SEAT NUMBER CONFIGURATION OF THE BOX-BALL SYSTEM, AND ITS RELATION TO THE 10-ELIMINATION AND INVARIANT MEASURES

HAYATE SUDA

ABSTRACT. The box-ball system (BBS) is a soliton cellular automaton introduced by [TS], and it is known that the dynamics of the BBS can be linearized by several methods. Recently, a new linearization method, called the seat number configuration, is introduced by [MSSS]. The aim of this paper is fourfold. First, we introduce the k-skip map $\Psi_k : \Omega \to \Omega$, where Ω is the state space of the BBS, and show that the k-skip map induces a shift operator of the seat number configuration. Second, we show that the k-skip map is a natural generalization of the 10-elimination, which was originally introduced by [MIT] to solve the initial value problem of the periodic BBS. Third, we generalize the notions and results of the seat number configuration and the k-skip map for the BBS on the whole-line. Finally, we investigate the distribution of $\Psi_k(\eta), \eta \in \Omega$ when the distribution of η belongs to a certain class of invariant measures of the BBS introduced by [FG]. As an application of the above results, we obtain the long-time behavior of the integrated current of $\Psi_k(\eta)$ with Markov stationary initial distributions.

1. INTRODUCTION

The box-ball system (BBS) is a soliton cellular automaton introduced by [TS]. In this paper, we consider the BBS(ℓ), $\ell \in \mathbb{N} \cup \{\infty\}$, which is a class of generalizations of the BBS introduced by [TM]. The configuration space Ω is either $\{0,1\}^{\mathbb{N}}$ or a certain subset of $\{0,1\}^{\mathbb{Z}}$, and depending on the configuration space, we refer to the BBS as the BBS on the half-line or the BBS on the whole-line, respectively. Here, for $\eta \in \Omega$, $\eta(x) = 0$ (resp. $\eta(x) = 1$) means that the site x is vacant (resp. occupied). The dynamics of the BBS(ℓ) on the half-line is given via the *carrier with capacity* ℓ, $W_\ell(\eta, \cdot) : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, which is recursively constructed as follows :

- An empty carrier starts from $x = 0$, i.e., $W_\ell(\eta, 0) = 0$.
- If there is a ball at x and the carrier is not full, then the carrier picks it up, i.e., if $\eta(x) = 1$ and $W_\ell(\eta, x-1) < \ell$, then $W_\ell(\eta, x) = W_\ell(\eta, x-1) + 1$.
- If x is vacant and the carrier is not empty, i.e., then the carrier puts down a ball, i.e., if $\eta(x) = 0$ and $W_\ell(\eta, x-1) > 0$ then $W_\ell(\eta, x) = W_\ell(\eta, x-1) - 1$.
- Otherwise, the carrier just goes through.

Then, by using $W_\ell(\eta, \cdot)$, the one-step time evolution of the BBS is described by the operator $T_\ell : \Omega \to \Omega$ defined as

$$T_\ell \eta(x) := \eta(x) + W_\ell(\eta, x-1) - W_\ell(\eta, x). \tag{1.1}$$

Note that for the case $\Omega \subset \{0,1\}^{\mathbb{Z}}$, the domain of $W_\ell(\eta, \cdot)$ can be extended to $\mathbb{Z}$, and the one-step time evolution of the BBS(ℓ) on the whole-line is also described by $T_\ell : \Omega \to \Omega$, which is defined via the same equation (1.1), see Section 4 for details.

Despite the simple description of the dynamics, it is known that the BBS exhibits solitonic behavior. In addition, the relationships between the BBS and classical /

quantum integrable systems have been well-studied, see [IKT] and references therein for details. As in the cases of some integrable systems, such as the Korteweg-de Vries equation, the initial value problem of the BBS can be solved via the explicit linearization methods [T, KOSTY, MIT, FNRW]. Recently, a new linearization of the BBS(ℓ) on the half-line, called the seat number configuration, is introduced by [MSSS], and by using the seat number configuration, the explicit relationships between the rigged configuration and the slot decomposition are investigated.

The aim of this paper is fourfold. The first is to define a natural shift operator for the seat number configuration, and this is done via the *k-skip map*, $\Psi_k : \Omega \to \Omega$, introduced in Section 2. The second is to describe the relationships between the seat number configuration and the 10-elimination, and this will be done in Section 3. The 10-elimination was originally introduced in [MIT] to solve the initial value problem of the periodic BBS, and in this paper, the 10-elimination will be defined as a map $\Phi_1 : \Omega_{<\infty} \to \Omega_{<\infty}$, where $\Omega_{<\infty}$ is the set of all finite ball configuration on the half-line :

$$\Omega_{<\infty} := \left\{ \eta \in \{0,1\}^{\mathbb{N}} \ ; \ \sum_{x \in \mathbb{N}} \eta(x) < \infty \right\}.$$

We will show that the 1-skip map is a natural generalization of the 10-elimination on Ω, i.e.,

$$\Psi_1(\eta) = \Phi_1(\eta) \text{ for any } \eta \in \Omega_{<\infty},$$

see Theorem 3.2 for the precise statement. This result reveals the explicit relationship between the seat number configuration and the 10-elimination. Note that, as mentioned later, Theorem 3.2 can be proved indirectly by using the results from [KS] and [MSSS]. However, in this paper we directly prove Theorem 3.2 and no such result is used in the proof. The third is to generalize the notions and results of the seat number configuration and the k-skip map for the BBS on the whole-line, and this is done in Section 4. Finally, we investigate several statistical properties of the BBS on the whole-line in Section 5. First we investigate the distribution of $\Psi_k(\eta)$ on Ω when the distribution of η belongs to a certain class of invariant measures of the BBS introduced by [FG]. In particular, we will show that if $(\eta(x))_{x \in \mathbb{Z}}$ is the sequence of i.i.d. Bernoulli(ρ) with $0 \le \rho < \frac{1}{2}$ or two-sided stationary Markov chain with transition matrix $Q = (Q(i,j))_{i,j=0,1}$ satisfying $Q(0,1) + Q(1,1) < 1$, then $(\Psi_k(\eta(x)))_{x \in \mathbb{Z}}$ is also a two-sided stationary Markov chain. Combining this and the results in [CKST], we will obtain the long-time behavior of integrated ball current $C^n(\Psi_k(\eta(x)))$, where $C^n(\eta)$ is the total number of balls crossing the origin $x = 0$ up to n-step time evolution. Note that since the seat number configuration is closely relates to the *local energy* [MSSS, Remark 4.1], C^n can be considered as the integrated energy current of the BBS configuration.

The contents of Section 2,3 and 4 are closely relate to [MSSS] and can be considered as a continuation of it. On the other hand, the contents of Section 5 are related to the field of the randomized BBS, which has been actively studied in recent years [CKST, CS, CS2, FG, KL, KLO, KMP, KMP2, KMP3, LLP, LLPS]. We expect that the statistical properties of the seat number configuration and the k-skip map introduced in this paper will be useful in the analysis of the randomized BBS. In fact, in the forthcoming paper [OSS], the space-time scaling limit of a tagged soliton is considered, and k-skip map plays an essential role in the proof. We look forward to seeing more studies utilizing the k-skip map in the future.

2. Seat number configuration

In this section, first we briefly recall the definition of the carrier with seat numbers and the corresponding seat number configuration introduced in [MSSS]. Then, in the subsequent subsection, we introduce the notion of the k-skip map, and show that the k-skip map induces a shift operator of the seat number configuration. Note that throughout this section, $\Omega = \{0,1\}^{\mathbb{N}}$.

2.1. **Carrier with seat numbers.** We consider a situation where the seats of the carrier are numbered by $\mathbb{N}$, and introduce functions $\mathcal{W}_k(\eta,\cdot) : \mathbb{Z}_{\geq 0} \to \{0,1\}$ which represents the number of ball sitting in No.k seat of the carrier, i.e., $\mathcal{W}_k(\eta,x) = 0$ (resp. $\mathcal{W}_k(\eta,x) = 1$) means that the No.$k$ seat is vacant (resp. occupied) at site x. Then, the refined construction of the carrier with seat numbers is given as follows :

- An empty carrier starts from $x = 0$, i.e., $\mathcal{W}_k(\eta,0) = 0$ for any $k \in \mathbb{N}$.
- If there is a ball at site x, then the carrier picks the ball and puts it at the empty seat with the smallest seat number, i.e, if $\eta(x) = 1$ and $\min\{k \in \mathbb{N} \ ; \ \mathcal{W}_k(\eta,x-1) = 0\} = \ell$, then

$$\mathcal{W}_k(\eta,x) = \begin{cases} 1 & k = \ell, \\ \mathcal{W}_k(\eta,x-1) & k \neq \ell. \end{cases}$$

- If the site x is empty, and if there is at least one occupied seat, then the carrier puts down the ball at the occupied seat with the smallest seat number, i.e, if $\eta(x) = 0$ and $\min\{k \in \mathbb{N} \ ; \ \mathcal{W}_k(\eta,x-1) = 1\} = \ell < \infty$, then

$$\mathcal{W}_k(\eta,x) = \begin{cases} 0 & k = \ell, \\ \mathcal{W}_k(\eta,x-1) & k \neq \ell. \end{cases}$$

- Otherwise, the carrier just goes through, i.e., if $\eta(x) = 0$ and $\mathcal{W}_k(\eta,x-1) = 0$ for any $k \in \mathbb{N}$, then

$$\mathcal{W}_k(\eta,x) = \mathcal{W}_k(\eta,x-1) = 0$$

  for any $k \in \mathbb{N}$.

In other words, we define $\mathcal{W}_k(\eta,\cdot) : \mathbb{Z}_{\geq 0} \to \{0,1\}$ recursively as follows : $\mathcal{W}_k(\eta,0) := 0$ for any $k \in \mathbb{N}$, and

$$\mathcal{W}_k(\eta,x) = \mathcal{W}_k(\eta,x-1) + \eta(x)(1-\mathcal{W}_k(\eta,x-1))\prod_{\ell=1}^{k-1}\mathcal{W}_\ell(\eta,x-1) \\ - (1-\eta(x))\mathcal{W}_k(\eta,x-1)\prod_{\ell=1}^{k-1}(1-\mathcal{W}_\ell(\eta,x-1)).$$

From the above construction of $\mathcal{W}_k$, we see that

$$W_\ell(\eta,x) = \sum_{k=1}^{\ell}\mathcal{W}_k(\eta,x), \tag{2.1}$$

for any $\ell \in \mathbb{N}$ and $x \in \mathbb{N}$, where W_ℓ is the carrier with capacity ℓ defined in Introduction. Then, the seat number configuration $\eta_k^\sigma \in \Omega, k \in \mathbb{N}, \sigma \in \{\uparrow,\downarrow\}$ is defined as

$$\eta_k^{\uparrow}(x) := \begin{cases} 1 & \text{if } \mathcal{W}_k(\eta, x) > \mathcal{W}_k(\eta, x-1), \\ 0 & \text{otherwise} \end{cases}$$
$$= \eta(x)(1 - \mathcal{W}_k(\eta, x-1)) \prod_{\ell=1}^{k-1} \mathcal{W}_\ell(\eta, x-1), \tag{2.2}$$

and

$$\eta_k^{\downarrow}(x) := \begin{cases} 1 & \text{if } \mathcal{W}_k(\eta, x) < \mathcal{W}_k(\eta, x-1), \\ 0 & \text{otherwise} \end{cases}$$
$$= (1 - \eta(x))\mathcal{W}_k(\eta, x-1) \prod_{\ell=1}^{k-1} (1 - \mathcal{W}_\ell(\eta, x-1)). \tag{2.3}$$

Here, $\eta_k^{\uparrow}(x) = 1$ (resp. $\eta_k^{\downarrow}(x) = 1$) means that a ball gets into (resp. off) No.k seat at site x. We also define $r(\eta, \cdot) \in \Omega$ as

$$r(\eta, x) := 1 - \sum_{k \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_k^{\sigma}(x),$$

and $r(\eta, x) = 1$ means that the carrier goes through site x. We note that $\mathcal{W}_k$ can be represented as

$$\mathcal{W}_k(\eta, x) = \sum_{y=1}^{x} \left(\eta_k^{\uparrow}(y) - \eta_k^{\downarrow}(y) \right), \tag{2.4}$$

for any $k \in \mathbb{N}$ and $x \in \mathbb{N}$. Also, we recall a useful lemma proved in [MSSS].

Lemma 2.1 (Lemma 3.1 in [MSSS])**.** *Suppose that $\eta \in \Omega$. Then, for any $k \in \mathbb{N}$ and $x \in \mathbb{N}$, we have the following.*

(1) $\eta_k^{\uparrow}(x) = 1$ implies $\sum_{y=1}^{x} \left(\eta_\ell^{\uparrow}(y) - \eta_\ell^{\downarrow}(y) \right) = 1$ for any $1 \le \ell \le k$.
(2) $\eta_k^{\downarrow}(x) = 1$ implies $\sum_{y=1}^{x} \left(\eta_\ell^{\uparrow}(y) - \eta_\ell^{\downarrow}(y) \right) = 0$ for any $1 \le \ell \le k$.
(3) $r(\eta, x) = 1$ implies $\sum_{y=1}^{x} \left(\eta_\ell^{\uparrow}(y) - \eta_\ell^{\downarrow}(y) \right) = 0$ for any $\ell \in \mathbb{N}$.

