Ordering results of extreme order statistics from multiple-outlier scale models with dependence

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Abstract: In this paper, we focus on stochastic comparisons of extreme order statistics stemming from multiple-outlier scale models with dependence. Archimedean copula is used to model dependence structure among nonnegative random variables. Sufficient conditions are obtained for comparison of the largest order statistics in the sense of the usual stochastic, reversed hazard rate, star and Lorenz orders. The smallest order statistics are also compared with respect to the usual stochastic, hazard rate, star and Lorenz orders. To illustrate the theoretical establishments, some examples are provided.

Keywords: Archimedean copula; Majorization; Stochastic orders; Star order; Multipleoutlier model.

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1. Introduction

Order statistics play a vital role in many fields such as statistical inference, economics, reliability theory and operations research. Consider a random sample X_1, \dots, X_n from a population. Then, the *i*th order statistic is denoted by $X_{i:n}$, where $i = 1, \dots, n$. In reliability theory, the *i*th order statistic represents the lifetime of a (n - i + 1)-out-of-*n* system, which functions if at least n - i + 1 of *n* components work. In particular, the order statistics $X_{1:n}$ and $X_{n:n}$ represent the lifetimes of series and parallel systems, respectively. Due to the correspondence between the order statistics and the systems' reliability, a lot of effort has been put to study ordering results between order statistics in terms of many well known stochastic orders. In this paper, we deal with the comparison of extreme order statistics arising from dependent multiple-outlier scale models in the sense of the usual stochastic, reversed hazard rate, hazard rate, star and Lorenz orders. A multiple-outlier model is a collection of random variables X_1, \dots, X_n such that $X_i \stackrel{st}{=} X$, $i = 1, \dots, p$ and $X_i = Y$, $i = p + 1, \dots, n$, where $1 \leq p < n$. Here, the notation $X_i \stackrel{st}{=} X$ means that the distributions of X_i and X are same. Due to the robustness of different estimators of model parameters, multiple-outlier models have been

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widely used by many researchers. Now, we present some developments on stochastic comparisons between order statistics arising from multiple-outlier models. Kochar and Xu (2011) considered multiple-outlier exponential models. They showed that more heterogeneity among the scale parameters of the model results more skewed order statistics. Zhao and Balakrishnan (2012) took similar model and obtained ordering results between the largest order statistics with respect to the likelihood ratio, reversed hazard rate, hazard rate and usual stochastic orderings. Zhao and Balakrishnan (2015) discussed stochastic comparisons of the largest order statistics from multiple-outlier gamma models in terms of various stochastic orderings such as the likelihood ratio, hazard rate, star and dispersive orders. Kochar and Torrado (2015) established likelihood ratio ordering between the largest order statistics arising from independent multiple-outlier scale models. Sufficient conditions for the comparison of the lifetimes of series systems with respect to dispersive order have been obtained by Fang et al. (2016). They considered that the components of the series systems follow multiple-outlier Weibull models. Amini-Seresht et al. (2016) studied multiple-outlier proportional hazard rate models and developed ordering results with respect to the star, Lorenz and dispersive orders. Further, they proved that more heterogeneity among the multiple-outlier components led to a more skewed lifetime of a k-out-of-n system consisting of these components. Wang and Cheng (2017) studied an open problem on mean residual life ordering between two parallel systems under multiple-outlier exponential models which was proposed by Balakrishnan and Zhao (2013).

Let $\{X_1, \dots, X_p, X_{p+1}, \dots, X_n\}$ be a collection of n independent random variables following the multiple-outlier exponential model, where X_i , $i = 1, \dots, p$ follow exponential distribution with parameter λ_1 and X_j , $j = p + 1, \dots, n$ follow exponential distribution with parameter λ_2 , with n = p + q. Further, let $\{Y_1, \dots, Y_{p^*}, Y_{p^*+1}, \dots, Y_{n^*}\}$ be a collection of n^* independent random variables following the multiple-outlier exponential model, where Y_i , $i = 1, \dots, p^*$ follow exponential distribution with parameter λ_1^* and Y_j , j = $p^* + 1, \dots, n^*$ follow exponential distribution with parameter λ_2^* , with $n^* = p^* + q^*$. Denote the largest order statistics by $X_{n:n}(p,q)$ and $Y_{n^*:n^*}(p^*,q^*)$ arising from $\{X_1, \dots, X_p, X_{p+1}, \dots, X_n\}$ and $\{Y_1, \dots, Y_{p^*}, Y_{p^*+1}, \dots, Y_{n^*}\}$, respectively. Under this set-up, Balakrishnan and Torrado (2016) obtained conditions under which the likelihood ratio order holds between $X_{n:n}(p,q)$ and $Y_{n^*:n^*}(p^*,q^*)$. In particular, for $(p^*,q^*) \succeq^w(p,q)$, they showed that

$$\underbrace{(\lambda_1,\cdots,\lambda_1}_p,\underbrace{\lambda_2,\cdots,\lambda_2}_q)\succeq^w(\underbrace{\lambda_1^*,\cdots,\lambda_1^*}_p,\underbrace{\lambda_2^*,\cdots,\lambda_2^*}_q)\Rightarrow X_{n:n}(p,q)\geq_{lr}Y_{n^*:n^*}(p^*,q^*),\qquad(1.1)$$

when $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$ and $1 \leq p^* \leq p \leq q \leq q^*$. The authors also extended the result given by (1.1) to the case of the proportional hazard rate models. Recently, Torrado (2017) developed the comparison result similar to (1.1) for the multiple-outlier scale models when the random variables are independent. It is noted that almost all concerned research in this area has been developed under the assumption of statistically independent component lifetimes. However, there are some practical situations, where the condition of statistically mutual independence among the component lifetimes is evidently unsuitable. For an example, let us consider a mechanical system. The components of the system are suffering a common stress. Then, it is of huge interest to include statistical dependence among component lifetimes into the study of stochastic comparison of the lifetimes of the series and parallel systems. Further, note that due to complexity of working with the dependent random variables, marginal effort was put to the study of dependent multiple-outlier models by the researchers (see Navarro et al. (2018)). These are the main motivations to investigate ordering properties of the extreme order statistics arising from multiple-outlier dependent scale components. The dependency structure among the random variables is modeled by the concept of Archimedean copulas. We recall that a nonnegative random variable X with distribution function F_X is said to follow the scale model if there exists $\lambda > 0$ such that $F_X(x) = F(\lambda x)$, where F is the baseline distribution function and λ is the scale parameter.

In this paper, we will develop different ordering results between the largest as well as the smallest order statistics stemming from multiple-outlier dependent scale models with respect to several stochastic orderings such as the usual stochastic, hazard rate, reversed hazard rate, star and Lorenz orders. Let $\{X_1, \dots, X_{n_1^*}, X_{n_1^*+1}, \dots, X_{n^*}\}$ be a set of dependent and heterogeneous random observations. The observations are sharing a common Archimedean copula with generator ψ_1 and are taken from the multiple-outlier scale model, where for $i = 1, \dots, n_1^*, X_i \sim F_1(\lambda_1 x)$ and for $j = n_1^* + 1, \dots, n^*, X_j \sim F_2(\lambda_2 x)$, where $\lambda_1, \lambda_2 > 0$. Note that $F_1(.)$ and $F_2(.)$ are two different baseline distribution functions. Also, let $\{Y_1, \dots, Y_{n_1^*}, Y_{n_1^*+1}, \dots, Y_{n^*}\}$ be another set of dependent and heterogeneous random observations sharing a common Archimedean copula with generator ψ_2 , drawn from the multiple-outlier scale model, where for $i = 1, \dots, n_1^*, Y_i \sim F_1(\mu_1 x)$ and for $j = n_1^* + 1, \dots, n^*$, $Y_j \sim F_2(\mu_2 x)$, where $\mu_1, \mu_2 > 0$. Denote by r_1, \tilde{r}_1 and r_2, \tilde{r}_2 the hazard rate and reversed hazard rate functions for F_1 and F_2 , respectively. Further, denote $X_{n:n}(n_1, n_2), Y_{n^*:n^*}(n_1^*, n_2^*)$ and $X_{1:n}(n_1, n_2)$, $Y_{1:n^*}(n_1^*, n_2^*)$ are the largest and the smallest order statistics, respectively arising from $\{X_1, \dots, X_{n_1}, X_{n_1+1}, \dots, X_n\}$ and $\{Y_1, \dots, Y_{n_1^*}, Y_{n_1^*+1}, \dots, Y_{n^*}\}$, where $1 \le n_1 \le n_1$ $n_1^* \leq n_2^* \leq n_2$, $n = n_1 + n_2$ and $n^* = n_1^* + n_2^*$. We aim to establish sufficient conditions, under which the following implications hold:

$$\underbrace{(\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\cdots,\lambda_2}_{n_2^*})}_{n_1^*} \succeq \underbrace{(\underbrace{\mu_1,\cdots,\mu_1}_{n_1^*},\underbrace{\mu_2,\cdots,\mu_2}_{n_2^*})}_{n_1^*} \Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*) \leq_{st} [\leq_{rh}]X_{n:n}(n_1,n_2),$$

$$\underbrace{(\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\cdots,\lambda_2}_{n_2^*})}_{n_1^*} \succeq \underbrace{(\underbrace{\mu_1,\cdots,\mu_1}_{n_1^*},\underbrace{\mu_2,\cdots,\mu_2}_{n_2^*})}_{n_2^*} \Rightarrow X_{1:n}(n_1,n_2) \leq_{st} Y_{1:n^*}(n_1^*,n_2^*)$$

and

$$(\underbrace{u_1, \cdots, u_1}_{n_1^*}, \underbrace{u_2, \cdots, u_2}_{n_2^*}) \succeq_w (\underbrace{v_1, \cdots, v_1}_{n_1^*}, \underbrace{v_2, \cdots, v_2}_{n_2^*}) \Rightarrow X_{1:n}(n_1, n_2) \leq_{hr} Y_{1:n^*}(n_1^*, n_2^*)$$

where $u_i = \log \lambda_i$ and $v_i = \log \mu_i$, i = 1, 2.

The remainder of the paper is rolled out as follows. Some basic definitions and important lemmas are provided in Section 2. Section 3 consists of two subsections. In Subsection 3.1, we obtain sufficient conditions, under which two largest order statistics are comparable according to the usual stochastic order, reversed hazard rate order, star order and Lorenz order, whereas in Subsection 3.2, we study the usual stochastic order, hazard rate order, star order and Lorenz order between two smallest order statistics. We also present some examples to illustrate the established results. Finally, we conclude the paper in Section 4.

Throughout the paper, we only concern about nonnegative random variables. Increasing and decreasing mean nondecreasing and nonincreasing, respectively. Also, the prime $\prime\prime$ stands for the first order derivative.

2. Basic notions

In this section, we recall some basic definitions and well known concepts of stochastic orders and majorization. Let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ be two *n*-dimensional vectors such that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{A}$, where $\mathbb{A} \subset \mathbb{R}^n$ and \mathbb{R}^n is an *n*-dimensional Euclidean space. Also, consider the order coordinates of the vectors \boldsymbol{x} and \boldsymbol{y} as $x_{1:n} \leq \dots \leq x_{n:n}$ and $y_{1:n} \leq \dots \leq y_{n:n}$, respectively.

Definition 2.1. A vector \boldsymbol{x} is said to be

- majorized by another vector \boldsymbol{y} , (denoted by $\boldsymbol{x} \preceq^m \boldsymbol{y}$), if for each $l = 1, \cdots, n-1$, we have $\sum_{i=1}^l x_{i:n} \ge \sum_{i=1}^l y_{i:n}$ and $\sum_{i=1}^n x_{i:n} = \sum_{i=1}^n y_{i:n}$;
- weakly submajorized by another vector \boldsymbol{y} , denoted by $\boldsymbol{x} \preceq_w \boldsymbol{y}$, if for each $l = 1, \dots, n$, we have $\sum_{i=l}^n x_{i:n} \leq \sum_{i=l}^n y_{i:n}$;
- weakly supermajorized by another vector \boldsymbol{y} , denoted by $\boldsymbol{x} \preceq^w \boldsymbol{y}$, if for each $l = 1, \dots, n$, we have $\sum_{i=1}^{l} x_{i:n} \geq \sum_{i=1}^{l} y_{i:n}$.

