

Gaussian Random Matrix Ensembles in Phase Space

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30th June, 2019

Abstract

A new class of Random Matrix Ensembles is introduced. The Gaussian orthogonal, unitary, and symplectic ensembles GOE, GUE, and GSE, of random matrices are analogous to the classical Gibbs ensemble governed by Boltzmann's distribution in the coordinate space. The proposed new class of Random Matrix ensembles is an extension of the above Gaussian ensembles and it is analogous to the canonical Gibbs ensemble governed by Maxwell-Boltzmann's distribution in phase space. The thermodynamical magnitudes of partition function, intrinsic energy, free energy of Helmholtz, free energy of Gibbs, enthalpy, as well as entropy, equation of state, and heat capacities, are derived for the new ensemble. The examples of nonideal gas with quadratic potential energy as well as ideal gas of quantum matrices are provided. The distribution function for the new ensembles is derived from the maximum entropy principle.

1 Introduction

The Random Matrix Theory RMT is a well established branch of mathematics and it studies random matrices (random matrix variables) defined over following fields \mathbf{F} : real \mathbf{R} , complex \mathbf{C} , and quaternion \mathbf{H} . The matrix elements are random variables with assumed distribution functions. Many random magnitudes derived from random matrices are also studied: their eigenvalues, eigenvectors, eigenphases, determinants, kernels, correlation functions etc. The application of RMT to nuclear physics began with works of von Neumann and Wigner [1, 2, 3, 4, 5] and also Landau and

Smorodinsky [6]. They assumed a statistical hypothesis for the many-body quantum Hamiltonian to explain observed nuclear spectra. They postulated that the Hamiltonian operator acting in truncated N -dimensional Hilbert space $V_{\mathcal{X}} = \mathbf{F}^N$ is a random matrix with matrix elements Gaussian distributed. So the definition of N -dimensional Gaussian orthogonal, unitary, and symplectic ensembles $\text{GOE}(N)$, $\text{GUE}(N)$, $\text{GSE}(N)$, was made, as well as Poisson ensemble PE. RMT is different from both classical statistical mechanics and quantum statistical mechanics. In the classical statistical mechanics the approach is based on dynamics of the considered system in the phase-space. In the quantum statistical mechanics the studied space is Fock's space on which the quantum operators act. Both statistical mechanics deal with three ensembles: microcanonical, canonical, and grand canonical. The hermitean random matrices in RMT are derived from the symmetry principle and not from dynamics. Hence, there are the three classes of ensembles: orthogonal, unitary, and symplectic, corresponding to three Lie's groups: orthogonal $\text{O}(N, \mathbf{F})$, unitary $\text{U}(N, \mathbf{F})$, and symplectic $\text{Sp}(N, \mathbf{F})$, that leave invariant the matrix Haar's measure. However if the random matrices are not hermitean, then the Lie's group is general linear over field \mathbf{F} , thence there are three Ginibre's ensembles. The principal difference between statistical mechanics and RMT is that in the former case the ensemble consists of systems with the same Hamiltonian and with different initial condition whereas in the latter case the matrix ensemble consists of systems with different quantum matrices, *e.g.*, with different quantum Hamiltonians, but with the same symmetry class. Since the 1950s the march of RMT through physics was imposing with applications to nuclear physics, atomic physics, condensed phase physics, field theory, quantum gravity, quantum chromodynamics, quantum chaos, disordered mesoscopic systems [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

