# Effect of shape asymmetry on percolation of aligned and overlapping objects on lattices

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We investigate the percolation transition of aligned, overlapping, non-symmetrical shapes on lattices. Using the recently proposed lattice version of excluded volume theory, we show that shape-asymmetry leads to some intriguing consequences regarding the percolation behavior of asymmetric shapes. We consider a prototypical asymmetric shape - rectangle - on a square lattice and show that for rectangles of width unity (sticks), the percolation threshold is a monotonically decreasing function of the stick length, whereas, for rectangles of width greater than two, it is a monotonically increasing function. Interestingly, for rectangles of width two, the percolation threshold is independent of its length. The limiting case of the length of the rectangles going to infinity shows that the limiting threshold value is finite and depends upon the width of the rectangle. Unlike the case of symmetrical shapes like squares, there seems to be no continuum percolation problem that corresponds to this limit. We show that similar results hold for other asymmetric shapes and lattices. The critical properties of the aligned and overlapping rectangles are evaluated using Monte Carlo simulations. We find that the threshold values given by the lattice-excluded volume theory are in good agreement with the simulation results, especially for larger rectangles. We verify the isotropy of the percolation threshold and also compare our results with models where rectangles of mixed orientation are allowed. Our simulation results show that alignment increases the percolation threshold. The calculation of critical exponents places the model in the standard percolation universality class. Our results show that shape-anisotropy of the aligned, overlapping percolating units has a marked influence on the percolation properties, especially when a subset of the dimensions of the percolation units are made to diverge.

Keywords: Lattice Percolation, Asymmetric shapes, Percolation threshold, Overlapping rectangles

## I. INTRODUCTION

Percolation problems have commanded enduring interest in the realm of statistical physics, finding diverse applications in fields ranging from material science and polymer chemistry to epidemic spreading and financial markets [1–8]. The fundamental focus of percolation theory lies in investigating the connectivity properties of random and disordered media. In the simplest model of lattice site percolation, each site of a lattice is randomly occupied with a probability $p$. Clusters are formed by the nearest occupied sites, and as the occupation probability is gradually increased from zero, the system eventually reaches a point where the largest cluster spans the entire lattice, signaling the occurrence of percolation [9, 10]. Percolation models can also be extended to continuum space, where geometric shapes or objects, which can partially or fully overlap, are randomly placed in the space with a specific number density. Some of the commonly considered shapes include discs, spheres, cubes, squares, and sticks [11–13]. Overlapping objects give rise to distinct clusters, and, similar to lattice percolation, the system exhibits a phase transition, marked by the emergence of a spanning cluster at a critical number density. Identifying the percolation threshold and characterizing the associated critical behavior constitute a major research theme in the study of various percolation systems [10]. The probability of forming a spanning cluster serves as one of the order parameters, and the critical phenomenon is characterized by a power-law divergence of specific quantities close to the percolation threshold. Also, the universality of critical exponents is seen across many models of percolation [14].

Among the diverse variants of percolation on lattices, many recent studies have focused on the percolation of extended shapes that do not overlap [15–26]. Compared to this, there are only a few studies on extended shapes that overlap [27–30]. The models with overlapping shapes introduce multisite occupancy and/or multiple occupancy of a site, also serving as a natural bridge between various lattice and continuum models. For example, the percolation of overlapping squares and cubes on lattices was studied by Koza et al. [27], who focused on calculating the percolation thresholds of these models and their transition from discrete to continuum values. Percolation of discrete overlapping hyperspheres on hypercubic lattices was studied by Brzeski and Kondrat [29]. They studied the discrete-to-continuum transition of hyperspheres which enabled the evaluation of the threshold for the 3D and higher dimensional continuum problems with greater accuracy. Apart from these, recent studies focus on site percolation with extended neighborhoods, which can be mapped onto the problem

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of lattice percolation of extended and overlapping shapes like discs, squares, etc [27, 31, 32]. Apart from their theoretical interest, these models are also important from an application point of view. Examples include the study of nano-composite materials consisting of conductive rod-like particles [33, 34] or systems consisting of conducting polymers [35, 36].

This paper delves into the percolation of overlapping and aligned asymmetrical shapes on lattices. More specifically, as a prototypical example of such problems, we consider the percolation of aligned and overlapping "rectangles"on a two-dimensional square lattice in detail. In the particular case of either side of the rectangle being of unit length, we can call the shapes rods or sticks (See Fig. 1). The motivation behind considering this and other extended non-symmetric shapes stems from the peculiar behavior of the percolation threshold of aligned and overlapping rectangles in the 2D continuum problem. In the latter, the percolation threshold remains independent of the rectangles' aspect ratio due to the affine symmetry of such systems [37]. In simple words, for such systems, scaling one or both directions will not change the connectivity properties of the system and hence will not change the percolation threshold. Similar invariance of percolation threshold under scaling also exists for disc percolation in continuum [38]. However, on lattices, such symmetry under scaling cannot be defined, and we show that this leads to intriguing percolation behavior, which depends on the aspect ratio of the overlapping rectangles and other shapes.

Employing the excluded volume theory adapted to a lattice setting [27] and employing Monte Carlo simulations, we analyze the percolation of aligned and overlapping rectangles on the two-dimensional square lattice. We show that the lattice version of the excluded volume theory gives several intriguing predictions regarding the percolation threshold of such systems. For the specific case of aligned and overlapping rectangles, the value of the percolation threshold as we increase the length of the rectangles depends on its width. For width one rectangles, the threshold monotonically decreases with the length, and for width greater than 2, it increases monotonically. For width 2, the threshold is independent of the length! Moreover, we get a non-zero percolation threshold, even in the limiting case where one side of the rectangle extends to infinity while keeping the other side finite. Similar results hold for other shapes and dimensions as well. The theory gives us good numerical estimates for the percolation thresholds for rectangles and other asymmetric shapes.

Our Monte Carlo simulation results confirm the theoretical predictions regarding the behavior of the percolation thresholds for aligned and overlapping rectangles. We verify that isotropy of the percolation threshold holds even with rectangles of large aspect ratios, just as in the continuum problem [37]. Additionally, critical exponents are obtained, demonstrating that the problem lies within the same universality class of lattice percolation. Furthermore, a comparison between results for aligned sticks and a mixture of sticks of varying orientations is presented, providing valuable insights into the impact of anisotropy on the percolation process.