For later use, we introduce some functions. For any $k \in \mathbb{N}$ and $x \in \mathbb{Z}_{\ge 0}$, we define $\xi_k(\eta, x)$ as

$$\xi_k(\eta, x) := x - \sum_{y=1}^{x} \sum_{\ell=1}^{k} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_\ell^{\sigma}(y)$$
$$= \sum_{y=1}^{x} \left(r(y) + \sum_{\ell \ge k+1} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_\ell^{\sigma}(y) \right).$$

Note that $\xi_k(\eta, \cdot)$ is non-decreasing, and

$$|\xi_k(\eta, x+1) - \xi_k(\eta, x)| \le 1,$$

for any $k \in \mathbb{N}$ and $x \in \mathbb{Z}_{\ge 0}$. Then, for any $k \in \mathbb{N}$ and $i \in \mathbb{Z}_{\ge 0}$, we define $s_k(\eta, i)$ as

$$s_k(\eta, i) := \min \{ x \in \mathbb{Z}_{\ge 0} \ ; \ \xi_k(\eta, x) = i \},$$

where $\min \varnothing := \infty$. Note that $s_k(\eta, 0) = 0$ for any $k \in \mathbb{N}$. Finally, for any $k \in \mathbb{N}$ and $i \in \mathbb{Z}_{\ge 0}$, we define $\zeta_k(\eta, i)$ as

$$\zeta_k(\eta, i) := \sum_{y = s_k(\eta, i)+1}^{s_k(\eta, i+1)} \left(\eta_k^{\uparrow}(y) - \eta_{k+1}^{\uparrow}(y) \right). \tag{2.5}$$

We note that thanks to Lemma 2.1, $\zeta_k(\eta, i)$ can be represented as

$$\zeta_k(\eta, i) = \sum_{y=s_k(\eta,i)+1}^{s_k(\eta,i+1)} \left(\eta_k^{\downarrow}(y) - \eta_{k+1}^{\downarrow}(y) \right). \tag{2.6}$$

Remark 2.1. *Note that ζ can be considered as a map $\zeta : \Omega_r \to \mathbb{Z}_{\geq 0}^{\mathbb{N} \times \mathbb{Z}_{\geq 0}}$, and it is shown in* [CS, MSSS] *that ζ is a bijection between $\Omega_r \subset \Omega$ and $\bar{\Omega} \subset \mathbb{Z}_{\geq 0}^{\mathbb{N} \times \mathbb{Z}_{\geq 0}}$, where Ω_r and $\bar{\Omega}$ are defined as*

$$\Omega_r := \left\{ \eta \in \Omega \ ; \ \sum_{x \in \mathbb{N}} r(\eta, x) = \infty \right\},$$

$$\bar{\Omega} := \left\{ \zeta \in \mathbb{Z}_{\geq 0}^{\mathbb{N} \times \mathbb{Z}_{\geq 0}} \ ; \ \max\{k \in \mathbb{N} \ ; \ \zeta_k(i) > 0\} \text{ for any } i \right\}.$$

We conclude this subsection by quoting a main result in [MSSS]. In [MSSS, Proposition 2.3 and Theorem 2.3], it is shown that the dynamics of the BBS(ℓ) is linearized in terms of ζ :

Theorem 2.1 (Proposition 2.3 and Theorem 2.3 in [MSSS])**.** *Suppose that $\eta \in \Omega$ and $s_k(\eta, i+1) < \infty$ for some $k \in \mathbb{N}$ and $i \in \mathbb{Z}_{\geq 0}$. Then, we have*

$$\zeta(T_\ell \eta, i) = \zeta(\eta, i - \min\{k, \ell\}),$$

with convention $\zeta(\eta, i) = 0$ for any $i < 0$.

Remark 2.2. *Note that if $\eta \in \Omega_r$, then $s_k(\eta, i) < \infty$ for any $k \in \mathbb{N}$ and $i \in \mathbb{Z}_{\geq 0}$. Therefore, by combining the bijectivity of ζ and Theorem 2.1, the initial value problem of the BBS(ℓ) can be solved when the initial configuration is an element of Ω_r.*

2.2. **The k-skip map.** To study the relationship between the 10-elimination and the seat number configuration, we introduce the *k-skip map*, $\Psi_k : \Omega \to \Omega$, $k \in \mathbb{N}$, defined as

$$\Psi_k(\eta)(x) := \eta(s_k(\eta, x)).$$

We may call $\Psi_k(\eta)$ the k-skipped configuration (of η). In this subsection, we investigate some basic properties of $(\Psi_k)_{k \in \mathbb{N}}$ from the viewpoint of the seat number configuration. A key observation is that Ψ_k shifts the seat numbers as follows :

Proposition 2.1. *Suppose that $\eta \in \Omega$. For any $k, \ell \in \mathbb{N}$, $\sigma \in \{\uparrow, \downarrow\}$ and $x \in \mathbb{N}$, we have*

$$\Psi_k(\eta)_\ell^\sigma(x) = \eta_{k+\ell}^\sigma(s_k(x)). \tag{2.7}$$

In particular,

$$r(\Psi_k(\eta), x) = r(\eta, s_k(x)).$$

As a consequence of Proposition 2.1, we obtain the following semigroup property of $(\Psi_k)_{k \in \mathbb{Z}_{\geq 0}}$:

Proposition 2.2. *Suppose that $\eta \in \Omega$. For any $k, \ell \in \mathbb{N}$ and $x \in \mathbb{Z}_{\geq 0}$, we have*

$$\Psi_k(\Psi_\ell(\eta))(x) = \Psi_{k+\ell}(\eta)(x).$$

In addition, we will also show that Ψ_k induces a *shift operator* of $(\zeta_k)_{k \in \mathbb{N}}$:

Proposition 2.3. *Suppose that $\eta \in \Omega$. For any $k \in \mathbb{N}$, $\ell \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{\geq 0}$, we have*

$$\zeta_k(\Psi_\ell(\eta), i) = \zeta_{k+\ell}(\eta, i).$$

From Theorem 2.1 and Proposition 2.2, we see that Ψ_k and T_ℓ are both shift operator of different variables, but in general, they do not commute. We can obtain the following formulas only in the special cases $T_\ell, \ell = 1, \infty$.

Proposition 2.4. *For any $\eta \in \Omega_r$ and $k \in \mathbb{N}$, we have*

$$\begin{aligned}\Psi_k (T_1 \eta) &= T_1 \Psi_k (\eta) \\ \Psi_k (T_\infty \eta) &= T_\infty \Psi_k (T_k \eta) .\end{aligned} \tag{2.8}$$

Proof of Proposition 2.4. For any $k, \ell, m \in \mathbb{N}$ and $i \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned}\zeta_m (\Psi_k (T_\ell \eta), i) &= \zeta_{m+k} (\eta, i - (m+k) \wedge \ell), \\ \zeta_m (T_\ell \Psi_k (\eta), i) &= \zeta_{m+k} (\eta, i - m \wedge \ell).\end{aligned}$$

Then, we see that

$$\begin{aligned}\zeta_m (\Psi_k (T_\ell \eta), i) &= \zeta_{m+k} (\eta, i - (m+k) \wedge \ell) \\ &= \zeta_{m+k} \left(T_{(m+k)\wedge\ell - m\wedge\ell} \eta, i - m \wedge \ell\right) \\ &= \zeta_m \left(T_\ell \Psi_k \left(T_{(m+k)\wedge\ell - m\wedge\ell} \eta\right), i\right),\end{aligned}$$

and that when $\ell = 1, \infty$, the quantity $(m+k) \wedge \ell - m \wedge \ell$ does not depend on $m \in \mathbb{N}$. For such cases, from the bijectivity of ζ, we have (2.8). □

In the rest of this subsection we will prove Propositions 2.1, 2.2 and 2.3. First, we prepare some lemmas. Then we will give the proofs of the propositions.

Lemma 2.2. *For any $k, \ell \in \mathbb{N}$, $z, w \in \mathbb{Z}_{\geq 0}, z \leq w$ and $\sigma \in \{\uparrow, \downarrow\}$, we have*

$$\sum_{y=z}^{w} \eta^{\sigma}_{k+\ell} (s_k (\eta, y)) = \sum_{y=s_k(\eta,z)}^{s_k(\eta,w)} \eta^{\sigma}_{k+\ell} (y) . \tag{2.9}$$

In particular, we have

$$\sum_{y=s_k(\eta,z)+1}^{s_k(\eta,z+1)-1} \eta^{\sigma}_{k+\ell} (y) = 0.$$

Proof of Lemma 2.2. Since all $(k+\ell, \sigma)$-seats in $[s_k (\eta, z), s_k (\eta, w)]$ are included in $(s_k (\eta, y) ; z \leq y \leq w)$, we have (2.9). □

Lemma 2.3. *For any $k \in \mathbb{N}$ and $x \in \mathbb{N}$, we have*

$$\begin{cases} W_\ell (\eta, s_k (\eta, x)) = \ell \text{ for any } 1 \leq \ell \leq k+1 \text{ if } \eta (s_k (\eta, x)) = 1, \\ W_\ell (\eta, s_k (\eta, x)) = 0 \text{ for any } 1 \leq \ell \leq k+1 \text{ if } \eta (s_k (\eta, x)) = 0. \end{cases} \tag{2.10}$$

In particular, if $W_\ell (\eta, s_k (\eta, x)) > 0$ for some $1 \leq \ell \leq k+1$, then $W_\ell (\eta, s_k (\eta, x)) = \ell$ for any $1 \leq \ell \leq k+1$. Also, if $W_\ell (\eta, s_k (\eta, x)) < \ell$ for some $1 \leq \ell \leq k+1$, then $W_\ell (\eta, s_k (\eta, x)) = 0$ for any $1 \leq \ell \leq k+1$.

Proof of Lemma 2.3. Observe that

$$r (\eta, s_k (\eta, x)) + \sum_{\ell \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta^{\sigma}_{k+\ell} (s_k (\eta, x)) = 1.$$

Then, (2.10) is a direct consequences of (2.1), (2.4) and Lemma 2.1. □

Lemma 2.4. *For any $k \in \mathbb{N}$, $\ell \geq 2$ and $x \in \mathbb{Z}_{\geq 0}$,*

$$\eta^{\uparrow}_{k+\ell}(s_k(\eta, x+1)) = 1 \text{ if and only if } \begin{cases} W_{k+\ell-1}(\eta, s_k(\eta, x)) = k+\ell-1, \\ \mathcal{W}_{k+\ell}(\eta, s_k(\eta, x)) = 0, \\ \eta(s_k(\eta, x+1)) = 1, \end{cases} \tag{2.11}$$

and

$$\eta^{\downarrow}_{k+\ell}(s_k(\eta, x+1)) = 1 \text{ if and only if } \begin{cases} W_{k+\ell-1}(\eta, s_k(\eta, x)) = 0, \\ \mathcal{W}_{k+\ell}(\eta, s_k(\eta, x)) = 1, \\ \eta(s_k(\eta, x+1)) = 0. \end{cases} \tag{2.12}$$

For the case $\ell = 1$,

$$\eta^{\uparrow}_{k+1}(s_k(\eta, x+1)) = 1 \text{ if and only if } \begin{cases} \mathcal{W}_{k+1}(\eta, s_k(\eta, x)) = 0, \\ \eta(s_k(\eta, x+1)) = 1, \end{cases} \tag{2.13}$$

and

$$\eta^{\downarrow}_{k+1}(s_k(\eta, x+1)) = 1 \text{ if and only if } \begin{cases} \mathcal{W}_{k+\ell}(\eta, s_k(\eta, x)) = 1, \\ \eta(s_k(\eta, x+1)) = 0. \end{cases} \tag{2.14}$$

Proof of Lemma 2.4. We only prove (2.11) and then (2.13). (2.12) and (2.14) can be proved in the similar way. Since the necessity ($\Leftarrow$) of (2.11) is clear from Lemma 2.2, we will show the sufficiency ($\Rightarrow$) of (2.11).

We observe that from Lemma 2.2, for any $m, n \in \mathbb{N}$, $z \in \mathbb{Z}_{\geq 0}$ and $s_m(\eta, z) \leq y \leq s_m(\eta, z+1) - 1$ we have

$$\mathcal{W}_{m+n}(\eta, s_m(\eta, z)) = \mathcal{W}_{m+n}(\eta, y). \tag{2.15}$$

If $\eta^{\uparrow}_{k+\ell}(s_k(x+1)) = 1$, then we have $\eta(s_k(x+1)) = 1$, $\mathcal{W}_{k+\ell}(\eta, s_k(\eta, x+1) - 1) = 0$, and

$$\mathcal{W}_m(\eta, s_k(\eta, x+1) - 1) = 1, \tag{2.16}$$

for any $1 \leq m \leq k + \ell - 1$. For the case $\ell \geq 2$, from (2.15) and (2.16) we see that $W_{k+1}(s_k(\eta, x)) > 0$, and thus from Lemma 2.3, we have

$$\eta^{\uparrow}_{k+\ell}(s_k(\eta, x+1)) = 1 \text{ implies } \begin{cases} W_{k+\ell-1}(\eta, s_k(\eta, x+1) - 1) = k+\ell-1, \\ \mathcal{W}_{k+\ell}(\eta, s_k(\eta, x+1) - 1) = 0, \\ \eta(s_k(\eta, x+1)) = 1, \end{cases}$$

and

$$\begin{cases} W_{k+\ell-1}(\eta, s_k(\eta, x+1) - 1) = k+\ell-1, \\ \mathcal{W}_{k+\ell}(\eta, s_k(\eta, x+1) - 1) = 0, \end{cases} \text{ if and only if } \begin{cases} W_{k+\ell-1}(\eta, s_k(\eta, x)) = k+\ell-1, \\ \mathcal{W}_{k+\ell}(\eta, s_k(\eta, x)) = 0. \end{cases}$$

Hence (2.11) is proved.