The aim of the present article is an extension of the Gaussian ensembles of RMT to a wider class of ensembles in matrix phase space. The ensembles (Gaussian or non-Gaussian) of RMT deal with quantum random matrices from the quantum matrix space analogous to the configuration space of classical statistical mechanics. The proposed extension will lead one the ensembles of pairs of random quantum matrices from the quantum matrix space analogous to the phase-space of classical statistical mechanics. As far as we consider the Gaussian ensembles of RMT, the probability density function $f_{\mathcal{X}}$ of random quantum Hamiltonian matrix X belonging to $\text{GOE}(N)$, $\text{GUE}(N)$, or $\text{GSE}(N)$ reads [25]:

$$f_{\mathcal{X}}(X) = \mathcal{C}_{X\beta} \exp \left[-\beta \cdot \frac{1}{2} \cdot \text{Tr}(X^\dagger X) \right], \quad (1)$$

$$\mathcal{C}_{X\beta} = \left(\frac{\beta}{2\pi}\right)^{\mathcal{N}_{X\beta}/2},$$

$$\mathcal{N}_{X\beta} = N + \frac{1}{2}N(N-1)D,$$

where β is an inverse of the temperature T measured in energetic scale, D is dimension of random matrix elements, the parameters β, D , assume values $\beta = 1, 2, 4$, and $D = 1, 2, 4$, for $\text{GOE}(N)$, $\text{GUE}(N)$, $\text{GSE}(N)$, respectively, and $\mathcal{N}_{X\beta}$ is number of independent matrix elements of hermitean Hamiltonian $X = X^\dagger$. The quantum Hamiltonian X is a random matrix variable and it is zero-centred Gaussian distributed with the diagonal covariance matrix $\text{Cov}(X_{ij}, X_{kl}) = \frac{1}{\beta}\delta_{ik}\delta_{jl}$. The matrix elements X_{ij} are independently Gaussian distributed, and $X_{ij} \in \mathbf{R}, \mathbf{C}, \mathbf{H}$, for GOE , GUE , GSE , respectively. The normalization of distribution of X is:

$$\int f_{\mathcal{X}}(X)dX = 1, \quad (2)$$

$$dX = \prod_{i=1}^N \prod_{j \geq i}^N \prod_{\gamma=0}^{D-1} dX_{ij}^{(\gamma)},$$

$$X_{ij} = (X_{ij}^{(0)}, \dots, X_{ij}^{(D-1)}) \in \mathbf{F},$$

where dX is Haar's measure in the matrix space. The Haar's measure dX is invariant under transformations from the orthogonal $\text{O}(N, \mathbf{F})$, unitary $\text{U}(N, \mathbf{F})$, and symplectic $\text{Sp}(N, \mathbf{F})$ Lie's groups of symmetries, respectively. The probability distribution $f_{\mathcal{X}}$ is invariant under the three Lie's groups, respectively. The Hamiltonian operators X act in given Hilbert space of eigenvectors $V_{\mathcal{X}} = \mathbf{F}^N$, so they belong themselves to the Hilbert space $W_{\mathcal{X}} = \text{Herm}(\mathbf{F}, N)$ of all $N \times N$ hermitean matrices with matrix elements belonging to the field \mathbf{F} . In the space $W_{\mathcal{X}}$ the scalar product of two operators X_1, X_2 , is given by formula

$$\langle X_1 | X_2 \rangle = \text{Tr}(X_1^\dagger X_2), \quad (3)$$

which yields

$$\langle X_1 | X_2 \rangle = \text{Tr}(X_1 X_2), \quad (4)$$

because $X_1^\dagger = X_1$. Since the space $W_{\mathcal{X}}$ is also Banach space, then the norm of operator X reads:

$$\|X\| = (|\langle X | X \rangle|)^{1/2} = (|\text{Tr}(X^\dagger X)|)^{1/2}. \quad (5)$$

Due to the hermiticity of X we have property $\langle X | X \rangle \geq 0$, and then

$$\|X\| = (\langle X | X \rangle)^{1/2} = [\text{Tr}(X^2)]^{1/2}. \quad (6)$$

It follows that $\|X\|$ is Euclidean norm, and the distance of two matrices X_1, X_2 , is given by

$$d(X_1, X_2) = \|X_1 - X_2\|. \quad (7)$$

Using Eqs. (3), (4), (5), (6), we rewrite the distribution Eq. (1):

$$f_{\mathcal{X}}(X) = \mathcal{C}_{X\beta} \exp[-\beta \cdot \frac{1}{2} \cdot \|X\|^2]. \quad (8)$$