The paper is organized as follows. In section II, we precisely define the model of overlapping rectangles on a square lattice. In Section III, we use excluded volume theory adapted to a lattice setting to analyze the problem. We use the theory to draw conclusions about the percolation of extended shapes other than rectangles and also in other dimensions. We give the results of simulation studies. The percolation threshold is evaluated, and results are compared. Isotropy of percolation threshold, critical exponents, and effect of mixed orientations are also given. We conclude in section IV.

## II. MODEL DEFINITION

In general, we look into the percolation problem of aligned, overlapping, non-symmetric shapes in lattice systems. For concreteness, we consider the case in which each unit of percolating objects is a rectangle with side lengths $k_1$ in the horizontal direction and $k_2$ in the vertical direction randomly positioned on a two-dimensional square lattice of size $L \times L$ (See Fig. 1). Without loss of generality, we will assume that $k_1 > k_2$ so that the rectangles are aligned in the horizontal direction.

The rectangles are allowed to overlap; hence, a site can be occupied more than once - a feature known as multiple occupancy. However, the complete overlapping of two rectangles is avoided (Note that the second rectangle doesn't contribute toward the total number of occupied sites). This will also retain the classical site percolation scenario when $k_1 = k_2 = 1$. Occupied neighboring sites are assumed to be connected, where the neighborhood is the Von-Neumann type. In other words, rectangles that share sites or are adjacent are considered connected entities, and rectangles that touch only at corners are considered non-connected (See Fig. 1).

As the density of occupied sites, denoted by $\phi$ (the ratio of the total number of occupied sites and $L^2$), is progressively increased by adding more rectangles to the lattice, a critical point is reached where a spanning cluster emerges. We define the spanning cluster as a connected path of rectangles in the vertical direction (perpendicular to the direction of alignment), signifying percolation in the system. Periodic boundary conditions are imposed in the horizontal direction. Snapshots of the final configurations for different values of $\phi$, with $k_1 = 3$ and $k_2 = 1$, are shown in Fig. 2. The percolation threshold is the specific value of $\phi$ where the spanning cluster first arises, indicating the transition from a non-percolating to a percolating state. $\phi$ may be considered the equivalent of the critical covered volume fraction (CCVF) in continuum percolation [11]. If $\eta$ represents the total number of sites occupied by all the objects, including multiple occupancy of sites, then we have the relation $\phi = 1 - \exp(-\eta)$.

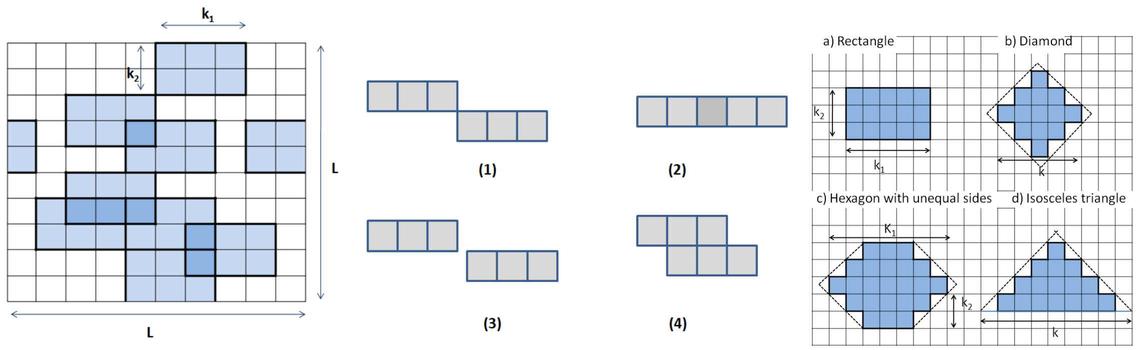


FIG. 1: (Left) Square lattice of size $L \times L$ with overlapping rectangles of length $k_1 = 3$ and width $k_2 = 2$ randomly distributed on it. Rectangles are aligned in the horizontal direction and can overlap. (Center) Different possible arrangements of two rectangles with $k_1 = 3$ and $k_2 = 1$ (1) Two rectangles touching at corners are considered unconnected. (2) Two overlapping rectangles are considered connected (3) Two non-overlapping rectangles (4) Two rectangles sharing edges are considered connected. (Right) A few of the extended shapes possible on a square lattice are shown with their defining parameters marked. The name for a shape is given based on the shape of its outline.

Again, $\eta$ is the equivalent of the volume density in continuum percolation. For convenience, we will use this terminology for the lattice percolation as well.

Note that when $k_1 = k_2 = k \to \infty$, the problem corresponds to the 2D continuum percolation problem of overlapping and aligned squares [27, 28]. The problem may be defined with shapes other than rectangles, and also for other types of lattices as well. For example, a few simple shapes that can be considered on the square lattice are shown in Fig. 1. We will discuss other extended shapes, lattices, and dimensions later in Sec.III.

## III. RESULTS AND DISCUSSION

### A. Lattice version of excluded volume theory

A major quantity of interest in the problem is the percolation threshold. i.e, the value of $\phi$ at which the system percolates for given values of $k_1$ and $k_2$. A useful analytical approximation technique to obtain the percolation threshold of systems for which the shapes of the percolating units are similar is the excluded volume theory [39, 40]. The excluded volume theory, initially proposed for continuum percolation systems, is based on the idea that the product of the number density of basic percolating units and the average excluded volume, which gives us the total excluded volume, is an invariant quantity for similar systems at the critical point. For continuum cases, the excluded volume is the volume around an object into which, if another object is placed, the two objects will overlap. Recently, the idea has been extended to lattice models as well, where the excluded volume is replaced by what is called the connectedness factor - the number of possible configurations of two basic percolating units such that they are connected - [27] (See Fig. 3). The connectedness factor will depend on the shape of the objects and also on their relative orientation. The lattice version of excluded volume theory thus says that the product of the number density of objects and the connectedness factor is a constant at criticality for similar systems where similarity refers to the shape of the basic percolating units. If $n$ is the number density of objects (total number of objects divided by the total number of lattice points) and $V_{ex}$ is the connectedness factor, then

$$n_c V_{ex} = B_c \tag{1}$$

where $n_c$ is the number density at the critical point and $B_c$ is the average number of connections an object has at criticality, which is expected to take the same value for similarly shaped objects. Here, the number density $n = \eta/V$ where $V$ is the volume (number of lattice sites occupied) of an object. Therefore, at criticality, we can write the CCVF,

$$\phi_c \approx 1 - \exp\left(-B_c \frac{V}{V_{ex}}\right) \tag{2}$$

Now for rectangles of sides $k_1$ and $k_2$, $V$ is simply $k_1 \times k_2$. We can find $V_{ex}$ by enumerating the configuration of two rectangles such that they are connected. It is easily seen that $V_{ex} = (2k_1 + 1)(2k_2 + 1) - 5$ (See Fig. 3). Therefore the covered volume fraction at the critical point,

$$\phi_c^{k1,k2} \approx 1 - \exp\left(-B_c \frac{k_1 k_2}{(2k_1+1)(2k_2+1) - 5}\right) \tag{3}$$

We can recover the continuum case in the limit of both $k_1$ and $k_2$ much greater than unity for which the above expression becomes independent of $k_1$ and $k_2$,

$$\phi_c \approx 1 - \exp(-B_c/4) \tag{4}$$

The value of $B_c$ here may be taken as that for systems of aligned squares or rectangles in the continuum $B_c = 4.3953711(5)$ [27].

