

# ULTRADISCRETE LIMIT OF THE SPECTRAL POLYNOMIAL OF THE $q$ -HEUN EQUATION

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ABSTRACT. It is known that the  $q$ -Heun equation has polynomial-type solutions in some special cases, and the condition for the accessory parameter  $E$  is described by the roots of the spectral polynomial. We investigate the spectral polynomial by considering the ultradiscrete limit.

## 1. INTRODUCTION

A  $q$ -difference analogue of Heun's differential equation was introduced by Hahn [2] in the form

$$(1.1) \quad a(x)g(x/q) + b(x)g(x) + c(x)g(qx) = 0$$

such that  $a(x)$ ,  $b(x)$ ,  $c(x)$  are polynomials such that  $\deg_x a(x) = \deg_x c(x) = 2$ ,  $a(0) \neq 0 \neq c(0)$  and  $\deg_x b(x) \leq 2$ . It was rediscovered in [9] by the fourth degeneration of Ruijsenaars-van Diejen system ([11, 7]) or by specialization of the linear difference equation associated to the  $q$ -Painlevé VI equation ([3]). We adopt the expression of the  $q$ -Heun equation as

$$(1.2) \quad (x - q^{h_1+1/2}t_1)(x - q^{h_2+1/2}t_2)g(x/q) \\ + q^{\alpha_1+\alpha_2}(x - q^{l_1-1/2}t_1)(x - q^{l_2-1/2}t_2)g(qx) \\ - \{(q^{\alpha_1} + q^{\alpha_2})x^2 + Ex + q^{(h_1+h_2+l_1+l_2+\alpha_1+\alpha_2)/2}(q^{\beta/2} + q^{-\beta/2})t_1t_2\}g(x) = 0,$$

which was employed in [10, 5]. The parameter  $E$  is called the accessory parameter. Note that we recover Heun's differential equation by the limit  $q \rightarrow 1$  ([2, 9]). Recently, the  $q$ -Heun equation appears in the study of degenerations of the Askey-Wilson algebra ([1]).

Solutions of the  $q$ -Heun equation were considered in [10, 5]. We investigate a solution of the  $q$ -Heun equation written as

$$(1.3) \quad f(x) = x^{\lambda_1} \sum_{n=0}^{\infty} c_n(E)x^n, \quad \lambda_1 = \frac{h_1 + h_2 - l_1 - l_2 - \alpha_1 - \alpha_2 - \beta + 2}{2}.$$

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Note that the value  $\lambda_1$  is one of the exponents of  $q$ -Heun equation about  $x = 0$ . Then the coefficients  $c_n(E)$  ( $n = 1, 2, \dots$ ) are determined recursive by

$$(1.4) \quad \begin{aligned} c_n(E)t_1t_2[q^{h_1+h_2}(1-q^n)(1-q^{n-\beta})] \\ = c_{n-1}(E)[Eq^{n-1+\lambda_1} + q^{1/2}(q^{h_1}t_1 + q^{h_2}t_2) + (q^{l_1}t_1 + q^{l_2}t_2)q^{2(n+\lambda_1)+\alpha_1+\alpha_2-5/2}] \\ - c_{n-2}(E)[q(1-q^{n-2+\lambda_1+\alpha_1})(1-q^{n-2+\lambda_1+\alpha_2})], \end{aligned}$$

with the initial condition  $c_0(E) = 1$  and  $c_{-1}(E) = 0$  (see [5]). If we regard  $E$  as an indeterminate, then  $c_n(E)$  is a polynomial of  $E$  of degree  $n$ . The polynomial type solution of the  $q$ -Heun equation, which is written as a terminating series, is described as follows.

**Proposition 1.1.** ([5]) *Let  $\lambda_1$  be the value in Eq.(1.3) and assume that  $-\lambda_1 - \alpha_1$  ( $:= N$ ) is a non-negative integer and  $\beta \notin \{1, 2, \dots, N, N+1\}$ . Set  $c_{-1}(E) = 0$ ,  $c_0(E) = 1$  and we determine the polynomials  $c_n(E)$  ( $n = 1, \dots, N+1$ ) recursively by Eq.(1.4) If  $E = E_0$  is a solution of the algebraic equation*

$$(1.5) \quad c_{N+1}(E) = 0,$$

then  $q$ -Heun equation defined in Eq.(1.2) has a non-zero solution of the form

$$(1.6) \quad f(x) = x^{\lambda_1} \sum_{n=0}^N c_n(E_0)x^n.$$

We call  $c_{N+1}(E)$  the spectral polynomial of the  $q$ -Heun equation.

In general it would be impossible to solve the roots of the spectral polynomial  $c_{N+1}(E)$  explicitly. To understand the roots of the spectral polynomial, we may apply the ultradiscrete limit  $q \rightarrow +0$ . In [5], the behaviour of the roots  $E = E_1, E_2, \dots, E_{N+1}$  of  $c_{N+1}(E) = 0$  by the ultradiscrete limit was studied in some cases. As  $q \rightarrow 0$ , the roots satisfy

$$(1.7) \quad E_k \sim -cq^{d-k}, \quad k = 1, 2, \dots, N+1$$

for some  $c \in \mathbb{R}_{>0}$  and  $d \in \mathbb{R}$  in those cases.

In this paper we investigate the roots of the spectral polynomial as  $q \rightarrow +0$  in more cases, which was partially done in [4, 8]. Namely we obtain results in the three cases in sections 3.1, 3.2 and 3.3. Note that the roots of the spectral polynomial do not satisfy the asymptotics as Eq.(1.7) in several cases.

This paper is organized as follows. In section 2, we introduce two kinds of equivalences on the limit  $q \rightarrow +0$ , which are used to analyze the coefficients of the polynomials  $c_j(E)$  ( $j = 1, 2, \dots, N+1$ ). In section 3, we investigate the polynomials  $c_j(E)$  ( $j = 1, 2, \dots, N+1$ ) and the roots of the spectral polynomial  $c_{N+1}(E)$  as  $q \rightarrow +0$  by dividing into three cases. In section 4, we give concluding remarks.

Throughout this paper, we assume  $0 < q < 1$ .

## 2. EQUIVALENCES ON THE ULTRADISCRETE LIMIT

As discussed in [5], we define the equivalence  $\sim$  of functions of the variable  $q$  by

$$(2.1) \quad a(q) \sim b(q) \Leftrightarrow \lim_{q \rightarrow +0} \frac{a(q)}{b(q)} = 1.$$

We also define the equivalence  $\sum_{j=0}^M a_j(q)E^j \sim \sum_{j=0}^M b_j(q)E^j$  of the polynomials of the variable  $E$  by  $a_j(q) \sim b_j(q)$  for  $j = 0, \dots, M$ . If  $\sum_{j=0}^M a_j(q)E^j \sim \sum_{j=0}^M c_j q^{\mu_j} E^j$  for some  $c_j \in \mathbb{R} \setminus \{0\}$  ( $j = 0, 1, \dots, M$ ), then we call  $\sum_{j=0}^M c_j q^{\mu_j} E^j$  the leading terms of  $\sum_{j=0}^M a_j(q)E^j$ . We introduce another equivalence by

$$(2.2) \quad a(q) \approx b(q) \Leftrightarrow \exists C > 0 \text{ such that } \lim_{q \rightarrow +0} \frac{a(q)}{b(q)} = C,$$

and the equivalence  $\sum_{j=0}^M a_j(q)E^j \approx \sum_{j=0}^M b_j(q)E^j$  by  $a_j(q) \approx b_j(q)$  for  $j = 0, \dots, M$ .

