Predictive Quantile Regression with Mixed Roots and Increasing Dimensions: The ALQR Approach*

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Abstract

In this paper we propose the adaptive lasso for predictive quantile regression (ALQR). Reflecting empirical findings, we allow predictors to have various degrees of persistence and exhibit different signal strengths. The number of predictors is allowed to grow with the sample size. We study regularity conditions under which stationary, local unit root, and cointegrated predictors are present simultaneously. We next show the convergence rates, model selection consistency, and asymptotic distributions of ALQR. We apply the proposed method to the out-of-sample quantile prediction problem of stock returns and find that it outperforms the existing alternatives. We also provide numerical evidence from additional Monte Carlo experiments, supporting the theoretical results.

Keywords: adaptive lasso, cointegration, forecasting, oracle property, quantile regression

JEL classification: C22, C53, C61

1 Introduction

Predictive quantile regression (QR) identifies the impact of predictors on a set of conditional quantiles of a response variable. It provides richer information on the heterogeneous distributional prediction. For example, the conditional quantile prediction of stock returns receives much attention in finance since the tail quantile information has a crucial role in measuring risk. Many

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Figure 1: Time-series Plots of Persistent and Stationary Predictors

Notes: All plots are based on 816 monthly observations ranging from January 1952 to December 2019. The acronym *dp* denotes the dividend price ratio and *dfr* denotes the default yield spread.

economic state variables are employed to predict stock returns and the number of candidate predictors is often large. When a large number of predictors are available, researchers encounter the inevitable model selection issue. A good model selection can improve forecasting performance but the opposite can also occur. Considering the importance of model selection in practice, we need constructive guidance for empirical applications.

In this paper we propose the adaptive lasso for predictive quantile regression (ALQR). Although there exists a large volume of literature on predictive mean regression of equity returns (see, e.g. Campbell (1987), Fama and French (1988), Hodrick (1992), Cenesizoglu and Timmermann (2012), Andersen et al. (2020) among others), predictive QR is relatively understudied. Cenesizoglu and Timmermann (2008) is an early paper on predictive QR, and Maynard et al. (2011), Lee (2016), Fan and Lee (2019), Gungor and Luger (2019), and Cai et al. (2022) recently develop inference methods in predictive QR with nonstationarity and heteroskedasticity. The proposed method is different from these approaches. We consider predictive QR with an increasing number of mixed root predictors and address two important problems raised in the stock returns data. First, the prediction power of each predictor can vary over different quantiles. By adapting the lasso, we allow the model selection based on real data not by the researcher's discretion. Second, the predictors widely used in predicting equity returns are composed of stationary, local unit root, and cointegrated processes. For example, we plot the time-series of two predictors, dividend price ratio (*dp*) and default yield spread (*dfr*), in Figure 1. Even a simple eyeballing test easily confirms their different levels of persistence. (We conduct more informative estimation procedures in Section 3). We provide a unified adaptive lasso framework that allows those mixed root predictors. We show that the estimator converges to the true parameter value at different convergence rates and that the faster convergence rates make the adaptive lasso more efficient in the model selection.

Naturally, some technical challenges arise. Since the seminal paper of Tibshirani (1996), the lasso has been intensively studied in various fields of statistical analysis. However, most studies have been focusing on the *i.i.d.* sample and it has not been a long time since more studies have been conducted with dependent data (see the references in the related literature section below). Furthermore, we allow nonstationary predictors, which impose an additional difficulty in the formal analysis of the proposed adaptive lasso. We tackle these issues by considering a simple model that only contains unit-root predictors first. Once establishing the desired properties of ALQR in this model, we generalize the model so that it includes all stationary, local unit root, and cointegrated predictors.

The contributions of this paper are two-fold. First, to the best of our knowledge, this is the first paper to study predictive QR with an increasing number of mixed root predictors. We propose the adaptive lasso for predictive quantile regression (ALQR) and derive the convergence rates, model selection consistency and asymptotic distributions of ALQR under some regularity conditions. As a by-product, we also prove that the standard QR estimator is consistent under both mixed roots (including the local unit roots) and the increasing dimension of the predictors, which is new in the literature. Second, we conduct an empirical analysis of the stock returns data and find that ALQR can improve prediction performance over existing alternatives across different quantiles. We apply the ALQR method along with the existing alternatives to the data set. The results confirm that ALQR shows better prediction performance across different quantiles, particularly at higher quantiles. As illustrated in Section 3, ALQR can be readily applicable to other applications.

The rest of the paper is organized as follows. This section finishes with a review of the relevant literature. In section 2, we introduce predictive QR models formally and define the ALQR estimator. In section 3, we investigate the performance of ALQR using the out-of-sample quantile prediction problem of stock returns. We study the theoretical properties of ALQR in sections 4 and 5. In section 7, we conduct some Monte Carlo simulation experiments. Section 7 concludes. All technical proofs are relegated to the appendix.

Related Literature The lasso has been extensively studied for cross-sectional data. Recently, there has been development in lasso procedures with dependent data. Basu and Michailidis (2015) exploit the spectral properties of the stationary time series design and investigate the regularity condition of the lasso that leads to the non-asymptotic bounds and the consistency results. Kock and Callot (2015) investigate the oracle property of the lasso in a stationary vector autoregression model. Adamek et al. (2020) provide an inference procedure based on the debiased/desparsified lasso under the near-epoch dependence assumption. Chernozhukov et al. (2021) propose a penalty selection algorithm with weakly dependent data and the post-selection inference procedure. Wu and Wu (2016) and Wong et al. (2020) analyze the lasso with non-sub-Gaussian processes that allow a heavy-tail distribution. Medeiros and Mendes (2016) show the asymptotic properties of the adaptive lasso for stationary high-dimensional time series models. For the cointegrated models, Kock (2016) shows the oracle property of the adaptive lasso in the autoregression model. Interestingly, he finds that the unit root test can be incorporated into the model selection procedure by adopting the Dickey-Fuller form of autoregression. Liao and Phillips (2015) propose a shrinkage estimator with multiple penalty terms to select the rank of cointegration and estimate the parameters simultaneously in a vector error correction model. Liang and Schienle (2019), Zhang et al. (2019), and Onatski and Wang (2018) investigate the same model in the high-dimensional setting.

Koo et al. (2020) recently use lasso to improve the prediction of stock returns. It is shown that lasso significantly reduces forecasting mean squared errors even with a mixture of stationary, unit-root, and cointegrated variables. However, the conventional lasso method may not have model selection consistency and the oracle property as shown by Meinshausen and Bühlmann (2004) and Fan and Li (2001). The adaptive lasso proposed by Zou (2006) improves the performance of the lasso. Instead of imposing the same penalty weight on all candidate parameters, the adaptive lasso penalizes each parameter proportionally to the inverse of its initial estimate. With a proper choice of the tuning parameter λ_n, the adaptive penalty weights for the irrelevant variables approach infinity, whereas those for the relevant variables converge to constants. Lee et al. (2021) apply the adaptive lasso to a predictive mean regression framework. Similar to Koo et al. (2020), predictors are allowed to have different degrees of persistence and cointegration. Lee et al. (2021) find that the adaptive lasso and a newly proposed twin adaptive lasso outperform the alternative methods in terms of predictor selection consistency and out-of-sample mean squared errors.

Some effort has been also made to investigate model selection and model estimation in QR under the *i.i.d.* samples. The ℓ_1-penalized method in the QR framework has been studied for high-dimensional data analysis (see, e.g. Portnoy (1984), Portnoy (1985), Knight and Fu (2000), Koenker (2005), Li and Zhu (2008), Lee et al. (2018) and Belloni and Chernozhukov (2011)). To overcome the problem of inconsistent model selection (Fan et al. (2014); Wang et al. (2012)), recent studies have further considered the adaptive lasso in QR. Wu and Liu (2009) discuss how to conduct model selection for QR models using SCAD and the adaptive lasso method. Zheng et al. (2013) establish the oracle property for an adaptive lasso QR model with heterogeneous error sequences. Zheng et al. (2015) study a globally adaptive lasso method for ultra high-dimensional QR models.

Notation Let $\|\cdot\|$ and $\|\cdot\|_0$ denote ℓ_2-norm and ℓ_0-norm, respectively. For a matrix S, $\|S\|$ represents the spectral norm. Let $F_a(\cdot)$ and $f_a(\cdot)$ denote a cumulative distribution function (CDF) and a probability density function (pdf) of a generic random variable a. Let $S > 0$ denote a generic positive definite matrix S. Let $\lambda_{\min}(S)$ and $\lambda_{\max}(S)$ denote the smallest and largest eigenvalues of S. We use $O_p(1)$ and $o_p(1)$ when a sequence is bounded in probability and converges to zero in probability, respectively. The $O(1)$ and $o(1)$ denote the non-stochastic counterparts.

2 Model and the ALQR Estimator

In this section, we introduce the predictive quantile regression model and the adaptive lasso for quantile regression (ALQR). We will develop the theory for the ALQR in two steps in Section 4. For a better exposition of the theory, we also propose the model and its estimator in two separate cases: (i) unit-root predictors; and (ii) mixed-root predictors.

2.1 QR Model with Unit-Root Predictors

Consider a predictive QR model with unit-root predictors:

$$\begin{aligned} Q_{y_t}(\tau|\mathcal{F}_{t-1}) &:= \mu_{0\tau} + x_{t-1}'\beta_{0\tau} \\ x_t &:= x_{t-1} + v_t, \end{aligned} \tag{1}$$

where $Q_{y_t}(\tau|\mathcal{F}_{t-1})$ is the conditional τ-quantile of y_t, $\mathcal{F}_t = \{z_j\}_{j=-\infty}^{t}$ with $z_j = (y_j, x_j')'$ is a natural filtration, x_t is a p_n-dimensional vector of unit-root predictors with a stationary $O_p(1)$ initialization of $v_0 = \sum_{j=0}^{\infty} D_{vj}\epsilon_{-j}$ following the innovation structure below, and $\beta_{0\tau}$ is the corresponding true parameter vector.

The innovation of unit root predictors, v_t, follows a linear process, which is commonly assumed in the predictive regression literature (see Phillips and Lee (2013, 2016), Cai et al. (2022), Lee et al. (2021) for a few recent papers):

$$v_t = \sum_{j=0}^{\infty} D_{vj}\epsilon_{t-j},\ \epsilon_t \sim mds(0, \Sigma),\ \Sigma > 0,$$

$$E||\epsilon_t||^{2+\kappa} < \infty,\ \kappa > 0,$$

$$D_{v0} = I_p,\ \sum_{j=0}^{\infty} ||D_{vj}|| < \infty,\ D_v(r) = \sum_{j=0}^{\infty} D_{vj}r^j, \text{ and } D_v(1) = \sum_{j=0}^{\infty} D_{vj} > 0,$$

$$\Sigma_{vv} = \sum_{h=-\infty}^{\infty} E(v_t v_{t-h}') = D_v(1)\Sigma D_v(1)',$$

where mds denotes martingale difference sequences with respect to the natural filtration.

We allow the dimension of predictors to increase, i.e. $p_n \to \infty$ as $n \to \infty$. To make notation simple, we omit subscript n and use p unless it may cause any confusion. We define $u_{t\tau} := y_t - Q_{y_t}(\tau|\mathcal{F}_{t-1})$, the deviation of y_t from the conditional τ-quantile. If the CDF of $u_{t\tau}$ is continuous, we have

$$F_{u_{t\tau}}^{-1}(\tau|\mathcal{F}_{t-1}) = F_{u_{t\tau}}^{-1}(\tau) = 0, \quad \text{for all } \tau$$

where $F_{u_{t\tau}}^{-1}$ is the inverse CDF of $u_{t\tau}$. It holds by construction that $\Pr(u_{t\tau} < 0|\mathcal{F}_{t-1}) = \tau$, which is equivalent to $F_{u_{t\tau}}^{-1}(\tau|\mathcal{F}_{t-1}) = 0$. To see the equivalence to the unconditional τ-quantile, we note that

$$\Pr(u_{t\tau} < 0) = E[\mathbf{1}(u_{t\tau} < 0)] = E[E[\mathbf{1}(u_{t\tau} < 0)|\mathcal{F}_{t-1}]] = E[\Pr(u_{t\tau} < 0|\mathcal{F}_{t-1})] = \tau,$$

where $\mathbf{1}(\cdot)$ denotes an indicator function. The equations above imply that both conditional and unconditional τ-quantiles of $u_{t\tau}$ are zero for any given τ. However, it does not mean that two distribution functions are the same, i.e. $F_{u_{t\tau_1}}^{-1}(\tau_2|\mathcal{F}_{t-1}) \neq F_{u_{t\tau_1}}^{-1}(\tau_2)$ for $\tau_1 \neq \tau_2$ in general.

Let $\psi_\tau(u_{t\tau}) := \tau - \mathbf{1}(u_{t\tau} < 0)$. It is easy to find that $\psi_\tau(u_{t\tau})$ is uncorrelated to any predetermined regressor. That is, $\psi_\tau(u_{t\tau})$ is uncorrelated with any past innovations, v_{t-j}, for $j \geq 1$:

$$\begin{aligned} Cov(\psi_\tau(u_{t\tau}), v_{t-j}) &= E[\psi_\tau(u_{t\tau}) \cdot v_{t-j}] - E[\psi_\tau(u_{t\tau})] \cdot E[v_{t-j}] \\ &= E[E[\psi_\tau(u_{t\tau})|\mathcal{F}_{t-1}] \cdot v_{t-j}] - E[E[\psi_\tau(u_{t\tau})|\mathcal{F}_{t-1}]] \cdot E[v_{t-j}] \\ &= 0, \end{aligned}$$

by the law of iterated expectations and the fact that $E\left(\psi_\tau(u_{t\tau})|\, \mathcal{F}_{t-1}\right) = \tau - \Pr\left(u_{t\tau} < 0|\mathcal{F}_{t-1}\right) = 0$. In our settings, however, we allow the QR-induced regression errors $\psi_\tau(u_{t\tau})$ to be contemporaneously correlated with the innovations of unit root sequences. This is commonly assumed in cointegration and predictive regression literature, inducing a potential second order bias arising from the one-sided correlation, see, e.g., Xiao (2009).

We now define the ALQR that minimizes the penalized QR objective function as follows:

$$(\hat{\mu}_\tau^{ALQR}, \hat{\beta}_\tau^{ALQR}) := \underset{\mu \in \mathbf{R}, \beta \in \mathbf{R}^p}{\arg\min} \sum_{t=1}^{n} \rho_\tau(y_t - \mu - x_{t-1}'\beta) + \sum_{j=1}^{p} \lambda_{n,j} |\beta_j|, \tag{2}$$

where $\rho_\tau(u) := u(\tau - \mathbf{1}(u < 0))$. Following Zou (2006), we define the tuning parameter of the penalty term as

$$\lambda_{n,j} := \frac{\lambda_n}{\omega_j} = \frac{\lambda_n}{|\tilde{\beta}_{\tau,j}|^\gamma},$$

where λ_n is a sequence converging to infinity, $\omega_j := |\tilde{\beta}_{\tau,j}|^\gamma$ with a positive constant γ, and $\tilde{\beta}_{\tau,j}$ is a first-step *consistent* estimator for $\beta_{0\tau,j}$. The penalty $\lambda_{n,j}$, which contrasts with the penalty in the standard lasso, is an adaptive weight and it has an inverse relationship with $\tilde{\beta}_{\tau,j}$. The idea of using a consistent first-step estimator as an adaptive weight in $\lambda_{n,j}$ is to avoid over-penalizing important predictors (or under-penalizing irrelevant predictors). In this paper, we use the ordinary QR estimator (Koenker and Bassett (1978)) below as a first-step estimator:

$$(\hat{\mu}_\tau^{QR}, \hat{\beta}_\tau^{QR}) := \underset{\mu \in \mathbf{R}, \beta \in \mathbf{R}^p}{\arg\min} \sum_{t=1}^{n} \rho_\tau(y_t - \mu - x_{t-1}'\beta). \tag{3}$$

The consistency of the ordinary QR estimator is provided in Section 4.

2.2 QR Model with Mixed Roots

We now introduce a QR model with mixed-root predictors. We assume the predictors of the model have different degrees of persistence: $I(0)$, local unit roots, and cointegration. This model is the most relevant in practice. The model in the previous section can be seen as a special case of it.

Let z_t, x_t^c, and x_t be vectors of stationary,[1] cointegrated, and local unit-root predictors whose

[1] In this paper, we use covariance (weak) stationarity unless noted otherwise. The linear process assumption below implies covariance stationarity of z_t. In addition, the first element of z_t is 1 so that $\beta_{0\tau,1}^z$ plays a role of the intercept.

lengths are p_z, p_c, and p_x, respectively. Let $p := p_z + p_c + p_x$. Given a sample of $\{y_t, z_t, x_t^c, x_t\}_{t=1}^n$, the QR model with mixed roots is defined as follows:

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = \; {z_{t-1}}'\beta_{0\tau}^z + {x_{t-1}^c}'\beta_{0\tau}^c + \; {x_{t-1}}'\beta_{0\tau}^x. \tag{4}$$

The cointegrated system in x_t^c has the triangular representation by Phillips (1991): for x_{1t}^c and x_{2t}^c whose dimensions are $p_1 \times 1$ and $p_2 \times 1$, respectively,

$$\begin{aligned} Ax_t^c = x_{1t}^c - A_1 x_{2t}^c &= v_{1t}^c, \\ (I_{p_2} - R_2 L)x_{2t}^c &= v_{2t}^c, \end{aligned} \tag{5}$$

where $R_2 = I_{p_2} + c_2/n$ with $c_2 = \text{diag}\,(\check{c}_1, \ldots, \check{c}_{p_2})$, L is the lag operator, and the vector v_{1t}^c is the vector of $I(0)$ cointegrating residuals. The cointegrated regressors x_t^c are decomposed into x_{1t}^c and x_{2t}^c. The local-to-unity parameter of x_{2t}^c is assumed to be $\check{c}_j \in (-\infty, \infty)$ for $j = 1, \ldots, p_2$. Thus, the cointegrated system in x_t^c includes both stationary and nonstationary local unit root regions. Using (5), we can easily characterize the cointegration relations in x_t^c by the matrices $A \equiv (I_{p_1}, -A_1)$ and A_1.

The near integrated process $x_t = (x_{t1}, ..., x_{tp_x})$ is defined in a similar way:

$$(I_{p_x} - R_x L)x_t = v_t, \tag{6}$$

where $R_x = I_{p_x} + c_x/n$ with $c_x = \text{diag}\,(\tilde{c}_1, ..., \tilde{c}_{p_x})$ and $\tilde{c}_j \in (-\infty, \infty)$. Again, the local-to-unity specification includes the unit root process as a special case when $\tilde{c}_j = 0$.

Let $X_t := (z_t', x_t^{c\prime}, x_t')'$ and $\beta^* := (\beta^{z\prime}, \beta^{c\prime}, \beta^{x\prime})'$. Define a p_c-dimensional vector $v_t^c := ({v_{1t}^c}', {v_{2t}^c}')'$ and a p-dimensional vector $e_t := (z_t', v_t^{c\prime}, v_t')'$. Abusing notation on ϵ_t, we assume that e_t follows a linear process:

$$e_t = D_e(L)\epsilon_t = \sum_{j=0}^{\infty} D_{ej}\epsilon_{t-j},$$

$$\epsilon_t = \begin{pmatrix} \epsilon_{zt} \\ \epsilon_{ct} \\ \epsilon_{vt} \end{pmatrix} \sim mds\,(0, \Sigma) \text{ with } \Sigma_{p\times p} = \begin{pmatrix} \Sigma_{zz} & \Sigma_{zc} & \Sigma_{zv} \\ \Sigma_{zc}' & \Sigma_{cc} & \Sigma_{cv} \\ \Sigma_{zv}' & \Sigma_{cv}' & \Sigma_{vv} \end{pmatrix} > 0,$$

$$E||\epsilon_t||^{2+\kappa} < \infty,\ \kappa > 0,$$

$$D_{e0} = I_p,\ \sum_{j=0}^{\infty} j||D_{ej}|| < \infty,\ D_e(z) = \sum_{j=0}^{\infty} D_{ej}z^j, \text{ and } D_e(1) = \sum_{j=0}^{\infty} D_{ej} > 0,$$

$$\Omega_{ee} = \sum_{h=-\infty}^{\infty} E(e_t e_{t-h}') = D_e(1)\Sigma D_e(1)'.$$

Note that this assumption implies a covariance stationary $O_p(1)$ initialization of $e_0 = (z_0', v_0^{c\prime}, v_0')' = \sum_{j=0}^{\infty} D_{ej}\epsilon_{-j}$.

Similar to Section 2.1, the QR-induced regression errors, $\psi_\tau(u_{t\tau})$, are uncorrelated to any predetermined regressor but allowed to be contemporaneously correlated with the innovations of unit root sequences x_{2t}^c and x_t, the stationary predictor z_t as well as the cointegrating residuals v_{2t}^c.

We define the ALQR for Model (4) as follows:

$$\hat{\beta}_\tau^{ALQR*} = \underset{\beta^* \in \mathbf{R}^p}{\arg\min} \sum_{t=1}^{n} \rho_\tau(y_t - X_{t-1}'\beta^*) + \sum_{j=1}^{p} \lambda_{n,j} |\beta_j^*|, \tag{7}$$

where $\lambda_{n,j} = \lambda_n/\omega_j$, $\omega_j = |\tilde{\beta}_{\tau,j}|^\gamma$, γ is a constant, and $\tilde{\beta}_{\tau,j}$ is a first-step *consistent* estimator. We recommend using the ordinary QR estimator for $\tilde{\beta}_{\tau,j}$. The consistency of QR will be discussed in Section 4.2.

3 Quantile Prediction of Stock Returns

In this section, we consider the quantile prediction problem of stock returns and illustrate the usefulness of the proposed ALQR method. We use an updated version of the data set in Welch and Goyal (2008), which ranges from January 1952 to December 2019. It is composed of 816 monthly observations of the US financial and macroeconomic variables. Our goal is to predict the quantiles of the excess stock returns (y_t) using 12 predictors. The variable names are summarized in Table 1. See also Welch and Goyal (2008) for more details on these variables.

Table 1: Variable Names and Persistency

Notation	Variable Name	High Persistence	AR(1) Coefficient
y	excess stock returns	No	0.102
dp	dividend price ratio	Yes	0.996
dy	dividend yield ratio	Yes	0.996
ep	earnings price ratio	Yes	0.990
bm	book-to-market ratio	Yes	0.994
dfy	default yield spread	Yes	0.969
ntis	net equity expansion	Yes	0.982
lty	long term yield	Yes	0.995
tbl	treasury bill rates	Yes	0.991
svar	stock variance	No	0.475
dfr	default return spread	No	-0.078
ltr	long term rate of returns	No	0.040
infl	inflation	No	0.545

We first check the mixed-root property of the predictors. In Figures 2–3, we plot each predictor using monthly observations. The predictors (dp, dy, ep, bm, dfy, ntis, lty, and tbl) collected in Figure 2 are highly persistent and have quite different patterns from the other four predictors (svar, dfr, ltr and infl) in Figure 3. The first-order autoregression coefficients reported in Table 1 also suggest that predictors have heterogeneous degrees of persistence. We also conduct the Johansen test for cointegration. The results show that the cointegrating rank is 3 in all of the 804-month rolling windows.[2] Thus, the data fit into the mixed-root model structure discussed in Section 2.2.