Next we will show (2.13). Since the sufficiency is clear from (2.15), we check the necessity. From the assumption $\mathcal{W}_{k+1}(\eta, s_k(\eta, x)) = 0$ and Lemma 2.3, we see that $W_{k+1}(\eta, s_k(\eta, x)) = 0$. Then, from $\eta(s_k(\eta, x+1)) = 1$, Lemmas 2.2 and 2.3, we get $\mathcal{W}_{k+1}(\eta, s_k(\eta, x+1) - 1) = 0$ and $W_k(\eta, s_k(\eta, x+1) - 1) = k$. Hence we have $\eta^{\uparrow}_{k+1}(s_k(\eta, x+1)) = 1$ and (2.13) is proved. $\square$

Proof of Proposition 2.1. We use induction on the space variable $x \in \mathbb{N}$. First we consider the case $x = 1$. Observe that in this case either $\eta^{\uparrow}_{k+1}(s_k(\eta,1)) = 1$ or $r(\eta, s_k(\eta,1)) = 1$ holds. On the other hand, we have

$$\mathcal{W}_1(\Psi_k(\eta), 1) = \Psi_k(\eta)(1) = \eta(s_k(1)).$$

Hence, (2.7) holds for $x = 1$.

From now on we assume that up to x, (2.7) holds for any $k, \ell \in \mathbb{N}$. Then, from (2.4) and Lemma 2.2, for any $k, \ell \in \mathbb{N}$ and $0 \le y \le x$ we obtain

$$\begin{aligned}\mathcal{W}_\ell(\Psi_k(\eta), y) &= \sum_{z=1}^{y} \left(\Psi_k(\eta)^{\uparrow}_{\ell}(z) - \Psi_k(\eta)^{\downarrow}_{\ell}(z)\right) \\ &= \sum_{z=1}^{y} \left(\eta^{\uparrow}_{k+\ell}(s_k(z)) - \eta^{\downarrow}_{k+\ell}(s_k(z))\right) \\ &= \sum_{z=1}^{s_k(y)} \left(\eta^{\uparrow}_{k+\ell}(z) - \eta^{\downarrow}_{k+\ell}(z)\right),\end{aligned}$$

and thus from (2.1) we get

$$\begin{aligned}W_\ell(\Psi_k(\eta), y) &= \sum_{m=1}^{\ell}\sum_{z=1}^{y} \left(\Psi_m(\eta)^{\uparrow}_{\ell}(z) - \Psi_m(\eta)^{\downarrow}_{\ell}(z)\right) \\ &= \sum_{m=1}^{\ell}\sum_{z=1}^{s_k(y)} \left(\eta^{\uparrow}_{k+m}(z) - \eta^{\downarrow}_{k+m}(z)\right) \\ &= W_{k+\ell}(\eta, s_k(y)) - W_k(\eta, s_k(y)).\end{aligned}$$

Therefore, from Lemmas 2.3 and 2.4, for $\sigma = \uparrow$ and $\ell \ge 2$ we have

$$\begin{aligned}\Psi_k(\eta)^{\uparrow}_{\ell}(x+1) = 1 \text{ if and only if } &\begin{cases} W_{\ell-1}(\Psi_k(\eta), x) = \ell - 1, \\ W_\ell(\Psi_k(\eta), x) = \ell - 1, \\ \Psi_k(\eta)(x+1) = 1, \end{cases} \\ \text{if and only if } &\begin{cases} W_{k+\ell-1}(\eta, s_k(x)) - W_k(\eta, s_k(x)) = \ell - 1, \\ W_{k+\ell}(\eta, s_k(x)) - W_k(\eta, s_k(x)) = \ell - 1, \\ \eta(s_k(x+1)) = 1, \end{cases} \\ \text{if and only if } &\begin{cases} W_{k+\ell-1}(\eta, s_k(x)) = k + \ell - 1, \\ W_{k+\ell}(\eta, s_k(x)) = k + \ell - 1, \\ \eta(s_k(x+1)) = 1, \end{cases} \\ \text{if and only if } &\eta^{\uparrow}_{k+\ell}(s_k(x+1)) = 1.\end{aligned}$$

For the case $\ell = 1$, we obtain

$$\begin{aligned}\Psi_k(\eta)^{\uparrow}_{1}(x+1) = 1 \text{ if and only if } &\begin{cases} \mathcal{W}_1(\Psi_k(\eta), x) = 0, \\ \Psi_k(\eta)(x+1) = 1, \end{cases} \\ \text{if and only if } &\begin{cases} \mathcal{W}_{k+1}(\eta, s_k(x)) = 0, \\ \eta(s_k(x+1)) = 1, \end{cases} \\ \text{if and only if } &\eta^{\uparrow}_{k+1}(s_k(x+1)) = 1.\end{aligned}$$

The case $\sigma = \downarrow$ can be proved in the similar way. ☐

Proof of Proposition 2.2. We observe that from Proposition 2.1, if $y = s_k(\Psi_\ell(\eta), x)$, for some $k \in \mathbb{N} \cup \{\infty\}$, $\ell \in \mathbb{N}$ and $x \in \mathbb{N}$, then $s_\ell(\eta, y) = s_{k+\ell}(\eta, x)$. Hence, for any $k \in \mathbb{N} \cup \{\infty\}$, $\ell \in \mathbb{N}$ and $x \in \mathbb{Z}$, we have

$$\begin{aligned} s_k\left(\Psi_\ell\left(\eta\right), x\right) &= \sum_{y=1}^{s_k(\Psi_\ell(\eta), x)} r\left(\Psi_\ell\left(\eta\right), y\right) + \left(\sum_{h \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \Psi_\ell\left(\eta\right)_h^\sigma\left(y\right)\right) \\ &= \sum_{y=1}^{s_k(\Psi_\ell(\eta), x)} r\left(\eta, s_\ell\left(\eta, y\right)\right) + \left(\sum_{h \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_{\ell+h}^\sigma\left(s_\ell\left(\eta, y\right)\right)\right) \\ &= \sum_{y=1}^{s_{k+\ell}(\eta, x)} r\left(\eta, y\right) + \left(\sum_{h \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_{\ell+h}^\sigma\left(y\right)\right) \\ &= \xi_\ell\left(\eta, s_{k+\ell}\left(\eta, x\right)\right), \end{aligned} \tag{2.17}$$

where at the third equality we use Lemma 2.2. From the above, we obtain

$$\begin{aligned} \Psi_k\left(\Psi_\ell\left(\eta\right)\right)\left(x\right) &= \Psi_\ell\left(\eta\right)\left(s_k\left(\Psi_\ell\left(\eta\right), x\right)\right) \\ &= \eta\left(s_\ell\left(\eta, s_k\left(\Psi_\ell\left(\eta\right), x\right)\right)\right) \\ &= \eta\left(s_\ell\left(\eta, \xi_\ell\left(\eta, s_{k+\ell}\left(\eta, x\right)\right)\right)\right) \\ &= \eta\left(s_{k+\ell}\left(\eta, x\right)\right) \\ &= \Psi_{k+\ell}\left(\eta\right)\left(x\right). \end{aligned}$$

□

Proof of Proposition 2.3. From Proposition 2.1, Lemma 2.2 and (2.17), for any $k, \ell \in \mathbb{N}$ and $i \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned} \zeta_k\left(\Psi_\ell\left(\eta\right), i\right) &= \sum_{y=s_k(\Psi_\ell(\eta), i)+1}^{s_k(\Psi_\ell(\eta), i+1)} \left(\Psi_\ell\left(\eta\right)_k^\uparrow\left(y\right) - \Psi_\ell\left(\eta\right)_{k+1}^\uparrow\left(y\right)\right) \\ &= \left(\sum_{y=s_k(\Psi_\ell(\eta), i)}^{s_k(\Psi_\ell(\eta), i+1)} \Psi_\ell\left(\eta\right)_k^\uparrow\left(y\right)\right) - \Psi_\ell\left(\eta\right)_{k+1}^\uparrow\left(s_k\left(\Psi_\ell\left(\eta\right), i+1\right)\right) \\ &= \left(\sum_{y=s_k(\Psi_\ell(\eta), i)}^{s_k(\Psi_\ell(\eta), i+1)} \eta_{k+\ell}^\uparrow\left(s_\ell\left(\eta, y\right)\right)\right) - \eta_{k+\ell+1}^\uparrow\left(s_\ell\left(\eta, \xi_\ell\left(\eta, s_{k+\ell}\left(\eta, i+1\right)\right)\right)\right) \\ &= \left(\sum_{y=s_{k+\ell}(\eta, i)}^{s_{k+\ell}(\eta, i+1)} \eta_{k+\ell}^\uparrow\left(y\right)\right) - \eta_{k+\ell+1}^\uparrow\left(s_{k+\ell}\left(\eta, i+1\right)\right) \\ &= \zeta_{k+\ell}\left(\eta, i\right). \end{aligned}$$

□

3. 10-elimination

In this section, we will first briefly recall 10-elimination. Throughout this section, $\Omega = \{0,1\}^{\mathbb{N}}$, but we often regard $\eta \in \Omega$ as an element in $\{0,1\}^{\mathbb{Z}}$ by setting $\eta(x) = 0$ for $x < 0$.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\eta(x)$	0	1	1	0	0	1	1	1	0	1	0	1	1	0	0	0	1	0	…
	0	1	~~1~~	~~0~~	0	1	1	~~1~~	~~0~~	~~1~~	~~0~~	1	~~1~~	~~0~~	0	0	~~1~~	~~0~~	…
$\Phi_1(\eta)$	0	$1^{(0)}$	0	1	$1^{(2)}$	$1^{(1)}$	0	0	$0^{(1)}$	0	0	0	0	0	0	0	0	0	…
	0	~~1~~	~~0~~	1	1	~~1~~	~~0~~	0	0	0	0	0	0	0	0	0	0	0	…
$\Phi_2(\eta)$	$0^{(1)}$	1	$1^{(0)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…
	0	1	~~1~~	~~0~~	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…
$\Phi_3(\eta)$	0	$1^{(0)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…
	0	~~1~~	~~0~~	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…
$\Phi_4(\eta)$	$0^{(1)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…

FIGURE 1. Recursive construction of $\Phi_k(\eta)$ for $k = 1, 2, 3, 4$.

3.1. **Definition of the 10-elimination.** For a given ball configuration $\eta \in \Omega_{<\infty}$, we will recursively construct a sequence $(\Phi_k(\eta)\ ;\ k \in \mathbb{N}) \subset \Omega_{<\infty}$ and corresponding 10-rigging $\left(J_k^{10}(\eta)\ ;\ k \in \mathbb{N}\right) \subset \mathbb{Z}_{\geq 0}^{\mathbb{N}}$. First, we regard η as a sequence of 1 and 0, i.e., $\eta = \eta(0)\eta(1)\eta(2)\ldots\eta(x)\ldots$, and we say that $\eta(x)\eta(x+1)$ is a 10-pair in η if $\eta(x) = 1$ and $\eta(x+1) = 0$. Then, we denote by $\Phi_1(\eta)$ the ball configuration obtained by removing all 10-pair in η and renumbering the remaining 1s and 0s from left to right. In Figure 1 we give an example of $\Phi_1(\eta)$ constructed from $\eta = 011001110101100011000\ldots$. Next, we explain how to construct $J_1^{10}(\eta)$ from η. We will denote by $(10)^m, m \in \mathbb{N}$ the m consecutive 10s, i.e., $(10)^1 = 10$, $(10)^2 = 1010$, $(10)^3 = 101010$, and so on. Observe that all $(10)^m$s are sandwiched by some $X = \eta(x), Y = \eta(x+2m+1)$ such that $\eta(x-1)\eta(x) \neq 10$ and $\eta(x+2m+1)\eta(x+2m+2) \neq 10$ where we use the convention $\eta(-1) = \eta(0) = 0$, and after one step time evolution, all such $(10)^m$s are removed, i.e.,

$$\eta = \ldots X(10)^m Y \ldots \overset{\text{10-elimination}}{\mapsto} \Phi_1(\eta) = \ldots XY \ldots.$$

Now, according to the values of X, Y and m, we write a number on the right shoulder of X as follows :

- if $XY = 11, 01, 00$, then we write $X^{(m)}$,
- if $XY = 10$, then we write $X^{(m-1)}$.

