

Extrapolating the precision of the Hypergeometric Resummation to Strong couplings with application to the \mathcal{PT} -Symmetric $i\phi^3$ Field Theory

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Abstract

In PRL 115, 143001 (2015), H. Mera et al. developed a new simple but precise Hypergeometric Resummation technique. In this work, we suggest to obtain half of the parameters of the Hypergeometric function from the strong coupling expansion of the physical quantity. Since these parameters are taking now their exact values they can improve the precision of the technique for the whole range of the coupling values. The second order approximant ${}_2F_1$ of the algorithm is applied to resum the perturbation series of the ground state energy of the \mathcal{PT} -symmetric $(i\phi^3)_{0+1}$ field theory. It gives accurate results compared to exact calculations from the literature specially for very large coupling values. The \mathcal{PT} -symmetry breaking of the Yang-Lee model has been investigated where third, fourth and fifth orders were able to get very accurate results when compared to other resummation methods involving 150 orders. The critical exponent ν of the $O(4)$ -symmetric model in three dimensions has been precisely obtained using only first order of perturbation series as input. The algorithm can be extended easily to accommodate any order of perturbation series in using the generalized Hypergeometric function ${}_{k+1}F_k$ as it shares the same analytic properties with ${}_2F_1$.

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In quantum field theories one is always confronted with perturbation series of zero radius of convergence [1]. For such cases Resummation techniques have been applied and successful results have been obtained[1–4]. Although these techniques can produce accurate results they sometimes use calculations for large number of loops as an input and most of the calculations go numerically. Recently, a precise as well as simple Resummation technique has been introduced [5] where it uses only four orders of the perturbation series as well as it is of analytic form. Such algorithm is very suitable for quantum field cases as it can give reasonable results with only few orders out of the perturbation series. The authors of Ref. [5] have extended the technique to employ higher orders of the perturbation series [6] via use of the generalized Hypergeometric functions followed by Borel transform that leads to a form of Meijer-G function [7]. Regarding the original Hypergeometric version in Ref.[5] and its upgrades, the prediction of the whole parameters of the Hypergeometric function is obtained from just weak-coupling data. Since for original version of the Hypergeometric approximants, the first two parameters of the Hypergeometric function are related to its asymptotic form for large values of the argument [7] (Strong-Coupling data), the prediction of these parameters from small coupling information might not lead to well known strong coupling behaviors of the physical quantity. To employ all the information from weak-coupling as well as strong coupling expansions, one needs to employ a generalized form of the Hypergeometric function. In this work, we introduce an algorithm that can lower the number of input perturbation terms to half of those needed by the Hypergeometric algorithm in Refs.[5] as well as guarantee the accuracy for very large coupling values.

To motivate for this work, we mention that in Ref.[2] the strong coupling behaviors have been stressed for both the \mathcal{PT} -symmetric and the real cubic anharmonic oscillator. In applying resummation techniques that involve 150 orders for the \mathcal{PT} - symmetric case the authors showed that:

$$\lim_{g \rightarrow \infty} \frac{E_0}{|g|^{\frac{1}{5}}} = 0.372545790452207098250601(1), \quad (1)$$

while for the cubic oscillator they obtained

$$\begin{aligned} \lim_{g \rightarrow -\infty} \frac{E_0}{|g|^{\frac{1}{5}}} &= 0.3013958756586835717823(7) \\ &+ 0.2189769214314493762936(0)i \end{aligned} \quad (2)$$

It is easy to check that the prediction of the original Hypergeometric Resummation in [5] gives zero in both cases. This is because there is like 6% error in the prediction of the second parameter in the Hypergeometric function which affects the precision of the algorithm for large coupling values. Accordingly, one needs to extrapolate the predictions of the algorithm to give accurate results for the whole coupling space.

In this work we apply the Hypergeometric Resummation algorithm to the \mathcal{PT} -symmetric Yang-Lee model but in guiding the Hypergeometric functions with parameters from the strong coupling behavior. In $0 + 1$ space-time dimension (quantum mechanics), one can follow a scaling as well as gauge canonical transformations to obtain the strong coupling expansion of the theory [2]. In higher dimensions (quantum field theory), for some cases one can obtain the strong-coupling expansions of a physical quantity [8, 9]. So feeding the resummation technique with two parameters (for the second order) from the perturbation series and the other two from the strong coupling expansion is possible for both quantum and quantum field problems. As we will see in this work, this algorithm lowers the number of orders from the perturbation series to two instead of four needed for the original algorithm in Ref.[5]. Besides the prediction is then more accurate for large couplings. The extension of the algorithm to higher orders is direct and shall be applied here to investigate \mathcal{PT} -symmetry breaking of the Yang-Lee model with increasing precision when moved from ${}_2F_1$ to ${}_3F_2$ and very high precision obtained from ${}_4F_3$ and ${}_5F_4$ when compared to resummation results from Ref.[2] where methods there involved 150 orders. It is worth mentioning that the strong coupling parameters have been used before in the literature to accelerate the convergence of resummation algorithms [12–18] and thus it is expected to have the same role in the Hypergeometric resummation algorithm too.