We next investigate the performance of ALQR in the quantile prediction problem of stock returns. For comparison, we also apply three alternative estimators in the literature: the QR estimator without selecting predictors (QR), the quantile lasso (LASSO), and the unconditional quantile without using any predictor (QUANT). For the tuning parameter of LASSO and ALQR, we use both the Bayesian information criteria (BIC) and the generalized information criteria (GIC), which are explained in detail in Section 6. We evaluate the performance of quantile prediction using the final prediction error (FPE) and the out-of-sample R^2 following Lu and Su (2015). The FPE measures the out-of-sample quantile prediction errors:

$$\text{FPE}(\tau) = \frac{1}{S}\sum_{s=1}^{S} \rho_\tau(y_s - \hat{y}_s), \tag{8}$$

where $\hat{y}_s$ is a prediction for τ-quantile of y_s and S is the number of out-of-sample predictions. Note that $\text{FPE}(\tau)$ averages the quantile loss function. It is different from the standard mean squared prediction error. At each quantile τ, a smaller FPE implies a better quantile prediction. The second measure of the performance is the out-of-sample R^2:

$$R^2(\tau) = 1 - \frac{\sum_{s=1}^{S} \rho_\tau(y_s - \hat{y}_s)}{\sum_{s=1}^{S} \rho_\tau(y_s - \bar{y}_s)}, \tag{9}$$

where $\bar{y}_s$ is the unconditional τ-quantile of y (QUANT) from the training sample. Note that the out-of-sample R^2 measures the performance of each method relative to QUANT. For example, a positive R^2 indicates that the prediction error is smaller than that of QUANT, and a larger R^2 implies a better prediction. By definition, R^2 of QUANT is always zero.

Tables 2–3 summarize the prediction results with BIC. They are based on the one-step-ahead prediction for the last 12 and 24 periods of the sample, respectively. The results with GIC are similar and left in the appendix. We consider 5 different quantiles, $\tau = 0.05, 0.1, 0.5, 0.9$, and 0.95. For each quantile, the best performance results, i.e., the smallest FPE and the highest R^2, are marked in bold. In addition to performance measures, we also report the average number of selected predictors and the corresponding tuning parameter value λ for LASSO and ALQR.

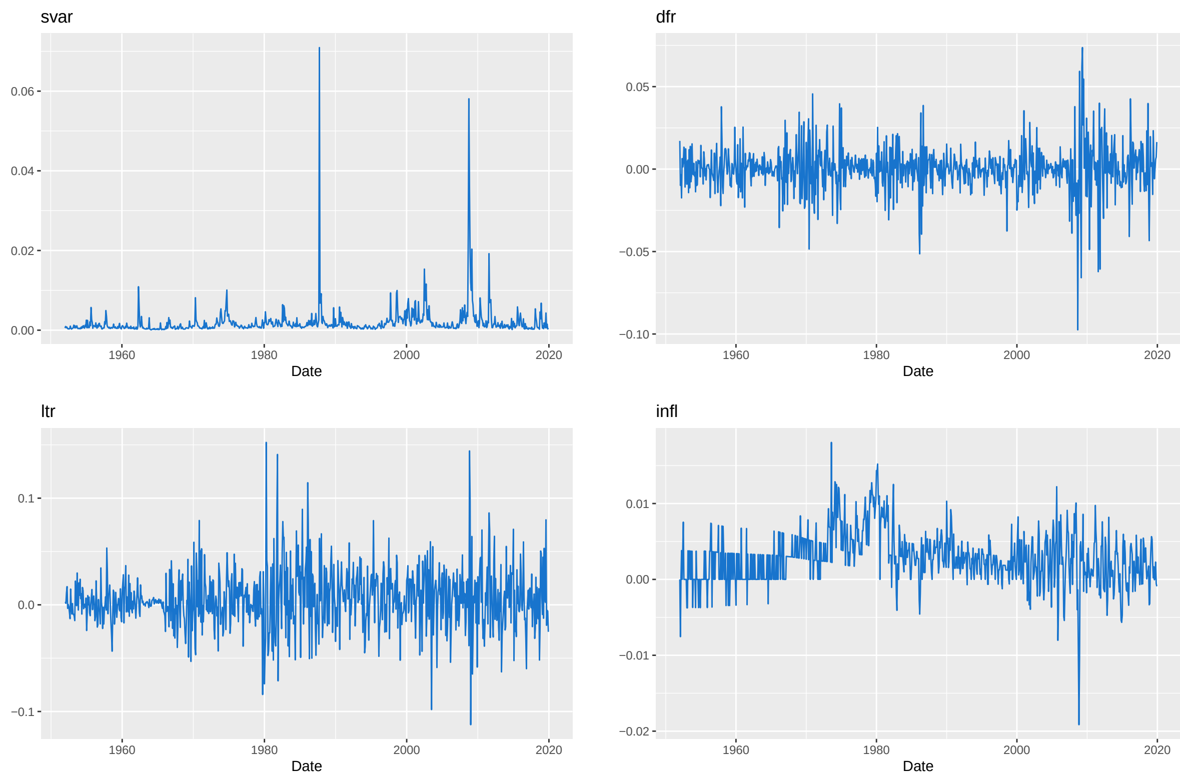
Overall, ALQR shows quite satisfactory results. First, ALQR performs the best in prediction across all quantiles, except for two designs: $\tau = 0.95$ at the 12-period experiment and $\tau = 0.05$ at the 24-period experiment. In the former design, LASSO performs slightly better than ALQR but the difference is negligible. In the latter design, QR performs better than the alternatives. Second, the relative performance of ALQR to QUANT improves substantially for higher quantiles. Looking at R^2, we find that ALQR predicts 44% ($\tau = 0.9$) and 32% ($\tau = 0.95$) better than QUANT at the 12-period design and 28% ($\tau = 0.9$) and 31% ($\tau = 0.95$) at the 24-period design, respectively.

[2] The Johansen cointegration test results are presented in the appendix, Table D.12.

Figure 2: Time-series Plots of Persistent Predictors

Notes: All plots are based on 816 monthly observations ranging from January 1952 to December 2019. The full predictor names are defined in Table 1.

Figure 3: Time-series Plots of Stationary Predictors



Notes: All plots are based on 816 monthly observations ranging from January 1952 to December 2019. The full predictor names are defined in Table 1.

Table 2: Prediction Results of Stock Returns: 12 one-period-ahead forecasts, BIC

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
		Final Prediction Error (FPE)			
QR	0.0047	0.0099	0.0134	0.0078	0.0060
LASSO	**0.0045**	0.0084	0.0123	0.0033	**0.0020**
ALQR	**0.0045**	**0.0083**	**0.0122**	**0.0031**	**0.0020**
QUANT	0.0046	**0.0083**	0.0124	0.0055	0.0029
		Out-of-Sample R^2			
QR	-0.0147	-0.1930	-0.0831	-0.4048	-1.0666
LASSO	0.0219	-0.0045	0.0094	0.4115	**0.3233**
ALQR	**0.0223**	**-0.0012**	**0.0167**	**0.4421**	0.3211
		Average # of Selected Predictors			
LASSO	11.00	9.42	12.00	10.92	10.58
ALQR	10.00	7.83	10.00	6.08	9.25
		Tuning Parameter (λ) by BIC			
LASSO ($\times 10^{-4}$)	82.07	684.27	0.04	263.25	27.66
ALQR ($\times 10^{-7}$)	5.26	89.29	1.46	283.17	0.88

Notes: For each quantile, the best performance result is written in bold font.

QR does not show particularly better prediction performance than QUANT, which is in line with the well-known result in the mean stock return prediction literature (see, e.g., Welch and Goyal (2008)). Third, ALQR selects fewer predictors than LASSO on average. This result is consistent with the report in literature that the standard lasso overselects relevant predictors in practice (See, e.g., Wasserman and Roeder (2009)). In Section 4, we prove the oracle properties of ALQR including model selection consistency. In Section 6, we provide further numerical evidence that the number of selected variables by ALQR is closer to the true sparsity via Monte Carlo simulations.

Finally, we make some remarks from the empirical perspective. In Figure 4, we plot the coefficient estimates of ALQR at the 12-period experiment. To make the graph readable, we divide the predictors into two groups (persistent and stationary) and restrict the quantiles into $\tau = 0.1, 0.5$, and 0.9. First, we observe that predictors have heterogeneous effects across quantiles. This provides useful information on predicting the distribution of stock returns. For high stock returns ($\tau = 0.9$), default yield spread (dfy), dividend price ratio (dp), dividend yield ratio (dy), stock variance (svar), and inflation (infl) show larger effects. However, long term yield (lty), net equity expansion (ntis), treasury bill rates (tbl), and stock variance (svar) show large coefficients for low stock returns ($\tau = 0.1$). Second, the magnitude of the coefficients at the median is relatively smaller than that at both tails. This result coincides with the previous findings in the literature (see, e.g., Welch and Goyal (2008) and Fan and Lee (2019)) that it is more difficult to predict the center part of the stock return distribution. The positive but small R^2's of ALQR at $\tau = 0.5$ in Tables 2–3 also reflect

Table 3: Prediction Results of Stock Returns: 24 one-period-ahead forecasts, BIC

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
			Final Prediction Error (FPE)		
QR	**0.0045**	0.0102	0.0147	0.0067	0.0050
LASSO	0.0058	0.0102	**0.0140**	**0.0045**	**0.0025**
ALQR	0.0058	**0.0097**	**0.0140**	**0.0045**	**0.0025**
QUANT	0.0055	0.0098	0.0148	0.0063	0.0036
			Out-of-Sample R^2		
QR	**0.1751**	-0.0388	0.0100	-0.0745	-0.3938
LASSO	-0.0590	-0.0349	0.0573	0.2767	0.2881
ALQR	-0.0526	**0.0103**	**0.0577**	**0.2798**	**0.3088**
			Average # of Selected Predictors		
LASSO	10.88	8.96	12.00	10.63	10.71
ALQR	9.96	8.88	10.63	10.25	9.67
			Tuning Parameter (λ) by BIC		
LASSO ($\times 10^{-4}$)	66.33	577.58	0.30	210.13	14.76
ALQR ($\times 10^{-7}$)	3.74	24.70	0.65	1.55	0.59

Notes: For each quantile, the best performance result is written in bold font.

such an aspect. Third, the coefficient of stock variance (svar) swings from a large positive value at $\tau = 0.9$ to a large negative number at $\tau = 0.1$. Since the return distribution would spread out when the market is volatile, this result is consistent with the stylized fact in the financial market.

4 Oracle Properties of ALQR Estimators

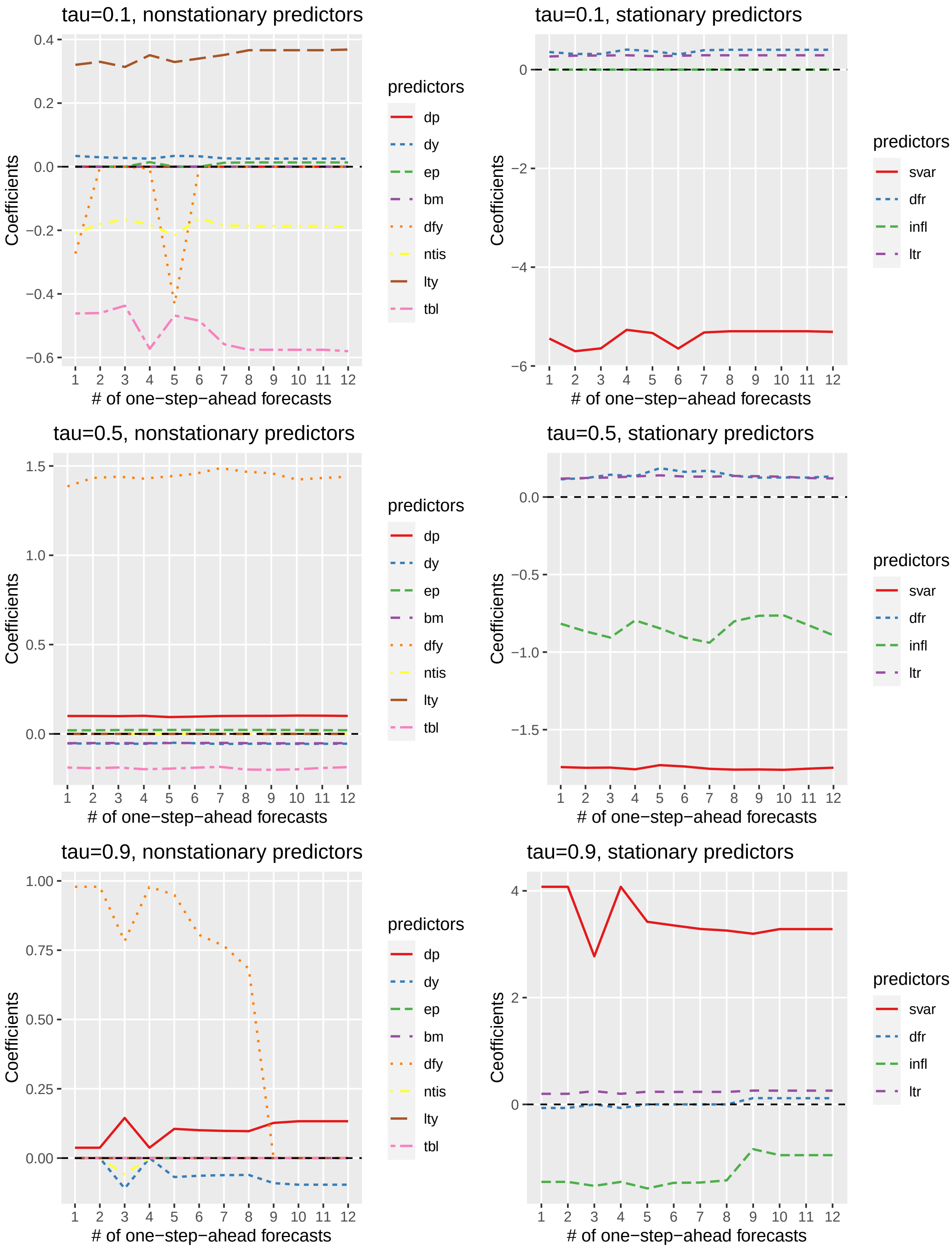
This section investigates the asymptotic properties of ALQR. We show that the oracle properties (Fan and Li (2001)) of the adaptive lasso remain valid in the QR framework when predictors have both increasing dimensions and various degrees of persistence. This result provides theoretical support for applying the ALQR method to stock returns prediction in Section 3. In Section 4.1, we discuss the case that all predictors are $I(1)$. Section 4.2 then generalizes the results to the case of mixed root predictors.

4.1 Unit-Root Predictors

Consider the QR model in Section 2.1 with unit-root predictors only. The following assumptions provide regularity conditions.

Assumption 4.1 (Assumption f) *(i) The distribution function of $u_{t\tau}$, $F(\cdot)$, has a continuous density $f(\cdot)$ with $f(a) > 0$ on $\{a : 0 < F(a) < 1\}$. (ii) The derivative of conditional distribution*

Figure 4: The ALQR (with BIC) Estimated Coefficients



function $F_{t-1}(a) = Pr[u_{t\tau} < a|\mathcal{F}_{t-1}]$, *which we denote* $f_{t-1}(\cdot)$, *is continuous and uniformly bounded above by a finite constant* c_f. *(iii) For any sequence* $\zeta_n \to F^{-1}(\tau)$, $f_{t-1}(\zeta_n)$ *is uniformly integrable, and* $E[f_{t-1}^{1+\eta}(F^{-1}(\tau))] < \infty$ *for some* $\eta > 0$.

For the bound and rate conditions, we define $A_{(t,p)} := E\left[f_{t-1}(0)x_{t-1}x_{t-1}'\right]$ and $B_{(t,p)} := E\left[\psi_\tau(u_{t\tau})^2 x_{t-1}x_{t-1}'\right]$.

Assumption 4.2 (Assumption $L1$**)** *For each* t *and* p, *there exist some constants* $\underline{c}_{A(t,p)}$ *and* $\underline{c}_{B(t,p)}$ *such that* $0 < \underline{c}_{A(t,p)} \leq \lambda_{\min}(\frac{A_{(t,p)}}{t})$ *and* $0 < \underline{c}_{B(t,p)} \leq \lambda_{\min}(\frac{B_{(t,p)}}{t})$.

Note that, for each t and p, $\lambda_{\max}(A_{(t,p)}/p)$ and $\lambda_{\max}(B_{(t,p)}/p)$ are bounded above since $|f_{t-1}(\cdot)|$ and $|\psi_\tau(\cdot)|$ are uniformly bounded and the variance of x_{t-1} is finite by the design of v_t. Let $\bar{c}_{A(t,p)} < \infty$ and $\bar{c}_{B(t,p)} < \infty$ be these finite bounds. We next define the following uniform bounds over t for each n: $\underline{c}_{A(n,p)} := \min_{1\leq t\leq n} \underline{c}_{A(t,p)}$, $\bar{c}_{A(n,p)} := \max_{1\leq t\leq n} \bar{c}_{A(t,p)}$, and $\bar{c}_{B(n,p)} := \max_{1\leq t\leq n} \bar{c}_{B(t,p)}$.

Assumption 4.3 (Assumption $U1$**)** *For some* $\alpha > 0$, *(i)* $p = n^\zeta$ *with* $0 < \alpha\zeta < 1$; *(ii)* $\frac{\bar{c}_{A(n,p)}^{1/2} n^{1/2}}{p^\alpha} \vee \frac{\bar{c}_{B(n,p)}^{1/2}}{p^\alpha} = o(\underline{c}_{A(n,p)})$; *(iii)* $\frac{p^{3/2}}{n^{2+\alpha\zeta}\bar{c}_{A(n,p)}^{1/2}} = \frac{n^{(3/2)\zeta}}{n^{2+\alpha\zeta}\bar{c}_{A(n,p)}^{1/2}} = o(1)$.

Assumption 4.4 (Assumption $\lambda 1$**)** λ_n *satisfies that (i)* $\frac{\lambda_n n^{\frac{1}{2}\zeta}}{n^{1+\alpha\zeta}\underline{c}_{A(n,p)}} \to 0$; *(ii)* $\frac{\lambda_n n^{(1-\alpha\zeta)\gamma}}{n^{2+\alpha\zeta}\bar{c}_{A(n,p)}^{1/2}} \to \infty$, *with* $\gamma > 0$.

Assumption 4.5 *Let* $q_n = \|\mathcal{A}_n\|_0$ *be the size of the true active set* $\mathcal{A}_n$. *There exist positive constants* c_q, c_β *and* c_0 *such that (i)* $q_n = O(n^{c_q})$ *with* $c_q < 1/3$;*(ii)* $2c_q < c_\beta < 1$ *and (iii)*

$$n^{(1-c_\beta)/2} \cdot \min_{1\leq j\leq q_n} \left|\beta_{0j}\right| \geq c_0.$$

We make some remarks on these assumptions. Assumption f is a standard restriction in the QR literature on the conditional density of regression residuals. Assumptions $L1$ and $U1$ modify the conventional conditions from stationary QR literature to allow for serial dependence in predictors and regression errors. In particular, we modify Assumption $A.2$ of Lu and Su (2015) to accommodate nonstationary x_t. Based on the model settings in Section 2.1, we find that

$$E\left[x_{t-1}x_{t-1}'\right] = t \cdot \Sigma_{vv} + o(t),$$

for $t = \lfloor rn \rfloor$ with $r \in (0,1)$, as $n \to \infty$. Therefore, we impose a similar set of restricted eigenvalue conditions for $\frac{E\left[x_{t-1}x_{t-1}'\right]}{t}$. This is a natural extension of the existing conditions in the *i.i.d.* or stationary QR setup to a nonstationary time series model with an increasing dimension. We allow the upper bounds $\bar{c}_{A(t,p)}$, $\bar{c}_{B(t,p)}$ and the lower bounds $\underline{c}_{A(t,p)}$, $\underline{c}_{B(t,p)}$ to depend on the dimension of predictors, hence the upper bounds can diverge to infinity and the lower bounds can converge to zero when $p \to \infty$ as $n \to \infty$. However, Assumption $U1$ (ii) imposes further restrictions on the upper bounds: The rate of convergence in p is slow enough so that (n/p^α)-consistency of the

ALQR estimator can be achieved. Specifically, Assumption $L1$ prevents high correlation between predictor innovations. Note that $\lambda_{\min}(E[x_{t-1}x_{t-1}]/t) = 0$ if there exits perfect multicollinearity between predictor innovations. Also, Assumption $U1$ restricts the strength of the contemporaneous correlation in v_t by limiting nonzero off-diagonal terms in Σ_{vv}.[3] In Assumption $U1$ (i), the condition $0 < \alpha\zeta < 1$ imposes $a_n = \frac{p^{\alpha}}{n} = \frac{n^{\alpha\zeta}}{n} \to 0$, indicating the restriction on the growth rate of the number of parameters. In the appendix, we discuss the technicality of these restrictions on p and λ in detail. Assumption $\lambda1$ collects a set of rate conditions on λ_n to show the asymptotic properties of the ALQR estimators. Assumption $\lambda1$ (i) is comparable to the condition of $\lambda_n/\sqrt{n} \to 0$ in a stationary time series model with fixed dimensions. This condition allows (n/p^{α})-rate under the lasso variable selection. Assumption $\lambda1$ (ii) presents a necessary condition for model selection consistency of ALQR, which depends on the consistency of the initial estimator for $\beta_{0\tau}$ (as shown in the appendix, proof of Theorem 4.2). Note that this condition is developed given that the ordinary QR estimator defined in (3) is used as an initial estimator for $\beta_{0\tau}$. It can be generalized further since we do not require a certain rate of consistency for the initial estimator. If another consistent estimator is used, condition (ii) may be adjusted to the corresponding rate of consistency. Assumption 4.5 is for the minimum signal strength of the coefficients in the true active set, which we adopted from Sherwood and Wang (2016).

Next we show that the ordinary QR estimator is consistent and can be a qualified initial estimator for ALQR.

Lemma 4.1 ***(Consistency of QR Estimator with unit-root predictors)*** *Under Assumptions 4.1–4.3, the QR estimator defined in (3) satisfies*

$$\left\|\hat{\beta}_{\tau}^{QR} - \beta_{0\tau}\right\| = O_p\left(\frac{p^{\alpha}}{n}\right).$$

It is well-known that the rate of convergence is $\frac{1}{n^{1/2}}$ with fixed p and $\frac{p^{1/2}}{n^{1/2}}$ with increasing p when predictors are $I(0)$. This indicates that the information loss (or the degrees of freedom) to estimate the increasing number of parameters is $p^{1/2}$. It is also known that when the predictors are $I(1)$, the rate of convergence becomes $\frac{1}{n}$ with fixed p. The super-consistency is caused by the stronger signal of $X'X$, where X is a $n \times p$ matrix of predictors. To allow for increasing p, Lemma 4.1 extends the existing result for nonstationary QR and finds that the rate becomes $\frac{p^{\alpha}}{n} = \frac{p^{1/2} \cdot p^{\alpha-1/2}}{n}$, where the additional rate loss $p^{\alpha-1/2}$ comes from the increasing singularity of $E[x_t x_t']$ as summarized in Assumptions $L1$ and $U1$. In contrast, we have $0 < [\lambda_{\min}(E[x_t x_t'])] < [\lambda_{\max}(E[x_t x_t'])] < \infty$ for $I(0)$ predictors with $p/n \to 0$.

Taking the QR estimator of Lemma 4.1 as an initial estimator for $\beta_{0\tau}$, we show the consistency

[3] In the i.i.d. mean regression model with highly correlated regressors, it is well known that the lasso performs worse than the ridge estimator. To accommodate the lasso with highly correlated regressors, researchers have also developed variants of the lasso such as the elastic net (Zou and Hastie, 2005) and the group lasso (Yuan and Lin, 2006). As shown in Figure D.1 in the appendix, however, most predictor innovations in our empirical application are not highly correlated to each other and satisfy this assumption. Furthermore, we investigate the performance of the proposed ALQR estimator when predictor innovations are highly correlated in the simulation experiments.

of ALQR and model selection consistency under unit roots.

