.5
- [29] R. Steinbauer. On the geometry of impulsive gravitational waves. *ArXiv:9809054[gr-qc]*, 1998. 3.6, 3.6, 3.7
- [30] R. Steinbauer. Every Lipschitz metric has C^1 -geodesics. *Classical Quantum Gravity*, 31(5):057001, 3, 2014. 3.3
- [31] W. Walter. *Ordinary differential equations*, volume 182 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998. Translated from the sixth German (1996) edition by Russell Thompson, Readings in Mathematics. 3.3
- [32] Y. B. Zel'dovic and A. G. Polnarev. Radiation of gravitational waves by a cluster of superdense stars. *Sov. Astron.*, 18:17, 1974. 1
- [33] P.-M. Zhang, C. Duval, G. W. Gibbons, and P. A. Horvathy. Soft gravitons and the memory effect for plane gravitational waves. *Phys. Rev.*, D96(6):064013, 2017. 1, 3.1, 3.7, 3.7
- [34] P.-M. Zhang, C. Duval, G. W. Gibbons, and P. A. Horvathy. The Memory Effect for Plane Gravitational Waves. *Phys. Lett.*, B772:743–746, 2017. 1, 3.1, 3.1, 3.7, 3.7
- [35] P.-M. Zhang, C. Duval, and P. A. Horvathy. Memory effect for impulsive gravitational waves. *Classical Quantum Gravity*, 35(6):065011, 20, 2018. (document), 1, 1, 2, 4, 8, 3, 9, 3.1, 3.2, 3.2, 3.2, 3.2, 3.3, 3.3, 3.3, 3.4, 3.5, 3.5, 3.6, 3.6, 3.7, 15
- [36] P.-M. Zhang, M. Elbistan, G. Gibbons, and P. A. Horvathy. Sturm-Liouville and Carroll: at the heart of the memory effect. *Gen. Relativity Gravitation*, 50(9):Art. 107, 9, 2018. 1