The Yang-Lee model or equivalently \mathcal{PT} -symmetric $i\phi^3$ field theory has been exposed to recent discussions because it has an imaginary potential but on the other hand has a real spectrum [20–25]. In fact, the ground state energy has a zero radius of convergence and thus non-perturbative Resummation algorithms are in a need to get reliable results from perturbative calculations as an input. Another aspect for which non-perturbative approaches of the Yang-Lee model are essential is because it represents the Landau-Ginzberg approximation of the Ising model near the Yang-Lee edge singularity [26–30]. Padé, Borel and other algorithms applied to the model in Refs.[31–34] . While Padé approximation can not account for the strong coupling behavior, most of Borel calculations are achieved

via numerical calculations. On the other hand, the recent Hypergeometric Resummation technique introduced in [5] is characterized by being simple, closed form as well as employs only few number of terms from the perturbation series as an input. In $0 + 1$ space-time dimensions (quantum mechanics) the Yang-Lee model has been stressed by the authors of Ref. [5] but we realized that the precision of the results for strong coupling (even for the real potential case) is questionable. As we suggested above, a way to have better fitting with available exact results is to provide the Hypergeometric function with known results from the strong coupling behavior. In fact, strong coupling expansion can be obtained in many cases. For instance, Hermitian theories like the ϕ^4 field theory has been extensively stressed in the literature and its strong coupling as well as large order behaviors are discussed and employed to accelerate the convergence of resummation techniques[1, 9]. Although for quantum field theory (dimension $d > 1$) the strong coupling parameters can be obtained using optimization methods, the employment of their approximate values lead to improvement of the resummation results [9–11].

Applying a simple and accurate Resummation algorithms to divergent series might have a strong impact on the field of \mathcal{PT} - symmetric field theories where one can resum the series from the known results of just the first few terms in the perturbation series and its strong coupling behavior. For instance, the \mathcal{PT} -symmetric $(-\phi)^4$ model is assumed to be asymptotically free [35–38] but up to the best of our knowledge no non-perturbative calculation for the Beta function appeared yet. Another application that the Hypergeometric Resummation can play a vital role is in the very recently introduced \mathcal{PT} -symmetric Higgs Mechanism and such Resummation technique may offer a non-perturbative tool that saves the effort and time for the calculation in such cases where high order of loop calculations is time consuming. A note to be mentioned is that it took the researchers like 25 years to move from the fifth order to sixth order of the perturbation series of the renormalization group functions for simple quantum field theories [39, 40]. So seeking a way to accelerate the convergence of a resummation algorithm is more than important in the field of quantum field theory. As we will see in this work, the critical exponent of the $O(4)$ -symmetric field theory is obtained at only the first order using the algorithm in this work. This theory can describe the finite temperature phase transition in QCD with two light flavors [41].

For the Hypergeometric Resummation there is another precision realization concerning the small coupling predictions where it has been realized by the authors themselves in Ref.

[19]. According to them, the series expansion of the Hypergeometric function does not have a zero radius of convergence while the aim is to sum a series of zero radius of convergence. To solve this problem, the authors set an algorithm that results in a Hypergeometric Resummation with zero radius of convergence [19]. In this work, we shall stress only the impact of employing the strong coupling behavior on the accuracy of the algorithm.

The Hypergeometric function ${}_2F_1$ can have a power law behavior near singular points [7] and thus in principle can account for the calculation of the critical exponents of the Yang-Lee model near the edge singularity but this will be out of the scope of this work.

To test the accuracy of algorithm before we go to the \mathcal{PT} -symmetric $i\phi^3$ theory, we consider the ${}_2F_1$ resummation of the ground state energy of the anharmonic oscillator where it has a perturbation series with zero radius of convergence [48]. We obtained the ground state energy from second order approximant as $E_0(g) = {}_2F_1\left(\frac{1}{3}, \frac{-1}{3}, c, -dg\right)$ and find $E_0(50) = 2.4484029106721046$. Although this result is better than Borel-Padé resummation in using 24^{th} order $BP12/12$ where it gives $E_0(50) = 2.3157388197$ [6], one can get better results in involving more terms from the perturbation series. When we use ${}_4F_3$ we get $E_0(50) = 2.4856072532925255$. Note that, for this theory, scaling properties can lead to the strong coupling expansion for the ground state energy of the form:

$$E_0 = g^{\frac{1}{3}} \sum_{i=0}^{\infty} m^2 \left(g^{\frac{-2}{3}}\right)^i. \quad (3)$$

Accordingly, the a_i parameters in the generalized Hypergeometric approximant ${}_pF_{p-1}(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_{p-1}; -\sigma z)$ are given by $a_1 = \frac{-1}{3}, a_2 = \frac{1}{3}, a_3 = 1, a_4 = \frac{5}{3}, a_5 = \frac{7}{3}, \dots$. Note also that the exact result for $E_0(50)$ is 2.4997087726 and the Hypergeometric resummation in Ref.[6] gives 2.4997107287 but in involving 25 orders. So it seems that the algorithm we use accelerates the convergence to the exact prediction as the second order in our algorithm have accuracy that lies between the 24^{th} order of Borel-Badé resummation and the 25^{th} of the generalized Hypergeometric (Meijer-G) resummation in Ref.[6] while our fourth order result is very close to the exact one.