Theorem 4.1 ***(Consistency of ALQR under Unit Roots)*** *Under Assumptions 4.1–4.4, the ALQR estimator defined in (2) satisfies*

$$\left\|\hat{\beta}_\tau^{ALQR} - \beta_{0\tau}\right\| = O_p\left(\frac{p^\alpha}{n}\right).$$

Theorem 4.1 confirms that both the ALQR estimator and the QR estimator of Lemma 4.1 are (n/p^α)-consistent. With a proper choice of the penalty parameter λ_n, the rate of consistency for the ALQR estimate is not affected. As we show in the proof of Theorem 4.1, this estimation rate is possible under Assumption λ1.

Theorem 4.2 ***(Model Selection Consistency under Unit Roots)*** *Let* $\hat{\mathcal{A}}_n := \{j : \hat{\beta}_{\tau,j}^{ALQR} \neq 0\}$ *and* $\mathcal{A}_0 := \{j : \beta_{0\tau,j} \neq 0\}$*. Under Assumptions 4.1–4.5, it holds that*

$$\Pr(j \in \hat{\mathcal{A}}_n) \longrightarrow 0, \quad \text{for } j \notin \mathcal{A}_0,$$

as $n \to \infty$.

Theorem 4.2 shows that the ALQR procedure is an oracle procedure. When the predictors with diverging dimensions are all unit root, the adaptive lasso is at least as good as other "oracle" procedures in terms of variable selection. This is a major difference from the standard lasso, which does not enjoy the oracle properties. As in Zou (2006), with a proper choice of λ_n, the adaptive lasso procedure can simultaneously estimate relevant coefficients and remove unimportant ones with probability approaching unity. In this study, we further develop the result of Zou (2006) and generalize the adaptive lasso procedure to a nonstationary QR model. Note that the consistency in model selection can be achieved only under Assumption λ1.

4.2 Mixed-Root Predictors

In this section, we consider a general QR model discussed in Section 2.2:

$$\begin{aligned} Q_{y_t}(\tau|\mathcal{F}_{t-1}) &= z_{t-1}'\beta_{0\tau}^z + x_{1t-1}^{c}{}'\beta_{0,1\tau}^c + x_{2t-1}^{c}{}'\beta_{0,2\tau}^c + x_{t-1}'\beta_{0\tau}^x \\ &= (z_{t-1}',\ x_{1t-1}^{c}{}', x_{2t-1}^{c}{}')\beta_{0\tau}^{(0)} + x_{t-1}'\beta_{0\tau}^{(1)} \\ &= X_{t-1}'\beta_\tau \end{aligned} \tag{10}$$

where $X_t = (z_t',\ x_{1t}^{c}{}', x_{2t}^{c}{}', x_t')'$ and $\beta_{0\tau} = (\beta_{0\tau}^{z'}, \beta_{0,1\tau}^{c'}, \beta_{0,2\tau}^{c'}, \beta_{0\tau}^{x'})'$ are p-dimensional vectors. This model allows predictors to have heterogeneous degrees of persistence as well as increasing dimensions. The mixed-root predictors in X_t include stationary (z_t), local unit root (x_t), and cointegrated (x_{1t}^c, x_{2t}^c) processes.

To capture the cointegration relation between x_{1t}^c and x_{2t}^c, we define a transformation matrix H as follows:

$$H_{(p\times p)} := \begin{pmatrix} I_{p_z} & 0 & 0 & 0 \\ 0 & I_{p_1} & 0 & 0 \\ 0 & A_1' & I_{p_2} & 0 \\ 0 & 0 & 0 & I_{p_x} \end{pmatrix}.$$

Using H, the original model can be rewritten as:

$$\begin{aligned} Q_{y_t}(\tau|\mathcal{F}_{t-1}) &= X_{t-1}'\beta_{0\tau} = \left(X_{t-1}'H^{-1}\right)H\beta_{0\tau} \\ &\equiv \tilde{x}_{t-1}'\tilde{\beta}_{0\tau}, \end{aligned} \tag{11}$$

where $\tilde{\beta}_{0\tau} := H\beta_{0\tau} = (\beta_{0\tau}^{z\prime}, \beta_{0,1\tau}^{c\prime}, (A_1'\beta_{0,1\tau}^c + \beta_{0,2\tau}^c)', \beta_{0\tau}^{x\prime})'$ and $\tilde{x}_{t-1}' := X_{t-1}'H^{-1} = (z_t',\ v_{1t}^{c}{}', x_{2t}^{c}{}', x_t') \equiv \left(w_t^{(0)\prime}, w_t^{(1)\prime}\right)$. The $w_t^{(0)} = (z_t', v_{1t}^{c}{}')'$ is a (p_z+p_1)-dimensional vector that collects all $I(0)$ processes, and $\tilde{\beta}_{0\tau}^{(0)}$ is the corresponding (p_z+p_1)-dimensional QR regression coefficient. The $w_t^{(1)} = (x_{2t}^{c}{}', x_t')'$ is a (p_2+p_x)-dimensional vector of all $I(1)$ processes, $\tilde{\beta}_{0\tau}^{(1)}$ is the corresponding (p_2+p_x)-dimensional QR regression coefficient. Since the cointegration rank of this system is p_1, we let $r := p_z + p_1$ be the number of $I(0)$ predictors in $\tilde{x}_{t-1}$. Note that the discussion in Section 4.1 can be considered as a special case of the mixed root model with $r = 0$. In this section, we generalize the results in Section 4.1 and examine a more practical model with $r > 0$.

The following assumptions provide additional regularity conditions for mixed roots. Abusing notation slightly, we will use the same notation for the sequences of bounds. Let M_n be a diagonal matrix of the form

$$M_n := \begin{pmatrix} \sqrt{n}I_r & 0 \\ 0 & I_{p-r} \end{pmatrix}.$$

Assumption 4.6 (Assumption $L2$) *There exist $\underline{c}_{A(n,p)}$ and $\underline{c}_{B(n,p)}$ such that*

$$0 < \underline{c}_{A(n,p)} \le \lambda_{\min}\left(\frac{M_n'\sum_{t=1}^n E\left[f_{t-1}(0)\tilde{x}_{t-1}\tilde{x}_{t-1}'\right]M_n}{n^2}\right)$$

and

$$0 < \underline{c}_{B(n,p)} \le \lambda_{\min}\left(\frac{M_n'\sum_{t=1}^n E\left[\psi_\tau(u_{t\tau})^2\tilde{x}_{t-1}\tilde{x}_{t-1}'\right]M_n}{n^2}\right)$$

as $n \to \infty$.

Assumption 4.7 (Assumption $U2$) *There exist $\bar{c}_{A(n,p)}$ and $\bar{c}_{B(n,p)}$ such that*

$$\lambda_{\max}\left(\frac{M_n'\sum_{t=1}^n E\left[f_{t-1}(0)\tilde{x}_{t-1}\tilde{x}_{t-1}'\right]M_n}{n^2}\right) \le \bar{c}_{A(n,p)} < \infty$$

and

$$\lambda_{\max}\left(\frac{M_n'\sum_{t=1}^n E\left[\psi_\tau(u_{t\tau})^2\tilde{x}_{t-1}\tilde{x}_{t-1}'\right]M_n}{n^2}\right)\le \bar{c}_{B(n,p)}<\infty$$

as $n\to\infty$.

Assumption 4.8 (Assumption $\lambda 2$) *When* $\lambda_n\to\infty$ *as* $n\to\infty$, *(i)* $\lambda_n\frac{n^{\frac{1}{2}\zeta}}{n^{\frac{1}{2}+\alpha\zeta}\underline{c}_{A(n,p)}}\to 0$; *(ii)* $\lambda_n\frac{n^{(1/2-\alpha\zeta)\gamma}}{n^{2+\alpha\zeta}\bar{c}_{A(n,p)}^{1/2}}\to\infty$.

Assumption $L2$, $U2$, and $\lambda 2$ are analogies to Assumption $L1$, $U1$ and $\lambda 1$ in Section 4.1. They are essential regularity conditions used to show ALQR's consistency in parameter estimation and model selection under mixed roots. Compared with Assumption $\lambda 1$, Assumption $\lambda 2$ are more restrictive on the choice of λ_n, because λ_n of Assumption $\lambda 2$ accommodates both stationary and local unit root predictors.

We first show the consistency of QR for the transformed model in (11).

Lemma 4.2 ***(Consistency of the transformed QR Estimator under Mixed Roots)*** *Under Assumptions 4.1, 4.3, 4.6 and 4.7, we have*

$$\left\|\hat{\beta}_\tau^{(0),QR}-\tilde{\beta}_{0\tau}^{(0)}\right\|=O_p\left(\frac{p^\alpha}{\sqrt{n}}\right),\ \textit{and}\ \left\|\hat{\beta}_\tau^{(1),QR}-\tilde{\beta}_{0\tau}^{(1)}\right\|=O_p\left(\frac{p^\alpha}{n}\right).$$

In a mixed root model, Lemma 4.2 shows that the QR estimators for the $I(0)$ predictors and $I(1)$ predictors have different convergence rates. As we discussed in the previous section, the faster convergence rate for the $I(1)$ predictors comes from the stronger signal of $E\left[w_t^{(1)}w_t^{(1)\prime}\right]$. For the later use of these rates, we define $a_n^{(0)}:=(p^\alpha/\sqrt{n})$ and $a_n^{(1)}:=(p^\alpha/n)$.

Based on the definition of the transformed model, we know that Lemma 4.2 can further imply consistency of the QR estimator in the original model (4). Since all parameters are identical in model (4) and (11) except $\tilde{\beta}_{2\tau}^c=A_1'\hat{\beta}_{1\tau}^c+\hat{\beta}_{2\tau}^c$, the consistency of QR estimator in (4) can be verified if we can show that $\hat{\beta}_{2\tau}^c$ is an $a_n^{(0)}$-consistent estimator for $\beta_{0,2\tau}^c$. To see the convergence rate of $\hat{\beta}_{2\tau}^c$, we have

$$\begin{aligned}
\hat{\beta}_{2\tau}^c &= \left(\tilde{\beta}_{2\tau}^c\right)^{QR}-A_1'\hat{\beta}_{1\tau}^c\\
&= \left(\left(\tilde{\beta}_{2\tau}^c\right)^{QR}-\tilde{\beta}_{0,2\tau}^c\right)-A_1'\left(\hat{\beta}_{1\tau}^c-\beta_{0,1\tau}^c\right)+\left(\tilde{\beta}_{0,2\tau}^c-A_1'\beta_{0,1\tau}^c\right)\\
&= O_p\left(a_n^{(1)}\right)+O_p\left(a_n^{(0)}\right)+\beta_{0,2\tau}^c\\
&= \beta_{0,2\tau}^c+O_p\left(a_n^{(0)}\right).
\end{aligned}$$

Thus, the $I(0)$ rate dominates, and $\left\|\hat{\beta}_{2\tau}^c-\beta_{0,2\tau}^c\right\|=O_p\left(a_n^{(0)}\right)$.

These results are summarized in the following corollary.

Corollary 4.3 ***(Consistency of the QR Estimator under Mixed Roots)*** *Under Assumptions 4.1, 4.3, 4.6 and 4.7,we have*

$$\left\|\hat{\beta}_{\tau}^{(0),QR*} - \beta_{0\tau}^{(0)}\right\| = O_p\left(a_n^{(0)}\right), \text{ and } \left\|\hat{\beta}_{\tau}^{(1),QR*} - \beta_{0\tau}^{(1)}\right\| = O_p\left(a_n^{(1)}\right).$$

Note that $\beta_{0\tau}^{(0)}$ is the $(p_z + p_1 + p_2)$-dimensional QR regression coefficient, and $\beta_{0\tau}^{(1)}$ is the p_x-dimensional QR regression coefficient, as given in Equation (10). We now investigate the oracle properties of ALQR with mixed roots.

Theorem 4.4 ***(Consistency of ALQR under Mixed Roots)*** *Under Assumptions 4.1, 4.3, 4.6-4.8, we have*

$$\left\|\hat{\beta}_{\tau}^{(0),ALQR*} - \beta_{0\tau}^{(0)}\right\| = O_p\left(a_n^{(0)}\right), \text{ and } \left\|\hat{\beta}_{\tau}^{(1),ALQR*} - \beta_{0\tau}^{(1)}\right\| = O_p\left(a_n^{(1)}\right).$$

Theorem 4.5 ***(Model Selection Consistency under Mixed Roots)*** *Under Assumptions 4.1, 4.3, 4.5-4.8, it holds that*

$$\Pr(j \in \hat{\mathcal{A}}_n) \longrightarrow 0, \quad \text{for } j \notin \mathcal{A}_0,$$

where $\hat{\mathcal{A}}_n = \{j : \hat{\beta}_{\tau,j}^{ALQR*} \neq 0\}$ *and* $\mathcal{A}_0 = \{j : \beta_{0\tau,j} \neq 0\}$.

It is worth emphasizing that the ALQR procedure does not require practitioners to conduct any pretest to identify different types of predictors. With the proper choice of λ_n along with consistent initial estimates of the parameters, ALQR can simultaneously estimate relevant regression coefficients and provide the consistent variable selection for the mixed root QR model.

5 Asymptotic Distributions of ALQR Estimators

In this section, we derive the asymptotic distributions of the proposed ALQR estimators. In QR literature, the Convexity Lemma (Pollard (1991)) typically provides a convenient limit theory, bypassing the stochastic equicontinuity argument, see Section 4 of Koenker (2005). Unfortunately, for the increasing dimension, we cannot use the Convexity Lemma which is defined in $\mathbb{R}^p$ with a fixed $p < \infty$. Thus we modify the classical chaining argument of Ruppert and Carroll (1980) to our ALQR framework. Lu and Su (2015) recently modified the proof of Ruppert and Carroll (1980) to show the asymptotic distribution of Jackknife Model Averaging QR with *iid* regressors of increasing dimensions. We follow a similar proof strategy but substantially refine the results to prove the distributional limit theory with weakly dependent regressors of increasing dimensions, along with the adaptive lasso penalty functions. For the asymptotic distribution with non-cointegrated local unit root predictors, we assume that the number of predictors (p_x) is fixed. Note that $p = p_z + p_1 + p_2 + p_x$ is still allowed to increase, even with this additional restriction of $p_x < \infty$.

We now state the additional assumptions we need to prove the asymptotic distributions of the ALQR estimators.

Assumption 5.1 *The $p_z \times 1$ vector of stationary predictors $z_t = \sum_{j=0}^{\infty} D_{zj}\epsilon_{t-j}$ satisfies the following condition from Corollary 2 of Withers (1981)*

$$\|D_{zj}\| = O\left(e^{-vj}\right) \text{ with } v > 0$$

along with the conditions (1), (2), (5) of Withers (1981), provided in the appendix for brevity, see (C.27)-(C.29) in Section C of the appendix. Then z_t is strong mixing with the strong mixing number $\alpha(j) = O(e^{-v\lambda j})$ with another constant $\lambda > 0$.

Assumption 5.2 $\sup_{j\geq 1} E(z_{t-1,j}^8) \leq c_z$ *for some $c_z \leq \infty$.*

Assumption 5.3 *As $n \to \infty$, $\underline{c}_B^{-1/2}\overline{c}_A^{1/2}p^2 n^{-1/2} \to 0$, and $n^{-3}p^{12}\underline{c}_B^{-4} \to 0$.*

Assumption 5.4 *We require p_x (the number of non-cointegrated local unit root predictors) to be finite: $p_x < \infty$.*

Assumption 5.1 is a strong mixing condition with the specific mixing rates to use the Bernstein Inequality of Merlevède et al. (2009). Assumption 5.2 is the moment condition we adopt from Lu and Su (2015). The moment condition is stronger than the typical assumptions when the number of the regressors is fixed. Assumption 5.3 is another set of combined rate conditions required to prove the asymptotic theories of ALQR estimators. Finally, Assumption 5.4 restricts the number of non-cointegrated local unit root predictors, which is also assumed in Koo et al. (2020) who study the mean regressions with increasing dimensions. The distributional QR limit theory with the increasing number of (non-cointegrated) unit-root regressors is an open question, and we leave it for a future research.

We are now ready to provide the asymptotic distributions of ALQR estimators for the coefficients in the true active set, which we denote $\hat{\beta}_\tau^{(i),ALQR*}(1)$ with $i = 1$ and 2 for $I(0)$ and $I(1)$ predictors, respectively. Let R be a generic constant $l \times p_i$ matrix, where l is fixed and p_i is defined conformably below. Notation $\Longrightarrow$ indicates the convergence in distribution.

Theorem 5.1 ***(Asymptotic Distributions of ALQR Estimators)*** *When Assumptions 4.1, 4.3, 4.5-4.8, 5.1-5.4 hold,*

(i) For I(0) active *predictors (the first $p_z + p_1$ predictors; hence $p_i = p_z + p_1$), the first q_n non-zero elements has the following limit theory:*

$$\sqrt{n}R\left[\hat{\beta}_\tau^{(0),ALQR*}(1) - \beta_{0\tau}^{(0)}(1)\right] \Longrightarrow N\left(0, RA_{(t,p)}^{-1}B_{(t,p)}A_{(t,p)}^{-1}R'\right)$$

(ii) For the middle p_2 predictors (the parameters with respect to the active *cointegrated predictors; hence $p_i = p_2$), the first q_n non-zero elements has the following limit theory:*

$$\sqrt{n}R\left[\hat{\beta}_\tau^{(1),ALQR*}(1) - \beta_{0\tau}^{(1)}(1)\right] \Longrightarrow N\left(0, RA_1'A_{(t,p)}^{-1}B_{(t,p)}A_{(t,p)}^{-1}A_1R'\right)$$

(iii) For the last p_x (non-cointegrated) active *local unit root predictors, with $p_x < \infty$, the first q_n non-zero elements has the following limit theory:*

$$n\left[\hat{\beta}_{\tau}^{(1),ALQR*}(1) - \beta_{0\tau}^{(1)}(1)\right] \Longrightarrow \left(M_{\beta_\tau}^x\right)^{-1} G_\tau^x,$$

where

$$G_\tau^x = \int J_x^c(r) dB_{\psi_\tau}, \text{ and } M_{\beta_\tau}^x = f_u(0)^{-1} \int J_x^c(r) J_x^c(r)' dr$$

and $B_{\psi_\tau} := BM(\tau(1-\tau))$ is a Brownian motion and $J_x^c(r) = \int_0^r e^{(r-s)C} dB_x(s)$ is an Ornstein-Uhlenbeck (OU) process.

The result of Theorem 5.1-(i) is in line with Theorem 2 of Medeiros and Mendes (2016) who studied the asymptotic distributions of adaptive lasso estimators in mean regressions. It is also interesting to see that there is an additional nuisance parameter A_1 (the cointegrating vector) in the asymptotic distribution of ALQR with the cointegrated predictors.

6 Monte Carlo Simulations

In this section, we conduct a set of Monte Carlo simulation studies to evaluate the forecasting performance of the proposed ALQR method. As is motivated by the data patterns of the stock returns application in Section 3, we construct a simulation environment that has 12 predictors. As seen in Table 1, we allow for mix-root predictors and let 8 of them be persistent.

Based on the monthly observations of the stationary predictors (*svar*, *dfr*, *infl* and *ltr*), we calibrate the structure of z_t by estimating a VAR(p) model with a Bayesian information criterion (BIC). An estimated VAR(2) model is obtained as follows:

$$\begin{aligned} z_t &= \begin{pmatrix} 0.427 & -0.059 & 0 & -0.017 \\ 0 & 0 & 0 & 0.074 \\ 0 & 0.022 & 0.538 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} z_{t-1} + \begin{pmatrix} 0.208 & 0 & 0 & 0 \\ 0.421 & -0.089 & -0.312 & 0 \\ 0.092 & 0.023 & 0.239 & 0 \\ 0 & 0 & 1.038 & 0 \end{pmatrix} z_{t-2} \\ &\quad + u_{zt}, \end{aligned}$$

where $u_{zt} \sim N(0,1)$. To identify the cointegrating relations among persistent predictors, we apply the Johansen test. The estimated cointegrating rank is 3. Thus, we generate a set of cointegrating predictors from the following model:

$$(x_{1t}^c, x_{2t}^c)' = \begin{pmatrix} 0.14 & 0.8495 & 0.0039 & -0.1545 \\ 0.19 & 0.8084 & 0.0033 & -0.0507 \\ -1.24 & 1.2438 & 0.9788 & 0.3206 \\ 0.02 & -0.0205 & 0.0009 & 0.9835 \end{pmatrix} (x_{1,t-1}^c, x_{2,t-1}^c)' + v_t^c,$$

where $v_t^c \sim N(0,1)$. For unit-root predictors, we use *i.i.d.* $N(0,1)$ innovations with stationary initializations.

We consider the following scenarios with different numbers of zero coefficients:
(1) 6 non-zero coefficients:

$$\beta^z = (-0.4, \mathbf{0}, \mathbf{0}, 0.1), \quad (\beta^{c\prime}, \beta^{x\prime}) = (0.2, -0.4, 0.1, 0.5, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0});$$

(2) 8 non-zero coefficients:

$$\beta^z = (-0.4, \mathbf{0}, \mathbf{0}, 0.1), (\beta^{c\prime}, \beta^{x\prime}) = (0.2, -0.4, 0.1, 0.5, 0.2, 0.15, \mathbf{0}, \mathbf{0});$$

(3) 12 non-zero coefficients (no zero coefficient):

$$\beta^z = (-0.4, 0.2, 0.15, 0.1), (\beta^{c\prime}, \beta^{x\prime}) = (0.2, -0.4, 0.1, 0.5, 0.2, 0.15, 0.1, -0.05).$$

The sample size is set to be $n = 1000$. The last 12 periods are used for out-of-sample prediction evaluation and the remaining periods are used for in-sample estimation. The quantiles of interest are 0.05, 0.1, 0.5, 0.9, 0.95. The prediction performance is measured by the final prediction error (FPE) and the out-of-sample R^2 defined in (8) and (9), respectively. The performance measures are computed by averaging over 1000 replications.

We briefly discuss the choice of tuning parameter λ. For ALQR, we recommend using the Bayesian information criterion (BIC) proposed by Wang and Leng (2007) or the generalized information criterion (GIC) proposed by Fan and Tang (2013) and Zheng et al. (2015). The objective function for choosing λ is defined as

$$\log\left(\frac{1}{n}\sum_{t=1}^{n}\rho_\tau(y_t - x_{t-1}'\hat{\beta}_\tau(\lambda))\right) + \Gamma_n \cdot \hat{S}(\lambda), \tag{12}$$

where $\hat{\beta}_\tau(\lambda)$ is ALQR given λ, Γ_n is a positive sequence converging to 0, and $\hat{S}(\lambda) := \|\hat{\mathcal{A}}_n\|_0$ is the number of active predictors selected by $\hat{\beta}_\tau(\lambda)$. We set $\Gamma_n = \log(n)/n$ for BIC, and $\Gamma_n = \log(p) \cdot \log(\log(n))/n$ for GIC and select $\hat{\lambda}$ that minimizes the information criterion (objective) function. We use the same procedures for LASSO. Different approaches like the k-fold cross-validation and the Akaike information criterion (AIC) are available but it is known that they sometimes fail to effectively identify the true model (see, e.g. Shao (1997); Wang et al. (2007); Zhang et al. (2010)). Zheng et al. (2013) also discussed that the statistical properties of the k-fold CV have not been well understood for high-dimensional regression with heavy-tailed errors, where QR is often applied. On the contrary, Wang et al. (2007) show the model selection consistency of BIC for the fixed dimension and Wang et al. (2009) do for the increasing dimension with $p < n$. Also, Fan and Tang (2013) and Zheng et al. (2015) show that GIC can identify the underlying true model consistently with probability approaching 1.