Note that $\mathbf{x} \leq^m \mathbf{y}$ implies both $\mathbf{x} \leq_w \mathbf{y}$ and $\mathbf{x} \leq^w \mathbf{y}$. For brief introduction of majorization orders and their applications, we refer to Marshall et al. (2011). Now, we present notions of stochastic orderings. Let X_1 and X_2 be two nonnegative random variables with probability density functions (PDFs) f_{X_1} and f_{X_2} , cumulative density functions (CDFs) F_{X_1} and F_{X_2} , survival functions $\bar{F}_{X_1} = 1 - F_{X_1}$ and $\bar{F}_{X_2} = 1 - F_{X_2}$, hazard rate functions $r_{X_1} = f_{X_1}/\bar{F}_{X_1}$ and $r_{X_2} = f_{X_2}/\bar{F}_{X_2}$ and reversed hazard rate functions $\tilde{r}_{X_1} = f_{X_1}/F_{X_1}$ and $\tilde{r}_{X_2} = f_{X_1}/F_{X_2}$, respectively.

Definition 2.2. A random variable X_1 is said to be smaller than X_2 in the

- hazard rate order (denoted by $X_1 \leq_{hr} X_2$) if $r_{X_1}(x) \geq r_{X_2}(x)$, for all x > 0;
- reversed hazard rate order (denoted by $X_1 \leq_{rh} X_2$) if $\tilde{r}_{X_1}(x) \leq \tilde{r}_{X_2}(x)$, for all x > 0;
- usual stochastic order (denoted by $X_1 \leq_{st} X_2$) if $\overline{F}_{X_1}(x) \leq \overline{F}_{X_2}(x)$, for all x;
- star order (denoted by $X_1 \leq_* X_2$ or $F_{X_1}(x) \leq_* F_{X_2}(x)$) if $F_{X_2}^{-1}F_{X_1}(x)$ is star shaped in the sense that $\frac{F_{X_2}^{-1}F_{X_1}(x)}{x}$ is increasing in x on the support of X_1 ;
- Lorenz order (denoted by $X_1 \leq_{Lorenz} X_2$) if

$$\frac{1}{E(X_1)} \int_0^{F_{X_1}^{-1}(u)} x dF_{X_1}(x) \ge \frac{1}{E(X_2)} \int_0^{F_{X_2}^{-1}(u)} x dF_{X_2}(x), \text{ for all } u \in (0,1].$$

Note that both the hazard rate and reversed hazard rate orderings imply the usual stochastic ordering. Also, star order implies Lorenz order (see Marshall and Olkin (2007)). One may refer to Shaked and Shanthikumar (2007) for a detailed discussion on various stochastic orderings. The next definition is for the Schur-convex and Schur-concave functions.

Definition 2.3. A function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex (Schur-concave) in \mathbb{R}^n if

$$\boldsymbol{x} \succeq^{m} \boldsymbol{y} \Rightarrow \Psi(\boldsymbol{x}) \ge (\le) \Psi(\boldsymbol{y}), \text{ for all } \boldsymbol{x}, \ \boldsymbol{y} \in \mathbb{R}^{n}.$$

Throughout the article, we will use the notations. (i) $\mathcal{D}_+ = \{(x_1, \dots, x_n) : x_1 \geq x_2 \geq \dots \geq x_n > 0\}$ and (ii) $\mathcal{E}_+ = \{(x_1, \dots, x_n) : 0 < x_1 \leq x_2 \leq \dots \leq x_n\}$. Denote by $h'(z) = \frac{dh(z)}{dz}$. The following consecutive lemmas due to Kundu et al. (2016) are useful to prove the results in the subsequent sections. The partial derivative of h with respect to its kth argument is denoted by $h_{(k)}(z) = \partial h(z)/\partial z_k$, for $k = 1, \dots, n$.

Lemma 2.1. Let $h : \mathcal{D}_+ \to \mathbb{R}$ be a function, continuously differentiable on the interior of \mathcal{D}_+ . Then, for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}_+$,

$$\boldsymbol{x} \succeq^{m} \boldsymbol{y} \text{ implies } h(\boldsymbol{x}) \ge (\le) h(\boldsymbol{y}),$$

if and only if $h_{(k)}(z)$ is decreasing (increasing) in $k = 1, \dots, n$.

Lemma 2.2. Let $h : \mathcal{E}_+ \to \mathbb{R}$ be a function, continuously differentiable on the interior of \mathcal{E}_+ . Then, for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}_+$,

 $\boldsymbol{x} \succeq^{m} \boldsymbol{y} \text{ implies } h(\boldsymbol{x}) \geq (\leq) h(\boldsymbol{y}),$

if and only if $h_{(k)}(\boldsymbol{z})$ is increasing (decreasing) in $k = 1, \cdots, n$.

The following lemma due to Saunders and Moran (1978) is useful to establish star order between the order statistics.

Lemma 2.3. Let $\{F_{\lambda}|\lambda \in \mathbb{R}\}$ be a class of distribution functions, such that F_{λ} is supported on some interval $(a,b) \subseteq (0,\infty)$ and has density f_{λ} which does not vanish on any subinterval of (a,b). Then,

$$F_{\lambda} \leq_* F_{\lambda^*}, \quad \lambda \leq \lambda^*$$

if and only if

$$\frac{F'_{\lambda}(x)}{xf_{\lambda}(x)}$$
 is decreasing in x,

where F'_{λ} is the derivative of F_{λ} with respect to λ .

To model the dependency structure among the random variables, the concept of copulas plays a vital role. One of the important characteristics of the copula is that it involves the information of the dependencies between the random variables apart from the behavior of the marginal distributions. Archimedean copulas are important class of copulas. These are used widely because of its simplicity. Let F and \overline{F} be the joint distribution function and the joint survival function of the random vector $\mathbf{X} = (X_1, \dots, X_n)$. Suppose there exist functions $C(\mathbf{z}) : [0, 1]^n \to [0, 1]$ and $\hat{C}(\mathbf{z}) : [0, 1]^n \to [0, 1]$ such that for all $x_i, i \in \mathcal{I}_n$, where \mathcal{I}_n is the index set

$$F(x_1,\cdots,x_n)=C(F_1(x_1),\cdots,F_n(x_n))$$

and

$$\bar{F}(x_1,\cdots,x_n)=\hat{C}(\bar{F}_1(x_1),\cdots,\bar{F}_n(x_n))$$

hold, where $\mathbf{z} = (z_1, \cdots, z_n)$. Then, $C(\mathbf{z})$ and $\hat{C}(\mathbf{z})$ are said to be the copula and survival copula of \mathbf{X} , respectively. Here, F_1, \cdots, F_n and $\bar{F}_1, \cdots, \bar{F}_n$ are the univariate marginal distribution functions and survival functions of the random variables X_1, \cdots, X_n , respectively. Now, let $\psi : [0, \infty) \to [0, 1]$ be a nonincreasing and continuous function, satisfying $\psi(0) = 1$ and $\psi(\infty) = 0$. Also, let $\psi = \phi^{-1} = \sup\{x \in \mathcal{R} : \phi(x) > v\}$ be the right continuous inverse. Further, suppose ψ satisfies the conditions $(i) \ (-1)^i \psi^i(x) \ge 0, \ i = 0, \ 1, \cdots, d-2$ and $(ii) \ (-1)^{d-2} \psi^{d-2}$ is nonincreasing and convex. That implies the generator ψ is d-monotone. Then, a copula C_{ψ} is said to be an Archimedean copula if it can be written as the following form

$$C_{\psi}(v_1, \cdots, v_n) = \psi(\phi(v_1), \cdots, \phi(v_n)), \text{ for all } v_i \in [0, 1], i \in \mathcal{I}_n.$$

For further discussion on Archimedean copulas, one may refer to Nelsen (2006) and McNeil and Nešlehová (2009).

Next lemma is taken from Li and Fang (2015), which has been used to prove the results in Theorems 3.1, 3.3, 3.8 and 3.10.

Lemma 2.4. For two n-dimensional Archimedean copulas C_{ψ_1} and C_{ψ_2} , if $\phi_2 \circ \psi_1$ is superadditive, then $C_{\psi_1}(\boldsymbol{z}) \leq C_{\psi_2}(\boldsymbol{z})$, for all $\boldsymbol{z} \in [0, 1]^n$. A function f is said to be super-additive, if $f(x) + f(y) \leq f(x+y)$, for all x and y in the domain of f.

3. Main Results

This section is completely devoted to establish sufficient conditions, under which the extreme order statistics arising from multiple outlier dependent scale models are comparable in different stochastic senses. The usual stochastic, hazard rate, reversed hazard rate, star and Lorenz orders are used in this sequel. Throughout this section, we denote two dimensional vectors by bold symbols. For example, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)$.

3.1 Orderings between the largest order statistics

This subsection addresses ordering results between the largest order statistics arising from multiple-outlier models. The following three consecutive theorems present different conditions, for which the usual stochastic order between the largest order statistics holds. Before presenting the first result, we state the following assumption.

Assumption 3.1. Let X_1, \dots, X_{n^*} be n^* dependent nonnegative random variables sharing Archimedean copula with generator ψ_1 , with $X_i \sim F_1(x\lambda_1)$, for $i = 1, \dots, n_1^*$ and $X_j \sim F_2(x\lambda_2)$, for $j = n_1^* + 1, \dots, n^*$. Also, let Y_1, \dots, Y_{n^*} be n^* dependent non-negative random variables sharing Archimedean copula with generator ψ_2 , with $Y_i \sim F_1(x\mu_1)$, for $i = 1, \dots, n_1^*$ and $Y_j \sim F_2(x\mu_2)$, for $j = n_1^* + 1, \dots, n^*$. Here, $n_1^* + n_2^* = n^*$, $\psi_1 = \phi_1^{-1}$ and $\psi_2 = \phi_2^{-1}$. **Theorem 3.1.** Under the set-up as in Assumption 3.1, let $\tilde{r}_1(x) \ge (\le)\tilde{r}_2(x)$ and $n_1^* \ge (\le)n_2^*$. Then,

$$(\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\cdots,\lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1,\cdots,\mu_1}_{n_1^*},\underbrace{\mu_2,\cdots,\mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*) \leq_{st} X_{n^*:n^*}(n_1^*,n_2^*),$$

provided λ , $\mu \in \mathcal{E}_+$ (\mathcal{D}_+) , $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex and $\tilde{r}_1(x)$ or $\tilde{r}_2(x)$ is decreasing.

Proof. The distribution functions of $X_{n^*:n^*}(n_1^*, n_2^*)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ are respectively given by

$$F_{X_{n^*:n^*}}(n_1^*, n_2^*)(x) = \psi_1 \left[n_1^* \phi_1 \left(F_1 \left(x \lambda_1 \right) \right) + n_2^* \phi_1 \left(F_2 \left(x \lambda_2 \right) \right) \right]$$

and

$$F_{Y_{n^*:n^*}}(n_1^*, n_2^*)(x) = \psi_2 \left[n_1^* \phi_2 \left(F_1 \left(x \mu_1 \right) \right) + n_2^* \phi_2 \left(F_2 \left(x \mu_2 \right) \right) \right].$$

Denote $A(\boldsymbol{\lambda}, \psi_1, x) = F_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)$ and $B(\boldsymbol{\mu}, \psi_2, x) = F_{Y_{n^*:n^*}}(n_1^*, n_2^*)(x)$. Using the fact that $\phi_2 \circ \psi_1$ is super-additive, one can easily obtain $A(\boldsymbol{\mu}, \psi_1, x) \leq B(\boldsymbol{\mu}, \psi_2, x)$. Therefore, to prove the desired result, we have to show that $A(\boldsymbol{\lambda}, \psi_1, x) \leq A(\boldsymbol{\mu}, \psi_1, x)$. This is equivalent to establish that the function $A(\boldsymbol{\lambda}, \psi_1, x)$ is increasing and Schur-concave with respect to $\boldsymbol{\lambda}$ (see Theorem A.8 of Marshall et al. (2011)). Further, on differentiating $A(\boldsymbol{\lambda}, \psi_1, x)$ with respect to λ_1 partially, we get

$$\frac{\partial A(\boldsymbol{\lambda},\psi_1,x)}{\partial \lambda_1} = x n_1^* \tilde{r}_1(x\lambda_1) \frac{\psi_1 \left[\phi_1 \left[F_1(x\lambda_1)\right]\right]}{\psi_1' \left[\phi_1 \left[F_1(x\lambda_1)\right]\right]} \psi_1' \left[n_1^* \phi_1 \left(F_1(x\lambda_1)\right) + n_2^* \phi_1 \left(F_2(x\lambda_2)\right)\right].$$
(3.1)

From (3.1), it is not difficult to check that $\frac{\partial A(\boldsymbol{\lambda},\psi_1,x)}{\partial \lambda_1} \geq 0$. Similarly, $\frac{\partial A(\boldsymbol{\lambda},\psi_1,x)}{\partial \lambda_2} \geq 0$. Thus, $A(\boldsymbol{\lambda},\psi_1,x)$ is increasing in λ_i , for i = 1, 2. To establish Schur-concavity of $A(\boldsymbol{\lambda},\psi_1,x)$, in view of Lemma 2.2 (Lemma 2.1), we only need to show that for $1 \leq i \leq j \leq n^*$, the following inequality holds:

$$\frac{\partial A(\boldsymbol{\lambda}, \psi_1, x)}{\partial \lambda_i} - \frac{\partial A(\boldsymbol{\lambda}, \psi_1, x)}{\partial \lambda_j} \ge (\le)0, \text{ for } \boldsymbol{\lambda} \in \mathcal{E}_+ (\mathcal{D}_+).$$
(3.2)

Next, consider three cases.