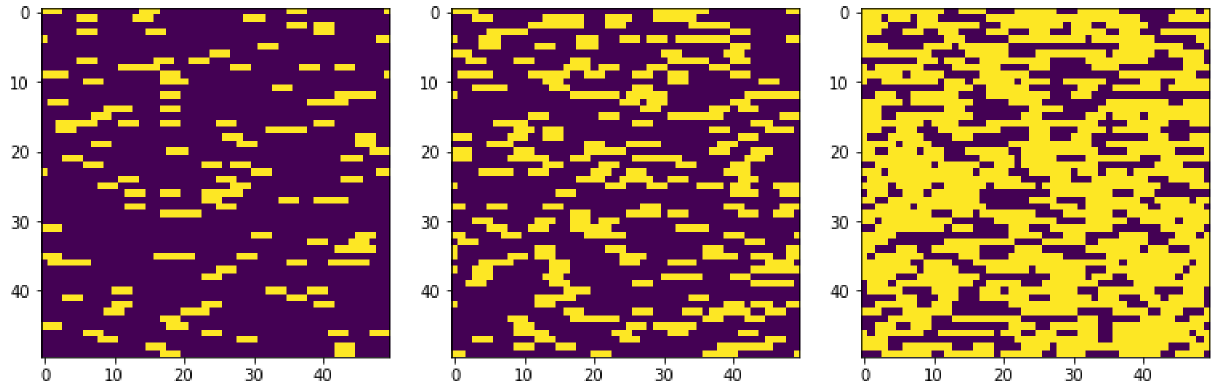


FIG. 2: Final configurations obtained with rectangles of side lengths $k_1 = 3, k_2 = 1$ for a $L = 50 \times 50$ square lattice and density of occupied sites $\phi = 0.14, 0.28$ and $0.64$. The lighter color corresponds to the rectangles. The first two configurations are non-percolating, while the third one has a spanning cluster in the vertical direction.

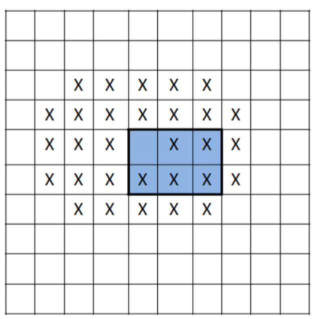

FIG. 3: If the top-left corner site of the rectangle is chosen as the index site (Site with no cross mark), the cross-marked cells constitute the excluded area of the rectangle shown which is of length $k_1 = 3$ and width $k_2 = 2$. i.e., if the index site of another rectangle falls within this area, it will overlap with the rectangle shown in the figure. The excluded volume is $V_{ex} = (2k_1 + 1)(2k_2 + 1) - 5$. Note that exact overlapping of rectangles is not allowed.

Several interesting inferences can be made from Eq. 3. Unlike the continuum case, the CCVF depends on the aspect ratio of the rectangles. Now consider the case in which we take $k_1$ to be very large for a fixed finite value of $k_2$. In the limit of $k_1 \to \infty$, the expression for CCVF becomes,

$$\phi_c^{k_1 \to \infty, k_2} \approx 1 - \exp\left(-B_c \frac{k_2}{2(2k_2 + 1)}\right) \tag{5}$$

This implies that for rectangles with finite width and length tending to infinity, the percolation threshold is finite and depends on the width of the rectangles. Moreover, unlike squares whose infinite limit gives us the continuum percolation problem, there seems to be no continuum problem corresponding to the case of rectangles with infinite length and finite width.

Eq. 3 gives how the percolation threshold approaches the limiting values given in Eq. 5 as we increase $k_1$. We can easily verify that for $k_2 = 1$, $\phi_c^{k_1,k_2}$ is a decreasing function of $k_1$ whereas for $k_2 > 2$, $\phi_c^{k_1,k_2}$ is increasing in $k_1$. Now for $k_2 = 2$, $\phi_c^{k_1,k_2}$ is independent of $k_1$, indicating that for overlapping and aligned rectangles of width two on a square lattice, the percolation threshold is independent of its length!

We can easily see that similar results will hold for extended shapes in other types of lattices and dimensions. For e.g., consider the problem of overlapping and aligned cuboids on a cubic lattice. If the side lengths of the cuboids are denoted by $k_1$, $k_2$ and $k_3$, then CCVF is given by,

$$\phi_c^{k1,k2,k3} \approx 1 - \exp\left(-B_c \frac{k_1 k_2 k_3}{V_{ex}^{k_1,k_2,k_3}}\right) \tag{6}$$

where $V_{ex}^{k_1,k_2,k_3}$ is,

$$\begin{aligned} V_{ex}^{k_1,k_2,k_3} = \; & (2k_1 - 1)(2k_2 - 1)(2k_3 - 1) + \\ & 2[(2k_1 - 1)(2k_2 - 1) + (2k_1 - 1)(2k_3 - 1) + \\ & (2k_2 - 1)(2k_3 - 1)] - 1 \end{aligned} \tag{7}$$

When all the side lengths tend to infinity, we recover the continuum percolation threshold of aligned and overlapping cubes in 3D, given by

$$\phi_c = 1 - \exp\left(\frac{-B_c}{8}\right) \tag{8}$$

with appropriate values of $B_c$. Also, from Eq. 7, we can verify that there are no integer values of $k_2$ and $k_3$, which will make the percolation threshold independent of $k_1$.