Then, the 10-rigging $J_1^{10}(\eta) = \left(J_1^{10}(\eta, i)\right)_{i \in \mathbb{Z}_{\geq 0}}, J_1^{10}(\eta, i) \subset \mathbb{N}$ is defined as

$$J_1^{10}(\eta, i) := \begin{cases} \{1, \ldots, m\} & \text{if } T\eta(i) = X^{(m)} \text{ and } m \in \mathbb{N}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Finally, we suppose that we have constructed $\Phi_k(\eta)$. Then, $\Phi_{k+1}(\eta)$ and $J_{k+1}^{10}(\eta)$ are defined as

$$\Phi_{k+1}(\eta) := \Phi_1(\Phi_k(\eta)), \quad J_{k+1}^{10}(\eta) := J_1^{10}(\Phi_k(\eta)).$$

For example, for the ball configuration used in Figure 1, $J_k^{10}(\eta)$ is given by

$$J_k^{10}(\eta,i)=\begin{cases}\{1\} & (k,i)=(1,5),(2,0),(4,0),\\ \{1,2\} & (k,i)=(1,4),\\ \varnothing & \text{otherwise.}\end{cases}$$

Note that since $\eta\in\Omega_{<\infty}$, $\Phi_k(\eta)=000\ldots$ for sufficiently large k.

It is known that BBS(ℓ) can be linearized via the 10-elimination as follows :

Theorem 3.1 (Theorem 2 in [MIT], Theorem 22 in [KNTW])**.** *Suppose that $\eta\in\Omega_{<\infty}$. Then for any $k\in\mathbb{N}$, $\ell\in\mathbb{N}\cup\{\infty\}$ and $i\in\mathbb{Z}_{\geq0}$, we have*

$$J_k^{10}(T_\ell\eta,i)=J_k^{10}(\eta,i-k\wedge\ell)$$

with convention $J_k^{10}(\eta,i)=\varnothing$ for $i<0$.

3.2. **On the relation to the seat number configuration.** In this subsection, we will show the following.

Theorem 3.2. *Suppose that $\eta\in\Omega_{<\infty}$. Then, for any $k\in\mathbb{N}$ and $i\in\mathbb{Z}_{\geq0}$, we have*

$$\Phi_k(\eta)=\Psi_k(\eta),\quad \left|J_k^{10}(\eta,i)\right|=\zeta_k(\eta,i). \tag{3.1}$$

From Theorem 3.2 and Proposition 2.3, we see that the 10-elimination can be considered as a shift operator on $\bar{\Omega}$.

Proof of Theorem 3.2. First we observe that it is sufficient to show (3.1) for $k=1$. Actually, if (3.1) holds for $k=1$, then from Propositions 2.2 and 2.3, we have

$$\Phi_2(\eta)=\Phi_1(\Phi_1(\eta))=\Phi_1(\Psi_1(\eta))=\Psi_1(\Psi_1(\eta))=\Psi_2(\eta),$$

and

$$\left|J_2^{10}(\eta,i)\right|=\left|J_1^{10}(\Phi_1(\eta),i)\right|=\left|J_1^{10}(\Psi_1(\eta),i)\right|=\zeta_1(\Psi_1(\eta),i)=\zeta_2(\eta,i).$$

Hence, by repeating the above argument, (3.1) can be proved for any $k\in\mathbb{N}$.

From now on we will show (3.1). Fix $\Omega\in\Omega_{<\infty}$. Observe that η can be decomposed as follows:

$$\eta=0^{\otimes m_0}1^{\otimes n_1}...0^{\otimes m_l}1^{\otimes n_l}0^{\otimes m_l},\quad m_l=\infty, \tag{3.2}$$

where $z^{\otimes r}$, $z=0,1$, $r\in\mathbb{Z}_{\geq0}\cup\{\infty\}$ is the r successive z's and $n_1,...,n_l\in\mathbb{N}$ satisfy

$$\sum_{i=1}^{l}n_i=\sum_{x\in\mathbb{Z}_{\geq0}}\eta(x).$$

Note that $z^{\otimes0}=\varnothing$. By using the above decomposition, we can see that the 10-elimination removes the rightmost "1" (resp. the leftmost "0") of all consecutive 1's (resp. 0's) except for the origin, and thus $\Phi_1(\eta)$ is expressed as

$$\Phi_1(\eta)=0^{\otimes m_0}1^{\otimes n_1-1}0^{\otimes m_1-1}...0^{\otimes m_l-1}1^{\otimes n_l-1}0^{\otimes\infty}.$$

On the other hand, $\Psi_1(\eta)$ skips the leftmost "1"(resp. "0") of all consecutive 1's (resp. 0's) except for the origin. Therefore we obtain

$$\begin{aligned}\Psi_1(\eta)&=0^{\otimes m_0}1^{\otimes n_1-1}0^{\otimes m_1-1}...0^{\otimes m_l-1}1^{\otimes n_l-1}0^{\otimes\infty}\\&=\Phi_1(\eta).\end{aligned}$$

Finally we will show $|J_1^{10}(\eta,i)|=\zeta_1(\eta,i)$. Assume that $|J_1^{10}(\eta,i)|=m, m\in\mathbb{Z}_{\geq0}$. Then, by considering the meanings of $\Phi_1(\eta)$ and $\Psi_1(\eta)$ via the decomposition (3.2) again, we have the following.

- For the case $\eta(s_k(i)) = \eta(s_k(i+1)) = 1$. Then, we obtain

$$\eta(s_k(i)-1)\eta(s_k(i))\dots\eta(s_k(i+1)-1)\eta(s_k(i+1)) = 1(10)^m 1,$$

where we use the convention $X(10)^0 Y = XY$ for $X, Y \in \{0,1\}$. Since all "0"s in $\eta(s_k(i)+1)\dots\eta(s_k(i+1))$ are $(1,\downarrow)$-seats, from (2.6) we have

$$\zeta_1(\eta, i) = \sum_{y=s_1(i)+1}^{s_i(i+1)} \eta_1^{\downarrow}(y) - \eta_2^{\downarrow}(y) = m.$$

- For the case $\eta(s_k(i)) = 0, \eta(s_k(i+1)) = 1$. Then, we obtain

$$\eta(s_k(i))\dots\eta(s_k(i+1)-1)\eta(s_k(i+1)) = 0(10)^m 1.$$

Since all "0"s in $\eta(s_k(i)+1)\dots\eta(s_k(i+1))$ are $(1,\downarrow)$-seats, from (2.6) we have

$$\zeta_1(\eta, i) = \sum_{y=s_1(i)+1}^{s_i(i+1)} \eta_1^{\downarrow}(y) - \eta_2^{\downarrow}(y) = m.$$

- For the case $\eta(s_k(i)) = 0, \eta(s_k(i+1)) = 0$. Then, we obtain

$$\eta(s_k(i))\dots\eta(s_k(i+1)-1)\eta(s_k(i+1)) = 0(10)^m 0.$$

Since all "1"s in $\eta(s_k(i)+1)\dots\eta(s_k(i+1))$ are $(1,\uparrow)$-seats, we have

$$\zeta_1(\eta, i) = \sum_{y=s_1(i)+1}^{s_i(i+1)} \eta_1^{\uparrow}(y) - \eta_2^{\uparrow}(y) = m.$$

- For the case $\eta(s_k(i)) = 1, \eta(s_k(i+1)) = 0$. Then, we obtain

$$\eta(s_k(i))\dots\eta(s_k(i+1)-1)\eta(s_k(i+1)) = 1(10)^{m+1} 0.$$

Since all "0"s in $\eta(s_k(i)+1)\dots\eta(s_k(i+1)-1)$ are $(1,\downarrow)$-seats and $s_k(i+1)$ is a $(2,\downarrow)$-seat, from (2.6) we have

$$\zeta_1(\eta, i) = \sum_{y=s_1(i)+1}^{s_i(i+1)} \eta_1^{\downarrow}(y) - \eta_2^{\downarrow}(y) = m.$$

From the above, we have (3.1) for $k = 1$. □

Remark 3.1. *We note that both the* 1*-skip map and the* 10*-elimination remove the same* 0*'s, but they may remove different* 1*'s, see Figure 2. In other words, the* 10*-elimination may remove* 1 *located at* x *such that* $\eta_\ell^{\uparrow}(x) = 1$ *for some* $\ell \ge 2$*. We also note that the* 1*-skip map is a more natural elimination than the* 10*-elimination in terms of solitons, see* [MSSS, Section 2.1] *for the relation between the seat number configuration and solitons identified via the Takahashi-Satsuma algorithm.*

4. BBS on the whole-line

Throughout this section, Ω is given by

$$\Omega = \left\{ \eta \in \{0,1\}^{\mathbb{Z}} \ ; \ \overline{\lim_{x\to-\infty}} \frac{1}{|x|} \sum_{y=-x}^{0} \eta(y) < \frac{1}{2} \right\}.$$

First, we recall the definition of the BBS(ℓ) on the whole-line, and then we introduce the seat number configuration and the k-skip map on Ω. Since in the rest of this paper, we mainly consider the BBS(∞), and thus we write $T := T_\infty$ for simplicity

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\eta(x)$	0	1	1	0	0	1	1	1	0	1	0	1	1	0	0	0	1	0	0	$\dots$
	0	1	~~1~~	~~0~~	0	1	1	~~1~~	~~0~~	~~1~~	~~0~~	1	~~1~~	~~0~~	0	0	~~1~~	~~0~~	0	$\dots$
$\Phi_1(\eta)$	0	$1^{(0)}$	0	1	$1^{(2)}$	$1^{(1)}$	0	0	$0^{(1)}$	0	0	0	0	0	0	0	0	0	0	$\dots$

$\eta(x)$	0	1	1	0	0	1	1	1	0	1	0	1	1	0	0	0	1	0	0	$\dots$
	0	~~1~~	1	~~0~~	0	~~1~~	1	1	~~0~~	~~1~~	~~0~~	~~1~~	1	~~0~~	0	0	~~1~~	~~0~~	0	$\dots$
$\Psi_1(\eta)$	0	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	$\dots$

FIGURE 2. Difference between the 1-skip map and the 10-elimination.

of notation. Next, we introduce the seat number configuration and the k-skip map for the whole-line case. We note that many notions are natural extensions of those introduced in Section 2, the same symbols will be used.

We also note that $\{0,1\}^{\mathbb{N}}$ can be considered as a subset of Ω in the following sense.

$$\{0,1\}^{\mathbb{N}} \cong \{\eta \in \Omega \ ; \ \eta(x) = 0 \text{ for any } x \le 0\}.$$

Thus, the half-line case is completely included in the whole-line case described below.

4.1. **Seat number configuration on the whole-line.** First, we introduce the carrier with capacity ℓ to define the one-step time evolution of the BBS(ℓ) on the whole-line. For any $\eta \in \Omega$, we define $s_\infty(\eta, x)$ recursively as follows :

$$s_\infty(\eta, 0) := \max\left\{ x \le 0 \ ; \ \sum_{y=z}^{x} (2\eta(x) - 1) < 0 \text{ for any } z \le x \right\}, \tag{4.1}$$

and

$$s_\infty(\eta, i) := \min\left\{ x > s_\infty(\eta, i-1) \ ; \ \sum_{y=z}^{x} (2\eta(x) - 1) < 0 \text{ for any } z \le x \right\},$$
$$s_\infty(\eta, -i) := \max\left\{ x < s_\infty(\eta, -i+1) \ ; \ \sum_{y=z}^{x} (2\eta(x) - 1) < 0 \text{ for any } z \le x \right\} \tag{4.2}$$

for any $i \in \mathbb{N}$, with convention $\min \varnothing = \infty$ and $\max \varnothing = -\infty$. Observe that if $\eta \in \Omega$, then $s_\infty(\eta, -i) > -\infty$ for any $i \in \mathbb{Z}_{\ge 0}$. Then, it is not difficult to check that by changing the starting point from 0 to $s_\infty(\eta, -i)$, $W_\ell(\eta^{(i)}, \cdot)$ can be defined on $[s_\infty(\eta, -i), \infty) \cap \mathbb{Z}$ by using the same construction described in Introduction, where

$$\eta^{(i)}(x) := \begin{cases} \eta(x) & x \ge s_\infty(\eta, -i), \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Then, for any $i, j \in \mathbb{N}$, $i \le j$, we see that

$$s_\infty(\eta^{(j)}, -i) = s_\infty(\eta, -i). \tag{4.4}$$

As a result, for any $i \in \mathbb{Z}_{\geq 0}$, $W_\ell\left(\eta^{(i)}, \cdot\right)$ and $W_\ell\left(\eta^{(i+1)}, \cdot\right)$ are consistent, i.e., for any $x \geq s_\infty(\eta, -i)$,

$$W_\ell\left(\eta^{(i)}, x\right) = W_\ell\left(\eta^{(i+1)}, x\right). \tag{4.5}$$

Hence, $W_\ell(\eta, x) := \lim_{i \to \infty} W_\ell\left(\eta^{(i)}, x\right)$ is well-defined for any $x \in \mathbb{Z}$. As a consequence, the one-step time evolution of the BBS (ℓ) on the whole-line can also be described by the operator $T_\ell : \Omega \to \Omega$ defined via (1.1). Similarly, for any $k \in \mathbb{N}$ and $i \in \mathbb{Z}_{\geq 0}$, $\mathcal{W}_k\left(\eta^{(i)}, \cdot\right) : [s_\infty(\eta, -i), \infty) \cap \mathbb{Z} \to \{0, 1\}$ can be defined by changing the starting point from 0 to $s_\infty(\eta, -i)$, and one can check the consistency of $\mathcal{W}_k\left(\eta^{(i)}, x\right)$ and $\mathcal{W}_k\left(\eta^{(i+1)}, x\right)$ for any $x \geq s_\infty(\eta, -i)$. Hence, $\mathcal{W}_k(\eta, x) := \lim_{i \to \infty} \mathcal{W}_k\left(\eta^{(i)}, x\right)$ is also well-defined for any $k \in \mathbb{N}$ and $x \in \mathbb{Z}$. In particular, from (2.1), for any $\ell \in \mathbb{N}$ and $x \in \mathbb{Z}$, we have

$$W_\ell(\eta, x) = \sum_{k=1}^{\ell} \mathcal{W}_\ell(\eta, x).$$

For later use, we state the above procedure as a lemma.