Overall, the performance of ALQR is satisfactory and confirms the theory developed in the previous section. Tables 4–6 summarize the simulation results from Scenarios 1–3. The upper and

Table 4: Simulation Results: Scenario 1 (6 non-zero coefficients)

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Tuning parameter selection: BIC					
		Final Prediction Error (FPE)			
QR	0.1821	0.2835	0.5807	0.2817	0.1795
LASSO	0.1091	0.1813	0.4048	0.1807	0.1084
ALQR	**0.1068**	**0.1795**	**0.4016**	**0.1793**	**0.1064**
QUANT	2.7366	4.9691	12.1063	4.8815	2.6289
		Out-of-Sample R^2			
QR	0.9335	0.9430	0.9520	0.9423	0.9317
LASSO	0.9601	0.9635	0.9666	0.9630	0.9588
ALQR	**0.9610**	**0.9639**	**0.9668**	**0.9633**	**0.9595**
		Average # of Selected Predictors			
LASSO	9.80	9.39	8.99	9.43	9.74
ALQR	6.24	6.06	5.97	6.05	6.26
		Tuning Parameter (λ)			
LASSO ($\times 10^0$)	8.27	15.68	36.46	15.17	8.56
ALQR ($\times 10^{-4}$)	1.71	2.38	3.99	2.34	1.69
Tuning parameter selection: GIC					
		Final Prediction Error (FPE)			
QR	0.1815	0.2834	0.5805	0.2817	0.1793
LASSO	0.1084	0.1817	0.4045	0.1805	0.1084
ALQR	**0.1065**	**0.1800**	**0.4017**	**0.1791**	**0.1068**
QUANT	2.7285	4.9162	12.0961	4.8751	2.6290
		Out-of-Sample R^2			
QR	0.9335	0.9424	0.9520	0.9422	0.9318
LASSO	0.9603	0.9630	0.9666	0.9630	0.9588
ALQR	**0.9610**	**0.9634**	**0.9668**	**0.9633**	**0.9594**
		Average # of Selected Predictors			
LASSO	10.14	9.79	9.35	9.80	10.13
ALQR	6.80	6.36	6.11	6.35	6.76
		Tuning Parameter (λ)			
LASSO ($\times 10^0$)	6.21	11.66	28.39	11.54	6.32
ALQR ($\times 10^{-4}$)	1.22	1.93	3.55	1.94	1.24

Table 5: Simulation Results: Scenario 2 (8 non-zero coefficients)

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Tuning parameter selection: BIC					
			Final Prediction Error (FPE)		
QR	0.1845	0.2864	0.5882	0.2878	0.1848
LASSO	0.1085	0.1818	0.4056	0.1800	0.1078
ALQR	**0.1074**	**0.1805**	**0.4033**	**0.1794**	**0.1069**
QUANT	2.6803	4.8142	12.1974	4.8994	2.5749
			Out-of-Sample R^2		
QR	0.9312	0.9405	0.9518	0.9413	0.9282
LASSO	0.9595	0.9622	0.9667	0.9633	0.9581
ALQR	**0.9599**	**0.9625**	**0.9669**	**0.9634**	**0.9585**
			Average # of Selected Predictors		
LASSO	10.33	10.09	9.74	10.09	10.27
ALQR	8.23	8.06	7.99	8.04	8.20
			Tuning Parameter (λ)		
LASSO ($\times 10^0$)	6.75	12.56	29.98	12.22	7.18
ALQR ($\times 10^{-4}$)	1.17	1.60	2.54	1.61	1.18
Tuning parameter selection: GIC					
			Final Prediction Error (FPE)		
QR	0.1852	0.2876	0.5880	0.2874	0.1848
LASSO	0.1090	0.1825	0.4035	0.1799	0.1076
ALQR	**0.1077**	**0.1810**	**0.4021**	**0.1792**	**0.1071**
QUANT	2.7245	4.8459	12.2354	4.8893	2.5796
			Out-of-Sample R^2		
QR	0.9320	0.9407	0.9519	0.9412	0.9283
LASSO	0.9600	0.9623	0.9670	0.9632	0.9583
ALQR	**0.9605**	**0.9627**	**0.9671**	**0.9633**	**0.9585**
			Average # of Selected Predictors		
LASSO	10.58	10.37	9.98	10.34	10.56
ALQR	8.58	8.28	8.09	8.29	8.55
			Tuning Parameter (λ)		
LASSO ($\times 10^0$)	5.18	9.33	23.82	9.46	5.27
ALQR ($\times 10^{-4}$)	0.85	1.30	2.30	1.29	0.87

Table 6: Simulation Results: Scenario 3 (12 non-zero coefficients)

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Tuning parameter selection: BIC					
			Final Prediction Error (FPE)		
QR	0.1927	0.2968	0.5992	0.2964	0.1906
LASSO	**0.1093**	**0.1820**	0.4046	**0.1802**	**0.1091**
ALQR	0.1099	0.1825	**0.4045**	0.1807	0.1092
QUANT	2.6958	4.9349	12.1341	4.8949	2.6794
			Out-of-Sample R^2		
QR	0.9285	0.9398	0.9506	0.9395	0.9289
LASSO	**0.9595**	**0.9631**	**0.9667**	**0.9632**	**0.9593**
ALQR	0.9592	0.9630	**0.9667**	0.9631	0.9592
			Average # of Selected Predictors		
LASSO	11.97	11.99	12.00	11.99	11.98
ALQR	11.77	11.86	11.95	11.86	11.78
			Tuning Parameter (λ)		
LASSO ($\times 10^{-1}$)	2.01	1.59	0.646	1.12	1.97
ALQR ($\times 10^{-5}$)	1.75	1.88	1.87	1.89	1.75
Tuning parameter selection: GIC					
			Final Prediction Error (FPE)		
QR	0.1945	0.2972	0.6002	0.2964	0.1906
LASSO	**0.1100**	**0.1818**	0.4048	**0.1802**	**0.1091**
ALQR	0.1105	0.1820	**0.4047**	0.1803	0.1092
QUANT	2.6718	4.9361	12.2134	4.8949	2.6794
			Out-of-Sample R^2		
QR	0.9272	0.9398	0.9509	0.9395	0.9289
LASSO	**0.9588**	**0.9632**	**0.9669**	**0.9632**	**0.9593**
ALQR	0.9586	0.9631	**0.9669**	**0.9632**	0.9592
			Average # of Selected Predictors		
LASSO	11.98	11.99	12.00	12.00	11.98
ALQR	11.86	11.92	11.98	11.93	11.85
			Tuning Parameter (λ)		
LASSO ($\times 10^{-1}$)	1.42	1.24	0.64	0.90	1.45
ALQR ($\times 10^{-5}$)	1.34	1.39	1.27	1.36	1.40

lower panels of each table contain the results of using BIC and GIC, respectively. First, when there are zero coefficients in the model (Scenarios 1 and 2), ALQR performs better than its alternatives in terms of FPE (or R^2). As predicted by existing theory in the literature, both LASSO and ALQR show better performance than QR. In the meantime, ALQR performs uniformly, though slightly, better than LASSO. When all predictors have non-zero coefficients (Scenario 3), LASSO performs slightly better than ALQR over all quantiles except $\tau = 0.5$. Second, the simulation results confirm the model selection consistency of ALQR derived in Theorem 4.5. When we take a look at the average number of selected predictors in each table, the results of ALQR are quite close to the number of the true non-zero coefficients (6, 8, and 12 in each scenario). As known in the literature, LASSO mostly overselects them in Scenarios 1–2. Even in Scenario 3 where there are no zero predictors, ALQR selects most of them successfully. These results show the robustness of ALQR in terms of model selection. Finally, both BIC and GIC perform well under different designs and we do not see much difference between these two methods.

To investigate the robustness of the simulation results above, we further conduct simulation experiments with predictors with highly correlated innovations. In section 4, the oracle properties of the ALQR estimator are developed under the assumption that most predictor innovations are not highly correlated to each other. Although our empirical application satisfies this assumption, we provide additional simulation results in Tables D.3-D.11 of the appendix to demonstrate the robustness of ALQR in various settings of correlated predictor innovations. Specifically, we generate the $I(1)$ predictors in Scenario 1-3 with correlated innovations. We consider correlation ρ =0.1, 0.5, and 0.9. We find that when the true DGP includes zero coefficients, ALQR is better than other alternative methods in most of the settings. Only when DGP has no zero coefficient ALQR is at risk of underperforming across quantiles, though the performance of the other methods is only slightly better than ALQR. An interesting finding is that ridge regression, commonly recommended when many predictors are highly correlated, has a mixed performance across quantiles. Based on the simulation results, we recommend using ALQR, especially in the case of sparse data. But ridge regression and other methods may be used if sparsity is not found in the data and the quantile of interest is around the center of the distribution.

In sum, the numerical experiments confirm that ALQR can provide satisfactory prediction performance in finite samples. The results are robust over different quantiles and simulation designs. Therefore, we can expect similar results in other quantile prediction applications when the predictors have mixed roots and are composed of $I(0)$, $I(1)$, and cointegrated processes.

7 Conclusion

In this paper, we show that the adaptive lasso for quantile regression (ALQR) is attractive in forecasting with stationary and nonstationary predictors as well as cointegrated predictors. The framework is general enough to include mixed roots but ALQR does not require any researchers' knowledge on the specific structure of each predictor nor the order of integration. In this general framework, we show that ALQR preserves the oracle properties. These advantages offer substantial convenience and robustness to empirical researchers working with quantile prediction using time series data.

We have focused on the case where the number of covariates, p, is allowed to grow as sample size increases, although p is smaller than n. This framework justifies a wide range of practical applications in economics, such as the stock return quantile prediction in this paper. It would be an interesting future research to allow p to be even larger than n, which has not been studied in a general time series framework with mixed roots.

Appendix

A Proofs for Section 4.1

For simplicity, we remove the intercept terms in Model (1) and define the dequantiled dependent variable (Lee (2016)) as

$$y_{t\tau} := y_t - \hat{\mu}_\tau^{QR}.$$

Note that, in a mixed-root model, the intercept term is included as one of $I(0)$ predictors (see Footnote 1), so there is no need to dequantile the dependent variable in Section 4.2.

Proof of Lemma 4.1: Let $a_n = p^\alpha/n$ and $c \in \mathbb{R}^p$ such that $||c|| = C$, where C is a finite constant. Denote the (unpenalized) quantile objective function as $Q_n^{QR}(\beta_\tau)$.

To show the result of consistency, it suffices to show that for any $\epsilon > 0$, there exists a sufficiently large C such that

$$P\left\{\inf_{||c||=C} Q_n^{QR}(\beta_{0\tau} + a_n c) > Q_n^{QR}(\beta_{0\tau})\right\} \geq 1 - \epsilon. \tag{A.1}$$

This inequality implies that with probability at least $1-\epsilon$, there is a local minimizer $\tilde{\beta}_\tau$ in the shrinking ball $\{\beta_{0\tau} + a_n c, ||c|| \leq C\}$ such that $||\tilde{\beta}_\tau - \beta_{0\tau}|| = O_p(a_n)$. Thus, the proof is completed if we show that the following term is positive:

$$Q_n^{QR}(\beta_{0\tau} + a_n c) - Q_n^{QR}(\beta_{0\tau}) = \sum_{t=1}^n \rho_\tau(u_{t\tau} - x_{t-1}' a_n c) - \sum_{t=1}^n \rho_\tau(u_{t\tau}) \tag{A.2}$$

By Knight's Identity,

$$\begin{aligned}
&\sum_{t=1}^n \left[\rho_\tau(u_{t\tau} - x_{t-1}' a_n c) - \rho_\tau(u_{t\tau})\right] \\
&= -a_n \sum_{t=1}^n x_{t-1}' c \cdot \psi_\tau(u_{t\tau}) + \sum_{t=1}^n \int_0^{x_{t-1}' a_n c} \left(\mathbf{1}(u_{t\tau} \leq s) - \mathbf{1}(u_{t\tau} \leq 0)\right) ds \\
&= -a_n \sum_{t=1}^n x_{t-1}' c \cdot \psi_\tau(u_{t\tau}) + \sum_{t=1}^n E\left[\int_0^{x_{t-1}' a_n c} \left(\mathbf{1}(u_{t\tau} \leq s) - \mathbf{1}(u_{t\tau} \leq 0)\right) ds\right] \\
&\quad + \sum_{t=1}^n \left\{\int_0^{x_{t-1}' a_n c} \left(\mathbf{1}(u_{t\tau} \leq s) - \mathbf{1}(u_{t\tau} \leq 0)\right) ds - E\left[\int_0^{x_{t-1}' a_n c} \left(\mathbf{1}(u_{t\tau} \leq s) - \mathbf{1}(u_{t\tau} \leq 0)\right) ds\right]\right\} \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

We will show that I_1 and I_3 are dominated by I_2 and that $I_2 > 0$.

First, we derive the upper bound of I_1.

$$\begin{aligned}
E|I_1|^2 &= a_n^2 \sum_{t=1}^{n} c' E\left[\psi_\tau(u_{t\tau})^2 x_{t-1} x_{t-1}'\right] c + 2 \sum_{t=2}^{n} \sum_{k=1}^{t-1} c' E\left[\psi_\tau(u_{t\tau}) \psi_\tau(u_{k\tau}) x_{t-1} x_{k-1}'\right] c \\
&= a_n^2 \sum_{t=1}^{n} c' E\left[\psi_\tau(u_{t\tau})^2 x_{t-1} x_{t-1}'\right] c \\
&\leq a_n^2 \sum_{t=1}^{n} t \bar{c}_{B(t,p)} C^2 \\
&\leq C^2 a_n^2 \bar{c}_{B(n,p)} \frac{n(n+1)}{2}.
\end{aligned}$$

The second equality holds since, for $k \leq t-1$,

$$E\left[\psi_\tau(u_{t\tau})^2 x_{t-1} x_{t-1}'\right] = E\left[E_{t-1}\left[\psi_\tau(u_{t\tau})\right] \psi_\tau(u_{k\tau}) x_{t-1} x_{k-1}'\right] = 0.$$

The third inequality holds by the definition of $\bar{c}_{B(t,p)}$. Therefore, the Chebyshev's inequality implies that

$$I_1 = O_p(a_n \bar{c}_{B(n,p)}^{1/2} n) = O_p(a_n^2 \bar{c}_{B(n,p)}^{1/2} n^2 p^{-\alpha}).$$

Next, we derive the lower bound of I_2.

$$\begin{aligned}
I_2 &= \sum_{t=1}^{n} E \int_0^{a_n x_{t-1}' c} \left(F_{t-1}(s) - F_{t-1}(0)\right) ds \\
&= \sum_{t=1}^{n} E \int_0^{a_n x_{t-1}' c} \left(f_{t-1}(0) \cdot s\right) ds \left\{1 + o_p(1)\right\} \\
&= \frac{1}{2} a_n^2 \sum_{t=1}^{n} c' E\left[f_{t-1}(0) x_{t-1} x_{t-1}'\right] c \left\{1 + o(1)\right\} \\
&\geq \frac{1}{4} C^2 a_n^2 \underline{c}_{A(n,p)} \frac{n(n+1)}{2} \\
&= O(a_n^2 n^2 \underline{c}_{A(n,p)}).
\end{aligned}$$

The first equality holds by the law of iterated expectations and the second does by the Taylor expansion. The inequality holds under Assumption $L1$.

Finally, we derive the upper bound of I_3.

$$
\begin{aligned}
Var(I_3) &= Var\left(\sum_{t=1}^{n}\int_0^{x'_{t-1}a_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds\right)\\
&\le E\left[\left(\sum_{t=1}^{n}\int_0^{x'_{t-1}a_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds\right)^2\right]\\
&= E\Bigg[\sum_{t=1}^{n}\left(\int_0^{x'_{t-1}a_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds\right)^2\\
&\quad +2\sum_{t=2}^{n}\sum_{k=1}^{t-1}\left(\int_0^{x'_{t-1}a_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds\right)\left(\int_0^{x'_{k-1}a_nc}\left(\mathbf{1}(u_{k\tau}\le s)-\mathbf{1}(u_{k\tau}\le 0)\right)ds\right)\Bigg]\\
&\le E\left[\sum_{t=1}^{n}\left(x'_{t-1}a_nc\right)^2+2\sum_{t=2}^{n}\sum_{k=1}^{t-1}\left|x'_{t-1}a_nc\right|\left|x'_{k-1}a_nc\right|\right]\\
&= a_n^2\sum_{t=1}^{n}c'E\left[x_{t-1}x'_{t-1}\right]c+2a_n^2\sum_{t=2}^{n}\sum_{k=1}^{t-1}E\left[\left|x'_{t-1}c\right|\left|x'_{k-1}c\right|\right]\\
&\equiv V_{3,1}+V_{3,2}
\end{aligned}
$$

Using the similar arguments in I_1, we have

$$
V_{3,1}\le a_n^2C^2\frac{\overline{c}_{A(n,p)}}{c_f}\frac{n(n+1)}{2}=O(a_n^2\overline{c}_{A(n,p)}n^2).
$$

By the Cauchy-Schwarz inequality, the definition of $\overline{c}_{A(n,p)}$, and $t>k$, we have

$$
\begin{aligned}
V_{3,2} &\le 2a_n^2\sum_{t=2}^{n}\sum_{k=1}^{t-1}\sqrt{E\left[(x'_{t-1}x_{t-1})\right]}\sqrt{E\left[(x'_{t-1}x_{t-1})\right]}\\
&\le 2a_n^2\sum_{t=2}^{n}\sum_{k=1}^{t-1}\sqrt{C^2t\frac{\overline{c}_{A(t,p)}}{c_f}}\sqrt{C^2t\frac{\overline{c}_{A(t,p)}}{c_f}}\\
&\le 2a_n^2C^2\frac{\overline{c}_{A(n,p)}}{c_f}\sum_{t=2}^{n}\sum_{k=1}^{t-1}t\\
&= O(a_n^2\overline{c}_{A(n,p)}n^3)
\end{aligned}
$$

Therefore, $Var(I_3)=O(a_n^2\overline{c}_An^3)$, and Chebyshev's inequality implies that

$$
I_3=O_p(\overline{c}_{A(n,p)}^{1/2}a_nn^{3/2})=O_p\left(\overline{c}_{A(n,p)}^{1/2}a_n^2n^{5/2}p^{-\alpha}\right).
$$

By Assumptions $U1$ and $U2$, we establish the desired result. ■

Proof of Theorem 4.1: For simplicity, in this proof, we use $\hat{\beta}_\tau$ to represent the ALQR estimator $\hat{\beta}_\tau^{ALQR}$. Without loss of generality, let the values of $\beta_{0\tau,1},\beta_{0\tau,2},...,\beta_{0\tau,q_n}$ be nonzero and $\beta_{0\tau,q_n+1}$,

$\beta_{0\tau,q_n+2},...,\beta_{0\tau,p}$ be zero. Let $E_{t-1}(\cdot)\equiv E(\cdot|\mathcal{F}_{t-1})$.

To show the result of consistency, it suffices to show that for any $\epsilon>0$, there exists a sufficiently large C such that

$$P\left\{\inf_{||c||=C} Q_n(\beta_{0\tau}+a_nc)>Q_n(\beta_{0\tau})\right\}\geq 1-\epsilon. \tag{A.3}$$

This inequality implies that with probability at least $1-\epsilon$, there is a local minimizer $\hat{\beta}_\tau$ in the shrinking ball $\{\beta_{0\tau}+a_nc, ||c||\leq C\}$ such that $||\hat{\beta}_\tau-\beta_{0\tau}||=O_p(a_n)$.

Since

$$\begin{aligned}
&Q_n(\beta_{0\tau}+a_nc)-Q_n(\beta_{0\tau}) \qquad\qquad (A.4)\\
&=\left(\sum_{t=1}^n\rho_\tau(u_{t\tau}-x'_{t-1}a_nc)-\sum_{t=1}^n\rho_\tau(u_{t\tau})\right)+\left(\sum_{j=1}^p\lambda_{n,j}|\beta_{0\tau,j}+a_nc_j|-\sum_{j=1}^p\lambda_{n,j}|\beta_{0\tau,j}|\right)\\
&=\left(\sum_{t=1}^n\rho_\tau(u_{t\tau}-x'_{t-1}a_nc)-\sum_{t=1}^n\rho_\tau(u_{t\tau})\right)\\
&+\left(\sum_{j=1}^{q_n}\lambda_{n,j}(|\beta_{0\tau,j}+a_nc_j|-|\beta_{0\tau,j}|)+\sum_{j=q_n+1}^{p}\lambda_{n,j}(|\beta_{0\tau,j}+a_nc_j|-|\beta_{0\tau,j}|)\right)\\
&=\left(\sum_{t=1}^n\rho_\tau(u_{t\tau}-x'_{t-1}a_nc)-\sum_{t=1}^n\rho_\tau(u_{t\tau})\right)\\
&+\left(\sum_{j=1}^{q_n}\lambda_{n,j}(|\beta_{0\tau,j}+a_nc_j|-|\beta_{0\tau,j}|)+\sum_{j=q_n+1}^{p}\lambda_{n,j}(|a_nc_j|)\right)\\
&\geq\left(\sum_{t=1}^n\rho_\tau(u_{t\tau}-x'_{t-1}a_nc)-\sum_{t=1}^n\rho_\tau(u_{t\tau})\right)+\sum_{j=1}^{q_n}\lambda_{n,j}(|\beta_{0\tau,j}+a_nc_j|-|\beta_{0\tau,j}|)\\
&\equiv D_1+D_2,
\end{aligned}$$

we need to show that D_1+D_2 is positive.

For D_2, we know $\big||\beta_{0\tau,j}+a_nc_j|-|\beta_{0\tau,j}|\big|\leq|a_nc_j|$. Given that $\tilde{\beta}_\tau$ is a (a_n^{-1})-consistent estimate of $\beta_{0\tau}$, we have $a_n^{-1}(\tilde{\beta}_{\tau,j}-\beta_{0\tau,j})=O_p(1)$, and then $\tilde{\beta}_{\tau,j}=O_p(a_n)+\beta_{0\tau,j}=o_p(1)+\beta_{0\tau,j}$. Thus,

$$\begin{aligned}
|D_2|\leq&\sum_{j=1}^{q_n}\lambda_{n,j}|a_nc_j|\\
=&\lambda_na_n\sum_{j=1}^{q_n}\left|\frac{1}{|\tilde{\beta}_{\tau,j}|^\gamma}c_j\right|\\
\leq&\lambda_na_n\left(\sum_{j=1}^{q_n}\frac{1}{(\tilde{\beta}_{\tau,j})^{2\gamma}}\right)^{1/2}\cdot||c||
\end{aligned}$$

$$
\begin{aligned}
&\leq C\lambda_n a_n \left(\sum_{j=1}^{q_n} \frac{1}{(o_p(1)+\beta_{0\tau,j})^{2\gamma}}\right)^{1/2} \\
&= C\lambda_n a_n \left(O_p(1)\sum_{j=1}^{q_n} 1\right)^{1/2} \\
&= O_p(\lambda_n a_n q_n^{1/2}) \\
&= O_p(\lambda_n a_n p^{1/2}). \qquad \text{(A.5)}
\end{aligned}
$$

We next consider D_1. Following the proof of Lemma 4.1, we have that the dominating term of D_1 is $O_p(a_n^2 n^2 \underline{c}_{A(n,p)})$ and that it is positive with probability approaching 1. Under Assumption $\lambda 1$ (i), D_1 dominates D_2. We complete the proof of Theorem 4.1. ■

Proof of Theorem 4.2: From Theorem 4.1 for a sufficiently large constant C, $\hat{\beta}_\tau^{ALQR}$ is a local minimizer lies in the ball $\{\beta_{0\tau}+a_n c, ||c|| \leq C\}$ with probability approaching 1 and $a_n = p^\alpha/n$. For simplicity, in this proof, we use $\hat{\beta}_\tau$ to represent the ALQR estimator $\hat{\beta}_\tau^{ALQR}$.

First, note that the subgradient of the unpenalized objective function, $s(\beta_\tau) = (s_1(\beta_\tau), ..., s_p(\beta_\tau))'$ is given by (Sherwood and Wang (2016), page 298 and Lemma 1):

$$
s_j(\beta_\tau) = -\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(y_{t\tau} - x'_{t-1}\beta_\tau\right) + \sum_{t=1}^{n} x_{t-1,j}k_t \quad \text{for } 1 \leq j \leq p,
$$

where $k_t = 0$ if $y_{t\tau} - x'_{t-1}\beta_\tau \neq 0$ and $k_t \in [-\tau, 1-\tau]$ if $y_{t\tau} - x'_{t-1}\beta_\tau = 0$.