Case I: For $1 \le i \le j \le n_1^*$, $\lambda_i = \lambda_j = \lambda_1$. In this case, $\frac{\partial A(\lambda,\psi_1,x)}{\partial \lambda_i} - \frac{\partial A(\lambda,\psi_1,x)}{\partial \lambda_j} = 0$. Case II: For $n_1^* + 1 \le i \le j \le n^*$, $\lambda_i = \lambda_j = \lambda_2$. Here, $\frac{\partial A(\lambda,\psi_1,x)}{\partial \lambda_i} - \frac{\partial A(\lambda,\psi_1,x)}{\partial \lambda_j} = 0$. Case III: For $1 \le i \le n_1^*$ and $n_1^* + 1 \le j \le n^*$, $\lambda_i = \lambda_1$ and $\lambda_j = \lambda_2$. For this case, consider $\lambda_1 \le (\ge)\lambda_2$, which implies $\phi_1(F_1(x\lambda_1)) \ge (\le)\phi_1(F_1(x\lambda_2))$. Further, under the given assumption, we get $\phi_1(F_1(x\lambda_2)) \ge (\le)\phi_1(F_2(x\lambda_2))$. Hence, $\phi_1(F_1(x\lambda_1)) \ge (\le)\phi_1(F_2(x\lambda_2))$. Again, ψ_1 is log-convex. Therefore, we have

$$-\frac{\psi_1(w)}{\psi_1'(w)}\Big|_{w=\phi_1[F_1(x\lambda_1)]} \ge (\le) -\frac{\psi_1(w)}{\psi_1'(w)}\Big|_{w=\phi_1[F_2(x\lambda_2)]}.$$
(3.3)

Moreover, $\tilde{r}_1(w)$ is decreasing in w > 0, hence

$$\tilde{r}_1(x\lambda_1) \ge (\le)\tilde{r}_1(x\lambda_2). \tag{3.4}$$

Also, $\tilde{r}_1(x) \ge (\le)\tilde{r}_2(x)$ gives

$$\tilde{r}_1(x\lambda_2) \ge (\le)\tilde{r}_2(x\lambda_2). \tag{3.5}$$

Equations (3.4), (3.5) and $n_1^* \ge (\le) n_2^*$, together imply

$$n_1^* \tilde{r}_1(x\lambda_1) \ge (\le) n_2^* \tilde{r}_2(x\lambda_2). \tag{3.6}$$

Finally, combining (3.3) and (3.6), we obtain (3.2). This completes the proof of the theorem.

In the previous result, we assume that the dependence structures of two sets of samples having multiple-outliers are different. Also, first n_1^* observations of $\{X_1, \dots, X_{n_1^*}, X_{n_1^*+1}, \dots, X_{n^*}\}$ have baseline distribution function F_1 and remaining observations have baseline distribution function F_2 . Similarly, for the other set of observations $\{Y_1, \dots, Y_{n_1^*}, Y_{n_1^*+1}, \dots, Y_{n^*}\}$. The following corollary, which is a direct consequence of Theorem 3.1 presents some special cases.

Corollary 3.1. In addition to Assumption 3.1, let $\psi_1 = \psi_2 = \psi$, $n_1^* \ge (\le)n_2^*$ and ψ be log-convex. Further, let λ , $\mu \in \mathcal{E}_+$ (\mathcal{D}_+). Then,

- $(i) \quad \underbrace{(\lambda_1, \cdots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \cdots, \lambda_2}_{n_2^*}) \succeq^w \underbrace{(\mu_1, \cdots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \cdots, \mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n^*:n^*}(n_1^*, n_2^*),$ $provided \quad \tilde{r}_1(x) \quad or \quad \tilde{r}_2(x) \quad is \ decreasing \ and \quad \tilde{r}_1(x) \geq (\leq)\tilde{r}_2(x);$
- (ii) for $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$, we have

$$(\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\cdots,\lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1,\cdots,\mu_1}_{n_1^*},\underbrace{\mu_2,\cdots,\mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*) \leq_{st} X_{n^*:n^*}(n_1^*,n_2^*),$$

provided $\tilde{r}(x)$ is decreasing.

The next theorem states that the ordering result holds between the largest order statistics $X_{n:n}(n_1, n_2)$ and $X_{n^*:n^*}(n_1^*, n_2^*)$ according to the usual stochastic ordering. Here, the samples are collected from multiple-outlier dependent scale models. Also, it is assumed that the samples are sharing Archimedean copula with a common generator.

Assumption 3.2. Let X_1, \dots, X_{n^*} be n^* dependent nonnegative random variables sharing Archimedean copula with generator ψ_1 , such that $X_i \sim F_1(x\lambda_1)$, for $i = 1, \dots, n_1^*$ and $X_j \sim F_2(x\lambda_2)$, for $j = n_1^* + 1, \dots, n^*$. We assume that there exist two natural numbers n_1 and n_2 such that $1 \leq n_1 \leq n_1^* \leq n_2^* \leq n_2$. Also, $n = n_1 + n_2$, $n^* = n_1^* + n_2^*$ and $\psi_1 = \phi_1^{-1}$.

Theorem 3.2. Let Assumption 3.2 hold with $F_1 \ge F_2$. Then, for $\lambda \in \mathcal{D}_+$, we have

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n:n}(n_1, n_2)$$

Proof. The distribution functions of $X_{n:n}(n_1, n_2)$ and $X_{n^*:n^*}(n_1^*, n_2^*)$ can be written respectively as

$$F_{X_{n:n}}(n_1, n_2)(x) = \psi_1[n_1\phi_1(F_1(x\lambda_1)) + n_2\phi_1(F_2(x\lambda_2))]$$

and

$$F_{X_{n^*:n^*}}(n_1^*, n_2^*)(x) = \psi_1[n_1^*\phi_1(F_1(x\lambda_1)) + n_2^*\phi_1(F_2(x\lambda_2))].$$

To obtain the desired result, one needs to show $F_{X_{n:n}}(n_1, n_2)(x) \leq F_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)$. Equivalently, we have to establish that $(n_1^* - n_1)\phi_1(F_1(x\lambda_1)) \leq (n_2 - n_2^*)\phi_1(F_2(x\lambda_2))$. Now, $(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow (n_1 + n_2) \geq (n_1^* + n_2^*) \Rightarrow (n_2 - n_2^*) \geq (n_1^* - n_1) \geq 0$. Also, $\lambda_1 \geq \lambda_2 \Rightarrow \phi_1(F_2(x\lambda_2)) \geq \phi_1(F_1(x\lambda_1)) \geq 0$. Using these arguments, we get the required inequality. Hence, the proof is completed.

In Theorem 3.2, if we take the same baseline distribution, then the following corollary is immediate.

Corollary 3.2. Let Assumption 3.2 hold with $F_1 = F_2$. Then, for $\lambda \in \mathcal{D}_+$, we have

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n:n}(n_1, n_2).$$

Next, we observe that two largest order statistics $X_{n:n}(n_1, n_2)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ are comparable with respect to the usual stochastic order. It is worth mentioning that the order statistics are constructed from two multiple-outlier dependent samples having sample sizes nand n^* . The pairs of the sizes of both the outliers (n_1, n_2) and (n_1^*, n_2^*) are assumed to be connected according to the weakly submajorization order. The following assumption is useful for the next theorem.

Assumption 3.3. Let X_1, \dots, X_n be *n* nonnegative dependent random variables sharing Archimedean copula with generator ψ_1 , such that $X_i \sim F_1(x\lambda_1)$, for $i = 1, \dots, n_1$ and $X_j \sim F_2(x\lambda_2)$, for $j = n_1 + 1, \dots, n$. Also, let Y_1, \dots, Y_{n^*} be n^* dependent nonnegative random variables sharing Archimedean copula with generator ψ_2 , such that $Y_i \sim F_1(x\mu_1)$, for $i = 1, \dots, n_1^*$ and $Y_j \sim F_2(x\mu_2)$, for $j = n_1^* + 1, \dots, n^*$. Here, $1 \leq n_1 \leq n_1^* \leq n_2^* \leq n_2$, $n = n_1 + n_2$ and $n^* = n_1^* + n_2^*$.

Theorem 3.3. Assume that Assumption 3.3 hold with $\tilde{r}_1(x) \leq \tilde{r}_2(x)$. Let $(n_1, n_2) \succeq_w (n_1^*, n_2^*)$. Then,

$$(\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\cdots,\lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1,\cdots,\mu_1}_{n_1^*},\underbrace{\mu_2,\cdots,\mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*) \leq_{st} X_{n:n}(n_1,n_2),$$

provided λ , $\mu \in \mathcal{D}_+$, $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex and $\tilde{r}_1(x)$ or $\tilde{r}_2(x)$ is decreasing.

Proof. By Theorem 3.1, we have

$$(\underbrace{\lambda_1, \cdots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \cdots, \lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1, \cdots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \cdots, \mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n^*:n^*}(n_1^*, n_2^*).$$
(3.7)

Further, from Theorem 3.2, we have

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n:n}(n_1, n_2).$$
(3.8)

Upon combining inequalities given by (3.7) and (3.8), the required result readily follows. \Box

The result stated in Theorem 3.3 is general. However, if we take some restrictions on the generators of the Archimedean copula and on the cumulative distribution functions, then we get some particular results. These are presented in the following corollary, which follows from Theorem 3.3.

Corollary 3.3. Let the set-up in Assumption 3.3 hold with $(n_1, n_2) \succeq_w (n_1^*, n_2^*)$. Also, let $\lambda, \mu \in \mathcal{D}_+, \psi_1 = \psi_2 = \psi$ and ψ be log-convex. Then,

$$(i) \quad \underbrace{(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2)}_{\substack{n_1^* \\ provided}} \succeq \underbrace{(\mu_1, \cdots, \mu_1, \mu_2, \cdots, \mu_2)}_{\substack{n_1^* \\ n_1^*}} \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n:n}(n_1, n_2),$$

$$(ii) \quad \underbrace{(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2)}_{\substack{n_1^* \\ n_1^*}} \succeq \underbrace{(\mu_1, \cdots, \mu_1, \mu_2, \cdots, \mu_2)}_{\substack{n_1^* \\ n_1^*}} \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n:n}(n_1, n_2),$$

$$(ii) \quad \underbrace{(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2)}_{\substack{n_2^* \\ n_1^*}} \succeq \underbrace{(\mu_1, \cdots, \mu_1, \mu_2, \cdots, \mu_2)}_{\substack{n_2^* \\ n_2^*}} \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n:n}(n_1, n_2),$$

$$(ii) \quad \underbrace{(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2)}_{\substack{n_2^* \\ n_2^*}} \succeq \underbrace{(\mu_1, \cdots, \mu_1, \mu_2, \cdots, \mu_2)}_{\substack{n_2^* \\ n_2^*}} \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \leq_{st} X_{n:n}(n_1, n_2),$$

We now present a numerical example, which provides an illustration of Theorem 3.3.