From the above properties of scalar product, norm, and distance, we infer the analogy between configuration space \mathbf{F}^N of x -components of generalized coordinates of any one-dimensional classical system of N particles, and the configuration space $W_{\mathcal{X}}$ of generalized "X-coordinate" of quantum system described by Hamiltonian operator X . From the formulae (2),(8), we deduce that the quantum Hamiltonian X has continuous (non-discrete) distribution that is analogous to the distribution of x -coordinate of one-dimensional classical particle in potential of harmonic oscillator $U(x) = \frac{1}{2}x^2$. The x -coordinate of the classical particle has the Boltzmann's distribution [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]:

$$f_{\mathcal{B}}(x) = \mathcal{C}_X \exp[-\frac{1}{k_B T} \cdot \frac{1}{2} k x^2], \quad (9)$$

The classical particle's statistics is governed by canonical ensemble in configuration space (coordinate space). In view of this analogy parameter β corresponds to $\frac{1}{k_B T}$. Hence, the discrete "temperature" T for the Gaussian ensembles reads:

$$T = \frac{1}{k_B \beta}. \quad (10)$$

The main point is that the quantum Hamiltonians X are governed by classical continuous distribution (2),(8), and not by quantum discrete distribution. The quantum Hamiltonians are belonging to the configuration space (generalized coordinates's space) $W_{\mathcal{X}}$. The Gaussian ensembles GOE, GUE, GSE, of the Hamiltonians describe the classical non-ideal gas of harmonic oscillators of quantum matrices in configuration space. We naturally extend this description by the introduction of momentum space and phase space for quantum Hamiltonians. Firstly, we concentrate on definition of momentum space of random matrices corresponding to configuration space. We define the quantum operator of generalized linear momentum P that is correlated with the quantum operator X of generalized "X-positon" ("X-coordinate"). The classical observables of generalized momentum and generalized position are classically canonically conjugated in classical mechanics and in classical statistical mechanics. The operator P acts on Hilbert space $V_{\mathcal{P}} = \mathbf{F}^N$,

and it belongs to Hilbert space $W_{\mathcal{P}} = \text{Herm}(\mathbf{F}, N)$ of momenta. The scalar product, norm, and distance in $W_{\mathcal{P}}$ are as follows:

$$\langle P_1 | P_2 \rangle = \text{Tr}(P_1^\dagger P_2), \quad (11)$$

$$\|P\| = (\langle P | P \rangle)^{1/2} = (|\text{Tr}(P^\dagger P)|)^{1/2}. \quad (12)$$

$$d(P_1, P_2) = \|P_1 - P_2\|. \quad (13)$$

We assume that P is hermitean $P^\dagger = P$, which yields $\langle P | P \rangle \geq 0$, and

$$\|P\| = (\langle P | P \rangle)^{1/2} = [\text{Tr}(P^2)]^{1/2}. \quad (14)$$

Again, the above properties of scalar product, norm, and distance in momentum space, conduct us to the analogy between momentum space \mathbf{F}^N of x -components of generalized linear momenta of any one-dimensional classical system of N particles, and the momentum space $W_{\mathcal{P}}$ of "P-momenta" of quantum system described by momentum operator P . The postulated distribution of momentum is analogous to classical Maxwell's distribution and it reads:

$$f_{\mathcal{P}}(P) = \mathcal{C}_{P\beta} \exp[-\beta \cdot \frac{1}{2M} \cdot \text{Tr}(P^\dagger P)], \quad (15)$$

$$\mathcal{C}_{P\beta} = \left(\frac{\beta}{2\pi M}\right)^{\mathcal{N}_{P\beta/2}},$$

$$\mathcal{N}_{P\beta} = N + \frac{1}{2}N(N-1)D,$$

$$\int f_{\mathcal{P}}(P) dP = 1, \quad (16)$$

$$dP = \prod_{k=1}^N \prod_{l \geq k}^N \prod_{\delta=0}^{D-1} dP_{kl}^{(\delta)},$$