Lemma 4.1. *Suppose that $\eta \in \Omega$. Then, for any $k \in \mathbb{N}$, $i \in \mathbb{Z}_{\geq 0}$ and $x \geq s_\infty(\eta, -i)$, we have*

$$\mathcal{W}_k\left(\eta^{(i)}, x\right) = \mathcal{W}_k\left(\eta^{(i+1)}, x\right).$$

In other words, the value of $\mathcal{W}_k(\eta, \cdot)$ on $[s_\infty(\eta, -i), \infty)$ is independent of $\eta(x), x \leq s_\infty(\eta, -i) - 1$.

Now, we can define the seat number configuration $\eta_k^\sigma \in \{0, 1\}^{\mathbb{Z}}$, $k \in \mathbb{N}$, $\sigma \in \{\uparrow, \downarrow\}$ by the same equations (2.2) and (2.3). Also, we define $r(\eta, \cdot) \in \{0, 1\}^{\mathbb{Z}}$ as

$$r(\eta, x) := 1 - \sum_{k \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_k^\sigma(x).$$

Then, from (2.1), (2.4) and Lemma 4.1, we have

$$\mathcal{W}_k(\eta, x) = \sum_{y = s_\infty(\eta, -i) + 1}^{x} \left(\eta_k^\uparrow(y) - \eta_k^\downarrow(y)\right), \tag{4.6}$$

for any $k \in \mathbb{N}$, $i \in \mathbb{Z}_{\geq 0}$ and $x \geq s_\infty(\eta, -i)$. Also, thanks to Lemma 4.1, as in the half-line case (Lemma 2.1), we obtain the following basic property of the seat number configuration.

Lemma 4.2. *Suppose that $\eta \in \Omega$. Then, for any $k \in \mathbb{N}$, $i \leq 0$ and $x \geq s_\infty(\eta, i)$, we have the following.*

(1) $\eta_k^\uparrow(x) = 1$ implies $\sum_{y = s_\infty(\eta, i) + 1}^{x} \left(\eta_\ell^\uparrow(y) - \eta_\ell^\downarrow(y)\right) = 1$ for any $1 \leq \ell \leq k$.
(2) $\eta_k^\downarrow(x) = 1$ implies $\sum_{y = s_\infty(\eta, i) + 1}^{x} \left(\eta_\ell^\uparrow(y) - \eta_\ell^\downarrow(y)\right) = 0$ for any $1 \leq \ell \leq k$.
(3) $r(\eta, x) = 1$ implies $\sum_{y = s_\infty(\eta, i) + 1}^{x} \left(\eta_\ell^\uparrow(y) - \eta_\ell^\downarrow(y)\right) = 0$ for any $\ell \in \mathbb{N}$.

For later use, we note the relationship between the seat number configuration and the notion of *slots* and corresponding slot configuration introduced by [FNRW]. Since we will not use the definition of slots in the subsequent sections, we do not give it in this paper, see [FNRW, Section 1] for the precise definition of slots. As in the half-line case, we can obtain the following.

Proposition 4.1. *Suppose that* $\eta \in \Omega$ *and* $|s_\infty(\eta, i)| < \infty$ *for any* $i \in \mathbb{Z}$. *Then, for any* $k \in \mathbb{N}$ *and* $x \in \mathbb{Z}$,

$$\eta_k^\uparrow(x) + \eta_k^\downarrow(x) = 1 \text{ if and only if } x \text{ is a } (k-1)\text{-slot but not a } k\text{-slot.}$$

Proof of Proposition 4.1. This is a direct consequence of [MSSS, Proposition 2.3] and Lemma 4.1. □

Then, for any $k \in \mathbb{N} \cup \{\infty\}$, we define $\xi_k(\eta, \cdot) : \mathbb{Z} \to \mathbb{Z}$ and $s_k(\eta, \cdot) : \mathbb{Z} \to \mathbb{Z} \cup \{\infty\}$ as

$$\xi_k(\eta, x) - \xi_k(\eta, x-1) := r(\eta, x) + \sum_{\ell \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_{k+\ell}^\sigma(x),$$
$$\xi_k(\eta, s_\infty(\eta, 0)) := 0,$$

and

$$s_k(\eta, x) := \min\{y \in \mathbb{Z} \; ; \; \xi_k(\eta, y) = x\}. \tag{4.7}$$

Note that s_∞ defined via (4.7) coincides with s_∞ defined via (4.1)-(4.2). For later use, we note that thanks to Lemma 4.1, Lemma 2.2 also holds for the whole-line case.

Lemma 4.3. *Suppose that* $\eta \in \Omega$. *Then, for any* $k, \ell \in \mathbb{N}$, $z, w \in \mathbb{Z}$, $z \leq w$ *and* $\sigma \in \{0, 1\}$, *we have*

$$\sum_{y=z}^{w} \eta_{k+\ell}^\sigma(s_k(\eta, y)) = \sum_{y=s_k(\eta,z)}^{s_k(\eta,w)} \eta_{k+\ell}^\sigma(y).$$

Finally, for any $k \in \mathbb{N}$ and $i \in \mathbb{Z}$, we define $\zeta_k(\eta, \cdot) : \mathbb{Z} \to \mathbb{Z} \cup \{\infty\}$ by (2.5). Thanks to Proposition 4.1, we see that our ζ coincides with the slot decomposition introduced by [FNRW], see also [MSSS, Proposition 2.3] for the half-line case.

The dynamics of the BBS (∞) can be linearized through ζ, but an offset is required.

Theorem 4.1. *Suppose that* $\eta \in \Omega$ *and* $s_k(\eta, i+1) < \infty$ *for some* $k \in \mathbb{N}$ *and* $i \in \mathbb{Z}$. *Then, we have*

$$\zeta_k(T\eta, i + k + o_k(\eta)) = \zeta_k(\eta, i),$$

where the offset o_k *is given by*

$$o_k(\eta) := s_\infty(\eta, 0) - s_\infty(T\eta, 0) + 2 \sum_{y=s_\infty(\eta,0)+1}^{0} \sum_{\ell=1}^{k} \eta_\ell^\downarrow(y) - 2 \sum_{y=s_\infty(T\eta,0)+1}^{0} \sum_{\ell=1}^{k} T\eta_\ell^\uparrow(y).$$

Remark 4.1. *From Proposition 4.2 which will be described in the next subsection, we see that the following quantity*

$$\sum_{y=s_\infty(\eta,0)+1}^{x} \sum_{\ell=1}^{k} \eta_\ell^\downarrow(y) - \sum_{y=s_\infty(T\eta,0)+1}^{x} \sum_{\ell=1}^{k} T\eta_\ell^\uparrow(y)$$

does not depend on $x \in \mathbb{Z}$. *Hence, by setting* $x = s_\infty(T\eta, 0), s_\infty(\eta, 0)$, *we have*

$$\begin{aligned} o_k(\eta) &= -\xi_k(\eta, s_\infty(T\eta, 0)) - W_k(\eta, s_\infty(T\eta, 0)) \\ &= \xi_k(T\eta, s_\infty(\eta, 0)) - W_k(T\eta, s_\infty(\eta, 0)). \end{aligned}$$

Remark 4.2. *Since we compute the offset with respect to the origin while* [FNRW] *computes the offset with respect to the tagged k-slots, the values of the offsets are different. The offset in* [FNRW] *is independent of $(\zeta_\ell)_{\ell \le k}$, but the formula* [FNRW, (3.1)] *is not very easy to compute. On the other hand, our offset o_k may depend on ζ_ℓ for some $\ell \le k$, but a simple formula is obtained. Moreover, the proof of Theorem 4.1 is much simpler than that of* [FNRW, Theorem 4.1].

The proof of Theorem 4.1 will be given in Section 4.2.

Remark 4.3. *Note that ζ can be considered as a map $\zeta : \Omega_r \to \mathbb{Z}_{\ge 0}^{\mathbb{N}\times\mathbb{Z}}$, and it is shown in* [FNRW] *that ζ is a bijection between $\Omega_r \subset \Omega$ and $\bar{\Omega} \subset \mathbb{Z}_{\ge 0}^{\mathbb{N}\times\mathbb{Z}}$, where Ω_r and $\bar{\Omega}$ are defined as*

$$\Omega_r := \{\eta \in \Omega \ ; \ |s_\infty(\eta, i)| < \infty \text{ for any } i, \ s_\infty(\eta, 0) = 0\},$$

$$\bar{\Omega} := \left\{\zeta \in \mathbb{Z}_{\ge 0}^{\mathbb{N}\times\mathbb{Z}} \ ; \ \sum_{k\in\mathbb{N}} \zeta_k(i) < \infty \text{ for any } i\right\}.$$

4.2. **Proof of Theorem 4.1.** To show Theorem 4.1, we need the following property of the seat number configuration.

Proposition 4.2. *For any $\eta \in \Omega$, $k \in \mathbb{N}$ and $x \in \mathbb{Z}$, we have*

$$\eta_k^{\downarrow}(x) = T\eta_k^{\uparrow}(x). \tag{4.8}$$

In addition, $\eta_k^{\uparrow}(x) = 1$ implies

$$r(T\eta, x) + \sum_{\ell \ge k} T\eta_\ell^{\downarrow}(x) = 1.$$

Proof of Proposition 4.2. First, we note that for the half-line case, the assertions of this proposition have been proven in [MSSS, Proposition 3.1]. In the proof of [MSSS, Proposition 3.1], the boundary condition of the function $\widetilde{\mathcal{W}}_k(\eta, x) := 1 - \mathcal{W}_k(T\eta, x)$ is given by $\widetilde{\mathcal{W}}_k(\eta, 0) = 1$ for any $k \in \mathbb{N}$, but one can check that the proof does not depend on the boundary condition. Instead, the following condition $\sum_{y=1}^{x}(1 - 2T\eta(y)) \le 0$, $x \in \mathbb{N}$ is essential, and this condition trivially holds for the half-line case. For the whole-line case, the following inequality

$$\sum_{y=s_\infty(\eta,i)+1}^{x} (1 - 2T\eta(y)) \le 0,$$

holds for any $i \in \mathbb{Z}$ and $x \ge s_\infty(\eta, i)$. Therefore, by following the strategy of [MSSS, Proposition 3.1], we obtain (4.2) for $\eta^{(i)}$. Hence, from Lemma 4.1, by taking the limit $i \to \infty$, we have (4.2) for any $\eta \in \Omega$. □

Proof of Theorem 4.1. First we note that from Lemma 4.2, $\zeta_k(\eta, i)$ can be represented as

$$\begin{aligned}\zeta_k(\eta, i) &= \left|\left\{x \in \mathbb{Z} \ ; \ \eta_k^{\uparrow}(x) = 1, \ \xi_k(\eta, x) = i\right\}\right| - \left|\left\{x \in \mathbb{Z} \ ; \ \eta_{k+1}^{\uparrow}(x) = 1, \ \xi_k(\eta, x) = i + 1\right\}\right| \\ &= \left|\left\{x \in \mathbb{Z} \ ; \ \eta_k^{\downarrow}(x) = 1, \ \xi_k(\eta, x) = i\right\}\right| - \left|\left\{x \in \mathbb{Z} \ ; \ \eta_{k+1}^{\downarrow}(x) = 1, \ \xi_k(\eta, x) = i + 1\right\}\right|.\end{aligned}$$

Hence, from Proposition 4.2, we have

$$\begin{aligned}\zeta_k(T\eta, i) = &\left|\left\{x \in \mathbb{Z} \ ; \ \eta_k^{\downarrow}(x) = 1, \ \xi_k(T\eta, x) = i\right\}\right| \\ &- \left|\left\{x \in \mathbb{Z} \ ; \ \eta_{k+1}^{\downarrow}(x) = 1, \ \xi_k(T\eta, x) = i + 1\right\}\right|.\end{aligned}$$

Now we show that for any $k \in \mathbb{N}$, the following quantity

$$\xi_k(T\eta, x) - \xi_k(\eta, x) - W_k(T\eta, x) - W_k(\eta, x)$$

is independent of $x \in \mathbb{Z}$ and equal to $o_k(\eta)$. Actually, from (4.5), (4.6) and proposition 4.2, we have

$$\begin{aligned}
&\xi_k(T\eta, x) - \xi_k(\eta, x) - W_k(T\eta, x) - W_k(\eta, x) - (s_\infty(\eta, 0) - s_\infty(T\eta, 0)) \\
&= \begin{cases} 2\sum_{\ell=1}^{k}\left(\sum_{y=s_\infty(\eta,0)+1}^{x} \eta_\ell^{\downarrow}(y) - \sum_{y=s_\infty(T\eta,0)+1}^{x} T\eta_\ell^{\uparrow}(y)\right) & x \geq s_\infty(\eta, 0), \\ 2\sum_{\ell=1}^{k}\left(\sum_{y=x+1}^{s_\infty(\eta,0)} \eta_\ell^{\downarrow}(y) - \sum_{y=x+1}^{s_\infty(T\eta,0)} T\eta_\ell^{\uparrow}(y)\right) & x \leq s_\infty(\eta, 0) - 1, \end{cases} \\
&= o_k(\eta) - (s_\infty(\eta, 0) - s_\infty(T\eta, 0)).
\end{aligned}$$