For the special case of thin sheets (say $k_2 \to \infty$ and $k_3 \to \infty$ with finite $k_1$), we get $\phi_c^{k_1} = 1 - \exp\left(-B_c \frac{k_1}{(8k_1 + 4)}\right)$. For the special case of sticks in 3D (say $k_2 = k_3 = 1$ and finite $k_1$), we get $\phi_c^{k_1} = 1 - \exp\left(-B_c \frac{k_1}{5(2k_1 - 1) + 1}\right)$. Letting the length of the

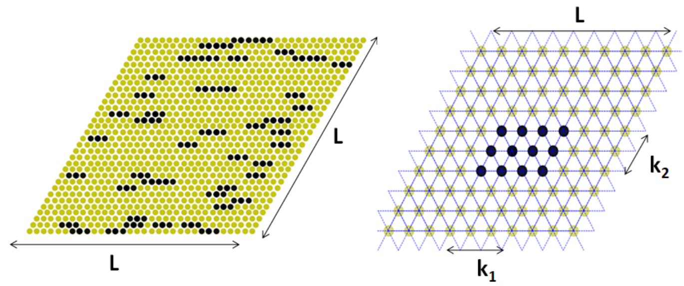


FIG. 4: (Left) Overlapping trimers (sticks of length $k_1 = 3$ and width $k_2 = 1$) on a triangular lattice. (Right) A parallelogram of $k_1 = 4$ and $k_2 = 3$ on a triangular lattice. The excluded volume will be $V_{ex} = (2k_1 + 1)(2k_2 + 1) - 3$.

sticks go to infinity, i.e., considering $k_1 \to \infty$, we get the finite limiting value $\phi_c = 1 - \exp(-B_c/10)$. As in the case of rectangles in a 2D square lattice, the limiting case of non-symmetric shapes does not give the corresponding continuum value.

Considering the general case of overlapping and aligned sticks of length $k_1$ randomly placed on a hypercubic lattice in $d$ dimension, we can write the dimension-dependent expression for the excluded volume of sticks,

$$V_{ex} = (2d - 1)(2k_1 - 1) + 1 \tag{9}$$

In this case, as the length of the sticks tends to infinity, we get the limiting threshold value in $d$ dimensions as,

$$\phi_c = 1 - \exp\left(-B_c \frac{1}{2(2d-1)}\right) \tag{10}$$

Now, for the continuum problem where aligned hypercubes are placed randomly in $d$ dimensional space, we have the percolation threshold $\phi_c = 1 - \exp(-B_c \frac{1}{2^d})$ [27]. Assuming $B_c$ to be the same, we have the curious observation that the percolation threshold of sticks whose length tends to infinity and that of hypercubes (squares in 2d) whose sides tend to infinity have the same value in $d \approx 3.66$ dimensions. In other words, for dimensions three and less, the percolation threshold of the infinite hypercubes is larger than that of infinite sticks whereas for dimensions greater than three, the percolation threshold of infinite sticks is larger than that of infinite hypercubes.

We can see that similar results, as discussed above, will also hold for other lattices. For e.g., on a triangular lattice, for objects of width $k_2$ and length $k_1$ (a "parallelogram"), (See Fig. 4), $V_{ex} = (2k_1 + 1)(2k_2 + 1) - 3$. In this case, we can see that $\phi_c$ is independent of $k_1$ for $k_2 = 1$ (sticks). Considering the limit $k_1 \to \infty$, we get $\phi_c = 1 - \exp\left(-B_c \frac{1}{6}\right)$. Whereas, if we consider parallelograms of equal side-lengths, say $k_1 = k_2 = k$, we get $\phi_c = 1 - \exp\left(-B_c \frac{1}{4}\right)$ in the limit of $k \to \infty$.

The above results show that on $N$-dimensional lattices when we are considering the percolation of aligned and extended shapes which can be described by $M$ independent length parameters, whenever we let a subset of these parameters go to infinity while keeping the remaining parameters finite, we expect a finite percolation threshold. Moreover, this limiting threshold value will differ from the corresponding continuum value.

We can also perform the excluded volume theory calculations for other shapes in Fig. 1. For diamonds of size $k$ (See Fig. 1), the area of the diamond is $V = \left(\frac{k^2+1}{2}\right)$ where $k$ must be odd. If we choose the central site of the diamond as the index site, the shape of the excluded area will be a bigger diamond of size $(2k + 1)$. We can calculate the areas and excluded areas of isosceles triangles, and hexagons in a similar way. If we choose the central site of the base of the isosceles triangle as the index site, the shape of the excluded area will be a hexagon of dimensions $(2k + 1)$ and $(k + 1)/2$. For the hexagon, the shape of the excluded area will also be a hexagon of dimensions $(2k_1 + 1)$ and $(2k_2 + 1)$.

In table I, we provide a summary of the expressions for the percolation thresholds obtained for a few different shapes using the discrete version of excluded volume theory. Wherever possible, we have included numerical values for the limiting cases, where $B_c$ values for different shapes are taken from existing studies of the continuum percolation problem of the corresponding shapes. Note that the diamond shape on an upright square lattice is equivalent to a square shape on a diagonal square lattice [41]. Hence, we expect that the $B_c$ values for the two are the same.

The following section presents simulation results for aligned and overlapping rectangles that closely agree with the predicted behavior. In particular, the agreement between the theory and simulation results seems exact for the limiting cases.

### B. Simulation results

We simulate the model of aligned and overlapping rectangles on a two-dimensional square lattice and study its percolation properties. To construct the model, aligned rectangles of size $k_1 \times k_2$ are distributed uniformly and randomly on a square lattice of size $L \times L$. The percolation probability is determined by detecting the presence of a spanning cluster in the vertical direction using the standard Hoshen-Kopelman algorithm [42]. Later in Sec. III B 1, we verify that the results remain unchanged if we consider the horizontal direction for defining the spanning cluster. For each system size and density of occupied sites, we generate a number of samples, and the percolation probability is evaluated as the fraction of samples that are percolating. The number of samples considered varies between $10^3$ for $L \leq 256$ and $10^2$ for higher values of $L$. Percolation probability is plotted against the den-