Example 3.1. Set $\lambda = (\lambda_1, \lambda_2) = (5, 2)$, $\mu = (\mu_1, \mu_2) = (6, 3)$, $(n_1, n_2) = (1, 11)$, $(n_1^*, n_2^*) = (5, 6)$, $\psi_1(x) = e^{-x^{\frac{1}{9}}}$, $\psi_2(x) = e^{-x^{\frac{1}{10}}}$, x > 0. Consider the baseline distribution functions as $F_2(x) = 1 - e^{1-(1+x^2)^{\frac{1}{5}}}$ and $F_1(x) = 1 - e^{-x}$, x > 0. Here, both the reversed hazard rate functions \tilde{r}_1 and \tilde{r}_2 are decreasing and satisfy $\tilde{r}_1(x) \leq \tilde{r}_2(x)$, for x > 0. Further, ψ_1 and ψ_2 are log-convex, $\phi_2 \circ \psi_1$ is super-additive. Thus, all the conditions of Theorem 3.3 are satisfied. Now, we plot the graphs of $F_{X_{12:12}}(1, 11)(x)$ and $F_{Y_{11:11}}(5, 6)(x)$ in Figure 1a, which shows that $Y_{11:11}(5, 6) \leq_{st} X_{12:12}(1, 11)$ holds.



Figure 1: (a) Plots of the distribution functions $F_{X_{12:12}}(1,11)(x)$ and $F_{Y_{11:11}}(5,6)(x)$ as in Example 3.1. (b) Plot of $F_{X_{9:9}}(1,8)(x) - F_{Y_{7:7}}(3,4)(x)$ as in Counterexample 3.1.

Next, we present a counterexample to illustrate that the result does not hold if $\tilde{r}_1(x) \geq \tilde{r}_2(x)$ and $\lambda \in \mathcal{E}_+$ in Theorem 3.3.

Counterexample 3.1. Consider $\lambda = (\lambda_1, \lambda_2) = (2, 6), \ \mu = (\mu_1, \mu_2) = (8, 2), \ (n_1, n_2) = (1, 8), \ (n_1^*, n_2^*) = (3, 4), \ \psi_1(x) = e^{-x^{\frac{1}{3}}}, \ \psi_2(x) = e^{-x^{\frac{1}{10}}}, \ x > 0.$ Baseline distribution functions are taken as $F_1(x) = 1 - e^{-x}$ and $F_2(x) = 1 - (1 + 2x)^{-0.5}, \ x > 0.$ It can be seen that all the conditions of Theorem 3.3 are satisfied except $\lambda \in \mathcal{D}_+$ and $\tilde{r}_1(x) \leq \tilde{r}_2(x)$. Now, we plot the graph of $F_{X_{9:9}}(1, 8)(x) - F_{Y_{7:7}}(3, 4)(x)$ in Figure 1b, which reveals that $Y_{7:7}(3, 4) \leq x X_{9:9}(1, 8)$.

In the preceding theorems, we have derived sufficient conditions, under which the largest order statistics from multiple-outlier dependent scale models obey the usual stochastic order. However, naturally, it is of interest to extend the ordering results to some other stronger concepts of the stochastic orders. In this part of the subsection, we establish sufficient conditions, under which the reversed hazard rate order holds between the largest order statistics. The following theorem shows that the largest order statistics $X_{n^*:n^*}(n_1^*, n_2^*)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ have the reversed hazard rate ordering when the scale parameters are associated with the weakly supermajorization order. The samples are heterogeneous and follow multiple-outlier dependent scale models.

Theorem 3.4. Let Assumption 3.1 hold with $r_1 = r_2 = r$, $n_1^* \ge (\le) n_2^*$ and $\psi_1 = \psi_2 = \psi$. Also, suppose ψ is log-concave, $\frac{1-\psi}{\psi'}$ is decreasing and $\frac{1-\psi}{\psi'} [\frac{1-\psi}{\psi'}]'$ is increasing. Then,

$$(\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\cdots,\lambda_2}_{n_2^*})\succeq^w(\underbrace{\mu_1,\cdots,\mu_1}_{n_1^*},\underbrace{\mu_2,\cdots,\mu_2}_{n_2^*})\Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*)\leq_{rh}X_{n^*:n^*}(n_1^*,n_2^*),$$

provided λ , $\mu \in \mathcal{E}_+$ (\mathcal{D}_+), r(x) is decreasing and xr(x) is decreasing and convex.

Proof. Under the given assumption $r_1 = r_2 = r$ implies $F_1 = F_2 = F$. The reversed hazard rate function of $X_{n^*:n^*}(n_1^*, n_2^*)$ is

$$\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x) = \frac{\psi' [n_1^* \phi \left(F(x\lambda_1)\right) + n_2^* \phi \left(F(x\lambda_2)\right)]}{\psi [n_1^* \phi \left(F(x\lambda_1)\right) + n_2^* \phi \left(F(x\lambda_2)\right)]} \left[\frac{n_1^* \lambda_1 f(x\lambda_1)}{\psi' [\phi \left(F(x\lambda_1)\right)]} + \frac{n_2^* \lambda_2 f(x\lambda_2)}{\psi' [\phi \left(F(x\lambda_2)\right)]}\right]$$

$$= \frac{\psi' [n_1^* \phi \left(F(x\lambda_1)\right) + n_2^* \phi \left(F(x\lambda_2)\right)]}{\psi [n_1^* \phi \left(F(x\lambda_1)\right) + n_2^* \phi \left(F(x\lambda_2)\right)]} \left[\frac{n_1^* \lambda_1 r(x\lambda_1) [1 - \psi [\phi \left(F(x\lambda_1)\right)]]}{\psi' [\phi \left(F(x\lambda_1)\right)]} + \frac{n_2^* \lambda_2 r(x\lambda_2) [1 - \psi [\phi \left(F(x\lambda_2)\right)]]}{\psi' [\phi \left(F(x\lambda_2)\right)]}\right], \qquad (3.9)$$

where f is the probability density function corresponding to F. Denote $z = n_1^* \phi(F(x\lambda_1)) + n_2^* \phi(F(x\lambda_2))$. The partial derivative of $\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)$ with respect to λ_1 is obtained as

$$\frac{\partial [\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)]}{\partial \lambda_1} = n_1^* x r(x\lambda_1) \frac{d}{dz} \left[\frac{\psi'(z)}{\psi(z)} \right] \left[\frac{1 - \psi \left[\phi \left[F(x\lambda_1) \right] \right]}{\psi'[\phi \left(F(x\lambda_1) \right)]} \right] \left[\sum_{i=1}^{n^*} \frac{\lambda_i f(x\lambda_i)}{\psi'[\phi \left(F(x\lambda_i) \right)]} \right]
+ n_1^* x \lambda_1 \left[r(x\lambda_1) \right]^2 \frac{\psi'(z)}{\psi(z)} \left[\frac{1 - \psi(v)}{\psi'(v)} \frac{d}{dv} \left[\frac{1 - \psi(v)}{\psi'(v)} \right] \right]_{v = \phi(F(x\lambda_1))}
+ n_1^* \frac{d}{dw} \left[w r(w) \right]_{w = x\lambda_1} \left[\frac{1 - \psi \left[\phi \left[F(x\lambda_1) \right] \right]}{\psi'[\phi \left[F(x\lambda_1) \right] \right]} \right] \frac{\psi'(z)}{\psi(z)}.$$
(3.10)

Utilizing Theorem A.8 of Marshall et al. (2011), to obtain the desired result, we need to prove that $\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)$ is decreasing and Schur-convex with respect to λ . Using the given assumptions and Equation (3.10), the decreasing property of $\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)$ with respect to λ is obvious. Further, according to Lemma 2.2 (Lemma 2.1), to show Schur-convexity of $\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)$, we have to establish that for $1 \leq i \leq j \leq n^*$,

$$\left[\frac{\partial [\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)]}{\partial \lambda_i} - \frac{\partial [\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)]}{\partial \lambda_j}\right] \le (\ge)0, \text{ for } \boldsymbol{\lambda} \in \mathcal{E}_+ (\mathcal{D}_+).$$
(3.11)

Now, consider the following three cases.

Case I: For $1 \le i \le j \le n_1^*$, $\lambda_i = \lambda_j = \lambda_1$. Here, $\frac{\partial [\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)]}{\partial \lambda_i} - \frac{\partial [\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)]}{\partial \lambda_j} = 0$. Case II: For $n_1^* + 1 \le i \le j \le n^*$, $\lambda_i = \lambda_j = \lambda_2$. Hence, $\frac{\partial [\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)]}{\partial \lambda_i} - \frac{\partial [\tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)]}{\partial \lambda_j} = 0$. Case III: For $1 \le i \le n_1^*$ and $n_1^* + 1 \le j \le n^*$, $\lambda_i = \lambda_1$ and $\lambda_j = \lambda_2$. Consider $\lambda_1 \le \lambda_2$, which gives $\phi(F(x\lambda_1)) \ge \phi(F(x\lambda_2))$. Here, we only discuss the proof when $\lambda_1 \le \lambda_2$. The other case when $\lambda_1 \ge \lambda_2$ can be proved in the similar way. The concavity property of $\ln \psi$ provides $\frac{d}{dz} \left[\frac{\psi'(z)}{\psi(z)} \right] \le 0$. Again, using decreasing property of $\frac{1-\psi}{\psi'}$, we have

$$\frac{1 - \psi(w)}{\psi'(w)}\Big|_{w = \phi[F(x\lambda_1)]} \le \frac{1 - \psi(w)}{\psi'(w)}\Big|_{w = \phi[F(x\lambda_2)]} \le 0.$$
(3.12)

Further, it has been assumed that r(x) is decreasing, xr(x) is decreasing and convex. Therefore, using $n_1^* \ge n_2^*$, we have

$$r(x\lambda_1) \ge r(x\lambda_2),\tag{3.13}$$

$$n_1^* x \lambda_1 r(x \lambda_1) \ge n_2^* x \lambda_2 r(x \lambda_2)$$
 and (3.14)

$$n_1^* \frac{d}{dw} \left[wr(w) \right]_{w=x\lambda_1} \le n_2^* \frac{d}{dw} \left[wr(w) \right]_{w=x\lambda_2} \le 0.$$
(3.15)

Moreover, $\frac{1-\psi(w)}{\psi'(w)}\frac{d}{dw}\left[\frac{1-\psi(w)}{\psi'(w)}\right]$ is increasing. Therefore, we obtain the following inequality

$$\left[\frac{1-\psi(w)}{\psi'(w)}\frac{d}{dw}\left[\frac{1-\psi(w)}{\psi'(w)}\right]\right]_{w=\phi[F(x\lambda_1)]} \ge \left[\frac{1-\psi(w)}{\psi'(w)}\frac{d}{dw}\left[\frac{1-\psi(w)}{\psi'(w)}\right]\right]_{w=\phi[F(x\lambda_2)]} \ge 0.$$
(3.16)

Now, combining (3.12)-(3.16) and the given assumptions, we obtain that the inequality given by (3.11) holds. This completes the proof.

In the next theorem, we show that the largest order statistics $X_{n:n}(n_1, n_2)$ and $X_{n^*:n^*}(n_1^*, n_2^*)$ are comparable according to the reversed hazard rate order.

Theorem 3.5. Let Assumption 3.2 hold with $\psi_1 = \psi$ and $r_1 = r_2 = r$. Then, for $\lambda \in D_+$, we have

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{n^*:n^*}(n_1^*, n_2^*) \leq_{rh} X_{n:n}(n_1, n_2),$$

provided $\ln \psi$ is concave, $\frac{1-\psi}{\psi'}$ and xr(x) are decreasing.

