# Data-Driven Logistic Regression Ensembles With Applications in Genomics

Anthony-Alexander Christidis Department of Statistics University of British Columbia 2207 Main Mall Vancouver, BC V6T 1Z4 anthony.christidis@stat.ubc.ca Stefan Van Aelst Department of Mathematics KU Leuven Celestijnenlaan 200B 3001 Leuven, Belgium stefan.vanaelst@kuleuven.be

Ruben Zamar Department of Statistics University of British Columbia 2207 Main Mall Vancouver, BC V6T 1Z4 ruben@stat.ubc.ca

#### Abstract

Advances in data collecting technologies in genomics have significantly increased the need for tools designed to study the genetic basis of many diseases. Statistical tools used to discover patterns between the expression of certain genes and the presence of diseases should ideally perform well in terms of both prediction accuracy and identification of key biomarkers. We propose a new approach for dealing with high-dimensional binary classification problems that combines ideas from regularization and ensembling. The ensembles are comprised of a relatively small number of highly accurate and interpretable models that are learned directly from the data by minimizing a global objective function. We derive the asymptotic properties of our method and develop an efficient algorithm to compute the ensembles. We demonstrate the good performance of our method in terms of prediction accuracy and identification of key biomarkers using several medical genomics datasets involving common diseases such as cancer, multiple sclerosis and psoriasis. In several applications our method could identify key biomarkers that were absent in state-of-the-art competitor methods. We develop a variable importance ranking tool that may guide the focus of researchers on the most promising genes. Based on numerical experiments we provide guidelines for the choice of the number of models in our ensembles.

*Keywords:* Consistent ensemble; Diversity penalty; High-dimensional classification; Logistic regression models; Split modeling.

### 1 Introduction

The surge of genomic data collection through ever-evolving technologies has necessitated the development of sophisticated statistical methods to analyze high-dimensional gene expression data, unlocking its potential for breakthroughs in healthcare and scientific discovery. In many medical genomics applications, the goal is to train a classifier that can accurately predict the presence of a disease or particular subtypes of a disease based on the genetic profile of a patient. Furthermore, in the medical sciences and other fields predictions often have important consequences in decisionmaking processes, so there is a high demand for interpretable learning methods (see e.g. Murdoch et al., 2019; Rudin, 2019; Rudin et al., 2022). It should thus be clear how a classifier arrives at its predictions so that decisions can be fully explained. In addition, medical researchers aim to unravel the relation between the genetic profile and occurrence of a disease, so an appropriate classification method should also be able to identify patterns between the expression of key biomarkers and the presence of a disease.

The vast richness and availability of medical genomics data is evident from large publicly accessible databases such as the Gene Expression Omnibus (GEO) database (Edgar et al., 2002; Barrett et al., 2012). In many of these datasets, a large number of gene expression measurements are collected for a relatively small number of (cell tissue) samples. The status of the sample is typically given as well, specifying e.g. whether the sample collected was affected by a disease or not. Classification methods can then be used to discover patterns between the expression level of certain genes and the presence of the disease. The two types of methods that are generally deployed in such applications are sparse and ensemble methods, respectively.

On the one hand, sparse methods yield interpretable models that only use a subset of the genes to make a decision (see e.g. Hastie et al., 2015, for a modern treatment of sparse methods). However, because the number of predictor genes is much larger than the number of samples, there may be several models comprised of different subsets of genes that are equally accurate in predicting the presence of the disease (a phenomenon coined "the multiplicity of good models" by McCullagh and Nelder, 1989). Thus, several potentially important genes may be erroneously discarded from the decision-making process when using a single sparse model. On the other hand, ensemble methods combine multiple diverse models and generally achieve superior prediction performance if the members of the ensemble are sufficiently diverse. Diversity is often achieved using randomization (see e.g. Ho, 1998; Breiman, 2001; Song et al., 2013) or sequentially fitting the residuals of the previous fit (see e.g. Friedman, 2001; Bühlmann and Yu, 2003; Schapire and Freund, 2012; Yu et al., 2020). Ensemble methods have been particularly successful in high-dimensional prediction tasks with genomic data (see e.g. Dorani et al., 2018; Zahoor and Zafar, 2020). However, while ad hoc methods have been developed to assess variable importance in some ensemble methods, such as the variable importance measure of Breiman (2001), interpretation of the resulting prediction rules and identification of important predictor genes (i.e. key biomarkers) is less straightforward.

We propose a new approach to learn a diverse ensemble of sparse classification models that is especially well suited for high-dimensional medical genomics applications. We demonstrate the advantage of our ensemble method in terms of both prediction and identification of key biomarkers using several datasets from the GEO database. These datasets involve different types of common diseases such as cancer, multiple sclerosis and psoriasis. In particular, we show that the prediction accuracy of the sparse models in our ensembles matches the prediction accuracy of standard singlemodel sparse methods. Since the models in the ensembles are learned simultaneously and directly form the data (free of randomization or other heuristics) by optimizing a global objective function, they each provide an alternative explanation for the relationship between the genes and disease. The objective function contains a penalty that promotes diversity in the obtained models so that individual models may be driven by potentially different biological mechanisms. With respect to the identification of important predictor genes, our examples demonstrate that the proposed ensemble method can identify key biomarkers that are discarded by state-of-the-art sparse methods and ensemble variable ranking methods. At the same time, we show that our method generally includes key biomarkers selected by sparse methods or flagged as important by ensemble variable ranking methods.

Each of the models in our proposed ensemble method is a penalized logistic regression with a sparsity inducing penalty such as the Lasso (Donoho and Johnstone, 1994; Tibshirani, 1996) or the elastic net (Zou and Hastie, 2005). Rather than resorting to randomization or other indirect methods to generate different models, we jointly learn the models in the ensemble on the training data and incorporate a diversity penalty (Christidis et al., 2020) in the objective function with the aim to diversify the models. The degree to which these models are sparse and diverse is driven directly by the data. In this way, the method efficiently exploits the so-called accuracy-diversity trade-off between the models, and generates an ensemble with high predictive performance that often matches or even outperforms popular state-of-the-art ensemble methods. Moreover, by ensembling the models at the level of their linear predictors we retain interpretability for the logistic regression coefficients in the ensembled model. We use several measures of diversity (Kuncheva and Whitaker, 2003) to study the accuracy-diversity trade-off in our ensemble method. This trade-off provides insight in the effect of the number of ensembled models on the performance of the resulting ensemble. This insight makes it possible to make some recommendations for the choice of the number of ensembled models.

The remainder of the paper is organized as follows. In Section 2 we briefly review the literature on sparse and ensemble methods, and we analyze lung and thyroid cancer genomic datasets to highlight the potential benefit of our hybrid approach over standard state-of-the-art sparse and ensemble methods. In Section 3 we introduce our method to learn a data-driven diverse ensemble of sparse logistic regression models. We show that this new ensemble method is consistent under mild regularity assumptions. We illustrate how the proposed method can uncover multiple mechanisms for the relationship between predictors and the response on the thyroid cancer genomics dataset. An efficient block coordinate descent algorithm to solve the multi-convex optimization problem is provided in Section 4. Section 5 presents the results of an extensive simulation study to compare our method with a large number of state-of-the-art competitors for high-dimensional binary classification. In Section 6 we further show the excellent performance of the proposed method in a numerical study using a large collection of high-dimensional medical genomic datasets. We also show how our method can be used to rank genes in order of importance. To provide insight in how to choose the number of models in our method, we study the accuracy-diversity trade-off and computational cost of our method in Section 7. Potential directions for further research are provided in Section 8.

### 2 Empirical Motivation

In a typical medical genomics data application the training data consist of thousands of gene expression level measurements on a limited number of samples, collected from some cancer tumor tissue or from some adjacent normal tissue for example. In such an application the main goal is to train a classification model that accurately identifies the tumor status of the tissue samples using the available gene expression data. At the same time, medical researchers wish to identify important subsets of differentially expressed genes and the way they operate.

As motivating examples we consider the lung and thyroid cancer genomic datasets, GSE10245

and GSE5364 respectively, from the Gene Expression Omnibus (GEO) database. GSE10245 consists of n = 58 non-small cell lung cancer (NSCLC) tissue samples of subtypes adenocarcinoma (AC) or squamous cell carcinoma (SCC) with p = 54,675 gene expression measurements. GSE5364 consists of n = 51 cell tissue samples of either thyroid cancer tumors or adjacent normal tissues with p = 22,283 gene expression measurements. The objective is to construct a classifier that can accurately specify the cancer tumor status of the samples, AC or SCC subtype for the lung cancer dataset and cancer tumor or adjacent tissue sample for the thyroid cancer dataset, using their gene expression levels. The classifier should also identify, for each dataset, all possible relevant biomarkers that play a role in the differentiation between the two types of cell tissue samples.

The data are pre-processed in a standard way by using a three step procedure similar to Dudoit et al. (2002): (1) thresholding of gene expression levels with a floor of 20 and a ceiling of 20,000; (2) exclusion of genes for which the ratio between the maximum and minimum expression level is less than 2; (3) exclusion of genes for which the difference between the maximum and minimum expression level is less than 200; (4) base-2 logarithmic transformation. For most methods we then retain from the remaining genes the p = 500 genes which are most important according to their q-value (Storey, 2002; Storey et al., 2023). The q-value is a popular alternative to the pvalue for high-dimensional data that controls for the false discovery rate and alleviates the issue of overestimating significance that arises from using multiple pairwise t-tests.

Training a classification model for these examples is particularly challenging due to the large number of gene expression measurements relative to the available number of tissue samples. The two main approaches proposed in the literature for dealing with such high-dimensional data are sparse methods and ensemble methods, respectively. We briefly review some state-of-the-art methods from both approaches and highlight their strengths and limitations on these datasets. We also use these examples to illustrate the good performance of our hybrid method, which uses an ensemble of sparse models as explained in Section 3, both in terms of prediction accuracy and identification of target biomarkers.

### 2.1 Sparse Methods

Let  $\mathcal{D} = (\mathbf{y}, \mathbf{X})$  denote the training data, where  $\mathbf{y} \in \mathbb{R}^n$  is the vector of class labels and  $\mathbf{X} \in \mathbb{R}^{n \times p}$  is the design matrix which consists of n measurements  $\mathbf{x}_i$  on p predictors. The classes are labeled by  $y_i \in \{-1, 1\}$  for convenience and the predictor variables have been standardized, i.e.  $\sum_{i=1}^n x_{ij}/n = 0$  and  $\sum_{i=1}^n x_{ij}^2/n = 1, 1 \le j \le p$ . Logistic regression models the class-conditional probabilities through

and  $\sum_{i=1} x_{ij}^2/n = 1, 1 \le j \le p$ . Logistic regression models the class-conditional probabilities through a nonlinear function of the predictor variables,

$$p_i = \mathbb{P}(y_i = 1 | \mathbf{x}_i) = S(\beta_0 + \mathbf{x}_i^T \boldsymbol{\beta}), \quad 1 \le i \le n,$$
(1)

where  $\beta_0$  and  $\boldsymbol{\beta} \in \mathbb{R}^p$  are the intercept and vector of regression coefficients, respectively. The function  $S(t) = e^t/(1+e^t)$  is the well-known logistic function. With  $f(\mathbf{x}_i) = \beta_0 + \mathbf{x}_i^T \boldsymbol{\beta}$ , let

$$\mathcal{L}\left(f(\mathbf{x}_{i}), y_{i}\right) = \mathcal{L}(\beta_{0}, \boldsymbol{\beta} \mid y_{i}, \mathbf{x}_{i}) = \log\left(1 + e^{-y_{i}f(\mathbf{x}_{i})}\right),$$
(2)

denote the logistic regression loss function. Minimizing the corresponding empirical loss

$$\mathcal{V}_n(f) = \frac{1}{n} \sum_{i=1}^n \mathcal{L}\left(f(\mathbf{x}_i), y_i\right),\tag{3}$$

leads to overfitting of the training data and poor out-of-sample prediction accuracy when  $p \gg n$ .

By restricting model complexity, sparse methods aim to find a single (sparse) model that achieves good prediction accuracy using only a small subset of the predictors. They have proven to be highly successful approaches for high-dimensional classification problems in the genomic sciences (see e.g. Zuo et al., 2017; Rejchel and Bogdan, 2020). Sparse regularization methods typically solve an optimization problem of the form

$$\min_{\beta_0 \in \mathbb{R}, \beta \in \mathbb{R}^p} \mathcal{V}_n(f) + \lambda P(\beta) \tag{4}$$

where  $P(\beta)$  is a penalty function that induces sparsity in the coefficient vector  $\beta$ . The tuning parameter $\lambda > 0$  is usually determined in a data-driven way, typically by cross-validation (CV). Next to lasso and elastic net, some of the more popular sparse regularization methods are the adaptive lasso (Zou, 2006), the relaxed lasso (Meinshausen, 2007), the smoothly clipped absolute deviation (SCAD) estimator (Fan and Li, 2001) and the minimum concave penalized (MCP) estimator (Zhang, 2010). A vast amount of asymptotic theory has been developed for a large class of regularized estimators, see e.g. Bühlmann and van de Geer (2011) for an extensive treatment. In summary, sparse regularization methods yield a single interpretable model with good prediction accuracy and extensive asymptotic theory that describes their behavior.

#### 2.2 Ensemble Methods

Ensemble methods have proven to be very successful in high-dimensional classification tasks, often yielding higher prediction accuracy than their sparse single-model competitors. To better understand the behavior of ensemble models, Ueda and Nakano (1996) first developed a decomposition of its generalization error for the regression case. Brown et al. (2005) provided an in-depth analysis of the bias-variance-covariance trade-off in regression ensembles. In particular, if the ensemble of a collection of estimators  $\hat{f}_1, \ldots, \hat{f}_G$  is their average  $\bar{f} = \sum_{g=1}^G \hat{f}_g/G$ , then its mean squared prediction error (MSPE) can be decomposed as  $MSPE[\bar{f}] = Bias[\bar{f}]^2 + Var[\bar{f}] + \sigma^2$ , where  $\sigma^2$  is the irreducible variance of the errors. The bias and variance of the regression ensemble can be decomposed further as

$$\operatorname{Bias}\left[\overline{f}\right] = \overline{\operatorname{Bias}}_{G},\tag{5}$$

$$\operatorname{Var}\left[\bar{f}\right] = \frac{1}{G} \,\overline{\operatorname{Var}}_G + \frac{G-1}{G} \,\overline{\operatorname{Cov}}_G,\tag{6}$$

where  $\overline{\text{Bias}}_G$ ,  $\overline{\text{Var}}_G$  and  $\overline{\text{Cov}}_G$  are the average of the biases, variances, and pairwise covariances of the *G* estimators in the ensemble, respectively. From (6) it becomes clear that if the number of estimators increases, their correlations play a much more critical role than their average variability to obtain a good ensemble estimator. A similar principle was derived for classifier ensembles by Tumer and Ghosh (1996) and later refined by Fumera and Roli (2003).

The need for diversity among the models in an ensemble has been studied more broadly, with random forests (Breiman, 2001) being a very popular example. In random forests the individual decision trees are usually weakened by random selection of candidate features at each node, resulting in de-correlated trees that strengthen the ensemble accuracy. Similarly, in the random generalized linear models (RGLM) method of Song et al. (2013), random sampling of the data (bagging) (Breiman, 1996) and the random predictor subspace method (Ho, 1998) are combined to create varying samples and predictor sets, resulting in weak but de-correlated individual models. In the (extreme) gradient boosting method of Chen and Guestrin (2016), diverse trees are generated by

sequentially fitting the residuals of the previous fit rather than the original responses, resulting in a collection of trees that are poor individual predictors but can produce ensembles with high accuracy.

In summary, current ensemble methods lack interpretability as they generally generate many weak models that are only useful if a large number of them are pooled together. However, ad hoc tools have been introduced in the literature to assess predictor importance in some blackbox ensemble methods such as the Gini importance index in random forests (Breiman, 2001). Finally, due to the heuristic nature of current ensemble methodology, less theory has been developed to study their asymptotic behavior.

### 2.3 Methods and Performance Measures

We use the thyroid (GSE5364) and lung cancer (GSE10245) genomic datasets to illustrate the performance of several existing state-of-the-art sparse and ensemble methods and to compare them to our new proposal. We use all methods as implemented in the CRAN (R Core Team, 2022) R packages given below:

- 1-2. Lasso and Elastic Net (EN) for logistic regression from the glmnet package (Friedman et al., 2010).
- 3-4. Adaptive Lasso and Relaxed Lasso for logistic regression from the gcdnet (Yang and Zou, 2017) and glmnet packages, respectively.
  - 5. Minimum concave penalized (MCP) logistic regression from the ncvreg package (Breheny and Huang, 2011).
  - 6. Sure Independence Screening (SIS) with the SCAD penalty from the SIS package (Saldana and Feng, 2018).
- 7–8. **Split-Lasso** and **Split-EN** logistic regression which will be introduced in the following sections as implemented in the SplitGLM package (Christidis et al., 2021).
  - 9. Random GLM (**RGLM**) from the RGLM package (Song and Langfelder, 2013).
- 10. Random Forest (**RF**) from the randomForest package (Liaw and Wiener, 2002).
- 11. Extreme Gradient Boosting (XGB) from the xgboost package (Chen et al., 2020).

Methods 1-6 are sparse methods while 9-11 are "blackbox" ensemble methods. Methods 7-8 are the split logistic regression ensembles proposed in Section 3 using either the Lasso or the elastic net as sparsity penalty. For all ensemble methods we consider ensembles of G = 5, 10 and 25 models. For RGLM and RF we also consider as number of models the default value in their respective R package, which is G = 100 for RGLM and G = 500 for RF. Since the default number of models for these methods may be suboptimal, we also consider G = 200 for RGLM and G = 1,000 for RF. The number of models in XGB is chosen by using the default CV procedure as implemented in **xgboost**. For all other tuning parameters we use the default selection procedure in their respective implementation in the R packages listed above. Except for RF, all the methods use the p = 500genes selected according to the q-value to train the models and make predictions. RF generally thrives in handling ultra high-dimensional data and therefore we use the number of genes retained after applying the gene thresholding and filtering steps, which is much closer to the original number of genes in each dataset.

To evaluate prediction performance we report the misclassification rate (MR) of all the methods. For the ensemble methods (Split-Lasso, Split-EN, RGLM, RF and XGB) we also report the average misclassification rate of the individual models ( $\overline{\text{MR}}$ ) which measures the prediction accuracy of the individual models used in an ensemble. To estimate MR and  $\overline{\text{MR}}$ , we randomly split the lung and thyroid datasets N = 50 times into training sets of size n = 29 and n = 26, respectively, to fit the models, and corresponding tests sets of size m = 29 and m = 25, respectively, to evaluate the prediction performance of the methods.

To evaluate the performance of the methods with respect to identification of target biomarkers, we investigate the inclusion or absence of several key genes identified in the literature for lung and thyroid cancer in the resulting models.

### 2.4 Prediction Performance

In Table 1 we report the MR of the sparse and ensemble methods averaged over the N = 50 random splits of the lung and thyroid cancer genomic datasets. For the ensemble methods we also report the average individual model  $\overline{\text{MR}}$ . In each column the best performances are marked in bold. If we leave the split logistic regression methods out of the comparison, then RGLM with G = 100 and G = 200 models yields the best performance in terms of MR for both datasets. For the thyroid cancer dataset, also RF with G = 500 and 1,000 models yields the best performances. However, the average individual model  $\overline{\text{MR}}$  reveals that at the same time the individual models in RF and RGLM yield a weak performance. Moreover, while RF and RGLM with a large number models perform very well, their performance deteriorates when a small number of models is used. In RF and RGLM diversity between the individual models. This approach requires a large number of decorrelated models to achieve a low MR.

As will be seen in Section 3, split logistic regression ensembles are modeled directly from the data without relying on randomization or any heuristics. Moreover, the degree to which the models are sparse and diverse is driven directly by the data. For both the lung and thyroid cancer genomic datasets, the split logistic regression methods obtained sparse and diverse individual models whose prediction accuracy  $\overline{\text{MR}}$  either match or nearly match the performance of the corresponding single model sparse estimator. Hence, each of the individual models in Split-Lasso/Split-EN is just as reliable as the single-model sparse Lasso/EN estimators. Hence, each individual model in split logistic regression provides a potential explanation for the relationship between the genetic covariates and the cancer diagnosis of the patient. By ensembling a relatively small number of these accurate, sparse and diverse models, the performance of split logistic regression ensembles is comparable to the performance of the best blackbox ensemble methods. The overall good predictive performance of split logistic regression compared to state-of-the-art competitors is shown further in Section 6 by a benchmark study using a large collection of high-dimensional genomics datasets.

Beyond the good predictive performance of split logistic regression ensembles and the individual models that comprise them, they can be used to uncover potentially different mechanisms that explain the relationship between the gene expression levels and the tumor status of the tissue samples. Over the N = 50 random splits, the individual models in Split-EN ensembles with G = 10 were comprised of 23 genes on average for both the lung and thyroid cancer datasets. In comparison, the logistic elastic net contained on average 44 and 38 genes, respectively. Since split logistic regression learns the ensembles directly from the data, the inclusion of a gene in any of the models is the result of its contribution to the minimization of the global objective function which includes a sparsity and a diversity penalty. Thus, each model in the ensemble is a sparse, fully interpretable model that can reveal the relation between the tumor status and a set of genes.

Table 1: Average MR and MR of sparse and ensemble methods estimated using $N = 50$ random	om
splits of the lung (GSE10245) and thyroid (GSE5364) cancer genomic datasets into training se	ets
with $n = 29$ and $n = 26$ samples respectively, and test sets with the remaining samples. T	'he
standard errors are in parenthesis.	

	GSE	10245	GSE5364			
Method	MR	$\overline{\mathbf{MR}}$	MR	$\overline{\mathbf{MR}}$		
Lasso	$0.09 \ (0.05)$	_	$0.13 \ (0.07)$	_		
EN	0.08 (0.05)	_	0.11 (0.06)	_		
Adaptive	$0.15 \ (0.08)$	_	0.14(0.10)	_		
Relaxed	$0.09 \ (0.05)$	_	0.14(0.08)	_		
MCP	0.16(0.09)	_	0.18(0.11)	_		
SIS-SCAD	$0.16\ (0.08)$	_	$0.19\ (0.11)$	_		
Split-Lasso-5	0.08  (0.05)	$0.09 \ (0.04)$	$0.10 \ (0.06)$	0.14 (0.		
Split-Lasso-10	$0.07 \ (0.04)$	0.09  (0.03)	$0.10\ (0.05)$	0.15 (0.		
Split-Lasso-25	0.07  (0.05)	$0.10\ (0.03)$	0.09  (0.05)	0.16 (0.		
Split-EN-5	0.07  (0.04)	$0.08 \ (0.04)$	$0.10\ (0.05)$	0.12 (0		
Split-EN-10	$0.07 \ (0.04)$	0.08  (0.03)	$0.10\ (0.05)$	0.13 (0		
Split-EN-25	0.07  (0.04)	0.09  (0.03)	$0.09\ (0.05)$	0.14 (0.		
RGLM-5	$0.10 \ (0.05)$	0.48(0.08)	$0.15 \ (0.09)$	0.55 (0.		
RGLM-10	$0.09 \ (0.05)$	0.49(0.07)	$0.12 \ (0.07)$	0.54(0.		
RGLM-25	$0.09 \ (0.05)$	0.49(0.05)	$0.11 \ (0.07)$	0.55 (0.		
RGLM-100	$0.08\ (0.05)$	0.49(0.04)	$0.10 \ (0.07)$	0.55 (0.		
RGLM-200	0.08  (0.05)	0.48~(0.04)	$0.09 \ (0.07)$	0.55 (0.		
RF-5	$0.15 \ (0.07)$	$0.25 \ (0.06)$	$0.21 \ (0.08)$	0.30 (0.		
RF-10	$0.12 \ (0.06)$	0.25~(0.04)	$0.16 \ (0.08)$	0.30 (0.		
RF-25	$0.11 \ (0.07)$	0.25~(0.03)	$0.11 \ (0.07)$	0.30 (0.		
RF-500	$0.10 \ (0.07)$	0.25~(0.02)	$0.10 \ (0.08)$	0.31 (0.		
RF-1,000	$0.10 \ (0.08)$	$0.25\ (0.02)$	$0.09 \ (0.07)$	0.30 (0.		
XGB	$0.19 \ (0.09)$	$0.25 \ (0.07)$	0.19(0.12)	0.22 (0.		

### 2.5 Identification of Target Biomarkers

An advantage of split logistic regression ensembles over their single model counterparts is that by combining information from several individual models, they may identify more relevant genes. For example, over the N = 50 random splits of the lung cancer genomic dataset, the Split-EN ensembles based on G = 10 models contained on average 231 genes compared to only 44 genes for the logistic elastic net. Since the inclusion of a gene in the Split-EN ensemble is due to its importance in the minimization of its objective function, it may be expected that these genes are relevant to differentiate between the AC and SCC lung cancer subtype. For example, both Split-EN with G = 10 models and the logistic elastic net included the gene CGN in all N = 50 random splits of the lung cancer dataset, a biomarker that has been identified as highly expressed in AC but significantly reduced in SCC (Paschoud et al., 2007). Also the genes PTGFRN, SOX2 and TMC5 have all been identified as key biomarkers to differentiate between AC and SCC cancer subtypes (see George et al., 2015; Karachaliou et al., 2013; Xiao et al., 2017, respectively). Split-EN with G = 10 models indeed includes these genes in the model more than 80% of the times, while they were included in less than 30% of the logistic elastic net models over the same random splits of the data. We also note that the genes selected by the logistic elastic net were always a subset of the genes selected by the Split-EN ensemble model with G = 10 models. Similarly, genes that were consistently ranked as one of the top 250 most important genes by RF based on the Gini index were often also selected by Split-EN with G = 10 models. For example, the gene IRF6 which has been found to drive the progression of NSCLC (Liu et al., 2021) was 88% of the times in the top 250 for RF. This gene was also included in 74% of Split-EN models with G = 10. On the other hand, several of the genes most often selected by Split-EN such as the key biomarkers CGN, PTGFRN, SOX2 and TMC5 mentioned above, were never included in the top 250 by RF.

In split logistic regression genes may appear in more than one individual model, albeit at a cost in the global objective function (see Section 3). Hence, genes appearing in multiple individual models are expected to be more important biomarkers than others in terms of predicting the response. This information can thus be exploited to rank the genes in order of importance as explained in Section 6 based on the thyroid cancer dataset.

### 3 Split Logistic Regression Models

We now introduce split logistic regression which combines the stability and interpretability of sparse methods with the high accuracy achieved by ensemble methods. Let  $(\mathbf{y}, \mathbf{X})$  denote the training data, as before. Using the logistic loss function (13), the split logistic regression objective function to simultaneously fit G models is given by

$$\mathcal{J}\left(\beta_0^1, \boldsymbol{\beta}^1, \dots, \beta_0^G, \boldsymbol{\beta}^G\right) = \sum_{g=1}^G \left[\frac{1}{n} \sum_{i=1}^n \mathcal{L}\left(\beta_0^g, \boldsymbol{\beta}^g | y_i, \mathbf{x}_i\right) + \lambda_s P_s(\boldsymbol{\beta}^g)\right] + \frac{\lambda_d}{2} \sum_{h \neq g} P_d(\boldsymbol{\beta}^h, \boldsymbol{\beta}^g), \tag{7}$$

which needs to be minimized with respect to all regression coefficients. The sparsity penalty function  $P_s$  regularizes each of the G individual model such that they contain a subset of the available predictors that work well together to model and predict the outcome. On the other hand, the goal of the diversity penalty  $P_d$  is to restrict the models by discouraging each variable from appearing in several models. Hence, the diversity penalty encourages the different models in the ensemble to complement each other well. Note that the diversity penalty  $P_d$  needs to have two desirable properties. First, it should encourage the selection of uncorrelated models. Secondly, it should be computationally tractable so that the objective function (17) can be minimized in a stable and timely manner. Finally, for moderate values of G the ensemble tends to be sparse in the sense that the set of predictors that appear in at least one of the models will be much smaller than the complete set of candidate predictors.

For the sparsity penalty we use the elastic net (Zou and Hastie, 2005),

$$P_{s}(\beta^{g}) = \frac{1-\alpha}{2} \|\beta^{g}\|_{2}^{2} + \alpha \|\beta^{g}\|_{1}, \quad \alpha \in [0,1],$$
(8)

but other penalties such as SCAD or MCP could be used as well. For the diversity penalty which

encourages the individual models to be sufficiently different, we use the proposal of Christidis et al. (2020),

$$P_d(\boldsymbol{\beta}^h, \boldsymbol{\beta}^g) = \sum_{j=1}^p |\beta_j^g| |\beta_j^h|.$$
(9)

Hence, models can share a predictor variable only if this sufficiently improves their fits.

The tuning constants  $\lambda_s, \lambda_d \geq 0$  control the amount of shrinkage and diversity between the models, respectively. Letting  $\lambda_d \to \infty$ , enforces that the diversity penalty  $P_d\left(\boldsymbol{\beta}^h, \boldsymbol{\beta}^g\right) \to 0$  for all  $g \neq h$  so that the active variables in each of the individual models are distinct. In this case the models in the ensemble are fully diverse in the sense that they do not share any predictor variables. On the other hand, it can be seen that for  $\lambda_d = 0$ , the solution for all G individual models is the same. In this case all models are equal to the logistic elastic net solution with penalty parameter  $\lambda_s$ , which is then also the split logistic regression ensemble solution. Hence, split logistic regression is a generalization of regularized logistic regression and allows for the selection of G > 1 accurate and potentially diverse models that can be combined well in an ensemble. Note that since both  $\lambda_s$  and  $\lambda_d$  are chosen by CV, the degree of sparsity of the models and diversity between the models is driven by the data.

Minimizing the split logistic regression objective function (17) yields solutions  $\hat{f}_g(\mathbf{x}) = \hat{\beta}_0^g + \mathbf{x}^T \hat{\boldsymbol{\beta}}^g$  for  $g = 1, \dots, G$  which are well-suited for creating an ensemble. We use the ensembling function

$$\hat{f}(\mathbf{x}) = S\left(\frac{1}{G}\sum_{g=1}^{G}\hat{f}_{g}(\mathbf{x})\right) = S\left(\frac{1}{G}\left(\sum_{g=1}^{G}\hat{\beta}_{0}^{g}\right) + \mathbf{x}^{T}\left(\frac{1}{G}\sum_{g=1}^{G}\hat{\beta}^{g}\right)\right).$$
(10)

The advantage of this ensembling function is that the ensemble also becomes a logistic transformation of a linear function. Hence the resulting model is still a highly interpretable logistic regression model.

For the ensembling function (10), consistency of split logistic regression is established in Theorem 1 below. The proof of Theorem 1 is provided in the supplementary material where we also study its prediction error in the more general case of model misspecification.

**Theorem 1** Assume the data  $(y_i, \mathbf{x}_i)$ ,  $1 \leq i \leq n$ , follow a logistic regression model for some  $\boldsymbol{\beta}^* \in \mathbb{R}^p$ , with  $\|\boldsymbol{\beta}^*\|_1$  and  $\|\boldsymbol{\beta}^*\|_2$  of order smaller than  $\sqrt{n/\log(p)}$  and  $\log(p)/n \to 0$ . Let  $\hat{f}_1, \ldots, \hat{f}_G$  be the solution of (17). If  $\lambda_s$  and  $\lambda_d$  are of order  $\sqrt{\log(p)/n}$ , then the ensemble prediction  $\hat{f}$  given in (10) is consistent.

As seen in Table 1, split logistic regression ensembles achieve an excellent performance on the lung and thyroid cancer genomic datasets. Ensembles with a sufficient number of groups yield an accuracy close to optimal among all methods considered. At the same time the accuracy of the individual models in the ensembles is comparable to the accuracy of the best single model sparse methods. For example, consider the Split-EN solution for the thyroid cancer genomic dataset based on G = 10 models which is the value selected by CV (as discussed in Section 7). The number of genes in these G = 10 individual models ranged between 23 and 48, which confirms their sparsity. For comparison, the logistic elastic net produced a model with 37 genes. All of these 37 genes also appear in the split logistic regression ensemble model which contains 206 genes. This suggests that the Split-EN solution may reveal more relevant genes because its individual models may represent different mechanisms that can explain the presence of thyroid cancerous cell tissue.