$$P_{kl} = (P_{kl}^{(0)}, \dots, P_{kl}^{(D-1)}) \in \mathbf{F},$$

where M is "mass" of the particle in matrix space. The Haar's measure dP is invariant under transformations from the orthogonal $O(N, \mathbf{F})$, unitary $U(N, \mathbf{F})$, and symplectic $Sp(N, \mathbf{F})$ Lie's groups of symmetries, respectively. Also the probability density function $f_{\mathcal{P}}$ is invariant under above Lie's groups. Formally momentum P belongs to Gaussian ensembles that we will denote by $\text{GOE}(N, P)$, $\text{GUE}(N, P)$, $\text{GSE}(N, P)$. Momentum is zero-centred Gaussian distributed with diagonal covariance matrix $\text{Cov}(P_{kl}, P_{mn}) = \frac{M}{\beta} \delta_{km} \delta_{ln}$. Secondly, we are able now to introduce a phase space W_{Γ} of generalized canonically conjugated operators of linear momenta

P and “ X -coordinates”. The phase space W_Γ is an *analogy* to classical phase space \mathbf{F}^{2N} of x -components of generalized coordinates and generalized linear momenta of classical one-dimensional system of N particles. The pair of operators $(X, P) = \Gamma$ (the direct sum of operators) composes a point in phase space of random matrices $W_\Gamma = W_X \times W_P$. The Haar’s measure $d\Gamma$ in the matrix phase space is given by:

$$d\Gamma = \frac{1}{\mathcal{N}_{\Gamma\beta}! \cdot h^{s_{\Gamma\beta}}} \cdot dX \cdot dP, \quad (17)$$

$$\mathcal{N}_{\Gamma\beta} = \mathcal{N}_{P\beta} = \mathcal{N}_{X\beta} = N + \frac{1}{2}N(N-1)D,$$

$$s_{\Gamma\beta} = \mathcal{N}_{\Gamma\beta}.$$

The Haar’s measure $d\Gamma$ is invariant under composite transformations $U_\Gamma = (U_X, U_P)$ from the direct sums of the Lie’s groups of symmetries: orthogonal $O(N, \mathbf{F}) \oplus O(N, \mathbf{F})$, unitary $U(N, \mathbf{F}) \oplus U(N, \mathbf{F})$, and symplectic $Sp(N, \mathbf{F}) \oplus Sp(N, \mathbf{F})$, respectively. The distribution of the pair Γ is postulated in the following form:

$$f_\Gamma(\Gamma) = \mathcal{C}_{P\beta} \exp \left[-\beta \cdot \left(\frac{1}{2M} \cdot \text{Tr}(P^\dagger P) + \frac{K}{2} \cdot \text{Tr}(X^\dagger X) \right) \right] = \quad (18)$$

$$= \mathcal{C}_{P\beta} \exp \left[-\beta \cdot \left(\frac{1}{2M} \cdot \|P\|^2 + \frac{K}{2} \cdot \|X\|^2 \right) \right],$$

$$\mathcal{C}_{\Gamma\beta} = \mathcal{C}_{P\beta} \cdot \mathcal{C}_{X\beta} = \left(\frac{\beta}{2\pi M} \right)^{\mathcal{N}_{\Gamma\beta}} \cdot \left(\frac{K}{M} \right)^{\mathcal{N}_{\Gamma\beta}/2},$$

$$\int f_\Gamma(X, P) dX \cdot dP = 1,$$

which is analog of classical Maxwell-Boltzmann’s distribution. The above distribution is also invariant under direct sums of orthogonal, unitary, and symplectic Lie’s groups of transformations of composite symmetry. Hence, we extended both the random quantum matrices X to direct sums Γ of random quantum matrices and the symmetry Lie’s groups to the direct sums of symmetry Lie’s groups. The Hamiltonian operators X and momentum operators P are independent random variables. We denote the Gaussian orthogonal, unitary, and symplectic ensembles in the phase space as follows: $\text{GOE}(N, \Gamma)$, $\text{GUE}(N, \Gamma)$, $\text{GSE}(N, \Gamma)$, whereas the standard Gaussian ensembles in configuration space might be symbolized by $\text{GOE}(N, X) = \text{GOE}(N)$, $\text{GUE}(N, X) = \text{GUE}(N)$, $\text{GSE}(N, X) = \text{GSE}(N)$, respectively.