Another example to test algorithm is to stress the critical exponent ν of the $O(4)$ -symmetric model. In Ref.[42], the ε - expansion of the critical exponent ν of the $O(4)$ model ($\varepsilon=4-D$) has been linked to the ϵ -expansion of the σ -model ($\epsilon = 2 - D$) using the fact that both lie in the same class of universality. In fact, the authors were able to write

one expansion as the strong coupling expansion of the other in writing the two expansions in terms of a new variable $\tilde{\varepsilon} = 2\frac{D-4}{D-2}$. They obtained the strong coupling expansion (N=4) as:

$$\nu^{-1} = 4\tilde{\varepsilon}^{-1} - 8\frac{N-4}{N-2}\tilde{\varepsilon}^{-2} + O(\tilde{\varepsilon}^{-3}) \quad (4)$$

while the weak coupling expansion is given by:

$$\nu^{-1} = 2 - \frac{1}{2}\tilde{\varepsilon} + 0.0833333\tilde{\varepsilon}^2 + O(\tilde{\varepsilon}^3). \quad (5)$$

In fact, the a_i parameters in the approximants ${}_pF_{p-1}(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_{p-1}; \sigma z)$ can be concluded from the strong coupling expansion to be 1, 2, 3, and so on. However, one can realize that in that case $a_i - a_j$ is an integer which means that the strong coupling expansion of ${}_pF_{p-1}(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_{p-1}; \sigma z)$ will include logarithmic dependance [7] which is not reflected in the strong coupling expansion above. The only exception is for the approximant ${}_1F_0(1, , \sigma\tilde{\varepsilon})$. Matching the expansion of this approximant with the weak-coupling expansion above one finds that $\sigma = -\frac{1}{4}$. Accordingly, the resummed ν exponent is given by

$$\nu = \frac{1}{2 {}_1F_0(1, / \frac{-1}{4}\tilde{\varepsilon})}.$$

In three dimensions ($\varepsilon = 1$) it gives the result $\nu = 0.75$ compared to 0.735 from the resummation of fifth order series in Ref.[42] and 0.7479 using MonteCarlo sc. [43, 44]. Note that we get this result using only first order from perturbation series as input which up to the best of our knowledge is the first time to get that result that fast. To compare with figure 3 in Ref. [42], we generated the graph in Fig.1 while the details of the critical exponents of $O(N)$ -symmetric model using the Hypergeometric algorithm is postponed to a separate work.

Informed by the precise results we obtained above from a very few orders of perturbation series as input, we stress a model that is can be considered as a hot research point [2, 23–25, 45–47]. That model is the \mathcal{PT} -symmetric $i\phi^3$ theory which has a Lagrangian density of the form:

$$\mathcal{L}[\phi] = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2(x) - \frac{i\sqrt{g}}{6}\phi^3(x), \quad (6)$$

with a corresponding Hamiltonian density:

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2(x) + \frac{i\sqrt{g}}{6}\phi^3(x). \quad (7)$$

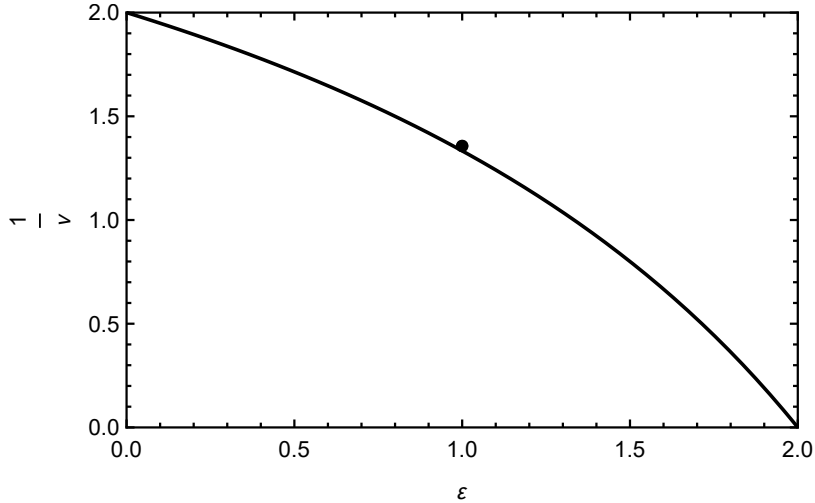


FIG. 1: The approximant $\nu^{-1} = 2 {}_1F_0(1, \frac{-1}{4}\bar{\epsilon})$ versus $\epsilon = 4 - D$ for the $O(4)$ -symmetric model with dot represents the six loops prediction for comparison with Figure 3 in Ref.[42].

The Hamiltonian operator is \mathcal{PT} -symmetric and thus the spectrum is real. This Hamiltonian is closely related to the Hamiltonian

$$H_J = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{i}{6}\phi^3(x) + iJ\phi, \quad (8)$$

where in 0+1 space-time dimensions, one can start from H and apply scaling as well as gauge transformation to get H_J [2, 32, 49]. Although H_J is \mathcal{PT} -symmetric, the \mathcal{PT} -symmetry is not exact for all real J values and the \mathcal{PT} -symmetry is broken for some critical J value [2]. According to Yang and Lee, the partition function or equivalently the vacuum to vacuum amplitude can have a zero for negative \mathcal{J} values known as Yang-Lee edge singularity [26–29]. This specific zero is associated with non-analyticity of the ground state energy and this is supposed to be associated with \mathcal{PT} -symmetry breaking. Near the edge singularity the theory is totally non-perturbative and one needs to apply non-perturbative techniques. The perturbation series for the ground state energy of the Hamiltonian H also has a zero radius of convergence and thus Resummation is needed anyway. We will stress the resummation of ground state energy of both Hamiltonians H and H_J and show that our second to fourth orders are very competitive to the results with the 150th order of resummation methods in Ref.[2]. For H_J , the important \mathcal{PT} -symmetry breaking near the edge singularity will be shown in our results while the first 20th orders of weak coupling expansion cannot account for it.