### 4 Algorithm

The difficulty of obtaining a global minimizer of the objective function (17) is primarily due to the non-convexity of the diversity penalty  $P_d$ . Note that a global minimum of the nonnegative objective function (17) exists for any  $\lambda_s > 0$  because  $\mathcal{J}\left(\beta_0^1, \beta^1, \ldots, \beta_0^G, \beta^G\right) \to \infty$  if  $\|\beta^g\| \to \infty$  for any  $1 \le g \le G$ .

To construct an efficient algorithm, we observe that the objective function is multi-convex. That is, the parameters of the objective function can be partitioned in such a way that the problem is convex on each set when the others are kept fixed. A modern rigorous treatment of multi-convex programming can be found in Shen et al. (2017). In our case, the optimization problem (17) for the parameters  $(\beta_0^g, \beta^g)$  of a particular model reduces to a penalized logistic regression problem with a weighted elastic net penalty. Indeed, ignoring constant terms, the objective function for  $(\beta_0^g, \beta^g)$  reduces to

$$\mathcal{J}(\beta_0^g, \beta^g) = \frac{1}{n} \sum_{i=1}^n \mathcal{L}(\beta_0^g, \beta^g | y_i, \mathbf{x}_i) + \lambda_s \frac{(1-\alpha)}{2} \|\beta^g\|_2^2 + \sum_{j=1}^p |\beta_j^g| u_{j,g},$$

where the weights  $u_{j,g}$  in the  $L_1$  penalty term are given by  $u_{j,g} = \alpha \lambda_s + \lambda_d/2 \sum_{h \neq g} |\beta_j^h|$ . For each model the problem thus reduces to a weighted elastic net optimization where the weights in the lasso penalty depend on the value of the coefficients in the other models. We exploit the multi-convex structure of the objective function to develop a block coordinate descent algorithm (Xu and Yin, 2013).

Recent work in non-convex optimization using block coordinate descent algorithms for applications in statistics and machine learning has been very promising, see Yang et al. (2019) for examples. The key idea is to sequentially update the current estimate for each model using a quadratic approximation  $\mathcal{L}_Q$  for the logistic loss function in the objective function. To update the coefficients  $(\beta_0^g, \beta^g)$  of a particular model g we thus need to solve

$$\min_{\boldsymbol{\beta}_{0}^{g} \in \mathbb{R}, \boldsymbol{\beta}^{g} \in \mathbb{R}^{p}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}_{Q}\left(\boldsymbol{\beta}_{0}^{g}, \boldsymbol{\beta}^{g} \mid y_{i}, \mathbf{x}_{i}\right) + \lambda_{s} P_{s}(\boldsymbol{\beta}^{g}) + \frac{\lambda_{d}}{2} \sum_{\substack{h=1\\h \neq g}}^{G} P_{d}\left(\boldsymbol{\beta}^{h}, \boldsymbol{\beta}^{g}\right) \right\}.$$
(11)

Using the quadratic approximation  $\mathcal{L}_Q$  for the logistic loss function (13) derived in the supplementary material, the update for each model is given in the proposition below.

**Proposition 1** Let  $(\tilde{\beta}_0^1, \tilde{\beta}^1), \ldots, (\tilde{\beta}_0^G, \tilde{\beta}^G)$  denote the current estimates. The coordinate descent updates for  $\tilde{\beta}_0^g$  and  $\tilde{\beta}^g = (\tilde{\beta}_1^g, \ldots, \tilde{\beta}_p^g)^T$  are given by

$$\hat{\beta}_{0}^{g} = \tilde{\beta}_{0}^{g} + \frac{\langle \mathbf{z} - \tilde{\mathbf{p}}^{g}, \mathbf{1}_{n} \rangle}{\langle \tilde{\mathbf{w}}^{g}, \mathbf{1}_{n} \rangle},$$

$$\hat{\beta}_{j}^{g} = \frac{Soft\left(\frac{1}{n}\left(\tilde{r}_{j}^{g} + \tilde{\beta}_{j}^{g}\langle \mathbf{x}_{j}^{2}, \tilde{\mathbf{w}}^{g} \rangle\right), \alpha\lambda_{s} + \frac{\lambda_{d}}{2}\sum_{h \neq g} |\tilde{\beta}_{j}^{h}|\right)}{\frac{1}{n}\langle \mathbf{x}_{j}^{2}, \tilde{\mathbf{w}}^{g} \rangle + (1 - \alpha)\lambda_{s}} \quad j = 1, \dots, p,$$

where  $Soft(\mu, \gamma) = sign(\mu) \times max(|\mu| - \gamma)$ ,  $\mathbf{1}_n = (1, \ldots, 1)^T \in \mathbb{R}^n$  and  $\tilde{r}_j^g = \langle \mathbf{x}_j, \mathbf{z} \rangle - \langle \mathbf{x}_j, \mathbf{\tilde{p}}^g \rangle$ . The elements of the n-dimensional vectors  $\mathbf{z}$ ,  $\mathbf{\tilde{p}}^g$  and  $\mathbf{\tilde{w}}^g$  are given by  $z_i = (y_i+1)/2$ ,  $\tilde{p}_i^g = S(\tilde{\beta}_0^g + \mathbf{x}_i^T \boldsymbol{\tilde{\beta}}^g)$  and  $\tilde{w}_i^g = \tilde{p}_i^g(1 - \tilde{p}_i^g)$ ,  $1 \le i \le n$ , respectively.

The algorithm cycles through the components of  $(\beta_0^1, \beta^1)$  by applying a single coordinate descent update to each parameter, then through those of  $(\beta_0^2, \beta^2)$ , and so on until we reach  $(\beta_0^G, \beta^G)$ . Then, we check for convergence. Convergence is declared when successive estimates of the coefficients in the ensemble model show little difference, i.e.  $\max_{0 \le j \le p} |\tilde{\beta}_j - \hat{\beta}_j|^2 < \delta$ , for some small tolerance level  $\delta > 0$ , with  $\tilde{\beta}_j = \sum_{g=1}^G \tilde{\beta}_j^g / G$  and  $\hat{\beta}_j = \sum_{g=1}^G \hat{\beta}_j^g / G$  the respective estimates for the ensemble model. The algorithm converges to a coordinatewise minimizer of (17) by Theorem 4.1 of Tseng (2001).

More details of the algorithm are given in the supplementary material.

To select the tuning parameters we alternate between a grid search for the sparsity penalty and a grid search for the diversity penalty, such that the cross-validated loss of the ensemble classifier is minimized. The details are available in the supplementary material. By default, 10-fold CV is used. Note that the value  $\lambda_d = 0$  is included in the grid search for the diversity penalty, such that the (single model) elastic net is a possible solution of split logistic regression. The warm-start and active-set cycling strategies proposed by Friedman et al. (2010) are well suited for our algorithm, and have been incorporated to speed up the algorithm. Our choice for the ensembling function (10) also allows the construction of coefficient solution paths for the ensembled model which is illustrated in the supplementary material.

#### Simulation Study $\mathbf{5}$

We investigate the performance of split logistic regression in an extensive simulation study which examines three scenarios. In Scenario 1 all predictors are equally correlated. In Scenario 2 the correlation between active and inactive predictors is lower than the other correlations. In Scenario 3, the active predictors follow a block-correlation structure.

#### 5.1Simulation Settings

For all three scenarios, the coefficients of the active variables are randomly generated as  $(-1)^{z}u$ where z is Bernoulli distributed with parameter 0.3 and u is uniformly distributed on the interval (0, 1/2). The predictor variables follow a multivariate normal distribution with mean vector zero, variances one and correlations as described below for the different scenarios. Let  $\zeta \in (0,1)$  denote the proportion of active variables, hence the number of active variables is given by  $\zeta p$ . We consider the case p = 1,000 and sparsity levels  $\zeta \in \{0.1, 0.2, 0.4\}$ . For all simulation settings we consider the sample sizes n = 50 and n = 100 for the training data with event probability  $\mathbb{P}(Y = 1) \in$  $\{0.2, 0.3, 0.4\}.$ 

Scenario 1: Data are generated according to the logistic model

$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \mathbf{x}_{A,i}^T \boldsymbol{\beta}_A, \quad 1 \le i \le n,$$

where  $\mathbf{x}_{A,i}^T$  are the active predictors and  $\boldsymbol{\beta}_A$  the corresponding regression coefficients. In this scenario all predictors are equally correlated with correlation equal to  $\rho = 0.2$ ,  $\rho = 0.5$ , or  $\rho = 0.8$ , respectively.

**Scenario 2**: Data are generated from the logistic model of Scenario 1 with correlation  $\rho_1$  between active and inactive predictors and  $\rho_2$  for all other correlations, where  $\rho_1$  and  $\rho_2$  are taken over all combinations  $\rho_1 \in \{0, 0.2, 0.5\}$  and  $\rho_2 \in \{0.2, 0.5, 0.8\}$  such that  $\rho_1 < \rho_2$ .

Scenario 3: Data are generated from a logistic model with active predictors in B disjoint blocks

$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \sum_{b=1}^B \mathbf{x}_{b,i}^T \boldsymbol{\beta}_b, \quad 1 \le i \le n,$$

where  $\mathbf{x}_{b,i}$  are the predictor variables for block b and  $\beta_b$  the corresponding regression coefficients. Each block contains 25 predictors, so the number of blocks in each setting equals  $B = \zeta p/25$ . The correlation between predictors in different blocks is given by  $\rho_1$  while  $\rho_2$  is the correlation between predictors in the same block. The correlations  $\rho_1$  and  $\rho_2$  are taken over all combinations of  $\rho_1 \in \{0.2, 0.5\}$  and  $\rho_2 \in \{0.5, 0.8\}$  such that  $\rho_1 < \rho_2$ .

### 5.2 Methods and Performance Measures

We compare the performance of split logistic regression to the nine methods listed in Section 2. For split logistic regression, we use G = 10 models, which may be a sub-optimal choice. For RGLM and RF we use the default number of models in their R package implementations, G = 100 and G = 500 models respectively.

To compare the methods, we measure for each method its misclassification rate (MR), sensitivity (SE), specificity (SP), test-sample loss (TL) using (13), recall (RC) and precision (PR) on an independent test set of size m = 2,000. Note that recall (RC) and precision (PR) are defined as

$$\mathrm{RC} = \frac{\sum_{j=1}^{p} \mathbb{I}(\beta_j \neq 0, \hat{\beta}_j \neq 0)}{\sum_{j=1}^{p} \mathbb{I}(\beta_j \neq 0)}, \quad \mathrm{PR} = \frac{\sum_{j=1}^{p} \mathbb{I}(\beta_j \neq 0, \hat{\beta}_j \neq 0)}{\sum_{j=1}^{p} \mathbb{I}(\hat{\beta}_j \neq 0)},$$

where  $\beta$  and  $\hat{\beta}$  are the true and estimated regression coefficients, respectively. Since RGLM and RF with their default number of models tend to use all the predictors, we do not compute their RC and PR. For each configuration, we randomly generate N = 50 training and test sets and measure average performance of each method over the test sets.

#### 5.3 Results

In Table 2 the simulation results are summarized by reporting for each performance metric the average rank of the competitors over all simulation settings. Lower ranks indicate better performance. Detailed simulation results are available in the supplementary material.

From Table 2 it can be seen that the split logistic regression methods overall performed best in terms of MR, with the black box ensemble methods RF and RGLM being the closest competitors. RF and the adaptive Lasso had the two best ranks in terms of SP followed by the split regression methods, but the split logistic regression methods outperformed the other methods in terms of SE (particularly for low event probabilities, i.e.  $\mathbb{P}(Y = 1) = 0.2$ ). The split logistic regression also performed favorably compared to the competitors when the proportion of active variables was high ( $\zeta = 0.2$  and  $\zeta = 0.4$ ). The scenario and correlation level  $\rho$  between the predictors did not have a strong effect on the performance of Split-Lasso and Split-EN relative to the competitors. The split logistic regression methods (unanimously) yielded the best RC. This comes at a price in PR, which is approximately on par with the single-model base estimators Lasso and EN.

To illustrate the competitive advantage of the split logistic regression methods, Figure 1 displays the average SE and SP of the sparse and ensemble methods in Scenario 3 with  $\rho_1 = 0.2$ ,  $\rho_2 = 0.5$ , p = 1,000, n = 50,  $\mathbb{P}(Y = 1) = 0.4$  and  $\zeta = 0.4$ . Split-Lasso and Split-EN clearly outperformed their competitors in terms of SE while maintaining a high SP. Figure 2 shows the corresponding RC

Method	$\mathbf{MR}$	SE	SP	$\mathbf{TL}$	RC	$\mathbf{PR}$
Lasso	5.86	6.04	7 14	5 26	4 70	6 41
EN	4.60	4.20	5.77	3.88	3.00	5.88
Adaptive	8.25	9.69	2.86	7.43	4.50	6.31
Relaxed	7.04	5.42	8.60	8.49	6.11	4.81
MCP	9.29	9.15	9.28	8.60	8.44	4.09
SIS-SCAD	9.94	10.06	8.41	9.56	8.51	2.85
Split-Lasso-10	2.34	2.15	4.30	1.81	2.00	6.08
Split-EN-10	1.95	1.88	3.88	1.45	1.00	7.01
RGLM-100	2.59	3.14	3.33	3.42	_	_
RF-500	3.82	6.10	1.54	5.41	_	_
XGB	10.31	8.17	10.90	10.69	6.74	1.57

Table 2: Average performance metrics ranks of the methods over all simulation configurations. The two best results for each criterion are highlighted in **bold**.

and PR of the methods, but without SIS-SCAD due to its poor performance. While the median PR of the sparse and split logistic regression methods are relatively similar, they are significantly better than the RC of the sparse logistic regression methods. The TL ranks in Table 2 show that split logistic regression also provided the best fits, whereas RF was outranked by the single-model sparse methods Lasso and EN for this criterion.

## 6 Medical Genomics Data Applications

To demonstrate the competitive performance of split logistic regression in practice, we apply split logistic regression and its competitors in Section 2 to ten medical genomics datasets available from the Gene Expression Omnibus (GEO) database. Some characteristics of the datasets are given in Table 3. The first eight datasets involve the classification of different types of cancerous cell tissues, whereas the last two datasets involve the separation of psoriasis and multiple sclerosis cell tissues from adjacent normal cell tissue. All datasets are first pre-processed as described in Section 2 using the choices p = 100, 250, 500, and 1,000, respectively, for the number of retained genes according to their q-value.

### 6.1 Classification Performances

Each dataset is randomly split N = 50 times into a training set and test set. For the proportion of training data we considered both 0.35 and 0.50, where proportion 0.35 is only used if the resulting training set contains at least 20 observations. Each of the methods is applied to the training data and evaluated on the test set using both MR and TL. For the split logistic regression methods we use CV to determine the optimal number of models G, as discussed further in the next section. We also use CV to choose the optimal number of models for RGLM and RF. The candidate number



Figure 1: SE and SP of the sparse and ensemble classification methods over N = 50 random training sets under Scenario 3 with  $\rho_1 = 0.2$ ,  $\rho_2 = 0.5$ , p = 1,000, n = 50,  $\mathbb{P}(Y = 1) = 0.4$  and  $\zeta = 0.4$ .



Figure 2: RC and PR of the sparse and split classification methods over N = 50 random training sets under Scenario 3 with  $\rho_1 = 0.2$ ,  $\rho_2 = 0.5$ , p = 1,000, n = 50,  $\mathbb{P}(Y = 1) = 0.4$  and  $\zeta = 0.4$ .

GEO ID	n	Genes	Description
GSE20347	34	$22,\!277$	Esophageal cancerous cell tissue.
GSE23400	106	$22,\!283$	Esophageal cancerous cell tissue.
GSE23400	102	$22,\!477$	Esophageal cancerous cell tissue.
GSE5364	29	22,283	Esophageal cancerous cell tissue.
GSE25869	75	$27,\!578$	Gastric cancerous cell tissue.
GSE10245	58	$54,\!675$	Lung cancerous cell tissue.
GSE5364	30	22,283	Lung cancerous cell tissue.
GSE5364	51	22,283	Thyroid cancerous cell tissue.
GSE21942	27	$54,\!675$	Multiple sclerosis cell tissue.
GSE14905	54	$54,\!675$	Psoriasis cell tissue

Table 3: GEO identification (ID) codes, sample sizes (n), number of genes and datasets descriptions.

of models for RGLM are G = 100 (R package default) and 200, while the candidate number of models for RF are G = 500 (R package default) and 1,000. Based on our numerical experiments no prediction improvements are observed beyond these numbers of models for RGLM and RF. For  $p \in \{100, 250, 500, 1,000\}$ , we report in Table 4 the average ranks of the methods across each of the ten gene expression datasets and both training data proportions. The detailed results of the simulation study are available in the accompanying supplementary material.

From Table 4 it can be seen that Split-EN and Split-Lasso show the best performance as they achieved the best two average ranks for both MR and TL across all number of genes retained. These methods are thus very accurate classifiers and also yield good predicted probabilities as indicated by their small test-sample losses. The logistic elastic net was the closest competitor for all values of p in terms of MR and TL, while RF and RGLM were the third and fourth best methods in terms of overall rank for both MR and TL. However, RF and RGLM were not competitive with split logistic regression or the logistic elastic net on half of the medical genomics datasets in Table 3. Moreover, RGLM is prone to include irrelevant genes due to the inherently random nature of the method and lack of sparsity.

For each dataset we computed the average ranks of the methods based on the results for each training proportion and  $p \in \{100, 250, 500, 1, 000\}$ . Table 5 summarizes the results by showing for each method the number of times it belonged to the top one, top three and lowest three average ranks among the ten datasets. It can be seen that the split logistic regression methods are the most stable methods since they are the only methods that appear in the top three for the majority of the datasets and never belong to the worst three methods for both performance measures. The closest competitors are the black box ensemble methods RF and RGLM, but they are less often in the top three. Moreover, for one of the datasets RGLM is also one of the three worst methods in terms of MR.

	100						0 Bank			
	p =	100	p =	p = 250		500	p = 1	1,000	Ra	ank
Method	MR	TL	MR	TL	MR	$\mathbf{TL}$	MR	$\mathbf{TL}$	MR	TL
Lasso	5.07	5.43	4.68	4.93	6.00	5.29	5.75	5.14	5.38	5.20
EN	3.79	3.36	3.68	3.14	3.86	3.36	3.86	3.71	3.79	3.39
Adaptive	6.86	6.00	7.00	6.50	6.93	6.93	6.89	7.14	6.92	6.64
Relaxed	6.36	10.43	6.79	9.79	6.82	10.29	7.07	10.14	6.76	10.16
MCP	9.36	8.29	8.61	8.14	9.04	7.93	9.04	7.86	9.01	8.05
SIS-SCAD	9.39	8.29	8.68	8.29	9.21	8.50	9.11	8.36	9.10	8.36
Split-Lasso-CV	3.46	3.29	3.14	3.07	2.61	2.14	2.71	2.29	2.98	2.70
Split-EN-CV	2.46	1.71	2.93	2.14	1.93	1.64	2.18	1.36	2.38	1.71
RGLM-CV	4.46	5.79	4.89	6.14	5.00	6.14	4.82	6.07	4.79	6.04
RF-CV	4.57	3.86	5.18	4.07	4.14	3.86	4.21	3.93	4.53	3.93
XGB	10.21	9.57	10.43	9.79	10.46	9.93	10.36	10.00	10.37	9.82

Table 4: Average ranks for MR and TL over the ten gene expression datasets in Table 3 for different number of genes retained after pre-processing. The two best results for each column are highlighted in bold.

Table 5: Number of top and lowest ranks for MR and TL over the ten gene expression datasets in Table 3.

	Top	<b>b</b> 1	Top	o 3	Low 3		
Method	MR	$\mathbf{TL}$	MR	$\mathbf{TL}$	MR	$\mathbf{TL}$	
Lasso	0	0	0	0	0	1	
EN	1	2	5	8	0	0	
Adaptive	1	0	1	0	1	1	
Relaxed	0	0	0	0	1	8	
MCP	0	0	0	0	9	3	
SIS-SCAD	0	0	0	0	8	6	
Split-Lasso	1	0	6	8	0	0	
Split-EN	3	7	9	10	0	0	
RGLM	3	0	5	2	1	0	
$\operatorname{RF}$	1	1	3	2	0	0	
XGB	0	0	0	0	9	9	

### 6.2 Ranking Genes

Beyond the good prediction accuracy of split logistic regression ensembles, the information of the individual models can be exploited to identify gene sets in order of importance. For a split logistic regression solution, let  $\mathcal{M}_g = \{j : \hat{\beta}_j^g \neq 0\}$  denote the set of the genes selected for model g  $(1 \leq g \leq G)$ . Now, by considering the sets

$$\mathcal{A}_k = \left\{ j : \sum_{g=1}^G \mathbb{I}\left(j \in \mathcal{M}_g\right) \ge k \right\}, \quad 1 \le k \le G,$$

where  $\mathcal{A}_G \subseteq \mathcal{A}_{G-1} \subseteq \cdots \subseteq \mathcal{A}_1$ , we can study the distribution of the genes across the individual models in the split logistic regression solution. These sets identify genes related to cancerous cell tissue in order of importance, because genes can only appear in several models if these genes allow to largely reduce the loss function to compensate for the diversity penalty in (17).

As an example, consider again the thyroid cancer dataset (GSE5364) from Section 2. With cross-validated values for  $\lambda_s$ ,  $\lambda_d$  and G on the full dataset, the Split-EN solution for these data consists of G = 10 sparse and diverse models. For this example, we find that  $|\mathcal{A}_7| = 0, |\mathcal{A}_6| =$  $1, |\mathcal{A}_5| = 5, |\mathcal{A}_4| = 9, |\mathcal{A}_3| = 39, |\mathcal{A}_2| = 88$  and  $|\mathcal{A}_1| = 206$ . Unsurprisingly, genes shared by multiple models consistently had the same sign, re-enforcing the understanding of their relationship with the thyroid cancer. To demonstrate that split logistic regression can identify important genes that may be missed by other methods, let us consider  $\mathcal{A}_5$ . This set contains five genes that appear in at least half of the 10 individual models in the Split-EN solution and thus yield an important contribution to the ensemble. Interestingly, four of these genes do not appear in the logistic elastic net model and thus would be considered irrelevant for the separation of the classes according to this method. One of these genes is SLC25A3 which is the only gene that appears in six models and encodes a protein that contains two isoforms, one of which contains an exon that regulates thyroid tissue function (Bhoj et al., 2015). The three other genes are EEF2, SNHG4 and TCEB2. The genes EEF2 and SNHG4 have been identified as target biomarkers in thyroid cancer (see Cai et al., 2022; Lan, 2022, respectively). Interestingly, TCEB2 has been identified as a target biomarker in kidney cancer (Long et al., 2017), however a deep bidirectional connection has been made between these two types of cancer due to genetic irregularities that are common to both types (Bellini et al., 2022). Finally, none of the genes in  $|\mathcal{A}_5|$  were in the top 250 genes ranked by RF on the full thyroid cancer data. On the other hand, most of the genes ranked in the top 250 by RF (particularly the ones ranked near the very top) also appear in the Split-EN ensemble.

### 7 The Number of Models

Constructing accurate and diverse models for an ensemble are opposite objectives (Krogh and Vedelsby, 1995). We perform an empirical study to explore this accuracy-diversity trade-off for split logistic regression, which will drive the choice for the number of models in real medical genomics data applications.

The analysis of genomics data via high-throughput technologies has generated the need for classification algorithms that can handle high-dimensional data containing correlated predictors (genes) within different pathways or networks, see Yousefi et al. (2011) and Zhang and Coombes (2012) for example. In light of this, to investigate the accuracy-diversity trade-off of split logistic regression, we use the high-dimensional block correlation setting of Scenario 3 in the previous section with configuration parameters  $(n, p) = (50, 1,000), (\rho_1, \rho_2) = (0.2, 0.5), \zeta \in \{0.1, 0.2, 0.4\}$ 

and  $\mathbb{P}(Y = 1) = 0.2$ .

To quantify diversity for ensemble classifiers, we adopt the entropy diversity measure of Kuncheva and Whitaker (2003). Given an ensemble comprised of G individual classifiers, the entropy measure (EM) for a given  $\mathbf{x}$  is defined as

$$\mathrm{EM}(\mathbf{x}) = \frac{1}{G - \lceil G/2 \rceil} \min\left(\ell(\mathbf{x}), G - \ell(\mathbf{x})\right), \tag{12}$$

where  $\ell(\mathbf{x})$  denotes the numbers of individual classifiers in the ensemble that correctly classify  $\mathbf{x}$ . The entropy measure ranges between 0 and 1, with  $\text{EM}(\mathbf{x}) = 0$  and  $\text{EM}(\mathbf{x}) = 1$  corresponding to no diversity and to the highest possible diversity between the individual classifiers, respectively. The overall entropy  $\mathbb{E}[\text{EM}(\mathbf{x})]$  can be estimated by averaging  $\text{EM}(\mathbf{x})$  over a test set.

### 7.1 Results

Table 6 shows the evolution of the ensemble misclassification rate (MR), the average misclassification rate of the individual models ( $\overline{\text{MR}}$ ) and the entropy diversity measure (EM) averaged over the test sets, as a function of the number of models for split logistic regression. It also contains the overlap (OVP) between the individual models in the ensemble, defined as

$$OVP = \frac{\sum_{j=1}^{p} o_j \mathbb{I}\{ o_j \neq 0 \}}{\sum_{j=1}^{p} \mathbb{I}\{ o_j \neq 0 \}}, \quad o_j = \frac{1}{G} \sum_{g=1}^{G} \mathbb{I}\{\hat{\beta}_j^g \neq 0 \}.$$

It can be seen that in all three settings  $\overline{MR}$  and EM increase with the number of models, while the ensemble MR decreases. Hence, as the number of models increases, the accuracy of the individual models has less impact on the ensemble MR compared to their level of diversity. Split logistic regression manages to achieve a proper balance for this trade-off, resulting in a low MR for the ensemble.

	$\zeta=0.1$				$oldsymbol{\zeta}=0.2$				$oldsymbol{\zeta}=0.4$			
G	MR	$\overline{\mathbf{MR}}$	$\mathbf{E}\mathbf{M}$	OVP	MR	$\overline{\mathbf{MR}}$	$\mathbf{E}\mathbf{M}$	OVP	$\mathbf{MR}$	$\overline{\mathbf{MR}}$	$\mathbf{E}\mathbf{M}$	OVP
1	0.14	_	_	_	0.19	_	_		0.12	_	_	
2	0.14 0.13	0.14	-0.05	-0.76	0.12 0.12	0.12	0.06	0.70	0.12	0.12	0.04	-0.77
5	0.13	0.14	0.10	0.54	0.11	0.12	0.13	0.36	0.10	0.12	0.11	0.37
7	0.13	0.14	0.11	0.48	0.11	0.12	0.13	0.34	0.10	0.12	0.13	0.24
10	0.12	0.15	0.14	0.31	0.11	0.13	0.14	0.21	0.10	0.12	0.13	0.19
15	0.12	0.16	0.20	0.19	0.10	0.14	0.19	0.09	0.10	0.13	0.17	0.09
20	0.12	0.17	0.21	0.11	0.10	0.14	0.20	0.05	0.10	0.13	0.18	0.05
25	0.12	0.17	0.24	0.09	0.10	0.14	0.22	0.05	0.09	0.14	0.20	0.05

Table 6: MR,  $\overline{MR}$ , EM and OVP as a function of the number of models under Scenario 3 averaged over the test sets.

When G is small, the average  $\overline{\text{MR}}$  of the individual models in Table 6 is close to the MR of the logistic elastic net (the case G = 1 in Table 6). The choice of diversity tuning parameter  $\lambda_d$  is driven by the data based on a CV criterion. For a small number of models, a smaller value of  $\lambda_d$ is selected such that the OVP of the models is large and they share a lot of important predictors. This results in accurate individual models but a relatively low diversity as seen from the EM values. As the number of models increases, the OVP becomes smaller, resulting in individual models that have a higher average EM. Indeed, for a large number of models it becomes beneficial to increase diversity between the models to decrease the misclassification rate of the ensemble. In this case the diversity penalty in split logistic regression reduces the overlap between individual models, leading to high diversity which results in high classification accuracy. In summary, split logistic regression thus successfully achieves the proper balance between individual model accuracy and diversity regardless of the number of models. Alternative diversity measures are considered in the supplementary material and lead to the same conclusions.

### 7.2 Computational Cost

Table 6 indicates that a larger number of models results in an ensemble with lower misclassification rate. However, MR stabilizes quickly, so there is a diminishing returns type of behavior in terms of prediction accuracy versus computational cost. Indeed, we also ran split logistic regression using G = 50 models, but in all cases there is hardly any improvement in MR compared to the ensemble with G = 25 models shown in Table 6. In fact, with G = 25 models split logistic regression already achieves nearly full diversity (OV  $\approx 1/G$ ), so little gain is expected by increasing the number of models further while computation time does grow.

Table 7 shows the average computation time (in CPU seconds) across all sparsity levels of scenario 3 as a function of the number of models. As G increases there is a price to pay in average individual  $\overline{\text{MR}}$  which may not be compensated by a lower ensemble MR. The computation time seems to depend linearly on the number of models and is approximately given by  $4.88 + 1.27 \times G$  for  $G \geq 2$ . In real data applications such as the gene expression applications in the previous section it is generally a good strategy to use a data-driven choice of the number of models in the ensemble by increasing G until the CV performance has stabilized.

Table 7: Computation time of R function call for split logistic regression in CPU seconds for varying number of models. CPU seconds are on a 2.7 GHz Intel Xeon processor in a machine running Linux 7.8 with 125 GB of RAM.

G	2	5	7	10	15	20	25
Time	6.48	10.49	14.38	19.29	23.39	31.04	35.59

### 8 Discussion and Future Directions

We presented a new approach to learn a diverse ensemble of sparse logistic regression models that is well suited for high-dimensional medical genomics data. The individual models for the ensemble are learned simultaneously by optimizing an objective function which balances between individual model strength and diversity between the models. The sparsity penalty in the objective function controls the stability of the individual models while the diversity penalty favorably exploits the accuracy-diversity trade-off to achieve excellent performance for the resulting ensemble. In contrast to other popular ensemble methods, split logistic regression models remain logistic regression models and thus are highly interpretable. Moreover, the individual models in the ensemble may be of interest in their own right because they each provide a relationship between the predictor genes and disease status that can provide insight to very complex biological mechanisms.

On several medical genomics datasets from the GEO database split logistic regression achieved state-of-the-art prediction accuracy. In several examples split logistic regression also identified unique key biomarkers that were not picked up by other methods, while at the same time selecting key biomarkers that were consistently picked up by other methods. A variable ranking method for split logistic regression was also developed to help guide researchers in which genes to investigate in the hope of developing effective drug therapies.

Due to the diversity penalty, split logistic regression makes use of different groups of variables in the individual models to build an ensemble. Allowing interactions among predictors can be beneficial to further improve the prediction performance of classifiers. Since split logistic regression can have much higher recall than single-model methods such as lasso and the elastic net, our methodology can also be useful to detect interaction effects that would be missed by such single-model methods. This is important for example for genomics data applications where it is known that gene interaction effects are common. The diversity penalty can also be combined more generally with different sparsity penalties such as the group lasso (Meier et al., 2008) for categorical variables or the fused lasso (Tibshirani et al., 2005) for data exhibiting spatial or temporal structures.

Block coordinate descent is an effective approach to solve the multi-convex optimization problem underlying split logistic regression. Multi-convex programming is an emerging field in optimization with many applications in statistics and machine learning, see e.g. Shen et al. (2017) and Pardalos et al. (2017). In future research we will investigate whether alternative approaches can further decrease the computational cost of the method.

In split logistic regression the models are ensembled at the level of the linear predictors. This guarantees high interpretability of the ensemble model, but is not necessarily optimal from a prediction point of view. In future research it will be examined whether alternative ensembling functions can improve on the prediction accuracy of the ensemble.

Ensemble methods are very popular to analyze small sample data with a large number of predictor variables, and our method provides a framework to build an optimal classification ensemble model. Similarly to logistic regression, the general split modeling framework could be applied to multi-class classification problems to obtain a powerful ensemble classifier. The split modeling framework could also be extended to generalized linear models in general.

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### Supplementary Material

The supplementary material contains the consistency proof, details of our algorithm, the analysis with the alternative diversity measures and the full results of our simulation and medical genomics data experiments. The R package SplitGLM along with its reference manual is publicly available on CRAN at https://CRAN.R-project.org/package=SplitGLM. The data and scripts to replicate the numerical experiments are available at https://doi.org/10.5281/zenodo.10250506.