2 The Thermodynamics of New Ensembles

The "classical" Hamiltonian $\mathcal{H}_\Gamma(X, P)$, in the matrix phase space is a sum of the "classical" kinetic energy $\mathcal{T}_\Gamma(P)$, and the "classical" potential energy $\mathcal{U}_\Gamma(X, P)$:

$$\mathcal{H}_\Gamma(X, P) = \mathcal{T}_\Gamma(P) + \mathcal{U}_\Gamma(X, P). \quad (19)$$

Firstly, let us consider the example of nonideal gas of harmonic oscillators in the matrix phase space. Then, $f_\Gamma(\Gamma)$ is Maxwell-Boltzmann's distribution with quadratic potential energy, and the considered three ensembles of pairs of random matrices are $\text{GOE}(N, \Gamma)$, $\text{GUE}(N, \Gamma)$, $\text{GSE}(N, \Gamma)$. The distribution $f_\Gamma(\Gamma)$ Eq.(18), and the ensemble average $\langle g(\Gamma) \rangle$ of magnitude $g(\Gamma)$ can be finally written in traditional form:

$$\begin{aligned} f_\Gamma(\Gamma) &= \mathcal{C}_{\Gamma\beta} \cdot \exp[-\beta \cdot \mathcal{H}_\Gamma(\Gamma)], \quad (20) \\ \int f_\Gamma(\Gamma) dX \cdot dP &= 1, \\ \langle g(\Gamma) \rangle &= \int g(\Gamma) \cdot \exp[-\beta \cdot \mathcal{H}_\Gamma(\Gamma)] d\Gamma / \int \exp[-\beta \cdot \mathcal{H}_\Gamma(\Gamma)] d\Gamma \\ &= \frac{1}{Z_\beta} \cdot \int \exp[-\beta \cdot \mathcal{H}_\Gamma(\Gamma)] d\Gamma, \end{aligned}$$

under following conditions:

$$\begin{aligned} \mathcal{T}_\Gamma(P) &= \frac{1}{2M} \cdot \|P\|^2, \quad (21) \\ \mathcal{U}_\Gamma(\Gamma) &= \frac{K}{2} \cdot \|X\|^2, \\ Z_\beta &= \int \exp[-\beta \cdot \mathcal{H}_\Gamma(\Gamma)] d\Gamma = (\mathcal{N}_{\Gamma\beta}! \cdot h^{s_{\Gamma\beta}} \cdot \mathcal{C}_{\Gamma\beta})^{-1}. \end{aligned}$$

The partition function Z_β for the new ensembles $\text{GOE}(N, \Gamma)$, $\text{GUE}(N, \Gamma)$, $\text{GSE}(N, \Gamma)$, can be easily calculated and it reads:

$$Z_\beta = (\mathcal{N}_{\Gamma\beta}! \cdot h^{s_{\Gamma\beta}})^{-1} \cdot \left(\frac{2\pi}{\beta}\right)^{\mathcal{N}_{\Gamma\beta}} \cdot \left(\frac{K}{M}\right)^{\mathcal{N}_{\Gamma\beta}/2}. \quad (22)$$

It follows that the Helmholtz's free energy F_β is equal to:

$$F_\beta = -\frac{1}{\beta} \ln Z_\beta = -\frac{1}{\beta} \ln[(\mathcal{N}_{\Gamma\beta}! \cdot h^{s_{\Gamma\beta}})^{-1} \cdot \left(\frac{2\pi}{\beta}\right)^{\mathcal{N}_{\Gamma\beta}} \cdot \left(\frac{K}{M}\right)^{\mathcal{N}_{\Gamma\beta}/2}]. \quad (23)$$