For the model represented by the above Lagrangian density, up to first order in g , the ground state energy receives only contributions from the sunset and dumbbell diagrams shown in Fig.2 while Mercedes and other four diagrams contribute to the g^2 order which are shown in Fig.3 (these diagrams are all listed in Ref.[20] too). Accordingly, up to g^2 order, the vacuum energy is given by:

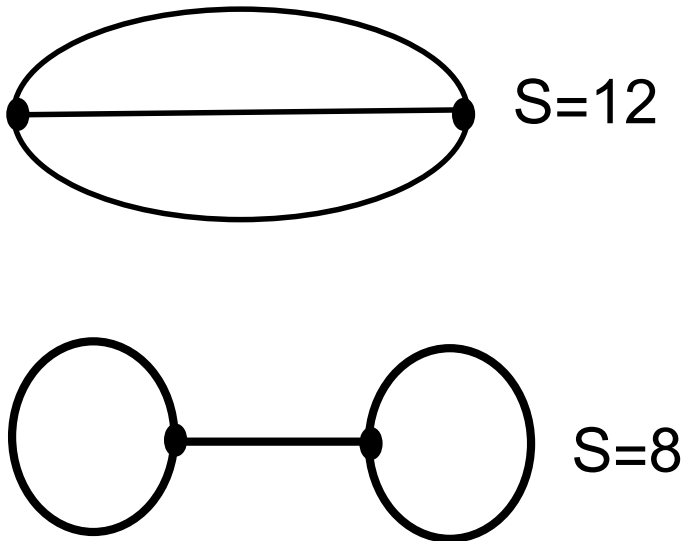


FIG. 2: The 2-vertices Feynman diagrams contributing to the first order in squared-coupling g of vacuum energy for the \mathcal{PT} -symmetric $(i\phi^3)_{0+1}$ field theory. The symmetry factor S is written for each diagram.

$$E_0 = \frac{1}{2} + \frac{11g}{288} - \frac{930}{288^2}g^2, \quad (9)$$

where we assumed in Eq.(6) that $m = 1$ and the space-time dimension is $0 + 1$. This result can be checked in many articles although some of them obtained it in different basis [2, 20]. It is well known that this series is Borel summable as well as having a zero radius of convergence. Accordingly, non-perturbative techniques are needed in order to get reasonable results for the vacuum energy. In many articles, different techniques applied ranging from Padé approximation [21] and Borel Resummation [2] to the recent Hypergeometric Resummation [5]. In fact, Padé approximation although can account for needed branch cuts of the

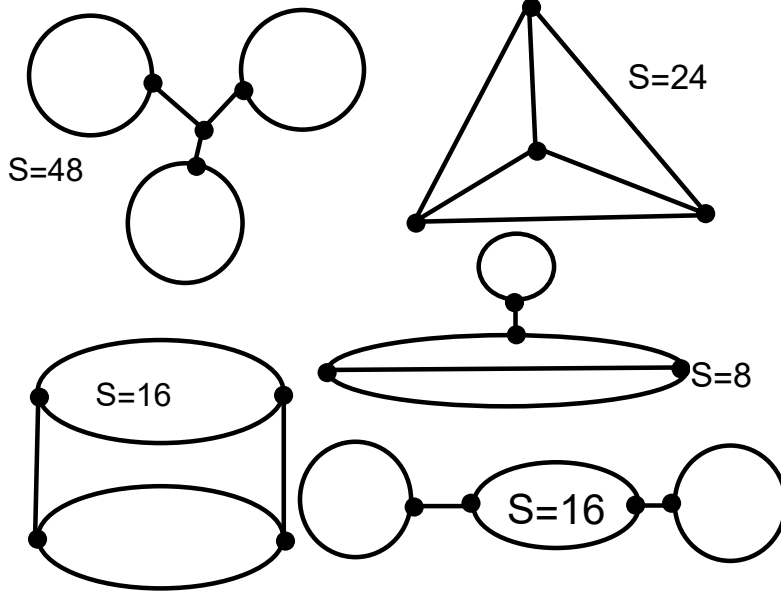


FIG. 3: The vacuum diagrams contribution to the order g^2 of the ground state energy of the \mathcal{PT} -symmetric $i\phi^3$ field theory.

divergent series, it fails to reproduce the strong coupling behavior of the physical quantity under consideration.

The Hypergeometric algorithm in Ref.[5] has features that recommend it for resummation of divergent series. The suggested ${}_2F_1(a_1, a_2; b_1; \sigma z)$ approximant needs information from four orders in perturbation series. According to the Hypergeometric Resummation of a divergent series, the vacuum energy of the Hamiltonian H in $0 + 1$ space-time dimensions is given by:

$$E_0 = \frac{1}{2} {}_2F_1\left(a, b, c, -\frac{g}{d}\right), \quad (10)$$

where ${}_2F_1$ is the Hypergeometric function and the parameters a, b, c and d can be obtained from the series expansion of the Hypergeometric function and then matching them with the first four terms in the perturbation series. The series expansion of Eq.(10) can be obtained as

$$E_0 = \frac{1}{2} - \frac{(ab)}{2(cd)}g + \frac{a(a+1)b(b+1)}{4c(c+1)d^2}g^2 - \frac{(a(a+1)(a+2)b(b+1)(b+2))}{12(c(c+1)(c+2)d^3)}g^3 + \frac{a(a+1)(a+2)(a+3)b(b+1)(b+2)(b+3)}{48c(c+1)(c+2)(c+3)d^4}g^4 + O(g^5)$$