### Appendix A: Ensemble Asymptotics

In this section, we prove a general result for the asymptotic behavior of the prediction error of the ensemble split regression method and show that it implies consistency of the prediction under the assumptions stated in Theorem 1 of the article.

#### 8.1 Preliminaries

Consider data  $\{y_i, \mathbf{x}_i\}_{i=1}^n$  where  $\mathbf{x}_i \in \mathbb{R}^p$  and  $y_i \in \{-1, 1\}$  for i = 1, 2, ..., n. Without loss of generality, we assume each column of the design matrix  $\mathbf{X} \in \mathbb{R}^{n \times p}$  has been scaled by its maximum value such that  $\max_{1 \le j \le p} \|\mathbf{x}_{\cdot j}\|_{\infty} \le 1$  where  $\mathbf{x}_{\cdot j}$  is the *j*-th column of  $\mathbf{X}$ . Let  $\mathcal{H}$  be a (rich) parameter space that includes the of space linear functions, and for each  $f \in \mathcal{H}$  we take the (convex) logistic loss function  $\mathcal{L} : \mathcal{H} \times \mathbb{R}^p \times \{-1, 1\} \mapsto \mathbb{R}$  as defined in the main article,

$$\mathcal{L}(f(\mathbf{x}_i), y_i) = \log(1 + e^{-y_i f(\mathbf{x}_i)}).$$
(13)

We denote the *empirical risk* by

$$\mathcal{V}_n(f) = \frac{1}{n} \sum_{i=1}^n \mathcal{L}(f(\mathbf{x}_i), y_i),$$

such that  $\mathcal{V}_n$  is the *empirical measure* that puts mass 1/n for each observation  $(y_i, \mathbf{x}_i)$ , and the *expected risk* by

$$\mathcal{V}(f) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \mathcal{L}(f(\mathbf{x}_i), y_i) \right]$$

We also denote the *target function* 

$$f^* = \mathop{\arg\min}_{f \in \mathcal{H}} \mathcal{V}(f)$$

as minimizer of the expected risk. For any  $f \in \mathcal{H}$ , the excess risk is given by

$$\mathcal{E}(f) = \mathcal{V}(f) - \mathcal{V}(f^*),$$

where by definition  $\mathcal{E}(f) \geq 0$  for all  $f \in \mathcal{H}$ . In the case of model misspecification where the target function  $f^*$  is not necessarily linear, we define the linear subspace  $\mathcal{H}_{\beta} = \{f_{\beta} : \beta \in \mathbb{R}^{p+1}\} \subset \mathcal{H}$ , where the map  $\beta \mapsto f_{\beta}$  is linear. As in Section 6.6 of Bühlmann and van de Geer (2011) we consider the notationally simpler case without intercept. We denote the best linear approximation of the target function  $f^*$  by

$$f_{\beta^*} = \operatorname*{arg\,min}_{f \in \mathcal{H}_{\beta}} \mathcal{V}(f),$$

We define the *empirical process* for the linear subspace as

$$\{\mathcal{P}_n(f_{\boldsymbol{\beta}}) = \mathcal{V}_n(f_{\boldsymbol{\beta}}) - \mathcal{V}(f_{\boldsymbol{\beta}}) : f_{\boldsymbol{\beta}} \in \mathcal{H}_{\boldsymbol{\beta}}\}.$$

For a fixed (and arbitrary)  $f_{\tilde{\beta}}$ , we define

$$Z_M = \sup_{\|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_1 \le M} \left| \mathcal{P}_n(f_{\boldsymbol{\beta}}) - \mathcal{P}_n(f_{\tilde{\boldsymbol{\beta}}}) \right|$$
(14)

and the set

$$\mathcal{B} = \{ Z_M \le \lambda_0 M \},\$$

where

$$\lambda_0 = 4T(n,p) + \frac{t}{3n} + \sqrt{\frac{2t}{n}}\sqrt{1 + 8T(n,p)},$$
$$T(n,p) = \sqrt{\frac{2\log(2p)}{n}} + \frac{\log(2p)}{3n}.$$

By Lemma 14.20 and Theorem 14.5 of Bühlmann and van de Geer (2011), we have the probability inequality

$$\mathbb{P}(\mathcal{B}) \ge 1 - \exp(-t),$$

i.e. for some M sufficiently small

$$\mathbb{P}(Z_M \le \lambda_0 M) \ge 1 - \eta$$

for some  $\lambda_0$  that depends on the sample size n, the dimensionality of the data p, and the confidence level  $1 - \eta$ .

We denote the total sparsity and diversity penalties of the split logistic regression parameters for any set of G linear functions  $f_{\beta^1}, \ldots, f_{\beta^G}$  by

$$P(f_{\beta^1}, \dots, f_{\beta^G}) = \sum_{g=1}^G P_s(\beta^g) = \sum_{g=1}^G \left[ \frac{1-\alpha}{2} \|\beta^g\|_2^2 + \alpha \|\beta^g\|_1 \right]$$
(15)

where  $\alpha \in [0, 1]$ , and

$$Q(f_{\beta^1}, \dots, f_{\beta^G}) = \sum_{h \neq g} P_d\left(\beta^g, \beta^h\right) = \sum_{h \neq g} \sum_{j=1}^p |\beta_j^g| |\beta_j^h|, \tag{16}$$

respectively.

### 8.2 Ensemble Consistency

Let the solution to the split logistic regression objective function be the collection of functions

$$\left(f_{\hat{\beta}^{1}},\ldots,f_{\hat{\beta}^{G}}\right) = \operatorname*{arg\,min}_{f_{\beta^{1}},\ldots,f_{\beta^{G}}\in\mathcal{H}_{\beta}} \left\{ \sum_{g=1}^{G} \left[\frac{1}{n}\sum_{i=1}^{n} \mathcal{L}\left(f_{\beta^{g}}(\mathbf{x}_{i}),y_{i}\right) + \lambda_{s}P_{s}(\beta^{g})\right] + \frac{\lambda_{d}}{2}\sum_{h\neq g} P_{d}(\beta^{h},\beta^{g}) \right\}, \quad (17)$$

and let

$$(f_{\tilde{\boldsymbol{\beta}}^{1}},\ldots,f_{\tilde{\boldsymbol{\beta}}^{G}}) = \operatorname*{arg\,min}_{f_{\boldsymbol{\beta}^{1}},\ldots,f_{\boldsymbol{\beta}^{G}}\in\mathcal{H}_{\boldsymbol{\beta}}} \left\{ \sum_{g=1}^{G} \left[ \mathcal{E}(f_{\boldsymbol{\beta}^{g}}) + \lambda_{s} P_{s}(\boldsymbol{\beta}^{g}) \right] + \frac{\lambda_{d}}{2} \sum_{h\neq g} P_{d}(\boldsymbol{\beta}^{h},\boldsymbol{\beta}^{g}) \right\},\tag{18}$$

For  $Z_M$  taken as (14) using the solution from (18), define

$$\widetilde{M} = \frac{1}{G\lambda_0} \left[ \sum_{g=1}^G \mathcal{E}(f_{\tilde{\beta}^g}) + 2\lambda_s P(f_{\tilde{\beta}^1}, \dots, f_{\tilde{\beta}^G}) + \frac{\lambda_d}{2} Q(f_{\tilde{\beta}^1}, \dots, f_{\tilde{\beta}^G}) \right].$$
(19)

Let the set

$$\widetilde{\mathcal{B}} = \left\{ Z_{\widetilde{M}} \le \lambda_0 \widetilde{M} \right\}$$
(20)

where  $\lambda_s \geq 4G\lambda_0/\alpha$  if  $\alpha \in (0,1]$  and  $\lambda_s \geq 8G^3\lambda_0/\widetilde{M}$  if  $\alpha = 0$ . Then, we will prove below that on the set  $\widetilde{\mathcal{B}}$ , it holds that

$$\mathcal{V}\left(\frac{1}{G}\sum_{g=1}^{G}f_{\hat{\beta}^{g}}\right) - \mathcal{V}(f^{*}) \leq 2\left[\mathcal{E}(f_{\beta^{*}}) + 2\alpha\lambda_{s}\|\beta^{*}\|_{1} + \frac{1-\alpha}{2}\lambda_{s}\|\beta^{*}\|_{2}^{2} + \frac{\lambda_{d}(G-1)}{2}\|\beta^{*}\|_{2}^{2}\right].$$
 (21)

Hence, if the target is linear, i.e.  $f^* = f_{\beta^*}$ , then it holds that

$$\mathcal{V}\left(\frac{1}{G}\sum_{g=1}^{G}f_{\hat{\boldsymbol{\beta}}^{g}}\right) - \mathcal{V}(f^{*}) \leq 4\alpha\lambda_{s}\|\boldsymbol{\beta}^{*}\|_{1} + (1-\alpha)\lambda_{s}\|\boldsymbol{\beta}^{*}\|_{2}^{2} + \lambda_{d}(G-1)\|\boldsymbol{\beta}^{*}\|_{2}^{2}.$$
 (22)

Therefore, if the data come from a logistic model it follows that if we take  $\lambda_s$  and  $\lambda_d$  to be or order  $\sqrt{\log(p)/n}$ , and we assume that  $\|\boldsymbol{\beta}^*\|_1$  and  $\|\boldsymbol{\beta}^*\|_2^2$  are of order smaller than  $\sqrt{n/\log(p)}$  and  $\log(p)/n \to 0$ , then the ensemble prediction  $(1/G) \sum_{g=1}^G f_{\hat{\boldsymbol{\beta}}^g}$  is consistent. In the more general case of model misspecification  $(f^* \neq f_{\boldsymbol{\beta}^*})$ , the prediction error converges to  $2 \mathcal{E}(f_{\boldsymbol{\beta}^*})$ .

### 8.3 Ensemble Consistency Proof

Let  $f_{\hat{\beta}^1}, \ldots, f_{\hat{\beta}^G}$  be the solution to split logistic regression with G groups for data  $\{y_i, \mathbf{x}_i\}_{i=1}^n$ . Then, for any  $f_{\beta^1}, \ldots, f_{\beta^G} \in \mathcal{H}_{\beta}$  it holds that

$$\sum_{g=1}^{G} \mathcal{V}_n(f_{\hat{\beta}^g}) + \lambda_s P(f_{\hat{\beta}^1}, \dots, f_{\hat{\beta}^G}) + \frac{\lambda_d}{2} Q(f_{\hat{\beta}^1}, \dots, f_{\hat{\beta}^G})$$
$$\leq \sum_{g=1}^{G} \mathcal{V}_n(f_{\beta^g}) + \lambda_s P(f_{\beta^1}, \dots, f_{\beta^G}) + \frac{\lambda_d}{2} Q(f_{\beta^1}, \dots, f_{\beta^G}).$$

Note that if  $\bar{\beta}^g = t\hat{\beta}^g + (1-t)\tilde{\beta}^g$  for any  $t \in [0,1]$ , by a convexity argument

$$\begin{split} &\sum_{g=1}^{G} \mathcal{V}_n(f_{\tilde{\beta}^g}) + \lambda_s P(f_{\tilde{\beta}^1}, \dots, f_{\tilde{\beta}^G}) \\ &\leq t \left[ \sum_{g=1}^{G} \mathcal{V}_n(f_{\hat{\beta}^g}) + \lambda_s P(f_{\hat{\beta}^1}, \dots, f_{\hat{\beta}^G}) \right] \\ &+ (1-t) \left[ \sum_{g=1}^{G} \mathcal{V}_n(f_{\tilde{\beta}^g}) + \lambda_s P(f_{\tilde{\beta}^1}, \dots, f_{\tilde{\beta}^G}) \right] \\ &\leq t \left[ \sum_{g=1}^{G} \mathcal{V}_n(f_{\hat{\beta}^g}) + \lambda_s P(f_{\hat{\beta}^1}, \dots, f_{\hat{\beta}^G}) + \frac{\lambda_d}{2} Q(f_{\hat{\beta}^1}, \dots, f_{\hat{\beta}^G}) \right] \\ &+ (1-t) \left[ \sum_{g=1}^{G} \mathcal{V}_n(f_{\tilde{\beta}^g}) + \lambda_s P(f_{\tilde{\beta}^1}, \dots, f_{\tilde{\beta}^G}) + \frac{\lambda_d}{2} Q(f_{\tilde{\beta}^1}, \dots, f_{\tilde{\beta}^G}) \right] \\ &\leq \sum_{g=1}^{G} \mathcal{V}_n(f_{\tilde{\beta}^g}) + \lambda_s P(f_{\tilde{\beta}^1}, \dots, f_{\tilde{\beta}^G}) + \frac{\lambda_d}{2} Q(f_{\tilde{\beta}^1}, \dots, f_{\tilde{\beta}^G}). \end{split}$$

We can write

$$\begin{split} &\sum_{g=1}^G \mathcal{E}(f_{\bar{\beta}^g}) + \lambda_s P(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G}) \\ &= \sum_{g=1}^G \mathcal{E}(f_{\bar{\beta}^g}) + \lambda_s P(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G}) \\ &+ \left[\sum_{g=1}^G \mathcal{V}_n(f_{\bar{\beta}^g}) - \sum_{g=1}^G \mathcal{V}_n(f_{\bar{\beta}^g})\right] + \left[\sum_{g=1}^G \mathcal{V}_n(f_{\bar{\beta}^g}) - \sum_{g=1}^G \mathcal{V}_n(f_{\bar{\beta}^g})\right] \\ &+ \left[\sum_{g=1}^G \mathcal{V}(f_{\bar{\beta}^g}) - \sum_{g=1}^G \mathcal{V}(f_{\bar{\beta}^g})\right] + \left[\lambda_s P(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G}) - \lambda_s P(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G})\right] \\ &+ \left[\frac{\lambda_d}{2}Q(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G}) - \frac{\lambda_d}{2}Q(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G})\right] \\ &= - \left[\sum_{g=1}^G \mathcal{P}_n(f_{\bar{\beta}^g}) - \sum_{g=1}^G \mathcal{P}_n(f_{\bar{\beta}^g})\right] + \sum_{g=1}^G \mathcal{E}(f_{\bar{\beta}^g}) \\ &+ \left[\left(\sum_{g=1}^G \mathcal{V}_n(f_{\bar{\beta}^g}) + \lambda_s P(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G})\right) - \left(\sum_{g=1}^G \mathcal{V}_n(f_{\bar{\beta}^g}) + \lambda_s P(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G}) + \frac{\lambda_d}{2}Q(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G})\right)\right] \end{split}$$

Thus we get the basic inequality

$$\begin{split} &\sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^{g}}) + \lambda_{s} P(f_{\bar{\beta}^{1}}, \dots, f_{\bar{\beta}^{G}}) \\ &\leq -\left[\sum_{g=1}^{G} \mathcal{P}_{n}(f_{\bar{\beta}^{g}}) - \sum_{g=1}^{G} \mathcal{P}_{n}(f_{\tilde{\beta}^{g}})\right] + \sum_{g=1}^{G} \mathcal{E}(f_{\tilde{\beta}^{g}}) + \lambda_{s} P(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}}) + \frac{\lambda_{d}}{2} Q(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}}) \\ &= -\sum_{g=1}^{G} \left[ \mathcal{P}_{n}(f_{\bar{\beta}^{g}}) - \mathcal{P}_{n}(f_{\tilde{\beta}^{g}}) \right] + \sum_{g=1}^{G} \mathcal{E}(f_{\tilde{\beta}^{g}}) + \lambda_{s} P(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}}) + \frac{\lambda_{d}}{2} Q(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}}). \end{split}$$

In other words, to bound the sum of the excess risk  $\sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^g})$  we need to control the sum of the increments of the empirical processes  $\mathcal{P}_n(f_{\bar{\beta}^g}) - \mathcal{P}_n(f_{\tilde{\beta}^g}), 1 \leq g \leq G$ . Let

$$t^g = \frac{\widetilde{M}}{\widetilde{M} + \|\hat{\boldsymbol{\beta}}^g - \tilde{\boldsymbol{\beta}}^g\|_1}.$$

If  $\bar{\boldsymbol{\beta}}^g = t^g \hat{\boldsymbol{\beta}}^g + (1 - t^g) \tilde{\boldsymbol{\beta}}^g$ , then  $\| \tilde{\boldsymbol{\beta}}^g - \bar{\boldsymbol{\beta}}^g \|_1 \leq \widetilde{M}$ . Then on the set  $\widetilde{\mathcal{B}}$ ,

$$\sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^{g}}) + \lambda_{s} P(f_{\bar{\beta}^{1}}, \dots, f_{\bar{\beta}^{G}})$$

$$\leq \sum_{g=1}^{G} Z_{\widetilde{M}} + \sum_{g=1}^{G} \mathcal{E}(f_{\tilde{\beta}^{g}}) + \lambda_{s} P(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}}) + \frac{\lambda_{d}}{2} Q(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}})$$

$$= \sum_{g=1}^{G} \lambda_{0} \widetilde{M} + \sum_{g=1}^{G} \mathcal{E}(f_{\tilde{\beta}^{g}}) + \lambda_{s} P(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}}) + \frac{\lambda_{d}}{2} Q(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}})$$

For the case  $\alpha \in (0, 1]$  we obtain

$$\sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^{g}}) + \lambda_{s} P(f_{\bar{\beta}^{1}}, \dots, f_{\bar{\beta}^{G}}) + \lambda_{s} P(f_{\bar{\beta}^{1}}, \dots, f_{\bar{\beta}^{G}})$$

$$\leq G\lambda_{0}\widetilde{M} + \sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^{g}}) + 2\lambda_{s} P(f_{\bar{\beta}^{1}}, \dots, f_{\bar{\beta}^{G}}) + \frac{\lambda_{d}}{2} Q(f_{\bar{\beta}^{1}}, \dots, f_{\bar{\beta}^{G}})$$

$$= 2G\lambda_{0}\widetilde{M} \leq \alpha\lambda_{s}\frac{\widetilde{M}}{2}$$

since  $\lambda_s \geq 4G\lambda_0/\alpha$ . Notice that

$$\sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^g}) + \lambda_s P(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G}) + \lambda_s P(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G})$$

$$\geq \sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^g}) + \lambda_s \alpha \sum_{g=1}^{G} \|\bar{\beta}^g - \tilde{\beta}^g\|_1 + \lambda_s \frac{1-\alpha}{2} \sum_{g=1}^{G} \|\bar{\beta}^g - \tilde{\beta}^g\|_2^2$$

$$\geq \sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^g}) + \lambda_s \alpha \sum_{g=1}^{G} \|\bar{\beta}^g - \tilde{\beta}^g\|_1.$$

This implies

$$\|\bar{\boldsymbol{\beta}}^g - \tilde{\boldsymbol{\beta}}^g\|_1 \le rac{\widetilde{M}}{2}$$

for all  $1 \leq g \leq G$ , which in turn implies

$$\|\hat{\boldsymbol{\beta}}^g - \tilde{\boldsymbol{\beta}}^g\|_1 \le \widetilde{M}$$

for all  $1 \leq g \leq G$ . In the case  $\alpha = 0$ , we obtain

$$\begin{split} &\sum_{g=1}^{G} \mathcal{E}(f_{\tilde{\beta}^{g}}) + \lambda_{s} P(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}}) + \lambda_{s} P(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}}) \\ &\leq G\lambda_{0}\widetilde{M} + \sum_{g=1}^{G} \mathcal{E}(f_{\tilde{\beta}^{g}}) + 2\lambda_{s} P(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}}) + \frac{\lambda_{d}}{2} Q(f_{\tilde{\beta}^{1}}, \dots, f_{\tilde{\beta}^{G}}) \\ &= 2G\lambda_{0}\widetilde{M} \leq \frac{\lambda_{s}}{G^{2}} \frac{\widetilde{M}^{2}}{4}, \end{split}$$

since  $\lambda_s \geq 8G^3 \lambda_0 / \widetilde{M}$ , and

$$\begin{split} &\sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^g}) + \lambda_s P(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G}) + \lambda_s P(f_{\bar{\beta}^1}, \dots, f_{\bar{\beta}^G}) \\ &\geq \sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^g}) + \frac{\lambda_s}{2} \sum_{g=1}^{G} \|\bar{\beta}^g - \tilde{\beta}^g\|_2^2 \\ &\geq \sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^g}) + \frac{\lambda_s}{2G} \sum_{g=1}^{G} \|\bar{\beta}^g - \tilde{\beta}^g\|_1^2 \\ &\geq \sum_{g=1}^{G} \mathcal{E}(f_{\bar{\beta}^g}) + \frac{\lambda_s}{2G^2} \left( \sum_{g=1}^{G} \|\bar{\beta}^g - \tilde{\beta}^g\|_1 \right)^2. \end{split}$$

This again implies that  $\|\bar{\beta}^g - \tilde{\beta}^g\|_1 \leq \widetilde{M}/2$  and  $\|\hat{\beta}^g - \tilde{\beta}^g\|_1 \leq \widetilde{M}$  for  $1 \leq g \leq G$ .

Repeating the argument with  $\bar{\beta}$  replaced by  $\hat{\beta}$  yields on the set  $\tilde{\beta}$  the inequality

$$\sum_{g=1}^{G} \mathcal{E}(f_{\hat{\beta}^g}) \leq 2 \left[ \sum_{g=1}^{G} \mathcal{E}(f_{\tilde{\beta}^g}) + 2\lambda_s P(f_{\tilde{\beta}^1}, \dots, f_{\tilde{\beta}^G}) + \frac{\lambda_d}{2} Q(f_{\tilde{\beta}^1}, \dots, f_{\tilde{\beta}^G}) \right]$$

Notice that for the best linear predictor  $f_{\beta^*}$  used for all G functions, we can rewrite (15) and (16) as

$$P(f_{\beta^*}, \dots, f_{\beta^*}) = \alpha G \|\beta^*\|_1 + \frac{1 - \alpha}{2} G \|\beta^*\|_2^2, \text{ and}$$
$$Q(f_{\beta^*}, \dots, f_{\beta^*}) = G(G - 1) \|\beta^*\|_2^2,$$

respectively. Thus

$$\frac{1}{G}\sum_{g=1}^{G}\mathcal{E}(f_{\hat{\boldsymbol{\beta}}^{g}}) \leq 2\left[\mathcal{E}(f_{\boldsymbol{\beta}^{*}}) + 2\alpha\lambda_{s}\|\boldsymbol{\beta}^{*}\|_{1} + \frac{1-\alpha}{2}\lambda_{s}\|\boldsymbol{\beta}^{*}\|_{2}^{2} + \frac{\lambda_{d}(G-1)}{2}\|\boldsymbol{\beta}^{*}\|_{2}^{2}\right].$$

By the convexity of (13),

$$\frac{1}{G}\sum_{g=1}^{G}\mathcal{E}(f_{\hat{\beta}^g}) = \frac{1}{G}\sum_{g=1}^{G}\mathcal{V}(f_{\hat{\beta}^g}) - \mathcal{V}(f^*) \ge \mathcal{V}\left(\frac{1}{G}\sum_{g=1}^{G}f_{\hat{\beta}^g}\right) - \mathcal{V}(f^*),$$

so we get the desired inequality (21),

$$\mathcal{V}\left(\frac{1}{G}\sum_{g=1}^{G}f_{\hat{\beta}^{g}}\right) - \mathcal{V}(f^{*}) \leq 2\left[\mathcal{E}(f_{\beta^{*}}) + 2\alpha\lambda_{s}\|\beta^{*}\|_{1} + \frac{1-\alpha}{2}\lambda_{s}\|\beta^{*}\|_{2}^{2} + \frac{\lambda_{d}(G-1)}{2}\|\beta^{*}\|_{2}^{2}\right].$$

### Appendix B: Details of the Algorithm

In this section, we provide the derivation for the quadratic approximation of the logistic regression loss, the high-level steps of the block coordinate descent algorithm, and a detailed description of the alternating grid search for the tuning parameters.

### 8.4 Quadratic Approximation

For the binary classification problem with the classes labeled as  $Y = \{-1, 1\}$ , let  $\mathbf{y} \in \mathbb{R}^n$  be the vector of class labels and  $\mathbf{X} \in \mathbb{R}^{n \times p}$  be the design matrix with sample size n and number of features p. The logistic regression loss function is given by

$$\mathcal{L}\left(f(\mathbf{x}_{i}), y_{i}\right) = \mathcal{L}(\beta_{0}, \boldsymbol{\beta} \mid y_{i}, \mathbf{x}_{i}) = \log\left(1 + e^{-y_{i}f(\mathbf{x}_{i})}\right), \quad 1 \le i \le n,$$
(23)

where  $f(\mathbf{x}_i) = \beta_0 + \mathbf{x}_i^T \boldsymbol{\beta}$  is a linear function of the predictor variables,  $\beta_0 \in \mathbb{R}$  and  $\boldsymbol{\beta} \in \mathbb{R}^p$  are the intercept and vector of regression coefficients.

We denote  $\mathbf{X}_A \in \mathbb{R}^{n \times (p+1)}$  the augmented design matrix whose first column is a column of ones and  $\boldsymbol{\beta}_A \in \mathbb{R}^{p+1} = (\beta_0, \boldsymbol{\beta}^T)^T$  the vector with all regression parameters. The quadratic approximation for the logistic regression loss in (23) at the current estimates  $\tilde{\beta}_A$  is given by

$$\begin{split} \frac{1}{n}\sum_{i=1}^{n}\mathcal{L}_{Q}(\beta_{0},\boldsymbol{\beta}\mid\boldsymbol{y}_{i},\mathbf{x}_{i}) &= \frac{1}{n}\sum_{i=1}^{n}\mathcal{L}(\tilde{\beta}_{0},\boldsymbol{\tilde{\beta}}\mid\boldsymbol{y}_{i},\mathbf{x}_{i}) + \mathcal{V}\left(\tilde{\beta}_{0},\boldsymbol{\tilde{\beta}}\mid\mathbf{y},\mathbf{X}_{A}\right)\left(\boldsymbol{\beta}_{A}-\boldsymbol{\tilde{\beta}}_{A}\right) \\ &+ \frac{1}{2}\left(\boldsymbol{\beta}_{A}-\boldsymbol{\tilde{\beta}}_{A}\right)^{T}\mathcal{H}\left(\tilde{\beta}_{0},\boldsymbol{\tilde{\beta}}\mid\mathbf{y},\mathbf{X}_{A}\right)\left(\boldsymbol{\beta}_{A}-\boldsymbol{\tilde{\beta}}_{A}\right), \end{split}$$

where the gradient vector and hessian matrix are given by

$$\begin{split} \mathcal{V}\left(\tilde{\beta}_{0}, \tilde{\boldsymbol{\beta}} \mid \mathbf{y}, \mathbf{X}\right) &= \nabla \left(\frac{1}{n} \sum_{i=1}^{n} \mathcal{L}(\beta_{0}, \boldsymbol{\beta} \mid y_{i}, \mathbf{x}_{i})\right) \Big|_{(\beta_{0}, \boldsymbol{\beta}) = (\tilde{\beta}_{0}, \tilde{\boldsymbol{\beta}})} \\ &= \frac{1}{n} \mathbf{X}_{A}^{T} (\mathbf{z} - \tilde{\boldsymbol{p}}), \\ \mathcal{H}\left(\tilde{\beta}_{0}, \tilde{\boldsymbol{\beta}} \mid \mathbf{y}, \mathbf{X}\right) &= \nabla \left(\frac{1}{n} \sum_{i=1}^{n} \mathcal{L}(\beta_{0}, \boldsymbol{\beta} \mid y_{i}, \mathbf{x}_{i})\right) \nabla^{T} \Big|_{(\beta_{0}, \boldsymbol{\beta}) = (\tilde{\beta}_{0}, \tilde{\boldsymbol{\beta}})} \\ &= -\frac{1}{n} \mathbf{X}_{A}^{T} \tilde{\mathbf{W}} \mathbf{X}_{A}. \end{split}$$

The elements of the *n*-dimensional vectors  $\mathbf{z}$ ,  $\tilde{\mathbf{p}}$  and  $\tilde{\mathbf{w}}$  are given by  $z_i = (y_i+1)/2$ ,  $\tilde{p}_i = S(\tilde{\beta}_0 + \mathbf{x}_i^T \tilde{\boldsymbol{\beta}})$ and  $\tilde{w}_i = p_i(1-p_i)$ ,  $1 \le i \le n$  respectively. The  $n \times n$  weight matrix at the current parameter estimates is given by  $\tilde{\mathbf{W}} = \text{diag}(\tilde{\mathbf{w}})$ . The quadratic approximation can subsequently be rewritten as a weighted least-squares problem

$$\frac{1}{n}\sum_{i=1}^{n} \mathcal{L}_{Q}(\beta_{0},\boldsymbol{\beta} \mid y_{i},\mathbf{x}_{i}) = \frac{1}{2n} \left(\boldsymbol{\tilde{y}} - \mathbf{X}_{A}\boldsymbol{\beta}_{A}\right)^{T} \boldsymbol{\tilde{W}} \left(\boldsymbol{\tilde{y}} - \mathbf{X}_{A}\boldsymbol{\beta}_{A}\right) + C\left(\boldsymbol{\tilde{\beta}}_{0},\boldsymbol{\tilde{\beta}}\right) \\
= \frac{1}{2n}\sum_{i=1}^{n} \tilde{w}_{i} \left(\boldsymbol{\tilde{y}}_{i} - f(\mathbf{x}_{i})\right)^{2} + C\left(\boldsymbol{\tilde{\beta}}_{0},\boldsymbol{\tilde{\beta}}\right),$$
(24)

where the elements of the *n*-dimensional vector  $\tilde{\mathbf{y}}$  are given by  $\tilde{y}_i = \tilde{\beta}_0 + \mathbf{x}_i^T \tilde{\boldsymbol{\beta}} + (z_i - \tilde{p}_i)/\tilde{w}_i, 1 \le i \le n$ , and  $C\left(\tilde{\beta}_0, \tilde{\boldsymbol{\beta}}\right)$  is a constant term.

#### 8.5 Block Coordinate Descent Algorithm

The objective function is multi-convex and can be written as a weighed elastic net problem for each individual model, where the  $L_1$  penalty depends on the parameters in the other models. In particular, for a given model g, the objective function is given by

$$\mathcal{J}\left(\beta_0^g, \boldsymbol{\beta}^g \mid \mathbf{y}, \mathbf{X}\right) = \frac{1}{n} \sum_{i=1}^n \mathcal{L}(\beta_0^g, \boldsymbol{\beta}^g \mid y_i, \mathbf{x}_i) + \lambda_s \frac{(1-\alpha)}{2} \|\boldsymbol{\beta}^g\|_2^2 + \sum_{j=1}^p |\beta_j^g| u_{j,g}, \quad 1 \le g \le G.$$
(25)

with weights

$$u_{j,g} = \alpha \lambda_s + \frac{\lambda_d}{2} \sum_{h \neq g} |\beta_j^h|.$$

We apply a block coordinate descent algorithm by cycling through the parameters of one model at a time and we apply the coordinate descent updates in a deterministic, cyclic order. When updating the parameters of an individual model, a single coordinate descent update is applied for each parameter as follows. For notational convenience, we denote by  $\tilde{\mathbf{p}}^g$ ,  $\tilde{\mathbf{w}}^g$  and  $\tilde{\mathbf{y}}^g$  the *n*dimensional vectors with elements  $\tilde{p}_i^g = S(\tilde{\beta}_0^{~g} + \mathbf{x}_i^T \tilde{\boldsymbol{\beta}}^g)$ ,  $\tilde{w}_i^g = \tilde{p}_i^g(1-\tilde{p}_i^g)$  and  $\tilde{y}_i^g = \tilde{\beta}_0^g + \mathbf{x}_i^T \tilde{\boldsymbol{\beta}}^g + (z_i - \tilde{p}_i^g)/\tilde{w}_i^g$ ,  $1 \leq i \leq n$ , respectively. To obtain the coordinate descent updates we replace the logistic loss in the objective function (25) by its quadratic approximation (24) at the current parameter estimates for the ensemble. For parameter j of a particular model g,  $1 \leq j \leq p$ , the coordinate descent update is then given by

$$\begin{split} \hat{\beta}_{j}^{g} &= \operatorname*{arg\,min}_{\beta_{j}^{g} \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}_{Q}(\beta_{0}^{g}, \boldsymbol{\beta}^{g} \mid y_{i}, \mathbf{x}_{i}) + \lambda_{s} \frac{(1-\alpha)}{2} \|\boldsymbol{\beta}^{g}\|_{2}^{2} + \sum_{j=1}^{p} |\beta_{j}^{g}| u_{j,g} \\ &= \operatorname*{arg\,min}_{\beta_{j}^{g} \in \mathbb{R}} \frac{1}{2n} \sum_{i=1}^{n} \tilde{w}_{i} \left( \tilde{y}_{i}^{g} - \beta_{0}^{g} - \sum_{i=1}^{n} \sum_{k \neq j}^{p} x_{ik} \tilde{\beta}_{k}^{g} - \beta_{j}^{g} x_{ij} \right)^{2} + \lambda_{s} \frac{(1-\alpha)}{2} \left( \beta_{j}^{g} \right)^{2} + |\beta_{j}^{g}| u_{j,g} \\ &= \frac{\operatorname{Soft} \left( \frac{1}{n} \left( \tilde{r}_{j}^{g} + \tilde{\beta}_{j}^{g} \langle \mathbf{x}_{j}^{2}, \tilde{\mathbf{w}}^{g} \rangle \right), \, \alpha \lambda_{s} + \frac{\lambda_{d}}{2} \sum_{h \neq g} |\tilde{\beta}_{j}^{h}| \right)}{\frac{1}{n} \langle \mathbf{x}_{j}^{2}, \tilde{\mathbf{w}}^{g} \rangle + (1-\alpha) \lambda_{s}}, \end{split}$$

where  $\tilde{r}_j^g = \langle \mathbf{x}_j, \mathbf{z} \rangle - \langle \mathbf{x}_j, \tilde{\mathbf{p}}^g \rangle$ , and the last equality follows from the optimality condition for subgradients. A similar derivation can be made for the coordinate descent update of the intercept term

$$\hat{\beta}_0^g = \tilde{\beta}_0^g + \frac{\langle \mathbf{z} - \tilde{\mathbf{p}}^g, \mathbf{1}_n \rangle}{\langle \tilde{\mathbf{w}}^g, \mathbf{1}_n \rangle},$$

which yields the results in Proposition 1 of the article. When all parameter estimates of model g have been updated, also the vectors  $\tilde{\mathbf{p}}^g$  and  $\tilde{\mathbf{w}}^g$  are updated. The active set cycling strategy (Friedman et al., 2010) is also adopted and available in our software implementation. In Algorithm 1 we provide the steps to generate solutions for split logistic regression when  $\lambda_s$  and  $\lambda_d$  are fixed.