The free energy does not depend on the volume \mathcal{V}_X , hence the pressure vanishes:

$$p_\beta = -\frac{\partial F_\beta}{\partial \mathcal{V}_X} = 0, \quad (24)$$

and Eq. (24) is equation of state, where $\mathcal{V}_X = \int dX$ is the volume of the subset of X -coordinate space to which the Hamiltonians X are confined. Entropy of the system S_β is proportional to Boltzmann's H function of the condition of the system and it is the ensemble average of negative η_β of phase operator:

$$\begin{aligned} S_\beta &= -k_B \mathbb{H} = \langle \eta_\beta \rangle, \\ \eta_\beta &= -k_B \ln f_\Gamma(\Gamma), \\ S_\beta &= \int (-k_B \ln f_\Gamma(\Gamma)) f_\Gamma(\Gamma) d\Gamma = k_B \ln Z_\beta^{\text{ID}} + k_B \mathcal{N}_{\Gamma\beta}. \end{aligned} \quad (25)$$

It follows that the intrinsic energy U_β is equal to:

$$U_\beta = \langle \mathcal{H}_\Gamma \rangle = F_\beta + T S_\beta = F_\beta + \frac{1}{\beta k_B} S_\beta = \frac{\mathcal{N}_{\Gamma\beta}}{\beta}. \quad (26)$$

The enthalpy $H_{X\beta} = U_\beta + p_\beta V$, and Gibbs's free energy $G_\beta = F_\beta + p_\beta V$, are given by formulae:

$$H_{X\beta} = U_\beta, G_\beta = F_\beta. \quad (27)$$

The averaged square $\langle \mathcal{H}_\Gamma^2 \rangle$ of "classical" energy and the variance $\text{Var}(\mathcal{H}_\Gamma)$ read:

$$\begin{aligned} \langle \mathcal{H}_\Gamma^2 \rangle &= \mathcal{N}_{\Gamma\beta} \cdot (\mathcal{N}_{\Gamma\beta} + 1) \cdot \beta^{-2}, \\ \text{Var}(\mathcal{H}_\Gamma) &= \langle \mathcal{H}_\Gamma^2 \rangle - \langle \mathcal{H}_\Gamma \rangle^2 = \mathcal{N}_{\Gamma\beta} \cdot \beta^{-2}. \end{aligned}$$

Finally, the heat capacity at constant pressure C_p , the heat capacity at constant volume C_V , and isentropic exponent (polytropic exponent) κ are:

$$\begin{aligned} C_p &= \mathcal{N}_{\Gamma\beta} \cdot k_B, \\ C_V &= \mathcal{N}_{\Gamma\beta} \cdot k_B, \\ \kappa &= \frac{C_p}{C_V} = 1. \end{aligned} \quad (28)$$

Secondly, we study the example of ideal gas in the phase space W_Γ of pairs of quantum random matrices. For that case the "classical" potential energy $\mathcal{U}_\Gamma^{\text{ID}}(X, P)$ vanishes:

$$\mathcal{H}_\Gamma^{\text{ID}}(X, P) = \mathcal{T}_\Gamma^{\text{ID}}(P), \mathcal{U}_\Gamma^{\text{ID}}(X, P) = 0. \quad (29)$$

The three ensembles with the "classical" Hamiltonian given by Eq. (29) will be denoted as follows IDEAL(N, β, Γ), IDEAL(N, β, Γ), IDEAL(N, β, Γ).