We are going to find simpler forms of the polynomials  $c_n(E)$  ( $n = 1, 2, \dots, N + 1$ ) determined by Eq.(1.4) with respect to the equivalence  $\sim$  or  $\approx$ . For simplicity, we assume

$$(2.3) \quad N = -\lambda_1 - \alpha_1 \in \mathbb{Z}_{\geq 0}, \beta < 1, \alpha_2 - \alpha_1 < 1, t_1 > 0, t_2 > 0, h_1 < h_2, l_1 < l_2,$$

throughout this paper, which was also assumed in [5]. It follows from Eq.(1.4) that the polynomials  $c_n(E)$  satisfy

$$(2.4) \quad c_n(E) \sim t_1^{-1} t_2^{-1} [E q^{n-1-h_1-h_2+\lambda_1} + t_1 q^{1/2-h_2} + t_1 q^{2n-1/2-l_2-\beta}] c_{n-1}(E) \\ - t_1^{-1} t_2^{-1} q^{2n-1-l_1-l_2-\beta} c_{n-2}(E)$$

for  $n = 1, 2, \dots, N + 1$  under the assumption that there are no cancellation of the leading terms of the coefficients of  $E^j$  ( $j = 0, 1, \dots, n - 1$ ) in the right hand side with respect to the limit  $q \rightarrow +0$ . In [5], the leading terms of  $c_n(E)$  ( $n = 1, 2, \dots, N + 1$ ) were investigated for the case ( $1 + h_2 - l_2 - \beta > 0$  and  $2 + 2h_2 - l_1 - l_2 - \beta > 0$ ) and the case ( $2N + 1 + h_2 - l_2 - \beta < 0$  and  $2N + l_1 - l_2 - \beta < 0$ ).

### 3. ANALYSIS OF THE SPECTRAL POLYNOMIAL BY THE ULTRADISCRETE LIMIT

We investigate the leading terms of  $c_n(E)$  ( $n = 1, 2, \dots, N + 1$ ) for three cases with weaker conditions in the sections 3.1, 3.2 and 3.3.

#### 3.1. The case $1 + h_2 - l_2 - \beta > 0$ .

If  $1 + h_2 - l_2 - \beta > 0$  and the condition in Eq.(2.3) is satisfied, then it follows from Eq.(2.4) that

$$(3.1) \quad c_n(E) \sim t_1^{-1} t_2^{-1} (E q^{n-1-h_1-h_2+\lambda_1} + t_1 q^{1/2-h_2}) c_{n-1}(E) \\ - t_1^{-1} t_2^{-1} q^{2n-1-l_1-l_2-\beta} c_{n-2}(E)$$

for  $n = 1, 2, \dots, N + 1$  under the assumption that there are no cancellation of the leading terms of the coefficients of  $E^j$  ( $j = 0, 1, \dots, n - 1$ ) in the right hand side with respect to the limit  $q \rightarrow +0$ . We investigate a sufficient condition that there are no cancellation of the leading terms of the coefficients of  $E^j$  ( $j = 0, 1, \dots, n - 1$ ).

Since  $c_0(E) = 1$  and  $c_{-1}(E) = 0$ , the leading terms of  $c_1(E)$  are described by

$$(3.2) \quad c_1(E) \sim t_1^{-1} t_2^{-1} (E q^{\lambda_1-h_1-h_2} + t_1 q^{-h_2+1/2}),$$

and we do not need any conditions for no cancellation of the leading terms. By applying Eq.(3.1), we have

$$(3.3) \quad c_2(E) \sim t_1^{-2}t_2^{-2}(Eq^{1-h_1-h_2+\lambda_1} + t_1q^{1/2-h_2})(Eq^{-h_1-h_2+\lambda_1} + t_1q^{1/2-h_2}) \\ - t_1^{-1}t_2^{-1}q^{3-l_1-l_2-\beta}$$

under the assumption that there are no cancellation of the leading terms. In this case, the cancellation of the leading terms may occur on the coefficient of  $E^0$  and the candidate for the cancellation is  $t_2^{-2}q^{1-2h_2} - t_1^{-1}t_2^{-1}q^{3-l_1-l_2-\beta}$ . If  $1-2h_2 \neq 3-l_1-l_2-\beta$ , i.e.  $2h_2 - l_1 - l_2 - \beta \neq -2$ , then there are no cancellation of the leading terms. If  $2 + 2h_2 - l_1 - l_2 - \beta > 0$ , then we may ignore the term  $t_1^{-1}t_2^{-1}q^{3-l_1-l_2-\beta}$  and we have

$$(3.4) \quad c_2(E) \sim t_1^{-2}t_2^{-2}(Eq^{1-h_1-h_2+\lambda_1} + t_1q^{1/2-h_2})(Eq^{-h_1-h_2+\lambda_1} + t_1q^{1/2-h_2}).$$

The leading terms of the polynomials  $c_n(E)$  were studied in [5] for the case  $1 + h_2 - l_2 - \beta > 0$  and  $2 + 2h_2 - l_1 - l_2 - \beta > 0$ . If  $2 + 2h_2 - l_1 - l_2 - \beta < 0$ , then we have

$$(3.5) \quad c_2(E) \sim t_1^{-2}t_2^{-2}[q^{1-2h_1-2h_2+2\lambda_1}E^2 + t_1q^{1/2-h_2}q^{-h_1-h_2+\lambda_1}E - t_1t_2q^{3-l_1-l_2-\beta}].$$

We assume  $2h_2 - l_1 - l_2 - \beta \neq -2$ . It follows from Eq.(3.1) for  $n = 3$  that

$$(3.6) \quad c_3(E) \sim c_2(E)t_1^{-1}t_2^{-1}(Eq^{2+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) - c_1(E)t_1^{-1}t_2^{-1}q^{5-l_1-l_2-\beta} \\ \sim t_1^{-3}t_2^{-3}(Eq^{2+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2})(Eq^{1+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) \\ (Eq^{\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) \\ - t_1^{-2}t_2^{-2}q^{5-l_1-l_2-\beta}(Eq^{\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) \\ - t_1^{-2}t_2^{-2}q^{3-l_1-l_2-\beta}(Eq^{2+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) \\ \sim t_1^{-3}t_2^{-3}[E^3q^{3+3(\lambda_1-h_1-h_2)} + E^2t_1q^{1+2(\lambda_1-h_1-h_2)}q^{-h_2+1/2} \\ + Et_1^2q^{\lambda_1-h_1-h_2}q^{2(-h_2+1/2)} + t_1^3q^{3(-h_2+1/2)} \\ - t_1t_2q^{-l_1-l_2-\beta}(2Eq^{5+\lambda_1-h_1-h_2} + t_1q^{3-h_2+1/2})]$$

under the assumption that there are no cancellations of the leading terms. In this case, the cancellation of the leading terms may occur on the coefficients of  $E^1$  and  $E^0$ . Hence if  $2h_2 - l_1 - l_2 - \beta \notin \{-4, -2\}$ , then the cancellation of the leading terms

does not occur. On the polynomial  $c_4(E)$ , we have

$$\begin{aligned}
(3.7) \quad c_4(E) &\sim c_3(E)t_1^{-1}t_2^{-1}(Eq^{3+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) - c_2(E)t_1^{-1}t_2^{-1}q^{7-l_1-l_2-\beta} \\
&\sim t_1^{-4}t_2^{-4}(Eq^{\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2})(Eq^{1+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) \\
&\quad (Eq^{2+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2})(Eq^{3+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) \\
&\quad - t_1^{-3}t_2^{-3}q^{3-l_1-l_2-\beta}(Eq^{2+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2})(Eq^{3+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) \\
&\quad - t_1^{-3}t_2^{-3}q^{5-l_1-l_2-\beta}(Eq^{\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2})(Eq^{3+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) \\
&\quad - t_1^{-3}t_2^{-3}q^{7-l_1-l_2-\beta}(Eq^{\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2})(Eq^{1+\lambda_1-h_1-h_2} + t_1q^{-h_2+1/2}) \\
&\quad + t_1^{-2}t_2^{-2}q^{7-l_1-l_2-\beta}q^{3-l_1-l_2-\beta} \\
&\sim t_1^{-4}t_2^{-4}[E^4q^{6+4(\lambda_1-h_1-h_2)} + E^3t_1q^{3(\lambda_1-h_1-h_2)}q^{3-h_2+1/2} \\
&\quad + E^2t_1^2q^{2(\lambda_1-h_1-h_2)}q^{1+2(-h_2+1/2)} + Et_1^3q^{\lambda_1-h_1-h_2}q^{3(-h_2+1/2)} \\
&\quad + t_1^4q^{4(-h_2+1/2)} - t_1t_2q^{-l_1-l_2-\beta}\{3E^2q^{8+2(\lambda_1-h_1-h_2)} \\
&\quad + 2Et_1q^{5-h_2+1/2}q^{\lambda_1-h_1-h_2} + t_1^2q^{3+2(-h_2+1/2)}\} + t_1^2t_2^2q^{10-2(l_1+l_2+\beta)}]
\end{aligned}$$