In particular, from (4.5), (4.6), Lemma 4.2 and proposition 4.2, if $x = s_{k-1}(\eta, i)$ for some $i \in \mathbb{Z}$ and $r(\eta, x) = 0$, then we obtain

$$W_k(\eta, x) + W_k(T\eta, x) = k,$$

and thus we get

$$\xi_k(T\eta, x) - \xi_k(\eta, x) = k + o_k(\eta). \tag{4.9}$$

From the above, we have

$$\begin{aligned}
&\zeta_k(\eta, i) \\
&= \left|\left\{x \in \mathbb{Z} \ ; \ \eta_k^{\downarrow}(x) = 1, \ \xi_k(\eta, x) = i\right\}\right| - \left|\left\{x \in \mathbb{Z} \ ; \ \eta_{k+1}^{\downarrow}(x) = 1, \ \xi_k(\eta, x) = i + 1\right\}\right| \\
&= \left|\left\{x \in \mathbb{Z} \ ; \ T\eta_k^{\uparrow}(x) = 1, \ \xi_k(T\eta, x) = i + k + o_k(\eta)\right\}\right| \\
&\quad - \left|\left\{x \in \mathbb{Z} \ ; \ T\eta_k^{\uparrow}(x) = 1, \ \xi_k(T\eta, x) = i + k + o_k(\eta) + 1\right\}\right| \\
&= \zeta_k(T\eta, i + k + o_k(\eta)),
\end{aligned}$$

and thus Theorem 4.1 is proved. □

4.3. **The k-skip map on the whole-line.** In this subsection, we will show how the propositions stated in section 2.2 can be generalized. For the whole-line case, we define the k-skip map $\Psi_k : \Omega \to \Omega$ as

$$\Psi_k(\eta)(x) := \eta(s_k(\eta, x + \xi_k(\eta, 0))).$$

Recall that $\eta^{(i)}$ is defined in (4.3). As we observed in Lemma 4.1, we also have the following.

Lemma 4.4. *Suppose that $\eta \in \Omega$. Then, for any $k \in \mathbb{N}$, $i \in \mathbb{Z}_{\geq 0}$ and $x \geq \xi_k(\eta, s_\infty(\eta, -i)) - \xi_k(\eta, 0)$, we have*

$$\Psi_k\left(\eta^{(i)}\right)(x) = \Psi_k\left(\eta^{(i+1)}\right)(x). \tag{4.10}$$

In particular, for any $x \in \mathbb{Z}$ we have

$$\Psi_k(\eta)(x) = \lim_{i \to \infty} \Psi_k\left(\eta^{(i)}\right)(x).$$

Proof. From (4.4), we see that $\xi_k(\eta^{(i)}, x) = \xi_k(\eta^{(i+1)}, x)$ for any $x \geq s_\infty(\eta, -i)$. Hence, we get $s_k(\eta^{(i)}, x) = s_k(\eta^{(i+1)}, x)$ for any $x \geq \xi_k(\eta, s_\infty(\eta, -i))$. Thus we obtain (4.10). □

Now, we generalize Propositions 2.1, 2.2 and 2.3 for the whole-line case as follows :

Proposition 4.3. *Suppose that* $\eta \in \Omega$. *Then, for any* $k, \ell \in \mathbb{N}$, $\sigma \in \{\uparrow, \downarrow\}$ *and* $x \in \mathbb{Z}$, *we have*

$$\Psi_k(\eta)^{\sigma}_{\ell}(x) = \eta^{\sigma}_{k+\ell}(s_k(\eta, x + \xi_k(\eta, 0))). \tag{4.11}$$

In addition, we have

$$\Psi_k(\Psi_\ell(\eta))(x) = \Psi_{k+\ell}(\eta)(x), \tag{4.12}$$

and

$$\zeta_k(\Psi_\ell(\eta), i) = \zeta_{k+\ell}(\eta, i). \tag{4.13}$$

Proof. First we show (4.11). Thanks to Lemma 4.4, it is sufficient to show that

$$\Psi_k\left(\eta^{(i)}\right)^{\sigma}_{\ell}(x) = \eta^{\sigma}_{k+\ell}(s_k(\eta, x + \xi_k(\eta, 0))) \tag{4.14}$$

for any $i \in \mathbb{Z}_{\geq 0}$, $k, \ell \in \mathbb{N}$, $\sigma \in \{\uparrow, \downarrow\}$ and $x \in \mathbb{Z}$. We fix $i \in \mathbb{Z}_{\geq 0}$ and define $\tilde{\eta}^{(i)} \in \Omega$ as

$$\tilde{\eta}^{(i)}(x) := \eta^{(i)}(x + s_\infty(\eta, -i)).$$

Since $r(\tilde{\eta}^{(i)}, x) = 1$ for any $x \leq 0$ and $\tilde{\eta}^{(i)}$ can be regarded as an element of $\{0,1\}^{\mathbb{N}}$, from Proposition 2.1, we get

$$\Psi_k\left(\tilde{\eta}^{(i)}\right)^{\sigma}_{\ell}(x) = \left(\tilde{\eta}^{(i)}\right)^{\sigma}_{k+\ell}\left(s_k\left(\tilde{\eta}^{(i)}, x\right)\right),$$

for any $k, \ell \in \mathbb{N}$, $\sigma \in \{\uparrow, \downarrow\}$ and $x \in \mathbb{Z}$. On the other hand, by direct computation, for any $x \in \mathbb{Z}$ we obtain

$$\xi_k\left(\tilde{\eta}^{(i)}, x\right) = \xi_k\left(\eta^{(i)}, x + s_\infty(\eta, -i)\right) - \xi_k\left(\eta^{(i)}, s_\infty(\eta, -i)\right),$$

and thus we have

$$s_k\left(\tilde{\eta}^{(i)}, x\right) = s_k\left(\eta^{(i)}, x + \xi_k\left(\eta^{(i)}, s_\infty(\eta, -i)\right)\right) - s_\infty(\eta, -i).$$

From the above, we obtain

$$\begin{aligned}
\Psi_k\left(\tilde{\eta}^{(i)}\right)(x) &= \tilde{\eta}^{(i)}\left(s_k\left(\tilde{\eta}^{(i)}, x\right)\right) \\
&= \eta^{(i)}\left(s_k\left(\tilde{\eta}^{(i)}, x\right) + s_\infty(\eta, -i)\right) \\
&= \eta^{(i)}\left(s_k\left(\eta^{(i)}, x + \xi_k\left(\eta^{(i)}, s_\infty(\eta, -i)\right)\right)\right) \\
&= \Psi_k\left(\eta^{(i)}\right)\left(x + \xi_k\left(\eta^{(i)}, s_\infty(\eta, -i)\right) - \xi_k\left(\eta^{(i)}, 0\right)\right),
\end{aligned}$$

for any $k, \ell \in \mathbb{N}$ and $x \in \mathbb{Z}$. In particular, we have

$$\begin{aligned}
&\left(\eta^{(i)}\right)^{\sigma}_{k+\ell}\left(s_k\left(\eta^{(i)}, x + \xi_k\left(\eta^{(i)}, s_\infty(\eta, -i)\right)\right)\right) \\
&= \left(\tilde{\eta}^{(i)}\right)^{\sigma}_{k+\ell}\left(s_k\left(\tilde{\eta}^{(i)}, x\right)\right) = \Psi_k\left(\tilde{\eta}^{(i)}\right)^{\sigma}_{\ell}(x) \\
&= \Psi_k\left(\eta^{(i)}\right)^{\sigma}_{\ell}\left(x + \xi_k\left(\eta^{(i)}, s_\infty(\eta, -i)\right) - \xi_k\left(\eta^{(i)}, 0\right)\right),
\end{aligned}$$

for any $k, \ell \in \mathbb{N}$, $\sigma \in \{\uparrow, \downarrow\}$ and $x \in \mathbb{Z}$. Therefore we have (4.14). For later use, we note some equations derived from (4.11). We observe that from (4.11), if $y = s_k(\Psi_\ell(\eta), x)$

for some $k \in \mathbb{N} \cup \{\infty\}$, $\ell \in \mathbb{N}$ and $x \in \mathbb{Z}$, then $s_\ell(\eta, y + \xi_\ell(\eta, 0)) = s_{k+\ell}(\eta, x)$. Hence, for any $k \in \mathbb{N} \cup \{\infty\}$, $\ell \in \mathbb{N}$ and $x \in \mathbb{Z}$, we have

$$\begin{aligned}
& s_k(\Psi_\ell(\eta), x) - s_k(\Psi_\ell(\eta), x-1) \\
&= \sum_{y = s_k(\Psi_\ell(\eta), x-1)+1}^{s_k(\Psi_\ell(\eta), x)} r(\Psi_\ell(\eta), y) + \left(\sum_{h \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \Psi_\ell(\eta)_h^\sigma(y) \right) \\
&= \sum_{y = s_k(\Psi_\ell(\eta), x-1)+1}^{s_k(\Psi_\ell(\eta), x)} r(\eta, s_\ell(\eta, y + \xi_\ell(\eta, 0))) + \left(\sum_{h \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_{\ell+h}^\sigma(s_\ell(\eta, y + \xi_\ell(\eta, 0))) \right) \\
&= \sum_{y = s_{k+\ell}(\eta, x-1)+1}^{s_{k+\ell}(\eta, x)} r(\eta, y) + \left(\sum_{h \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_{\ell+h}^\sigma(y) \right) \\
&= \xi_\ell(\eta, s_{k+\ell}(\eta, x)) - \xi_\ell(\eta, s_{k+\ell}(\eta, x-1)),
\end{aligned}$$

where at the third equality we use the fact that $(s_{k+h}(\eta, x))_{x \in \mathbb{Z}} \subset (s_k(\eta, x))_{x \in \mathbb{Z}}$ for any $k \in \mathbb{N}$ and $h \in \mathbb{N} \cup \{\infty\}$. By using the same computation, for any $k \in \mathbb{N} \cup \{\infty\}$ and $\ell \in \mathbb{N}$, we obtain

$$\begin{aligned}
s_k(\Psi_\ell(\eta), 0) &= s_\infty(\Psi_\ell(\eta), 0) \\
&= -\sum_{y = s_\infty(\Psi_\ell(\eta), 0)+1}^{0} \sum_{h \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \Psi_\ell(\eta)_h^\sigma(y) \\
&= -\sum_{y = s_\infty(\eta, 0)+1}^{0} \sum_{h \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_{\ell+h}^\sigma(y) \\
&= -\xi_\ell(\eta, 0),
\end{aligned}$$

and for any $k, \ell \in \mathbb{N}$, we also get

$$\begin{aligned}
\xi_k(\Psi_\ell(\eta), 0) &= \sum_{y = s_\infty(\Psi_\ell(\eta), 0)+1}^{0} \sum_{h \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \Psi_\ell(\eta)_{k+h}^\sigma(y) \\
&= \sum_{y = s_\infty(\eta, 0)+1}^{0} \sum_{h \in \mathbb{N}} \sum_{\sigma \in \{\uparrow, \downarrow\}} \eta_{k+\ell+h}^\sigma(y) \\
&= \xi_{k+\ell}(\eta, 0). \qquad (4.15)
\end{aligned}$$

In particular, for any $k \in \mathbb{N} \cup \{\infty\}$, $\ell \in \mathbb{N}$ and $x \in \mathbb{Z}$, we have

$$s_k(\Psi_\ell(\eta), x) = \xi_\ell(\eta, s_{k+\ell}(\eta, x)) - \xi_\ell(\eta, 0). \qquad (4.16)$$

Next we show (4.12). From (4.15) and (4.16), for any $k, \ell \in \mathbb{N}$ and $x \in \mathbb{Z}$ we have

$$\begin{aligned}
\Psi_k(\Psi_\ell(\eta))(x) &= \Psi_\ell(\eta)(s_k(\Psi_\ell(\eta), x + \xi_k(\Psi_\ell(\eta), 0))) \\
&= \Psi_\ell(\eta)(\xi_\ell(\eta, s_{k+\ell}(\eta, x + \xi_{k+\ell}(\eta, 0))) - \xi_\ell(\eta, 0)) \\
&= \eta(s_\ell(\eta, \xi_\ell(\eta, s_{k+\ell}(\eta, x + \xi_{k+\ell}(\eta, 0))))) \\
&= \eta(s_{k+\ell}(\eta, x + \xi_{k+\ell}(\eta, 0))) \\
&= \Psi_{k+\ell}(\eta)(x),
\end{aligned}$$

and thus we obtain (4.12).