Hence, $f_\Gamma(\Gamma)$ Eq.(20) is Maxwell-Boltzmann's distribution with vanishing potential energy:

$$\begin{aligned} f_\Gamma^{\text{ID}}(\Gamma) &= \mathcal{C}_{\Gamma\beta} \cdot \exp[-\beta \cdot \mathcal{H}_\Gamma^{\text{ID}}(\Gamma)] = \mathcal{C}_{\Gamma\beta} \cdot \exp[-\beta \cdot \mathcal{T}_\Gamma(\Gamma)] = \quad (30) \\ &= \mathcal{C}_{\Gamma\beta} \cdot \exp\left[-\beta \cdot \frac{1}{2M} \cdot \|P\|^2\right], \\ \int f_\Gamma^{\text{ID}}(\Gamma) dX \cdot dP &= 1. \end{aligned}$$

The partition function Z_β^{ID} for the ideal gas reads:

$$Z_\beta^{\text{ID}} = (\mathcal{N}_{\Gamma\beta}! \cdot h^{s_{\Gamma\beta}})^{-1} \cdot \left(\frac{2\pi M}{\beta}\right)^{\mathcal{N}_{\Gamma\beta}/2} \cdot \mathcal{V}_X^{\mathcal{N}_{\Gamma\beta}}. \quad (31)$$

It implies that the Helmholtz's free energy F_β^{ID} equals:

$$F_\beta^{\text{ID}} = -\frac{1}{\beta} \ln Z_\beta^{\text{ID}} = -\frac{1}{\beta} \ln\left[(\mathcal{N}_{\Gamma\beta}! \cdot h^{s_{\Gamma\beta}})^{-1} \cdot \left(\frac{2\pi M}{\beta}\right)^{\mathcal{N}_{\Gamma\beta}/2} \cdot \mathcal{V}_X^{\mathcal{N}_{\Gamma\beta}}\right]. \quad (32)$$

For the ideal gas, the free energy depends on the volume \mathcal{V}_X , hence the pressure is:

$$p_\beta^{\text{ID}} = -\frac{\partial F_\beta^{\text{ID}}}{\partial \mathcal{V}_X} = \frac{1}{\beta \mathcal{V}_X} \cdot \mathcal{N}_{\Gamma\beta}, \quad (33)$$

and Eq. (33) is equation of state. We observe that the entropy of the gas S_β^{ID} is:

$$S_\beta^{\text{ID}} = k_B \ln Z_\beta^{\text{ID}} + k_B \frac{1}{2} \mathcal{N}_{\Gamma\beta}. \quad (34)$$

These results readily lead to the formula for intrinsic energy U_β^{ID} :

$$U_\beta^{\text{ID}} = \frac{\mathcal{N}_{\Gamma\beta}}{2\beta}. \quad (35)$$

Consequently, the enthalpy $H_{X\beta}^{\text{ID}}$, and Gibbs's free energy G_β^{ID} , are given by formulae:

$$H_{X\beta}^{\text{ID}} = \frac{3\mathcal{N}_{\Gamma\beta}}{2\beta}, G_\beta^{\text{ID}} = \frac{1}{\beta} \cdot [-\ln Z_\beta^{\text{ID}} + \mathcal{N}_{\Gamma\beta}]. \quad (36)$$

Clearly, the averaged square $\langle (\mathcal{H}_\Gamma^{\text{ID}})^2 \rangle$ of "classical" energy and the variance $\text{Var}(\mathcal{H}_\Gamma^{\text{ID}})$ are:

$$\begin{aligned} \langle (\mathcal{H}_\Gamma^{\text{ID}})^2 \rangle &= \frac{1}{2} \cdot \mathcal{N}_{\Gamma\beta} \cdot \left(\frac{1}{2} \cdot \mathcal{N}_{\Gamma\beta} + 1\right) \cdot \beta^{-2}, \\ \text{Var}(\mathcal{H}_\Gamma^{\text{ID}}) &= \frac{1}{2} \cdot \mathcal{N}_{\Gamma\beta} \cdot \beta^{-2}. \end{aligned}$$