under the assumption that there are no cancellations of the leading terms. The cancellation of the leading terms may occur on the coefficients of  $E^2$ ,  $E^1$  and  $E^0$ , and the cancellation of the leading terms does not occur under the condition  $2h_2 - l_1 - l_2 - \beta \notin \{-6, -4, -2\}$ . On the polynomial  $c_n(E)$ , we have the following proposition.

**Proposition 3.1.** *Let  $M \in \{1, 2, \dots, N+1\}$ . A sufficient condition that the cancellation of the leading terms in the right hand side of Eq.(3.1) does not occur for  $n = 1, 2, \dots, M$  is written as  $2h_2 - l_1 - l_2 - \beta \notin \{-2M+2, -2M+4, \dots, -4, -2\}$ .*

*Proof.* We can write the recursive relation as

$$(3.8) \quad c_n(E) = (p_n E + q_n)c_{n-1}(E) - r_n c_{n-2}(E),$$

where

$$(3.9) \quad r_n \sim t_1^{-1}t_2^{-1}q^{2n-1-l_1-l_2-\beta}, \quad p_n \sim t_1^{-1}t_2^{-1}q^{n-1+\lambda_1-h_1-h_2}, \quad q_n \sim t_2^{-1}q^{-h_2+1/2}.$$

By applying Eq.(3.8) repeatedly, we have

$$(3.10) \quad c_M(E) \sim \sum_{0 \leq k \leq M/2} (-1)^k \sum' \prod_{l=1}^k r_{n_l+1} \prod'' (p_m E + q_m),$$

where the summation  $\sum'$  is over the integers  $1 \leq n_1 < n_2 < \dots < n_k \leq M-1$  such that  $n_l - n_{l-1} \geq 2$  for  $l = 2, \dots, k$  and the product  $\prod''_m$  is over the integer  $1 \leq m \leq M$  such that  $m \notin \{n_l, n_l+1\}$  for  $l = 1, \dots, k$ . The term  $r_n(p_{n+1}E + q_{n+1}) \sim t_1^{-2}t_2^{-2}q^{2n-1-l_1-l_2-\beta}(q^{n+\lambda_1-h_1-h_2}E + t_1q^{-h_2+1/2})$  is stronger than  $(p_{n-1}E + q_{n-1})r_{n+1} \sim t_1^{-2}t_2^{-2}q^{2n-1-l_1-l_2-\beta}(q^{n+\lambda_1-h_1-h_2}E + t_1q^{-h_2+5/2})$ . By applying this relation repeatedly,

we find that the strongest term in Eq.(3.10) for the fixed  $k$  is contained in

$$\begin{aligned}
(3.11) \quad & r_2 r_4 \cdots r_{2k} (p_{2k+1} E + q_{2k+1}) (p_{2k+2} E + q_{2k+2}) \cdots (p_M E + q_M) \\
& \approx \sum_{0 \leq l \leq M-2k} r_2 r_4 \cdots r_{2k} p_{2k+1} p_{2k+2} \cdots p_{2k+l} q_{2k+l+1} \cdots q_M E^l \\
& \approx \sum_{0 \leq l \leq M-2k} q^{d(k,l)} E^l,
\end{aligned}$$

where  $d(k, l) = k(2k + 1 - l_1 - l_2 - \beta) + l(l - 1)/2 + l(2k - h_1 - h_2 + \lambda_1) + (M - 2k - l)(-h_2 + 1/2)$ . Hence we have

$$(3.12) \quad c_M(E) \approx \sum_{l, k \geq 0, 0 \leq l + 2k \leq M} (-1)^k q^{d(k,l)} E^l.$$

If the cancellation of the leading terms occur, then we have  $d(k, l) = d(k', l)$  for  $k \neq k'$ , i.e.,  $2(k + k') + 2l + 2h_2 - (l_1 + l_2 + \beta) = 0$  for  $k \neq k'$ ,  $0 \leq 2k + l \leq M$  and  $0 \leq 2k' + l \leq M$ . Since  $k \neq k'$ , we have  $0 < k + k' + l < M$  and a sufficient condition that the cancellation of the leading terms does not occur is written as  $2h_2 - l_1 - l_2 - \beta \notin \{-2M + 2, -2M + 4, \dots, -4, -2\}$ .  $\square$

We consider the expression of the leading terms roughly by using the equivalence  $\approx$  under the condition of Proposition 3.1. On the polynomials  $c_2(E)$  and  $c_3(E)$ , we have

$$\begin{aligned}
(3.13) \quad & c_2(E) \sim t_1^{-2} t_2^{-2} (E q^{1+\lambda_1-h_1-h_2} + t_1 q^{-h_2+1/2}) (E q^{\lambda_1-h_1-h_2} + t_1 q^{-h_2+1/2}) \\
& \quad - t_1^{-1} t_2^{-1} q^{3-l_1-l_2-\beta} \\
& \approx (E q^{1+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) (E q^{\lambda_1-h_1-h_2} + q^{-h_2+1/2}) - q^{3-l_1-l_2-\beta}, \\
& c_3(E) \sim t_1^{-3} t_2^{-3} (E q^{2+\lambda_1-h_1-h_2} + t_1 q^{-h_2+1/2}) (E q^{1+\lambda_1-h_1-h_2} + t_1 q^{-h_2+1/2}) \\
& \quad (E q^{\lambda_1-h_1-h_2} + t_1 q^{-h_2+1/2}) \\
& \quad - t_1^{-2} t_2^{-2} q^{3-l_1-l_2-\beta} (2E q^{2+\lambda_1-h_1-h_2} + t_1 q^{-h_2+1/2}) \\
& \approx (E q^{2+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) (E q^{1+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) (E q^{\lambda_1-h_1-h_2} + q^{-h_2+1/2}) \\
& \quad - q^{3-l_1-l_2-\beta} (E q^{2+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) \\
& \approx (E q^{2+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) c_2(E).
\end{aligned}$$

We also have

$$\begin{aligned}
(3.14) \quad & c_4(E) \sim c_3(E) t_1^{-1} t_2^{-1} (E q^{3+\lambda_1-h_1-h_2} + t_1 q^{-h_2+1/2}) - c_2(E) t_1^{-1} t_2^{-1} q^{7-l_1-l_2-\beta} \\
& \approx c_2(E) [(E q^{3+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) (E q^{2+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) - q^{7-l_1-l_2-\beta}].
\end{aligned}$$

These relations are generalized as follows.