Finally we show (4.13). From (4.11) and (4.16), for any $k, \ell \in \mathbb{N}$ and $i \in \mathbb{Z}$, we have

$$\begin{aligned}
&\zeta_k\left(\Psi_\ell(\eta), i\right) \\
&= \sum_{y=s_k(\Psi_\ell(\eta),i)+1}^{s_k(\Psi_\ell(\eta),i+1)} \left(\Psi_\ell(\eta)_k^{\uparrow}(y) - \Psi_\ell(\eta)_{k+1}^{\uparrow}(y)\right) \\
&= \left(\sum_{y=s_k(\Psi_\ell(\eta),i)}^{s_k(\Psi_\ell(\eta),i+1)} \Psi_\ell(\eta)_k^{\uparrow}(y)\right) - \Psi_\ell(\eta)_{k+1}^{\uparrow}\left(s_k\left(\Psi_\ell(\eta), i+1\right)\right) \\
&= \left(\sum_{y=\xi_\ell(\eta,s_{k+\ell}(\eta,i))-\xi_\ell(\eta,0)}^{\xi_\ell(\eta,s_{k+\ell}(\eta,i+1))-\xi_\ell(\eta,0)} \eta_{k+\ell}^{\uparrow}\left(s_\ell\left(\eta, y+\xi_\ell(\eta,0)\right)\right)\right) - \eta_{k+\ell}^{\uparrow}\left(s_\ell\left(\eta, \xi_\ell\left(\eta, s_{k+\ell}(\eta, i+1)\right)\right)\right) \\
&= \left(\sum_{y=s_{k+\ell}(\eta,i)}^{s_{k+\ell}(\eta,i+1)} \eta_{k+\ell}^{\uparrow}(y)\right) - \eta_{k+\ell}^{\uparrow}\left(s_{k+\ell}(\eta, i+1)\right) \\
&= \zeta_{k+\ell}(\eta, i),
\end{aligned}$$

and thus we obtain (4.13).

□

We conclude this section by describing the relation between T and Ψ_k.

Proposition 4.4. *Suppose that $\eta \in \Omega$. Then, for any $k \in \mathbb{N}$ and $x \in \mathbb{Z}$, we have*

$$T\Psi_k(\eta)\left(x + \sum_{\ell=1}^{k} r\left(\Psi_{\ell-1}(\eta), 0\right)\right) = \Psi_k(T\eta)(x)$$

Proof. Thanks to (4.12), it is sufficient to consider the case $k = 1$. Suppose that $T\Psi_1(\eta)(x + r(\eta, 0)) = 1$. Then, there exists $\ell \in \mathbb{N}$ such that

$$T\Psi_1(\eta)_\ell^{\uparrow}(x + r(\eta, 0)) = 1.$$

From (4.8) and (4.11), we have

$$\eta_{\ell+1}^{\downarrow}\left(s_1\left(\eta, x + r(\eta, 0) + \xi_1(\eta, 0)\right)\right) = \Psi_1(\eta)_\ell^{\downarrow}(x + r(\eta, 0)) = 1.$$

Again by using (4.8), we obtain

$$T\eta_{\ell+1}^{\uparrow}\left(s_1\left(\eta, x + r(\eta, 0) + \xi_1(\eta, 0)\right)\right) = 1.$$

On the other hand, from (4.9) we get

$$\begin{aligned}
&\xi_1\left(T\eta, s_1\left(\eta, x + r(\eta, 0) + \xi_1(\eta, 0)\right)\right) \\
&= x + r(\eta, 0) + \xi_1(\eta, 0) + 1 + o_1(\eta) \\
&= x + r(\eta, 0) + \xi_1(T\eta, 0) + W_1(\eta, 0) + W_1(T\eta, 0) \\
&= x + \xi_1(T\eta, 0).
\end{aligned}$$

Since the site $s_1(\eta, x + r(\eta, 0) + \xi_1(\eta, 0))$ is a $(\ell + 1, \uparrow)$-seat in $T\eta$, we have

$$s_1\left(T\eta, x + \xi_1(T\eta, 0)\right) = s_1\left(\eta, x + r(\eta, 0) + \xi_1(\eta, 0)\right).$$

Hence we have

$$\begin{aligned}\Psi_1(T\eta)^{\uparrow}_{\ell}(x) &= T\eta^{\uparrow}_{\ell+1}(s_1(T\eta, x+\xi_1(T\eta,0)))\\ &= T\eta^{\uparrow}_{\ell+1}(s_1(\eta, x+r(\eta,0)+\xi_1(\eta,0)))\\ &= 1,\end{aligned}$$

and thus we see that $T\Psi_1(\eta)(x+r(\eta,0)) = 1$ implies $\Psi_1(T\eta)(x) = 1$. By the same computation, we can also show that $\Psi_1(T\eta)(x) = 1$ implies $T\Psi_1(\eta)(x+r(\eta,0)) = 1$.

□

5. Distribution of k-skipped configuration and long-time behavior of integrated ball current

In this section we investigate the distribution of $\Psi_k(\eta)$ when the distribution of η belongs to a class of invariant measures introduced by [FG]. In addition, we derive the long-time behavior of the integrated current of $\Psi_k(\eta)$. Throughout this subsection, we restrict the state space $\Omega_* \subset \Omega$, defined as

$$\Omega_* := \{\eta \in \Omega \ ; \ |s_\infty(\cdot, i)| < \infty \text{ for any } i \in \mathbb{Z}\}.$$

Also, we define $\widehat{\Omega}_* \subset \Omega_*$ as

$$\widehat{\Omega}_* := \{\eta \in \Omega_* \ ; \ s_\infty(\eta, 0) = 0\}.$$

In Section 5.1 we prepare some notions and then we describe our results in Section 5.2.

5.1. **Excursion.** First, we introduce the notion of *excursion*, which will be used to define a class of invariant measures of the BBS. For any $n \in \mathbb{Z}_{\geq 0}$, we say that a sequence $(e_j)_{j=0}^{2n}, e_j \in \{0,1\}$ is a excursion with length $2n+1$ if

$$e_0 = 0, \quad \sum_{j=1}^{m}(2e_j - 1) > 0 \text{ for any } 1 \leq m < 2n, \quad \sum_{j=1}^{2n}(2e_j - 1) = 0.$$

We denote by $\mathcal{E}_n$ the set of all excursions with length $2n+1$, and denote by $\mathcal{E} := \bigcup_{n\in\mathbb{Z}_{\geq 0}} \mathcal{E}_n$ the set of all excursions. There is a natural injection $\iota : \mathcal{E} \to \{0,1\}^{\mathbb{Z}}$ given by

$$\iota(\varepsilon)(x) := \begin{cases} \varepsilon_x & 1 \leq x \leq |\varepsilon|, \\ 0 & \text{otherwise}, \end{cases}$$

and $\Omega_1 := \iota(\mathcal{E})$ is written as

$$\Omega_1 = \left\{\eta \in \{0,1\}^{\mathbb{Z}} \ ; \ \eta(x) = 0 \text{ for any } x < 0 \text{ or } x \geq s_\infty(\eta, 1)\right\}.$$

Observe that for any $\varepsilon \in \mathcal{E}$ and $k \in \mathbb{N}$, we have $\Psi_k(\iota(\varepsilon)) \in \Omega_1$. Hence, the following map

$$\widetilde{\Psi}_k(\varepsilon) := \iota^{-1}(\Psi_k(\iota(\varepsilon))), \tag{5.1}$$

is well-defined for any $\mathcal{E}$, and we call $\widetilde{\Psi}_k : \mathcal{E} \to \mathcal{E}$ the k-skip map for excursions. Also, we extend the notion of ζ for excursions. For any $\varepsilon \in \mathcal{E}$ and $k \in \mathbb{N}$ we define

$$\zeta_k(\varepsilon) := \sum_{i\in\mathbb{Z}} \zeta_k(\iota(\varepsilon), i).$$

Note that we can obtain the following formula of $|\varepsilon|$.

$$|\varepsilon| = s_\infty(\iota(\varepsilon), 1) = 1 + 2\sum_{k\in\mathbb{N}} k\zeta_k(\varepsilon). \tag{5.2}$$

In addition, from Proposition 2.3, we see that $\widetilde{\Psi}_k$ is a shift operator of ζ :

Proposition 5.1. *For any $\varepsilon \in \mathcal{E}$ and $k, \ell \in \mathbb{N}$, we have*

$$\zeta_k\left(\widetilde{\Psi}_\ell(\varepsilon)\right) = \zeta_{k+\ell}(\varepsilon).$$

Now we introduce a family of probability measures on $\mathcal{E}$ via $\zeta_k : \mathcal{E} \to \mathbb{Z}_{\ge 0}$. For any $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, we define

$$\mathcal{A} := \left\{\alpha = (\alpha_k)_{k\in\mathbb{N}} \subset [0,1)^{\mathbb{N}}\ ;\ Z_\alpha := \sum_{\varepsilon\in\mathcal{E}}\prod_{k\in\mathbb{N}} \alpha_k^{\zeta_k(\varepsilon)} < \infty\right\}.$$

Then, for any $\alpha \in \mathcal{A}$, we define a canonical probability measure ν_α on $\mathcal{E}$ as

$$\nu_\alpha(\varepsilon) := \frac{1}{Z_\alpha}\prod_{k\in\mathbb{N}} \alpha_k^{\zeta_k(\varepsilon)}.$$

Observe that from (5.2), for any $\varepsilon \in \mathcal{E}$, we get

$$\rho(\alpha) := \mathbb{E}_{\nu_\alpha}[|\varepsilon|] = 1 + 2\sum_{\varepsilon\in\mathcal{E}}\sum_{k\in\mathbb{N}} k\zeta_k(\varepsilon)\nu_\alpha(\varepsilon).$$

We denote by $\mathcal{A}_+ \subset \mathcal{A}$ the set of all $\alpha \in \mathcal{A}$ such that $\rho(\alpha) < \infty$, i.e.,

$$\mathcal{A}^+ := \left\{\alpha = (\alpha_k)_{k\in\mathbb{N}} \in \mathcal{A}\ ;\ \sum_{\varepsilon\in\mathcal{E}}\sum_{k\in\mathbb{N}} k\zeta_k(\varepsilon)\nu_\alpha(\varepsilon) < \infty\right\}.$$

We introduce a shift operator $\theta : \mathcal{A} \to \mathcal{A}$ defined as

$$(\theta\alpha)_k := \frac{\alpha_{k+1}}{(1-\alpha_1)^{2k}}. \tag{5.3}$$

It is known that if $\alpha \in \mathcal{A}_+$, then $\theta\alpha \in \mathcal{A}_+$, see [FG, (3.1), Theorem 3.1] for details. Now we compute the distribution of $\Psi_k(\varepsilon)$ when the distribution of ε is given by ν_α.

Lemma 5.1. *Suppose that $\alpha \in \mathcal{A}$. Then, for any $k \in \mathbb{N}$ and $\varepsilon' \in \mathcal{E}$ we have*

$$\nu_\alpha(\{\varepsilon \in \mathcal{E}\ ;\ \Psi_k(\varepsilon) = \varepsilon'\}) = \nu_{\theta^k\alpha}(\varepsilon'). \tag{5.4}$$

Proof of Lemma 5.1. From Proposition 5.1 and the uniformity of ν_α, it is sufficient to show that

$$\nu_\alpha(\zeta_k(\Psi_1(\varepsilon)) = n_k \text{ for any } k \in \mathbb{N}) = \nu_{\theta\alpha}(\zeta_k(\varepsilon) = n_k \text{ for any } k \in \mathbb{N})$$

for $\mathbf{n} = (n_k)_{k\in\mathbb{Z}_{\ge 0}} \subset (\mathbb{Z}_{\ge 0})^{\mathbb{N}}$ such that $n_k = 0$ for sufficiently large k. For such $\mathbf{n}$, we define

$$\mathcal{E}(\mathbf{n}) := \{\varepsilon \in \mathcal{E}\ ;\ \zeta_k(\varepsilon) = n_k \text{ for any } k \in \mathbb{N}\}.$$