| Shape | Lattice | CCVF $\phi_c$ from discrete excluded volume theory | Limiting Values of $\phi_c$ |
|---|---|---|---|
| Rectangles of size $k_1 \times k_2$ | 2D Square | $1-\exp\left(-B_c \frac{k_1 k_2}{(2k_1+1)(2k_2+1)-5}\right)$ | $1-\exp\left(-B_c \frac{k_2}{4k_2+2}\right)$ ($k_1 \to \infty$, finite $k_2$) |
| Squares ($k_1 = k_2 = k$) | 2D Square | $1-\exp\left(-B_c \frac{k^2}{(2k+1)^2-5}\right)$ | 0.6667 ($k \to \infty$), $B_c$ = 4.3953711(5)[27] |
| Diamonds of linear size $k$ | 2D Square | $1-\exp\left(-B_c \frac{k^2+1}{4k(k+1)}\right)$ | $1-\exp(-B_c/4)$ ($k \to \infty$) |
| Triangles (isosceles of base $k$) | 2D Square | $1-\exp\left(-B_c \frac{k^2+2k+1}{2(3k^2+6k-1)}\right)$ | $1-\exp(-B_c/6)$ ($k \to \infty$) |
| Hexagon with dimensions $k_1$ and $k_2$ | 2D Square | $1-\exp\left(-B_c \frac{k_1+2k_2(k_1-k_2-1)}{(2k_1+1)+2(2k_2+1)(2k_1-2k_2-1)-1}\right)$ | $1-\exp\left(-B_c \frac{k_2}{4k_2+2}\right)$ ($k_1 \to \infty$, finite $k_2$) |
| Cubes ($k_1 = k_2 = k_3 = k$) | 3D Cubic | $1-\exp\left(-B_c \frac{k^3}{(2k-1)^3+6(2k-1)^2-1}\right)$ | 0.2773 ($k \to \infty$), $B_c$ = 2.5978(5)[27] |
| Sticks of length $k_1$ | 3D Cubic | $1-\exp\left(-B_c \frac{k_1}{5(2k_1-1)+1}\right)$ | 0.22877 ($k_1 \to \infty$) , $B_c$ = 2.5978(5)[27] |
| Parallelograms of size $k_1 \times k_2$ | 2D Triangular | $1-\exp\left(-B_c \frac{k_1 k_2}{(2k_1+1)(2k_2+1)-3}\right)$ | $1-\exp\left(-B_c \frac{k_2}{4k_2+2}\right)$ ($k_1 \to \infty$, finite $k_2$) |

TABLE I: Expressions for CCVF $\phi_c$ for various shapes and lattices from discrete excluded volume theory. For obtaining the numerical values in the last column, the $B_c$ value for a shape is assumed to be that of the continuum percolation problem of the same shape.

sity of occupied sites for different system sizes $L$, and by fitting each curve with the function $\frac{1+\mathrm{erf}[\phi-\phi_c(L)]/\Delta(L)}{2}$, we obtain the effective percolation threshold $\phi_c(L)$ for system size $L$ and width of the transition region $\Delta(L)$ [43]. We have the linear scaling relation,

$$\phi_c(L) = B * \Delta(L) + \phi_c(\infty) \tag{11}$$

where $\phi_c(\infty)$ is the percolation threshold in the limit of infinite system size $L \to \infty$, and $B$ is a constant [10]. A typical example of the plot of percolation probability against the density of occupied sites for $k_1 = 3, k_2 = 1$ rectangles for different values of $L$ and the corresponding plot of $\phi_c(L)$ against $\Delta(L)$ is shown in figure 5. $Y-$intercept of the latter plot gives the percolation threshold in the $L \to \infty$ limit. The percolation threshold is determined for rectangles of widths $k_2 = 1, 2$ and 3 for increasing values of $k_1$, and is shown in table II along with the values based on the lattice version of excluded volume theory discussed in Sec.III A.

Plots of the percolation threshold against $k_1$ for rectangles of width $k_2 = 1, 2$ and 3 are shown in Fig. 6. As predicted by the excluded volume theory, for $k_2 = 1$, the CCVF is monotonically decreasing, whereas for $k_2 = 3$, it is monotonically increasing with $k_1$. For $k_2 = 2$, critical density is nearly a constant. Percolation thresholds for the limiting case of $k_1 \to \infty$ are also shown in the figure.

For the particular case of aligned overlapping sticks ($k_2 = 1$), we saw that the dependence of percolation threshold on stick length $k_1$ is $\phi_c \sim 1 - \exp\left(-B_c \dfrac{k_1}{3(2k_1+1) - 5}\right)$. We may compare this with earlier studies on non-overlapping stick percolation. The latter often predicts a power-law dependence of the percolation threshold on stick length $\phi_c \sim 1/k_1^\alpha$ [15, 19] or an exponentially decreasing dependence [21].

### 1. *Isotropy of percolation threshold*

When considering $k_1 \times k_2$ rectangles on the square lattice, we fixed $k_2$ and varied $k_1$. For $k_1 > k_2$, this means that there is a preferred direction for the percolation to happen in finite systems. i.e., along the horizontal direction. In the last section, we defined the system as percolating when there is a spanning cluster in a direction perpendicular to the direction of alignment. A natural question is whether the percolation point is dif-

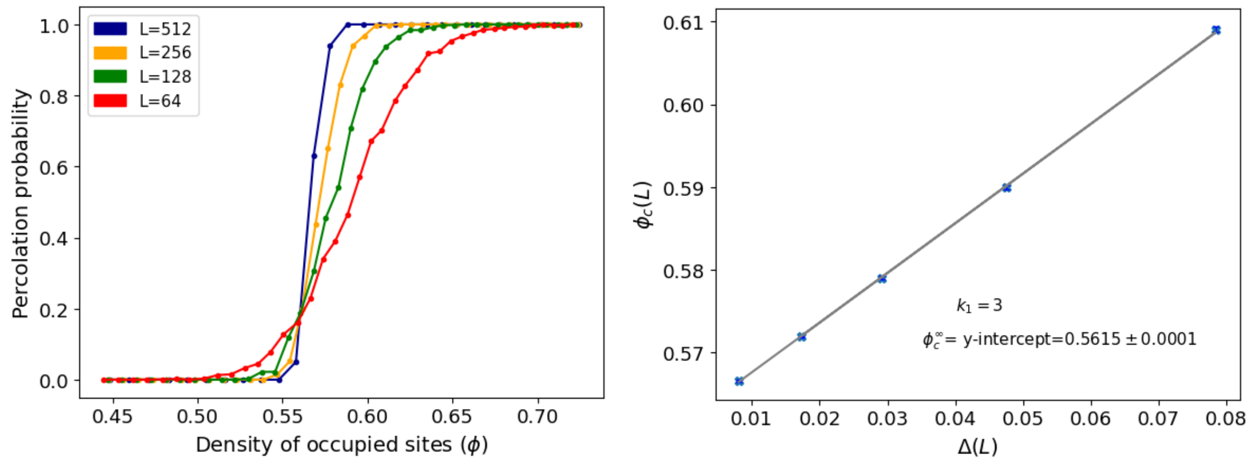


FIG. 5: (Left) Variation of the percolation probability with the density of occupied sites $\phi$ for $k_1 = 3, k_2 = 1$ rectangles for different system sizes $L$. (Right) Variation of the corresponding effective percolation threshold $\phi_c^L$ with the width of the transition region $\Delta(L)$. $Y-$intercept of the graph yields the threshold in the limit of infinite system size.