**Proposition 3.2.** *Assume that  $1 + h_2 - l_2 - \beta > 0$  and  $2h_2 - l_1 - l_2 - \beta \notin \{-2N, -2N + 2, \dots, -4, -2\}$ .*

(i) If  $n \in \mathbb{Z}_{\geq 1}$  and  $2n < N + 1$ , then

$$(3.15) \quad c_{2n}(E) \approx c_{2n-2}(E) [(Eq^{2n-1+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) \\ (Eq^{2n-2+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) - q^{4n-1-l_1-l_2-\beta}].$$

(ii) If  $n \in \mathbb{Z}_{\geq 1}$  and  $2n < N$ , then

$$(3.16) \quad c_{2n+1}(E) \approx c_{2n}(E)(Eq^{2n+\lambda_1-h_1-h_2} + q^{-h_2+1/2}).$$

*Proof.* The case  $n = 1$  in (i) and (ii) was shown by Eq.(3.13). We show the formulas for  $n = m + 1$  by assuming those for  $n = m$ . It follows from  $2h_2 - l_1 - l_2 - \beta \notin \{-2N, -2N + 2, \dots, -4, -2\}$  that there are no cancellations of the leading terms of the coefficients  $E^j$  ( $j = 0, 1, \dots$ ) on the right hand side of Eq.(3.1) for  $n = 2m + 2$ . Hence it follows from Eq.(3.1) and  $c_{2m+1}(E) \approx c_{2m}(E)(Eq^{2m+\lambda_1-h_1-h_2} + q^{-h_2+1/2})$  that

$$(3.17) \quad c_{2m+2}(E) \approx c_{2m+1}(E)(Eq^{2m+1+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) - c_{2m}(E)q^{4m+3-l_1-l_2-\beta} \\ \approx c_{2m}(E) [(Eq^{2m+1+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) \\ (Eq^{2m+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) - q^{4m+3-l_1-l_2-\beta}].$$

Therefore we obtain (i) for  $n = m + 1$ .

It follows from Eq.(3.1) for  $n = 2m+3, 2m+2$  and  $c_{2m+1}(E) \approx c_{2m}(E)(Eq^{2m+\lambda_1-h_1-h_2} + q^{-h_2+1/2})$  that

$$(3.18) \quad c_{2m+3}(E) \approx c_{2m+2}(E)(Eq^{2m+2+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) - c_{2m+1}(E)q^{4m+5-l_1-l_2-\beta} \\ \approx c_{2m+1}(E)(Eq^{2m+1+\lambda_1-h_1-h_2} + q^{-h_2+1/2})(Eq^{2m+2+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) \\ - c_{2m}(E)q^{4m+3-l_1-l_2-\beta}(Eq^{2m+2+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) \\ - c_{2m}(E)q^{4m+5-l_1-l_2-\beta}(Eq^{2m+\lambda_1-h_1-h_2} + q^{-h_2+1/2}).$$

Since the term  $q^{4m+3-l_1-l_2-\beta}(Eq^{2m+2+\lambda_1-h_1-h_2} + q^{-h_2+1/2})$  is stronger than the term  $q^{4m+5-l_1-l_2-\beta}(Eq^{2m+\lambda_1-h_1-h_2} + q^{-h_2+1/2})$ , we may neglect the term  $c_{2m}(E)q^{4m+5-l_1-l_2-\beta}(Eq^{2m+\lambda_1-h_1-h_2} + q^{-h_2+1/2})$  under the equivalence  $\approx$  and we have  $c_{2m+3}(E) \approx c_{2m+2}(E)(Eq^{2m+2+\lambda_1-h_1-h_2} + q^{-h_2+1/2})$ .  $\square$

By applying Proposition 3.2 repeatedly, we obtain the approximation of the spectral polynomial  $c_{N+1}(E)$  as follows.

**Theorem 3.3.** *We assume Eq.(2.3),  $1 + h_2 - l_2 - \beta > 0$  and  $2h_2 - l_1 - l_2 - \beta \notin \{-2N, -2N + 2, \dots, -4, -2\}$ . Set*

$$(3.19) \quad p_n(E) = (Eq^{2n-1+\lambda_1-h_1-h_2} + q^{-h_2+1/2})(Eq^{2n-2+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) \\ - q^{4n-1-l_1-l_2-\beta},$$

$$\tilde{c}_{N+1}(E) = \begin{cases} \prod_{n=1}^{(N+1)/2} p_n(E), & N \text{ is odd,} \\ (Eq^{N+\lambda_1-h_1-h_2} + q^{-h_2+1/2}) \prod_{n=1}^{N/2} p_n(E), & N \text{ is even.} \end{cases}$$

Then we have  $c_{N+1}(E) \approx \tilde{c}_{N+1}(E)$ .

We investigate the zeros of  $p_n(E)$  as  $q \rightarrow +0$ . If  $2h_2 - l_1 - l_2 - \beta > 2 - 4n$  (i.e.,  $-2h_2 + 1 < 4n - 1 - l_1 - l_2 - \beta$ ), then

$$(3.20) \quad p_n(E) \sim q^{4n-3+2(\lambda_1-h_1-h_2)}(E + q^{-2n+3/2-\lambda_1+h_1})(E + q^{-2n+5/2-\lambda_1+h_1}).$$

If  $2h_2 - l_1 - l_2 - \beta < 2 - 4n$ , then we have  $p_n(E) \sim q^{4n-3+2(\lambda_1-h_1-h_2)}(E^2 + Eq^{-2n+3/2-\lambda_1+h_1} - q^{2-l_1-l_2-\beta-2\lambda_1+2h_1+2h_2})$ . Moreover, if  $1 < 4n + 2h_2 - l_1 - l_2 - \beta < 2$ , then

$$(3.21) \quad p_n(E) \sim q^{4n-3+2(\lambda_1-h_1-h_2)}(E + q^{-2n+3/2-\lambda_1+h_1})(E - q^{2n+1/2-\lambda_1-l_1-l_2-\beta+h_1+2h_2}).$$

In the case  $4n + 2h_2 - l_1 - l_2 - \beta < 1$ , the polynomial  $p_n(E)$  is not factorized as Eq.(3.21) as  $q \rightarrow +0$ , and we solve the quadratic equation  $E^2 + Eq^{-2n+3/2-\lambda_1+h_1} - q^{2-l_1-l_2-\beta-2\lambda_1+2h_1+2h_2} = 0$  as  $q \rightarrow 0$  by the quadratic formula. It follows from  $4n + 2h_2 - l_1 - l_2 - \beta < 1$  that

$$(3.22) \quad \begin{aligned} E &= \frac{q^{-2n+3/2-\lambda_1+h_1} \pm \sqrt{q^{2(-2n+3/2-\lambda_1+h_1)} + 4q^{2-l_1-l_2-\beta-2\lambda_1+2h_1+2h_2}}}{2} \\ &\sim \frac{q^{-2n+3/2-\lambda_1+h_1} \pm \sqrt{4q^{2-l_1-l_2-\beta-2\lambda_1+2h_1+2h_2}}}{2} \\ &\sim \pm q^{1-l_1/2-l_2/2-\beta/2-\lambda_1+h_1+h_2} = \pm q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, \end{aligned}$$

where we used  $\lambda_1 = (h_1 + h_2 - l_1 - l_2 - \alpha_1 - \alpha_2 - \beta + 2)/2$ .

We describe the zeros of polynomial  $\tilde{c}_{N+1}(E)$  defined in Eq.(3.19) as  $q \rightarrow +0$  by dividing into parts.