#### **Algorithm 1** Split Logistic Regression for Fixed $\lambda_s$ and $\lambda_d$

**Input:** Design matrix  $\mathbf{X} \in \mathbb{R}^{n \times p}$ , response vector  $\mathbf{y} \in \mathbb{R}^n$ , current solutions  $\tilde{\boldsymbol{\beta}}_{1:G}$ ,  $\ell_1 - \ell_2$  mixing parameter  $\alpha \in [0, 1]$ , sparsity and diversity tuning parameters  $\lambda_s, \lambda_d \geq 0$ , and convergence tolerance parameter  $\delta > 0$ .

- 1: Compute the current probabilities  $\tilde{p}_i^g = S(\tilde{\beta_0}^g + \mathbf{x}_i^T \tilde{\boldsymbol{\beta}}^g)$ , weights  $\tilde{w}_i^g = \tilde{p}_i^g(1 \tilde{p}_i^g)$  and residuals  $\tilde{r}_j^g = \langle \mathbf{x}_j, \mathbf{z} \rangle \langle \mathbf{x}_j, \tilde{\mathbf{p}}^g \rangle$ ,  $1 \le i \le n$ ,  $1 \le j \le p$ ,  $1 \le g \le G$ .
- 2: Repeat the following steps until convergence.
  - 2.1: For each model  $g, 1 \leq g \leq G$ :
    - 2.1.1: Perform a single (block) coordinate descent update for the intercept and each predictor  $j, 1 \le j \le p$ .
      - 2.1.1.1: Compute the new intercept in model g,

$$\hat{\beta}_0^g = \tilde{\beta}_0^g + \frac{\langle \mathbf{z} - \tilde{\mathbf{p}}^g, \mathbf{1}_n \rangle}{\langle \tilde{\mathbf{w}}^g, \mathbf{1}_n \rangle}$$

- 2.1.1.2: If  $\hat{\beta}_0^g \neq \tilde{\beta}_0^g$ , then update the probabilities  $\tilde{\mathbf{p}}^g$ , weights  $\tilde{\mathbf{w}}^g$  and residuals  $\tilde{\mathbf{r}}^g$  for model g.
- 2.1.1.3: Update j-th coefficient in model g,

$$\hat{\beta}_{j}^{g} = \frac{\text{Soft}\left(\frac{1}{n}\left(\tilde{r}_{j}^{g} + \tilde{\beta}_{j}^{g}\langle \mathbf{x}_{j}^{2}, \tilde{\mathbf{w}}^{g} \rangle\right), \, \alpha\lambda_{s} + \frac{\lambda_{d}}{2}\sum_{h \neq g} |\tilde{\beta}_{j}^{h}|\right)}{\frac{1}{n}\langle \mathbf{x}_{j}^{2}, \tilde{\mathbf{w}}^{g} \rangle + (1 - \alpha)\lambda_{s}}$$

2.1.1.4: If  $\hat{\beta}_j^g \neq \tilde{\beta}_j^g$ , then update the probabilities  $\tilde{\mathbf{p}}^g$ , weights  $\tilde{\mathbf{w}}^g$  and residuals  $\tilde{\mathbf{r}}^g$  for model g.

3: If successive estimates of the coefficients in the ensemble model show little difference, i.e.

$$\max_{1 \le j \le p} \left( \frac{1}{G} \sum_{g=1}^G \tilde{\beta}_j^g - \frac{1}{G} \sum_{g=1}^G \hat{\beta}_j^g \right)^2 < \delta,$$

then convergence is declared.

4: Return the coefficients for each model  $(\hat{\beta}_0^g, \hat{\beta}^g), 1 \le g \le G$ .

### 8.6 Alternating Grid Search for Tuning Parameters

The selection of the sparsity and diversity tuning parameters,  $\lambda_s$  and  $\lambda_d$ , is done by an alternating grid search. The first grid search is over  $\lambda_s$  with the diversity tuning parameter fixed at  $\lambda_d^{(0)} = 0$ , which yields a first value  $\lambda_s^{\text{opt}}$  minimizing the cross-validated loss. Keeping the sparsity parameter fixed at value  $\lambda_s^{\text{opt}}$ , we now perform a grid search over  $\lambda_d$  which yields  $\lambda_d^{\text{opt}}$ . This process is repeated until the cross-validated loss no longer decreases. The high-level steps of the alternating grid search are given in Algorithm 2.

To construct a grid for  $\lambda_s$ , we estimate a value  $\lambda_s^{\max}$  that makes all models null. In the special case where  $\lambda_d = 0$  and  $\alpha > 0$ , it can easily be shown that  $\lambda_s^{\max} = \frac{1}{2\alpha} \max_{1 \le j \le p} |\bar{\mathbf{x}}_j|$ . For  $\lambda_d > 0$ , we

estimate the smallest  $\lambda_s^{\max}$  that makes all models null by performing an internal grid search. Based on this maximal value  $\lambda_s^{\max}$  we then construct the grid for the sparsity penalty  $\lambda_s$  similarly to the case of (single-model) penalized logistic regression. that is, we use (by default) 100 log-equispaced points between  $\epsilon \lambda_s^{\max}$  and  $\lambda_s^{\max}$ , where  $\epsilon = 10^{-4}$  if p < n and  $10^{-2}$  otherwise.

The smallest diversity penalty  $\lambda_d^{\max}$  that makes the models fully disjoint for some fixed  $\lambda_s \geq 0$  is similarly estimated via a grid search. We then analogously generate the diversity penalty grid using (by default) 100 log-equispaced points between  $\epsilon \lambda_d^{\max}$  and  $\lambda_d^{\max}$ . For a grid search over one of the tuning parameters while keeping the other one fixed, we use warm-starts by computing solutions for a decreasing sequence of  $\lambda_s$  or  $\lambda_d$ , leading to a more stable algorithm.

### Algorithm 2 Alternating CV Procedure

**Input:** Design matrix  $\mathbf{X} \in \mathbb{R}^{n \times p}$ , response vector  $\mathbf{y} \in \mathbb{R}^n$ ,  $\ell_1 - \ell_2$  mixing parameter  $\alpha \in [0, 1]$  and convergence tolerance parameter  $\delta > 0$ .

- 1: Set  $\lambda_d^{\text{opt}} = 0$  and the next search is for the sparsity tuning parameter  $\lambda_s^{\text{opt}}$ .
- 2: Alternate between a search for  $\lambda_s^{\text{opt}}$  or  $\lambda_d^{\text{opt}}$  until CV MSPE no longer decreases.
  - 2.1: If the search is for  $\lambda_s^{\text{opt}}$ :
    - 2.1.1: If  $\lambda_d^{\text{opt}} = 0$ , set  $\lambda_s^{\max} = (1/2\alpha) \max_{1 \le j \le p} |\bar{\mathbf{x}}_j|$ . Otherwise perform a grid search to find the smallest  $\lambda_s^{\max}$  such that each model is null.
    - 2.1.2: Generate the log-equispaced grid between  $\epsilon \lambda_s$  and  $\lambda_s^{\text{max}}$ .
    - 2.1.3: For each  $\lambda_s$  in the log-equispaced grid compute  $\hat{\boldsymbol{\beta}}_{1:G}(\lambda_s) = (\hat{\boldsymbol{\beta}}^1(\lambda_s), \dots, \hat{\boldsymbol{\beta}}^G(\lambda_s))$  with Algorithm 1, using the previous solution in the grid as a warm-start.
    - 2.1.4: Set  $\lambda_s^{\text{opt}}$  using the value in the grid that minimized the CV MSPE.

Otherwise if the search is for  $\lambda_d^{\text{opt}}$ :

- 2.1.1: Perform a grid search to find the smallest  $\lambda_d^{\text{max}}$  such that makes models fully disjoint.
- 2.1.2: Generate the log-equispaced grid between  $\epsilon \lambda_d$  and  $\lambda_d^{\max}$ .
- 2.1.3: For each  $\lambda_d$  in the log-equispaced grid compute  $\hat{\boldsymbol{\beta}}_{1:G}(\lambda_d) = (\hat{\boldsymbol{\beta}}^1(\lambda_d), \dots, \hat{\boldsymbol{\beta}}^G(\lambda_d))$  with Algorithm 1, using the previous solution in the grid as a warm-start.
- 2.1.4: Set  $\lambda_d^{\text{opt}}$  using the value in the grid that minimized the CV MSPE.
- 3: For  $\lambda_d^{\text{opt}}$  and the smallest  $\lambda_s^{\max}$  such that each model is null, generate the log-equispaced grid between  $\epsilon \lambda_s$  and  $\lambda_s^{\max}$ .
- 4: For each  $\lambda_s$  in the log-equispaced grid compute  $\hat{\boldsymbol{\beta}}_{1:G}(\lambda_s) = (\hat{\boldsymbol{\beta}}^1(\lambda_s), \dots, \hat{\boldsymbol{\beta}}^G(\lambda_s))$  with Algorithm 1, using the previous solution in the grid as a warm-start.
- 5: Return the coefficients of the models  $\hat{\boldsymbol{\beta}}_{1:G}(\lambda_s) = (\hat{\boldsymbol{\beta}}^1(\lambda_s), \dots, \hat{\boldsymbol{\beta}}^G(\lambda_s))$  for each  $\lambda_s$  in the grid.

## **Appendix C: Alternative Diversity Measures**

In this section, we investigate the accuracy-diversity trade-off using several alternative diversity measures to complement and consolidate the results obtained in Section 6 of the main article based on the entropy diversity measure.

#### 8.7 Disagreement Measure

The disagreement (DIS) diversity measure (Skalak et al., 1996; Ho, 1998) of an ensemble comprised of G individual classifiers for a given input  $\mathbf{x}$  is defined as

$$DIS(\mathbf{x}) = \frac{1}{G(G-1)} \sum_{g=1}^{G} \sum_{h \neq g}^{G} DIS_{g,h}(\mathbf{x})$$

where the disagreement between between classifiers g and h is given by

$$DIS_{g,h}(\mathbf{x}) = \begin{cases} 1, & \text{if classifiers } g \text{ and } h \text{ disagree on the class of } \mathbf{x}, \\ 0, & \text{if classifiers } g \text{ and } h \text{ agree on the class of } \mathbf{x}. \end{cases}$$

The disagreement measure is a pairwise diversity measure and ranges between 0 and 1, where  $DIS(\mathbf{x}) = 0$  corresponds to no disagreement and increasing values of  $DIS(\mathbf{x})$  correspond to more disagreement between the individual classifiers.

#### 8.8 Double-Fault Measure

The double-fault (DF) diversity measure (Giacinto and Roli, 2001) of an ensemble comprised of G individual classifiers for some a given  $\mathbf{x}$  is defined as

$$\mathrm{DF}(\mathbf{x}) = \frac{1}{G(G-1)} \sum_{g=1}^{G} \sum_{h \neq g}^{G} \mathrm{DF}_{g,h}(\mathbf{x})$$

where the double-fault between between classifiers g and h is given by

$$DF_{g,h}(\mathbf{x}) = \begin{cases} 1, & \text{if classifiers } g \text{ and } h \text{ both misclassify } \mathbf{x}, \\ 0, & \text{if at most one of classifiers } g \text{ and } h \text{ misclassify } \mathbf{x}. \end{cases}$$

The double-fault measure is a pairwise diversity measure and ranges between 0 and 1, where  $DF(\mathbf{x}) = 0$  corresponds to no double-faults and increasing values of  $DF(\mathbf{x})$  correspond to more double-faults between the individual classifiers.

#### 8.9 Kohavi-Wolpert Variance

The Kohavi-Wolpert variance (KW) diversity measure (Kohavi et al., 1996) of an ensemble comprised of G individual classifiers for a given input  $\mathbf{x}$  is defined as

$$KW(\mathbf{x}) = \frac{1}{G^2} \ell(\mathbf{x})(G - \ell(\mathbf{x}))$$

where  $\ell(\mathbf{x})$  denotes the numbers of individual classifiers that correctly classified input  $\mathbf{x}$ . The Kohavi-Wolpert variance is a non-pairwise diversity measure with  $KW(\mathbf{x}) = 0$  corresponding to no diversity and increasing values of  $KW(\mathbf{x})$  corresponding to more diversity between the individual classifiers. Kuncheva and Whitaker (2003) have shown that

$$KW(\mathbf{x}) = \frac{G-1}{2G} DIS(\mathbf{x}).$$

### 8.10 Generalized Diversity

The generalized diversity (GD) measure (Partridge and Krzanowski, 1997) of an ensemble comprised of G individual classifiers for a given input  $\mathbf{x}$  is defined as

$$GD(\mathbf{x}) = 1 - \frac{\sum_{g=1}^{G} \frac{g(g-1)}{G(G-1)} \mathbb{P}(\ell(\mathbf{x}) = g)}{\sum_{g=1}^{G} \frac{g}{G} \mathbb{P}(\ell(\mathbf{x}) = g)}$$

where  $\ell(\mathbf{x})$  denotes the numbers of individual classifiers from the ensemble that correctly classify input  $\mathbf{x}$ . The generalized diversity is a non-pairwise diversity measure and ranges between 0 and 1, where  $GD(\mathbf{x}) = 0$  corresponds to no diversity and  $GD(\mathbf{x}) = 1$  corresponds to maximum diversity between the individual classifiers.

### 8.11 Results

In Table 8 we report the results of the alternative diversity measures for the same simulation settings as in Table 2 in the article. Similarly to the entropy diversity in the article, the DIS, DF, KW and GD measures are reported as a function of the number of models in split logistic regression, averaged over the test sets. It can be seen that the DIS, KW and GD diversity measures all increase with the number of models, while the DF diversity measure decreases. Hence, all measures confirm that the individual models become more diverse when the number of models increases.

Table 8: DIS, DF, KW and GD as a function of the number of models under	r Scenario 3.
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	$oldsymbol{\zeta}=0.1$				$oldsymbol{\zeta}=0.2$				$\zeta=0.4$			
$\overline{\mathbf{G}}$	DIS	DF	KW	GD	DIS	DF	KW	GD	DIS	DF	KW	GD
$\overline{2}$	0.08	0.78	0.02	0.39	0.07	0.82	0.02	0.43	0.07	0.84	0.02	0.45
5	0.11	0.76	0.04	0.58	0.09	0.81	0.03	0.60	0.11	0.81	0.05	0.69
7	0.12	0.75	0.05	0.63	0.10	0.81	0.04	0.66	0.14	0.79	0.06	0.75
10	0.13	0.74	0.06	0.67	0.11	0.80	0.05	0.70	0.14	0.79	0.07	0.76
15	0.17	0.71	0.08	0.72	0.18	0.74	0.08	0.77	0.18	0.75	0.08	0.79
20	0.22	0.67	0.10	0.75	0.20	0.72	0.10	0.79	0.19	0.75	0.09	0.80
25	0.24	0.65	0.11	0.75	0.21	0.71	0.10	0.79	0.20	0.74	0.10	0.80

### Appendix D: Full Results of Simulation Study

In this section, the full results of the simulation study in Section 5 of the article are reported. For the three scenarios described in the article and the considered values of the correlation parameters  $(\rho, \rho_1 \text{ and } \rho_2)$ , sample size (n), sparsity level  $(\zeta)$ , and probability of positive events  $(\pi_1)$ , the tables report the misclassification rate (MR), sensitivity (SE), specificity (SP), test-sample loss (TL), recall (RC) and precision (PR) for the eleven methods listed in Section 5.2 of the article.

- Scenario 1:
  - Tables 2-7 contain the results for the MR, SE and SP.
  - Tables 8-13 contain the results for the TL, RC and PR.
- Scenario 2:

- Tables 14-19 contain the results for the MR, SE and SP.
- Tables 20-25 contain the results for the TL, RC and PR.
- Scenario 3:
  - Tables 26-31 contain the results for the MR, SE and SP.
  - Tables 32-37 contain the results for the TL, RC and PR.

## Appendix E: Full Results for Medical Genomics Data

Tables 38-51 contain the full results for the gene expression data benchmark study in Section 7 of the article. For each data set we report the performance of the eleven methods listed in Section 2 of the article for the considered training set proportions and the different numbers of genes p preserved after the pre-processing step, where we considered p = 100, 250, 500 and 1,000. To simplify the comparison for each setting, we report relative performance of the methods in terms of MR and TL. That is, the performance reported for each method is relative to the best performing method, i.e. the ratio of its performance to the best performing method. Thus, the best performing method in each column of the tables corresponds the value 1.00.

Table 9: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho=0.2,\,n=50,\,p=1,000.$ 

		Ģ	$\zeta = 0.1$	L	Ģ	$\zeta = 0.2$	2	$oldsymbol{\zeta}=0.4$		
$\pi_1$	Method	$\mathbf{MR}$	SE	$\mathbf{SP}$	MR	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.29	0.53	0.84	0.20	0.68	0.87	0.17	0.74	0.89
	Elastic Net	0.27	0.56	0.85	0.18	0.72	0.89	0.14	0.78	0.91
	Adaptive	0.35	0.33	0.88	0.26	0.49	0.91	0.22	0.59	0.90
	Relaxed	0.29	0.52	0.83	0.21	0.68	0.86	0.18	0.74	0.88
	MCP	0.33	0.46	0.82	0.27	0.53	0.85	0.25	0.60	0.85
0.4	SIS-SCAD	0.33	0.50	0.78	0.30	0.56	0.80	0.28	0.61	0.79
	Split-Lasso-10	0.24	0.61	0.86	0.15	0.77	0.91	0.10	0.84	0.94
	Split-EN-10	0.23	0.62	0.87	0.15	0.77	0.91	0.10	0.85	0.94
	<b>RGLM-100</b>	0.23	0.63	0.86	0.16	0.70	0.92	0.13	0.76	0.94
	RF-500	0.24	0.56	0.89	0.17	0.66	0.95	0.13	0.73	0.96
	XGB	0.34	0.52	0.76	0.31	0.56	0.77	0.28	0.59	0.79
$\pi_1$	Method	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.25	0.42	0.91	0.19	0.56	0.92	0.17	0.55	0.95
	Elastic Net	0.24	0.42	0.92	0.17	0.60	0.93	0.14	0.62	0.96
	Adaptive	0.29	0.13	0.97	0.24	0.29	0.96	0.22	0.29	0.98
	Relaxed	0.26	0.41	0.89	0.19	0.57	0.91	0.17	0.58	0.93
	MCP	0.28	0.31	0.90	0.25	0.40	0.90	0.24	0.34	0.94
0.3	SIS-SCAD	0.29	0.36	0.87	0.26	0.42	0.88	0.25	0.38	0.90
	Split-Lasso-10	0.22	0.48	0.92	0.14	0.67	0.95	0.10	0.71	0.97
	Split-EN-10	0.21	0.50	0.92	0.13	0.67	0.95	0.10	0.73	0.97
	RGLM-100	0.21	0.49	0.93	0.16	0.55	0.97	0.15	0.52	0.99
	RF-500	0.23	0.37	0.96	0.17	0.46	0.98	0.17	0.43	0.99
	XGB	0.30	0.41	0.83	0.27	0.45	0.85	0.26	0.44	0.86
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.19	0.25	0.96	0.15	0.39	0.96	0.14	0.39	0.97
	Elastic Net	0.19	0.25	0.97	0.14	0.44	0.97	0.12	0.48	0.98
	Adaptive	0.21	0.05	0.99	0.19	0.11	0.99	0.18	0.14	0.99
	Relaxed	0.19	0.25	0.96	0.16	0.41	0.95	0.14	0.46	0.95
	MCP	0.21	0.14	0.96	0.19	0.21	0.96	0.19	0.20	0.96
0.2	SIS-SCAD	0.22	0.14	0.96	0.20	0.20	0.95	0.19	0.20	0.96
	Split-Lasso-10	0.17	0.30	0.97	0.12	0.53	0.97	0.09	0.58	0.99
	Split-EN-10	0.17	0.31	0.97	0.11	0.55	0.97	0.09	0.61	0.99
	RGLM-100	0.17	0.27	0.98	0.14	0.34	0.99	0.14	0.31	1.00
	RF-500	0.18	0.18	0.99	0.16	0.25	1.00	0.15	0.21	1.00
	XGB	0.23	0.27	0.91	0.20	0.32	0.92	0.20	0.33	0.91
		C	$\zeta = 0.1$	-	Ç	5 = 0.2	2		$\zeta = 0.4$	4
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$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.16	0.76	0.89	0.12	0.84	0.91	0.10	0.86	0.93
	Elastic Net	0.15	0.78	0.89	0.10	0.86	0.92	0.08	0.89	0.94
	Adaptive	0.20	0.65	0.91	0.14	0.78	0.92	0.11	0.83	0.93
	Relaxed	0.17	0.77	0.87	0.13	0.84	0.90	0.11	0.87	0.91
	MCP	0.21	0.69	0.86	0.17	0.76	0.87	0.16	0.76	0.89
0.4	SIS-SCAD	0.22	0.66	0.85	0.20	0.72	0.85	0.19	0.72	0.87
	Split-Lasso-10	0.14	0.80	0.89	0.08	0.88	0.93	0.06	0.92	0.96
	Split-EN-10	0.14	0.81	0.89	0.08	0.89	0.94	0.05	0.92	0.96
	RGLM-100	0.14	0.79	0.90	0.09	0.86	0.94	0.07	0.88	0.96
	RF-500	0.14	0.78	0.91	0.09	0.85	0.95	0.07	0.87	0.98
	XGB	0.23	0.66	0.84	0.20	0.72	0.85	0.19	0.71	0.87
$\pi_1$	Method	MR	SE	$\mathbf{SP}$	MR	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.15	0.64	0.94	0.11	0.75	0.95	0.09	0.80	0.95
	Elastic Net	0.14	0.67	0.94	0.10	0.78	0.96	0.08	0.84	0.96
	Adaptive	0.20	0.40	0.97	0.15	0.58	0.97	0.11	0.71	0.96
	Relaxed	0.16	0.64	0.92	0.12	0.75	0.93	0.10	0.82	0.93
	MCP	0.21	0.50	0.92	0.17	0.58	0.93	0.16	0.66	0.91
0.3	SIS-SCAD	0.22	0.44	0.93	0.19	0.51	0.93	0.18	0.57	0.92
	Split-Lasso-10	0.13	0.70	0.94	0.08	0.82	0.96	0.06	0.88	0.97
	Split-EN-10	0.13	0.71	0.94	0.08	0.83	0.96	0.05	0.89	0.97
	RGLM-100	0.13	0.68	0.95	0.09	0.75	0.97	0.07	0.80	0.98
	RF-500	0.13	0.65	0.96	0.10	0.71	0.98	0.08	0.77	0.99
	XGB	0.21	0.56	0.89	0.18	0.59	0.91	0.19	0.61	0.90
$\pi_1$	Method	MR	$\mathbf{SE}$	$\mathbf{SP}$	MR	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SP
	Lasso	0.13	0.55	0.95	0.09	0.70	0.96	0.08	0.70	0.97
	Elastic Net	0.12	0.59	0.96	0.08	0.74	0.97	0.07	0.75	0.98
	Adaptive	0.16	0.30	0.98	0.12	0.48	0.98	0.11	0.51	0.98
	Relaxed	0.13	0.57	0.95	0.10	0.71	0.95	0.09	0.72	0.96
	MCP	0.17	0.35	0.95	0.16	0.44	0.95	0.16	0.38	0.96
0.2	SIS-SCAD	0.18	0.27	0.97	0.16	0.36	0.96	0.16	0.32	0.97
	Split-Lasso-10	0.11	0.63	0.96	0.07	0.80	0.97	0.05	0.81	0.98
	Split-EN-10	0.11	0.63	0.96	0.06	0.80	0.97	0.05	0.82	0.98
	RGLM-100	0.11	0.57	0.97	0.08	0.69	0.98	0.07	0.65	0.99
	RF-500	0.12	0.52	0.98	0.08	0.64	0.99	0.08	0.58	1.00
	XGB	0.18	0.48	0.90	0.16	0.54	0.91	0.15	0.51	0.93

Table 10: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho=0.5,\,n=50,\,p=1,000.$ 

 $\zeta = 0.4$  $\zeta = 0.1$  $\zeta = 0.2$  $\mathbf{SE}$ Method  $\mathbf{MR}$  $\mathbf{SE}$  $\mathbf{SP}$  $\mathbf{MR}$  $\mathbf{SE}$  $\mathbf{SP}$  $\mathbf{MR}$  $\mathbf{SP}$  $\pi_1$ Lasso 0.110.860.910.070.900.950.060.920.95Elastic Net 0.970.100.870.920.060.910.960.050.94Adaptive 0.120.810.920.080.880.950.070.900.95Relaxed 0.860.910.080.900.930.070.920.930.11MCP 0.150.780.890.130.790.910.130.820.910.4SIS-SCAD 0.910.780.920.120.820.920.150.770.13Split-Lasso-10 0.09 0.920.050.930.960.030.96 0.970.88Split-EN-10 0.960.090.880.930.050.930.960.030.97**RGLM-100** 0.910.040.100.870.930.060.960.940.97RF-500 0.090.860.930.060.900.970.040.94 0.98XGB 0.820.160.790.870.150.800.890.150.88Method  $\mathbf{MR}$  $\mathbf{SE}$  $\mathbf{SP}$  $\mathbf{MR}$  $\mathbf{SE}$  $\mathbf{SP}$  $\mathbf{MR}$  $\mathbf{SE}$  $\mathbf{SP}$  $\pi_1$ 0.060.90Lasso 0.100.810.940.070.860.960.96Elastic Net 0.090.820.940.890.920.970.060.970.04Adaptive 0.060.960.120.680.960.080.810.970.87Relaxed 0.110.810.930.080.870.940.070.910.94MCP 0.670.930.720.930.130.750.930.150.130.3SIS-SCAD 0.150.630.950.120.690.950.120.730.950.910.040.940.97Split-Lasso-10 0.090.840.940.050.97Split-EN-10 0.09 0.850.940.050.920.970.030.950.97**RGLM-100** 0.950.050.890.970.040.920.090.830.98RF-500 0.090.820.950.050.880.980.040.900.99XGB 0.150.730.900.140.760.910.130.780.90 $\mathbf{SP}$  $\mathbf{SP}$ Method  $\mathbf{MR}$ SE  $\mathbf{SP}$  $\mathbf{MR}$  $\mathbf{SE}$  $\mathbf{MR}$  $\mathbf{SE}$  $\pi_1$ 0.830.08 Lasso 0.740.970.060.970.050.810.98Elastic Net 0.070.760.970.050.850.970.040.850.99Adaptive 0.550.980.080.980.070.680.990.110.69Relaxed 0.080.750.960.070.830.950.070.810.97MCP 0.120.550.960.110.620.950.110.600.96 SIS-SCAD 0.20.130.460.970.120.540.970.110.550.97Split-Lasso-10 0.070.780.970.050.880.970.040.870.99Split-EN-10 0.070.790.970.040.880.970.030.880.99**RGLM-100** 0.070.750.970.050.810.980.050.800.99**RF-500** 0.740.050.810.040.790.070.980.991.00XGB0.120.650.940.700.930.110.700.940.11

Table 11: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho = 0.8$ , n = 50, p = 1,000.

Table 12: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho = 0.2$ , n = 100, p = 1,000.

			$\zeta = 0.1$	L		5 = 0.2	2		$\zeta = 0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.25	0.66	0.82	0.18	0.73	0.88	0.14	0.79	0.91
	Elastic Net	0.24	0.67	0.82	0.17	0.75	0.89	0.12	0.81	0.92
	Adaptive	0.27	0.53	0.86	0.21	0.63	0.90	0.15	0.74	0.92
	Relaxed	0.25	0.65	0.81	0.18	0.73	0.87	0.14	0.79	0.90
	MCP	0.27	0.62	0.80	0.23	0.64	0.85	0.20	0.69	0.87
0.4	SIS-SCAD	0.30	0.58	0.79	0.26	0.61	0.83	0.24	0.64	0.83
	Split-Lasso-10	0.22	0.69	0.84	0.14	0.79	0.90	0.09	0.86	0.94
	Split-EN-10	0.22	0.69	0.84	0.14	0.79	0.91	0.09	0.87	0.94
	RGLM-100	0.22	0.70	0.83	0.15	0.76	0.92	0.11	0.80	0.95
	RF-500	0.22	0.64	0.87	0.16	0.69	0.94	0.11	0.75	0.97
	XGBoost	0.31	0.59	0.76	0.27	0.60	0.82	0.24	0.63	0.84
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.22	0.54	0.89	0.16	0.67	0.91	0.13	0.71	0.94
	Elastic Net	0.22	0.54	0.89	0.15	0.69	0.92	0.11	0.74	0.95
	Adaptive	0.25	0.30	0.95	0.19	0.49	0.95	0.16	0.54	0.96
	Relaxed	0.22	0.54	0.88	0.16	0.67	0.91	0.13	0.72	0.93
	MCP	0.25	0.44	0.89	0.21	0.52	0.91	0.19	0.54	0.92
0.3	SIS-SCAD	0.26	0.43	0.88	0.23	0.49	0.89	0.21	0.53	0.89
	Split-Lasso-10	0.20	0.58	0.91	0.13	0.73	0.93	0.08	0.81	0.96
	Split-EN-10	0.20	0.58	0.91	0.13	0.74	0.93	0.08	0.81	0.96
	RGLM-100	0.20	0.60	0.90	0.14	0.66	0.95	0.11	0.67	0.98
	RF-500	0.21	0.46	0.95	0.16	0.54	0.98	0.14	0.54	1.00
	XGBoost	0.27	0.47	0.84	0.23	0.51	0.87	0.22	0.50	0.90
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SP
	Lasso	0.18	0.38	0.94	0.13	0.53	0.95	0.11	0.54	0.97
	Elastic Net	0.17	0.38	0.95	0.12	0.56	0.96	0.10	0.59	0.98
	Adaptive	0.20	0.11	0.99	0.17	0.26	0.98	0.15	0.28	0.99
	Relaxed	0.18	0.38	0.94	0.13	0.54	0.95	0.11	0.57	0.97
	MCP	0.20	0.27	0.95	0.18	0.33	0.95	0.16	0.31	0.96
0.2	SIS-SCAD	0.20	0.30	0.93	0.18	0.33	0.94	0.17	0.31	0.96
	Split-Lasso-10	0.16	0.44	0.95	0.10	0.63	0.96	0.07	0.68	0.98
	Split-EN-10	0.16	0.44	0.95	0.10	0.64	0.96	0.07	0.70	0.98
	RGLM-100	0.16	0.46	0.95	0.12	0.48	0.98	0.11	0.44	1.00
	RF-500	0.17	0.28	0.98	0.14	0.33	0.99	0.14	0.28	1.00

Table 13: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho = 0.5$ , n = 100, p = 1,000.