The order of the series in Eq.(9) does not have enough information to solve for the four

unknown parameters. So one have to add contributions from vacuum diagrams up to eight vertices. Although this is possible but in applying the method to a more realistic field theory like QCD, it will be time consuming. It would be better to seek a way to lower the number of orders in the perturbation series needed to find the different parameters. Rather than this, the parameters when all are obtained from just the first few orders in perturbation series, the resummed function does not reproduce well known strong coupling limits of the Yang-Lee model shown in Eqs(1) & (2). So it is very necessary to feed the Hypergeometric function with parameters obtained from strong coupling behavior. In fact, when $a - b$ is not an integer and for large values of $|g|$, the Hypergeometric function has the following asymptotic form[7];

$${}_2F_1(a, b, c, g) \sim \lambda_1 g^{-a} + \lambda_2 g^{-b}, |g| \gg 1.$$

The strong coupling behavior of a physical quantity can be obtained exactly in some cases. In such cases, the parameters a and b are known exactly and only a second order of the perturbation series is sufficient to predict the other two parameters. In 0+1, the Hamiltonian takes the form

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2(x) + \frac{i\sqrt{g}}{6}\phi^3(x). \quad (11)$$

A symmetry transformation of the form

$$\begin{aligned} \phi &\rightarrow \exp(-w\pi)\phi\exp(w\pi) = \phi - w[\pi, \phi] \\ &= \phi + iw, \quad w = \frac{m^2}{\sqrt{g}}, \end{aligned}$$

leads to:

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \left(\frac{1}{6}i\sqrt{g}\right)\phi^3 + \frac{im^4}{2\sqrt{g}}\phi - \frac{m^6}{3g} \quad (12)$$

Note that this transformation changes the metric operator but keeping the spectrum invariant [50, 51]. If we follow this by a scaling transformation of the form $\exp(i \ln \beta)$ it scales ϕ by a factor β [50]. Taking $\beta = g^{-\frac{1}{10}}$ leads to the result:

$$H = \sqrt[5]{g} \left(\frac{\pi^2}{2} + \frac{i\phi^3}{6} + \frac{1}{2} \frac{im^4}{g^{\frac{4}{5}}} \phi \right) - \frac{m^6}{3g}. \quad (13)$$

This form suggests an expansion for the energy in the form

$$E_0 = -\frac{m^6}{3g} + \sum_{l=0}^{\infty} c_l g^{-\frac{4l-1}{5}}. \quad (14)$$

This result has been shown in Ref.[2] and it suggests that $a = 1$ while b equals $\frac{-1}{5}$. Accordingly, the Hypergeometric Resummation of the perturbation series in Eq.(9) takes the form

$$E_0 = \frac{1}{2} {}_2F_1 \left(1, \frac{-1}{5}, c, -\frac{g}{d} \right). \quad (15)$$

The parameters c and d can be found from matching the coefficients of the first two terms from series expansion of this equation with those in Eq.(9) and then we get:

$$E_0 = \frac{1}{2} {}_2F_1 \left(1, -\frac{1}{5}, \frac{465}{19}, -\frac{8525}{912}g \right).$$

The resummed form in Eq.(15) is fed with information from small coupling (parameters $c&d$) and strong coupling (parameters $a&b$) behaviors. Accordingly, one expect to give accurate results for the whole range of the coupling space. To test that expectation, let us check the accuracy of the Resummation formula in Eq.(15). For $g = \frac{1}{2}$, we get $E_0 = 0.516915$ compared to the best of the resummation algorithms at 150th order in Ref.[2] which gives $E_0 = 0.516892$. Also, for $g = 1$, we have $E_0 = 0.530886$ compared to $E_0 = 0.5307818$ form Ref.[2]. Now we compare with some larger values of g . For $g = \frac{288}{49}$, one gets $E_0 = 0.614319$ while the result in Ref.[2] gives $E_0 = 0.612738$. Also, for $g = 4 \times 288$, we get $E_0 = 1.55851$ while the exact value (reported in the last row in table III in Ref.([21])) is $E_0 = 1.53078$. So it seems that the simple method of Hypergeometric Resummation gives precise results though it has been fed with information of the first two terms in the perturbation series. However, the original version introduced in Ref.[5] gives precise results for a wide range of coupling values but not for very large coupling values. For instance when $g = 4 \times 288$ it gives $E_0 = 1.48104$ which is not as accurate as our prediction when both are compared with the exact value above. For more tests of our results and also the original version introduced in Ref.[5] one needs to check for the limit at $g \rightarrow \pm\infty$. For $g \rightarrow -\infty$ our prediction is

$$\lim_{g \rightarrow -\infty} \frac{E_0}{|g|^{\frac{1}{5}}} = 0.30738 + 0.223325i$$

while the prediction of the original form in Ref.([5]) is zero and the methods in Ref.([2]) gives $0.30139588 + 0.2189769214i$. This is expected because any tiny difference in the parameters a and b will ruin the strong coupling behavior of the resummed function. In fact, the imaginary part of the vacuum energy for a real potential can be obtained by non-perturbative techniques only and thus it is always a good test for any Resummation tool. On the other hand, for the \mathcal{PT} -symmetric case ($g \rightarrow \infty$), we can find the result:

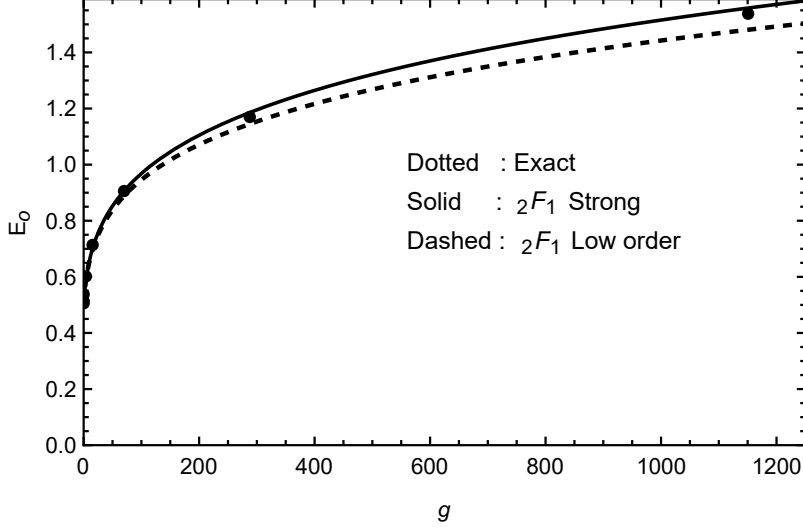


FIG. 4: Comparison between our Resummation formula ${}_2F_1$ for the ground state energy of the Hamiltonian in Eq.(7)(solid), the original form in Ref.[5] (dashed) and exact results from Ref.[21] (dots). Note that the coupling in our work is rescaled from that in Ref.[21] where g in our work is equivalent to $288\lambda^2$ in that reference.

$$\lim_{g \rightarrow \infty} \frac{E_0}{|g|^{\frac{1}{5}}} = 0.379943,$$

while the result from Ref.[2] is

$$\lim_{g \rightarrow \infty} \frac{E_0}{|g|^{\frac{1}{5}}} = 0.3723,$$

but as expected the original Hypergeometric Resummation algorithm gives zero again. These results show clearly that feeding the Hypergeometric Resummation with parameters from the strong coupling behavior is necessary to extrapolate the prediction to the large coupling behavior of the resummed function. The accuracy of our results for large coupling values over the predictions from the original algorithm in Ref.[5] is clear from Fig.4. In this figure one can realize that both of our formula and original one give reasonable results compared to exact results for not so large values of the coupling. For very large values however, one can realize that our formula fits well with exact results but the original formula deviates from the exact results. This is expected as the parameters in Ref.[5] are all predicted from the first four terms in the perturbation series and thus expected to loose memory for strong coupling predictions.

The extension of the method to higher orders is direct as one suggests the resummation function to be ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; -\sigma z)$. When $p = q + 1$, the set of functions ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; -\sigma z)$ are all sharing the same analytic properties. In our algorithm, the a_i parameters are determined exactly from the strong coupling expansion of the theory under consideration while b_i and σ parameters are determined from $q + 1$ set of algebraic equations obtained by comparing the series expansion of ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; -\sigma z)$ with the perturbation series of the physical quantity. In fact, this algorithm reduces the non linearity of the parameters equations to half. One can even get an equivalent set of equations which are all linear in σ and all the equations consider only powers of one in each parameters. This strategy avoids troubles faced in the generalized Hypergeometric resummation technique in the literature in solving the set of equations of the N parameters. For the Hamiltonian H in Eq.(7) we resummed the ground state energy using ${}_2F_1$, ${}_3F_2$, and ${}_4F_3$ and the results are listed in table I compared to exact results and the 150^{th} resummation techniques from Ref.[2]. It is clear that our fourth order resummation (${}_4F_3$) gives accurate results and the accuracy is improved systemically when moving to higher orders. The plot of the fifth order (${}_5F_4$) versus exact results for the same quantity is shown in Fig.5 where one can realize that the precision of the algorithm is improving systematically from second (${}_2F_1$) to fifth order (${}_5F_4$).

TABLE I: The Hypergeometric resummation ${}_2F_1$, ${}_3F_2$ and ${}_4F_3$ for the ground state function in Eq.(7) compared to the 150^{th} order of resummation methods in Ref.[2] and exact results. It is very clear that the second order ${}_2F_1$ gives accurate results and we get higher precision in going to higher orders where our 4^{th} order resummation ${}_4F_3$ gives results competitive to the the 150^{th} order of resummation methods in Ref.[2].

g	${}_2F_1$	${}_3F_2$	${}_4F_3$	E_0^{150} [2]	E_{exact}
0.5	0.516915482	0.51689308	0.516891566	0.516891764	—
1	0.530885535	0.53079024	0.53077974	0.5307817593	0.53078176
$\frac{288}{49}$	0.614318594	0.61296986	0.61260464	0.61273810639	0.612738106

It is well known that the series ${}_pF_{p-1}(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_{p-1}; \sigma z)$ has a finite radius of convergence while it has been used to resum a divergent series of zero radius of convergence and this issue has been stressed in Ref.[19]. However, the parameter σ is taking large values that accounts for very small radius of convergence (but non-zero) and thus the resummation