Let $\mathcal{D} = \{t : y_{t\tau} - x'_{t-1}\hat{\beta}_\tau = 0\}$, then

$$
\begin{aligned}
s_j(\hat{\beta}_\tau) &= -\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(y_{t\tau} - x'_{t-1}\hat{\beta}_\tau\right) + \sum_{t\in\mathcal{D}} x_{t-1,j}(k_t^* + (1-\tau)), \\
&= -\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(y_{t\tau} - x'_{t-1}\hat{\beta}_\tau\right) + h_n,
\end{aligned}
$$

where $h_n := \sum_{t\in\mathcal{D}} x_{t-1,j}(k_t^* + (1-\tau))$ and

$$
k_t^* = \begin{cases} 0, & \text{if } y_{t\tau} - x'_{t-1}\hat{\beta}_\tau \neq 0 \\ \in [-\tau, 1-\tau], & \text{if } y_{t\tau} - x'_{t-1}\hat{\beta}_\tau = 0. \end{cases}
$$

With probability one (Koenker (2005, Section 2.2)), $|\mathcal{D}| = p$. Thus $h_n = O_p(p^{3/2}) = O_p(n^{(3/2)\zeta})$.

Next, define the subgradient of the penalized objective function as $S_j(\beta_\tau)$:

$$
\begin{aligned}
S_j(\beta_\tau) &= -\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(y_{t\tau} - x'_{t-1}\beta_\tau\right) + h_n + \frac{\lambda_n}{|\tilde{\beta}_{j,\tau}|^\gamma} sgn(\beta_{j,\tau}) \\
&= -\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau} - x'_{t-1}\delta_\tau\right) + h_n + \frac{\lambda_n}{|\tilde{\beta}_{j,\tau}|^\gamma} sgn(\beta_{j,\tau}),
\end{aligned}
$$

where $u_{t\tau} = u_{0t} - F_{u_0}^{-1}(\tau)$ and $\delta_\tau = \beta_\tau - \beta_{0\tau}$. The subgradient condition requires that at the optimum, $\hat{\beta}_\tau$,

$$0 \in S_j(\hat{\beta}_\tau).$$

That is,

$$\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau} - x_{t-1}'\hat{\delta}_\tau\right) - h_n = \frac{\lambda_n}{|\tilde{\beta}_{j,\tau}|^\gamma} sgn(\hat{\beta}_{j,\tau}). \tag{A.6}$$

If $j = 1,..,q_n$, it implies that $|sgn(\hat{\beta}_{j,\tau})| = 1$. Then we can write the subgradient condition (A.6) as:

$$\left|\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau} - \hat{\delta}_\tau' x_{t-1}\right) - h_n\right| = \frac{\lambda_n}{|\tilde{\beta}_{j,\tau}|^\gamma}. \tag{A.7}$$

In the following, we show that this subgradient condition does not hold for $j \notin \mathcal{A}_0$, i.e., $j = q_n + 1, ..., p$. It suffices to show:

(a) $(n^{\alpha\zeta+2}\bar{c}_{A(n,p)}^{1/2})^{-1}\frac{\lambda_n}{|\tilde{\beta}_{j,\tau}|^\gamma} \to \infty$;

(b) $\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau} - x_{t-1}'\delta_\tau\right) - h_n \le O_p(n^{\alpha\zeta+2}\bar{c}_{A(n,p)}^{1/2})$.

For (b), we first prove that the first term on the left-hand side of (A.7) is dominated by $O_p(n^{\alpha\zeta+2}\bar{c}_{A(n,p)}^{1/2})$:

$$\begin{aligned}
&\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right) \\
=\ &\sum_{t=1}^{n} x_{t-1,j}\Big[\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right) - E_{t-1}\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right) - \psi_\tau\left(u_{t\tau}\right) + E_{t-1}\psi_\tau\left(u_{t\tau}\right)\Big] \\
&+ \sum_{t=1}^{n} x_{t-1,j}E_{t-1}\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right) + \sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau}\right) \\
\equiv\ &I_1 + I_2 + I_3.
\end{aligned} \tag{A.8}$$

For I_1, let $I_1 = A + B$, where

$$\begin{aligned}
A &\equiv \sum_{t=1}^{n} x_{t-1,j}\Big[\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right) - E_{t-1}\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right)\Big], \\
B &\equiv \sum_{t=1}^{n} x_{t-1,j}\Big[\psi_\tau\left(u_{t\tau}\right) - E_{t-1}\psi_\tau\left(u_{t\tau}\right)\Big].
\end{aligned}$$

Note that

$$E_{t-1}\psi_\tau\left(u_{t\tau}-\delta_\tau' x_{t-1}\right) = E_{t-1}\psi_\tau\left(u_{t\tau}\right)+\left.\frac{\partial E_{t-1}\psi_\tau\left(u_{t\tau}-\delta_\tau' x_{t-1}\right)}{\partial \delta_\tau'}\right|_{\delta_\tau=\mathbf{0}}\delta_\tau+o_p(a_n)$$

$$= -x_{t-1}' f_{t-1}(0)\delta_\tau+o_p(a_n), \quad \text{(A.9)}$$

$$E_{t-1}\psi_\tau^2\left(u_{t\tau}-\delta_\tau' x_{t-1}\right) = E_{t-1}\psi_\tau^2\left(u_{t\tau}\right)+\left.\frac{\partial E_{t-1}\psi_\tau^2\left(u_{t\tau}-\delta_\tau' x_{t-1}\right)}{\partial \delta_\tau'}\right|_{\delta_\tau=\mathbf{0}}\delta_\tau+o_p(a_n)$$

$$= \tau(1-\tau)-x_{t-1}' f_{t-1}^2(0)\delta_\tau+o_p(a_n). \quad \text{(A.10)}$$

Then, we have

$$E(A)=EE_{t-1}(A) = E\left\{\sum_{t=1}^n x_{t-1,j}\Big[E_{t-1}\psi_\tau\left(u_{t\tau}-\delta_\tau' x_{t-1}\right)-E_{t-1}\psi_\tau\left(u_{t\tau}-\delta_\tau' x_{t-1}\right)\Big]\right\}=0,$$

and

$$\begin{aligned}
E_{t-1}(A^2) =& \sum_{t=1}^n x_{t-1,j}^2 E_{t-1}\Big[\psi_\tau\left(u_{t\tau}-\delta_\tau' x_{t-1}\right)-E_{t-1}\psi_\tau\left(u_{t\tau}-\delta_\tau' x_{t-1}\right)\Big]^2+0\\
=& \sum_{t=1}^n x_{t-1,j}^2\Big[E_{t-1}\psi_\tau^2\left(u_{t\tau}-\delta_\tau' x_{t-1}\right)-\left(E_{t-1}\psi_\tau\left(u_{t\tau}-\delta_\tau' x_{t-1}\right)\right)^2\Big]\\
=& \sum_{t=1}^n x_{t-1,j}^2\Big(\tau(1-\tau)-x_{t-1}' f_{t-1}^2(0)\delta_\tau+o_p(a_n)\Big)\\
&-\sum_{t=1}^n x_{t-1,j}^2\Big(-x_{t-1}' f_{t-1}(0)\delta_\tau+o_p(\gamma_n)\Big)^2 \quad \text{(by (A.9) and (A.10))}\\
=& O_p(n^2)-\sum_{t=1}^n x_{t-1,j}^2 x_{t-1}' c f_{t-1}^2(0) O_p(a_n)+O_p(n^2)o_p(a_n)\\
&-\sum_{t=1}^n x_{t-1,j}^2\Big(c' x_{t-1}x_{t-1}' c f_{t-1}^2(0)O_p(a_n^2)-2x_{t-1}' c f_{t-1}(0)O_p(a_n)o_p(a_n)+o_p(a_n^2)\Big).\\
&\text{(since } \delta_\tau=\beta_\tau-\beta_{0\tau}=O_p(a_n)c \text{ and } x_{t-1}'\delta_\tau=x_{t-1}' c O_p(a_n)\text{)} \qquad \text{(A.11)}
\end{aligned}$$

Under Assumption f and the definition of $\bar{c}_{A(n,p)}$, we have:

$$E\left[\sum_{t=1}^n c' f_{t-1}(0)x_{t-1}x_{t-1}' c\right]^{1/2} \le \left[\sum_{t=1}^n E c' f_{t-1}(0)x_{t-1}x_{t-1}' c\right]^{1/2} \le O(\bar{c}_{A(n,p)}^{1/2} n), \quad \text{(by Jensen's inequality)}$$

$$
\begin{aligned}
E(\max_{1\le t\le n}|x'_{t-1}cf_{t-1}(0)|)^2 &\le E(\sum_{t=1}^{n}|x'_{t-1}cf_{t-1}(0)|)^2\\
&= E\left[\sum_{t=1}^{n}(x'_{t-1}cf_{t-1}(0))^2+2\sum_{t=2}^{n}\sum_{k=1}^{t-1}|x'_{t-1}cf_{t-1}(0)|\cdot|x'_{k-1}cf_{k-1}(0)|\right]\\
&\le c_f\sum_{t=1}^{n}E(c'x_{t-1}x'_{t-1}cf_{t-1}(0))+2E\left[\sum_{t=2}^{n}\sum_{k=1}^{t-1}|x'_{t-1}cf_{t-1}(0)|\cdot|x'_{k-1}cf_{k-1}(0)|\right]\\
&\le O(c_f\overline{c}_{A(n,p)}n^2)+2\sum_{t=2}^{n}\sum_{k=1}^{t-1}\sqrt{E\left[(x'_{t-1}cf_{t-1}(0))^2\right]}\cdot\sqrt{E\left[(x'_{k-1}cf_{k-1}(0))^2\right]}\\
&\le O(c_f\overline{c}_{A(n,p)}n^2)+O(c_f\overline{c}_{A(n,p)}n^3)=O(c_f\overline{c}_{A(n,p)}n^3),\text{(by Cauchy-Schwarz inequality)}
\end{aligned}
$$

and

$$
E(\max_{1\le t\le n}|x'_{t-1}cf_{t-1}(0)|)\le\left[E(\max_{1\le t\le n}|x'_{t-1}cf_{t-1}(0)|)^2\right]^{1/2}\le\left[O(c_f\overline{c}_{A(n,p)}n^3)\right]^{1/2}=O(c_f^{1/2}\overline{c}_{A(n,p)}^{1/2}n^{3/2}).
$$

Using the above results and the Cauchy-Schwarz inequality, we can show that:

$$
\begin{aligned}
EE_{t-1}(A^2)\le\ & O(n^2)-O(a_n)E[\sum_{t=1}^{n}x_{t-1,j}^2x'_{t-1}cf_{t-1}^2(0)]+o(n^2a_n)-O(a_n^2)E[\sum_{t=1}^{n}x_{t-1,j}^2c'x_{t-1}x'_{t-1}cf_{t-1}^2(0)]\\
&+o(a_n^2)E[\sum_{t=1}^{n}x_{t-1,j}^2x'_{t-1}cf_{t-1}(0)]-o(a_n^2)O(n^2)\\
\le\ & O(n^2)+O(a_n)E\left(\max_{1\le t\le n}|x'_{t-1}cf_{t-1}^2(0)|\sum_{t=1}^{n}x_{t-1,j}^2\right)+o(n^2a_n)\\
&+O(a_n^2)E\left((\max_{1\le t\le n}|x'_{t-1}cf_{t-1}(0)|)^2\sum_{t=1}^{n}x_{t-1,j}^2\right)+o(a_n^2)E\left(\max_{1\le t\le n}|x'_{t-1}cf_{t-1}(0)|\sum_{t=1}^{n}x_{t-1,j}^2\right)\\
&+o(n^2a_n^2)\\
\le\ & O(n^2)+O(a_n)O(n^2)O(c_fc_f^{1/2}\overline{c}_{A(n,p)}^{1/2}n^{3/2})+o(n^2a_n)+O(a_n^2)O(n^2)O(c_f\overline{c}_{A(n,p)}n^3)\\
&+o(a_n^2)O(n^2)O(c_f^{1/2}\overline{c}_{A(n,p)}^{1/2}n^{3/2})+o(n^2a_n^2)\\
=\ & O(n^2)+O(a_nn^{7/2}\overline{c}_{A(n,p)}^{1/2})+o(a_nn^2)+O(a_n^2n^5\overline{c}_{A(n,p)})+o(a_n^2n^{7/2}\overline{c}_{A(n,p)}^{1/2})+o(a_n^2n^2)\\
=\ & O(n^2)+O(n^{\alpha\zeta+(5/2)}\overline{c}_{A(n,p)}^{1/2})+o(n^{\alpha\zeta+1})\\
&+O(n^{2\alpha\zeta+3}\overline{c}_{A(n,p)})+o(n^{2\alpha\zeta+(3/2)}\overline{c}_{A(n,p)}^{1/2})+o(n^{2\alpha\zeta}).
\end{aligned}
\tag{A.12}
$$

Given the conditions: $0<\alpha\zeta<1$, it is easy to verify that the fourth term, $O(n^{2\alpha\zeta+3}\overline{c}_{A(n,p)})$, dominates the other terms. Thus, $E(A^2)\le O(n^{2\alpha\zeta+3}\overline{c}_{A(n,p)})$. The Chebyshev's inequality implies that

$$
A\le O_p(n^{\alpha\zeta+(3/2)}\overline{c}_{A(n,p)}^{1/2}). \tag{A.13}
$$

For B, it is easy to obtain that

$$E_{t-1}[\psi_\tau(u_{t\tau})] = 0, \quad E_{t-1}[\psi_\tau^2(u_{t\tau})] = \tau(1-\tau). \tag{A.14}$$

Then we can show:

$$E_{t-1}(B) = \sum_{t=1}^{n} x_{t-1,j}\Big[E_{t-1}\psi_\tau(u_{t\tau}) - E_{t-1}\psi_\tau(u_{t\tau})\Big] = 0,$$
$$E_{t-1}(B^2) = \sum_{t=1}^{n} x_{t-1,j}^2 E_{t-1}\Big[\psi_\tau(u_{t\tau}) - E_{t-1}\psi_\tau(u_{t\tau})\Big]^2 = \sum_{t=1}^{n} x_{t-1,j}^2 E_{t-1}\psi_\tau^2(u_{t\tau}) = O_p(n^2),$$

which implies that

$$B \le O_p(n). \tag{A.15}$$

By the results of (A.13) and (A.15), we have

$$I_1 \le O_p(n^{\alpha\zeta+(3/2)}\bar{c}_{A(n,p)}^{-1/2}). \tag{A.16}$$

For I_2, by (A.9),

$$\sum_{t=1}^{n} x_{t-1,j}E_{t-1}\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right) = -\sum_{t=1}^{n} x_{t-1,j}x_{t-1}' f_{t-1}(0)\delta_\tau + o_p(a_n)\sum_{t=1}^{n} x_{t-1,j}.$$

Thus,

$$\begin{aligned}
& E\left[\sum_{t=1}^{n} x_{t-1,j}E_{t-1}\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right)\right] \\
= \; & -E\sum_{t=1}^{n}[x_{t-1,j}x_{t-1}' c f_{t-1}(0) O_p(a_n)] + E[o_p(a_n n^{3/2})] \\
\le \; & O(a_n)E\left[\sum_{t=1}^{n} x_{t-1,j}x_{t-1}' c f_{t-1}(0)\right] + o(a_n n^{3/2}) \\
\le \; & O(a_n)E\left(\max_{1\le t\le n}|x_{t-1}' c f_{t-1}(0)|\sum_{t=1}^{n} x_{t-1,j}\right) + o(a_n n^{3/2}) \\
\le \; & O(a_n)O(n^{3/2})O(c_f^{1/2}\bar{c}_{A(n,p)}^{-1/2} n^{3/2}) + o(a_n n^{3/2}) \\
= \; & O(a_n n^3 \bar{c}_{A(n,p)}^{-1/2}) + o(a_n n^{3/2}) \\
= \; & O(n^{\alpha\zeta+2}\bar{c}_{A(n,p)}^{-1/2}),
\end{aligned} \tag{A.17}$$

and

$$E\left[\left(\sum_{t=1}^{n} x_{t-1,j}E_{t-1}\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right)\right)^2\right]$$

$$
\begin{aligned}
&= E\left[\left(-\sum_{t=1}^{n} x_{t-1,j}x'_{t-1}f_{t-1}(0)\delta_\tau + o_p(a_n)\sum_{t=1}^{n} x_{t-1,j}\right)^2\right] \\
&= E\Bigg[\left(\sum_{t=1}^{n} x_{t-1,j}x'_{t-1}f_{t-1}(0)\delta_\tau\right)^2 + \left(o_p(a_n)\sum_{t=1}^{n} x_{t-1,j}\right)^2 \\
&\quad - 2\left(\sum_{t=1}^{n} x_{t-1,j}x'_{t-1}f_{t-1}(0)\delta_\tau\right)\cdot\left(o_p(a_n)\sum_{t=1}^{n} x_{t-1,j}\right)\Bigg] \\
&\le O(a_n^2)E\left[\sum_{t=1}^{n}\left(x_{t-1,j}x'_{t-1}cf_{t-1}(0)\right)^2 + 2\sum_{t=2}^{n}\sum_{k=1}^{t-1} x_{t-1,j}x_{k-1,j}x'_{t-1}cf_{t-1}(0)\cdot x'_{k-1}cf_{k-1}(0)\right] \\
&\quad + o(a_n^2)E\left[\sum_{t=1}^{n} x_{t-1,j}^2 + 2\sum_{t=2}^{n}\sum_{k=1}^{t-1} x_{t-1,j}x_{k-1,j}\right] \\
&\quad + o(a_n^2)E\left[\left(\sum_{t=1}^{n} x_{t-1,j}x'_{t-1}cf_{t-1}(0)\right)\cdot\left(\sum_{t=1}^{n} x_{t-1,j}\right)\right] \\
&\le \quad O(a_n^2)E\left[\sum_{t=1}^{n} x_{t-1,j}^2 c' x_{t-1}x'_{t-1}cf_{t-1}^2(0) + 2\sum_{t=2}^{n}\sum_{k=1}^{t-1} x_{t-1,j}x_{k-1,j}x'_{t-1}cf_{t-1}(0)\cdot x'_{k-1}cf_{k-1}(0)\right] \\
&\quad + o(a_n^2)\left[O(n^2)+O(n^3)\right] + o(a_n^2)E\left[\left(\max_{1\le t\le n}|x'_{t-1}cf_{t-1}(0)|\sum_{t=1}^{n} x_{t-1,j}\right)\cdot O_p(n^{3/2})\right] \\
&\le O(a_n^2)E\left((\max_{1\le t\le n}|x'_{t-1}cf_{t-1}(0)|)^2\sum_{t=1}^{n} x_{t-1,j}^2\right) + O(a_n^2)E\left[(\max_{1\le t\le n}|x'_{t-1}cf_{t-1}(0)|)^2\sum_{t=2}^{n}\sum_{k=1}^{t-1} x_{t-1,j}x_{k-1,j}\right] \\
&\quad + o(a_n^2 n^3) + o(a_n^2)E\left[\left(\max_{1\le t\le n}|x'_{t-1}cf_{t-1}(0)|\sum_{t=1}^{n} x_{t-1,j}\right)\cdot O_p(n^{3/2})\right] \\
&\le O(a_n^2)O(n^2)O(c_f\overline{c}_{A(n,p)}n^3) + O(a_n^2)O(n^3)O(c_f\overline{c}_{A(n,p)}n^3) + o(a_n^2n^3) + o(a_n^2)O(n^{3/2})O(n^{3/2})O(c_f^{1/2}\overline{c}_{A(n,p)}^{1/2}n^{3/2}) \\
&= O(a_n^2n^5\overline{c}_{A(n,p)}) + O(a_n^2n^6\overline{c}_{A(n,p)}) + o(a_n^2n^3) + o(a_n^2n^{9/2}\overline{c}_{A(n,p)}^{1/2}) \\
&= O(a_n^2n^6\overline{c}_{A(n,p)}) = O(n^{2\alpha\zeta+4}\overline{c}_{A(n,p)}), \qquad \text{(A.18)}
\end{aligned}
$$

where the second inequality holds using Cauchy-Schwarz inequality and the following result:

$$
\begin{aligned}
\sum_{t=2}^{n}\sum_{k=1}^{t-1} x_{t-1,j}x_{k-1,j} &\leq \sum_{t=2}^{n}\left[x_{t-1,j}\left(\sum_{k=1}^{t-1} x_{k-1,j}\right)\right] \\
&\leq \left(\sum_{t=2}^{n} x_{t-1,j}^2\right)^{1/2} \cdot \left[\sum_{t=2}^{n}\left(\sum_{k=1}^{t-1} x_{k-1,j}\right)^2\right]^{1/2} \\
&= \left(O_p(n^2)\right)^{1/2} \cdot \left[\sum_{t=2}^{n} O_p(t^3)\right]^{1/2} \\
&= O_p(n) \cdot O_p(1) \left[\sum_{t=2}^{n} t^3\right]^{1/2} \\
&= O_p(n) \cdot O_p(1) \left[O(n^4)\right]^{1/2} \\
&= O_p(n^3).
\end{aligned}
$$

By the results of (A.17) and (A.18) , we apply Chebyshev's inequality and obtain that:

$$
I_2 \leq O_p(n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}). \tag{A.19}
$$

For I_3, it can be shown that

$$
\begin{aligned}
E_{t-1}\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau}\right) &= \sum_{t=1}^{n} x_{t-1,j}E_{t-1}\psi_\tau\left(u_{t\tau}\right) = 0, \\
E_{t-1}\left[\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau}\right)\right]^2 &= \sum_{t=1}^{n} x_{t-1,j}^2 E_{t-1}\psi_\tau^2\left(u_{t\tau}\right) + 0 = \tau(1-\tau)O_p(n^2).
\end{aligned}
$$

This implies that

$$
I_3 \leq O_p(n). \tag{A.20}
$$

By the results of (A.16), (A.19) and (A.20), we find the upper bound of the first term in (b):

$$
\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right) \leq O_p(n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}).
$$

For the second term in (b), under Assumption $U1$ (iii), we have

$$
\frac{h_n}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}} = \frac{O_p(p^{3/2})}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}} = O_p(1)o(1) = o_p(1).
$$

Thus the result in (b) is shown:

$$\sum_{t=1}^{n} x_{t-1,j}\psi_\tau \left(u_{t\tau} - x'_{t-1}\hat{\delta}_\tau\right) - h_n \leq O_p(n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}). \tag{A.21}$$

For (a), under Assumption λ1 (ii) that $\frac{\lambda_n n^{(1-\alpha\zeta)\gamma}}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}} \to \infty$, we obtain:

$$(n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2})^{-1}\frac{\lambda_n}{|\tilde{\beta}_{j,\tau}|^\gamma} = \frac{\lambda_n n^{(1-\alpha\zeta)\gamma}}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}}\frac{1}{|(n/p^\alpha)\tilde{\beta}_{j,\tau}|^\gamma} = \frac{\lambda_n n^{(1-\alpha\zeta)\gamma}}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}}\frac{1}{O_p(1)} \to \infty. \tag{A.22}$$

By (A.7), (A.21), and (A.22), we can show that, for any $j \notin \mathcal{A}_0$,

$$\begin{aligned}
& \Pr\left(j \in \hat{\mathcal{A}}_n\right) \\
\leq\; & \Pr\left(\left|\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau} - \delta'_\tau x_{t-1}\right) - h_n\right| = \frac{\lambda_n}{|\tilde{\beta}_{j,\tau}|^\gamma}\right) \\
=\; & \Pr\left(\frac{1}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}}\left|\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau} - \delta'_\tau x_{t-1}\right) - h_n\right| = \frac{\lambda_n n^{(1-\alpha\zeta)\gamma}}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}}\frac{1}{|(n/p^\alpha)\tilde{\beta}_{j,\tau}|^\gamma}\right) \\
\longrightarrow\; & 0.
\end{aligned}$$

■

B Proofs for Section 4.2

Proof of Lemma 4.2: Let $Q_n^{QR}(\beta_\tau)$ be the (unpenalized) QR objective function. To show the consistency of QR estimator, it suffices to show that for any $\epsilon > 0$, there exists a sufficiently large C such that

$$P\left\{\inf_{||c||=C} Q_n^{QR}(\tilde{\beta}_\tau + a_n M_n c) > Q_n^{QR}(\tilde{\beta}_\tau)\right\} \geq 1 - \epsilon, \tag{B.23}$$

where $a_n M_n = \begin{pmatrix} a_n^{(0)} I_r & 0 \\ 0 & a_n^{(1)} I_{p-r} \end{pmatrix}$ so that $a_n M_n c = \left(a_n^{(0)} c_1, ..., a_n^{(0)} c_r, a_n^{(1)} c_{r+1}, ...a_n^{(1)} c_p\right)'$.