		(	$\zeta = 0.1$	L	(	$\zeta = 0.2$	2		$\zeta=0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SP
	Lasso	0.15	0.79	0.89	0.10	0.84	0.93	0.08	0.88	0.94
	Elastic Net	0.15	0.80	0.89	0.09	0.85	0.94	0.07	0.90	0.95
	Adaptive	0.16	0.74	0.91	0.11	0.80	0.94	0.09	0.86	0.95
	Relaxed	0.16	0.78	0.89	0.11	0.84	0.93	0.09	0.89	0.93
	MCP	0.18	0.75	0.87	0.14	0.77	0.91	0.13	0.80	0.91
0.4	SIS-SCAD	0.19	0.73	0.86	0.16	0.75	0.90	0.15	0.78	0.90
	Split-Lasso-10	0.14	0.82	0.90	0.08	0.89	0.94	0.05	0.93	0.96
	Split-EN-10	0.14	0.82	0.90	0.08	0.89	0.94	0.05	0.93	0.96
	RGLM-100	0.14	0.81	0.90	0.09	0.86	0.95	0.06	0.90	0.97
	RF-500	0.14	0.80	0.91	0.09	0.84	0.96	0.06	0.88	0.98
	XGBoost	0.20	0.74	0.85	0.16	0.75	0.90	0.16	0.77	0.89
$\pi_1$	Method	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	SE	SP
	Lasso	0.14	0.74	0.92	0.09	0.81	0.94	0.08	0.83	0.96
	Elastic Net	0.13	0.75	0.92	0.08	0.83	0.95	0.06	0.86	0.97
	Adaptive	0.15	0.62	0.94	0.10	0.74	0.96	0.09	0.78	0.97
	Relaxed	0.14	0.74	0.91	0.10	0.82	0.94	0.09	0.83	0.94
	MCP	0.17	0.66	0.91	0.14	0.70	0.93	0.14	0.68	0.93
0.3	SIS-SCAD	0.18	0.63	0.91	0.14	0.68	0.93	0.15	0.65	0.93
	Split-Lasso-10	0.12	0.77	0.92	0.07	0.87	0.95	0.05	0.90	0.97
	Split-EN-10	0.12	0.78	0.92	0.07	0.87	0.96	0.05	0.91	0.97
	RGLM-100	0.12	0.76	0.93	0.08	0.82	0.97	0.06	0.82	0.98
	RF-500	0.12	0.73	0.94	0.08	0.78	0.98	0.07	0.79	0.99
	XGBoost	0.18	0.65	0.89	0.15	0.68	0.92	0.15	0.68	0.91
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	SE	SP
	Lasso	0.11	0.63	0.95	0.08	0.72	0.97	0.07	0.75	0.98
	Elastic Net	0.11	0.65	0.95	0.07	0.75	0.97	0.06	0.78	0.98
	Adaptive	0.13	0.44	0.98	0.10	0.54	0.99	0.09	0.63	0.99
	Relaxed	0.11	0.64	0.95	0.09	0.72	0.96	0.07	0.76	0.97
	MCP	0.15	0.49	0.94	0.14	0.48	0.96	0.13	0.51	0.96
0.2	SIS-SCAD	0.15	0.43	0.95	0.14	0.44	0.97	0.14	0.45	0.97
	Split-Lasso-10	0.10	0.69	0.95	0.06	0.81	0.97	0.04	0.85	0.99
	Split-EN-10	0.10	0.69	0.95	0.06	0.81	0.97	0.04	0.85	0.99
	RGLM-100	0.10	0.66	0.96	0.07	0.71	0.99	0.06	0.71	0.99
	RF-500	0.10	0.61	0.97	0.08	0.65	0.99	0.07	0.65	1.00
	XGBoost	0.15	0.56	0.93	0.13	0.56	0.95	0.13	0.57	0.94

Table 14: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho=0.8,\,n=100,\,p=1,000.$ 

		(	$\zeta = 0.1$	_	C	$\zeta = 0.2$	2		$oldsymbol{\zeta} = 0.\mathbf{z}$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.10	0.87	0.92	0.07	0.91	0.95	0.05	0.94	0.95
	Elastic Net	0.09	0.87	0.93	0.06	0.92	0.96	0.04	0.95	0.97
	Adaptive	0.10	0.85	0.93	0.07	0.89	0.95	0.05	0.93	0.95
	Relaxed	0.10	0.86	0.92	0.07	0.91	0.94	0.06	0.93	0.94
	MCP	0.12	0.83	0.91	0.10	0.85	0.93	0.09	0.88	0.92
0.4	SIS-SCAD	0.12	0.83	0.91	0.10	0.86	0.93	0.09	0.88	0.93
	Split-Lasso-10	0.09	0.88	0.93	0.05	0.94	0.96	0.03	0.97	0.97
	Split-EN-10	0.09	0.88	0.93	0.05	0.94	0.96	0.03	0.97	0.97
	RGLM-100	0.09	0.88	0.93	0.05	0.93	0.96	0.03	0.96	0.97
	RF-500	0.09	0.87	0.94	0.05	0.92	0.97	0.03	0.96	0.98
	XGBoost	0.14	0.82	0.89	0.11	0.85	0.92	0.10	0.87	0.92
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SP
	Lasso	0.09	0.84	0.94	0.06	0.89	0.96	0.05	0.93	0.96
	Elastic Net	0.09	0.85	0.94	0.05	0.90	0.97	0.04	0.94	0.97
	Adaptive	0.10	0.79	0.95	0.06	0.85	0.97	0.05	0.92	0.97
	Relaxed	0.09	0.84	0.94	0.06	0.89	0.95	0.06	0.93	0.95
	MCP	0.13	0.76	0.92	0.12	0.74	0.94	0.11	0.81	0.93
0.3	SIS-SCAD	0.12	0.74	0.94	0.11	0.71	0.96	0.10	0.80	0.95
	Split-Lasso-10	0.08	0.87	0.94	0.05	0.93	0.97	0.03	0.97	0.97
	Split-EN-10	0.08	0.87	0.94	0.04	0.93	0.97	0.03	0.97	0.97
	RGLM-100	0.08	0.86	0.94	0.05	0.90	0.97	0.03	0.94	0.98
	RF-500	0.08	0.85	0.95	0.05	0.89	0.98	0.03	0.93	0.99
	XGBoost	0.12	0.79	0.92	0.10	0.80	0.94	0.10	0.83	0.94
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SP
	Lasso	0.08	0.77	0.96	0.05	0.83	0.98	0.04	0.88	0.98
	Elastic Net	0.07	0.79	0.96	0.04	0.85	0.98	0.03	0.90	0.99
	Adaptive	0.09	0.64	0.98	0.06	0.75	0.99	0.05	0.82	0.99
	Relaxed	0.08	0.78	0.96	0.05	0.83	0.97	0.05	0.88	0.97
	MCP	0.12	0.64	0.94	0.11	0.63	0.96	0.10	0.66	0.96
0.2	SIS-SCAD	0.12	0.51	0.97	0.12	0.48	0.98	0.10	0.55	0.98
	Split-Lasso-10	0.07	0.82	0.96	0.04	0.89	0.98	0.02	0.94	0.99
	Split-EN-10	0.07	0.82	0.96	0.04	0.89	0.98	0.02	0.94	0.99
	RGLM-100	0.07	0.81	0.96	0.04	0.84	0.99	0.03	0.88	0.99
	RF-500	0.07	0.79	0.97	0.04	0.82	0.99	0.03	0.86	1.00
	XGBoost	0.10	0.73	0.94	0.09	0.71	0.96	0.09	0.75	0.96

			$\zeta=0.1$	L		$\zeta=0.2$	2		$oldsymbol{\zeta}=0.$	4
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	PR
	Lasso	1.13	0.03	0.16	0.89	0.03	0.24	0.76	0.03	0.44
	Elastic Net	1.08	0.04	0.16	0.80	0.05	0.24	0.64	0.05	0.43
	Adaptive	1.23	0.03	0.17	1.04	0.03	0.23	0.93	0.03	0.45
	Relaxed	1.18	0.02	0.20	1.16	0.02	0.23	0.97	0.03	0.42
	MCP	1.26	0.01	0.17	1.11	0.01	0.21	1.07	0.01	0.43
0.4	SIS-SCAD	1.37	0.01	0.21	1.31	0.00	0.23	1.23	0.00	0.49
	Split-Lasso-10	1.00	0.22	0.15	0.66	0.29	0.25	0.46	0.27	0.45
	Split-EN-10	0.97	0.26	0.14	0.66	0.31	0.24	0.45	0.35	0.44
	RGLM-100	0.97	_	_	0.81	_	_	0.76	_	_
	RF-500	1.03	_	_	0.89	_	_	0.84	_	_
	XGB	1.24	0.03	0.36	1.19	0.01	0.48	1.13	0.01	0.59
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	1.04	0.02	0.14	0.84	0.02	0.22	0.74	0.02	0.43
	Elastic Net	1.01	0.04	0.14	0.74	0.04	0.22	0.62	0.05	0.43
	Adaptive	1.16	0.02	0.14	0.97	0.02	0.22	0.92	0.02	0.43
	Relaxed	1.49	0.02	0.15	0.99	0.02	0.22	0.91	0.02	0.44
	MCP	1.17	0.01	0.12	1.06	0.01	0.22	1.02	0.00	0.34
0.3	SIS-SCAD	1.28	0.00	0.16	1.13	0.00	0.21	1.08	0.00	0.38
	Split-Lasso-10	0.92	0.18	0.15	0.62	0.25	0.25	0.45	0.26	0.47
	Split-EN-10	0.90	0.27	0.13	0.61	0.30	0.25	0.43	0.35	0.45
	RGLM-100	0.89	_	_	0.74	_	_	0.70	_	_
	RF-500	0.94	_	_	0.81	_	_	0.76	_	_
	XGB	1.19	0.03	0.41	1.12	0.01	0.46	1.10	0.01	0.65
$\pi_1$	Method	$\mathbf{TL}$	RC	PR	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	PR
	Lasso	0.90	0.02	0.15	0.70	0.02	0.25	0.64	0.02	0.42
	Elastic Net	0.87	0.04	0.14	0.63	0.04	0.23	0.53	0.04	0.44
	Adaptive	0.98	0.02	0.15	0.87	0.02	0.24	0.82	0.02	0.41
	Relaxed	1.12	0.02	0.14	0.96	0.02	0.25	0.92	0.02	0.43
	MCP	1.03	0.00	0.12	0.90	0.00	0.24	0.88	0.00	0.43
0.2	SIS-SCAD	1.08	0.00	0.17	0.98	0.00	0.19	0.96	0.00	0.44
	Split-Lasso-10	0.79	0.17	0.15	0.52	0.24	0.27	0.40	0.24	0.48
	Split-EN-10	0.77	0.20	0.14	0.50	0.30	0.25	0.37	0.33	0.46
	RGLM-100	0.76	_	_	0.63	_	_	0.59	_	_
	RF-500	0.79	_	_	0.67	_	_	0.62	_	_
	XGB	1.05	0.02	0.40	0.98	0.02	0.55	1.00	0.01	0.71

Table 15: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho = 0.2$ , n = 50, p = 1,000.

			$oldsymbol{\zeta} = 0.1$	L	(	$\zeta = 0.2$	2		$oldsymbol{\zeta} = 0.$	4
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.76	0.02	0.13	0.54	0.02	0.23	0.45	0.02	0.43
	Elastic Net	0.69	0.04	0.12	0.46	0.05	0.22	0.35	0.05	0.41
	Adaptive	0.87	0.02	0.13	0.65	0.03	0.24	0.56	0.03	0.42
	Relaxed	1.05	0.02	0.13	0.84	0.02	0.23	0.71	0.02	0.44
	MCP	0.92	0.01	0.15	0.75	0.01	0.22	0.73	0.01	0.41
0.4	SIS-SCAD	0.98	0.00	0.15	0.85	0.00	0.28	0.82	0.00	0.47
	Split-Lasso-10	0.64	0.17	0.14	0.39	0.23	0.25	0.27	0.23	0.47
	Split-EN-10	0.63	0.23	0.13	0.38	0.30	0.24	0.26	0.31	0.45
	RGLM-100	0.66	_	_	0.55	_	_	0.52	_	_
	RF-500	0.70	_	_	0.59	_	_	0.56	_	—
	XGB	1.05	0.02	0.35	0.98	0.02	0.52	0.95	0.01	0.63
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.69	0.02	0.13	0.50	0.02	0.19	0.43	0.02	0.42
	Elastic Net	0.65	0.04	0.13	0.43	0.04	0.21	0.34	0.05	0.42
	Adaptive	0.86	0.02	0.12	0.65	0.02	0.19	0.56	0.02	0.41
	Relaxed	1.29	0.02	0.14	0.82	0.02	0.19	0.70	0.02	0.41
	MCP	0.90	0.00	0.12	0.77	0.00	0.19	0.73	0.00	0.45
0.3	SIS-SCAD	0.90	0.00	0.15	0.83	0.00	0.16	0.80	0.00	0.40
	Split-Lasso-10	0.61	0.20	0.16	0.36	0.24	0.27	0.26	0.24	0.50
	Split-EN-10	0.59	0.27	0.14	0.35	0.31	0.26	0.25	0.32	0.47
	RGLM-100	0.61	_	_	0.50		_	0.48		_
	RF-500	0.64	_	_	0.54	_	_	0.52	_	—
	XGB	1.03	0.02	0.45	0.94	0.01	0.54	0.96	0.01	0.57
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	PR	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.60	0.02	0.11	0.44	0.02	0.22	0.38	0.02	0.37
	Elastic Net	0.54	0.03	0.11	0.37	0.04	0.20	0.30	0.04	0.40
	Adaptive	0.71	0.02	0.11	0.56	0.02	0.22	0.50	0.02	0.38
	Relaxed	0.87	0.01	0.13	0.73	0.02	0.23	0.54	0.02	0.39
	MCP	0.81	0.00	0.08	0.71	0.00	0.30	0.73	0.00	0.48
0.2	SIS-SCAD	0.82	0.00	0.10	0.71	0.00	0.25	0.71	0.00	0.40
	Split-Lasso-10	0.50	0.22	0.15	0.30	0.23	0.28	0.24	0.23	0.52
	Split-EN-10	0.49	0.26	0.15	0.29	0.30	0.26	0.22	0.32	0.49
	RGLM-100	0.51	_	_	0.43	_	_	0.41	_	—
	RF-500	0.53	_	_	0.45	_	_	0.44	_	_
	XGB	0.97	0.01	0.33	0.95	0.01	0.45	0.93	0.00	0.55

Table 16: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho = 0.5$ , n = 50, p = 1,000.

			$oldsymbol{\zeta} = 0.1$	L	(	$\zeta=0.2$	2		$oldsymbol{\zeta} = 0.$	4
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.51	0.02	0.13	0.34	0.02	0.22	0.29	0.02	0.39
	Elastic Net	0.46	0.04	0.12	0.28	0.05	0.22	0.23	0.04	0.39
	Adaptive	0.63	0.02	0.14	0.43	0.02	0.21	0.41	0.02	0.39
	Relaxed	0.79	0.02	0.14	0.53	0.01	0.19	0.73	0.01	0.40
	MCP	0.68	0.00	0.13	0.62	0.00	0.26	0.61	0.00	0.43
0.4	SIS-SCAD	0.69	0.00	0.14	0.63	0.00	0.16	0.60	0.00	0.44
	Split-Lasso-10	0.44	0.18	0.15	0.25	0.21	0.28	0.19	0.19	0.49
	Split-EN-10	0.44	0.27	0.14	0.24	0.30	0.25	0.19	0.32	0.46
	RGLM-100	0.45	_	_	0.35	_	_	0.33	_	_
	RF-500	0.46	_	_	0.38	_	_	0.36	_	_
	XGB	1.01	0.01	0.26	1.04	0.00	0.44	1.03	0.00	0.71
$\pi_1$	Method	$\mathbf{TL}$	RC	PR	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.47	0.01	0.11	0.32	0.02	0.22	0.27	0.02	0.41
	Elastic Net	0.43	0.03	0.11	0.26	0.04	0.20	0.22	0.04	0.41
	Adaptive	0.61	0.01	0.11	0.42	0.02	0.22	0.36	0.02	0.41
	Relaxed	0.82	0.01	0.14	0.66	0.01	0.20	0.49	0.01	0.44
	MCP	0.67	0.00	0.16	0.59	0.00	0.27	0.57	0.00	0.54
0.3	SIS-SCAD	0.68	0.00	0.12	0.61	0.00	0.21	0.59	0.00	0.47
	Split-Lasso-10	0.41	0.20	0.17	0.23	0.23	0.31	0.18	0.22	0.55
	Split-EN-10	0.41	0.27	0.14	0.23	0.33	0.28	0.18	0.33	0.51
	RGLM-100	0.42	_	_	0.32	_	_	0.31	_	_
	RF-500	0.42	_	_	0.34	_	_	0.33	_	_
	XGB	0.99	0.01	0.36	1.02	0.00	0.46	1.02	0.00	0.76
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.39	0.01	0.10	0.28	0.01	0.22	0.25	0.01	0.38
	Elastic Net	0.35	0.03	0.10	0.23	0.04	0.21	0.19	0.04	0.39
	Adaptive	0.49	0.01	0.10	0.39	0.01	0.22	0.38	0.01	0.40
	Relaxed	0.63	0.01	0.09	0.59	0.01	0.23	0.68	0.01	0.39
	MCP	0.56	0.00	0.23	0.61	0.00	0.30	0.51	0.00	0.53
0.2	SIS-SCAD	0.60	0.00	0.15	0.56	0.00	0.27	0.53	0.00	0.41
	Split-Lasso-10	0.33	0.26	0.20	0.21	0.28	0.36	0.18	0.24	0.60
	Split-EN-10	0.32	0.36	0.17	0.20	0.40	0.32	0.17	0.35	0.56
	RGLM-100	0.35	—	_	0.29	_	—	0.28	_	—
	RF-500	0.35	_	_	0.29	_	_	0.28	_	—
	XGB	1.01	0.00	0.34	1.01	0.00	0.59	1.01	0.00	0.72

Table 17: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho=$  0.8, n= 50, p= 1,000.

		(	$oldsymbol{\zeta}=0.1$	L	(	$\zeta = 0.2$	2		$\zeta = 0.$	4
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	1.01	0.06	0.19	0.79	0.05	0.26	0.62	0.05	0.45
	Elastic Net	0.99	0.07	0.18	0.74	0.07	0.26	0.55	0.07	0.45
	Adaptive	1.09	0.05	0.19	0.89	0.05	0.25	0.71	0.05	0.45
	Relaxed	1.04	0.05	0.19	0.84	0.05	0.28	0.72	0.05	0.45
	MCP	1.10	0.03	0.21	0.97	0.02	0.27	0.86	0.01	0.44
0.4	SIS-SCAD	1.16	0.01	0.25	1.12	0.01	0.28	1.05	0.01	0.44
	Split-Lasso-10	0.92	0.30	0.16	0.63	0.41	0.24	0.40	0.41	0.44
	Split-EN-10	0.92	0.32	0.15	0.62	0.46	0.23	0.39	0.49	0.43
	<b>RGLM-100</b>	0.92	—	_	0.72	—	_	0.64	—	—
	RF-500	0.99	—	—	0.84	_	—	0.78	—	—
	XGB	1.17	0.05	0.28	1.07	0.03	0.38	1.01	0.03	0.55
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.95	0.06	0.19	0.72	0.05	0.25	0.59	0.04	0.44
	Elastic Net	0.93	0.08	0.19	0.68	0.07	0.25	0.52	0.07	0.43
	Adaptive	1.01	0.06	0.19	0.82	0.05	0.25	0.70	0.04	0.44
	Relaxed	0.98	0.06	0.19	0.79	0.05	0.26	0.69	0.04	0.44
	MCP	1.03	0.02	0.20	0.89	0.01	0.26	0.83	0.01	0.43
0.3	SIS-SCAD	1.09	0.01	0.16	1.01	0.01	0.28	0.96	0.01	0.46
	Split-Lasso-10	0.84	0.31	0.16	0.58	0.39	0.25	0.37	0.41	0.45
	Split-EN-10	0.84	0.34	0.15	0.57	0.46	0.24	0.37	0.48	0.44
	RF-500	0.91	_	_	0.77	—	_	0.71	—	—
	RGLM-100	0.84	_	_	0.66	—	_	0.59	—	—
	XGB	1.10	0.05	0.28	1.00	0.03	0.35	0.96	0.02	0.58
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.79	0.05	0.17	0.62	0.04	0.26	0.52	0.04	0.42
	Elastic Net	0.77	0.06	0.17	0.58	0.06	0.25	0.46	0.06	0.44
	Adaptive	0.89	0.04	0.17	0.74	0.04	0.25	0.65	0.04	0.43
	Relaxed	0.83	0.04	0.17	0.64	0.04	0.26	0.59	0.04	0.42
	MCP	0.90	0.02	0.19	0.79	0.01	0.30	0.79	0.01	0.40
0.2	SIS-SCAD	1.00	0.01	0.17	0.88	0.01	0.25	0.81	_	0.38
	Split-Lasso-10	0.71	0.31	0.15	0.48	0.38	0.26	0.33	0.38	0.48
	Split-EN-10	0.71	0.33	0.14	0.47	0.45	0.25	0.32	0.46	0.46
	RF-500	0.76	_	_	0.64	_	_	0.58	_	_
	RGLM-100	0.72	_	_	0.56	_	_	0.51	_	_
	XGB	0.99	0.04	0.32	0.93	0.02	0.39	0.87	0.02	0.59

Table 18: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho = 0.2$ , n = 100, p = 1,000.

		(	$\zeta = 0.1$	L	(	$\zeta = 0.2$	2		$\zeta=0.4$	4
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.70	0.04	0.15	0.47	0.04	0.24	0.37	0.04	0.42
	Elastic Net	0.67	0.06	0.15	0.43	0.06	0.24	0.31	0.06	0.41
	Adaptive	0.76	0.04	0.15	0.56	0.04	0.25	0.45	0.04	0.42
	Relaxed	0.75	0.04	0.18	0.55	0.04	0.24	0.48	0.03	0.42
	MCP	0.79	0.01	0.14	0.64	0.01	0.24	0.59	0.01	0.37
0.4	SIS-SCAD	0.84	0.01	0.14	0.71	0.01	0.26	0.67	0.01	0.43
	Split-Lasso-10	0.62	0.25	0.15	0.36	0.33	0.25	0.24	0.33	0.46
	Split-EN-10	0.61	0.28	0.14	0.36	0.40	0.24	0.23	0.41	0.45
	RF-500	0.66	_	_	0.54	_	_	0.51	_	_
	RGLM-100	0.62	_	_	0.47	_	_	0.44	_	_
	XGB	0.90	0.04	0.31	0.78	0.03	0.38	0.77	0.02	0.56
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.62	0.04	0.15	0.44	0.03	0.23	0.36	0.03	0.42
	Elastic Net	0.60	0.06	0.15	0.39	0.06	0.24	0.30	0.06	0.42
	Adaptive	0.69	0.04	0.15	0.52	0.03	0.23	0.45	0.03	0.42
	Relaxed	0.69	0.04	0.15	0.49	0.03	0.23	0.52	0.03	0.43
	MCP	0.79	0.01	0.16	0.64	0.01	0.26	0.65	0.01	0.43
0.3	SIS-SCAD	0.85	0.01	0.14	0.66	0.01	0.26	0.66	0.00	0.40
	Split-Lasso-10	0.56	0.27	0.15	0.32	0.34	0.26	0.22	0.34	0.49
	Split-EN-10	0.56	0.33	0.14	0.32	0.40	0.25	0.22	0.43	0.48
	RF-500	0.60	_	_	0.49	_	_	0.46	_	_
	RGLM-100	0.56	_	_	0.43	_	—	0.41	_	_
	XGB	0.89	0.04	0.32	0.78	0.03	0.41	0.80	0.02	0.62
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.53	0.03	0.14	0.37	0.03	0.24	0.31	0.03	0.44
	Elastic Net	0.51	0.05	0.14	0.33	0.05	0.23	0.26	0.06	0.43
	Adaptive	0.60	0.03	0.14	0.47	0.03	0.24	0.39	0.03	0.44
	Relaxed	0.59	0.03	0.13	0.89	0.03	0.25	0.42	0.03	0.43
	MCP	0.70	0.01	0.14	0.64	0.00	0.32	0.61	0.00	0.47
0.2	SIS-SCAD	0.78	0.00	0.12	0.64	0.01	0.30	0.65	0.00	0.46
	Split-Lasso	0.46	0.30	0.16	0.27	0.36	0.30	0.19	0.34	0.52
	Split-EN	0.46	0.35	0.16	0.27	0.45	0.27	0.19	0.42	0.50
	RF-500	0.50	_	_	0.41	_	_	0.39	_	_
	<b>RGLM-100</b>	0.48	_	_	0.37	_	_	0.35	_	_
	XGB	0.80	0.03	0.35	0.71	0.02	0.49	0.79	0.01	0.65

Table 19: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho = 0.5$ , n = 100, p = 1,000.

		(	$\zeta=0.1$	L	(	$\zeta = 0.2$	2		$\zeta = 0.$	4
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.47	0.03	0.14	0.30	0.02	0.21	0.24	0.02	0.42
	Elastic Net	0.44	0.05	0.14	0.26	0.05	0.22	0.20	0.06	0.42
	Adaptive	0.54	0.03	0.14	0.37	0.02	0.21	0.30	0.03	0.42
	Relaxed	0.53	0.02	0.13	0.46	0.02	0.22	0.49	0.02	0.41
	MCP	0.56	0.01	0.15	0.47	0.01	0.26	0.44	0.00	0.40
0.4	SIS-SCAD	0.57	0.00	0.10	0.48	0.00	0.21	0.43	0.00	0.40
	Split-Lasso-10	0.42	0.21	0.16	0.23	0.26	0.26	0.16	0.26	0.49
	Split-EN-10	0.42	0.26	0.15	0.23	0.35	0.24	0.16	0.38	0.46
	RF-500	0.43	—	—	0.33	—	—	0.31	—	—
	RGLM-100	0.43	_	_	0.30	_	_	0.28	_	—
	XGB	0.82	0.03	0.36	0.71	0.02	0.57	0.68	0.01	0.70
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.43	0.02	0.11	0.27	0.02	0.22	0.21	0.02	0.38
	Elastic Net	0.40	0.04	0.11	0.24	0.05	0.21	0.18	0.05	0.41
	Adaptive	0.50	0.02	0.11	0.34	0.02	0.22	0.27	0.02	0.39
	Relaxed	0.48	0.02	0.10	0.37	0.02	0.20	0.36	0.02	0.37
	MCP	0.59	0.00	0.08	0.54	0.00	0.27	0.50	0.00	0.48
0.3	SIS-SCAD	0.59	0.00	0.14	0.54	0.00	0.21	0.50	0.00	0.38
	Split-Lasso-10	0.38	0.24	0.15	0.21	0.33	0.31	0.15	0.30	0.55
	Split-EN-10	0.38	0.31	0.15	0.21	0.42	0.28	0.15	0.42	0.52
	RF-500	0.39	_	_	0.30	—	_	0.28	—	—
	RGLM-100	0.39	—	—	0.28	—	—	0.26	—	—
	XGB	0.75	0.03	0.41	0.67	0.02	0.48	0.68	0.01	0.66
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.35	0.02	0.13	0.24	0.02	0.21	0.18	0.02	0.42
	Elastic Net	0.33	0.04	0.12	0.20	0.05	0.21	0.16	0.05	0.40
	Adaptive	0.44	0.02	0.12	0.31	0.02	0.20	0.25	0.02	0.41
	Relaxed	0.51	0.02	0.12	0.37	0.02	0.19	0.34	0.02	0.40
	MCP	0.55	0.00	0.20	0.50	0.00	0.17	0.48	0.00	0.51
0.2	SIS-SCAD	0.57	0.00	0.14	0.53	0.00	0.13	0.50	0.00	0.36
	Split-Lasso-10	0.32	0.31	0.18	0.18	0.42	0.36	0.13	0.34	0.60
	Split-EN-10	0.31	0.37	0.16	0.18	0.53	0.33	0.14	0.45	0.56
	RF-500	0.33	_	_	0.26	_	_	0.24	_	_
	RGLM-100	0.33	_	_	0.24	_	_	0.22	_	_
	XGB	0.76	0.03	0.45	0.78	0.01	0.45	0.72	0.01	0.62

Table 20: Mean misclassification rate, sensitivity and specificity for Scenario 1,  $\rho=0.8,\,n=100,\,p=1,000.$ 

		(	$\zeta = 0.1$	L	G	$\zeta = 0.2$	2		$\zeta=0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SI
	Lasso	0.19	0.71	0.88	0.13	0.81	0.90	0.11	0.85	0.92
	Elastic Net	0.18	0.73	0.89	0.12	0.83	0.92	0.09	0.87	0.9
	Adaptive	0.22	0.59	0.91	0.15	0.74	0.92	0.12	0.80	0.9
	Relaxed	0.20	0.70	0.87	0.14	0.80	0.89	0.12	0.85	0.9
	MCP	0.23	0.66	0.85	0.17	0.74	0.88	0.16	0.76	0.8
0.4	SIS-SCAD	0.23	0.67	0.84	0.20	0.72	0.86	0.20	0.71	0.8
	Split-Lasso-10	0.16	0.76	0.90	0.10	0.86	0.93	0.06	0.90	0.9
	Split-EN-10	0.16	0.76	0.90	0.10	0.86	0.93	0.06	0.91	0.9
	RGLM-100	0.16	0.76	0.90	0.10	0.85	0.94	0.07	0.88	0.9
	RF-500	0.17	0.71	0.92	0.10	0.81	0.95	0.07	0.86	0.9
	XGB	0.23	0.69	0.82	0.20	0.72	0.85	0.19	0.72	0.86
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\overline{\mathbf{MR}}$	$\mathbf{SE}$	SI
	Lasso	0.18	0.56	0.93	0.13	0.69	0.95	0.10	0.78	0.9
	Elastic Net	0.17	0.56	0.94	0.12	0.72	0.95	0.09	0.81	0.9
	Adaptive	0.21	0.36	0.97	0.17	0.50	0.97	0.12	0.67	0.9
	Relaxed	0.18	0.56	0.92	0.14	0.70	0.93	0.11	0.80	0.92
	MCP	0.22	0.48	0.92	0.18	0.57	0.93	0.17	0.64	0.9
0.3	SIS-SCAD	0.22	0.46	0.92	0.19	0.52	0.92	0.18	0.58	0.9
	Split-Lasso-10	0.16	0.60	0.95	0.10	0.75	0.97	0.07	0.85	0.9
	Split-EN-10	0.15	0.60	0.95	0.10	0.75	0.97	0.06	0.86	0.9'
	RGLM-100	0.15	0.61	0.95	0.10	0.73	0.97	0.07	0.80	0.9
	RF-500	0.17	0.48	0.98	0.12	0.63	0.99	0.08	0.74	0.9
	XGB	0.22	0.57	0.87	0.19	0.58	0.91	0.19	0.63	0.89
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SI
	Lasso	0.15	0.43	0.96	0.11	0.62	0.96	0.09	0.65	0.9'
	Elastic Net	0.14	0.44	0.97	0.10	0.65	0.97	0.08	0.70	0.93
	Adaptive	0.18	0.15	0.99	0.14	0.35	0.98	0.13	0.42	0.99
	Relaxed	0.15	0.46	0.95	0.12	0.63	0.94	0.10	0.67	0.9
	MCP	0.17	0.35	0.95	0.15	0.48	0.94	0.15	0.39	0.9
0.2	SIS-SCAD	0.18	0.25	0.97	0.16	0.36	0.96	0.16	0.31	0.9'
	Split-Lasso-10	0.13	0.48	0.97	0.08	0.69	0.98	0.06	0.74	0.9
	Split-EN-10	0.13	0.46	0.97	0.08	0.69	0.98	0.06	0.74	0.99
	RGLM-100	0.13	0.46	0.98	0.09	0.64	0.98	0.08	0.63	0.9
		0.15	0.91	0.00	0 10	0 59	0.00	0.00	054	1 0
	RF-500	0.15	0.51	0.99	0.10	0.32	0.99	0.09	0.34	1.0

Table 21: Mean misclassification rate, sensitivity and specificity for Scenario 2b,  $\rho_1 = 0.5$ ,  $\rho_2 = 0.2$ , n = 50, p = 1,000.