(i) If  $2h_2 - l_1 - l_2 - \beta > -2$ , then

$$(3.23) \quad E \sim q^{-j+3/2-\lambda_1+h_1} \quad (1 \leq j \leq N+1).$$

(ii-1) If  $2h_2 - l_1 - l_2 - \beta < -2N - 1$  and  $N$  is odd, then

$$(3.24) \quad E \sim -q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, q^{(\alpha_1+\alpha_2+h_1+h_2)/2} \quad ((N+1)/2\text{-ple}).$$

(ii-2) If  $2h_2 - l_1 - l_2 - \beta < -2N + 1$ ,  $2h_2 - l_1 - l_2 - \beta \neq -2N$  and  $N$  is even, then

$$(3.25) \quad \begin{aligned} E &\sim -q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, q^{(\alpha_1+\alpha_2+h_1+h_2)/2} \quad (N/2\text{-ple}), \\ &\quad -q^{-N+1/2-\lambda_1+h_1}. \end{aligned}$$

(iii-1) If  $-4m + 1 < 2h_2 - l_1 - l_2 - \beta < -4m + 2$  for some  $m \in \mathbb{Z}$  such that  $1 \leq m \leq (N+1)/2$ , then

$$(3.26) \quad \begin{aligned} E &\sim -q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, \quad ((m-1)\text{-ple}) \\ &\quad q^{2m-3/2+\lambda_1+\alpha_1+\alpha_2+h_2}, \\ &\quad -q^{-j+3/2-\lambda_1+h_1} \quad (2m \leq j \leq N+1). \end{aligned}$$

(iii-2) If  $-4m - 2 < 2h_2 - l_1 - l_2 - \beta < -4m + 1$  for some  $m \in \mathbb{Z}$  such that  $1 \leq m \leq (N-1)/2$  and  $2h_2 - l_1 - l_2 - \beta \neq -4m$ , then

$$(3.27) \quad \begin{aligned} E &\sim -q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, \quad (m\text{-ple}) \\ &\quad -q^{-j+3/2-\lambda_1+h_1} \quad (2m+1 \leq j \leq N+1). \end{aligned}$$



We discuss the zeros of the polynomials  $c_{N+1}(E)$  as  $q \rightarrow +0$  by comparing with the zeros of  $\tilde{c}_{N+1}(E)$  described above. If  $2h_2 - l_1 - l_2 - \beta > -2$ , then it was shown in [5] that the zeros of the polynomial  $c_{N+1}(E)$  are written as  $E_j(q)$  ( $j = 1, \dots, N+1$ ) such that  $E_j(q) \sim -t_1 q^{-j+3/2-\lambda_1+h_1}$  for sufficiently small  $q (> 0)$ . Hence we have  $E_j(q) \approx -q^{-j+3/2-\lambda_1+h_1}$  for  $j = 1, \dots, N+1$  and it is compatible with Eq.(3.23).

Although the multiplicity of the roots of  $\tilde{c}_{N+1}(E)$  for  $q = 0$  appears in the case  $2h_2 - l_1 - l_2 - \beta > -7$ , the roots of  $\tilde{c}_{N+1}(E)$  for  $q > 0$  do not have multiplicity, which follows from the real root property of the spectral polynomial discussed in [5]. We give an example. In the case  $N = 3$  and  $2h_2 - l_1 - l_2 - \beta > -7$ , Eq.(3.7) is written as

$$(3.28) \quad c_4(E) \sim t_1^{-4} t_2^{-4} [E^4 q^{6+4(\lambda_1-h_1-h_2)} + E^3 t_1 q^{3(\lambda_1-h_1-h_2)} q^{3-h_2+1/2} \\ - t_1 t_2 q^{-l_1-l_2-\beta} \{3E^2 q^{8+2(\lambda_1-h_1-h_2)} + 2Et_1 q^{5-h_2+1/2} q^{\lambda_1-h_1-h_2}\} \\ + t_1^2 t_2^2 q^{10-2(l_1+l_2+\beta)}],$$

and the zeros of the right hand side of Eq.(3.28) satisfy  $E \approx +q^{(\alpha_1+\alpha_2+h_1+h_2)/2}$ ,  $-q^{(\alpha_1+\alpha_2+h_1+h_2)/2}$  with multiplicity two (see Eq.(3.24)). To obtain more detailed asymptotics, we set  $E = s q^{(\alpha_1+\alpha_2+h_1+h_2)/2}$ , substitute it into the right hand side and observe the condition that the leading term disappear. Then we have

$$(3.29) \quad s^4 - 3t_1 t_2 s^2 + t_1^2 t_2^2 = 0.$$

Hence  $E = \pm q^{(\alpha_1+\alpha_2+h_1+h_2)/2} (t_1 t_2)^{1/2} (\sqrt{5}+1)/2$  and  $E = \pm q^{(\alpha_1+\alpha_2+h_1+h_2)/2} (t_1 t_2)^{1/2} (\sqrt{5}-1)/2$  are the roots of the right hand side of Eq.(3.28), which do not have multiplicity.

### 3.2. The case $2N + 1 + h_2 - l_2 - \beta < 0$ .

If  $2N + 1 + h_2 - l_2 - \beta < 0$  and the condition in Eq.(2.3) is satisfied, then

$$(3.30) \quad c_n(E) \sim t_1^{-1} t_2^{-1} (E q^{n-1-h_1-h_2+\lambda_1} + q^{2n-1/2-l_2-\beta} t_1) c_{n-1}(E) \\ - q^{2n-1-l_1-l_2-\beta} t_1^{-1} t_2^{-1} c_{n-2}(E)$$

for  $n = 1, 2, \dots, N+1$  under the assumption of no cancellations. Then we have

$$(3.31) \quad c_1(E) \sim t_1^{-1} t_2^{-1} (E q^{-h_1-h_2+\lambda_1} + q^{3/2-l_2-\beta} t_1), \\ c_2(E) \sim t_1^{-2} t_2^{-2} (E q^{-h_1-h_2+\lambda_1} + q^{3/2-l_2-\beta} t_1) (E q^{1-h_1-h_2+\lambda_1} + q^{7/2-l_2-\beta} t_1) \\ - q^{3-l_1-l_2-\beta} t_1^{-1} t_2^{-1},$$

and the candidate of the cancellation is the term  $t_2^{-2} q^{5-2l_2-2\beta} - t_1^{-1} t_2^{-1} q^{3-l_1-l_2-\beta}$ . Hence there are no cancellation of the leading terms for  $c_2(E)$  in the case  $l_1 - l_2 - \beta \neq -2$ . If  $2 + l_1 - l_2 - \beta < 0$ , then we may ignore the term  $t_1^{-1} t_2^{-1} q^{3-l_1-l_2-\beta}$  and we have

$$(3.32) \quad c_2(E) \sim t_1^{-2} t_2^{-2} (E q^{-h_1-h_2+\lambda_1} + q^{3/2-l_2-\beta} t_1) (E q^{1-h_1-h_2+\lambda_1} + q^{7/2-l_2-\beta} t_1).$$

If  $2 + l_1 - l_2 - \beta > 0$ , then

$$(3.33) \quad c_2(E) \sim t_1^{-2} t_2^{-2} [q^{1-2h_1-2h_2+2\lambda_1} E^2 + t_1 q^{5/2-h_1-h_2+\lambda_1-l_2-\beta} E - t_1 t_2 q^{3-l_1-l_2-\beta}].$$

We assume  $l_1 - l_2 - \beta \neq -2$  and apply Eq.(3.30) for  $n = 3$ . Then

$$\begin{aligned}
(3.34) \quad c_3(E) &\sim c_2(E)t_1^{-1}t_2^{-1}(Eq^{2+\lambda_1-h_1-h_2} + t_1q^{11/2-l_2-\beta}) - c_1(E)t_1^{-1}t_2^{-1}q^{5-l_1-l_2-\beta} \\
&\sim t_1^{-3}t_2^{-3}(Eq^{-h_1-h_2+\lambda_1} + q^{3/2-l_2-\beta}t_1)(Eq^{1-h_1-h_2+\lambda_1} + q^{7/2-l_2-\beta}t_1) \\
&\quad (Eq^{2+\lambda_1-h_1-h_2} + t_1q^{11/2-l_2-\beta}) \\
&\quad - t_1^{-2}t_2^{-2}q^{3-l_1-l_2-\beta}(Eq^{2-h_1-h_2+\lambda_1} + t_1q^{11/2-l_2-\beta}) \\
&\quad - t_1^{-2}t_2^{-2}q^{5-l_1-l_2-\beta}(Eq^{-h_1-h_2+\lambda_1} + t_1q^{3/2-l_2-\beta}) \\
&\sim t_1^{-3}t_2^{-3}[E^3q^{3+3(\lambda_1-h_1-h_2)} + E^2t_1q^{2(\lambda_1-h_1-h_2)}q^{9/2-(l_2+\beta)} \\
&\quad + Et_1^2q^{\lambda_1-h_1-h_2}q^{7-2(l_2+\beta)} + t_1^3q^{21/2-3(l_2+\beta)} \\
&\quad - t_1t_2q^{5-l_1-l_2-\beta}(2Eq^{\lambda_1-h_1-h_2} + t_1q^{3/2-l_2-\beta})]
\end{aligned}$$

under the assumption that there are no cancellation of the leading terms. Then the condition  $l_1 - l_2 - \beta \notin \{-4, -2\}$  is sufficient for non-cancellation of the leading terms, and it is generalized as follows.