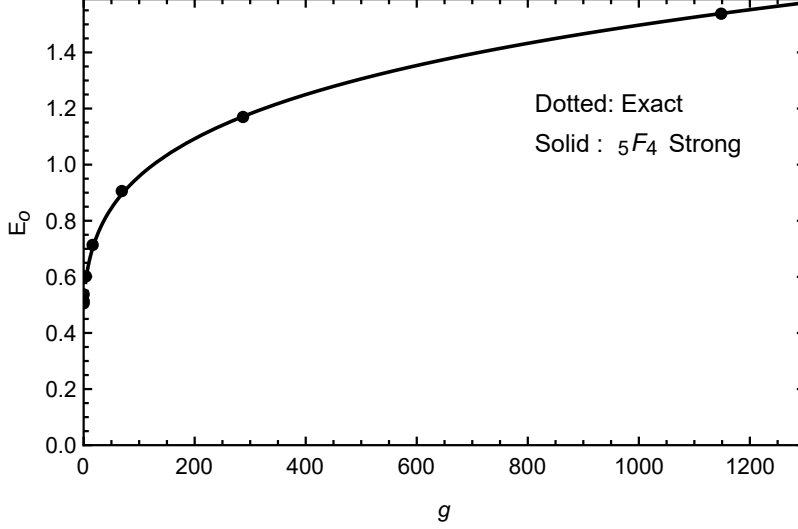


FIG. 5: Comparison between our Resummation formula ${}_5F_4$ for the ground state energy of the Hamiltonian in Eq.(7)(solid) and exact results from Ref.[21] (dots). The accuracy looks improved when compared with the second order (${}_2F_1$) in Fig. 4.

gives good results specially for not so small couplings. On the other hand, we have in physics divergent series that do have finite radius of convergence for which the Hypergeometric resummation is more suitable and expected to give more precise results. Examples of these kind are the strong coupling expansion of a physical quantity. The vacuum energy of the Yang-Lee model is of that type.

For the investigation of \mathcal{PT} -symmetry breaking in the Yang-Lee model represented by the Hamiltonian H in Eq(8), the ground state energy up to second order in J is given by [2];

$$E_J = .3725457904522070982506011 + 0.3675358055441936035304J \\ + 0.1437877004150665158339J^2 + O(J^3)$$

The resummation for this perturbation series is then obtained as:

$$E_J = {}_2F_1(-3/2, -1/4, b_1; -\sigma J).$$

We compared our results with the 20th order of the perturbation series from Ref.[2] in Fig.6. The resumed series gives reasonable agreement with this relatively high order of perturbation series but deviate from each other near the critical region where the resumed formula starts

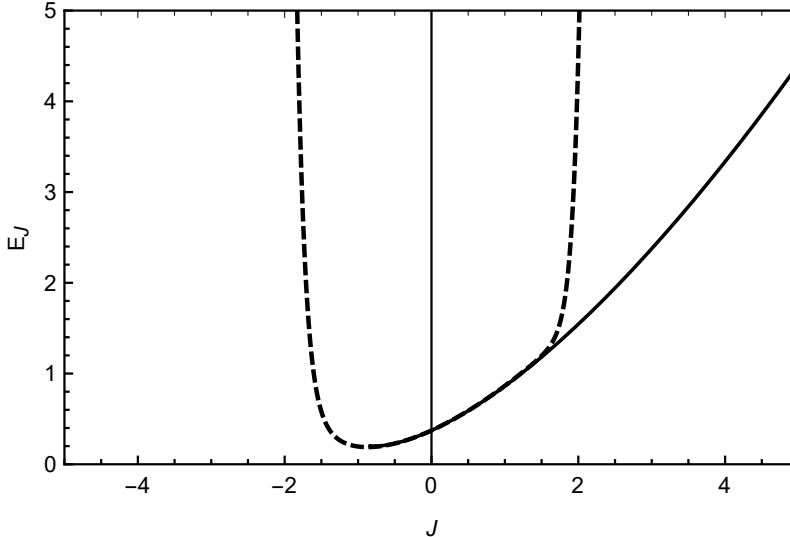


FIG. 6: Comparison between our Resummation formula ${}_2F_1$ for E_J (solid) and the 20th order of the perturbation series (dashed) from Ref.[2]. While the agreement is good for a range of the coupling J , the perturbation series fails (as expected) to produce the \mathcal{PT} -symmetry breaking expected as well as fails to fit with strong coupling behavior.

to be complex (\mathcal{PT} -symmetry breaking) and also separated for a relatively strong coupling. A note to be mentioned that the resummed formula using four terms from the perturbation series as in Ref.[5], agrees well with our result but they deviates at very strong couplings as expected. To get more accurate results to be compared with 150th order of resummation methods in Ref.[2], we obtained also the resummation approximations ${}_3F_2$, ${}_4F_3$ and ${}_5F_4$ where we listed them in table II. It is very clear that our resummation formula are giving precise results although we used only low orders of calculations compared to the the 150th order of resummation methods in Ref.[2]. Note that in this table for $J = -1$, ${}_2F_1$ results in a tiny imaginary part to the ground state energy and this is because it predicts a smaller critical coupling than ${}_3F_2$, ${}_4F_3$ and ${}_5F_4$ and the methods in Ref.[2]. In fact this is acceptable because near the critical point the theory is highly non-perturbative and thus higher orders like ${}_3F_2$, ${}_4F_3$ and ${}_5F_4$ are expected to give more accurate result for the critical coupling.