| Length $k_1$ | $\phi_c^{k_1}$, $k_2 = 1$ | $\phi_c^{k_1}$, $k_2 = 1$ Theory | $\phi_c^{k_1}$, $k_2 = 2$ | $\phi_c^{k_1}$, $k_2 = 3$ | $\phi_c^{k_1}$, $k_2 = 3$ Theory |
|---|---|---|---|---|---|
| 1 | 0.5927(4) | 0.667 | 0.5723(6) | 0.557(2) | 0.561 |
| 2 | 0.5715(18) | 0.585 | 0.5837(2) | 0.5838(8) | 0.585 |
| 3 | 0.5615(1) | 0.561 | 0.5876(3) | 0.5948(6) | 0.593 |
| 4 | 0.5523(9) | 0.550 | 0.5903(5) | 0.5965(4) | 0.597 |
| 5 | 0.5497(19) | 0.544 | 0.5916(5) | 0.597(2) | 0.599 |
| 7 | 0.5422(6) | 0.536 | 0.5909(26) | 0.603(3) | 0.602 |
| 9 | 0.5402(46) | 0.532 | 0.5933(8) | 0.6048(7) | 0.604 |
| 11 | 0.540(1) | 0.530 | 0.5922(27) | 0.605(2) | 0.605 |
| 13 | 0.539(4) | 0.529 | 0.5950(16) | 0.606(2) | 0.606 |
| 15 | 0.538(4) | 0.527 | 0.592(4) | 0.608(2) | 0.606 |
| 17 | 0.536(4) | 0.526 | 0.593(7) | 0.609(2) | 0.607 |
| 19 | 0.529(6) | 0.526 | 0.5927(41) | 0.609(1) | 0.607 |
| 21 | 0.528(7) | 0.525 | 0.5928(14) | 0.6102(1) | 0.607 |

TABLE II: Percolation thresholds of aligned and overlapping rectangles of length $k_1$ and width $k_2 = 1, 2$ and 3 on a square lattice determined from simulations. The corresponding results from the discrete excluded volume theory for $k_2 = 2$ and 3 are also given for comparison. For $k_2 = 2$, the theory gives a constant value for $\phi_c^{k_1} = 0.5848$.

ferent if we define a configuration as percolating when there is a spanning cluster in the direction of alignment.

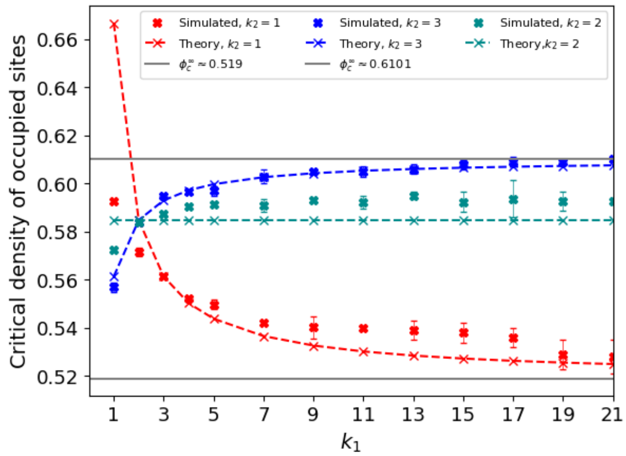


FIG. 6: Percolation threshold $\phi_c$ vs $k_1$ for rectangles of width $k_2 = 1, 2,$ and 3. The dashed lines represent the corresponding predicted values based on the discrete excluded volume theory, and the continuous horizontal lines represent the thresholds in the limit $k_1 \to \infty$.

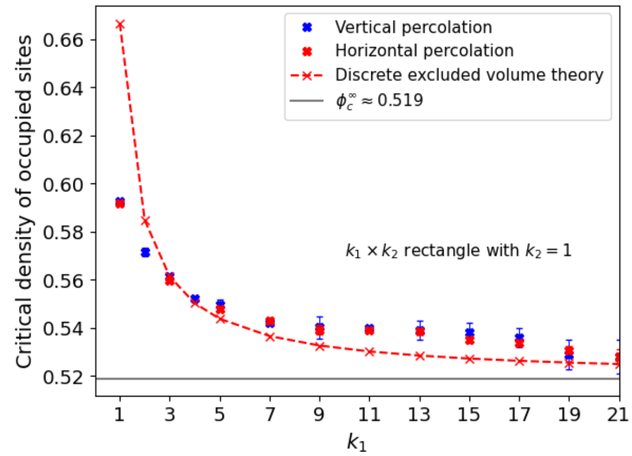


FIG. 7: Percolation threshold versus stick length $k_1$ for two different definitions of percolation. In one case, percolation is defined as the emergence of a spanning cluster in the vertical direction (perpendicular to the direction of alignment of sticks). In the other scenario, it is defined as the emergence of a spanning cluster in the horizontal direction (in the direction of alignment of sticks). Results from excluded volume theory and the limiting value of the threshold for sticks of infinite length are also shown.

In continuum percolation models, previous studies suggest isotropy of the percolation threshold even for highly anisotropic systems like rectangles with large aspect ratios [37]. It is found that the effective percolation threshold for a finite system size in the direction of alignment will always be smaller compared to that in the orthogonal direction. However, this difference will decrease with increasing system size and finally vanish in the infinite system size limit. We verify that this isotropy of the percolation threshold holds for aligned rectangles on the square lattice as well. The critical density of occupied sites evaluated (See figure 7) shows that the percolation threshold is also isotropic in the case of lattices. Just as in the continuum case, this isotropy may be explained based on the uniqueness of the spanning cluster in percolation problems [37].

### 2. *Critical exponents for aligned stick model*

Critical exponents are evaluated using finite size scaling methods [9, 10]. If $S_{max}$ is the size of the largest cluster and $S$ is the total size of the remaining clusters, we have the scaling relations,

$$S_{max} \sim L^{d_f} \tag{12}$$

$$S \sim L^{\frac{\gamma}{\nu}} \tag{13}$$

where $d_f$ is the fractal dimension, $\gamma$ is the exponent corresponding to mean cluster size and $\nu$ is the correlation length exponent. The plots of $S_{max}$ against system size $L$ and $S$ against $L$ for $k_1 = 3, k_2 = 1$ are shown in Fig. 8. The slopes of log-log plots give the fractal dimension $d_f = 1.90 \pm 0.04$ and the ratio $\frac{\gamma}{\nu} = 1.77 \pm 0.06$. The percolation probability $P(\phi, L)$ is expected to scale with the system size $L$ as

$$P(\phi) = f((\phi - \phi_c)L^{\frac{1}{\nu}}) \tag{14}$$

where $f$ is the scaling function. This implies that the curves of percolation probability for different $L$ when plotted as a function of $(\phi - \phi_c)L^{\frac{1}{\nu}}$ with the correct value for $\phi_c$ and $\nu$, will collapse to a single curve. In Fig. 9, we verify that curves for various system sizes fall on top of each other for $\frac{1}{\nu} = 0.75$ and $\phi_c = 0.5615$.