		Ģ	$\zeta = 0.1$	L	G	$\zeta = 0.2$	2		$oldsymbol{\zeta}=0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SI
	Lasso	0.13	0.80	0.91	0.10	0.85	0.94	0.08	0.88	0.9
	Elastic Net	0.12	0.81	0.92	0.09	0.86	0.95	0.07	0.90	0.9
	Adaptive	0.13	0.77	0.93	0.10	0.82	0.96	0.08	0.86	0.9
	Relaxed	0.13	0.81	0.90	0.10	0.86	0.92	0.10	0.87	0.9
	MCP	0.16	0.76	0.89	0.14	0.78	0.91	0.14	0.81	0.9
0.4	SIS-SCAD	0.15	0.76	0.90	0.13	0.77	0.93	0.12	0.83	0.9
	Split-Lasso-10	0.11	0.84	0.92	0.07	0.90	0.96	0.05	0.94	0.9
	Split-EN-10	0.11	0.84	0.93	0.07	0.90	0.96	0.05	0.94	0.9
	RGLM-100	0.10	0.86	0.93	0.06	0.91	0.96	0.04	0.94	0.9
	RF-500	0.10	0.85	0.94	0.06	0.90	0.97	0.04	0.94	0.9
	XGB	0.16	0.79	0.88	0.14	0.81	0.89	0.15	0.81	0.88
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SP	$\overline{\mathbf{MR}}$	$\mathbf{SE}$	SI
	Lasso	0.13	0.71	0.94	0.09	0.79	0.95	0.08	0.85	0.9
	Elastic Net	0.12	0.71	0.95	0.08	0.81	0.96	0.07	0.87	0.9
	Adaptive	0.15	0.57	0.97	0.10	0.73	0.97	0.09	0.79	0.9
	Relaxed	0.13	0.73	0.92	0.10	0.81	0.94	0.09	0.86	0.9
	MCP	0.16	0.67	0.92	0.13	0.71	0.94	0.13	0.75	0.9
0.3	SIS-SCAD	0.15	0.63	0.94	0.13	0.69	0.95	0.12	0.71	0.9
	Split-Lasso-10	0.11	0.76	0.95	0.06	0.86	0.97	0.05	0.91	0.9
	Split-EN-10	0.11	0.75	0.95	0.06	0.86	0.97	0.05	0.91	0.9
	RGLM-100	0.10	0.80	0.95	0.06	0.88	0.97	0.04	0.92	0.9
	RF-500	0.09	0.78	0.96	0.05	0.87	0.98	0.04	0.90	0.9
	XGB	0.14	0.76	0.90	0.13	0.77	0.91	0.13	0.78	0.90
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SI
	Lasso	0.11	0.54	0.97	0.09	0.67	0.97	0.07	0.71	0.9
	Elastic Net	0.11	0.56	0.98	0.08	0.68	0.98	0.06	0.74	0.9
	Adaptive	0.14	0.34	0.99	0.12	0.47	0.98	0.10	0.53	0.9
	Relaxed	0.11	0.60	0.96	0.09	0.73	0.95	0.08	0.74	0.9
	MCP	0.13	0.52	0.96	0.12	0.60	0.95	0.11	0.59	0.9
0.2	SIS-SCAD	0.13	0.48	0.97	0.12	0.50	0.97	0.12	0.50	0.9
	Split-Lasso-10	0.09	0.62	0.98	0.07	0.74	0.98	0.05	0.78	0.9
	Split-EN-10	0.09	0.60	0.98	0.07	0.74	0.98	0.05	0.79	0.9
	RGLM-100	0.08	0.68	0.98	0.06	0.78	0.98	0.05	0.80	0.9
	<b>RE 500</b>	0.08	0.66	0.98	0.05	0.78	0.99	0.05	0.79	1.0
	111-500	0.00	0.00	0.00	0.00	00	0.00	0.00	00	1.0

Table 22: Mean misclassification rate, sensitivity and specificity for Scenario 2b,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.2$ , n = 50, p = 1,000.

			<b>5</b> = 0.1		_ (	<b>5</b> = 0.2	2		$\zeta = 0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SF
	Lasso	0.21	0.72	0.85	0.13	0.82	0.90	0.10	0.86	0.94
	Elastic Net	0.20	0.72	0.85	0.12	0.82	0.91	0.09	0.87	0.94
	Adaptive	0.22	0.62	0.89	0.14	0.76	0.93	0.11	0.81	0.95
	Relaxed	0.21	0.72	0.84	0.14	0.81	0.90	0.10	0.86	0.92
	MCP	0.22	0.70	0.84	0.16	0.77	0.88	0.14	0.79	0.91
0.4	SIS-SCAD	0.21	0.70	0.85	0.15	0.78	0.91	0.13	0.78	0.93
	Split-Lasso-10	0.19	0.75	0.86	0.11	0.84	0.92	0.07	0.90	0.95
	Split-EN-10	0.19	0.74	0.86	0.11	0.84	0.92	0.07	0.90	0.95
	<b>RGLM-100</b>	0.19	0.76	0.85	0.10	0.85	0.92	0.06	0.90	0.96
	RF-500	0.19	0.74	0.85	0.11	0.84	0.93	0.06	0.90	0.9'
	XGB	0.23	0.70	0.81	0.15	0.79	0.88	0.14	0.81	0.89
$\pi_1$	Method	MR	SE	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SF
	Lasso	0.19	0.60	0.91	0.13	0.74	0.93	0.09	0.81	0.95
	Elastic Net	0.19	0.61	0.91	0.12	0.75	0.93	0.08	0.83	0.95
	Adaptive	0.23	0.37	0.96	0.15	0.58	0.96	0.11	0.72	0.9'
	Relaxed	0.19	0.61	0.90	0.13	0.75	0.92	0.10	0.82	0.94
	MCP	0.20	0.57	0.90	0.15	0.68	0.92	0.13	0.71	0.94
0.3	SIS-SCAD	0.21	0.51	0.93	0.15	0.66	0.94	0.13	0.69	0.95
	Split-Lasso-10	0.18	0.63	0.91	0.11	0.78	0.94	0.07	0.87	0.9
	Split-EN-10	0.18	0.64	0.91	0.11	0.77	0.94	0.07	0.87	0.90
	<b>RGLM-100</b>	0.17	0.68	0.89	0.10	0.80	0.94	0.06	0.87	0.9'
	RF-500	0.18	0.63	0.91	0.10	0.78	0.95	0.06	0.86	0.98
	XGB	0.21	0.65	0.86	0.14	0.75	0.90	0.13	0.77	0.91
$\pi_1$	Method	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SF
	Lasso	0.16	0.45	0.95	0.10	0.61	0.97	0.08	0.73	0.96
	Elastic Net	0.16	0.45	0.95	0.10	0.63	0.97	0.08	0.75	0.9'
	Adaptive	0.19	0.18	0.99	0.13	0.39	0.99	0.11	0.52	0.98
	Relaxed	0.16	0.46	0.94	0.11	0.64	0.96	0.09	0.77	0.94
	MCP	0.18	0.41	0.94	0.13	0.52	0.96	0.12	0.58	0.95
0.2	SIS-SCAD	0.18	0.29	0.97	0.13	0.47	0.97	0.12	0.49	0.9'
	Split-Lasso-10	0.15	0.49	0.95	0.09	0.67	0.97	0.06	0.79	0.9'
	Split-EN-10	0.15	0.49	0.95	0.09	0.67	0.97	0.06	0.79	0.9'
	RGLM-100	0.15	0.56	0.94	0.08	0.69	0.97	0.06	0.77	0.98
	RF-500	0.15	0.50	0.95	0.09	0.66	0.98	0.06	0.76	0.99

Table 23: Mean misclassification rate, sensitivity and specificity for Scenario 2b,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.5$ , n = 50, p = 1,000.

		Ģ	$\zeta = 0.1$	-	Ç	$\zeta = 0.2$	2		$\zeta=0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SF
	Lasso	0.16	0.77	0.89	0.11	0.82	0.93	0.09	0.87	0.94
	Elastic Net	0.16	0.77	0.89	0.10	0.84	0.93	0.08	0.89	0.95
	Adaptive	0.16	0.73	0.91	0.12	0.78	0.94	0.09	0.85	0.94
	Relaxed	0.17	0.76	0.88	0.12	0.82	0.92	0.10	0.87	0.93
	MCP	0.19	0.73	0.87	0.15	0.76	0.91	0.13	0.81	0.9
0.4	SIS-SCAD	0.19	0.73	0.87	0.16	0.74	0.90	0.15	0.79	0.89
	Split-Lasso-10	0.14	0.79	0.90	0.09	0.86	0.95	0.06	0.92	0.9
	Split-EN-10	0.14	0.79	0.90	0.09	0.86	0.95	0.05	0.92	0.9
	RGLM-100	0.14	0.80	0.90	0.09	0.86	0.95	0.06	0.90	0.9'
	RF-500	0.15	0.75	0.92	0.09	0.81	0.97	0.06	0.87	0.93
	XGB	0.21	0.71	0.85	0.18	0.73	0.88	0.15	0.77	0.89
$\pi_1$	Method	MR	SE	SP	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	MR	$\mathbf{SE}$	SI
	Lasso	0.15	0.69	0.92	0.10	0.77	0.95	0.08	0.82	0.9
	Elastic Net	0.14	0.69	0.93	0.10	0.79	0.95	0.07	0.84	0.9
	Adaptive	0.16	0.59	0.95	0.11	0.70	0.96	0.09	0.76	0.9'
	Relaxed	0.15	0.69	0.92	0.11	0.77	0.94	0.09	0.82	0.9
	MCP	0.17	0.64	0.91	0.14	0.69	0.93	0.14	0.69	0.93
0.3	SIS-SCAD	0.17	0.63	0.91	0.15	0.66	0.93	0.15	0.68	0.93
	Split-Lasso-10	0.13	0.72	0.93	0.08	0.83	0.96	0.05	0.88	0.9'
	Split-EN-10	0.13	0.72	0.93	0.08	0.83	0.96	0.05	0.88	0.9'
	RGLM-100	0.13	0.73	0.93	0.08	0.81	0.97	0.06	0.83	0.98
	RF-500	0.14	0.63	0.96	0.09	0.74	0.98	0.07	0.77	0.99
	XGB	0.18	0.64	0.89	0.16	0.68	0.91	0.15	0.68	0.92
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SF
	Lasso	0.12	0.55	0.96	0.09	0.66	0.97	0.07	0.73	0.98
	Elastic Net	0.12	0.55	0.96	0.08	0.68	0.98	0.06	0.75	0.98
	Adaptive	0.15	0.34	0.98	0.12	0.48	0.99	0.09	0.59	0.99
	Relaxed	0.13	0.55	0.95	0.10	0.67	0.96	0.08	0.74	0.9'
	MCP	0.15	0.45	0.95	0.13	0.52	0.95	0.13	0.54	0.9
0.2	SIS-SCAD	0.15	0.45	0.95	0.14	0.44	0.96	0.14	0.48	0.9
	Split-Lasso-10	0.11	0.59	0.97	0.07	0.74	0.98	0.05	0.81	0.99
	Split-EN-10	0.11	0.58	0.97	0.07	0.73	0.98	0.05	0.81	0.99
	RGLM-100	0.11	0.61	0.96	0.07	0.70	0.99	0.06	0.73	0.99
			0.44	0.00	0.00		1 00	0.00	0.00	1 0
	RF-500	0.12	0.44	0.99	0.09	0.55	1.00	0.08	0.62	1.00

Table 24: Mean misclassification rate, sensitivity and specificity for Scenario 2b,  $\rho_1 = 0.5$ ,  $\rho_2 = 0.2$ , n = 100, p = 1,000.

		(	$\zeta = 0.1$		Ç	$\zeta = 0.2$	2		$\zeta = 0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SI
	Lasso	0.11	0.83	0.92	0.08	0.89	0.94	0.06	0.93	0.9
	Elastic Net	0.11	0.84	0.93	0.07	0.90	0.95	0.05	0.94	0.9
	Adaptive	0.11	0.82	0.94	0.08	0.88	0.95	0.06	0.92	0.9
	Relaxed	0.12	0.83	0.92	0.09	0.89	0.93	0.07	0.92	0.9
	MCP	0.14	0.80	0.90	0.11	0.84	0.92	0.10	0.87	0.9
0.4	SIS-SCAD	0.13	0.81	0.91	0.10	0.85	0.92	0.09	0.88	0.9
	Split-Lasso-10	0.09	0.86	0.93	0.06	0.92	0.96	0.03	0.96	0.9
	Split-EN-10	0.10	0.86	0.93	0.06	0.92	0.96	0.03	0.96	0.9
	RGLM-100	0.09	0.87	0.93	0.05	0.93	0.96	0.03	0.96	0.9
	RF-500	0.09	0.86	0.94	0.05	0.92	0.97	0.03	0.96	0.9
	XGB	0.13	0.82	0.90	0.10	0.86	0.92	0.10	0.87	0.92
$\pi_1$	Method	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\overline{\mathbf{MR}}$	$\mathbf{SE}$	$\mathbf{SP}$	$\overline{\mathbf{MR}}$	$\mathbf{SE}$	SI
	Lasso	0.10	0.79	0.94	0.07	0.84	0.96	0.06	0.90	0.9
	Elastic Net	0.10	0.79	0.94	0.07	0.85	0.97	0.05	0.92	0.9
	Adaptive	0.10	0.74	0.96	0.08	0.80	0.97	0.06	0.88	0.9
	Relaxed	0.11	0.79	0.93	0.08	0.84	0.95	0.07	0.90	0.9
	MCP	0.14	0.73	0.92	0.12	0.75	0.94	0.11	0.80	0.9
0.3	SIS-SCAD	0.12	0.73	0.94	0.11	0.73	0.96	0.10	0.80	0.9
	Split-Lasso-10	0.09	0.83	0.95	0.05	0.89	0.97	0.03	0.95	0.9
	Split-EN-10	0.09	0.83	0.95	0.05	0.89	0.97	0.03	0.95	0.9
	RGLM-100	0.08	0.84	0.95	0.05	0.90	0.97	0.03	0.94	0.9
	RF-500	0.08	0.83	0.96	0.05	0.88	0.98	0.03	0.94	0.9
	XGB	0.12	0.79	0.92	0.10	0.81	0.93	0.09	0.84	0.94
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SI
	Lasso	0.09	0.68	0.97	0.07	0.75	0.98	0.05	0.83	0.9
	Elastic Net	0.09	0.68	0.97	0.06	0.76	0.98	0.04	0.84	0.9
	Adaptive	0.11	0.54	0.98	0.07	0.67	0.99	0.06	0.76	0.9
	Relaxed	0.10	0.69	0.96	0.07	0.76	0.97	0.06	0.84	0.9
	MCP	0.12	0.61	0.95	0.11	0.60	0.96	0.10	0.65	0.9
0.2	SIS-SCAD	0.13	0.47	0.98	0.12	0.44	0.98	0.11	0.54	0.9
	Split-Lasso-10	0.08	0.74	0.97	0.05	0.81	0.98	0.03	0.89	0.9
	Split-EN-10	0.08	0.73	0.97	0.05	0.81	0.99	0.03	0.89	0.9
	RGLM-100	0.07	0.76	0.97	0.04	0.84	0.99	0.03	0.89	0.9
				0.00	0.04	0.01	0.00	0.00	0.00	1 0
	RF-500	0.07	0.75	0.98	0.04	0.81	0.99	0.03	0.86	1.0

Table 25: Mean misclassification rate, sensitivity and specificity for Scenario 2b,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.2$ , n = 100, p = 1,000.

		Ģ	$\zeta = 0.1$	L	Ç	$\zeta = 0.2$	2		$\zeta=0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SI
	Lasso	0.18	0.75	0.86	0.11	0.84	0.92	0.08	0.89	$0.9^{-1}$
	Elastic Net	0.18	0.75	0.87	0.11	0.85	0.92	0.07	0.90	0.9
	Adaptive	0.19	0.70	0.89	0.11	0.83	0.93	0.08	0.88	0.9
	Relaxed	0.19	0.75	0.85	0.12	0.83	0.92	0.09	0.89	0.9
	MCP	0.20	0.72	0.85	0.13	0.81	0.90	0.11	0.85	0.9
0.4	SIS-SCAD	0.20	0.74	0.85	0.13	0.82	0.91	0.10	0.85	0.9
	Split-Lasso-10	0.18	0.76	0.87	0.10	0.86	0.93	0.06	0.92	0.9
	Split-EN-10	0.18	0.76	0.87	0.10	0.86	0.93	0.06	0.92	0.9
	RGLM-100	0.17	0.77	0.86	0.09	0.87	0.93	0.05	0.92	0.9
	RF-500	0.18	0.76	0.86	0.10	0.86	0.94	0.05	0.92	0.9
	XGB	0.21	0.74	0.83	0.13	0.82	0.90	0.10	0.86	0.92
$\pi_1$	Method	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\overline{\mathbf{MR}}$	SE	SP	$\overline{\mathbf{MR}}$	$\mathbf{SE}$	SI
	Lasso	0.17	0.66	0.91	0.10	0.80	0.94	0.07	0.85	0.9
	Elastic Net	0.17	0.66	0.91	0.10	0.81	0.94	0.07	0.87	0.9
	Adaptive	0.18	0.56	0.94	0.10	0.75	0.96	0.08	0.81	0.9'
	Relaxed	0.17	0.67	0.90	0.11	0.79	0.93	0.08	0.86	0.9
	MCP	0.19	0.63	0.90	0.13	0.75	0.92	0.12	0.76	0.9
0.3	SIS-SCAD	0.18	0.62	0.91	0.12	0.74	0.94	0.11	0.73	0.9
	Split-Lasso-10	0.16	0.68	0.91	0.09	0.84	0.94	0.05	0.90	0.9
	Split-EN-10	0.16	0.68	0.91	0.09	0.84	0.94	0.05	0.90	0.9'
	RGLM-100	0.16	0.72	0.90	0.09	0.84	0.94	0.05	0.90	0.9'
	RF-500	0.16	0.69	0.90	0.09	0.82	0.95	0.05	0.88	0.93
	XGB	0.19	0.68	0.87	0.12	0.79	0.92	0.10	0.81	0.94
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SI
	Lasso	0.14	0.53	0.95	0.09	0.70	0.96	0.06	0.78	0.93
	Elastic Net	0.14	0.53	0.95	0.09	0.71	0.96	0.06	0.79	0.93
	Adaptive	0.17	0.27	0.99	0.11	0.54	0.98	0.07	0.70	0.99
	Relaxed	0.14	0.54	0.95	0.09	0.71	0.96	0.07	0.79	0.9'
	MCP	0.16	0.51	0.94	0.12	0.60	0.95	0.11	0.61	0.9
0.2	SIS-SCAD	0.16	0.39	0.96	0.12	0.51	0.97	0.12	0.46	0.99
	Split-Lasso-10	0.14	0.56	0.95	0.08	0.75	0.97	0.05	0.84	0.93
	Split-EN-10	0.14	0.55	0.95	0.08	0.75	0.97	0.05	0.83	0.9
	RGLM-100	0.13	0.64	0.93	0.07	0.77	0.96	0.04	0.83	0.9
							~ ~ -	~ ~ ~		
	RF-500	0.14	0.57	0.95	0.08	0.74	0.97	0.05	0.80	0.9

Table 26: Mean misclassification rate, sensitivity and specificity for Scenario 2b,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.5$ , n = 100, p = 1,000.

 $\zeta = 0.1$  $\zeta = 0.2$  $\zeta = 0.4$ Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\pi_1$ Lasso 0.840.090.610.610.070.720.500.050.850.79Elastic Net 0.150.560.540.130.680.400.090.80Adaptive 0.950.090.630.720.070.730.620.050.84Relaxed 0.950.08 0.690.980.06 0.790.840.040.89MCP 0.970.030.660.780.020.810.720.010.890.4SIS-SCAD 1.020.020.810.840.010.910.860.010.950.38Split-Lasso-10 0.710.450.440.460.470.310.340.660.570.31Split-EN-10 0.710.580.330.440.430.450.64**RGLM-100** 0.73— \_ 0.58— — 0.52— \_ \_ \_ \_ RF-500 0.860.70\_ 0.61\_ \_\_\_\_ XGB 1.030.050.760.960.030.880.930.020.93Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\pi_1$ Lasso 0.820.080.560.590.070.740.470.040.82Elastic Net 0.560.520.770.140.120.680.390.080.79Adaptive 0.890.090.580.730.070.720.610.040.81Relaxed 1.270.080.700.870.060.820.890.040.88MCP 0.940.020.680.790.010.830.730.010.900.3SIS-SCAD 0.950.020.840.830.010.950.790.010.94Split-Lasso-10 0.700.460.370.430.430.490.300.320.65Split-EN-10 0.530.430.550.300.680.360.440.440.63**RGLM-100** 0.530.480.69— \_ — \_ — \_ **RF-500** 0.80 \_ \_ 0.64\_ \_ 0.56\_ \_\_\_\_ XGB 1.050.040.781.000.020.900.940.010.92Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\pi_1$ Lasso 0.670.070.550.520.060.660.440.03 0.82Elastic Net 0.640.130.520.450.110.650.360.070.79Adaptive 0.800.070.550.660.060.660.580.040.82Relaxed 0.87 0.070.610.850.050.760.640.030.86MCP 0.870.020.730.760.010.870.710.000.930.2SIS-SCAD 0.790.020.920.720.010.970.730.000.970.280.29Split-Lasso-10 0.580.440.340.370.400.440.64Split-EN-10 0.590.480.360.370.490.430.280.400.62**RGLM-100** 0.59— 0.46— 0.42\_ \_ \_ \_ **RF-500** 0.540.67\_ \_ \_\_\_ \_ 0.47\_ \_\_\_\_ XGB 0.940.040.910.020.890.930.840.010.88

Table 27: Mean misclassification rate, sensitivity and specificity for Scenario 2,  $\rho_1 = 0.5$ ,  $\rho_2 = 0.2$ , n = 50, p = 1,000.

 $\zeta = 0.1$  $\zeta = 0.2$  $\zeta = 0.4$  $\mathbf{RC}$ Method  $\mathbf{TL}$  $\mathbf{PR}$  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\pi_1$ Lasso 0.630.080.570.460.050.570.400.030.66Elastic Net 0.590.170.570.400.130.630.340.080.73Adaptive 0.720.080.530.560.050.570.530.030.66Relaxed 1.050.070.721.210.840.790.040.790.02MCP 0.730.010.670.640.010.760.630.000.820.4SIS-SCAD 0.730.020.910.630.010.970.580.011.000.490.490.280.380.25Split-Lasso-10 0.310.410.250.550.25Split-EN-10 0.500.720.290.320.610.440.450.61RGLM-1000.50— 0.37— — 0.32— — — \_ \_ \_ RF-500 0.560.42\_ 0.37\_ \_\_\_\_ XGB 0.970.030.880.970.010.951.020.000.98Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\pi_1$ Lasso 0.610.070.510.440.040.540.380.030.67Elastic Net 0.570.540.160.390.120.610.320.080.73Adaptive 0.720.070.500.520.050.520.470.030.67Relaxed 0.870.060.681.040.040.790.730.020.84MCP 0.700.010.670.590.010.810.580.000.790.3SIS-SCAD 0.700.020.940.610.010.990.610.001.00Split-Lasso-10 0.480.490.280.300.370.400.240.270.57Split-EN-10 0.290.310.590.240.480.710.430.440.61**RGLM-100** 0.350.310.48— \_ — \_ — \_ **RF-500** 0.52\_ \_ 0.38\_ \_ 0.34\_ \_\_\_\_ 0.94XGB 0.030.890.960.010.961.020.000.96Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\pi_1$ Lasso 0.510.060.500.420.040.580.340.020.62Elastic Net 0.480.160.530.380.100.620.300.070.72Adaptive 0.510.650.060.500.560.040.560.020.62Relaxed 0.900.050.740.830.030.770.770.020.80MCP 0.600.010.800.540.010.880.510.000.900.2SIS-SCAD 0.640.010.940.580.010.980.550.001.000.420.320.260.25Split-Lasso-10 0.490.310.340.420.55Split-EN-10 0.430.690.310.310.540.440.260.420.59**RGLM-100** 0.400.32— 0.29\_ \_ \_ \_ \_ **RF-500** 0.43\_ \_ 0.33\_\_\_ \_ 0.29\_ \_\_\_\_ XGB 0.930.020.980.950.990.000.880.010.91

Table 28: Mean misclassification rate, sensitivity and specificity for Scenario 2,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.2$ , n = 50, p = 1,000.

 $\zeta = 0.1$  $\zeta = 0.2$  $\zeta = 0.4$  $\mathbf{RC}$ Method  $\mathbf{TL}$  $\mathbf{PR}$  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\pi_1$ Lasso 0.890.090.390.610.070.490.450.050.60Elastic Net 0.870.160.400.580.150.520.390.120.64Adaptive 1.000.090.380.710.070.490.570.050.60Relaxed 1.040.071.020.060.860.750.580.580.04MCP 0.940.020.510.720.010.570.620.010.670.4SIS-SCAD 0.97 0.030.760.700.020.890.630.010.970.820.250.520.33Split-Lasso-10 0.510.460.350.370.470.52Split-EN-10 0.830.590.270.630.320.330.550.45**RGLM-100** 0.83— 0.51— \_ 0.37— — \_ \_ \_ RF-500 0.86\_ 0.57\_ 0.42\_ \_\_\_\_ XGB 1.060.050.560.950.030.870.970.010.95Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\pi_1$ Lasso 0.830.080.360.590.060.490.430.040.55Elastic Net 0.820.560.150.400.140.500.380.110.59Adaptive 0.960.080.370.720.060.470.520.040.55Relaxed 0.870.070.920.060.880.040.700.510.57MCP 0.550.010.590.010.740.890.020.680.670.3SIS-SCAD 0.960.030.740.690.020.860.620.010.97Split-Lasso-10 0.790.420.300.500.430.330.320.340.44Split-EN-10 0.790.570.240.500.320.320.610.530.45**RGLM-100** 0.790.480.36— \_ — \_ — — **RF-500** 0.81\_ \_ 0.53\_ \_ 0.39\_ \_\_\_\_ XGB 1.060.050.650.940.030.860.970.010.96Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\pi_1$ Lasso 0.740.060.330.490.060.490.390.040.55Elastic Net 0.720.120.330.460.130.520.360.090.59Adaptive 0.520.860.060.340.620.060.480.040.58Relaxed 1.160.060.390.920.050.640.850.030.69MCP 0.800.020.560.600.010.770.560.000.810.2SIS-SCAD 0.800.020.660.650.010.860.590.910.010.250.29Split-Lasso-10 0.680.430.410.470.320.350.44Split-EN-10 0.680.520.250.410.640.290.290.520.44**RGLM-100** 0.69\_ 0.400.33\_ \_ — \_ \_ **RF-500** 0.340.69\_ \_ 0.43\_\_\_ \_ \_ \_\_\_\_ XGB 0.990.040.620.920.020.970.860.010.95

Table 29: Mean misclassification rate, sensitivity and specificity for Scenario 2,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.5$ , n = 50, p = 1,000.

			$oldsymbol{\zeta} = 0.1$	L	(	$\zeta=0.2$	2		$oldsymbol{\zeta} = 0.$	4
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.72	0.16	0.58	0.52	0.11	0.71	0.40	0.07	0.82
	Elastic Net	0.71	0.21	0.53	0.48	0.16	0.67	0.35	0.11	0.80
	Adaptive	0.79	0.15	0.59	0.61	0.11	0.71	0.48	0.07	0.82
	Relaxed	1.01	0.13	0.70	0.65	0.10	0.77	0.57	0.06	0.89
	MCP	0.81	0.06	0.73	0.67	0.04	0.83	0.59	0.02	0.91
0.4	SIS-SCAD	0.83	0.04	0.85	0.74	0.02	0.93	0.69	0.01	0.96
	Split-Lasso-10	0.64	0.66	0.25	0.39	0.61	0.38	0.27	0.47	0.62
	Split-EN-10	0.64	0.72	0.25	0.39	0.70	0.35	0.26	0.59	0.58
	RGLM-100	0.66	_	_	0.48	_	_	0.42	_	_
	RF-500	0.80	_	_	0.63	_	_	0.55	_	_
	XGB	0.95	0.11	0.72	0.85	0.06	0.89	0.78	0.03	0.95
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.66	0.15	0.58	0.48	0.10	0.73	0.39	0.06	0.82
	Elastic Net	0.65	0.20	0.56	0.44	0.15	0.68	0.34	0.11	0.80
	Adaptive	0.72	0.15	0.59	0.57	0.10	0.73	0.46	0.06	0.81
	Relaxed	0.71	0.14	0.62	0.63	0.09	0.78	0.55	0.06	0.88
	MCP	0.75	0.05	0.74	0.62	0.03	0.88	0.67	0.01	0.91
0.3	SIS-SCAD	0.78	0.04	0.86	0.67	0.02	0.90	0.72	0.01	0.94
	Split-Lasso-10	0.58	0.66	0.28	0.36	0.58	0.39	0.25	0.44	0.60
	Split-EN-10	0.59	0.73	0.25	0.36	0.68	0.36	0.25	0.55	0.58
	RGLM-100	0.60	_	_	0.44	_	_	0.40	_	—
	RF-500	0.72	_	_	0.57	_	_	0.50	_	—
	XGB	0.90	0.10	0.73	0.84	0.05	0.89	0.79	0.03	0.95
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.57	0.12	0.55	0.41	0.09	0.73	0.33	0.06	0.85
	Elastic Net	0.56	0.18	0.51	0.38	0.14	0.70	0.29	0.10	0.83
	Adaptive	0.65	0.12	0.56	0.51	0.09	0.75	0.42	0.06	0.85
	Relaxed	0.66	0.12	0.62	0.54	0.08	0.80	0.45	0.05	0.89
	MCP	0.70	0.03	0.76	0.66	0.02	0.86	0.62	0.01	0.95
0.2	SIS-SCAD	0.75	0.03	0.82	0.64	0.02	0.94	0.63	0.01	0.97
	Split-Lasso-10	0.50	0.62	0.28	0.31	0.56	0.39	0.22	0.42	0.62
	Split-EN-10	0.51	0.63	0.30	0.31	0.65	0.37	0.22	0.53	0.59
	RGLM-100	0.51	—	—	0.38	—	—	0.34	—	—
	RF-500	0.61	—	—	0.48	—	—	0.42	—	—
	XGB	0.81	0.08	0.75	0.79	0.04	0.88	0.72	0.02	0.93

Table 30: Mean misclassification rate, sensitivity and specificity for Scenario 2,  $\rho_1 = 0.5$ ,  $\rho_2 = 0.2$ , n = 100, p = 1,000.