To conclude, we stressed the recently introduced Hypergeometric Resummation algorithm [5]. We realized that when applying the algorithm to the \mathcal{PT} -symmetric $i\phi^3$ field theory it gives accurate results for a range of the coupling values but for very large coupling values

TABLE II: The Hypergeometric resummation ${}_2F_1$, ${}_3F_2$, ${}_4F_3$ and ${}_5F_4$ for the ground state function of the Hamiltonian in Eq.(8) compared to the 150^{th} order of resummation methods in Ref.[2] and the 20^{th} order of the perturbation series (E_{per}) from Ref.[2]. Our resummation formulae all show up \mathcal{PT} -symmetry Breaking and precision is improved using higher orders. Our third (${}_3F_2$), fourth (${}_4F_3$) and fifth (${}_5F_4$) orders are showing results with competitive precision to the 150^{th} order of resummation methods in Ref.[2].

J	${}_2F_1$	${}_3F_2$	${}_4F_3$	${}_5F_4$	E_0^{150} [2]	E_{per}
$-2\frac{4}{5}$	0.289111-0.345157 i	0.388417-0.328504i	0.394688-0.3560448i	0.388902-0.358021i	0.389 8-0.3644i	2.52955
-1	0.229176 - 0.029543 i	0.19719967	0.19580355	0.195741	0.1957508	0.19574
$-5\frac{-4}{5}$	0.282836	0.282700	0.282699	0.28269926	0.282699	0.282699
$-21.6\frac{-4}{5}$	0.342161	0.342158	0.342158	0.342158	0.342158	0.342158

the results are deviated from expected ones either from exact calculations or from strong coupling limits where both are known from the literature. We expected that the reason behind this is that the four parameters of the Hypergeometric function are all predicted from the first four perturbative terms in the divergent series of the ground state energy and thus the resummed function has no guidance for strong coupling values. In fact, there exist well known techniques to obtain the strong coupling expansion of a physical quantity either for the quantum mechanical case and some times for quantum field cases [2, 8, 9]. Accordingly, we suggested to feed the Hypergeometric function with two parameters that can be predicted from the strong coupling behavior and the other two parameters from the first two terms in the perturbation series. In that way we obtained a Hypergeometric function that bears information from weak coupling as well as strong coupling behaviors and thus expected to give accurate results for the whole range of the coupling space.

We tested the algorithm by obtaining a very precise value for the critical exponent ν of the $O(4)$ field theoretic model from the first order in perturbation series and the asymptotic strong coupling data as input. Up to the best of our knowledge, this is the first time to get such accurate result from that low order of perturbation series. This result assures that the effect of involving exact parameters from strong coupling expansion shall accelerate the convergence of the resummation function toward exact results.

We showed that the extension of the algorithm is direct and one can use as a resummation function the generalized Hypergeometric function ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; -\sigma z)$ where the parameters a_i are all obtained exactly from the strong coupling expansion of the physical quantity under consideration. In fact, this algorithm reduces the non-linearity issue which

one faces when trying to find all the parameters from just perturbation series as it lowers the number of equations in the parameters by a factor of half. Besides, one can get an equivalent set of equations where are all linear in the parameter σ as well as having only the power one in each parameter. Also, the algorithm guarantees reliable results even for very large coupling values.

We tested the idea for the anharmonic oscillator for $g = 50$ and found that our second order (${}_2F_1$) result is better than the 24th order for Borel- Padé (BP_{12-12}) but is not of same precision as the 25th order of the generalized Hypergeometric (Meijer-G approximants) algorithm in Ref.[6]. However, our fourth order calculation ${}_4F_3(a_1, a_2, \dots, a_4; b_1, b_2, \dots, b_3; -\sigma z)$ shows good results in comparison with exact ones.

Since our main problem is to resum the ground state function of the \mathcal{PT} -symmetric $i\phi^3$, we tested the prediction of the modified algorithm and found that the second order calculation ${}_2F_1(a_1, a_2; b_1; -\sigma z)$ gives accurate results compared to exact results for a wide range of the coupling. Of course as the coupling increases our predictions go better than those from the original version of the algorithm as explained above. The modified algorithm introduced here is successful to reproduce the well known limit of the ground state energy as $g \rightarrow \pm\infty$. Also, the precision is improved when we use higher order where the fourth order predictions give competitive results compared to the 150th order of the resummation algorithms used in Ref.[2] (Table I).

The \mathcal{PT} -symmetry breaking of the Yang-Lee model has been tested where the perturbation series can not account for it at any order. The second order shows good agreement with 20th order of the perturbation series of the ground state energy but far from the critical region as well as large values of the coupling, the perturbation series fails to give reliable results. While the second order of our calculation gives reasonable results and accounts for \mathcal{PT} -symmetry breaking at a negative coupling as it was predicted by the work in Ref.[2], it predicts smaller (in value) critical coupling. The accuracy is highly increased in using higher orders where the third, fourth and fifth orders showed great results compared to the 150th resummation algorithm used in Ref.[2] (Table II).

The algorithm introduced here gives accurate results with less effort as one is not in a need to obtain more than the second order of the strong coupling expansion of a physical quantity and can then conclude all the strong coupling parameters in the series since the coefficients in the strong coupling expansion do not matter here. We think that this might be the most

simple, accurate and time saving resummation algorithm as it uses the already summed huge set of functions ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; -\sigma z)$ where the variety of parameters can fit with huge number of problems in physics.

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