As is shown in the proof of Lemma 4.1, this proof can be completed if we show that the following term is positive:

$$Q_n^{QR}(\tilde{\beta}_\tau + a_n M_n c) - Q_n^{QR}(\tilde{\beta}_\tau) = \sum_{t=1}^{n}\rho_\tau(u_{t\tau} - \tilde{x}'_{t-1} a_n M_n c) - \sum_{t=1}^{n}\rho_\tau(u_{t\tau}).$$

By Knight's Identity,

$$
\begin{aligned}
&\sum_{t=1}^{n}\left[\rho_\tau(u_{t\tau}-\tilde{x}_{t-1}'a_nM_nc)-\rho_\tau(u_{t\tau})\right]\\
&\quad=-a_n\sum_{t=1}^{n}\tilde{x}_{t-1}'M_nc\cdot\psi_\tau(u_{t\tau})+\sum_{t=1}^{n}\int_0^{\tilde{x}_{t-1}'a_nM_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds\\
&\quad=-a_n\sum_{t=1}^{n}\tilde{x}_{t-1}'M_nc\cdot\psi_\tau(u_{t\tau})+\sum_{t=1}^{n}E\left[\int_0^{\tilde{x}_{t-1}'a_nM_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds\right]\\
&\quad+\sum_{t=1}^{n}\left\{\int_0^{\tilde{x}_{t-1}'a_nM_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds-E\left[\int_0^{\tilde{x}_{t-1}'a_nM_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds\right]\right\}\\
&\quad\equiv I_1+I_2+I_3.
\end{aligned}
$$

We will show that I_1 and I_2 are dominated by I_2 and that $I_2>0$.

First, we derive the upper bound of I_1. For a large n,

$$
\begin{aligned}
E|I_1|^2&=a_n^2\sum_{t=1}^{n}c'M_n'E\left[\psi_\tau(u_{t\tau})^2\tilde{x}_{t-1}\tilde{x}_{t-1}'\right]M_nc+2a_n^2\sum_{t=2}^{n}\sum_{k=1}^{t-1}c'M_n'E\left[\psi_\tau(u_{t\tau})\psi_\tau(u_{k\tau})\tilde{x}_{t-1}\tilde{x}_{k-1}'\right]M_nc\\
&=a_n^2\sum_{t=1}^{n}c'M_n'E\left[\psi_\tau(u_{t\tau})^2\tilde{x}_{t-1}\tilde{x}_{t-1}'\right]M_nc\\
&=a_n^2c'M_n'\sum_{t=1}^{n}E\left[\psi_\tau(u_{t\tau})^2\tilde{x}_{t-1}\tilde{x}_{t-1}'\right]M_nc\\
&\le a_n^2n^2\bar{c}_{B(n,p)}C^2\ ,\quad\text{(from Assumption U2)}
\end{aligned}
$$

so that

$$
I_1=O\left(a_nn\bar{c}_{B(n,p)}^{1/2}\right).
$$

For I_2, for a large n,

$$
\begin{aligned}
I_2&=\sum_{t=1}^{n}E\int_0^{\tilde{x}_{t-1}'a_nM_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds\\
&=\sum_{t=1}^{n}E\int_0^{\tilde{x}_{t-1}'a_nM_nc}\left(f_{t-1}(0)\cdot s\right)ds\left\{1+o_p(1)\right\}\\
&=\frac{1}{2}a_n^2\sum_{t=1}^{n}c'M_n'E\left[f_{t-1}(0)\tilde{x}_{t-1}\tilde{x}_{t-1}'\right]M_nc\left\{1+o_p(1)\right\}\\
&=\frac{1}{2}a_n^2c'\left(M_n'\sum_{t=1}^{n}E\left[f_{t-1}(0)\tilde{x}_{t-1}\tilde{x}_{t-1}'\right]M_n\right)c\left\{1+o_p(1)\right\}\\
&\ge\frac{1}{4}a_n^2\underline{f}\underline{c}_{A(n,p)}C^2n^2\quad\text{(by Assumption L2)}\\
&=O(a_n^2n^2\underline{c}_{A(n,p)}).
\end{aligned}
$$

Finally, for I_3, for a large n,

$$\begin{aligned} Var(I_3) &= Var\left(\sum_{t=1}^{n}\int_0^{\tilde{x}'_{t-1}a_nM_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds\right) \\ &\le E\left[\left(\sum_{t=1}^{n}\int_0^{\tilde{x}'_{t-1}a_nM_nc}\left(\mathbf{1}(u_{t\tau}\le s)-\mathbf{1}(u_{t\tau}\le 0)\right)ds\right)^2\right] \\ &\le a_n^2\sum_{t=1}^{n}c'M_n'E\left[x_{t-1}x'_{t-1}\right]M_nc+2a_n^2\sum_{t=2}^{n}\sum_{k=1}^{t-1}E\left[\left|x'_{t-1}M_nc\right|\left|x'_{k-1}M_nc\right|\right] \\ &= V_{3,1}+V_{3,2}. \end{aligned}$$

By the proof of Lemma 4.1, we can easily show that the following bounds hold

$$V_{3,1}\le O(a_n^2\bar{c}_{A(n,p)}n^2),\ \text{and } V_{3,2}\le O(a_n^2\bar{c}_{A(n,p)}n^3).$$

Based on the results above, we have

$$I_3=O_p(\bar{c}_{A(n,p)}^{1/2}a_nn^{3/2}),$$

which establishes the desired result. ■

Remark B.1 *Define $H^{-1}M_n=M_n^*$. The normalizing matrix then is*

$$\begin{pmatrix} \sqrt{n}I_{p_z} & 0 & 0 & 0 \\ 0 & \sqrt{n}I_{p_1} & 0 & 0 \\ 0 & -\sqrt{n}A_1' & I_{p_2} & 0 \\ 0 & 0 & 0 & I_{p_x} \end{pmatrix},$$

and

$$\begin{aligned} a_n^{(1)}M_n^*c &= \frac{p^\alpha}{n}\begin{pmatrix} \sqrt{n}I_{p_z} & 0 & 0 & 0 \\ 0 & \sqrt{n}I_{p_1} & 0 & 0 \\ 0 & -\sqrt{n}A_1' & I_{p_2} & 0 \\ 0 & 0 & 0 & I_{p_x} \end{pmatrix}c=\begin{pmatrix} \frac{p^\alpha}{\sqrt{n}}I_{p_z} & 0 & 0 & 0 \\ 0 & \frac{p^\alpha}{\sqrt{n}}I_{p_1} & 0 & 0 \\ 0 & -\frac{p^\alpha}{\sqrt{n}}A_1' & \frac{p^\alpha}{n}I_{p_2} & 0 \\ 0 & 0 & 0 & \frac{p^\alpha}{n}I_{p_x} \end{pmatrix}c \\ &= \left(\frac{p^\alpha}{\sqrt{n}}(c_1,...,c_{p_z}),\frac{p^\alpha}{\sqrt{n}}(c_{p_z+1},...,c_{p_z+p_1}),\ \frac{p^\alpha}{\sqrt{n}}(c_1^*,...,c_{p_2}^*)+O\left(\frac{p^\alpha}{n}\right),\frac{p^\alpha}{n}(c_{p_z+p_1+p_2+1},...,c_p)\right), \end{aligned}$$

where

$$(c_1^*,...,c_{p_2}^*)'=A_1'(c_{p_z+1},...,c_{p_z+p_1})'.$$

Therefore, the reduced rate for the 3rd block component is well accommodated. We define the *convergence rates* $\tilde{a}^*_{n,j}$ *by*

$$\tilde{a}^*_{n,j} = \begin{cases} \frac{p^\alpha}{\sqrt{n}} = a_n^{(0)}, & \textit{for } j = 1, ..., r + p_2, \\ \frac{p^\alpha}{n} = a_n^{(1)}, & \textit{for } j = r + p_2 + 1, ..., p. \end{cases}$$

Proof of Theorem 4.4: Following the proof of Theorem 4.1, it suffices to show that for any $\epsilon > 0$, there exists a sufficiently large C such that

$$P\left\{\inf_{||c||\le C} Q_n(\beta_{0\tau} + a_n^{(1)} M_n^* c) > Q_n(\beta_{0\tau})\right\} \ge 1 - \epsilon,$$

where $M_n^* = Q^{-1} M_n$ as in Remark B.1.

This inequality implies that with probability at least $1-\epsilon$, there is a local minimizer $\hat{\beta}^*_\tau$ in the shrinking ball $\{\beta_{0\tau} + a_n^{(1)*} M_n^* c, ||c|| \le C\}$ such that $||\hat{\beta}^*_\tau - \beta_{0\tau}|| = O_p(\tilde{a}^*_{n,j})$, where $\tilde{a}^*_{n,j}$ is the j-th dominating rates from $a_n^{(1)} M_n^* c$, i.e.,

$$\tilde{a}^*_{n,j} = \begin{cases} \frac{p^\alpha}{\sqrt{n}} = a_n^{(0)}, & \text{for } j = 1, ..., r + p_2, \\ \frac{p^\alpha}{n} = a_n^{(1)}, & \text{for } j = r + p_2 + 1, ..., p. \end{cases}$$

Then we obtain

$$\begin{aligned} & Q_n(\beta_{0\tau} + a_n^{(1)} M_n^* c) - Q_n(\beta_{0\tau}) \\ \ge & \left(\sum_{t=1}^n \rho_\tau(u_{t\tau} - X'_{t-1} a_n^{(1)} M_n^* c) - \sum_{t=1}^n \rho_\tau(u_{t\tau})\right) + \sum_{j=1}^{q_n} \lambda_{n,j}(|\beta_{0\tau,j} + \tilde{a}^*_{n,j} c_j| - |\beta_{0\tau,j}|) \\ = & d_1^* + d_2^*. \end{aligned}$$

For d_1^*, similarly to Theorem 4.1, we have

$$\begin{aligned} & \sum_{t=1}^n \rho_\tau(u_{t\tau} - X'_{t-1} a_n^{(1)} M_n^* c) - \sum_{t=1}^n \rho_\tau(u_{t\tau}) \\ = & -a_n^{(1)} \sum_{t=1}^n X'_{t-1} M_n^* c \cdot \psi_\tau(u_{t\tau}) + \sum_{t=1}^n \int_0^{X'_{t-1} a_n^{(1)*} M_n^* c} (\mathbf{1}(u_{t\tau} \le s) - \mathbf{1}(u_{t\tau} \le 0))\, ds. \end{aligned}$$

Note that

$$\begin{aligned} M_n^{*\prime} \sum_{t=1}^n E\left[f_{t-1}(0) X_{t-1} X'_{t-1}\right] M_n^* &= M'_n \left(H^{-1}\right)' \sum_{t=1}^n E\left[f_{t-1}(0) X_{t-1} X'_{t-1}\right] H^{-1} M_n \\ &= M'_n \sum_{t=1}^n E\left[f_{t-1}(0) \left(H^{-1}\right)' X_{t-1} X'_{t-1} H^{-1}\right] M_n \\ &= M'_n \sum_{t=1}^n E\left[f_{t-1}(0) \tilde{x}_{t-1} \tilde{x}'_{t-1}\right] M_n, \end{aligned}$$

which is controlled by Assumptions $L2$ and $U2$. Thus, using the exactly same proof of Theorem 4.1, the dominating order in d_1^* is

$$\begin{aligned}
& \sum_{t=1}^{n} E \int_0^{x'_{t-1} a_n^{(1)} M_n^* c} \left(\mathbf{1}(u_{t\tau} \le s) - \mathbf{1}(u_{t\tau} \le 0)\right) ds \\
= & \sum_{t=1}^{n} E \int_0^{x'_{t-1} a_n^{(1)} M_n^* c} \left(f_{t-1}(0) \cdot s\right) ds \left\{1 + o_p(1)\right\} \\
= & \frac{1}{2} \left(a_n^{(1)}\right)^2 c' \left(M_n^{*\prime} \sum_{t=1}^{n} E\left[f_{t-1}(0) X_{t-1} X'_{t-1}\right] M_n^* \right) c \left\{1 + o_p(1)\right\} \\
\ge & O_p(\left(a_n^{(1)}\right)^2 n^2 \underline{c}_{A(n,p)}) = O_p\left(p^{2\alpha} \underline{c}_{A(n,p)}\right).
\end{aligned}$$

For d_2^*, the only differences with the proof of Theorem 4.1 are the rate of divergence in $\lambda_{n,j}$'s for $j = 1, ..., r + p_2$, and for $j = r + p_2 + 1, ..., p$. From Corollary 4.3,

$$\hat{\beta}_{\tau,j} = \begin{cases} O_p(a_n^{(0)}) + \beta_{0\tau,j} = o_p(1) + \beta_{0\tau,j}, & j = 1, ..., r + p_2, \\ O_p(a_n^{(1)}) + \beta_{0\tau,j} = o_p(1) + \beta_{0\tau,j}, & j = r + p_2 + 1, ..., p, \end{cases}$$

and clearly $a_n^{(0)} > a_n^{(1)}$ for any given n. Since $\beta_{0\tau,j} \neq 0$ for $j = 1, ..., q_n$, we have

$$\begin{aligned}
\sum_{j=1}^{q_n} \lambda_{n,j}(|\beta_{0\tau,j} + \tilde{a}^*_{n,j} c_j| - |\beta_{0\tau,j}|) & \le \sum_{j=1}^{q_n} \lambda_{n,j} \tilde{a}^*_{n,j} |c_j| = \lambda_n \sum_{j=1}^{q_n} \tilde{a}^*_{n,j} |c_j| \frac{1}{|\hat{\beta}_{\tau,j}|^\gamma} \\
& \le \lambda_n \max_j \left(\tilde{a}^*_{n,j}\right) \left(\sum_{j=1}^{q_n} \frac{1}{(o_p(1) + \beta_{0\tau,j})^{2\gamma}} \right)^{1/2} ||c|| \\
& = \lambda_n a_n^{(0)} q_n^{1/2} O_p(1) \\
& = O_p\left(\lambda_n (p^\alpha / \sqrt{n}) p^{1/2}\right) \\
& = O_p\left(\lambda_n \frac{p^{\frac{1}{2}+\alpha}}{n^{1/2}}\right),
\end{aligned}$$

so is dominated by $d_1^* = O_p\left(p^{2\alpha} \underline{c}_{A(n,p)}\right)$ under the condition

$$\lambda_n \frac{p^{\frac{1}{2}-\alpha}}{n^{1/2} \underline{c}_{A(n,p)}} = \lambda_n \frac{n^{\left(\frac{1}{2}-\alpha\right)\zeta}}{n^{1/2} \underline{c}_{A(n,p)}} = \lambda_n \frac{n^{\frac{1}{2}\zeta}}{n^{\frac{1}{2}+\alpha\zeta} \underline{c}_{A(n,p)}} \to 0.$$

■

Proof of Theorem 4.5: Among $\hat{\beta}_\tau^{ALQR*} = \hat{\beta}_\tau^* = (\hat{\beta}_\tau^{z*\prime}, \hat{\beta}_{1\tau}^{c*\prime}, \hat{\beta}_{2\tau}^{c*\prime}, \hat{\beta}_\tau^{x*\prime})'$, we just to need to show ALQR sparsity for $\hat{\beta}_{2\tau}^{c*}$, since other parts are shown by Theorem 4.2 and the existing proof for the $I(0)$ case. Among $j = r + 1, ..., r + p_2$, for any $j \notin \mathcal{A}_0$, $\left|\hat{\beta}_{j,\tau}^* - \beta_{j,0\tau}\right| = \left|\hat{\beta}_{j,\tau}^*\right| = O_p\left(\frac{p^\alpha}{\sqrt{n}}\right)$, so following the proof of Theorem 4.2,

$$
\begin{aligned}
& \Pr\left(j \in \hat{\mathcal{A}}_n\right) \\
\leq\; & \Pr\left(\left|\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right) - h_n\right| = \frac{\lambda_n}{|\hat{\beta}_{j,\tau}^*|^\gamma}\right) \\
=\; & \Pr\left(\frac{1}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}}\left|\sum_{t=1}^{n} x_{t-1,j}\psi_\tau\left(u_{t\tau} - \delta_\tau' x_{t-1}\right) - h_n\right| = \frac{\lambda_n\frac{\sqrt{n}}{p^\alpha}}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}}\frac{1}{|\frac{\sqrt{n}}{p^\alpha}\hat{\beta}_{j,\tau}^*|^\gamma}\right) \\
\longrightarrow\; & 0,
\end{aligned}
$$

as long as

$$
\frac{\lambda_n\left(\frac{\sqrt{n}}{p^\alpha}\right)^\gamma}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}} = \frac{\lambda_n n^{(1/2-\alpha\zeta)\gamma}}{n^{\alpha\zeta+2}\overline{c}_{A(n,p)}^{1/2}} \to \infty.
$$

■

C Proof for Section 5

Proof of Theorem 5.1: We first show the limit theory of ALQR estimator with I(0) predictors. For simplicity, we use A and B to represent $A_{(t,p)}$ and $B_{(t,p)}$, respectively. For an $m \times n$ real matrix A, we denote its Frobenius norm as $||A||_F := \sqrt{trace(A'A)}$. Recall that the I(0) part of our predictors is a linear process of the form of $z_t = \sum_{j=0}^{\infty} D_{zj}\epsilon_{t-j}$ with the conditions given in Section 2.2. In this proof, we use notation $x_t = \sum_{j=0}^{\infty} D_j\epsilon_{t-j}$ for this I(0) predictors for simplicity. Moreover, for simple notation, we drop some superscript/subscript and write $\hat{\beta}_\tau^{(0),ALQR*}$ as $\hat{\beta}$ unless we need some clarity. Also, without loss of generality, we set the first q_n elements of $\hat{\beta}$ be non-zero. Let $\delta = \sqrt{n}(\beta - \beta_0)$ and subscript $\mathcal{A}$ denote elements of the active set. Thus, the oracle (local) estimator is $\hat{\delta} = \left(\hat{\delta}_{\mathcal{A}}, 0\right)$. The oracle property of the ALQR estimator in Section 4 implies that $\hat{\delta}$ is a minimizer of

$$
\sum_{t=1}^{n}\rho_\tau\left(y_t - x_t'\left(n^{-1/2}\delta + \beta_0\right)\right) + \sum_{j=1}^{p}\lambda_{n,j}\left|n^{-1/2}\delta + \beta_0\right|
$$

with probability approaching 1. Define

$$
V_j(\delta) = -n^{-1/2}\sum_{t=1}^{n}\psi_\tau\left(y_t - x_t'\left(n^{-1/2}\delta + \beta_0\right)\right)x_{t,j} + n^{-1/2}\lambda_{n,j}sgn\left(n^{-1/2}\delta_j + \beta_{0,j}\right).
$$

Using vector notation, we have

$$
V(\hat{\delta}) - V(0) = -n^{-1/2}\sum_{t=1}^{n}\left[\psi_\tau\left(y_t - x_t'\left(n^{-1/2}\hat{\delta} + \beta_0\right)\right) - \psi_\tau\left(u_t\right)\right]x_t + n^{-1/2}\omega_n(\hat{\delta}),
$$

where $\omega_n\left(\hat{\delta}\right) = \left(\lambda_{n,1}\left|\tilde{\beta}_{\tau,1}\right|^{-\gamma}sgn\left(n^{-1/2}\hat{\delta}_1 + \beta_{0,1}\right), \ldots, \lambda_{n,q}\left|\tilde{\beta}_{\tau,q}\right|^{-\gamma}sgn\left(n^{-1/2}\hat{\delta}_q + \beta_{0,q}\right), 0, \ldots, 0\right)$.

Note that, except the penalty term $n^{-1/2}\omega_n(\hat{\delta})$,

$$V(\hat{\delta}) - V(0) = \left[V(\hat{\delta}) - V(0) - E\left[V(\hat{\delta})\right] + E\left[V(0)\right]\right] + \left[E\left[V(\hat{\delta})\right] - E\left[V(0)\right] + A\delta\right] - A\delta$$

and by rearranging

$$A\delta - V(0) = \left[V(\hat{\delta}) - V(0) - E\left[V(\hat{\delta})\right] + E\left[V(0)\right]\right] + \left[E\left[V(\hat{\delta})\right] - E\left[V(0)\right] + A\delta\right] - V(\hat{\delta}). \quad \text{(C.24)}$$

Define the weighted norm $||\cdot||_c$ by $||A||_c = ||cA||$ where c is an arbitrary $l \times p$ matrix with $||c|| \leq \underline{c}_B^{-1/2} L_c$ for a large constant $L_c < \infty$. Since l is fixed, without loss of generality we assume that $l = 1$.

Using this weighted norm $||\cdot||_c$, we will show

$$\begin{aligned} \|A\delta - V(0)\|_c \leq \sup_{||\hat{\delta}|| \leq \sqrt{p}L} \Big[& \left\|V(\hat{\delta}) - V(0) - E\left[V(\hat{\delta})\right] + E\left[V(0)\right]\right\|_c \\ & + \left\|E\left[V(\hat{\delta})\right] - E\left[V(0)\right] + A\delta\right\|_c + \left\|V\left(\hat{\delta}\right)\right\|_c \Big] = o_p(1), \end{aligned} \quad \text{(C.25)}$$

for a large constant L.

We denote the terms on the RHS of (C.25) as:

$$\begin{aligned} \text{(CA.6)} &:= \sup_{||\hat{\delta}|| \leq \sqrt{p}L} \left\|V(\hat{\delta}) - V(0) - E\left[V(\hat{\delta})\right] + E\left[V(0)\right]\right\|_c \\ \text{(CA.7)} &:= \sup_{||\hat{\delta}|| \leq \sqrt{p}L} \left\|E\left[V(\hat{\delta})\right] - E\left[V(0)\right] + A\delta\right\|_c \\ \text{(CA.8)} &:= \sup_{||\hat{\delta}|| \leq \sqrt{p}L} \left\|V\left(\hat{\Delta}\right)\right\|_c . \end{aligned}$$

For (CA.6), we need to show that for any large constant $L < \infty$,

$$\sup_{||\hat{\delta}|| \leq \sqrt{p}L} \left\|V(\hat{\delta}) - V(0) - E\left[V(\hat{\delta}) - V(0)\right]\right\|_c = o_p(1).$$

First, we write $a_t \equiv cx_t = a_t^+ - a_t^-$ where $a_t^+ = max\{a_t, 0\}$ and $a_t^- = max\{-a_t, 0\}$. By Minkowski's inequality, we obtain

$$\begin{aligned} & \sup_{||\hat{\delta}|| \leq \sqrt{p}L} \left\|V(\hat{\delta}) - V(0) - E\left[V(\hat{\delta}) - V(0)\right]\right\|_c \\ & \quad \leq \sup_{||\hat{\delta}|| \leq \sqrt{p}L} \left|V^+(\hat{\delta}) - V^+(0) - E\left[V^+(\hat{\delta}) - V^+(0)\right]\right| \\ & \quad + \sup_{||\hat{\delta}|| \leq \sqrt{p}L} \left|V^-(\hat{\delta}) - V^-(0) - E\left[V^-(\hat{\delta}) - V^-(0)\right]\right|, \end{aligned} \quad \text{(C.26)}$$

where $V^{+}(\hat{\delta}) \equiv n^{-1/2}\sum_{t=1}^{n}\psi_{\tau}\left(y_t - x'_{t-1}\left(n^{-1/2}\hat{\delta}+\beta_0\right)\right)a^{+}_{t-1}$, and $V^{-}(\hat{\delta})$ is analogously defined. Thus, it suffices to show that each term on the right hand side of (C.26) is $o_p(1)$.