**Proposition 3.4.** *Let  $M \in \{1, 2, \dots, N+1\}$ . A sufficient condition that the cancellation of the leading terms in the right hand side of Eq.(3.30) for  $n = 1, 2, \dots, M$  does not occur is written as  $l_1 - l_2 - \beta \notin \{-2M+2, -2M+4, \dots, -4, -2\}$ .*

*Proof.* We can prove the proposition similarly to Proposition 3.1, although we need a slight modification. The recursive relation for  $c_n(E)$  is written as Eq.(3.8) where

$$(3.35) \quad r_n \sim t_1^{-1}t_2^{-1}q^{2n-1-l_1-l_2-\beta}, \quad p_n \sim t_1^{-1}t_2^{-1}q^{n-1+\lambda_1-h_1-h_2}, \quad q_n \sim t_2^{-1}q^{2n-1/2-l_2-\beta}.$$

By applying Eq.(3.8) repeatedly, we have the expression as Eq.(3.10). In this case, the term  $(p_{n-1}E + q_{n-1})r_{n+1} \sim t_1^{-2}t_2^{-2}q^{2n+1-l_1-l_2-\beta}(q^{n-2+\lambda_1-h_1-h_2}E + t_1q^{2n-5/2-l_2-\beta})$  is stronger than  $r_n(p_{n+1}E + q_{n+1}) \sim t_1^{-2}t_2^{-2}q^{2n-1-l_1-l_2-\beta}(q^{n+\lambda_1-h_1-h_2}E + t_1q^{2n+3/2-l_2-\beta})$ . By applying this relation repeatedly, we find that the strongest term in Eq.(3.10) for the fixed  $k$  is contained in

$$\begin{aligned}
(3.36) \quad &(p_1E + q_1)(p_2E + q_2) \dots (p_{M-2k}E + q_{M-2k})r_{M-2k+2}r_{M-2k+4} \dots r_M \\
&\approx \sum_{0 \leq l \leq M-2k} q_1q_2 \dots q_{M-2k-l}p_{M-2k-l+1} \dots p_{M-2k}r_{M-2k+2}r_{M-2k+4} \dots r_M E^l.
\end{aligned}$$

By repeating the argument in the proof of Proposition 3.1, we can obtain Proposition 3.4.  $\square$

We consider the expression of the leading terms roughly by using the equivalence  $\approx$  under the condition of Proposition 3.4. On the polynomials  $c_2(E)$  and  $c_3(E)$ , we

have

$$\begin{aligned}
(3.37) \quad c_2(E) &\sim t_1^{-2}t_2^{-2}(Eq^{-h_1-h_2+\lambda_1} + q^{3/2-l_2-\beta}t_1)(Eq^{1-h_1-h_2+\lambda_1} + q^{7/2-l_2-\beta}t_1) \\
&\quad - q^{3-l_1-l_2-\beta}t_1^{-1}t_2^{-1} \\
&\approx (Eq^{-h_1-h_2+\lambda_1} + q^{3/2-l_2-\beta})(Eq^{1-h_1-h_2+\lambda_1} + q^{7/2-l_2-\beta}) - q^{3-l_1-l_2-\beta}, \\
c_3(E) &\sim t_1^{-3}t_2^{-3}(Eq^{-h_1-h_2+\lambda_1} + q^{3/2-l_2-\beta}t_1)(Eq^{1-h_1-h_2+\lambda_1} + q^{7/2-l_2-\beta}t_1) \\
&\quad (Eq^{2+\lambda_1-h_1-h_2} + q^{11/2-l_2-\beta}t_1) \\
&\quad - t_1^{-2}t_2^{-2}q^{5-l_1-l_2-\beta}(2Eq^{\lambda_1-h_1-h_2} + q^{3/2-l_2-\beta}t_1) \\
&\approx (Eq^{-h_1-h_2+\lambda_1} + q^{3/2-l_2-\beta})(Eq^{1-h_1-h_2+\lambda_1} + q^{7/2-l_2-\beta})(Eq^{2+\lambda_1-h_1-h_2} + q^{11/2-l_2-\beta}) \\
&\quad - q^{5-l_1-l_2-\beta}(Eq^{\lambda_1-h_1-h_2} + q^{3/2-l_2-\beta}) \\
&\approx c_1(E)\{(Eq^{1-h_1-h_2+\lambda_1} + q^{7/2-l_2-\beta})(Eq^{2+\lambda_1-h_1-h_2} + q^{11/2-l_2-\beta}) - q^{5-l_1-l_2-\beta}\}.
\end{aligned}$$

These are generalized as follows.

**Proposition 3.5.** *Assume that  $2N+1+h_2-l_2-\beta < 0$  and  $l_1-l_2-\beta \notin \{-2N, -2N+2, \dots, -4, -2\}$ .*

*Then we have*

$$\begin{aligned}
(3.38) \quad c_n(E) &\approx c_{n-2}(E)[(Eq^{n-1+\lambda_1-h_1-h_2} + q^{2n-1/2-l_2-\beta}) \\
&\quad (Eq^{n-2+\lambda_1-h_1-h_2} + q^{2n-2-1/2-l_2-\beta}) - q^{2n-1-l_1-l_2-\beta}]
\end{aligned}$$

for  $n = 2, 3, \dots, N+1$ .

*Proof.* The formula for  $n = 2$  was shown in Eq.(3.37). We show Eq.(3.38) for  $n = m+1$  by assuming the case  $n = m$ . It follows from the assumption that there are no cancellations of the leading terms on the right hand side of Eq.(3.30) for  $n = m+1$ . Hence it follows from Eq.(3.30) that

$$\begin{aligned}
(3.39) \quad c_{m+1}(E) &\approx c_m(E)(Eq^{m+\lambda_1-h_1-h_2} + q^{2m+2-l_2-\beta-1/2}) - c_{m-1}(E)q^{2m+1-l_1-l_2-\beta} \\
&\approx c_{m-1}(E)(Eq^{m+\lambda_1-h_1-h_2} + q^{2m+2-l_2-\beta-1/2})(Eq^{m-1+\lambda_1-h_1-h_2} + q^{2m-l_2-\beta-1/2}) \\
&\quad - c_{m-2}(E)q^{2m-1-l_1-l_2-\beta}(Eq^{m+\lambda_1-h_1-h_2} + q^{2m+2-l_2-\beta-1/2}) \\
&\quad - c_{m-1}(E)q^{2m+1-l_1-l_2-\beta}.
\end{aligned}$$

It follows from Eq.(3.30) for  $n = m-1$  that

$$(3.40) \quad c_{m-1}(E) \approx c_{m-2}(E)(Eq^{m-2+\lambda_1-h_1-h_2} + q^{2m-2-l_2-\beta-1/2}) - c_{m-3}(E)q^{2m-3-l_1-l_2-\beta}.$$

Since there are no cancellations of the leading terms on the right hand side of Eq.(3.40), the term  $c_{m-1}(E)q^{2m+1-l_1-l_2-\beta}$  is stronger than  $c_{m-2}(E)q^{2m+1-l_1-l_2-\beta}(Eq^{m-2+\lambda_1-h_1-h_2} + q^{2m-2-l_2-\beta-1/2})$  and it is stronger than  $c_{m-2}(E)q^{2m-1-l_1-l_2-\beta}(Eq^{m+\lambda_1-h_1-h_2} + q^{2m+2-l_2-\beta-1/2})$ .