Proof. The stated result will be proved, if we show that $\tilde{r}_{X_{n:n}}(n_1, n_2)(x) \geq \tilde{r}_{X_{n^*:n^*}}(n_1^*, n_2^*)(x)$. Equivalently,

$$\frac{\psi'\left[\sum_{i=1}^{n}\phi\left(F\left(x\lambda_{i}\right)\right)\right]}{\psi\left[\sum_{i=1}^{n}\phi\left(F\left(x\lambda_{i}\right)\right)\right]} \times \frac{\psi\left[\sum_{i=1}^{n^{*}}\phi\left(F\left(x\lambda_{i}\right)\right)\right]}{\psi'\left[\sum_{i=1}^{n^{*}}\phi\left(F\left(x\lambda_{i}\right)\right)\right]} \ge \frac{\sum_{i=1}^{n^{*}}\frac{\lambda_{i}r(x\lambda_{i})[1-\psi[\phi\left(F\left(x\lambda_{i}\right)\right)]]}{\psi'[\phi\left(F\left(x\lambda_{i}\right)\right)]]}}{\sum_{i=1}^{n}\frac{\lambda_{i}r(x\lambda_{i})[1-\psi[\phi\left(F\left(x\lambda_{i}\right)\right)]]}{\psi'[\phi\left(F\left(x\lambda_{i}\right)\right)]]}}.$$
(3.17)

The preceding inequality holds if the following two inequalities are satisfied,

$$\frac{\psi'\left[\sum_{i=1}^{n^*}\phi\left(F\left(x\lambda_i\right)\right)\right]}{\psi\left[\sum_{i=1}^{n^*}\phi\left(F\left(x\lambda_i\right)\right)\right]} \ge \frac{\psi'\left[\sum_{i=1}^{n}\phi\left(F\left(x\lambda_i\right)\right)\right]}{\psi\left[\sum_{i=1}^{n}\phi\left(F\left(x\lambda_i\right)\right)\right]} \Leftrightarrow (n_1^* - n_1)\phi(F(x\lambda_1)) \le (n_2 - n_2^*)\phi(F(x\lambda_2))$$
(3.18)

and

$$\sum_{i=1}^{n^*} \frac{\lambda_i r(x\lambda_i)[1 - \psi[\phi(F(x\lambda_i))]]}{\psi'[\phi(F(x\lambda_i))]} \geq \sum_{i=1}^n \frac{\lambda_i r(x\lambda_i)[1 - \psi[\phi(F(x\lambda_i))]]}{\psi'[\phi(F(x\lambda_i))]}$$

$$\Leftrightarrow (n_1^* - n_1) \frac{\lambda_1 r(x\lambda_1)[1 - \psi[\phi(F(x\lambda_1))]]}{\psi'[\phi(F(x\lambda_1))]} \geq (n_2 - n_2^*) \frac{\lambda_2 r(x\lambda_2)[1 - \psi[\phi(F(x\lambda_2))]]}{\psi'[\phi(F(x\lambda_2))]} (3.19)$$

Furthermore,

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow (n_1 + n_2) \ge (n_1^* + n_2^*) \Rightarrow (n_2 - n_2^*) \ge (n_1^* - n_1) \ge 0.$$
 (3.20)

Also,

$$\lambda_1 \ge \lambda_2 \Rightarrow \phi(F(x\lambda_2)) \ge \phi(F(x\lambda_1)) \ge 0.$$

Moreover, $\frac{1-\psi}{\psi'}$ is decreasing. Thus,

$$\frac{1 - \psi(w)}{\psi'(w)}\Big|_{w = \phi[F(x\lambda_2)]} \le \frac{1 - \psi(w)}{\psi'(w)}\Big|_{w = \phi[F(x\lambda_1)]} \le 0.$$
(3.21)

Using the decreasing property of xr(x), we have

$$x\lambda_1 r(x\lambda_1) \le x\lambda_2 r(x\lambda_2). \tag{3.22}$$

Combining Equations (3.20), (3.21) and (3.22), the inequality in (3.19) can be obtained. Using (3.20) and the assumption that ψ is log-concave, we get the inequality (3.18). Hence, the proof follows.

Now, we are ready to state a result which shows that the largest order statistics $X_{n:n}(n_1, n_2)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ can be compared with respect to the reversed hazard rate order. **Theorem 3.6.** Let the set-up in Assumption 3.3 hold with $\psi_1 = \psi_2 = \psi$ and $r_1 = r_2 = r$. Also, assume λ , $\mu \in \mathcal{D}_+$ and $(n_1, n_2) \succeq_w (n_1^*, n_2^*)$. Then,

$$(\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\cdots,\lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1,\cdots,\mu_1}_{n_1^*},\underbrace{\mu_2,\cdots,\mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*) \leq_{rh} X_{n:n}(n_1,n_2),$$

provided ψ is log-concave, $\frac{1-\psi}{\psi'}$ is decreasing, $\frac{1-\psi}{\psi'} [\frac{1-\psi}{\psi'}]'$ is increasing, xr(x) is decreasing, convex and r(x) is decreasing.

Proof. According to Theorem 3.4, we have

$$(\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\cdots,\lambda_2}_{n_2^*}) \succeq^w (\underbrace{\mu_1,\cdots,\mu_1}_{n_1^*},\underbrace{\mu_2,\cdots,\mu_2}_{n_2^*}) \Rightarrow Y_{n^*:n^*}(n_1^*,n_2^*) \leq_{rh} X_{n^*:n^*}(n_1^*,n_2^*).$$
(3.23)

Also, from Theorem 3.5, we get

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{n^*:n^*}(n_1^*, n_2^*) \leq_{rh} X_{n:n}(n_1, n_2).$$
 (3.24)

Thus, the proof of the theorem follows after combining the inequalities given by (3.23) and (3.24).

Below, we consider an example to illustrate Theorem 3.6.

Example 3.2. Consider $\lambda = (3,2), \ \mu = (6,5), \ (n_1,n_2) = (2,10), \ (n_1^*,n_2^*) = (3,4), \ \psi = e^{\frac{1}{0.2}(1-e^x)}, \ x > 0.$ Also, let the baseline distribution be $F(x) = 1 - (\frac{x}{b})^{-a}, \ x \ge b > 0, \ a > 0.$ It is not hard to see that for a = 5 and b = 1, all the conditions of Theorem 3.6 are satisfied. Further, we plot the graph of $F_{X_{12:12}}(2,10)(x)/F_{Y_{7:7}}(3,4)(x)$ in Figure 2a. This shows that the result in Theorem 3.6 holds.

Now, we derive conditions such that the star order holds between $X_{n^*:n^*}(n_1^*, n_2^*)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$. Denote $\lambda_{2:2} = \max\{\lambda_1, \lambda_2\}, \ \lambda_{1:2} = \min\{\lambda_1, \lambda_2\}, \ \mu_{2:2} = \max\{\mu_1, \mu_2\}$ and $\mu_{1:2} = \min\{\mu_1, \mu_2\}.$

Theorem 3.7. Under the set-up as in Assumption 3.1, with $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$ and $\psi_1 = \psi_2 = \psi$, we have

$$\frac{\lambda_{2:2}}{\lambda_{1:2}} \ge \frac{\mu_{2:2}}{\mu_{1:2}} \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \le_* X_{n^*:n^*}(n_1^*, n_2^*),$$

provided $\frac{\psi}{\psi'}$ is decreasing, convex, $\frac{x\tilde{r}'(x)}{\tilde{r}(x)}$ is decreasing and $x\tilde{r}(x)$ is increasing.

Proof. Under the assumption, $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$ gives $F_1 = F_2 = F$. The distribution functions of $X_{n^*:n^*}(n_1^*, n_2^*)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ are respectively given by

$$F_{X_{n^*:n^*}}(n_1^*, n_2^*)(x) = \psi \left[n_1^* \phi \left(F(x\lambda_1) \right) + n_2^* \phi \left(F(x\lambda_2) \right) \right]$$

and

$$F_{Y_{n^*:n^*}}(n_1^*, n_2^*)(x) = \psi \left[n_1^* \phi \left(F(x\mu_1) \right) + n_2^* \phi \left(F(x\mu_2) \right) \right].$$



Figure 2: (a) Plot of the ratio of two distribution functions $F_{X_{12:12}}(2, 10)(x)/F_{Y_{7:7}}(3, 4)(x)$ in Example 3.2. (b) Plot of $\bar{F}_{Y_{1:10}}(3, 7)(x)/\bar{F}_{X_{1:13}}(2, 11)(x)$ as in Example 3.4.

To obtain the required result, we consider two cases.

Case-I: $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$.

For convenience, we assume that $\lambda_1 + \lambda_2 = \mu_1 + \mu_2 = 1$. For this case, it is clear that $(\lambda_1, \lambda_2) \succeq^m (\mu_1, \mu_2)$. Now, take $\lambda_2 = \lambda \ge \lambda_1$, $\mu_2 = \mu \ge \mu_1$. Hence, $\lambda_1 = 1 - \lambda$ and $\mu_1 = 1 - \mu$. Based on this, the distribution functions of $X_{n^*:n^*}(n_1^*, n_2^*)$ and $Y_{n^*:n^*}(n_1^*, n_2^*)$ can be written in the following form

$$F_{X_{n^*:n^*}}(n_1^*, n_2^*)(x) \stackrel{def}{=} F_{\lambda}(x) = \psi \left[n_1^* \phi \left(F \left(x(1-\lambda) \right) \right) + n_2^* \phi \left(F \left(x\lambda \right) \right) \right]$$

and

$$F_{Y_{n^*:n^*}}(n_1^*, n_2^*)(x) \stackrel{def}{=} F_{\mu}(x) = \psi \left[n_1^* \phi \left(F \left(x(1-\mu) \right) + n_2^* \phi \left(F \left(x\mu \right) \right) \right] \right].$$

Now, according to Lemma 2.3, we have to show that $\frac{F'_{\lambda}(x)}{xf_{\lambda}(x)}$ is decreasing in $x \in \mathbb{R}^+$, for $\lambda \in (1/2, 1]$. The derivative of F_{λ} , with respect to λ is given by

$$F_{\lambda}'(x) = \begin{bmatrix} -xn_1^*\tilde{r}(x(1-\lambda))\frac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]} + xn_2^*\tilde{r}(x\lambda)\frac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]} \end{bmatrix}$$
$$\times \psi'[n_1^*\phi(F(x(1-\lambda))) + n_2^*\phi(F(x\lambda))].$$
(3.25)

Also, the probability density function corresponding to F_{λ} is

$$f_{\lambda}(x) = \left[(1-\lambda)n_1^* \tilde{r}(x(1-\lambda)) \frac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]} + \lambda n_2^* \tilde{r}(x\lambda) \frac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]} \right] \\ \times \psi' \left[n_1^* \phi(F(x(1-\lambda))) + n_2^* \phi(F(x\lambda)) \right].$$
(3.26)

Therefore,

$$\frac{F_{\lambda}'(x)}{xf_{\lambda}(x)} = \left(\lambda + \left[\frac{n_2^*\tilde{r}(x\lambda)\frac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]}}{n_1^*\tilde{r}(x(1-\lambda))\frac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]}} - 1\right]^{-1}\right)^{-1}$$

Thus, it suffices to show that $L(x) = \left(\tilde{r}(x\lambda)\frac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]}\right) / \left(\tilde{r}(x(1-\lambda))\frac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]}\right)$ is decreasing in $x \in \mathbb{R}^+$, for $\lambda \in (1/2, 1]$. The derivative of L(x) with respect to x is obtained as

$$L'(x) \stackrel{sign}{=} \frac{\lambda \tilde{r}'(x\lambda)}{\tilde{r}(x\lambda)} + \lambda \tilde{r}(x\lambda) \left[\frac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]} \right]' - \frac{(1-\lambda)\tilde{r}'(x(1-\lambda))}{\tilde{r}((1-\lambda)x)} - (1-\lambda)\tilde{r}(x(1-\lambda)) \left[\frac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]} \right]'.$$

Under the assumptions made, $\frac{x\tilde{r}'(x)}{\tilde{r}(x)}$ is decreasing and $x\tilde{r}(x)$ is increasing. Therefore, for $\lambda \in (1/2, 1]$,

$$\frac{x\lambda\tilde{r}'(x\lambda)}{\tilde{r}(x\lambda)} \le \frac{x(1-\lambda)\tilde{r}'(x(1-\lambda))}{\tilde{r}(x(1-\lambda))} \le 0 \text{ and } x\lambda\tilde{r}(x\lambda) \ge x(1-\lambda)\tilde{r}(x(1-\lambda)) \ge 0.$$
(3.27)

Also, since $\frac{\psi}{\psi}$ is decreasing and convex, we have

$$\left[\frac{\psi[\phi(F(x\lambda))]}{\psi'[\phi(F(x\lambda))]}\right]' \le \left[\frac{\psi[\phi(F(x(1-\lambda)))]}{\psi'[\phi(F(x(1-\lambda)))]}\right]' \le 0.$$
(3.28)

Now, combining Equations (3.27) and (3.28), we get $L'(x) \leq 0$, for $x \in \mathbb{R}^+$. Case-II. $\lambda_1 + \lambda_2 \neq \mu_1 + \mu_2$.