		(	$\zeta = 0.1$	L	(	$\zeta = 0.2$	2		$\zeta=0.4$	4
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.54	0.12	0.49	0.38	0.08	0.58	0.29	0.05	0.68
	Elastic Net	0.51	0.21	0.49	0.34	0.16	0.61	0.25	0.10	0.72
	Adaptive	0.60	0.12	0.47	0.44	0.08	0.55	0.36	0.05	0.68
	Relaxed	0.61	0.10	0.64	0.60	0.06	0.79	0.49	0.04	0.90
	MCP	0.65	0.03	0.59	0.52	0.02	0.64	0.46	0.01	0.77
0.4	SIS-SCAD	0.64	0.04	0.84	0.49	0.02	0.91	0.43	0.01	0.98
	Split-Lasso-10	0.44	0.65	0.22	0.27	0.51	0.35	0.19	0.36	0.54
	Split-EN-10	0.43	0.83	0.20	0.27	0.71	0.36	0.20	0.54	0.58
	RGLM-100	0.45	_	_	0.31	_	_	0.27	_	_
	RF-500	0.51	_	_	0.38	_	_	0.33	_	_
	XGB	0.79	0.08	0.81	0.69	0.04	0.93	0.68	0.02	0.92
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.50	0.11	0.48	0.34	0.07	0.55	0.27	0.05	0.66
	Elastic Net	0.48	0.21	0.49	0.31	0.16	0.60	0.24	0.10	0.72
	Adaptive	0.56	0.11	0.48	0.41	0.07	0.55	0.33	0.05	0.66
	Relaxed	0.71	0.09	0.64	0.47	0.06	0.73	0.43	0.04	0.86
	MCP	0.65	0.02	0.60	0.55	0.01	0.76	0.51	0.00	0.70
0.3	SIS-SCAD	0.61	0.03	0.85	0.53	0.02	0.95	0.50	0.01	0.99
	Split-Lasso-10	0.41	0.67	0.23	0.25	0.50	0.35	0.18	0.36	0.53
	Split-EN-10	0.41	0.78	0.25	0.25	0.69	0.36	0.19	0.53	0.57
	RGLM-100	0.42	_	_	0.28	_		0.25		_
	RF-500	0.47	_	_	0.35	_	_	0.30	_	_
	XGB	0.77	0.07	0.82	0.75	0.03	0.92	0.67	0.02	0.90
$\pi_1$	Method	TL	RC	PR	TL	RC	PR	TL	RC	$\mathbf{PR}$
	Lasso	0.44	0.10	0.47	0.31	0.06	0.59	0.24	0.04	0.68
	Elastic Net	0.43	0.19	0.48	0.29	0.14	0.63	0.21	0.09	0.73
	Adaptive	0.50	0.10	0.49	0.37	0.06	0.57	0.31	0.04	0.66
	Relaxed	0.59	0.08	0.67	0.47	0.05	0.74	0.37	0.04	0.84
	MCP	0.55	0.01	0.76	0.49	0.01	0.93	0.47	0.00	0.98
0.2	SIS-SCAD	0.59	0.03	0.92	0.55	0.01	1.00	0.50	0.01	1.00
	Split-Lasso-10	0.35	0.60	0.22	0.23	0.50	0.37	0.17	0.34	0.52
	Split-EN-10	0.36	0.77	0.22	0.23	0.67	0.38	0.17	0.49	0.55
	RGLM-100	0.36	_	_	0.24	_	_	0.21	_	_
	RF-500	0.40	_	_	0.29	_	_	0.25	_	_
	XGB	0.74	0.05	0.86	0.76	0.02	0.91	0.71	0.01	0.92

Table 31: Mean misclassification rate, sensitivity and specificity for Scenario 2,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.2$ , n = 100, p = 1,000.

			$\zeta = 0.1$	L		$\zeta = 0.2$	2		$\zeta=0.4$	4
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.82	0.15	0.46	0.52	0.11	0.53	0.36	0.07	0.62
	Elastic Net	0.81	0.23	0.47	0.50	0.19	0.50	0.33	0.14	0.65
	Adaptive	0.88	0.15	0.48	0.58	0.11	0.53	0.44	0.07	0.64
	Relaxed	0.86	0.10	0.78	0.80	0.09	0.67	0.52	0.06	0.76
	MCP	0.89	0.03	0.53	0.62	0.03	0.53	0.49	0.02	0.66
0.4	SIS-SCAD	0.95	0.06	0.76	0.62	0.04	0.83	0.49	0.02	0.90
	Split-Lasso-10	0.77	0.63	0.25	0.45	0.62	0.27	0.28	0.48	0.42
	Split-EN-10	0.78	0.70	0.24	0.45	0.72	0.26	0.28	0.67	0.40
	RGLM-100	0.80	_	_	0.45	_	_	0.31	_	_
	RF-500	0.83	_	_	0.51	_	_	0.38	_	_
	XGB	0.97	0.11	0.42	0.78	0.07	0.78	0.69	0.04	0.91
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.77	0.13	0.38	0.49	0.10	0.51	0.33	0.07	0.61
	Elastic Net	0.77	0.20	0.35	0.47	0.18	0.51	0.30	0.14	0.63
	Adaptive	0.82	0.13	0.41	0.56	0.10	0.49	0.41	0.07	0.62
	Relaxed	0.82	0.10	0.68	0.75	0.08	0.64	0.45	0.06	0.68
	MCP	0.83	0.03	0.47	0.65	0.02	0.48	0.54	0.01	0.69
0.3	SIS-SCAD	0.91	0.06	0.77	0.58	0.03	0.84	0.53	0.01	0.90
	Split-Lasso-10	0.72	0.59	0.22	0.41	0.63	0.25	0.25	0.50	0.39
	Split-EN-10	0.72	0.66	0.23	0.42	0.72	0.25	0.25	0.65	0.38
	RGLM-100	0.73	_	_	0.43	—	_	0.29	—	_
	RF-500	0.77	_	_	0.47	—	_	0.34	—	_
	XGB	0.92	0.10	0.45	0.80	0.06	0.80	0.74	0.03	0.88
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.66	0.11	0.38	0.43	0.08	0.45	0.29	0.06	0.62
	Elastic Net	0.65	0.19	0.40	0.42	0.16	0.46	0.27	0.13	0.63
	Adaptive	0.75	0.12	0.41	0.51	0.09	0.49	0.36	0.06	0.61
	Relaxed	0.66	0.10	0.59	0.51	0.07	0.56	0.44	0.05	0.73
	MCP	0.70	0.02	0.70	0.58	0.01	0.77	0.49	0.01	0.86
0.2	SIS-SCAD	0.77	0.05	0.86	0.59	0.03	0.88	0.53	0.01	0.98
	Split-Lasso-10	0.62	0.58	0.24	0.36	0.56	0.26	0.22	0.50	0.39
	Split-EN-10	0.62	0.64	0.26	0.36	0.71	0.22	0.22	0.64	0.38
	RGLM-100	0.65	_	_	0.36	_	_	0.25	_	—
	RF-500	0.66	_	_	0.40	_	_	0.29	_	_
	XGB	0.91	0.07	0.42	0.77	0.05	0.86	0.75	0.02	0.91

Table 32: Mean misclassification rate, sensitivity and specificity for Scenario 2,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.5$ , n = 100, p = 1,000.

		Ç	$\zeta = 0.1$	-	G	$\zeta = 0.2$	2		$\zeta=0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SP
	Lasso	0.25	0.62	0.84	0.21	0.67	0.87	0.17	0.74	0.88
	Elastic Net	0.24	0.64	0.85	0.19	0.70	0.88	0.15	0.78	0.90
	Adaptive	0.31	0.38	0.90	0.26	0.49	0.90	0.21	0.61	0.90
	Relaxed	0.25	0.63	0.83	0.22	0.67	0.86	0.18	0.75	0.87
	MCP	0.29	0.54	0.82	0.27	0.56	0.84	0.25	0.61	0.84
0.4	SIS-SCAD	0.30	0.59	0.78	0.29	0.57	0.80	0.28	0.60	0.80
	Split-Lasso-10	0.22	0.67	0.86	0.15	0.76	0.91	0.11	0.83	0.93
	Split-EN-10	0.22	0.67	0.86	0.15	0.76	0.91	0.10	0.84	0.93
	RGLM-100	0.22	0.68	0.85	0.17	0.69	0.92	0.14	0.75	0.94
	RF-500	0.23	0.60	0.88	0.18	0.64	0.94	0.14	0.71	0.96
	XGB	0.31	0.58	0.77	0.30	0.55	0.80	0.29	0.60	0.79
$\pi_1$	Method	MR	$\mathbf{SE}$	SP	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	MR	SE	SF
	Lasso	0.23	0.45	0.91	0.19	0.51	0.93	0.16	0.58	0.94
	Elastic Net	0.22	0.51	0.91	0.17	0.56	0.94	0.14	0.64	0.95
	Adaptive	0.28	0.17	0.97	0.26	0.21	0.97	0.22	0.28	0.98
	Relaxed	0.24	0.46	0.90	0.20	0.53	0.92	0.17	0.59	0.93
	MCP	0.27	0.34	0.91	0.24	0.38	0.92	0.24	0.36	0.93
0.3	SIS-SCAD	0.27	0.42	0.87	0.26	0.41	0.88	0.25	0.40	0.90
	Split-Lasso-10	0.20	0.55	0.92	0.14	0.64	0.95	0.10	0.74	0.96
	Split-EN-10	0.20	0.55	0.92	0.14	0.66	0.95	0.10	0.76	0.96
	RGLM-100	0.20	0.53	0.92	0.16	0.55	0.96	0.14	0.56	0.98
	RF-500	0.22	0.40	0.95	0.18	0.43	0.98	0.16	0.45	1.00
	XGB	0.28	0.47	0.83	0.27	0.44	0.86	0.26	0.45	0.86
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SF
	Lasso	0.18	0.29	0.96	0.16	0.40	0.96	0.14	0.44	0.9'
	Elastic Net	0.17	0.30	0.97	0.14	0.44	0.97	0.12	0.51	0.98
	Adaptive	0.20	0.06	0.99	0.19	0.13	0.99	0.18	0.14	0.99
	Relaxed	0.19	0.31	0.95	0.16	0.46	0.94	0.14	0.49	0.95
	MCP	0.20	0.22	0.96	0.20	0.24	0.95	0.19	0.21	0.96
0.2	SIS-SCAD	0.21	0.18	0.96	0.20	0.23	0.95	0.19	0.23	0.95
	Split-Lasso-10	0.16	0.34	0.97	0.12	0.51	0.97	0.09	0.61	0.99
	Split-EN-10	0.16	0.33	0.97	0.12	0.52	0.97	0.09	0.63	0.98
	RGLM-100	0.17	0.31	0.97	0.14	0.37	0.99	0.13	0.36	1.00
	RF-500	0.18	0.17	0.99	0.16	0.24	1.00	0.15	0.25	1.00

Table 33: Mean misclassification rate, sensitivity and specificity for Scenario 3b,  $\rho_1 = 0.5$ ,  $\rho_2 = 0.2$ , n = 50, p = 1,000.

		C	$\zeta = 0.1$	L	C	$\zeta = 0.2$	2		$oldsymbol{\zeta}=0.4$	4
$\pi_1$	Method	MR	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.22	0.68	0.84	0.20	0.68	0.88	0.18	0.72	0.88
	Elastic Net	0.21	0.69	0.85	0.18	0.71	0.89	0.16	0.76	0.90
	Adaptive	0.25	0.54	0.89	0.26	0.47	0.91	0.24	0.55	0.90
	Relaxed	0.23	0.68	0.83	0.21	0.67	0.87	0.18	0.74	0.87
	MCP	0.24	0.64	0.84	0.24	0.61	0.86	0.25	0.58	0.86
0.4	SIS-SCAD	0.25	0.64	0.82	0.27	0.59	0.82	0.28	0.59	0.80
	Split-Lasso-10	0.19	0.73	0.87	0.14	0.78	0.91	0.12	0.82	0.92
	Split-EN-10	0.19	0.72	0.87	0.14	0.78	0.91	0.12	0.83	0.92
	RGLM-100	0.18	0.74	0.87	0.16	0.72	0.91	0.15	0.74	0.92
	RF-500	0.20	0.66	0.90	0.18	0.66	0.93	0.15	0.72	0.94
	XGB	0.26	0.65	0.80	0.29	0.59	0.80	0.29	0.60	0.78
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.19	0.55	0.92	0.18	0.57	0.93	0.17	0.58	0.93
	Elastic Net	0.19	0.56	0.92	0.16	0.61	0.93	0.15	0.62	0.94
	Adaptive	0.25	0.25	0.97	0.24	0.28	0.97	0.23	0.29	0.97
	Relaxed	0.21	0.55	0.90	0.18	0.59	0.92	0.17	0.61	0.92
	MCP	0.22	0.49	0.91	0.22	0.48	0.91	0.23	0.44	0.92
0.3	SIS-SCAD	0.23	0.45	0.91	0.24	0.45	0.90	0.25	0.42	0.89
	Split-Lasso-10	0.17	0.62	0.93	0.13	0.69	0.94	0.11	0.72	0.96
	Split-EN-10	0.17	0.61	0.93	0.13	0.70	0.94	0.11	0.74	0.96
	RGLM-100	0.17	0.62	0.93	0.15	0.62	0.95	0.14	0.58	0.97
	RF-500	0.19	0.47	0.96	0.17	0.49	0.98	0.16	0.50	0.99
	XGB	0.24	0.57	0.84	0.26	0.48	0.85	0.25	0.49	0.86
$\pi_1$	Method	MR	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.17	0.36	0.96	0.15	0.39	0.97	0.14	0.45	0.96
	Elastic Net	0.16	0.36	0.97	0.14	0.43	0.97	0.12	0.51	0.97
	Adaptive	0.20	0.13	0.99	0.19	0.11	0.99	0.17	0.18	0.99
	Relaxed	0.17	0.37	0.95	0.15	0.40	0.96	0.15	0.50	0.94
	MCP	0.19	0.31	0.95	0.18	0.25	0.96	0.18	0.24	0.96
0.2	SIS-SCAD	0.19	0.23	0.96	0.19	0.22	0.96	0.19	0.22	0.95
	Split-Lasso-10	0.14	0.45	0.97	0.12	0.52	0.98	0.10	0.60	0.97
	Split-EN-10	0.15	0.44	0.97	0.12	0.53	0.98	0.09	0.62	0.98
	RGLM-100	0.15	0.43	0.97	0.14	0.41	0.98	0.13	0.40	0.99
	RF-500	0.17	0.27	0.99	0.16	0.25	0.99	0.14	0.28	1.00
	XGB	0.20	0.43	0.91	0.20	0.35	0.92	0.20	0.36	0.91

Table 34: Mean misclassification rate, sensitivity and specificity for Scenario 3b,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.2$ , n = 50, p = 1,000.

		(	<b>5</b> = 0.1	L	C	<b>5</b> = 0.2	2		$\zeta=0.4$	4
$\pi_1$	Method	MR	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.24	0.66	0.83	0.16	0.78	0.88	0.13	0.83	0.90
	Elastic Net	0.23	0.68	0.83	0.15	0.79	0.89	0.11	0.85	0.91
	Adaptive	0.29	0.53	0.85	0.18	0.67	0.91	0.15	0.76	0.92
	Relaxed	0.25	0.66	0.82	0.17	0.77	0.87	0.13	0.83	0.89
	MCP	0.26	0.63	0.81	0.19	0.72	0.87	0.17	0.76	0.87
0.4	SIS-SCAD	0.27	0.62	0.81	0.21	0.69	0.86	0.20	0.72	0.86
	Split-Lasso-10	0.22	0.69	0.84	0.14	0.82	0.89	0.10	0.87	0.92
	Split-EN-10	0.22	0.70	0.84	0.14	0.82	0.89	0.10	0.87	0.92
	RGLM-100	0.21	0.72	0.83	0.14	0.81	0.90	0.11	0.84	0.93
	RF-500	0.22	0.71	0.84	0.14	0.79	0.91	0.10	0.83	0.94
	XGB	0.27	0.65	0.80	0.22	0.69	0.83	0.21	0.70	0.85
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.22	0.55	0.90	0.15	0.69	0.92	0.12	0.76	0.93
	Elastic Net	0.21	0.56	0.90	0.14	0.72	0.92	0.11	0.78	0.94
	Adaptive	0.26	0.28	0.96	0.18	0.52	0.95	0.15	0.60	0.96
	Relaxed	0.22	0.54	0.89	0.15	0.70	0.91	0.12	0.79	0.92
	MCP	0.24	0.48	0.89	0.18	0.62	0.91	0.17	0.65	0.91
0.3	SIS-SCAD	0.25	0.46	0.89	0.19	0.57	0.91	0.19	0.55	0.92
	Split-Lasso-10	0.20	0.58	0.90	0.13	0.76	0.92	0.09	0.83	0.95
	Split-EN-10	0.20	0.58	0.90	0.13	0.76	0.92	0.09	0.83	0.95
	RGLM-100	0.20	0.63	0.89	0.13	0.74	0.93	0.10	0.77	0.96
	RF-500	0.20	0.59	0.90	0.13	0.70	0.94	0.10	0.73	0.97
	XGB	0.25	0.55	0.85	0.20	0.62	0.88	0.20	0.60	0.89
$\pi_1$	Method	MR	$\mathbf{SE}$	$\mathbf{SP}$	MR	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.18	0.44	0.94	0.13	0.56	0.95	0.10	0.66	0.96
	Elastic Net	0.17	0.46	0.94	0.12	0.59	0.96	0.09	0.71	0.96
	Adaptive	0.21	0.16	0.98	0.16	0.25	0.99	0.13	0.43	0.98
	Relaxed	0.18	0.45	0.93	0.13	0.56	0.94	0.11	0.69	0.95
	MCP	0.20	0.33	0.94	0.17	0.33	0.96	0.16	0.43	0.95
0.2	SIS-SCAD	0.21	0.28	0.94	0.17	0.28	0.97	0.17	0.35	0.96
	Split-Lasso-10	0.17	0.48	0.94	0.10	0.65	0.96	0.08	0.74	0.97
	Split-EN-10	0.16	0.48	0.94	0.10	0.65	0.96	0.07	0.75	0.97
	RGLM-100	0.16	0.54	0.93	0.11	0.58	0.97	0.09	0.64	0.98
	RF-500	0.16	0.49	0.94	0.12	0.53	0.98	0.09	0.59	0.99
	XGB	0.21	0.46	0.89	0.16	0.48	0.93	0.17	0.50	0.92

Table 35: Mean misclassification rate, sensitivity and specificity for Scenario 3b,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.5$ , n = 50, p = 1,000.

			5 = 0.1		(	<b>;</b> = 0.2	2		$\zeta = 0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.22	0.69	0.84	0.17	0.76	0.88	0.14	0.79	0.91
	Elastic Net	0.21	0.69	0.85	0.16	0.77	0.89	0.13	0.81	0.91
	Adaptive	0.24	0.60	0.88	0.19	0.68	0.90	0.15	0.74	0.91
	Relaxed	0.22	0.68	0.84	0.17	0.76	0.87	0.14	0.80	0.90
	MCP	0.24	0.65	0.82	0.21	0.69	0.86	0.20	0.70	0.86
0.4	SIS-SCAD	0.25	0.63	0.83	0.24	0.65	0.83	0.24	0.65	0.83
	Split-Lasso-10	0.20	0.71	0.86	0.14	0.80	0.90	0.09	0.86	0.94
	Split-EN-10	0.20	0.71	0.86	0.14	0.80	0.90	0.09	0.86	0.94
	RGLM-100	0.20	0.72	0.85	0.15	0.77	0.91	0.11	0.81	0.94
	RF-500	0.22	0.64	0.88	0.16	0.70	0.93	0.12	0.76	0.97
	XGB	0.28	0.60	0.80	0.25	0.63	0.83	0.24	0.66	0.83
$\pi_1$	Method	$\mathbf{MR}$	SE	SP	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	SE	SF
	Lasso	0.20	0.56	0.90	0.15	0.68	0.92	0.13	0.71	0.94
	Elastic Net	0.20	0.57	0.91	0.14	0.70	0.92	0.11	0.74	0.94
	Adaptive	0.23	0.38	0.95	0.17	0.56	0.95	0.15	0.58	0.96
	Relaxed	0.20	0.56	0.90	0.16	0.69	0.91	0.13	0.72	0.93
	MCP	0.23	0.49	0.89	0.19	0.58	0.90	0.19	0.56	0.91
0.3	SIS-SCAD	0.24	0.46	0.90	0.21	0.53	0.90	0.21	0.55	0.89
	Split-Lasso-10	0.18	0.60	0.91	0.13	0.74	0.93	0.08	0.82	0.95
	Split-EN-10	0.18	0.60	0.91	0.13	0.74	0.93	0.08	0.83	0.96
	RGLM-100	0.18	0.62	0.90	0.14	0.67	0.95	0.11	0.70	0.97
	RF-500	0.20	0.47	0.95	0.16	0.54	0.97	0.13	0.57	0.99
	XGB	0.25	0.50	0.86	0.23	0.52	0.88	0.22	0.52	0.89
$\pi_1$	Method	$\mathbf{MR}$	SE	SP	$\mathbf{MR}$	SE	SP	$\mathbf{MR}$	SE	SF
	Lasso	0.16	0.43	0.95	0.13	0.55	0.95	0.11	0.57	0.97
	Elastic Net	0.16	0.44	0.95	0.12	0.58	0.96	0.10	0.61	0.97
	Adaptive	0.19	0.17	0.98	0.16	0.30	0.98	0.14	0.35	0.98
	Relaxed	0.17	0.43	0.94	0.13	0.56	0.95	0.12	0.57	0.96
	MCP	0.18	0.36	0.94	0.17	0.40	0.94	0.16	0.35	0.96
0.2	SIS-SCAD	0.18	0.34	0.95	0.18	0.38	0.94	0.17	0.34	0.95
	Split-Lasso-10	0.15	0.46	0.95	0.11	0.64	0.96	0.07	0.70	0.98
	Split-EN-10	0.15	0.47	0.95	0.11	0.64	0.96	0.07	0.71	0.98
	RGLM-100	0.15	0.49	0.95	0.12	0.52	0.98	0.11	0.48	0.99
	RF-500	0.16	0.29	0.98	0.14	0.35	0.99	0.13	0.32	1.00

Table 36: Mean misclassification rate, sensitivity and specificity for Scenario 3b,  $\rho_1 = 0.5$ ,  $\rho_2 = 0.2$ , n = 100, p = 1,000.

			$\zeta = 0.1$	L	Ç	$\zeta = 0.2$	2		$\zeta = 0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$
	Lasso	0.19	0.74	0.87	0.14	0.79	0.90	0.14	0.80	0.90
	Elastic Net	0.18	0.74	0.87	0.14	0.79	0.91	0.13	0.82	0.91
	Adaptive	0.19	0.68	0.90	0.16	0.72	0.92	0.15	0.74	0.91
	Relaxed	0.19	0.73	0.86	0.15	0.78	0.89	0.14	0.80	0.90
	MCP	0.20	0.71	0.85	0.17	0.75	0.89	0.19	0.71	0.87
0.4	SIS-SCAD	0.20	0.71	0.86	0.20	0.71	0.86	0.24	0.64	0.84
	Split-Lasso-10	0.16	0.77	0.88	0.11	0.83	0.93	0.09	0.87	0.93
	Split-EN-10	0.16	0.77	0.88	0.11	0.83	0.92	0.09	0.87	0.94
	RGLM-100	0.16	0.78	0.88	0.12	0.81	0.92	0.11	0.81	0.93
	RF-500	0.17	0.72	0.90	0.15	0.72	0.94	0.12	0.77	0.96
	XGB	0.23	0.69	0.82	0.23	0.67	0.84	0.24	0.64	0.84
$\pi_1$	Method	MR	SE	$\mathbf{SP}$	$\mathbf{MR}$	SE	$\mathbf{SP}$	MR	$\mathbf{SE}$	SP
	Lasso	0.17	0.61	0.92	0.14	0.69	0.93	0.13	0.72	0.93
	Elastic Net	0.17	0.62	0.93	0.13	0.71	0.94	0.12	0.75	0.94
	Adaptive	0.19	0.45	0.96	0.17	0.54	0.96	0.16	0.56	0.96
	Relaxed	0.18	0.61	0.92	0.14	0.70	0.93	0.13	0.73	0.93
	MCP	0.19	0.59	0.91	0.16	0.63	0.92	0.18	0.60	0.92
0.3	SIS-SCAD	0.18	0.57	0.92	0.19	0.58	0.91	0.21	0.55	0.89
	Split-Lasso-10	0.15	0.66	0.93	0.10	0.78	0.95	0.09	0.82	0.95
	Split-EN-10	0.15	0.65	0.93	0.11	0.77	0.95	0.08	0.83	0.95
	RGLM-100	0.14	0.69	0.93	0.12	0.73	0.95	0.11	0.71	0.96
	RF-500	0.17	0.54	0.96	0.15	0.57	0.97	0.13	0.58	0.99
	XGB	0.21	0.59	0.88	0.21	0.57	0.88	0.21	0.52	0.90
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	MR	$\mathbf{SE}$	SP
	Lasso	0.14	0.49	0.96	0.12	0.60	0.95	0.11	0.56	0.97
	Elastic Net	0.14	0.49	0.96	0.11	0.62	0.96	0.10	0.60	0.97
	Adaptive	0.17	0.25	0.99	0.15	0.37	0.98	0.15	0.30	0.99
	Relaxed	0.14	0.48	0.95	0.12	0.61	0.95	0.11	0.58	0.96
	MCP	0.15	0.46	0.95	0.15	0.49	0.94	0.16	0.34	0.96
0.2	SIS-SCAD	0.16	0.44	0.95	0.16	0.44	0.94	0.17	0.32	0.96
	Split-Lasso-10	0.12	0.55	0.97	0.09	0.69	0.97	0.08	0.70	0.98
	Split-EN-10	0.13	0.53	0.97	0.09	0.69	0.97	0.07	0.72	0.98
	RGLM-100	0.12	0.57	0.96	0.10	0.61	0.97	0.10	0.51	0.99
		0 1 5	0.90	0 00	0 10	0.90	0 00	0 10	0.00	1 00
	RF-500	0.15	0.30	0.99	0.13	0.39	0.99	0.13	0.33	1.00

Table 37: Mean misclassification rate, sensitivity and specificity for Scenario 3b,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.2$ , n = 100, p = 1,000.

			5 = 0.1			5 = 0.2	2		$\zeta = 0.4$	4
$\pi_1$	Method	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SF
	Lasso	0.22	0.73	0.82	0.14	0.81	0.89	0.11	0.86	0.92
	Elastic Net	0.21	0.74	0.82	0.14	0.81	0.89	0.10	0.86	0.93
	Adaptive	0.22	0.69	0.84	0.15	0.77	0.91	0.11	0.82	0.93
	Relaxed	0.22	0.73	0.82	0.14	0.81	0.89	0.11	0.86	0.92
	MCP	0.23	0.72	0.81	0.16	0.78	0.88	0.13	0.81	0.9
0.4	SIS-SCAD	0.23	0.71	0.81	0.17	0.77	0.88	0.15	0.78	0.89
	Split-Lasso-10	0.20	0.75	0.83	0.13	0.83	0.90	0.08	0.89	0.94
	Split-EN-10	0.21	0.75	0.83	0.13	0.83	0.90	0.08	0.89	0.9
	RGLM-100	0.20	0.76	0.82	0.13	0.83	0.90	0.09	0.87	0.94
	RF-500	0.21	0.75	0.82	0.14	0.80	0.90	0.09	0.84	0.95
	XGB	0.25	0.71	0.79	0.19	0.76	0.85	0.16	0.76	0.89
$\pi_1$	Method	$\mathbf{MR}$	SE	SP	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\overline{\mathbf{MR}}$	$\mathbf{SE}$	SI
	Lasso	0.20	0.61	0.89	0.13	0.74	0.93	0.10	0.81	$0.9^{-1}$
	Elastic Net	0.20	0.61	0.89	0.13	0.75	0.93	0.09	0.83	0.94
	Adaptive	0.23	0.41	0.95	0.14	0.65	0.95	0.11	0.75	0.9
	Relaxed	0.21	0.60	0.89	0.14	0.73	0.92	0.10	0.82	0.93
	MCP	0.22	0.57	0.88	0.15	0.69	0.91	0.13	0.73	0.92
0.3	SIS-SCAD	0.22	0.54	0.89	0.16	0.66	0.92	0.15	0.69	0.93
	Split-Lasso-10	0.19	0.63	0.90	0.12	0.77	0.93	0.08	0.86	0.9
	Split-EN-10	0.19	0.63	0.90	0.12	0.77	0.93	0.08	0.86	0.95
	<b>RGLM-100</b>	0.19	0.67	0.88	0.11	0.77	0.94	0.09	0.81	0.9
	RF-500	0.19	0.63	0.90	0.12	0.73	0.94	0.09	0.77	0.9'
	XGB	0.23	0.61	0.85	0.17	0.69	0.90	0.15	0.71	0.91
$\pi_1$	Method	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	SE	$\mathbf{SP}$	$\mathbf{MR}$	$\mathbf{SE}$	SI
	Lasso	0.17	0.47	0.94	0.11	0.66	0.95	0.08	0.74	0.9
	Elastic Net	0.17	0.47	0.94	0.10	0.66	0.96	0.08	0.76	0.9'
	Adaptive	0.20	0.20	0.98	0.13	0.44	0.98	0.11	0.56	0.98
	Relaxed	0.17	0.48	0.94	0.11	0.65	0.95	0.09	0.75	0.9
	MCP	0.19	0.41	0.94	0.13	0.54	0.95	0.14	0.53	0.95
0.2	SIS-SCAD	0.19	0.36	0.95	0.14	0.48	0.96	0.13	0.51	0.9
	Split-Lasso-10	0.16	0.51	0.94	0.09	0.71	0.96	0.06	0.81	0.9'
	Split-EN-10	0.16	0.51	0.94	0.09	0.70	0.96	0.06	0.81	0.9'
	RGLM-100	0.16	0.59	0.92	0.10	0.68	0.96	0.07	0.73	0.93
		0 1 0	0 50	0.04	0 10	0.01	0.07	0.00	0.07	0.0
	RF-500	0.16	0.52	0.94	0.10	0.01	0.97	0.08	0.67	0.9

Table 38: Mean misclassification rate, sensitivity and specificity for Scenario 3b,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.5$ , n = 100, p = 1,000.

 $\zeta = 0.1$  $\zeta = 0.2$  $\zeta = 0.4$  $\mathbf{RC}$ Method  $\mathbf{TL}$  $\mathbf{PR}$  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\pi_1$ Lasso 1.040.060.330.920.040.350.770.030.47Elastic Net 0.990.080.310.840.06 0.350.650.060.47Adaptive 1.170.050.341.050.040.350.910.030.47Relaxed 1.330.351.180.031.000.050.370.030.49MCP 1.160.020.371.100.010.331.030.010.491.220.4SIS-SCAD 1.250.010.491.320.010.470.000.580.920.230.680.32Split-Lasso-10 0.310.320.480.300.49Split-EN-10 0.920.340.220.680.370.320.460.380.48**RGLM-100** 0.93— 0.81— \_ 0.76— — \_ \_ \_ RF-500 1.02\_ 0.91\_ 0.84\_ \_ XGB 1.180.040.491.170.020.591.130.010.72Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\pi_1$ Lasso 1.020.050.280.850.040.360.720.030.47Elastic Net 0.270.940.070.780.060.350.610.050.47Adaptive 1.11 0.050.291.040.040.370.940.030.47Relaxed 1.100.040.321.150.030.860.020.460.41MCP 0.010.311.020.010.381.030.010.521.110.3SIS-SCAD 1.220.010.441.180.010.451.160.000.520.32Split-Lasso-10 0.850.210.630.310.330.450.280.50Split-EN-10 0.330.210.620.390.850.300.430.370.48**RGLM-100** 0.740.690.86— \_ — \_ — \_ **RF-500** 0.94\_ \_ 0.82\_ \_ 0.75\_ \_\_\_\_ XGB 1.150.040.521.130.020.581.100.010.72Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\pi_1$ Lasso 0.820.040.300.710.030.370.620.020.46Elastic Net 0.790.060.280.650.060.360.520.040.47Adaptive 0.280.960.040.870.030.370.800.020.47Relaxed 0.940.040.341.040.030.380.910.020.48MCP 0.980.010.360.920.010.420.890.000.490.2SIS-SCAD 0.960.010.450.970.010.420.940.000.540.230.550.28Split-Lasso-10 0.730.260.340.400.270.50Split-EN-10 0.730.290.220.540.340.320.380.340.49**RGLM-100** 0.73\_ 0.640.58\_ — \_ \_ \_ **RF-500** 0.78\_ \_ 0.69\_\_\_ \_ 0.63\_ XGB 1.040.030.021.000.591.010.660.010.76

Table 39: Mean misclassification rate, sensitivity and specificity for Scenario 3,  $\rho_1 = 0.5$ ,  $\rho_2 = 0.2$ , n = 50, p = 1,000.