Hence we may neglect the term  $c_{m-2}(E)q^{2m-1-l_1-l_2-\beta}(Eq^{m+\lambda_1-h_1-h_2} + q^{2m+2-l_2-\beta-1/2})$  on the equivalence  $\approx$  in Eq.(3.39) and we obtain (3.38) for  $n = m+1$ .  $\square$

By applying Proposition 3.5 repeatedly, we obtain the approximation of the spectral polynomial  $c_{N+1}(E)$  as follows.

**Theorem 3.6.** *We assume Eq.(2.3),  $2N + 1 + h_2 - l_2 - \beta < 0$  and  $l_1 - l_2 - \beta \notin \{-2N, -2N + 2, \dots, -4, -2\}$ . Set*

$$(3.41) \quad p_n(E) = (Eq^{n-1+\lambda_1-h_1-h_2} + q^{2n-1/2-l_2-\beta}) \\ (Eq^{n-2+\lambda_1-h_1-h_2} + q^{2n-2-1/2-l_2-\beta}) - q^{2n-1-l_1-l_2-\beta},$$

$$\tilde{c}_{N+1}(E) = \begin{cases} \prod_{n=1}^{(N+1)/2} p_{2n}(E), & N \text{ is odd,} \\ (Eq^{\lambda_1-h_1-h_2} + q^{3/2-l_2-\beta}) \prod_{n=1}^{N/2} p_{2n+1}(E), & N \text{ is even.} \end{cases}$$

Then we have  $c_{N+1}(E) \approx \tilde{c}_{N+1}(E)$ .

We investigate the zeros of  $p_n(E)$  and  $\tilde{c}_{N+1}(E)$  as  $q \rightarrow +0$ . If  $l_1 - l_2 - \beta < 2 - 2n$ , then

$$(3.42) \quad p_n(E) \sim q^{2n-3+2(\lambda_1-h_1-h_2)}(E + q^{n-3/2+\lambda_1+l_1+\alpha_1+\alpha_2})(E + q^{n-5/2+\lambda_1+l_1+\alpha_1+\alpha_2}).$$

If  $2 - 2n < l_1 - l_2 - \beta < 3 - 2n$ , then

$$(3.43) \quad p_n(E) \sim q^{2n-3+2(\lambda_1-h_1-h_2)}(E + q^{n-5/2+\lambda_1+l_1+\alpha_1+\alpha_2})(E - q^{-n+5/2-\lambda_1-l_1+h_1+h_2}).$$

If  $3 - 2n < l_1 - l_2 - \beta$ , then the solutions of the quadratic equation  $p_n(E) = 0$  satisfy

$$(3.44) \quad E = \frac{q^{n-5/2+\lambda_1+l_1+\alpha_1+\alpha_2} \pm \sqrt{q^{2(n-5/2+\lambda_1+l_1+\alpha_1+\alpha_2)} + 4q^{\alpha_1+\alpha_2+h_1+h_2}}}{2} \\ \sim \pm q^{(\alpha_1+\alpha_2+h_1+h_2)/2},$$

as  $q \rightarrow +0$ .

We describe the zeros of polynomial  $\tilde{c}_{N+1}(E)$  as  $q \rightarrow +0$  by dividing into parts.

(i) If  $l_1 - l_2 - \beta < -2N$ , then

$$(3.45) \quad E \sim q^{j-3/2+\lambda_1+l_1+\alpha_1+\alpha_2} \quad (1 \leq j \leq N+1).$$

(ii-1) If  $l_1 - l_2 - \beta > -1$  and  $N$  is odd, then

$$(3.46) \quad E \sim -q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, q^{(\alpha_1+\alpha_2+h_1+h_2)/2} \quad ((N+1)/2\text{-ple}).$$

(ii-2) If  $l_1 - l_2 - \beta > -3$ ,  $l_1 - l_2 - \beta \neq -2$  and  $N$  is even, then

$$(3.47) \quad E \sim -q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, q^{(\alpha_1+\alpha_2+h_1+h_2)/2} \quad (N/2\text{-ple}), \\ -q^{-1/2+\lambda_1+l_1+\alpha_1+\alpha_2}.$$

(iii-1) If  $-2N + 4m < l_1 - l_2 - \beta < -2N + 4m + 1$  for some  $m \in \mathbb{Z}$  such that  $0 \leq m \leq (N-1)/2$ , then

$$(3.48) \quad E \sim -q^{j-3/2+\lambda_1+l_1+\alpha_1+\alpha_2}, \quad (1 \leq j \leq N-2m), \\ q^{-N+2m+3/2-\lambda_1-l_1+h_1+h_2}, \\ -q^{(l_3+l_4+h_1+h_2)/2}, q^{(l_3+l_4+h_1+h_2)/2} \quad (m\text{-ple}).$$

(iii-2) If  $-2N + 4m - 3 < l_1 - l_2 - \beta < -2N + 4m$  for some  $m \in \mathbb{Z}$  such that  $1 \leq m \leq (N - 1)/2$  and  $l_1 - l_2 - \beta \neq -2N + 4m - 2$ , then

$$(3.49) \quad E \sim -q^{j-3/2+\lambda_1+l_1+\alpha_1+\alpha_2}, \quad (1 \leq j \leq N - 2m + 1), \\ -q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, \quad q^{(\alpha_1+\alpha_2+h_1+h_2)/2}, \quad (m\text{-ple}).$$

We can also discuss the zeros of the polynomials  $c_{N+1}(E)$  as  $q \rightarrow +0$  by comparing with the zeros of  $\tilde{c}_{N+1}(E)$  as the case  $1 + h_2 - l_2 - \beta > 0$ . In particular, if  $l_1 - l_2 - \beta < -2N$ , then the zeros of the polynomial  $c_{N+1}(E)$  are written as  $E_j(q)$  ( $j = 1, \dots, N+1$ ) such that  $E_j(q) \sim -t_1 q^{j-3/2+\lambda_1+l_1+\alpha_1+\alpha_2}$  for sufficiently small  $q (> 0)$  (see [5]), and it is compatible with the equivalence  $c_{N+1}(E) \approx \tilde{c}_{N+1}(E)$  and Eq.(3.45).

### 3.3. The case $-2N < 1 + h_2 - l_2 - \beta < 0$ .

We consider the polynomials  $c_n(E)$  ( $n = 1, 2, \dots, N+1$ ) for the case  $-2N < 1 + h_2 - l_2 - \beta < 0$  and  $1 + h_2 - l_2 - \beta \notin \{-2N + 2, \dots, -4, -2\}$  with the condition in Eq.(2.3). In this case there exists  $K \in \{1, 2, \dots, N\}$  such that  $-2K < 1 + h_2 - l_2 - \beta < -2K + 2$ . It follows from Eq.(2.4) that the polynomials  $c_n(E)$  satisfy

$$(3.50) \quad c_n(E) \sim t_1^{-1} t_2^{-1} (E q^{n-1-h_1-h_2+\lambda_1} + t_1 q^{2n-1/2-l_2-\beta}) c_{n-1}(E) - t_1^{-1} t_2^{-1} q^{2n-1-l_1-l_2-\beta} c_{n-2}(E)$$

for  $n = 1, 2, \dots, K$  and

$$(3.51) \quad c_n(E) \sim t_1^{-1} t_2^{-1} (E q^{n-1-h_1-h_2+\lambda_1} + t_1 q^{1/2-h_2}) c_{n-1}(E) - t_1^{-1} t_2^{-1} q^{2n-1-l_1-l_2-\beta} c_{n-2}(E)$$

for  $n = K + 1, K + 2, \dots, N + 1$  under the assumption that there are no cancellations of the leading terms of the coefficients of  $E^j$  ( $j = 0, 1, \dots$ ) on the right hand sides.