The other critical exponents are evaluated by making use of the relations,

$$d_f = d - \frac{\beta}{\nu} \tag{15}$$

$$\sigma = \frac{1}{\nu d_f} \tag{16}$$

$$\tau = 1 + \frac{d}{d_f} \tag{17}$$

We obtain $\nu \approx 1.33$, $\beta \approx 0.133$, $\sigma \approx 0.3948$, $\gamma \approx 2.36$ and $\tau \approx 2.05$. Finally, if $P_{max}(\phi)$ is the probability of a site belonging to the largest cluster, it is expected to scale with the system size $L$ as,

$$P_{max}(\phi) = L^{-\frac{\beta}{\nu}} F((\phi - \phi_c)L^{\frac{1}{\nu}}) \tag{18}$$

where $F$ is the scaling function and $\beta$ is the order parameter exponent. When $P_{max}L^{\frac{\beta}{\nu}}$ is plotted against $\phi$

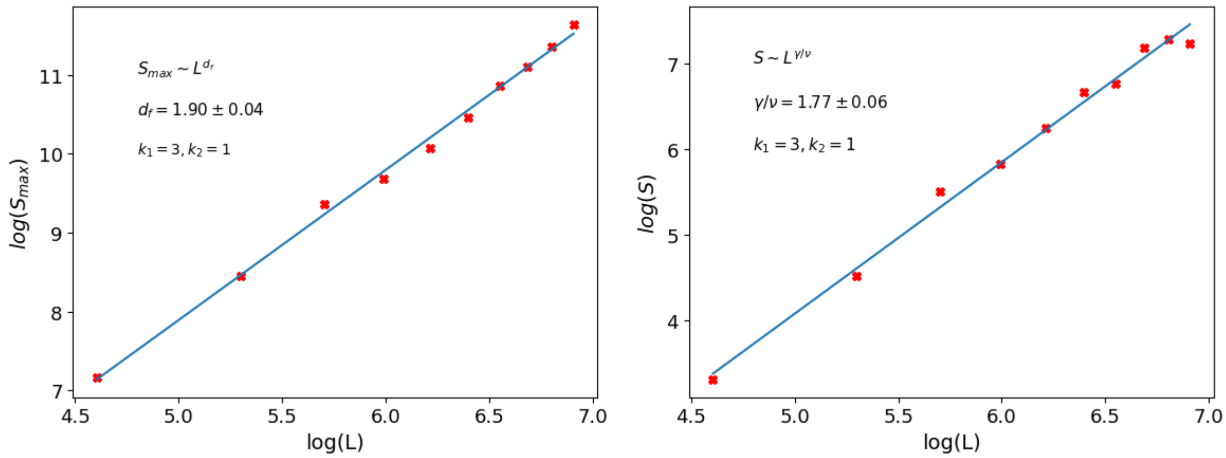


FIG. 8: (Left) Variation of the size of the largest cluster $S_{max}$ with system size $L$ along with the best straight line fit for $k_1 = 3, k_2 = 1$. The slope of the log-log plot gives the fractal dimension $d_f$. (Right) Variation of the size of the finite clusters $S$ with system size $L$ along with the best straight line fit. The slope of the log-log plot gives the ratio of critical exponents $\frac{\gamma}{\nu}$.

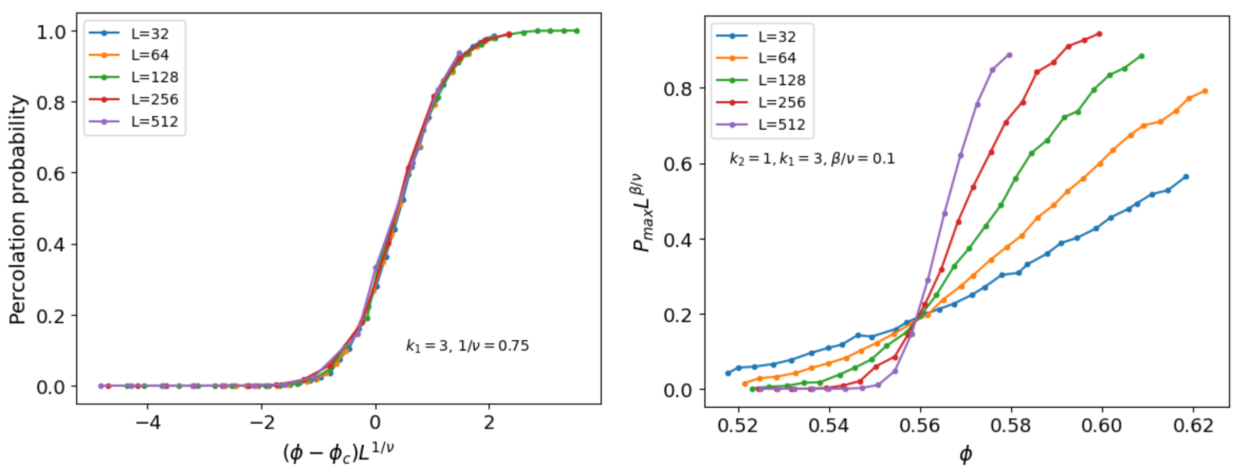


FIG. 9: (Left)Plots of percolation probability against $(\phi - \phi_c)L^{\frac{1}{\nu}}$ for $k_1 = 3, k_2 = 1$ with $\frac{1}{\nu} = 0.75$ and $\phi_c = 0.5615$. (Right) Plot of $P_{max}L^{\beta/\nu}$ vs $\phi$ for $k_1 = 3$ and $k_2 = 1$ for $\frac{\beta}{\nu} = 0.1$, $\frac{1}{\nu} = 0.75$

for different $L$, all curves will pass through a single point which is at $\phi = \phi_c$ as verified in Fig. 9.