 $\zeta = 0.1$  $\zeta = 0.2$  $\zeta = 0.4$  $\mathbf{RC}$ Method  $\mathbf{TL}$  $\mathbf{PR}$  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\pi_1$ Lasso 0.960.050.300.870.040.360.800.020.42Elastic Net 0.920.100.330.790.08 0.420.710.050.47Adaptive 1.050.050.301.040.040.380.990.030.43Relaxed 1.300.040.441.200.030.420.990.020.44MCP 1.020.020.341.000.010.471.040.010.46SIS-SCAD 1.220.41.060.020.621.280.010.640.000.560.810.380.240.620.390.29Split-Lasso-10 0.360.550.500.52Split-EN-10 0.820.460.220.630.490.330.400.50**RGLM-100** 0.81— \_ 0.77— — 0.76— \_ \_ \_ RF-500 0.94— 0.89\_ 0.84\_ \_ XGB 1.11 0.040.551.140.020.621.160.010.72Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\pi_1$ Lasso 0.860.050.300.820.040.370.750.020.39Elastic Net 0.320.740.830.090.070.400.660.050.44Adaptive 1.020.050.300.970.040.370.960.020.39Relaxed 1.170.040.430.930.030.400.900.020.39MCP 0.920.010.440.950.010.990.010.450.450.3SIS-SCAD 0.990.020.661.050.010.621.090.000.52Split-Lasso-10 0.740.360.240.610.350.350.510.270.49Split-EN-10 0.210.60 0.470.730.460.450.330.380.49**RGLM-100** 0.710.700.75— — \_ \_ — \_ **RF-500** 0.86\_ \_ 0.81\_ \_ 0.76\_ \_\_\_\_ XGB 1.090.030.631.11 0.020.641.060.010.65Method  $\mathbf{TL}$  $\mathbf{RC}$  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\mathbf{TL}$ RC  $\mathbf{PR}$  $\pi_1$ Lasso 0.780.040.310.700.030.310.640.020.37Elastic Net 0.750.070.310.650.060.360.560.040.42Adaptive 0.320.320.790.910.040.870.030.020.37Relaxed 1.010.040.400.920.030.350.940.020.39MCP 0.920.010.390.950.010.420.850.000.430.2SIS-SCAD 0.960.010.620.910.010.560.900.000.510.240.540.25Split-Lasso-10 0.650.330.310.360.450.51Split-EN-10 0.650.390.240.530.390.350.420.350.51**RGLM-100** 0.67— 0.62\_ 0.58\_ \_ \_ \_ **RF-500** 0.74\_ \_ 0.68\_\_\_ 0.62\_ \_\_\_\_ \_\_\_\_ XGB 1.000.030.710.020.710.981.020.010.71

Table 40: Mean misclassification rate, sensitivity and specificity for Scenario 3,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.2$ , n = 50, p = 1,000.

		(	$oldsymbol{\zeta} = 0.1$	L	(	$\zeta = 0.2$	2		$oldsymbol{\zeta} = 0.$	4
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	PR
0.4	Lasso	1.03	0.04	0.16	0.73	0.04	0.23	0.58	0.03	0.32
	Elastic Net	0.97	0.06	0.15	0.68	0.07	0.24	0.52	0.06	0.34
	Adaptive	1.12	0.04	0.15	0.88	0.04	0.24	0.74	0.03	0.32
	Relaxed	1.07	0.03	0.18	1.02	0.03	0.27	0.82	0.03	0.34
	MCP	1.08	0.01	0.22	0.84	0.01	0.28	0.76	0.01	0.34
	SIS-SCAD	1.14	0.01	0.28	0.92	0.01	0.36	0.84	0.01	0.45
	Split-Lasso-10	0.93	0.23	0.14	0.62	0.27	0.23	0.45	0.27	0.33
	Split-EN-10	0.93	0.27	0.15	0.62	0.31	0.22	0.44	0.35	0.31
	RGLM-100	0.94	_	_	0.64	_	_	0.56	_	_
	RF-500	0.93	_	_	0.70	_	_	0.61	_	_
	XGB	1.11	0.05	0.35	1.05	0.03	0.61	1.00	0.02	0.65
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	PR
	Lasso	0.96	0.04	0.13	0.69	0.03	0.22	0.55	0.02	0.25
	Elastic Net	0.93	0.06	0.14	0.64	0.07	0.25	0.49	0.05	0.27
	Adaptive	1.05	0.04	0.14	0.83	0.04	0.23	0.70	0.02	0.24
	Relaxed	1.02	0.03	0.16	0.85	0.03	0.22	0.72	0.02	0.25
	MCP	1.04	0.01	0.13	0.85	0.01	0.32	0.76	0.01	0.27
0.3	SIS-SCAD	1.06	0.01	0.17	0.84	0.01	0.47	0.86	0.00	0.29
	Split-Lasso-10	0.88	0.21	0.14	0.58	0.31	0.23	0.40	0.27	0.31
	Split-EN-10	0.88	0.27	0.15	0.58	0.37	0.23	0.40	0.38	0.30
	RGLM-100	0.88	_	_	0.60	_	_	0.52		_
	RF-500	0.86	_	_	0.65	_	_	0.56	_	—
	XGB	1.08	0.04	0.34	0.98	0.03	0.61	1.00	0.01	0.57
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.85	0.03	0.11	0.60	0.03	0.19	0.48	0.02	0.26
	Elastic Net	0.80	0.05	0.13	0.55	0.06	0.22	0.41	0.05	0.29
	Adaptive	0.92	0.03	0.11	0.74	0.03	0.19	0.60	0.02	0.27
	Relaxed	1.02	0.02	0.13	0.85	0.02	0.18	0.87	0.02	0.26
	MCP	0.90	0.01	0.18	0.87	0.00	0.23	0.76	0.00	0.35
0.2	SIS-SCAD	1.05	0.01	0.18	0.86	0.01	0.33	0.78	0.00	0.39
	Split-Lasso-10	0.75	0.23	0.13	0.49	0.28	0.22	0.35	0.28	0.35
	Split-EN-10	0.75	0.27	0.13	0.48	0.38	0.21	0.35	0.39	0.32
	RGLM-100	0.75	_	_	0.51	_	_	0.45	_	_
	RF-500	0.73	_	_	0.54	_	_	0.47	_	_
	XGB	1.05	0.03	0.38	0.95	0.02	0.53	0.98	0.01	0.59

Table 41: Mean misclassification rate, sensitivity and specificity for Scenario 3,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.5$ , n = 50, p = 1,000.

		(	$\zeta = 0.1$	L	(	$\zeta = 0.2$	2		$\zeta=0.4$	4
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.93	0.10	0.35	0.75	0.08	0.41	0.64	0.05	0.51
	Elastic Net	0.92	0.12	0.32	0.71	0.11	0.41	0.57	0.09	0.52
	Adaptive	1.00	0.09	0.34	0.85	0.07	0.40	0.72	0.05	0.50
	Relaxed	0.96	0.09	0.41	0.87	0.07	0.42	0.71	0.05	0.51
	MCP	1.03	0.04	0.39	0.89	0.03	0.44	0.85	0.02	0.53
0.4	SIS-SCAD	1.03	0.03	0.64	1.03	0.01	0.54	1.05	0.01	0.62
	Split-Lasso-10	0.86	0.42	0.23	0.61	0.47	0.31	0.42	0.44	0.49
	Split-EN-10	0.86	0.43	0.23	0.61	0.52	0.29	0.41	0.52	0.47
	RGLM-100	0.86	_	_	0.70	_	_	0.64	_	_
	RF-500	0.97	_	_	0.85	_	_	0.78	_	_
	XGB	1.12	0.07	0.43	1.03	0.05	0.51	1.00	0.03	0.65
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$
	Lasso	0.86	0.09	0.34	0.69	0.07	0.41	0.58	0.05	0.48
	Elastic Net	0.85	0.11	0.32	0.65	0.10	0.40	0.51	0.08	0.48
	Adaptive	0.94	0.09	0.33	0.76	0.07	0.41	0.68	0.05	0.47
	Relaxed	0.88	0.09	0.35	0.78	0.07	0.42	0.64	0.05	0.47
	MCP	0.97	0.03	0.39	0.84	0.02	0.44	0.80	0.01	0.49
0.3	SIS-SCAD	0.98	0.03	0.60	0.97	0.02	0.63	0.94	0.01	0.59
	Split-Lasso-10	0.79	0.39	0.23	0.56	0.47	0.32	0.38	0.43	0.49
	Split-EN-10	0.79	0.42	0.22	0.57	0.51	0.31	0.37	0.52	0.47
	RGLM-100	0.80	_	_	0.65	_	_	0.59	_	_
	RF-500	0.89	_	_	0.78	_	_	0.70	_	_
	XGB	1.05	0.07	0.44	0.98	0.04	0.52	0.97	0.02	0.61
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$
	Lasso	0.74	0.08	0.32	0.60	0.06	0.40	0.52	0.04	0.49
	Elastic Net	0.72	0.10	0.29	0.57	0.09	0.40	0.46	0.07	0.50
	Adaptive	0.82	0.07	0.32	0.71	0.06	0.40	0.61	0.04	0.48
	Relaxed	0.78	0.07	0.39	0.65	0.06	0.41	0.60	0.04	0.49
	MCP	0.83	0.03	0.41	0.79	0.02	0.47	0.74	0.01	0.51
0.2	SIS-SCAD	0.86	0.03	0.60	0.91	0.01	0.54	0.80	0.01	0.57
	Split-Lasso- $10$	0.66	0.37	0.23	0.48	0.46	0.31	0.33	0.41	0.51
	Split-EN-10	0.66	0.44	0.21	0.48	0.51	0.29	0.32	0.49	0.49
	RGLM-100	0.66	—	—	0.56	_	—	0.50	_	—
	RF-500	0.75	—	—	0.66	—	—	0.58	—	—
	XGB	0.94	0.06	0.48	0.91	0.04	0.57	0.90	0.02	0.68

Table 42: Mean misclassification rate, sensitivity and specificity for Scenario 3,  $\rho_1 = 0.5$ ,  $\rho_2 = 0.2$ , n = 100, p = 1,000.

		(	$\zeta = 0.1$	L	(	$\zeta = 0.2$	2	$oldsymbol{\zeta}=0.4$			
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	PR	
	Lasso	0.81	0.09	0.32	0.65	0.07	0.39	0.63	0.04	0.43	
	Elastic Net	0.79	0.14	0.35	0.62	0.12	0.44	0.57	0.08	0.50	
0.4	Adaptive	0.89	0.09	0.32	0.77	0.07	0.39	0.73	0.04	0.43	
	Relaxed	0.84	0.07	0.49	0.74	0.06	0.44	0.70	0.04	0.44	
	MCP	0.88	0.02	0.31	0.73	0.02	0.48	0.84	0.01	0.52	
	SIS-SCAD	0.90	0.04	0.75	1.02	0.02	0.81	1.01	0.01	0.64	
	Split-Lasso-10	0.71	0.56	0.18	0.51	0.53	0.31	0.42	0.45	0.50	
	Split-EN-10	0.72	0.63	0.18	0.51	0.62	0.30	0.41	0.55	0.50	
	RGLM-100	0.71	_	_	0.60	_	_	0.62	_	_	
	RF-500	0.87	_	_	0.81	_	_	0.77	_	_	
	XGB	1.02	0.07	0.49	0.97	0.06	0.60	1.00	0.03	0.62	
$\pi_1$	Method	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	$\mathbf{TL}$	$\mathbf{RC}$	$\mathbf{PR}$	
	Lasso	0.75	0.08	0.30	0.62	0.06	0.40	0.59	0.04	0.43	
	Elastic Net	0.74	0.12	0.33	0.60	0.11	0.42	0.53	0.07	0.49	
	Adaptive	0.83	0.08	0.32	0.74	0.06	0.40	0.72	0.04	0.43	
	Relaxed	0.78	0.07	0.47	0.73	0.06	0.42	0.64	0.04	0.43	
	MCP	0.85	0.02	0.34	0.73	0.02	0.47	0.78	0.01	0.47	
0.3	SIS-SCAD	0.85	0.03	0.69	0.87	0.02	0.68	0.94	0.01	0.64	
	Split-Lasso-10	0.66	0.52	0.21	0.48	0.52	0.32	0.39	0.43	0.49	
	Split-EN-10	0.67	0.56	0.21	0.48	0.61	0.30	0.38	0.53	0.49	
	RGLM-100	0.65	_	_	0.57	_	_	0.58	_	_	
	RF-500	0.79	_	_	0.74	_	—	0.70	_	_	
	XGB	0.97	0.07	0.50	0.95	0.05	0.60	0.95	0.03	0.64	
$\pi_1$	Method	$\mathbf{TL}$	RC	PR	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	PR	
	Lasso	0.66	0.07	0.27	0.56	0.06	0.40	0.52	0.03	0.42	
	Elastic Net	0.64	0.12	0.32	0.52	0.11	0.43	0.47	0.07	0.48	
	Adaptive	0.73	0.07	0.29	0.66	0.06	0.40	0.64	0.03	0.42	
	Relaxed	0.71	0.06	0.42	0.63	0.05	0.40	0.60	0.03	0.42	
	MCP	0.75	0.02	0.41	0.68	0.02	0.53	0.73	0.01	0.48	
0.2	SIS-SCAD	0.77	0.02	0.59	0.81	0.01	0.66	0.77	0.01	0.53	
	Split-Lasso-10	0.56	0.50	0.20	0.42	0.50	0.33	0.35	0.38	0.51	
	Split-EN-10	0.57	0.56	0.20	0.42	0.59	0.31	0.34	0.48	0.50	
	RGLM-100	0.56	—	—	0.50	—	—	0.49	—	—	
	RF-500	0.67	—	_	0.63	—	_	0.58	_	_	
	XGB	0.91	0.05	0.55	0.84	0.04	0.56	0.86	0.02	0.64	

Table 43: Mean misclassification rate, sensitivity and specificity for Scenario 3,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.2$ , n = 100, p = 1,000.

			$\zeta = 0.1$	L	c	$\zeta = 0.2$	2	$oldsymbol{\zeta}=0.4$			
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	
	Lasso	0.92	0.07	0.21	0.65	0.07	0.29	0.48	0.05	0.35	
	Elastic Net	0.91	0.11	0.21	0.63	0.11	0.30	0.45	0.09	0.38	
0.4	Adaptive	1.00	0.07	0.21	0.72	0.07	0.29	0.57	0.05	0.36	
	Relaxed	0.94	0.07	0.32	0.68	0.07	0.29	0.56	0.05	0.35	
	MCP	1.01	0.03	0.22	0.71	0.02	0.33	0.61	0.01	0.38	
	SIS-SCAD	0.99	0.04	0.51	0.75	0.03	0.55	0.67	0.01	0.55	
	Split-Lasso-10	0.88	0.39	0.18	0.58	0.42	0.24	0.37	0.42	0.33	
	Split-EN-10	0.88	0.45	0.16	0.58	0.46	0.23	0.38	0.48	0.32	
	RGLM-100	0.90	_	_	0.59	_		0.47		_	
	RF-500	0.90	_	_	0.66	_	_	0.55	_	_	
	XGB	1.06	0.09	0.30	0.91	0.06	0.45	0.82	0.04	0.54	
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	PR	$\mathbf{TL}$	RC	$\mathbf{PR}$	
	Lasso	0.87	0.07	0.21	0.60	0.06	0.26	0.46	0.05	0.34	
	Elastic Net	0.86	0.10	0.20	0.57	0.10	0.28	0.42	0.09	0.36	
	Adaptive	0.95	0.07	0.22	0.67	0.06	0.26	0.52	0.05	0.34	
	Relaxed	0.93	0.06	0.33	0.72	0.05	0.37	0.57	0.04	0.35	
	MCP	0.98	0.02	0.27	0.71	0.02	0.33	0.62	0.01	0.40	
0.3	SIS-SCAD	0.97	0.04	0.45	0.73	0.02	0.46	0.67	0.01	0.46	
	Split-Lasso-10	0.82	0.34	0.18	0.53	0.36	0.23	0.35	0.43	0.33	
	Split-EN-10	0.82	0.41	0.16	0.53	0.44	0.22	0.35	0.53	0.30	
	RGLM-100	0.86	_	_	0.54	_	_	0.44	_	_	
	RF-500	0.84	_	_	0.60	_	_	0.51	_	_	
	XGB	1.02	0.08	0.28	0.86	0.06	0.50	0.77	0.03	0.57	
$\pi_1$	Method	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	$\mathbf{TL}$	RC	$\mathbf{PR}$	
	Lasso	0.75	0.06	0.19	0.49	0.05	0.24	0.40	0.04	0.31	
	Elastic Net	0.74	0.10	0.21	0.47	0.10	0.27	0.36	0.08	0.35	
	Adaptive	0.86	0.06	0.20	0.59	0.05	0.25	0.50	0.04	0.32	
	Relaxed	0.77	0.05	0.29	0.56	0.05	0.30	0.50	0.04	0.33	
	MCP	0.82	0.02	0.32	0.65	0.01	0.38	0.62	0.01	0.40	
0.2	SIS-SCAD	0.87	0.03	0.39	0.67	0.02	0.48	0.63	0.01	0.45	
	Split-Lasso-10	0.71	0.39	0.15	0.43	0.44	0.21	0.29	0.41	0.34	
	Split-EN-10	0.71	0.44	0.15	0.43	0.51	0.19	0.29	0.50	0.31	
	RGLM-100	0.74	_	_	0.45	_	_	0.38	_	—	
	RF-500	0.72	_	_	0.50	_	_	0.43	_	_	
	XGB	0.95	0.06	0.29	0.85	0.04	0.54	0.78	0.03	0.59	

Table 44: Mean misclassification rate, sensitivity and specificity for Scenario 3,  $\rho_1 = 0.8$ ,  $\rho_2 = 0.5$ , n = 100, p = 1,000.

	p = 100		p =	<b>250</b>	p =	500	p = 1000	
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	1.92	1.54	2.45	1.61	2.25	1.64	2.15	1.64
EN	1.15	1.07	1.55	1.07	1.42	1.11	1.31	1.10
Adaptive	3.08	3.82	4.18	4.48	4.08	4.70	4.08	5.21
Relaxed	2.46	3.84	3.00	3.51	3.00	3.71	2.69	3.65
MCP	3.38	3.88	3.91	3.78	3.42	3.73	3.15	3.65
SIS-SCAD	3.23	4.41	3.64	4.20	3.25	4.25	2.92	4.14
Split-Lasso-CV	1.46	1.00	1.73	1.01	1.67	1.04	1.31	1.06
Split-EN-CV	1.23	1.03	1.64	1.00	1.33	1.00	1.31	1.00
RGLM-CV	2.00	1.76	2.27	1.76	1.83	1.94	1.92	2.00
RF-CV	1.00	3.88	1.00	3.77	1.00	3.77	1.00	3.75
XGB	3.85	5.22	4.55	5.09	4.17	5.12	3.85	5.08

Table 45: MR and TL relative performances for GSE20347 and training proportion 0.5.

Table 46: MR and TL relative performances for GSE23400 (part one) and training proportion 0.35.

	p = 100		p =	p = 250		p = 500		1000
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	1.06	1.12	1.06	1.12	1.14	1.17	1.08	1.21
EN	1.02	1.03	1.02	1.04	1.05	1.05	1.02	1.08
Adaptive	1.31	1.53	1.25	1.50	1.08	1.40	1.09	1.46
Relaxed	1.16	3.88	1.18	3.25	1.23	4.13	1.23	3.75
MCP	1.35	1.54	1.40	1.60	1.43	1.68	1.40	1.69
SIS-SCAD	1.50	1.77	1.57	1.88	1.64	1.97	1.62	1.98
Split-Lasso-CV	1.03	1.06	1.00	1.04	1.00	1.00	1.00	1.02
Split-EN-CV	1.00	1.00	1.01	1.00	1.02	1.00	1.01	1.00
RGLM-CV	1.10	1.15	1.09	1.15	1.15	1.20	1.08	1.24
RF-CV	1.05	1.36	1.09	1.45	1.14	1.50	1.10	1.54
XGB	1.74	2.25	1.86	2.43	2.03	3.19	1.98	3.24
	p =	100	p =	250	p = 500		p = 1000	
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Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	1.14	1.08	1.12	1.13	1.15	1.14	1.13	1.18
$\mathbf{EN}$	1.07	1.02	1.10	1.05	1.05	1.04	1.04	1.08
Adaptive	1.35	1.43	1.09	1.41	1.04	1.34	1.02	1.44
Relaxed	1.21	3.08	1.23	3.07	1.26	3.02	1.19	3.76
MCP	1.41	1.46	1.38	1.52	1.42	1.61	1.38	1.62
SIS-SCAD	1.42	1.63	1.46	1.77	1.41	1.88	1.41	1.94
Split-Lasso	1.06	1.06	1.00	1.03	1.03	1.02	1.00	1.01
Split-EN	1.00	1.00	1.02	1.00	1.00	1.00	1.03	1.00
RGLM-CV	1.10	1.09	1.02	1.13	1.01	1.18	1.05	1.19
RF-CV	1.12	1.40	1.09	1.47	1.12	1.53	1.11	1.55
XGB	1.48	1.82	1.42	1.91	1.71	2.82	1.76	2.95

Table 47: MR and TL relative performances for GSE23400 (part one) and training proportion 0.5.

Table 48: MR and TL relative performances for GSE23400 (part two) and training proportion 0.35.

	p=100		p =	<b>250</b>	$oldsymbol{p}=500$		p=1	p = 1000	
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	
Lasso	1.07	1.12	1.06	1.03	1.08	1.07	1.08	1.06	
EN	1.07	1.07	1.11	1.00	1.08	1.05	1.07	1.05	
Adaptive	1.00	1.16	1.00	1.10	1.00	1.20	1.00	1.33	
Relaxed	1.09	2.94	1.13	2.50	1.10	2.61	1.19	2.73	
MCP	1.10	1.27	1.10	1.20	1.05	1.26	1.07	1.27	
SIS-SCAD	1.11	1.45	1.13	1.44	1.10	1.49	1.13	1.52	
Split-Lasso-CV	1.10	1.18	1.08	1.03	1.03	1.00	1.03	1.00	
Split-EN-CV	1.06	1.10	1.08	1.00	1.03	1.01	1.01	1.00	
RGLM-CV	1.05	1.00	1.13	1.01	1.07	1.09	1.10	1.12	
RF-CV	1.10	1.39	1.11	1.34	1.12	1.40	1.10	1.42	
XGB	1.64	2.01	1.69	1.92	1.61	2.03	1.77	2.67	

	p = 100		p =	250	$oldsymbol{p}=500$		p=1000	
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	1.10	1.04	1.03	1.01	1.09	1.03	1.03	1.04
EN	1.20	1.06	1.03	1.00	1.03	1.00	1.04	1.03
Adaptive	1.00	1.19	1.00	1.21	1.05	1.23	1.00	1.35
Relaxed	1.14	2.91	1.14	3.18	1.09	2.89	1.05	2.76
MCP	1.12	1.26	1.04	1.21	1.08	1.24	1.05	1.24
SIS-SCAD	1.08	1.40	1.02	1.40	1.06	1.43	1.02	1.44
Split-Lasso-CV	1.24	1.11	1.06	1.03	1.00	1.02	1.02	1.01
Split-EN-CV	1.24	1.11	1.07	1.03	1.00	1.01	1.02	1.00
RGLM-CV	1.20	1.00	1.16	1.03	1.13	1.06	1.11	1.08
RF-CV	1.23	1.44	1.19	1.41	1.23	1.41	1.18	1.43
XGB	1.45	1.59	1.65	2.39	1.57	2.30	1.67	2.43

Table 49: MR and TL relative performances for GSE23400 (part two) and training proportion 0.5.

Table 50: MR and TL relative performances for GSE5364 (esophageal) and training proportion 0.5.

	p = 100		p =	250	p = 500		p=1	1000
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	2.47	1.95	2.56	2.21	2.67	2.25	2.78	2.23
EN	1.39	1.17	1.41	1.32	1.42	1.33	1.47	1.31
Adaptive	3.03	2.80	3.68	3.19	3.82	3.07	4.50	3.04
Relaxed	2.67	7.00	2.94	7.57	3.33	9.40	3.19	7.97
MCP	4.83	2.83	5.35	3.24	5.42	3.33	5.62	3.30
SIS-SCAD	4.61	2.82	5.09	3.30	5.27	3.35	5.41	3.29
Split-Lasso-CV	1.19	1.08	1.18	1.22	1.36	1.26	1.44	1.26
Split-EN-CV	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RGLM-CV	2.33	1.70	2.00	1.84	1.85	1.83	1.97	1.81
RF-CV	1.08	2.15	1.18	2.46	1.12	2.48	1.25	2.43
XGB	4.03	2.95	4.26	3.37	4.33	3.41	4.53	3.35

	p =	100	p =	<b>250</b>	p =	500	p=1	1000
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	1.13	1.42	1.24	1.38	1.24	1.34	1.24	1.37
EN	1.10	1.27	1.15	1.29	1.15	1.26	1.17	1.29
Adaptive	1.22	1.24	1.32	1.31	1.33	1.39	1.33	1.36
Relaxed	1.20	2.73	1.24	3.36	1.29	2.22	1.28	2.44
MCP	1.37	1.34	1.49	1.40	1.49	1.42	1.45	1.51
SIS-SCAD	1.40	1.31	1.51	1.40	1.50	1.45	1.51	1.49
Split-Lasso-CV	1.05	1.23	1.08	1.23	1.03	1.19	1.06	1.18
Split-EN-CV	1.05	1.11	1.03	1.12	1.07	1.12	1.02	1.11
RGLM-CV	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RF-CV	1.08	1.13	1.13	1.20	1.13	1.23	1.11	1.23
XGB	1.33	1.38	1.51	1.57	1.45	1.56	1.52	1.61

Table 51: MR and TL relative performances for GSE25869 and training proportion 0.5.

Table 52: MR and TL relative performances for GSE10245 and training proportion 0.35.

	p = 100		p =	<b>250</b>	$m{p}=500$		p=1	1000
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	1.21	1.31	1.17	1.33	1.24	1.37	1.27	1.40
EN	1.07	1.11	1.03	1.10	1.05	1.11	1.01	1.15
Adaptive	1.76	1.68	1.59	1.69	1.86	1.81	1.86	1.76
Relaxed	1.36	3.57	1.25	3.14	1.34	3.67	1.32	2.80
MCP	1.99	1.71	1.92	2.44	1.99	1.66	2.02	1.75
SIS-SCAD	2.07	1.70	2.01	1.72	2.06	1.69	2.04	1.69
Split-Lasso-CV	1.06	1.10	1.01	1.05	1.06	1.05	1.00	1.09
Split-EN-CV	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RGLM-CV	1.39	1.59	1.09	1.30	1.06	1.31	1.08	1.34
RF-CV	1.46	1.50	1.50	1.54	1.54	1.55	1.56	1.57
XGB	2.50	1.95	2.51	2.03	2.63	2.03	2.60	2.38

	p =	100	p =	<b>250</b>	p =	500	p=1	1000
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	1.29	1.31	1.29	1.37	1.36	1.33	1.29	1.35
EN	1.09	1.07	1.11	1.12	1.17	1.08	1.07	1.08
Adaptive	1.72	1.61	2.04	1.83	1.81	1.72	1.85	1.84
Relaxed	1.25	3.00	1.40	4.07	1.49	3.30	1.37	3.67
MCP	2.08	1.67	2.30	1.95	2.50	2.01	2.28	2.03
SIS-SCAD	2.20	1.75	2.45	2.09	2.54	2.13	2.38	2.18
Split-Lasso-CV	1.00	1.06	1.00	1.04	1.00	1.02	1.02	1.05
Split-EN-CV	1.02	1.00	1.01	1.00	1.02	1.00	1.00	1.00
RGLM-CV	1.36	1.84	1.24	1.52	1.26	1.54	1.13	1.55
RF-CV	1.33	1.40	1.51	1.58	1.63	1.61	1.49	1.66
XGB	2.61	2.17	2.80	2.51	3.10	2.77	2.92	2.86

Table 53: MR and TL relative performances for GSE10245 and training proportion 0.5.

Table 54: MR and TL relative performances for GSE5364 (lung) and training proportion 0.5.

	p = 100		p =	<b>250</b>	p = 500		p = 1000	
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	2.34	1.68	2.04	1.60	2.36	1.57	2.31	1.58
EN	1.75	1.23	1.45	1.20	1.51	1.12	1.57	1.13
Adaptive	3.25	1.87	3.07	1.92	2.92	1.78	3.06	1.74
Relaxed	2.57	5.31	2.29	4.60	2.51	4.32	2.59	4.64
MCP	3.47	1.98	3.36	2.05	3.53	1.98	3.74	2.02
SIS-SCAD	3.35	1.87	3.36	1.95	3.60	1.89	3.65	1.88
Split-Lasso-CV	1.00	1.05	1.16	1.07	1.17	1.03	1.13	1.03
Split-EN-CV	1.04	1.00	1.00	1.00	1.08	1.00	1.13	1.00
RGLM-CV	1.63	1.47	1.16	1.38	1.02	1.25	1.09	1.24
RF-CV	1.12	1.50	1.00	1.51	1.00	1.42	1.00	1.41
XGB	3.14	1.87	2.98	1.87	3.09	1.77	3.37	2.09

	p =	100	$oldsymbol{p}=$	<b>250</b>	p =	500	p=1	1000
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	1.39	1.24	1.43	1.22	1.47	1.36	1.35	1.29
EN	1.16	1.10	1.28	1.10	1.25	1.21	1.18	1.21
Adaptive	1.22	1.08	1.43	1.14	1.79	1.33	2.00	1.37
Relaxed	1.57	3.07	1.66	3.38	1.57	2.85	1.56	2.88
MCP	1.70	1.22	1.83	1.25	1.98	1.36	1.88	1.34
SIS-SCAD	1.84	1.20	1.97	1.21	1.98	1.32	1.92	1.30
Split-Lasso-CV	1.03	1.13	1.05	1.03	1.04	1.12	1.08	1.14
Split-EN-CV	1.00	1.00	1.00	1.00	1.00	1.08	1.02	1.06
RGLM-CV	1.11	1.08	1.08	1.04	1.03	1.00	1.00	1.00
RF-CV	1.01	1.03	1.03	1.04	1.12	1.15	1.02	1.13
XGB	1.94	1.21	2.18	1.23	2.21	1.61	2.07	1.55

Table 55: MR and TL relative performances for GSE5364 (thyroid) and training proportion 0.5.

Table 56: MR and TL relative performances for GSE21942 and training proportion 0.5.

	p = 100		p =	<b>250</b>	p = 500		p = 1	p = 1000	
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	
Lasso	1.82	1.32	1.50	1.25	1.75	1.36	1.75	1.40	
EN	1.00	1.00	1.00	1.00	1.00	1.01	1.00	1.05	
Adaptive	4.91	2.69	4.75	2.86	5.08	3.07	4.42	3.02	
Relaxed	2.64	3.20	2.17	2.61	2.67	3.14	2.42	2.39	
MCP	7.73	4.23	7.17	4.09	7.08	3.85	6.50	3.55	
SIS-SCAD	7.73	4.31	6.75	4.14	6.83	3.90	6.75	3.78	
Split-Lasso-CV	2.18	1.18	1.33	1.10	1.17	1.00	1.25	1.02	
Split-EN-CV	2.00	1.18	1.92	1.12	1.75	1.04	1.58	1.00	
RGLM-CV	2.45	2.58	2.33	2.55	2.92	2.47	3.25	2.40	
RF-CV	4.09	3.75	3.75	3.67	3.67	3.45	3.58	3.34	
XGB	9.45	5.23	8.67	5.09	8.58	4.76	5.67	5.98	

	p = 100		p=250		p = 500		p = 1000	
Method	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$	$\mathbf{MR}$	$\mathbf{TL}$
Lasso	1.82	1.65	1.78	1.67	1.72	1.60	2.00	1.61
EN	1.32	1.11	1.30	1.12	1.53	1.12	1.64	1.13
Adaptive	2.13	2.48	2.11	2.34	1.83	2.26	1.91	2.36
Relaxed	1.79	5.06	1.97	4.50	1.72	4.44	1.91	3.79
MCP	2.74	3.26	2.81	3.37	2.86	3.25	3.12	3.19
SIS-SCAD	2.58	3.58	2.41	3.50	2.44	3.39	2.58	3.26
Split-Lasso-CV	1.34	1.04	1.30	1.10	1.47	1.06	1.85	1.10
Split-EN-CV	1.18	1.00	1.14	1.00	1.22	1.00	1.52	1.00
RGLM-CV	1.95	1.62	1.65	1.47	1.67	1.39	1.85	1.41
RF-CV	1.00	2.59	1.00	2.63	1.00	2.56	1.00	2.53
XGB	2.39	3.47	2.46	3.53	2.44	7.49	2.67	7.33

Table 57: MR and TL relative performances for GSE14905 and training proportion 0.5.

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