In this case, we can take $\lambda_1 + \lambda_2 = k(\mu_1 + \mu_2)$, where k is a scalar. Hence, $(k\mu_1, k\mu_2) \preceq^m (\lambda_1, \lambda_2)$. Let us consider n^* dependent nonnegative random variables sharing Archimedean copula with generator ψ , such that $Z_i \sim F(k\mu_1 x)$, for $i = 1, \cdots, n_1^*$ and $Z_j \sim F(k\mu_2 x)$, for $j = n_1^* + 1, \cdots, n^*$. Here, $n_1^* + n_2^* = n^*$. Then, from Case-I, we have $Z_{n^*:n^*}(n_1^*, n_2^*) \leq_* X_{n^*:n^*}(n_1^*, n_2^*)$. Further, star order is scale invariant, and hence we obtain $Y_{n^*:n^*}(n_1^*, n_2^*) \leq_* X_{n^*:n^*}(n_1^*, n_2^*)$. This completes the proof of the theorem.

Using the fact that the star order implies the Lorenz order, the following result is a direct consequence of Theorem 3.7. Further, since the Lorenz order is mainly used to compare the income distributions, the following corollary is more interesting from the point of its applications in the study of incomes.

Corollary 3.4. Under the set-up as in Theorem 3.7,

$$\frac{\lambda_{2:2}}{\lambda_{1:2}} \ge \frac{\mu_{2:2}}{\mu_{1:2}} \Rightarrow Y_{n^*:n^*}(n_1^*, n_2^*) \le_{Lorenz} X_{n^*:n^*}(n_1^*, n_2^*),$$

provided $\frac{\psi}{\psi'}$ is decreasing, convex, $\frac{x\tilde{r}'(x)}{\tilde{r}(x)}$ is decreasing and $x\tilde{r}(x)$ is increasing.

3.2 Orderings between the smallest order statistics

In the previous subsection, we focus on the conditions, under which the largest order statistics are comparable according to various stochastic orders. Here, we develop conditions such that the usual stochastic, hazard rate, star and Lorenz orders hold between the smallest order statistics. In the following theorems, we consider that the samples are heterogeneous and taken from the multiple-outlier dependent scale models. We now consider the following assumption.

Assumption 3.4. Let X_1, \dots, X_{n^*} be n^* dependent nonnegative random variables sharing Archimedean survival copula with generator ψ_1 , with $X_i \sim F_1(x\lambda_1)$, for $i = 1, \dots, n_1^*$ and $X_j \sim F_2(x\lambda_2)$, for $j = n_1^* + 1, \dots, n^*$. Also, let Y_1, \dots, Y_{n^*} be n^* dependent non-negative random variables sharing Archimedean copula with generator ψ_2 , with $Y_i \sim F_1(x\mu_1)$, for i = $1, \dots, n_1^*$ and $Y_j \sim F_2(x\mu_2)$, for $j = n_1^* + 1, \dots, n^*$. Here, $n_1^* + n_2^* = n^*$, $\psi_1 = \phi_1^{-1}$ and $\psi_2 = \phi_2^{-1}$.

Theorem 3.8. Under the set-up as in Assumption 3.4, with $r_1(x) \leq (\geq)r_2(x)$ and $n_1^* \leq (\geq)n_2^*$,

$$(\underbrace{\lambda_{1}, \cdots, \lambda_{1}}_{n_{1}^{*}}, \underbrace{\lambda_{2}, \cdots, \lambda_{2}}_{n_{2}^{*}}) \succeq_{w} (\underbrace{\mu_{1}, \cdots, \mu_{1}}_{n_{1}^{*}}, \underbrace{\mu_{2}, \cdots, \mu_{2}}_{n_{2}^{*}}) \Rightarrow X_{1:n^{*}}(n_{1}^{*}, n_{2}^{*}) \leq_{st} Y_{1:n^{*}}(n_{1}^{*}, n_{2}^{*}),$$

provided λ , $\mu \in \mathcal{E}_+$ (\mathcal{D}_+) , $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex and $r_1(x)$ or $r_2(x)$ is increasing.

Proof. The reliability functions of $X_{1:n^*}(n_1^*, n_2^*)$ and $Y_{1:n^*}(n_1^*, n_2^*)$ are respectively given by

$$\bar{F}_{X_{1:n^*}}(n_1^*, n_2^*)(x) = \psi_1 \left[n_1^* \phi_1 \left(\bar{F}_1 \left(x \lambda_1 \right) \right) + n_2^* \phi_1 \left(\bar{F}_2 \left(x \lambda_2 \right) \right) \right]$$

and

$$\bar{F}_{Y_{1:n^*}}(n_1^*, n_2^*)(x) = \psi_2 \left[n_1^* \phi_2 \left(\bar{F}_1 \left(x \mu_1 \right) \right) + n_2^* \phi_2 \left(\bar{F}_2 \left(x \mu_2 \right) \right) \right].$$

Let us denote $C(\lambda, \psi_1, x) = \overline{F}_{X_{1:n^*}}(n_1^*, n_2^*)(x)$ and $D(\mu, \psi_2, x) = \overline{F}_{Y_{1:n^*}}(n_1^*, n_2^*)(x)$. According to Lemma 2.4, super-additivity property of $\phi_2 \circ \psi_1$ provides $C(\mu, \psi_1, x) \leq D(\mu, \psi_2, x)$. In order to prove the desired result, we need to show that $C(\lambda, \psi_1, x) \leq C(\mu, \psi_1, x)$. This is equivalent to show that $C(\lambda, \psi_1, x)$ is decreasing and Schur-concave with respect to λ . Taking derivative of $C(\lambda, \psi_1, x)$ with respect to λ_1 , we get

$$\frac{\partial C(\boldsymbol{\lambda},\psi_{1},x)}{\partial\lambda_{1}} = -n_{1}^{*}xr_{1}(x\lambda_{1})\frac{\psi_{1}\left[\phi_{1}\left(\bar{F}_{1}\left(x\lambda_{1}\right)\right)\right]}{\psi_{1}^{\prime}\left[\phi_{1}\left(\bar{F}_{1}\left(x\lambda_{1}\right)\right)\right]}\psi_{1}^{\prime}\left[n_{1}^{*}\phi_{1}\left(\bar{F}_{1}\left(x\lambda_{1}\right)\right) + n_{2}^{*}\phi_{1}\left(\bar{F}_{2}\left(x\lambda_{2}\right)\right)\right].$$
(3.29)

Equation (3.29) shows that $C(\boldsymbol{\lambda}, \psi_1, x)$ is decreasing in λ_1 , since $\frac{\partial C(\boldsymbol{\lambda}, \psi_1, x)}{\partial \lambda_1} \leq 0$. Also, $\frac{\partial C(\boldsymbol{\lambda}, \psi_1, x)}{\partial \lambda_2} \leq 0$. Therefore, $C(\boldsymbol{\lambda}, \psi_1, x)$ is decreasing in λ_i , for i = 1, 2. Further, to establish Schur-concavity of $C(\boldsymbol{\lambda}, \psi_1, x)$, we need to show that for $1 \leq i \leq j \leq n^*$, the following inequality holds (see Lemma 2.2 (Lemma 2.1)):

$$\left[\frac{\partial C(\boldsymbol{\lambda}, \psi_1, x)}{\partial \lambda_i} - \frac{\partial C(\boldsymbol{\lambda}, \psi_1, x)}{\partial \lambda_j}\right] \ge (\le)0, \text{ for } \boldsymbol{\lambda} \in \mathcal{E}_+ \ (\mathcal{D}_+).$$
(3.30)

Now, we study the following cases.

Case I: For $1 \le i \le j \le n_1^*$, $\lambda_i = \lambda_j = \lambda_1$. Thus, $\left[\frac{\partial C(\boldsymbol{\lambda},\psi_1,x)}{\partial \lambda_i} - \frac{\partial C(\boldsymbol{\lambda},\psi_1,x)}{\partial \lambda_j}\right] = 0$. Case II: For $n_1^* + 1 \le i \le j \le n^*$, $\lambda_i = \lambda_j = \lambda_2$. So, $\left[\frac{\partial C(\boldsymbol{\lambda},\psi_1,x)}{\partial \lambda_i} - \frac{\partial C(\boldsymbol{\lambda},\psi_1,x)}{\partial \lambda_j}\right] = 0$. Case III: For $1 \le i \le n_1^*$ and $n_1^* + 1 \le j \le n^*$, $\lambda_i = \lambda_1$ and $\lambda_j = \lambda_2$. Suppose $\lambda_1 \le (\ge)\lambda_2$ which implies $\phi_1(\bar{F}_1(x\lambda_1)) \le (\ge)\phi_1(\bar{F}_2(x\lambda_2))$ in view of $F_1 \le (\ge)F_2$. Now, applying the convexity property of $\ln \psi_1$, we can write

$$\frac{\psi_1(w)}{\psi_1'(w)}\Big|_{w=\phi_1[\bar{F}_1(x\lambda_1)]} \ge (\le)\frac{\psi_1(w)}{\psi_1'(w)}\Big|_{w=\phi_1[\bar{F}_2(x\lambda_2)]}.$$
(3.31)

Under the assumptions made, further, we obtain

$$n_1^* r_1(x\lambda_1) \le (\ge) n_2^* r_2(x\lambda_2).$$
 (3.32)

Finally, combining Equations (3.31) and (3.32), we observe that the inequality in (3.30) holds. This completes the proof. \Box

The following corollary is immediate from Theorem 3.8.

Corollary 3.5. Let the set-up as in Assumption 3.4 hold with $\psi_1 = \psi_2 = \psi$ and $n_1^* \leq (\geq)n_2^*$. Also, let ψ be log-convex and λ , $\mu \in \mathcal{E}_+$ (\mathcal{D}_+). Then,

$$(i) \underbrace{(\lambda_1, \cdots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \cdots, \lambda_2}_{n_2^*}) \succeq_w \underbrace{(\mu_1, \cdots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \cdots, \mu_2}_{n_2^*}) \Rightarrow X_{1:n^*}(n_1^*, n_2^*) \leq_{st} Y_{1:n^*}(n_1^*, n_2^*),$$

provided $r_1(x)$ or $r_2(x)$ is increasing with $r_1(x) \leq (\geq)r_2(x).$

$$(ii) \quad \underbrace{(\lambda_1, \cdots, \lambda_1}_{n_1^*}, \underbrace{\lambda_2, \cdots, \lambda_2}_{n_2^*}) \succeq_w \quad \underbrace{(\mu_1, \cdots, \mu_1}_{n_1^*}, \underbrace{\mu_2, \cdots, \mu_2}_{n_2^*}) \Rightarrow X_{1:n^*}(n_1^*, n_2^*) \leq_{st} Y_{1:n^*}(n_1^*, n_2^*),$$

$$provided \ r(x) \ is \ increasing, \ where \ r_1(x) = r_2(x) = r(x).$$

The next result reveals that the smallest order statistics $X_{1:n^*}(n_1^*, n_2^*)$ dominates $X_{1:n}(n_1, n_2)$ in the sense of the usual stochastic order under the condition that (n_1^*, n_2^*) is weakly submajorized by (n_1, n_2) . The following assumption will be helpful to prove the next two results.

Assumption 3.5. Let X_1, \dots, X_{n^*} be n^* dependent nonnegative random variables sharing Archimedean survival copula with generator ψ_1 , such that $X_i \sim F_1(x\lambda_1)$, for $i = 1, \dots, n_1^*$ and $X_j \sim F_2(x\lambda_2)$, for $j = n_1^* + 1, \dots, n^*$. We assume that there exist two natural numbers n_1 and n_2 such that $1 \leq n_1 \leq n_1^* \leq n_2^* \leq n_2$. Also, $n = n_1 + n_2$, $n^* = n_1^* + n_2^*$ and $\psi_1 = \phi_1^{-1}$.