We add the condition  $h_2 - l_1 + 1 > 0$  to avoid difficulty. Then we obtain the following proposition.

**Proposition 3.7.** *If  $h_2 - l_1 + 1 > 0$  and there exists  $K \in \{1, 2, \dots, N\}$  such that  $-2K < 1 + h_2 - l_2 - \beta < -2K + 2$ , then  $c_n(E)$  satisfies*

$$(3.52) \quad c_n(E) \sim \begin{cases} t_1^{-1} t_2^{-1} (E q^{n-1-h_1-h_2+\lambda_1} + q^{2n-1/2-l_2-\beta} t_1) c_{n-1}(E), & n = 1, 2, \dots, K, \\ t_1^{-1} t_2^{-1} (E q^{n-1-h_1-h_2+\lambda_1} + q^{1/2-h_2} t_1) c_{n-1}(E), & n = K + 1, K + 2, \dots, N + 1. \end{cases}$$

*Proof.* Eq.(3.52) for  $n = 1$  follows from Eq.(3.50). We show Eq.(3.52) for  $n = m$  ( $m = 2, \dots, K$ ) by assuming Eq.(3.52) for  $n = m - 1$ . It follows from Eq.(3.52) for  $n = m - 1$  that the right hand side of Eq.(3.50) for  $n = m$  is written as

$$(3.53) \quad t_1^{-2} t_2^{-2} \{ (E q^{m-1-h_1-h_2+\lambda_1} + q^{2m-1/2-l_2-\beta} t_1) (E q^{m-2-h_1-h_2+\lambda_1} + q^{2m-5/2-l_2-\beta} t_1) \\ - t_1 t_2 q^{2m-1-l_1-l_2-\beta} \} c_{m-2}(E).$$

Then it follows from  $4m - 3 - 2l_2 - 2\beta - (2m - 1 - l_1 - l_2 - \beta) = 2m + l_1 - 2 - l_2 - \beta < 2K + h_2 - 1 - l_2 - \beta < 0$  that we may ignore the term  $t_1^{-1} t_2^{-1} q^{2m-1-l_1-l_2-\beta} c_{m-2}(E)$  in Eq.(3.53) and that in Eq.(3.50) for  $n = m$ . Thus we have shown Eq.(3.52) for  $n = m$ . Hence we obtain Eq.(3.52) for  $n = 1, \dots, K$ .

Next we show Eq.(3.52) for  $n = K + 1$ . It follows from Eq.(3.52) for  $n = K$  that the right hand side of Eq.(3.51) for  $n = K + 1$  is written as

$$(3.54) \quad \begin{aligned} & t_1^{-1}t_2^{-1}[Eq^{K-h_1-h_2+\lambda_1} + t_1q^{1/2-h_2}]c_K(E) - t_1^{-1}t_2^{-1}q^{2K+1-l_1-l_2-\beta}c_{K-1}(E) \\ & \sim t_1^{-2}t_2^{-2}[(Eq^{K-h_1-h_2+\lambda_1} + t_1q^{1/2-h_2})(Eq^{K-1-h_1-h_2+\lambda_1} + q^{2K-1/2-l_2-\beta}t_1) \\ & \quad - t_1t_2q^{2K+1-l_1-l_2-\beta}]c_{K-1}(E). \end{aligned}$$

It follows from  $h_2 - l_1 + 1 > 0$  that we may ignore the term  $t_1^{-1}t_2^{-1}q^{2K+1-l_1-l_2-\beta}c_{K-1}(E)$  in Eq.(3.54) and that in Eq.(3.51) for  $n = K + 1$ . Hence we obtain Eq.(3.52) for  $n = K + 1$ . Let  $m \in \{K + 1, \dots, N\}$  and assume that Eq.(3.52) holds for  $n = m$ . Then we have

$$(3.55) \quad \begin{aligned} & t_1^{-2}t_2^{-2}\{(Eq^{m-h_1-h_2+\lambda_1} + t_1q^{1/2-h_2})c_m(E) - q^{2m+1-l_1-l_2-\beta}c_{m-1}(E)\} \\ & \sim t_1^{-1}t_2^{-1}\{(Eq^{m-h_1-h_2+\lambda_1} + t_1q^{1/2-h_2})(Eq^{m-h_1-h_2+\lambda_1} + t_1q^{1/2-h_2}) \\ & \quad - t_1t_2q^{2m+1-l_1-l_2-\beta}\}c_{m-1}(E). \end{aligned}$$

Since  $1 - 2h_2 - (2m + 1 - l_1 - l_2 - \beta) < -2h_2 - 2K - 2 + l_1 + l_2 + \beta < -h_2 - 2K - 1 + l_2 + \beta < 0$ , we may neglect the term  $t_1^{-1}t_2^{-1}q^{2m+1-l_1-l_2-\beta}c_{m-1}(E)$  and we have Eq.(3.52) for  $n = m + 1$ .  $\square$

**Theorem 3.8.** *We assume Eq.(2.3),  $h_2 - l_1 + 1 > 0$  and there exists  $K \in \{1, 2, \dots, N\}$  such that  $-2K < 1 + h_2 - l_2 - \beta < -2K + 2$ .*

(i) *The spectral polynomial  $c_{N+1}(E)$  satisfies*

$$(3.56) \quad \begin{aligned} c_{N+1}(E) & \sim (t_1t_2)^{-N-1} \prod_{n=1}^K (Eq^{n-1-h_1-h_2+\lambda_1} + q^{2n-1/2-l_2-\beta}t_1) \\ & \prod_{n=K+1}^{N+1} (Eq^{n-1-h_1-h_2+\lambda_1} + q^{1/2-h_2}t_1) \\ & = (t_1t_2)^{-N-1} q^{(N/2+\lambda_1-h_1-h_2)(N+1)} \prod_{n=1}^K (E + q^{n-3/2+\lambda_1+l_1+\alpha_1+\alpha_2}t_1) \\ & \prod_{n=K+1}^{N+1} (E + q^{-n+3/2+h_1-\lambda_1}t_1) \end{aligned}$$

(ii) *There exist solutions  $E_j(q)$  ( $j = 1, 2, \dots, N + 1$ ) to the equation  $c_{N+1}(E) = 0$  for sufficiently small  $q$  such that*

$$(3.57) \quad E_j(q) \sim \begin{cases} -q^{j-3/2+\lambda_1+l_1+\alpha_1+\alpha_2}t_1, & j = 1, \dots, K, \\ -q^{-j+3/2+h_1-\lambda_1}t_1, & j = K + 1, \dots, N + 1. \end{cases}$$

*Proof.* We obtain (i) by applying Proposition 3.7. Then it follows from the assumption  $-2K < 1 + h_2 - l_2 - \beta < -2K + 2$  that all of the zeros of the right hand side of Eq.(3.56) are different by the ultradiscrete limit ( $q \rightarrow +0$ ). Hence (ii) follows from the theorem in the appendix of [5].  $\square$

## 4. CONCLUDING REMARKS

In this paper, we investigated roots of the spectral polynomial  $c_{N+1}(E)$  as  $q \rightarrow +0$  for three cases in the sections 3.1, 3.2 and 3.3. Recall that the polynomial-type solution of the  $q$ -Heun equation exists, if the accessory parameter  $E$  is a root of the spectral polynomial  $c_{N+1}(E)$ , but we did not investigate the polynomial-type solutions of the  $q$ -Heun equation in this paper, which we leave as a problem.

Another problem is to consider polynomial-type solutions to degenerations of the  $q$ -Heun equation. Ultradiscrete limit would be applicable to those cases. Note that degenerations of the  $q$ -Heun equation would be obtained similarly to the degenerations of Heun's differential equation (see [6]).

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