We show the first term of (C.26) is $o_p(1)$. Let $\mathbf{D} \equiv \{\hat{\delta} \in \mathcal{R}^p : ||\hat{\delta}|| \leq \sqrt{p}L\}$ for some finite constant L. Let $|t|_{\infty}$ denote the maximum of the absolute values of the coordinates of t. We select $N_1 = (2n^2)^p$ grid points, $\hat{\delta}_1, ..., \hat{\delta}_{N_1}$ and cover $\mathbf{D}$ by cubes $\mathbf{D}_s = \left\{\hat{\delta} \in \mathcal{R}^p : |\hat{\delta} - \hat{\delta}_s|_{\infty} \leq \delta_{\epsilon n}\right\}$ with sides of length $\delta_{\epsilon n}$ where $\delta_{\epsilon n} = Lp^{1/2}/n^2$. Since $\psi_{\tau}(\cdot)$ is monotone, by the Minkowski's inequality we obtain that

$$
\begin{aligned}
& \sup_{||\hat{\delta}||\leq\sqrt{p}L}\left|V^{+}(\hat{\delta}) - V^{+}(0) - E\left[V^{+}(\hat{\delta}) - V^{+}(0)\right]\right| \\
\leq \;& \max_{1\leq s\leq N_1}\left|V^{+}(\hat{\delta}) - V^{+}(0) - E\left[V^{+}(\hat{\delta}) - V^{+}(0)\right]\right| \\
& + \max_{1\leq s\leq N_1}\left|n^{-1/2}\sum_{t=1}^{n}E\left[\psi_{st}(\delta_{\epsilon n})a^{+}_{t-1}\right] - E\left[\psi_{st}(-\delta_{\epsilon n})a^{+}_{t-1}\right]\right| \\
& + \max_{1\leq s\leq N_1}\left|n^{-1/2}\sum_{t=1}^{n}\left[\left[\psi_{st}(\delta_{\epsilon n}) - \psi_{st}(0)\right]a^{+}_{t-1} - E\left\{\left[\psi_{st}(\delta_{\epsilon n}) - \psi_{st}(0)\right]a^{+}_{t-1}\right\}\right]\right| \\
\equiv \;& I_1 + I_2 + I_3,
\end{aligned}
$$

where $\psi_{st}(\delta) = \psi_{\tau}\left(u_{t\tau} - n^{-1/2}\hat{\delta}'_s x_{t-1} + n^{-1/2}\delta||x_{t-1}||\right)$.

For I_1, note that

$$
\begin{aligned}
& V^{+}(\hat{\delta}) - V^{+}(0) - E\left[V^{+}(\hat{\delta}) - V^{+}(0)\right] \\
= \;& n^{-1/2}\sum_{t=1}^{n}\left[\psi_{\tau}\left(y_t - x'_{t-1}\left(n^{-1/2}\hat{\delta}+\beta_0\right)\right)a^{+}_{t-1} - \psi_{\tau}\left(y_t - x'_{t-1}\beta_0\right)a^{+}_{t-1}\right] \\
& -n^{-1/2}\sum_{t=1}^{n}E\left[\psi_{\tau}\left(y_t - x'_{t-1}\left(n^{-1/2}\hat{\delta}+\beta_0\right)\right)a^{+}_{t-1} - \psi_{\tau}\left(y_t - x'_{t-1}\beta_0\right)a^{+}_{t-1}\right] \\
= \;& n^{-1}\sum_{t=1}^{n}\eta_{ts} \\
= \;& n^{-1}\sum_{t=1}^{n}\eta_{ts}\mathbf{1}\left\{a^{+}_{t-1} \leq e_{1n}\right\} + n^{-1}\sum_{t=1}^{n}\eta_{ts}\mathbf{1}\left\{a^{+}_{t-1} > e_{1n}\right\} \\
\equiv \;& D_{1s} + D_{2s},
\end{aligned}
$$

where $\eta_{ts} \equiv n^{1/2}[\eta_{ts,0} - E(\eta_{ts,0})]$, $\eta_{ts,0} = [\psi_{\tau}(u_{t\tau} - n^{-1/2}\hat{\delta}'_s x_{t-1}) - \psi_{\tau}(u_{t\tau})]a^{+}_{t-1}$, and $e_{1n} = (np^4\underline{c}_B^{-4})^{1/8}$. To prove that $I_1 = o_p(1)$, it suffices to show

$$
\max_{1\leq s\leq N_1}|D_{ks}| = o_p(1) \quad \text{for k= 1 and 2.}
$$

For the I(0) predictors $x_t = \sum_{j=0}^{\infty}D_j\epsilon_{t-j}$, we have $D_j = O\left(e^{-vj}\right)$ with $v > 0$ along with the conditions (1), (2), (5) of Withers (1981): $\{\epsilon_j\}_{j=-\infty}^{\infty}$ are independent r.v's with characteristic functions $\{\phi_j\}$ such that

$$(1): \quad (2\pi\ )^{-1} \max_j \int \left|\phi_j(t)\right| dt < \infty, \tag{C.27}$$

$$(2): \quad \max_j E\left|\epsilon_j\right|^{\delta} < \infty \text{ for some } \delta > 0, \tag{C.28}$$

$$(5): \quad \sup_{m,s,k\geq 1} \sup_{\alpha,\beta,v} \max_t \left|\frac{\partial}{\partial v_t} P\left(W + v \in \cup_1^s D_j\right)\right| < \infty, \tag{C.29}$$

where

$$D_j = \mathbb{X}_{t=k}^{k+m-1}\left(\alpha_{jt}, \beta_{jt}\right),\ v = (v_k, ..., v_{k+m-1}),\ W = (W_k, ..., W_{k+m-1})$$

with $W_t = \sum_{j=0}^{t} D_j \epsilon_{t-1-j}$ and $\mathbb{X}$ represents a product space.

As a result, x_t is strong mixing with $\alpha(j) = O(e^{-v\lambda j})$ with another constant $\lambda > 0$ by Corollary 2 of Withers (1981). By the invariance property of the strong mixing processes, $\frac{\eta_{ts}}{n^{1/2}} = [\eta_{ts,0} - E(\eta_{ts,0})]$ is also strong mixing with $\alpha(j) = O(e^{-v\lambda j})$, and

$$\frac{\eta_{ts}}{n^{1/2}} \mathbf{1}\left\{a_{t-1}^{+} \leq e_{1n}\right\} = [\eta_{ts,0} - E(\eta_{ts,0})]\mathbf{1}\left\{a_{t-1}^{+} \leq e_{1n}\right\} \leq 2e_{1n} \text{ for all } t \text{ and } s.$$

From Theorem 2, Equation (2.3) of Merlevède et al. (2009), there is a constant C_3 depending only on $\widetilde{c}$ such that, for all $n \geq 2$,

$$P\left(\left|\sum_{t=1}^{n} X_t\right| \geq \eta\right) \leq \exp\left(-\frac{C_3\eta^2}{v^2 n + M^2 + \eta M \left(\log n\right)^2}\right).$$

with $v^2 = \sup_{i>0}\left(Var(X_i) + 2\sum_{j>i}\left|Cov\left(X_i, X_j\right)\right|\right) = Var(X_0) + 2\sum_{j=1}^{\infty}\left|Cov\left(X_j, X_0\right)\right|$ (using covariance stationarity).

In our case,

$$P\left(\left|\sum_{t=1}^{n} \frac{\eta_{ts}}{n^{1/2}} \mathbf{1}\left\{a_{t-1}^{+} \leq e_{1n}\right\}\right| \geq \varepsilon n^{1/2}\right),$$

so $X_t = \frac{\eta_{ts}}{n^{1/2}} \mathbf{1}\left\{a_{t-1}^{+} \leq e_{1n}\right\}$, $\eta = \varepsilon n^{1/2}$, and $M = 2e_{1n}$, thereby providing

$$P\left(\left|\sum_{t=1}^{n} \frac{\eta_{ts}}{n^{1/2}} \mathbf{1}\left\{a_{t-1}^{+} \leq e_{1n}\right\}\right| \geq \varepsilon n^{1/2}\right) \leq \exp\left(-\frac{C_3\varepsilon^2 n}{v^2 n + 4e_{1n}^2 + 2e_{1n}\varepsilon n^{1/2}\left(\log n\right)^2}\right).$$

Thus, it suffices to show $v^2 n$ is slower than n.

Note that

$$\begin{aligned}
& Var\left(\eta_{ts}\mathbf{1}\left\{a_{t-1}^{+}\leq e_{1n}\right\}\right) \\
= \ & E\left[\eta_{ts}^{2}\mathbf{1}\left\{a_{t-1}^{+}\leq e_{1n}\right\}\right]-E\left[\eta_{ts}\mathbf{1}\left\{a_{t-1}^{+}\leq e_{1n}\right\}\right]^{2} \\
\leq \ & E\left[\eta_{ts}^{2}\mathbf{1}\left\{a_{t-1}^{+}\leq e_{1n}\right\}\right] \\
= \ & E\left[n[\eta_{ts,0}-E(\eta_{ts,0})]^{2}\mathbf{1}\left\{a_{t-1}^{+}\leq e_{1n}\right\}\right] \\
\leq \ & nE\left[[\eta_{ts,0}]^{2}\right]=nE\left[[\left(\psi_{\tau}(u_{t\tau}-n^{-1/2}\hat{\delta}_{s}^{\prime}x_{t-1})-\psi_{\tau}(u_{t\tau})\right)\left(a_{t-1}^{+}\right)^{2}\right] \\
= \ & nE\left[[E\left[\left(\psi_{\tau}(u_{t\tau}-n^{-1/2}\hat{\delta}_{s}^{\prime}x_{t-1})-\psi_{\tau}(u_{t\tau})\right)|x_{t-1}\right]\left(a_{t-1}^{+}\right)^{2}\right] \\
= \ & nE\left[\left(F_{u}\left(n^{-1/2}\hat{\delta}_{s}^{\prime}x_{t-1}|x_{t-1}\right)-F_{u}\left(0|x_{t-1}\right)\right)\left(a_{t-1}^{+}\right)^{2}\right] \\
= \ & nE\left[\left(n^{-1/2}\hat{\delta}_{s}^{\prime}x_{t-1}f_{u}\left(0|x_{t-1}\right)+o_{p}(n^{-1/2}\right)\left(a_{t-1}^{+}\right)^{2}\right]=Cn^{1/2}p^{1/2}E[\left(a_{t-1}^{+}\right)^{2}],
\end{aligned}$$

where we used the fact $\left\|\hat{\delta}_{s}^{\prime}x_{t-1}\right\|\leq O_{p}(p^{1/2})$ (see (C.31) below). Therefore, we have $Var\left(X_{t}\right)=Var\left(\frac{\eta_{ts}}{n}\mathbf{1}\left\{a_{t-1}^{+}\leq e_{1n}\right\}\right)=\frac{1}{n^{2}}Var\left(\eta_{ts}\mathbf{1}\left\{a_{t-1}^{+}\leq e_{1n}\right\}\right)\leq Cn^{-3/2}\underline{c}_{B}^{-1}p^{3/2}$.

$$Var\left(\frac{\eta_{ts}}{n^{1/2}}\mathbf{1}\left\{a_{t-1}^{+}\leq e_{1n}\right\}\right)\leq Cn^{-1/2}\underline{c}_{B}^{-1}p^{3/2}. \tag{C.30}$$

It remains to show the covariance terms $\sum_{j=1}^{\infty}|Cov\left(X_{j},X_{0}\right)|$ is same or smaller order than the term in (C.30). By the inequality of the strong mixing processes, see e.g., Corollary 14.3 of Davidson (1994), for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$,

$$|Cov\left(X_{j},X_{0}\right)|\preceq\alpha(j)^{1/r}\left\|X_{j}\right\|_{p}\left\|X_{0}\right\|_{r}.$$

where $\preceq$ is $\leq$ up to a fixed constant term. Letting $p=q=2+\zeta,$ we have $r=\frac{2+\zeta}{\zeta}$. Note that

$$\left\|X_{j}\right\|_{2+\zeta}=\left(E\left[|X_{j}|^{2+\zeta}\right]\right)^{\frac{1}{2+\zeta}}$$

and

$$\begin{aligned}
& E\left[|X_{j}|^{2+\zeta}\right] \\
= \ & E\left[\left|\frac{\eta_{js}}{n^{1/2}}\mathbf{1}\left\{a_{j-1}^{+}\leq e_{1n}\right\}\right|^{2+\zeta}\right]=E\left[\left|[\eta_{ts,0}-E(\eta_{ts,0})]\mathbf{1}\left\{a_{t-1}^{+}\leq e_{1n}\right\}\right|^{2+\zeta}\right] \\
= \ & E\Big[\Big|\Big\{[\psi_{\tau}(u_{t\tau}-n^{-1/2}\hat{\delta}_{s}^{\prime}x_{t-1})-\psi_{\tau}(u_{t\tau})]a_{t-1}^{+}-E\left[[\psi_{\tau}(u_{t\tau}-n^{-1/2}\hat{\delta}_{s}^{\prime}x_{t-1})-\psi_{\tau}(u_{t\tau})]a_{t-1}^{+}\right]\Big\}\Big|^{2+\zeta} \\
& \times\mathbf{1}\left\{a_{j-1}^{+}\leq e_{1n}\right\}\Big] \\
\leq \ & E\left[\left|\psi_{\tau}(u_{t\tau}-n^{-1/2}\hat{\delta}_{s}^{\prime}x_{t-1})-\psi_{\tau}(u_{t\tau})\right|\left(a_{t-1}^{+}\right)^{2+\zeta}\right] \\
= \ & E\left[E\left[\left|\psi_{\tau}(u_{t\tau}-n^{-1/2}\hat{\delta}_{s}^{\prime}x_{t-1})-\psi_{\tau}(u_{t\tau})\right||x_{t-1}\right]\left(a_{t-1}^{+}\right)^{2+\zeta}\right]
\end{aligned}$$

From the same argument to derive (C.30), we have

$$E\left[\left|\psi_\tau(u_{t\tau}-n^{-1/2}\hat{\delta}_s'x_{t-1})-\psi_\tau(u_{t\tau})\right||x_{t-1}\right]=n^{-1/2}\hat{\delta}_s'x_{t-1}+o_p\left(n^{-1/2}\right).$$

Therefore, we get

$$E\left[|X_j|^{2+\zeta}\right]\preceq n^{-1/2}p^{1/2}E\left[\left(a_{t-1}^+\right)^{2+\zeta}\right]$$

and

$$\|X_j\|_{2+\zeta}\preceq n^{-\frac{1}{2(2+\zeta)}}p^{\frac{1}{2(2+\zeta)}}\left\|a_{t-1}^+\right\|_{2+\zeta}.$$

The stationarity assumption implies that $\|X_0\|_{2+\zeta}$ is the same order. Therefore,

$$|Cov\left(X_j,X_0\right)|\preceq\alpha(j)^{1/r}n^{-\frac{1}{(2+\zeta)}}p^{\frac{1}{(2+\zeta)}}\left\|a_{t-1}^+\right\|_{2+\zeta}^2$$

and

$$\sum_{j=1}^{\infty}|Cov\left(X_j,X_0\right)|\preceq Cn^{-\frac{1}{(2+\zeta)}}\underline{c}_B^{-1}p^{1+\frac{1}{(2+\zeta)}}.$$

In other words, the leading term of $v^2=\sup_{i>0}\left(Var(X_i)+2\sum_{j>i}|Cov\left(X_i,X_j\right)|\right)$ is $Var\left(\frac{\eta_{ts}}{n^{1/2}}\mathbf{1}\left\{a_{t-1}^+\leq e_{1n}\right\}\right)$, which is $Cn^{-1/2}\underline{c}_B^{-1}p^{3/2}$. Therefore, we conclude that

$$P\left(\left|\sum_{t=1}^{n}\frac{\eta_{ts}}{n^{1/2}}\mathbf{1}\left\{a_{t-1}^+\leq e_{1n}\right\}\right|\geq\varepsilon n^{1/2}\right)\leq\exp(-\frac{C_3\varepsilon^2 n}{Cn^{-1/2}\underline{c}_B^{-1}p^{3/2}+4e_{1n}^2+2e_{1n}\varepsilon n^{1/2}\left(\log n\right)^2})\to 0$$

using the same argument from Lu and Su (2015, p. 53). Note that, by the weak dependence assumption (asymptotic independence), we achieve the same order of magnitudes (in terms of n) in the exponent of the Bernstein's inequality.

The proofs of the other terms D_2, I_2 and I_3 do not rely on the independence assumption, such that the same argument of Lu and Su (2015) can carry over under our additional assumptions. Thus the proofs are omitted.

For (CA.7), we need to show:

$$\sup_{||\hat{\delta}||\leq\sqrt{p}L}||E[V(\delta)]-E[V(0)]+A\delta||_c=o(1).$$

By Assumption 4.1 and 4.6,

$$\begin{aligned}
&\sup_{||\hat{\delta}||\leq\sqrt{p}L}||E[V(\delta)]-E[V(0)]+A\delta||_c\\
&=\sup_{||\hat{\delta}||\leq\sqrt{p}L}||n^{-1/2}\sum_{t=1}^{n}E\left[\psi_\tau(u_t-n^{-1/2}\delta'x_{t-1})x_{t-1}\right]-n^{-1/2}\sum_{t=1}^{n}E\left[\psi_\tau(u_t)x_{t-1}\right]+A\delta||_c\\
&=\sup_{||\hat{\delta}||\leq\sqrt{p}L}||n^{-1/2}\sum_{t=1}^{n}E\left[\left(\tau-\mathbf{1}(u_t-n^{-1/2}\delta'x_{t-1}\leq 0)\right)x_{t-1}-\left(\tau-\mathbf{1}(u_t\leq 0)\right)x_{t-1}\right]+A\delta||_c
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{||\hat{\delta}|| \le \sqrt{p}L} ||n^{-1/2} \sum_{t=1}^{n} E\left[\left(-\mathbf{1}(u_t - n^{-1/2}\delta' x_{t-1} \le 0) + \mathbf{1}(u_t \le 0) \right) x_{t-1} \right] + A\delta||_c \\
&= \sup_{||\hat{\delta}|| \le \sqrt{p}L} ||n^{-1/2} \sum_{t=1}^{n} E\left[F(-u_t - n^{-1/2}\delta' x_{t-1}|x_{t-1})x_{t-1} - F(u_t|x_{t-1})x_{t-1} \right] - A\delta||_c \\
&= \sup_{||\hat{\delta}|| \le \sqrt{p}L} ||n^{-1/2} \sum_{t=1}^{n} E\left[n^{-1/2}\delta' x_{t-1} \left(\int_0^1 f(-u_t + sn^{-1/2}\delta' x_{t-1}|x_{t-1}) ds \right) x_{t-1} \right] - E\left[f(-u_t|x_{t-1})x_{t-1}x_{t-1}' \right] \delta||_c \\
&= \sup_{||\hat{\delta}|| \le \sqrt{p}L} ||n^{-1} \sum_{t=1}^{n} E\left[\left(\int_0^1 \left[f(-u_t + sn^{-1/2}\delta' x_{t-1}|x_{t-1}) - f(-u_t|x_{t-1}) \right] ds \right) x_{t-1}x_{t-1}'\delta \right] ||_c \quad \text{(stationary } x_{t-1}\text{)} \\
&\le C \sup_{||\hat{\delta}|| \le \sqrt{p}L} n^{-3/2} \sum_{t=1}^{n} E||\delta' x_{t-1} x_{t-1} x_{t-1}' \delta||_c \quad \text{(Taylor expansion)} \\
&\le C n^{-3/2} ||c|| \sup_{||\hat{\delta}|| \le \sqrt{p}L} \sum_{t=1}^{n} \left(E||\delta' x_{t-1}||_F^2 \right)^{1/2} \left(E||x_{t-1}x_{t-1}'\delta||_F^2 \right)^{1/2} \quad \text{(by Cauchy-Schwarz inequality)} \\
&\le C n^{-3/2} \underline{c}_B^{-1/2} L_c \sup_{||\hat{\delta}|| \le \sqrt{p}L} \sum_{t=1}^{n} \left(E||\delta' x_{t-1}||_F^2 \right)^{1/2} \left(E||x_{t-1}x_{t-1}'\delta||_F^2 \right)^{1/2} \quad \text{(since } ||c|| \le \underline{c}_B^{1/2} L_c\text{)} \\
&= C n^{-3/2} \underline{c}_B^{-1/2} L_c \sum_{t=1}^{n} O(\bar{c}_A^{1/2} p^{1/2}) O(p^{3/2}) \\
&= O(n^{-1/2} \underline{c}_B^{-1/2} \bar{c}_A^{1/2} p^2) = o_p(1),
\end{aligned}
$$

where the second last equality uses the facts that

$$
\begin{aligned}
E||\delta' x_{t-1}||_F^2 &\le E||x_{t-1}||^2 ||\delta||_F^2 \\
&= \lambda_{\max} \left(E[x_{t-1}x_{t-1}'] \right) ||\delta||_F^2 \\
&\le O(\bar{c}_A p) \quad \text{(Assumption 4.6)} \qquad \text{(C.31)}
\end{aligned}
$$

and

$$
\begin{aligned}
E||x_{t-1}x_{t-1}'\delta||_F^2 &\le E||x_{t-1}x_{t-1}'||_F^2 ||\delta||_F^2 \\
&= E\left[\left(\sum_{j=1}^{p} x_{t-1,j}^2 \right)^2 ||\delta||_F^2 \right] \\
&\le O(p^2) O(p) = O(p^3).
\end{aligned}
$$

Next, we show (CA.8):

$$
||V(\hat{\delta})||_c = o_p(1).
$$

Note that

$$\begin{aligned}
||V(\hat{\delta})||_c &= \|n^{-1/2}\sum_{t=1}^{n}\psi_\tau(u_t - n^{-1/2}\delta' x_{t-1})x_{t-1}\|_c \\
&= \|n^{-1/2}\sum_{t=1}^{n}\psi_\tau(y_t - \hat{\beta}' x_{t-1})x_{t-1}\|_c \\
&= \left|n^{-1/2}\sum_{t=1}^{n}\psi_\tau(y_t - \hat{\beta}' x_{t-1})cx_{t-1}\right| \\
&\leq n^{-1/2}\sum_{t=1}^{n}|cx_{t-1}|\,\mathbf{1}(y_t - \hat{\beta}' x_{t-1} = 0) \quad \text{(by the proof of Lemma A2 in Ruppert and Carroll (1980))} \\
&\leq n^{-1/2}\max_{1\leq t\leq n}|cx_{t-1}|\,p \\
&= o_p(1),
\end{aligned}$$

where we use the fact that

$$\begin{aligned}
P\left(\max_{1\leq t\leq n}|cx_{t-1}| \geq n^{1/2}p^{-1}\right) &\leq nP\left(|cx_{t-1}| \geq n^{1/2}p^{-1}\right) \quad \text{(by Boole's inequality)} \\
&\leq n\frac{E\,|cx_{t-1}|^8}{n^4p^{-8}} \quad \text{(by Markov's inequality)} \\
&\leq n^{-3}p^8\|c\|^8E\|x_{t-1}\|^8 \\
&= O(n^{-3}p^8\underline{c}_B^{-4}p^4) \quad (\text{since } \sup_{j\geq 1}E(x_{t-1,j}^8) \leq c_x \text{ for some } c_x \leq \infty) \\
&= O(n^{-3}p^{12}\underline{c}_B^{-4}) \\
&= o(1),
\end{aligned}$$

by the assumption that $n^{-3}p^{12}\underline{c}_B^{-4} \to 0$ as $n \to \infty$.