Theorem 3.9. Let Assumption 3.5 hold. Then, for $\lambda = (\lambda_1, \lambda_2) \in \mathcal{E}_+$ and $F_1 \leq F_2$, we have

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{1:n}(n_1, n_2) \leq_{st} X_{1:n^*}(n_1^*, n_2^*).$$

Proof. To obtain the desired result, it is sufficient to show that $\psi_1[\sum_{i=1}^n \phi_1(\bar{F}_i(x\lambda_i))] \leq \psi_1[\sum_{i=1}^{n^*} \phi_1(\bar{F}_i(x\lambda_i))]$. Equivalently,

$$(n_1^* - n_1)\phi_1\left(\bar{F}_1(x\lambda_1)\right) \le (n_2 - n_2^*)\phi_1\left(\bar{F}_2(x\lambda_2)\right).$$
(3.33)

Further, $(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow (n_1 + n_2) \ge (n_1^* + n_2^*) \Rightarrow (n_2 - n_2^*) \ge (n_1^* - n_1) \ge 0$ and $\lambda_1 \le \lambda_2 \Rightarrow \phi_1(\bar{F}_2(x\lambda_2)) \ge \phi_1(\bar{F}_1(x\lambda_1)) \ge 0$. Using these arguments, we get the inequality given by (3.33). Hence, the proof is completed.

In Theorem 3.9, if we take the same baseline distribution functions, then we have the following corollary.

Corollary 3.6. Let Assumption 3.5 hold. Then, for $\lambda \in \mathcal{E}_+$, $F_1 = F_2$, we have

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{1:n}(n_1, n_2) \leq_{st} X_{1:n^*}(n_1^*, n_2^*).$$

In the following, we develop conditions such that the usual stochastic order holds between the smallest order statistics $X_{1:n}(n_1, n_2)$ and $Y_{1:n^*}(n_1^*, n_2^*)$. The assumption in below is required for the next theorem.

Assumption 3.6. Let X_1, \dots, X_n be *n* nonnegative dependent random variables sharing Archimedean survival copula with generator ψ_1 , such that $X_i \sim F_1(x\lambda_1)$, for $i = 1, \dots, n_1$ and $X_j \sim F_2(x\lambda_2)$, for $j = n_1 + 1, \dots, n$. Also, let Y_1, \dots, Y_{n^*} be n^* dependent nonnegative random variables sharing Archimedean copula with generator ψ_2 , such that $Y_i \sim F_1(x\mu_1)$, for $i = 1, \dots, n_1^*$ and $Y_j \sim F_2(x\mu_2)$, for $j = n_1^* + 1, \dots, n^*$. Here, $1 \leq n_1 \leq n_1^* \leq n_2^* \leq n_2$, $n = n_1 + n_2$ and $n^* = n_1^* + n_2^*$.

Theorem 3.10. Let Assumption 3.6 hold, with $r_1(x) \leq r_2(x)$. Also, let λ , $\mu \in \mathcal{E}_+$. Then, for $(n_1, n_2) \succeq_w (n_1^*, n_2^*)$,

$$(\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1^*},\underbrace{\lambda_2,\cdots,\lambda_2}_{n_2^*})\succeq_w(\underbrace{\mu_1,\cdots,\mu_1}_{n_1^*},\underbrace{\mu_2,\cdots,\mu_2}_{n_2^*})\Rightarrow X_{1:n}(n_1,n_2)\leq_{st}Y_{1:n^*}(n_1^*,n_2^*),$$

provided $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex and $r_1(x)$ or $r_2(x)$ is increasing.

Proof. The theorem can be proved using Theorems 3.8 and 3.9. Thus, it is omitted. \Box

As an illustration of Theorem 3.10, we present the following example.

Example 3.3. Take $\lambda = (2,6)$, $\mu = (1,3)$, $(n_1,n_2) = (4,8)$, $(n_1^*,n_2^*) = (6,7)$, $\psi_1(x) = e^{-x^{\frac{1}{9}}}$ and $\psi_2(x) = e^{-x^{\frac{1}{10}}}$, x > 0. Also, let $F_1(x) = \left(\frac{x}{a}\right)^l$, $0 < x \le a$ and $F_2(x) = 1 - e^{-x}$, x > 0. It can be seen that for a = 400 and l = 2 all the conditions of Theorem 3.10 are satisfied. Now, we plot the graph of $\bar{F}_{X_{1:12}}(4,8)(x) - \bar{F}_{Y_{1:13}}(6.7)(x)$, given in Figure 3a. The figure suggests that $X_{1:12}(4,8) \le_{st} Y_{1:13}(6,7)$ holds.

Next, we present a counterexample, which shows that the stated usual stochastic order in Theorem 3.10 does not hold if the conditions $r_1(x) \leq r_2(x)$ and r_2 is increasing are dropped out.

Counterexample 3.2. Take $\lambda = (1.2, 3.6), \ \mu = (1.4, 3), \ (n_1, n_2) = (2, 11), \ (n_1^*, n_2^*) = (3, 9), \ \psi_1(x) = e^{-x^{\frac{1}{4.5}}} and \ \psi_2(x) = e^{-x^{\frac{1}{5}}}, \ x > 0.$ Also, suppose $F_1(x) = 1 - e^{-x}$ and $F_2(x) = 1 - (1 + 2x)^{-0.5}, \ x > 0.$ Clearly, all the conditions of Theorem 3.10 are satisfied except $r_1 \leq r_2$ and r_2 is increasing. Now, the graphs of $\overline{F}_{X_{1:13}}(2, 11)(x)$ and $\overline{F}_{Y_{1:12}}(3, 9)(x)$ are depicted in Figure 3b. It reveals that the usual stochastic order in Theorem 3.10 does not hold.

Upon using Theorem 3.10, one can easily conclude the following corollary.



Figure 3: (a) Plot of $\bar{F}_{X_{1:12}}(4,8)(x) - \bar{F}_{Y_{1:13}}(6,7)(x)$ as in Example 3.3. (b) Plots of $\bar{F}_{X_{1:13}}(2,11)(x)$ and $\bar{F}_{Y_{1:12}}(3,9)(x)$ as in Counterexample 3.2.

Corollary 3.7. Let Assumption 3.6 hold with $\psi_1 = \psi_2 = \psi$. Also, let λ , $\mu \in \mathcal{E}_+$, ψ is log-convex and $(n_1, n_2) \succeq_w (n_1^*, n_2^*)$. Then,

$$(i) \underbrace{(\lambda_{1}, \cdots, \lambda_{1}, \lambda_{2}, \cdots, \lambda_{2})}_{n_{1}^{*}} \succeq_{w} \underbrace{(\mu_{1}, \cdots, \mu_{1}, \mu_{2}, \cdots, \mu_{2})}_{n_{1}^{*}} \Rightarrow X_{1:n}(n_{1}, n_{2}) \leq_{st} Y_{1:n^{*}}(n_{1}^{*}, n_{2}^{*}),$$

$$(ii) \underbrace{(\lambda_{1}, \cdots, \lambda_{1}, \lambda_{2}, \cdots, \lambda_{2})}_{n_{1}^{*}} \succeq_{w} \underbrace{(\mu_{1}, \cdots, \mu_{1}, \mu_{2}, \cdots, \mu_{2})}_{n_{1}^{*}} \Rightarrow X_{1:n}(n_{1}, n_{2}) \leq_{st} Y_{1:n^{*}}(n_{1}^{*}, n_{2}^{*}),$$

$$(ii) \underbrace{(\lambda_{1}, \cdots, \lambda_{1}, \lambda_{2}, \cdots, \lambda_{2})}_{n_{1}^{*}} \succeq_{w} \underbrace{(\mu_{1}, \cdots, \mu_{1}, \mu_{2}, \cdots, \mu_{2})}_{n_{1}^{*}} \Rightarrow X_{1:n}(n_{1}, n_{2}) \leq_{st} Y_{1:n^{*}}(n_{1}^{*}, n_{2}^{*}),$$

$$(ii) \underbrace{(\lambda_{1}, \cdots, \lambda_{1}, \lambda_{2}, \cdots, \lambda_{2})}_{n_{1}^{*}} \succeq_{w} \underbrace{(\mu_{1}, \cdots, \mu_{1}, \mu_{2}, \cdots, \mu_{2})}_{n_{1}^{*}} \Rightarrow X_{1:n}(n_{1}, n_{2}) \leq_{st} Y_{1:n^{*}}(n_{1}^{*}, n_{2}^{*}),$$

$$(ii) \underbrace{(\lambda_{1}, \cdots, \lambda_{1}, \lambda_{2}, \cdots, \lambda_{2})}_{n_{1}^{*}} \succeq_{w} \underbrace{(\mu_{1}, \cdots, \mu_{1}, \mu_{2}, \cdots, \mu_{2})}_{n_{1}^{*}} \Rightarrow X_{1:n}(n_{1}, n_{2}) \leq_{st} Y_{1:n^{*}}(n_{1}^{*}, n_{2}^{*}),$$

$$(ii) \underbrace{(\lambda_{1}, \cdots, \lambda_{1}, \lambda_{2}, \cdots, \lambda_{2})}_{n_{2}^{*}} \succeq_{w} \underbrace{(\mu_{1}, \cdots, \mu_{1}, \mu_{2}, \cdots, \mu_{2})}_{n_{1}^{*}} \Rightarrow X_{1:n}(n_{1}, n_{2}) \leq_{st} Y_{1:n^{*}}(n_{1}^{*}, n_{2}^{*}),$$

$$(ii) \underbrace{(\lambda_{1}, \cdots, \lambda_{1}, \lambda_{2}, \cdots, \lambda_{2})}_{n_{2}^{*}} \succeq_{w} \underbrace{(\mu_{1}, \cdots, \mu_{1}, \mu_{2}, \cdots, \mu_{2})}_{n_{1}^{*}} \Rightarrow X_{1:n}(n_{1}, n_{2}) \leq_{st} Y_{1:n^{*}}(n_{1}^{*}, n_{2}^{*}),$$

Next, we provide three consecutive theorems, which deal with the hazard rate ordering between the smallest order statistics.

Theorem 3.11. Let Assumption 3.4 hold with $n_1^* \leq (\geq)n_2^*$, $\psi_1 = \psi_2 = \psi$, $F_1 = F_2 = F$, $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$ and $r_1 = r_2 = r$. Also, suppose ψ is log-concave, $\frac{1-\psi}{\psi'}$ is decreasing, $[\frac{1-\psi}{\psi'}]'/\frac{\psi}{\psi'}$ is increasing and λ , $\mu \in \mathcal{E}_+$ (\mathcal{D}_+). Then,

$$(\underbrace{m_1, \cdots, m_1}_{n_1^*}, \underbrace{m_2, \cdots, m_2}_{n_2^*}) \succeq_w (\underbrace{v_1, \cdots, v_1}_{n_1^*}, \underbrace{v_2, \cdots, v_2}_{n_2^*}) \Rightarrow X_{1:n^*}(n_1^*, n_2^*) \leq_{hr} Y_{1:n^*}(n_1^*, n_2^*),$$

where $m_i = \log \lambda_i$ and $v_i = \log \mu_i$, i = 1, 2, provided r(x) is increasing, $x\tilde{r}(x)$ is increasing and convex.