Immediately, we have that the heat capacities C_p^{ID} , C_V^{ID} , and isentropic exponent κ^{ID} are:

$$\begin{aligned} C_p^{\text{ID}} &= \frac{3}{2} \mathcal{N}_{\Gamma\beta} \cdot k_B, \\ C_V^{\text{ID}} &= \frac{1}{2} \cdot \mathcal{N}_{\Gamma\beta} \cdot k_B, \\ \kappa^{\text{ID}} &= 3. \end{aligned} \tag{37}$$

3 The Maximum Entropy Principle

In order to derive the probability distribution in matrix phase space W_Γ we apply the maximum entropy principle:

$$\max\{S_\beta(f_\Gamma) : \langle 1 \rangle = 1, \langle \mathcal{H}_\Gamma \rangle = U_\beta\}, \tag{38}$$

which yields:

$$\max\left\{\int (-k_B \ln f_\Gamma(\Gamma)) f_\Gamma(\Gamma) d\Gamma : \int f_\Gamma(\Gamma) d\Gamma = 1, \int \mathcal{H}_\Gamma(\Gamma) f_\Gamma(\Gamma) d\Gamma = U_\beta\right\}, \tag{39}$$

The maximization of entropy S_β under two additional constraints of normalization of the probability density function, and of equality of its first momentum and intrinsic energy, is equivalent to the minimization of the following functional $\mathcal{F}(f_\Gamma)$ with the use of Lagrange multipliers α_1, β_1 :

$$\begin{aligned} \min\{\mathcal{F}(f_\Gamma)\}, \\ \mathcal{F}(f_\Gamma) = \int (k_B \ln f_\Gamma(\Gamma)) f_\Gamma(\Gamma) d\Gamma + \alpha_1 \int f_\Gamma(\Gamma) d\Gamma + \beta_1 \int \mathcal{H}_\Gamma(\Gamma) f_\Gamma(\Gamma) d\Gamma. \end{aligned} \tag{40}$$

It follows, that the first variational derivative of $\mathcal{F}(f_\Gamma)$ must vanish:

$$\frac{\delta \mathcal{F}(f_\Gamma)}{\delta f_\Gamma} = 0, \tag{41}$$

which produces:

$$k_B(\ln f_\Gamma(\Gamma) + 1) + \alpha_1 + \beta_1 \mathcal{H}_\Gamma(\Gamma) = 0, \tag{42}$$

and equivalently:

$$\begin{aligned} f_\Gamma(\Gamma) &= \mathcal{C}_{\Gamma\beta} \cdot \exp[-\beta \cdot \mathcal{H}_\Gamma(\Gamma)] \\ \mathcal{C}_{\Gamma\beta} &= \exp[-(\alpha_1 + 1) \cdot k_B^{-1}], \beta = \beta_1 \cdot k_B^{-1}. \end{aligned} \tag{43}$$

The variational principle of maximum entropy does not force additional condition on functional form of energy $\mathcal{H}_\Gamma(\Gamma)$. Therefore, the distribution Eq. (43) defines a very large class of random matrix ensembles in phase space of generalized matrix coordinates and matrix momenta. The β parameter can assume any value. We can perform threefold restriction: either β is equal to 1, 2, 4, or $\mathcal{H}_\Gamma(\Gamma)$ is given by Eqs (19), (21), or finally both conditions are fulfilled. In the latter case we regain new Gaussian ensembles in phase space Eq. (18). In order to conclude, the derivation of the probability density function Eq. (43) is a new approach in Random Matrix Theory, since it defines a huge class of ensembles of direct sums of quantum operators of generalized coordinates and momenta in the new matrix phase space which are distributed according to classical continuous probability density. The ordinary Lie's groups of symmetries of both the probability densities and of the Haar's measures are extended to the direct sums of Lie's groups of symmetries. The studied new ensembles of random matrices describe one-dimensional nonideal gas with quadratic potential of quantum operators and ideal gas of quantum operators.

4 Acknowledgements

It is my pleasure to deeply thank Professor Antoni Ostoj-Gajewski for his continuous help.

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