From (C.25), the limit theory of $A\hat{\delta}$ follows $V(0)$ because the penalty term $n^{-1/2}\omega_n(\delta)$ for the non-zero q_n element is asymptotically negligible under our rate conditions. Note that $V(0) := n^{-1/2}\sum_{t=1}^{n}\psi_\tau(u_{t\tau})\,x_t$, then under our strong mixing condition of Assumption 5.1, Proposition B.2 of the supplement of Li and Liao (2020) provides:

$$\left\|V(0) - \tilde{S}_n\right\|_F = o_p(1),$$

where $\tilde{S}_n$ is a q_n-dimensional random vector with distribution $N(0, B)$. Therefore, $\hat{\delta} := \sqrt{n}(\hat{\beta} - \beta_0)$ converges to a q_n-dimensional random vector with distribution $N\left(0, A^{-1}BA^{-1}\right)$ in $\|\cdot\|_F$-norm, implying Theorem 5.1-(i), from the relation $\|M\| \leq \|M\|_F \leq \sqrt{r}\,\|M\|$ for a matrix M of rank r.

For the proof of Theorem 5.1-(ii), following the argument in Section 4.2, we have

$$R\left[\hat{\beta}_\tau^{(1),QR*}(1) - \beta_{0\tau}^{(1)}(1)\right] = R\left[\left(\tilde{\beta}_\tau^{(1),QR}(1) - \tilde{\beta}_{0\tau}^{(1)}(1)\right) - A_1'\left(\hat{\beta}_\tau^{(1),QR}(1) - \beta_{0\tau}^{(1)}(1)\right)\right],$$

so that

$$\begin{aligned}
\sqrt{n}R\left[\hat{\beta}_{\tau}^{(1),QR*}(1)-\beta_{0\tau}^{(1)}(1)\right] &= -\sqrt{n}RA_1'\left(\hat{\beta}_{\tau}^{(1),QR}(1)-\beta_{0\tau}^{(1)}(1)\right)+\frac{1}{\sqrt{n}}nR\left(\tilde{\beta}_{\tau}^{(1),QR}(1)-\tilde{\beta}_{0\tau}^{(1)}(1)\right) \\
&= -\sqrt{n}RA_1'\left(\hat{\beta}_{\tau}^{(1),QR}(1)-\beta_{0\tau}^{(1)}(1)\right)+O_p\left(\frac{1}{\sqrt{n}}\right) \\
&\Longrightarrow N\left(0, RA_1'A^{-1}BA^{-1}A_1R'\right).
\end{aligned}$$

Finally, for the fixed dimension of $p_x < \infty$, Theorem 5.1-(iii) follows from Lee (2016). Unlike the cointegrated parts of the system, this non-cointegrated local unit root regressors have their own limit theory from the block-wise diagonal structure of H and H^{-1} matrix transformations given in Section 4.2.

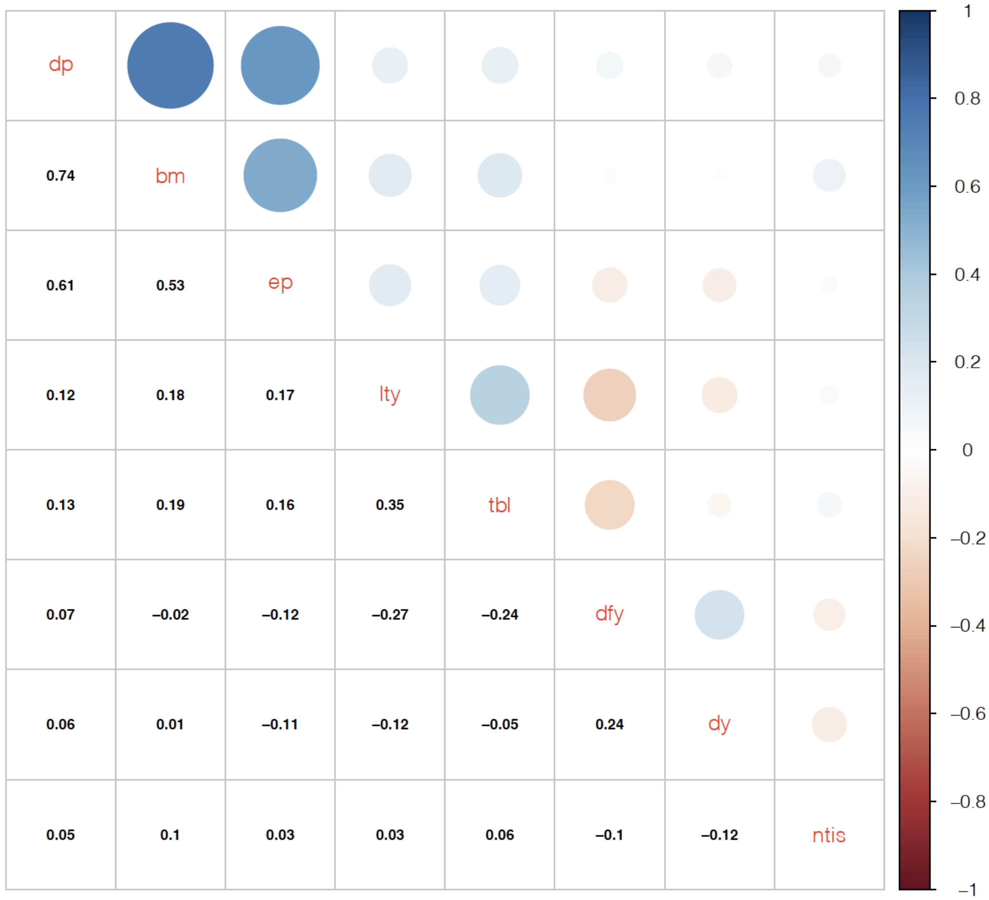
■

D Additional Tables and Figures

Table D.1: Prediction Results of Stock Returns: 12 one-period-ahead forecasts, GIC

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
	Final Prediction Error (FPE)				
QR	0.0047	0.0099	0.0134	0.0078	0.0060
LASSO	**0.0045**	**0.0083**	0.0123	**0.0032**	**0.0020**
ALQR	**0.0045**	**0.0083**	**0.0122**	**0.0032**	**0.0020**
QUANT	0.0046	**0.0083**	0.0124	0.0055	0.0029
	Out-of-Sample R^2				
QR	-0.0147	-0.1930	-0.0831	-0.4048	-1.0666
LASSO	0.0219	**-0.0009**	0.0094	0.4219	**0.3233**
ALQR	**0.0223**	-0.0029	**0.0167**	**0.4288**	0.3211
	Average # of Selected Predictors				
LASSO	11.00	10.00	12.00	11.42	10.58
ALQR	10.00	7.92	10.00	6.33	9.25
	Tuning Parameter (λ) by GIC				
LASSO ($\times 10^{-4}$)	82.07	225.20	0.04	49.17	27.66
ALQR ($\times 10^{-7}$)	5.26	85.12	1.46	197.67	0.88

Figure D.1: The Correlation Heat Map of the Persistent Predictors



Note: The correlation heat map is based on 816 monthly observations ranging from January 1952 to December 2019. All correlations are computed using the first difference of predictors. The full predictor names are defined in Table 1.

Table D.2: Prediction Results of Stock Returns: 24 one-period-ahead forecasts, GIC

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
	Final Prediction Error (FPE)				
QR	**0.0045**	0.0102	0.0147	0.0067	0.0050
LASSO	0.0058	0.0101	**0.0140**	**0.0045**	**0.0025**
ALQR	0.0058	**0.0097**	**0.0140**	**0.0045**	**0.0025**
QUANT	0.0055	0.0098	0.0148	0.0063	0.0036
	Out-of-Sample R^2				
QR	**0.1751**	-0.0388	0.0100	-0.0745	-0.3938
LASSO	-0.0590	-0.0245	0.0573	0.2747	0.2881
ALQR	-0.0526	**0.0103**	**0.0577**	**0.2798**	**0.3088**
	Average # of Selected Predictors				
LASSO	10.88	9.63	12.00	10.67	10.71
ALQR	9.96	8.88	10.63	10.25	9.67
	Tuning Parameter (λ) by GIC				
LASSO ($\times 10^{-4}$)	66.33	174.85	0.30	197.25	14.76
ALQR ($\times 10^{-7}$)	3.74	24.70	0.65	1.55	0.59

Table D.3: Simulation Results: Scenario 1 with Dependent Predictors

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Dependency Rate: $\rho = 0.1$					
		Final Prediction Error (FPE)			
QR	0.1076	0.1809	0.4072	0.1812	0.1081
LASSO	0.1073	0.1811	0.4084	0.1809	0.1077
ALQR	**0.1052**	**0.1785**	**0.4048**	**0.1794**	**0.1060**
RIDGE	0.2288	0.3050	0.4053	0.3031	0.2259
QUANT	2.2753	4.2477	11.4435	4.3347	2.3361
		Out-of-Sample R^2			
QR	0.9527	0.9574	0.9644	0.9582	0.9537
LASSO	0.9528	0.9574	0.9643	0.9583	0.9539
ALQR	**0.9537**	**0.9580**	**0.9646**	**0.9586**	**0.9546**
RIDGE	0.8994	0.9282	0.9646	0.9301	0.9033
		Average # of Selected Predictors			
LASSO	10.42	10.15	9.67	10.14	10.47
ALQR	6.56	6.35	6.38	6.35	6.50
		Tuning Parameter (λ)			
LASSO ($\times 10^0$)	8.57	15.18	37.39	15.33	8.43
ALQR ($\times 10^{-4}$)	1.68	2.39	3.81	2.43	1.73
RIDGE ($\times 10^{-5}$)	0.8	1.7	1.6	1.7	1.0

Table D.4: Simulation Results: Scenario 1 with Dependent Predictors

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Dependency Rate: $\rho = 0.5$					
	Final Prediction Error (FPE)				
QR	0.1064	0.1800	0.4074	0.1805	0.1074
LASSO	0.1069	0.1808	0.4094	0.1810	0.1078
ALQR	**0.1053**	**0.1785**	**0.4055**	**0.1790**	**0.1060**
RIDGE	0.2234	0.3046	0.4057	0.3142	0.2296
QUANT	2.5896	4.6670	11.7537	4.5306	2.4595
	Out-of-Sample R^2				
QR	0.9589	0.9614	0.9653	0.9601	0.9563
LASSO	0.9587	0.9613	0.9652	0.9600	0.9562
ALQR	**0.9593**	**0.9617**	**0.9655**	**0.9605**	**0.9569**
RIDGE	0.9137	0.9347	**0.9655**	0.9306	0.9067
	Average # of Selected Predictors				
LASSO	9.90	9.67	9.34	9.78	10.00
ALQR	6.46	6.32	6.33	6.29	6.46
	Tuning Parameter (λ)				
LASSO ($\times 10^0$)	9.82	17.20	37.28	16.12	9.37
ALQR ($\times 10^{-4}$)	1.50	2.08	3.25	2.09	1.53
RIDGE ($\times 10^{-5}$)	0.9	1.8	2.2	1.6	0.8

Table D.5: Simulation Results: Scenario 1 with Dependent Predictors

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Dependency Rate: $\rho = 0.9$					
			Final Prediction Error (FPE)		
QR	0.1067	0.1804	0.4077	0.1809	0.1083
LASSO	0.1060	0.1800	0.4095	0.1810	0.1075
ALQR	**0.1052**	**0.1783**	**0.4055**	**0.1787**	**0.1065**
RIDGE	0.2157	0.2955	0.4058	0.2990	0.2193
QUANT	2.2630	4.1512	11.0832	4.3207	2.4073
			Out-of-Sample R^2		
QR	0.9528	0.9565	0.9632	0.9581	0.9550
LASSO	0.9532	0.9566	0.9630	0.9581	0.9553
ALQR	**0.9535**	**0.9570**	**0.9634**	**0.9586**	**0.9558**
RIDGE	0.9047	0.9288	**0.9634**	0.9308	0.9089
			Average # of Selected Predictors		
LASSO	8.33	7.74	7.17	7.71	8.26
ALQR	6.17	6.01	6.07	5.97	6.13
			Tuning Parameter (λ)		
LASSO ($\times 10^0$)	13.09	23.81	52.67	23.93	13.39
ALQR ($\times 10^{-4}$)	1.36	1.67	2.11	1.57	1.38
RIDGE ($\times 10^{-5}$)	0.6	1.3	1.5	1.2	0.9

Table D.6: Simulation Results: Scenario 2 with Dependent Predictors

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Dependency Rate: $\rho = 0.1$					
			Final Prediction Error (FPE)		
QR	0.1075	0.1809	0.4074	0.1811	0.1081
LASSO	0.1074	0.1810	0.4087	0.1807	0.1080
ALQR	**0.1067**	**0.1794**	0.4068	**0.1799**	**0.1068**
RIDGE	0.2363	0.3123	**0.4057**	0.3123	0.2349
QUANT	2.3787	4.4253	11.8534	4.4989	2.4148
			Out-of-Sample R^2		
QR	0.9548	0.9591	0.9656	0.9597	0.9552
LASSO	0.9549	0.9591	0.9655	0.9598	0.9553
ALQR	**0.9552**	**0.9595**	0.9657	**0.9600**	**0.9558**
RIDGE	0.9007	0.9294	**0.9658**	0.9306	0.9027
			Average # of Selected Predictors		
LASSO	10.82	10.67	10.35	10.66	10.86
ALQR	8.44	8.33	8.31	8.34	8.45
			Tuning Parameter (λ)		
LASSO ($\times 10^0$)	6.95	12.49	30.84	12.25	6.89
ALQR ($\times 10^{-4}$)	1.23	1.61	2.52	1.62	1.18
RIDGE ($\times 10^{-5}$)	0.7	1.7	1.6	1.4	0.6

Table D.7: Simulation Results: Scenario 2 with Dependent Predictors

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Dependency Rate: $\rho = 0.5$					
			Final Prediction Error (FPE)		
QR	0.1072	0.1805	0.4081	0.1813	0.1082
LASSO	0.1073	0.1811	0.4099	0.1815	0.1085
ALQR	**0.1068**	**0.1799**	0.4070	**0.1805**	**0.1072**
RIDGE	0.2484	0.3313	**0.4063**	0.3406	0.2554
QUANT	2.7837	5.0151	12.9604	4.9791	2.6430
			Out-of-Sample R^2		
QR	0.9615	0.9640	0.9685	0.9636	0.9591
LASSO	0.9614	0.9639	0.9684	0.9635	0.9589
ALQR	**0.9616**	**0.9641**	**0.9686**	**0.9637**	**0.9594**
RIDGE	0.9108	0.9339	**0.9686**	0.9316	0.9034
			Average # of Selected Predictors		
LASSO	10.58	10.45	10.27	10.54	10.66
ALQR	8.33	8.32	8.32	8.27	8.39
			Tuning Parameter (λ)		
LASSO ($\times 10^0$)	7.23	12.75	28.74	12.08	6.94
ALQR ($\times 10^{-4}$)	1.19	1.41	2.07	1.44	1.05
RIDGE ($\times 10^{-5}$)	0.9	1.5	1.7	1.4	0.8

Table D.8: Simulation Results: Scenario 2 with Dependent Predictors

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Dependency Rate: $\rho = 0.9$					
			Final Prediction Error (FPE)		
QR	0.1065	0.1803	0.4074	0.1808	0.1082
LASSO	0.1065	0.1804	0.4103	0.1817	0.1078
ALQR	**0.1063**	**0.1797**	0.4065	**0.1798**	**0.1077**
RIDGE	0.2571	0.3418	**0.4058**	0.3469	0.2593
QUANT	2.7380	5.0229	13.4626	5.2381	2.8857
			Out-of-Sample R^2		
QR	0.9611	0.9641	0.9697	0.9655	0.9625
LASSO	0.9611	0.9641	0.9695	0.9653	**0.9627**
ALQR	**0.9612**	**0.9642**	0.9698	**0.9657**	**0.9627**
RIDGE	0.9061	0.9320	**0.9699**	0.9338	0.9101
			Average # of Selected Predictors		
LASSO	9.55	9.14	8.62	9.13	9.50
ALQR	8.04	7.98	8.05	7.94	8.05
			Tuning Parameter (λ)		
LASSO ($\times 10^0$)	9.47	17.32	40.78	17.24	9.63
ALQR ($\times 10^{-4}$)	1.06	1.32	1.52	1.21	0.98
RIDGE ($\times 10^{-5}$)	0.6	1.3	1.4	1.2	0.6

Table D.9: Simulation Results: Scenario 3 with Dependent Predictors

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Dependency Rate: $\rho = 0.1$					
	Final Prediction Error (FPE)				
QR	**0.1076**	**0.1808**	0.4074	0.1811	0.1082
LASSO	**0.1076**	**0.1808**	0.4074	**0.1810**	**0.1081**
ALQR	0.1079	0.1812	0.4074	0.1814	0.1086
RIDGE	0.2394	0.3157	**0.4057**	0.3122	0.2347
QUANT	2.3833	4.4624	11.9144	4.5103	2.4318
	Out-of-Sample R^2				
QR	**0.9549**	**0.9595**	0.9658	**0.9599**	**0.9555**
LASSO	**0.9549**	**0.9595**	0.9658	**0.9599**	**0.9555**
ALQR	0.9547	0.9594	0.9658	0.9598	0.9553
RIDGE	0.8996	0.9292	**0.9660**	0.9308	0.9035
	Average # of Selected Predictors				
LASSO	12.00	12.00	12.00	12.00	12.00
ALQR	11.83	11.92	12.00	11.93	11.82
	Tuning Parameter (λ)				
LASSO ($\times 10^{-1}$)	1.67	1.82	1.92	1.80	1.73
ALQR ($\times 10^{-5}$)	1.1	0.9	0.2	0.8	1.0
RIDGE ($\times 10^{-5}$)	0.6	1.1	1.0	1.1	0.6

Table D.10: Simulation Results: Scenario 3 with Dependent Predictors

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Dependency Rate: $\rho = 0.5$					
			Final Prediction Error (FPE)		
QR	0.1065	0.1799	0.4076	**0.1804**	0.1074
LASSO	**0.1064**	**0.1798**	0.4076	0.1805	**0.1073**
ALQR	0.1070	0.1805	0.4077	0.1810	0.1077
RIDGE	0.2514	0.3381	**0.4059**	0.3487	0.2597
QUANT	2.8938	5.2864	13.4026	5.2607	2.8399
			Out-of-Sample R^2		
QR	**0.9632**	**0.9660**	0.9696	**0.9657**	**0.9622**
LASSO	**0.9632**	**0.9660**	0.9696	**0.9657**	**0.9622**
ALQR	0.9630	0.9659	0.9696	0.9656	0.9621
RIDGE	0.9131	0.9360	**0.9697**	0.9337	0.9086
			Average # of Selected Predictors		
LASSO	12.00	12.00	12.00	12.00	12.00
ALQR	11.78	11.90	12.00	11.91	11.76
			Tuning Parameter (λ)		
LASSO ($\times 10^{-1}$)	1.47	1.50	1.68	1.60	1.57
ALQR ($\times 10^{-5}$)	1.2	1.0	0.2	1.0	1.2
RIDGE ($\times 10^{-5}$)	0.6	1.0	1.2	1.1	0.6

Table D.11: Simulation Results: Scenario 3 with Dependent Predictors

	Quantile (τ)				
	0.05	0.1	0.5	0.9	0.95
Dependency Rate: $\rho = 0.9$					
			Final Prediction Error (FPE)		
QR	**0.1069**	**0.1803**	0.4079	**0.1810**	**0.1083**
LASSO	**0.1069**	**0.1803**	0.4079	**0.1810**	**0.1083**
ALQR	0.1075	0.1818	0.4084	0.1823	0.1089
RIDGE	0.2638	0.3498	**0.4064**	0.3538	0.2660
QUANT	2.8993	5.1589	13.7720	5.3428	2.9630
			Out-of-Sample R^2		
QR	**0.9631**	**0.9651**	0.9704	**0.9661**	**0.9634**
LASSO	**0.9631**	**0.9651**	0.9704	**0.9661**	**0.9634**
ALQR	0.9629	0.9648	0.9703	0.9659	0.9632
RIDGE	0.9090	0.9322	**0.9705**	0.9338	0.9102
			Average # of Selected Predictors		
LASSO	11.95	11.99	12.00	11.98	11.96
ALQR	11.01	11.28	11.74	11.31	11.07
			Tuning Parameter (λ)		
LASSO ($\times 10^{-1}$)	1.51	1.25	1.02	1.20	1.49
ALQR ($\times 10^{-5}$)	2.7	3.4	2.1	3.2	2.6
RIDGE ($\times 10^{-5}$)	0.6	1.0	1.0	1.1	0.5

Table D.12: The Johansen Cointegration Test

Hypothesised number of CE(s)	Trace statistic												0.05 critical value
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	
$r = 0$	2.21	2.40	2.36	2.25	2.26	2.29	2.34	2.42	2.44	2.43	2.47	2.27	157.11
$r \leq 1$	10.25	10.66	11.18	10.84	10.52	10.41	10.66	10.91	11.00	10.39	10.04	10.36	124.25
$r \leq 2$	20.70	20.51	21.52	21.23	21.11	20.64	20.79	21.17	20.75	20.21	19.84	20.57	90.39
$r \leq 3$	41.47	41.45	41.99	41.25	40.85	40.48	41.01	41.81	41.28	40.87	40.36	40.96	70.6
$r \leq 4$	83.53	83.44	83.91	83.11	82.61	82.17	82.42	82.91	82.16	81.13	80.63	81.38	48.28
$r \leq 5$	143.67	142.99	143.71	142.78	142.42	142.09	142.33	142.95	142.65	141.95	141.40	141.37	31.52
$r \leq 6$	221.96	219.42	219.94	219.65	219.20	218.94	218.71	217.84	218.05	216.82	216.47	216.03	17.95
$r \leq 7$	355.85	353.34	353.44	352.96	352.51	351.99	352.06	351.59	356.59	356.55	357.69	352.80	8.18
Hypothesised number of CE(s)	Maximum eigenvalue statistic												0.05 critical value
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	
$r \leq 0$	2.21	2.40	2.36	2.25	2.26	2.29	2.34	2.42	2.44	2.43	2.47	2.27	51.07
$r \leq 1$	8.03	8.26	8.81	8.59	8.26	8.12	8.33	8.49	8.56	7.96	7.57	8.09	44.91
$r \leq 2$	10.45	9.84	10.34	10.39	10.59	10.23	10.12	10.26	9.75	9.82	9.80	10.21	39.43
$r \leq 3$	20.77	20.94	20.47	20.02	19.74	19.84	20.23	20.64	20.53	20.65	20.53	20.38	33.32
$r \leq 4$	42.06	41.99	41.92	41.85	41.76	41.68	41.41	41.10	40.89	40.27	40.27	40.43	27.14
$r \leq 5$	60.14	59.55	59.80	59.68	59.80	59.92	59.91	60.04	60.48	60.81	60.76	59.99	21.07
$r \leq 6$	78.29	76.43	76.23	76.87	76.78	76.85	76.38	74.89	75.40	74.87	75.07	74.66	14.9
$r \leq 7$	133.90	133.92	133.50	133.31	133.31	133.05	133.36	133.75	138.55	139.74	141.23	136.77	8.18

Notes: 1. r indicates the number of cointegrating vectors. 2. From Column (1) to Column (12), each column states the value of test statistic using data based on the rolling fixed window scheme for 12 one-step-ahead forecasts, respectively.

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