Proof. Denote by f the probability density function corresponding to the distribution function

F. The hazard rate function of $X_{1:n^*}(n_1^*, n_2^*)$ is given by

$$r_{X_{1:n^{*}}}(n_{1}^{*}, n_{2}^{*})(x) \stackrel{def}{=} \mathcal{E}(\boldsymbol{m}) = \frac{\psi'[z]}{\psi[z]} \left[\frac{n_{1}^{*}e^{m_{1}}f(xe^{m_{1}})}{\psi'[\phi(\bar{F}(xe^{m_{1}}))]} + \frac{n_{2}^{*}e^{m_{2}}f(xe^{m_{2}})}{\psi'[\phi(\bar{F}(xe^{m_{2}}))]} \right] \\ = \frac{\psi'[z]}{\psi[z]} \left[\frac{n_{1}^{*}e^{m_{1}}\tilde{r}(xe^{m_{1}})[1 - \psi[\phi(\bar{F}(xe^{m_{1}}))]]}{\psi'[\phi(\bar{F}(xe^{m_{1}}))]} + \frac{n_{2}^{*}e^{m_{2}}\tilde{r}(xe^{m_{2}})[1 - \psi[\phi(\bar{F}(xe^{m_{2}}))]]}{\psi'[\phi(\bar{F}(xe^{m_{2}}))]} \right] \\ + \frac{n_{2}^{*}e^{m_{2}}\tilde{r}(xe^{m_{2}})[1 - \psi[\phi(\bar{F}(xe^{m_{2}}))]]}{\psi'[\phi(\bar{F}(xe^{m_{2}}))]} \right],$$
(3.34)

where $z = n_1^* \phi\left(\bar{F}\left(xe^{m_1}\right)\right) + n_2^* \phi\left(\bar{F}\left(xe^{m_2}\right)\right)$, $m_i = \log \lambda_i$, for i = 1, 2 and $\boldsymbol{m} = (m_1, m_2)$. Also, f is the probability density function of F. The partial derivative of $\mathcal{E}(\boldsymbol{m})$ with respect to m_1 is given by

$$\frac{\partial \mathcal{E}(\boldsymbol{m})}{\partial m_{1}} = -n_{1}^{*}xe^{m_{1}}\tilde{r}(xe^{m_{1}})\frac{d}{dz}\left[\frac{\psi'(z)}{\psi(z)}\right]\left[\frac{1-\psi\left[\phi\left[\bar{F}\left(xe^{m_{1}}\right)\right]\right]}{\psi'\left[\phi\left(\bar{F}\left(xe^{m_{1}}\right)\right)\right]}\right]\left[\sum_{i=1}^{n^{*}}\frac{e^{m_{i}}f\left(xe^{m_{i}}\right)}{\psi'\left[\phi\left(\bar{F}\left(xe^{m_{i}}\right)\right)\right]}\right] \\
-n_{1}^{*}r(xe^{m_{1}})\left[x[e^{m_{1}}]^{2}\tilde{r}(xe^{m_{1}})\right]\frac{\psi'(z)}{\psi(z)}\left[\left[\frac{\psi(v)}{\psi'(v)}\right]^{2}\left[\frac{d}{dv}\left[\frac{1-\psi(v)}{\psi'(v)}\right]\right]\right] \\
+n_{1}^{*}\frac{d}{dw}\left[w\tilde{r}(w)\right]_{w=xe^{m_{1}}}\frac{1-\psi\left[\phi\left[\bar{F}\left(xe^{m_{1}}\right)\right]\right]}{\psi'\left[\phi\left[\bar{F}\left(xe^{m_{1}}\right)\right]\right]}\frac{\psi'(z)}{\psi(z)}.$$
(3.35)

From (3.35), it is easy to see that $\mathcal{E}(\boldsymbol{m})$ is increasing in m_1 . Similarly, $\mathcal{E}(\boldsymbol{m})$ is also increasing in m_2 . Hence, $\mathcal{E}(\boldsymbol{m})$ is increasing with respect to \boldsymbol{m} . Now, we only need to show the Schurconvexity of $\mathcal{E}(\boldsymbol{m})$ with respect to \boldsymbol{m} . This is equivalent to show that for $1 \leq i \leq j \leq n^*$,

$$\left[\frac{\partial \mathcal{E}(\boldsymbol{m})}{\partial m_i} - \frac{\partial \mathcal{E}(\boldsymbol{m})}{\partial m_j}\right] \le (\ge)0, \text{ for } \boldsymbol{m} \in \mathcal{E}_+ (\mathcal{D}_+).$$
(3.36)

Utilizing the assumptions made, the rest of the proof follows from the similar arguments of Theorem 3.8. Thus, it is omitted for the sake of conciseness. \Box

The following theorem demonstrates that under some conditions, the hazard rate ordering between $X_{1:n}(n_1, n_2)$ and $X_{1:n^*}(n_1^*, n_2^*)$ exists.

Theorem 3.12. Let Assumption 3.5 hold with $\psi_1 = \psi$ and $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$. Then, for $\lambda \in \mathcal{E}_+$, we have

$$(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow X_{1:n}(n_1, n_2) \leq_{hr} X_{1:n^*}(n_1^*, n_2^*)$$

provided $x\tilde{r}(x)$ is increasing, ψ'/ψ and $\frac{1-\psi}{\psi'}$ are decreasing.

Proof. The required result can be proved if we show that $r_{X_{1:n}}(n_1, n_2)(x) \ge r_{X_{1:n^*}}(n_1^*, n_2^*)(x)$ and equivalently,

$$\frac{\psi'\left[\sum_{i=1}^{n}\phi\left(\bar{F}(x\lambda_{i})\right)\right]}{\psi\left[\sum_{i=1}^{n}\phi\left(\bar{F}(x\lambda_{i})\right)\right]}\left[\sum_{i=1}^{n}\frac{\lambda_{i}\tilde{r}(x\lambda_{i})\left[1-\psi\left[\phi\left(\bar{F}(x\lambda_{i})\right)\right]\right]}{\psi'\left[\phi\left(\bar{F}(x\lambda_{i})\right)\right]}\right] \ge \frac{\psi'\left[\sum_{i=1}^{n^{*}}\phi\left(\bar{F}(x\lambda_{i})\right)\right]}{\psi\left[\sum_{i=1}^{n^{*}}\phi\left(\bar{F}(x\lambda_{i})\right)\right]}\left[\sum_{i=1}^{n^{*}}\frac{\lambda_{i}\tilde{r}(x\lambda_{i})\left[1-\psi\left[\phi\left(\bar{F}(x\lambda_{i})\right)\right]\right]}{\psi'\left[\phi\left(\bar{F}(x\lambda_{i})\right)\right]}\right].$$

$$(3.37)$$

To prove inequality (3.37), it is sufficient to show that the following two inequalities hold:

$$(n_1^* - n_1)\phi(\bar{F}(x\lambda_1)) \le (n_2 - n_2^*)\phi(\bar{F}(x\lambda_2))$$
(3.38)

and

$$(n_1^* - n_1) \frac{\lambda_1 \tilde{r}(x\lambda_1) [1 - \psi[\phi\left(\bar{F}(x\lambda_1)\right)]]}{\psi'[\phi\left(\bar{F}(x\lambda_1)\right)]} \ge (n_2 - n_2^*) \frac{x\lambda_2 \tilde{r}(x\lambda_2) [1 - \psi[\phi\left(\bar{F}(x\lambda_2)\right)]]}{\psi'[\phi\left(\bar{F}(x\lambda_2)\right)]}.$$
 (3.39)

Further, $(n_1, n_2) \succeq_w (n_1^*, n_2^*) \Rightarrow (n_1 + n_2) \ge (n_1^* + n_2^*) \Rightarrow (n_2 - n_2^*) \ge (n_1^* - n_1) \ge 0$. Also $\lambda_1 \le \lambda_2 \Rightarrow \phi(\bar{F}(x\lambda_2)) \ge \phi(\bar{F}(x\lambda_1)) \ge 0$. By the help of decreasing property of $\frac{1-\psi}{\psi'}$, we obtain

$$\frac{1 - \psi(w)}{\psi'(w)}|_{w = \phi[\bar{F}(x\lambda_2)]} \le \frac{1 - \psi(w)}{\psi'(w)}|_{w = \phi[\bar{F}(x\lambda_1)]} \le 0.$$
(3.40)

Since $x\tilde{r}(x)$ is increasing,

$$x\lambda_1 \tilde{r}(x\lambda_1) \le x\lambda_2 \tilde{r}(x\lambda_2). \tag{3.41}$$

Thus, the proof is completed from Equations (3.40), (3.41) and the given assumptions.

The next theorem states that if the scale parameters are connected with the weakly submajorized order and the sample size pairs (n_1, n_2) and (n_1^*, n_2^*) have weakly submajorized order, then the smallest order statistics of $X_{1:n}(n_1, n_2)$ is dominated by $Y_{1:n^*}(n_1^*, n_2^*)$ according to the hazard rate order.

Theorem 3.13. Let Assumption 3.6 hold with $\psi_1 = \psi_2 = \psi$, $r_1 = r_2 = r$ and $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$. Then, for λ , $\mu \in \mathcal{E}_+$ and $(n_1, n_2) \succeq_w (n_1^*, n_2^*)$,

$$(\underbrace{m_1, \cdots, m_1}_{n_1^*}, \underbrace{m_2, \cdots, m_2}_{n_2^*}) \succeq_w (\underbrace{v_1, \cdots, v_1}_{n_1^*}, \underbrace{v_2, \cdots, v_2}_{n_2^*}) \Rightarrow X_{1:n}(n_1, n_2) \leq_{hr} Y_{1:n^*}(n_1^*, n_2^*),$$

provided ψ is log-concave, $\frac{1-\psi}{\psi'}$ is decreasing, $\left[\frac{1-\psi}{\psi'}\right]'/\frac{\psi}{\psi'}$ and r(x) are increasing, $x\tilde{r}(x)$ is increasing and convex, where $m_i = \log \lambda_i$ and $v_i = \log \mu_i$, i = 1, 2.

Proof. The proof of the theorem follows from Theorem 3.11 and Theorem 3.12. Thus, it is omitted. \Box

To illustrate Theorem 3.13, we now consider the following example.

Example 3.4. Set $\lambda = (e^{0.5}, e^{0.6}), \ \mu = (e^{0.2}, e^{0.3}), \ (n_1, n_2) = (2, 11), \ (n_1^*, n_2^*) = (3, 7) \ and \ \psi(x) = e^{\frac{1}{0.99}(1-e^x)}, \ x > 0.$ Further, let $F(x) = \left(\frac{x}{a}\right)^l, \ 0 < x \le a.$ It can be easily shown that for $a = 1000 \ and \ l = 2, \ all \ the \ conditions \ of \ Theorem \ 3.13 \ are \ satisfied.$ Now, we plot the ratio $\frac{\bar{F}_{Y_{1:10}}(3,7)(x)}{\bar{F}_{X_{1:13}}(2,11)(x)}$ in Figure 2b, which is consistent with the result in Theorem 3.13.

In the next theorem, we develop some conditions under which two smallest order statistics are comparable according to the star order. **Theorem 3.14.** Under the set-up as in Assumption 3.4, with $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}$ and $\psi_1 = \psi_2 = \psi$,

$$\frac{\lambda_{2:2}}{\lambda_{1:1}} \ge \frac{\mu_{2:2}}{\mu_{1:2}} \Rightarrow Y_{1:n^*}(n_1^*, n_2^*) \le_* X_{1:n^*}(n_1^*, n_2^*),$$

provided $\frac{\psi}{\psi'}$ is decreasing, convex, $\frac{xr'(x)}{r(x)}$ and xr(x) are decreasing.

Proof. The distribution functions of $X_{1:n^*}(n_1^*, n_2^*)$ and $Y_{1:n^*}(n_1^*, n_2^*)$ are respectively given by

$$F_{X_{1:n^*}}(n_1^*, n_2^*)(x) = 1 - \psi \left[n_1^* \phi \left(\bar{F} \left(x \lambda_1 \right) \right) + n_2^* \phi \left(\bar{F} \left(x \lambda_2 \right) \right) \right]$$

and

$$F_{Y_{1:n^*}}(n_1^*, n_2^*)(x) = 1 - \psi \left[n_1^* \phi \left(\bar{F} \left(x \mu_1 \right) \right) + n_2^* \phi \left(\bar{F} \left(x \mu_2 \right) \right) \right].$$

Now, the rest of the proof follows using similar arguments as in Theorem 3.7. Thus, it is omitted. $\hfill \Box$

The following result is a direct consequence of Theorem 3.14.

Corollary 3.8. Under the assumptions as in Theorem 3.14,

$$\frac{\lambda_{2:2}}{\lambda_{1:1}} \ge \frac{\mu_{2:2}}{\mu_{1:2}} \Rightarrow Y_{1:n^*}(n_1^*, n_2^*) \le_{Lorenz} X_{1:n^*}(n_1^*, n_2^*).$$

provided $\frac{\psi}{\psi'}$ is decreasing, convex, $\frac{xr'(x)}{r(x)}$ and xr(x) are decreasing.

4. Concluding remarks

Due to simplicity in tackling the expressions/terms, most of the researchers have concentrated on the multiple-outlier models and studied ordering results between the order statistics under the set-up of independent random variables. However, the assumption of independent random variables is not feasible in many situations. So, it is required to assume dependent structure among the random observations. In this paper, we discussed some comparison results between the lifetimes of both parallel and series systems consisting of multiple-outlier dependent scale components in the sense of the usual stochastic, reversed hazard rate, hazard rate, star and Lorenz orders. The dependence structure has been modeled by Archimedean copulas. Sufficient conditions have been established for the purpose of the comparisons of the order statistics. Several examples and counterexamples are presented to illustrate the established results.

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