The obtained values of various critical exponents align with that of the standard percolation problem [9]. Similar values are obtained for other values of $k_1$ and $k_2$ as well. Hence we can conclude that the percolation problem of aligned overlapping rectangles belongs to the standard percolation universality class. Note that the above results were obtained only for a limited range of $k_1$ and $k_2$ values. There remains the possibility that the value of the critical exponents could be different for higher values of $k_1$ and $k_2$, especially in the limiting cases of $k_1 \to \infty$ with finite $k_2$.

### 3. *Comparison with rectangles of mixed orientation*

We consider the percolation of overlapping sticks having both orientations (rectangles with $k_1 = 1$ or $k_2 = 1$ in our notation) here. The problem has been considered earlier in [30]. To study the percolation as we vary the relative fraction of sticks having horizontal and vertical orientations, we can define the parameter [19]

$$s = \frac{|\,(n_v - n_h)\,|}{|\,(n_v + n_h)\,|} \tag{19}$$

where $n_v$ and $n_h$ represent the fraction of sticks that are vertically and horizontally oriented, respectively (See Fig. 10). Thus $s = 1$ corresponds to the fully aligned case, and $s = 0$ corresponds to the isotropic case where the average fraction of sticks that are vertically and horizontally oriented is equal. The values of the percolation threshold obtained for a few different values of the parameter $s$ are given in Table. III. We can see that the percolation threshold increases with an increase in the degree of alignment, which is also seen in earlier results for non-overlapping sticks [19].

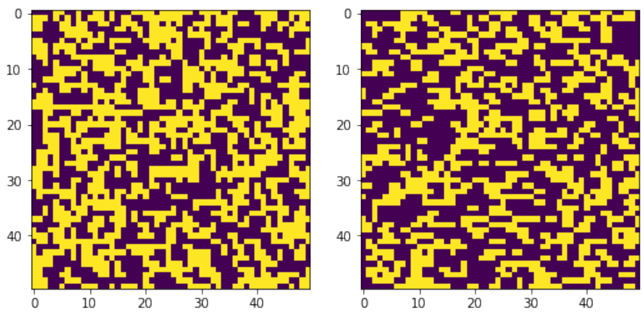

FIG. 10: a) Horizontal and vertical dimers on $50 \times 50$ square lattice with the density of occupied sites 0.5. $s \approx 0$. b)Horizontal and vertical dimers on $50 \times 50$ square lattice with $s \approx 0.66$

| $s$ | Length of the sticks | Critical density of occupied sites |
|---|---|---|
| $s = 0$ | 2 | 0.5483(2) |
| $s = 0.66$ | 2 | 0.55885(14) |
| $s = 1$ | 2 | 0.5715(18) |
| $s = 0$ | 3 | 0.50004(64) |
| $s = 0.33$ | 3 | 0.5325(38) |
| $s = 1$ | 3 | 0.5576(15) |

TABLE III: Effect of alignment in dimer ($k = 2$ stick) and trimer ($k = 3$ stick)percolation. The critical density of occupied sites for different values of $s$ is shown.

#### *4. Discrete to continuum transition*

Models of extended shapes on lattices interpolate between discrete and continuum percolation models. With symmetric shapes like squares, letting the size of the squares go to infinity - which is equivalent to letting the lattice spacing go to zero - for an infinite lattice - yields the continuum limit [28]. The convergence of discrete hypercubes and hyperspheres to corresponding continuum models and the universality of the convergence rate is discussed in [28, 29]. However, with non-symmetric shapes like rectangles considered here, letting the length of one side of the shape go to infinity while keeping the other side fixed - similar to letting the lattice spacing go to zero only along one direction - does not seem to correspond to any continuum picture. Note that the continuum percolation threshold for aligned rectangles of all aspect ratios is the same as that of aligned squares [37]. However, as we saw, no corresponding result exists for lattices. Just to emphasize the difference in values, we find the limiting value of $2D$ stick percolation model as $\phi_c^{\infty} \approx 0.519$, whereas the continuum percolation threshold for aligned rectangles is $\phi_c \approx 0.668$ [37]. Similarly, in three dimensions, we find the limiting value of the $3D$ stick model as $\phi_c \approx 0.229$, whereas the continuum percolation threshold value for overlapping cubes is $\phi_c \approx 0.277$ [13].

## IV. CONCLUSIONS

Percolation of overlapping and extended shapes on lattices that interpolate between discrete and continuum models is a topic of recent interest. This work investigates the percolation of aligned, overlapping, and non-symmetric shapes on lattices. First, we use a lattice version of excluded volume theory to show that shape-asymmetry leads to some intriguing consequences regarding the percolation behavior. As a quintessential example of an asymmetric object, we investigate the case of aligned and overlapping rectangles on a square lattice in detail and show that for rectangles of width unity (sticks), the percolation threshold is a monotonically decreasing function of the stick length. In contrast, for rectangles of width greater than two, the percolation threshold is a monotonically increasing function of the length. The percolation threshold is independent of its length for rectangles of width two. The limiting case of the length of the rectangles going to infinity shows some remarkable behavior. It is found that the limiting value depends upon the rectangle's width. This is in stark contrast to the continuum percolation of aligned overlapping rectangles for which the percolation threshold is independent of the aspect ratio. The general notion that continuum percolation of aligned objects can be considered the limiting case of corresponding discrete model [28] is only valid for symmetric shapes. Expressions for percolation threshold for a few different asymmetric shapes are also obtained using the lattice version of excluded volume theory.

We present simulation results for the percolation thresholds, which verify the predictions of the excluded volume theory. The percolation thresholds from simulations show that the results of the lattice-excluded volume theory give very good results, and its accuracy increases with the size of the extended shapes. We also determine the critical exponents for the model using simulations, which confirm that the model with finite rectangles belongs to the same universality class as the usual lattice percolation.

In addition, we verify the isotropy of the percolation threshold by obtaining the thresholds for percolation transition with the spanning cluster considered in the direction of alignment and the orthogonal direction. We also compare our results with models where rectangles of mixed orientation are allowed and models where non-overlapping sticks with varying degrees of anisotropy are considered. Our simulation results show that alignment increases the percolation threshold even for overlapping shapes.

The findings of this study contribute to a deeper understanding of percolation behavior in lattice systems with overlapping shapes and shed light on the intricate interplay between discrete and continuum models. Our results show that the lattice version of the excluded volume theory is remarkably accurate in describing the behavior of percolation of aligned and overlapping shapes on lattices.

There are many avenues for further research. Other factors like polydispersity of asymmetric shapes will affect the critical behavior of the percolation model [18]. Especially the effect of unbounded size distribution on critical behavior is a problem of interest [44, 45]. Another potential area for further investigation is the relation to extended neighborhood percolation models [31].

## Acknowledgments

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