It is known that $|\mathcal{E}(\mathbf{n})|$ is given via the so-called *Fermionic formula*,

$$|\mathcal{E}(\mathbf{n})| = \prod_{k=1}^{\infty}\binom{2\sum_{\ell\ge k+1}(\ell-k)n_\ell + n_k}{n_k},$$

see [KTT] for details. Then, from Proposition 5.1 we have

$$\begin{aligned}
&\nu_\alpha\left(\zeta_k\left(\Psi_1\left(\varepsilon\right)\right)=n_k \text{ for any } k\in\mathbb{N}\right)\\
&=\sum_{n=0}^{\infty}\nu_\alpha\left(\zeta_{k+1}\left(\varepsilon\right)=n_k \text{ for any } k\in\mathbb{N},\zeta_1\left(\varepsilon\right)=n\right)\\
&=\frac{1}{Z_\alpha}\prod_{k\in\mathbb{N}}\alpha_{k+1}^{n_k}\prod_{m=2}^{\infty}\binom{2\sum_{\ell\geq m+1}(\ell-m)n_{\ell-1}+n_{m-1}}{n_{m-1}}\sum_{n=0}^{\infty}\binom{2\sum_{\ell\geq 2}(\ell-1)n_{\ell-1}+n}{n}\alpha_1^n\\
&=\frac{1}{Z_\alpha}\prod_{k\in\mathbb{N}}\alpha_{k+1}^{n_k}\prod_{m=1}^{\infty}\binom{2\sum_{\ell\geq m+1}(\ell-m)n_{\ell}+n_{m}}{n_{m}}\left(\frac{1}{1-\alpha_1}\right)^{2\sum_{\ell\geq 2}(\ell-1)n_{\ell-1}+1}\\
&=\frac{1}{Z_\alpha(1-\alpha)}\prod_{k\in\mathbb{N}}(\theta a)_k^{n_k}\prod_{m=1}^{\infty}\binom{2\sum_{\ell\geq m+1}(\ell-m)n_{\ell}+n_{m}}{n_{m}}\\
&=\nu_{\theta\alpha}\left(\zeta_k\left(\varepsilon\right)=n_k \text{ for any } k\in\mathbb{N}\right),
\end{aligned}$$

where we use the equation $Z_{\theta\alpha}=Z_\alpha(1-\alpha_1)$ [FG, (3.22), (3.29)], and

$$\sum_{n=0}^{\infty}\binom{x+n}{n}y^n=\left(\frac{1}{1-y}\right)^{x+1},$$

for any $x\in\mathbb{Z}_{\geq 0}$ and $y\in(0,1)$. □

Next we observe that $\eta\in\widehat{\Omega}_*$ can be constructed from excursions as follows. For any $(\varepsilon_i)_{i\in\mathbb{Z}}\in\mathcal{E}^{\mathbb{Z}}$, we define $I\left((\varepsilon_i)_{i\in\mathbb{Z}}\right)$ as

$$I\left((\varepsilon_i)_{i\in\mathbb{Z}}\right)(x):=\begin{cases}\iota\left(\varepsilon_0\right)(x) & \text{if } 0\leq x\leq|\varepsilon_0|-1,\\ \iota\left(\varepsilon_{-1}\right)(x) & \text{if } |\varepsilon_{-1}|\leq x\leq -1,\\ \iota\left(\varepsilon_i\right)(x) & \text{if } i\geq 1 \text{ and } \sum_{m=0}^{i-1}|\varepsilon_m|\leq x\leq\sum_{m=0}^{i}|\varepsilon_m|-1,\\ \iota\left(\varepsilon_i\right)(x) & \text{if } i\leq -2 \text{ and } -\sum_{m=i}^{-1}|\varepsilon_m|\leq x\leq -\sum_{m=i+1}^{-1}|\varepsilon_m|-1,\end{cases}$$

then we can check that I is injective and $I\left(\mathcal{E}^{\mathbb{Z}}\right)=\widehat{\Omega}_*$. For later use, we prepare the following formula.

Lemma 5.2. *Suppose that $(\varepsilon_i)_{i\in\mathbb{Z}}\in\mathcal{E}^{\mathbb{Z}}$. Then for any $k\in\mathbb{N}$, we have*

$$\Psi_k\left(I\left((\varepsilon_i)_{i\in\mathbb{Z}}\right)\right)=I\left(\left(\tilde{\Psi}_k\left(\varepsilon_i\right)\right)_{i\in\mathbb{Z}}\right). \tag{5.5}$$

Proof of Lemma 5.2. This is a direct consequence of Lemma 4.1, (4.11), (5.1) and

$$s_\infty\left(I\left((\varepsilon_i)_{i\in\mathbb{Z}}\right),i\right)=\begin{cases}0 & \text{if } i=0,\\ \sum_{m=0}^{i-1}|\varepsilon_m| & \text{if } i\geq 1,\\ -\sum_{m=i}^{-1}|\varepsilon_m| & \text{if } i\leq -1.\end{cases}$$

□

We then denote by $\hat{\mu}_\alpha$, $\alpha\in\mathcal{A}^+$, the probability measure on $\widehat{\Omega}_*$ induced by the product probability measure $\prod_{\varepsilon_i\in\mathbb{Z}}\nu_\alpha\left(\varepsilon_i\right), e_i\in\mathcal{E}$ on $\mathcal{E}^{\mathbb{Z}}$ via the map I. In addition, for any $\alpha \in \mathcal{A}^+$, we define a probability measure μ_α on Ω_* as

$$\int_\Omega d\mu_\alpha(\eta) f(\eta) := \frac{1}{\rho(\alpha)} \int_{\widehat{\Omega}_*} d\hat{\mu}_\alpha(\eta) \sum_{y=0}^{s_\infty(\eta,1)-1} \tau_y f(\eta),$$

for any local function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, where $\tau_y, y \in \mathbb{Z}$ is a spatial shift operator defined as $\tau_y f(\eta) := f(\eta(\cdot + y))$. Then, the following result is shown by [FG].

Theorem 5.1 (Theorem 4.5 in [FG]). *Suppose that $\alpha \in \mathcal{A}^+$. Then, μ_α is a shift-stationary invariant measure of the BBS(∞), i.e., for any $x \in \mathbb{Z}$ and local function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$,*

$$\int_\Omega d\mu_\alpha(\eta) f(\eta(\cdot + x)) = \int_\Omega d\mu_\alpha(\eta) f(\eta), \quad \int_\Omega d\mu_\alpha(\eta) f(T\eta) = \int_\Omega d\mu_\alpha(\eta) f(\eta).$$

In the rest of this subsection, we consider another family of invariant measures introduced in [FG]. We define parameter sets $\mathcal{Q}, \mathcal{Q}^+$ as

$$\mathcal{Q} := \left\{ q = (q_k)_{k\in\mathbb{N}} \subset [0,1)^{\mathbb{N}} \; ; \; \sum_{k\in\mathbb{N}} k q_k < \infty \right\},$$
$$\mathcal{Q}^+ := \left\{ q = (q_k)_{k\in\mathbb{N}} \in \mathcal{Q} \; ; \; \sum_{k\in\mathbb{N}} k q_k < \infty \right\}.$$

For any $q \in \mathcal{Q}$, we define a probability measure φ_q on $\mathcal{E}$ as

$$\varphi_q(\varepsilon) := \prod_{k\in\mathbb{N}} q_k^{\zeta_k(\varepsilon)} (1 - q_k)^{2\sum_{\ell\in\mathbb{N}} \ell \zeta_{k+\ell}(\varepsilon)}.$$

Then, the one-to-one correspondence between ν_α and φ_q has been shown by [FG]. To state their result, for any $\alpha \in \mathcal{A}$ and $q \in \mathcal{Q}$, we define $q(\alpha)$ and $\alpha(q)$ as

$$q(\alpha)_k := \begin{cases} \alpha_1 & k = 1, \\ \dfrac{\alpha_k}{\prod_{\ell=1}^{k-1} (1 - q_\ell(\alpha))^{2(k-\ell)}} & k \geq 2, \end{cases} \tag{5.6}$$

$$\alpha(q)_k := q_k \prod_{\ell=1}^{k-1} (1 - q_\ell)^{2(k-\ell)}. \tag{5.7}$$

Theorem (Theorem 3.1 in [FG]). *The maps* (5.6) *and* (5.7) *are the inverse of each other. Moreover, we have*

$$q(\mathcal{A}) = \mathcal{Q}, \quad \alpha(\mathcal{Q}) = \mathcal{A},$$

and

$$q(\mathcal{A}^+) = \mathcal{Q}^+, \quad \alpha(\mathcal{Q}^+) = \mathcal{A}^+.$$

In particular, for any $q \in \mathcal{Q}$, we have

$$\varphi_q = \nu_{\alpha(q)}. \tag{5.8}$$

Thanks to (5.4) and (5.8), we have the distribution of $\Psi_k(\varepsilon)$ under φ_q. We define a shift operator $\tilde{\theta} : \mathcal{Q} \to \mathcal{Q}$ as $\tilde{q}_k := q_{k+1}$.

Lemma 5.3. *Suppose that $q \in \mathcal{Q}$. Then, for any $k \in \mathbb{N}$ and $\varepsilon' \in \mathcal{E}$, we have*

$$\varphi_q(\{\varepsilon \in \mathcal{E} \; ; \; \Psi_k(\varepsilon) = \varepsilon'\}) = \varphi_{\tilde{\theta}^k q}(\varepsilon').$$

Proof of Lemma 5.3. We observe that for any $k \in \mathbb{N}$,

$$\begin{aligned}\alpha\left(\tilde{\theta}q\right)_k &= q_{k+1}\prod_{\ell=1}^{k-1}\left(1-q_{\ell+1}\right)^{2(k-\ell)}\\ &= \frac{\alpha(q)_{k+1}}{\left(1-\alpha(q)_1\right)^{2k}}\\ &= \left(\theta\alpha(q)\right)_k.\end{aligned}$$

Hence we have $\alpha\left(\tilde{\theta}q\right) = \theta\alpha(q)$. By using this relation k times, we have $\alpha\left(\tilde{\theta}^k q\right) = \theta^k\alpha(q)$ for any $k \in \mathbb{N}$. Thus from (5.4) and (5.8), we have

$$\begin{aligned}\varphi_q\left(\{\varepsilon \in \mathcal{E}\ ;\ \Psi_k(\varepsilon) = \varepsilon'\}\right) &= \nu_{\alpha(q)}\left(\{\varepsilon \in \mathcal{E}\ ;\ \Psi_k(\varepsilon) = \varepsilon'\}\right)\\ &= \nu_{\theta^k\alpha(q)}\left(\varepsilon'\right)\\ &= \nu_{\alpha(\tilde{\theta}^k q)}\left(\varepsilon'\right)\\ &= \varphi_{\tilde{\theta}^k q}\left(\varepsilon'\right).\end{aligned}$$

□

We note that φ_q can be extended over $\mathcal{E}^{\mathbb{Z}}$ and Ω_* in the same way as we did for ν_α. We denote by $\hat{\phi}_q$, $q \in \mathcal{Q}^+$, the probability measure on $\widehat{\Omega}_*$ induced by the product probability measure $\prod_{\varepsilon_i \in \mathbb{Z}}\varphi_q(\varepsilon_i), e_i \in \mathcal{E}$ on $\mathcal{E}^{\mathbb{Z}}$ via the map I. In addition, for any $q \in \mathcal{Q}^+$, we define a probability measure ϕ_q on Ω_* as

$$\int_\Omega d\phi_q(\eta) f(\eta) := \frac{1}{\rho(\alpha(q))}\int_{\widehat{\Omega}_*} d\hat{\phi}_q(\eta)\sum_{y=0}^{s_\infty(\eta,1)-1}\tau_y f(\eta),$$

for any local function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$. We note that from (5.8), we have

$$\phi_q = \mu_{\alpha(q)}, \tag{5.9}$$

for any $q \in \mathcal{Q}^+$.

5.2. **Distribution of k-skipped configuration.** If the distribution of η is μ_α, then we can compute the distribution of $\Psi_k(\eta)$. Recall that $\theta : \mathcal{A} \to \mathcal{A}$ is defined by (5.3).

Theorem 5.2. *Suppose that $\alpha = (\alpha_\ell)_{\ell\in\mathbb{N}} \in \mathcal{A}^+$. Then, for any $k \in \mathbb{N}$ and local function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, we have*

$$\int_\Omega d\mu_\alpha(\eta) f(\Psi_k(\eta)) = \int_\Omega d\mu_{\theta^k\alpha}(\eta) f(\eta), \tag{5.10}$$

and

$$\int_\Omega d\mu_\alpha(\eta | s_\infty(\eta,0) = 0) f(\Psi_k(\eta)) = \int_\Omega d\mu_{\theta^k\alpha}(\eta | s_\infty(\eta,0) = 0) f(\eta). \tag{5.11}$$

Proof of Theorem 5.2. First we show (5.10). From Proposition (4.12), it is sufficient to consider the case $k = 1$. First we observe that for any $\eta \in \widehat{\Omega}_*$ and $0 \le y \le s_\infty(\eta,1) - 1$, we have

$$\begin{aligned}\Psi_1(\eta(x+y)) &= \eta\left(s_1\left(\eta(\cdot+y), x + \xi_1(\eta(\cdot+y),0)\right) + y\right)\\ &= \eta\left(s_1\left(\eta, x + \xi_1(\eta,y)\right)\right).\end{aligned}$$

In addition, for any $i \in \mathbb{Z}$ and local function $g : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, we have

$$\int_{\widehat{\Omega}} d\hat{\mu}_{\alpha}(\eta)\, \tau_{s_{\infty}(\eta,i)} g(\eta) = \int_{\widehat{\Omega}} d\hat{\mu}_{\alpha}(\eta)\, g(\eta).$$

Since $\tilde{\Psi}_1(\eta)$ does not depend on $\zeta_1(\iota(\varepsilon),\cdot)$, from the above observations and (5.5), we obtain

$$\begin{aligned}
&\int_{\Omega} d\mu_{\alpha}(\eta)\, f(\Psi_1(\eta)) \\
&= \frac{1}{\rho(\alpha)} \int_{\widehat{\Omega}_*} d\hat{\mu}_{\alpha}(\eta) \sum_{y=0}^{s_{\infty}(\eta,1)-1} f(\eta(s_1(\eta,\cdot+\xi_1(\eta,y)))) \\
&= \frac{1}{\rho(\alpha)} \int_{\widehat{\Omega}_*} d\hat{\mu}_{\alpha}(\eta) \sum_{j=0}^{\xi_1(\eta,s_{\infty}(\eta,1))-1} \sum_{y=s_1(\eta,j)}^{s_1(\eta,j+1)-1} f(\eta(s_1(\eta,\cdot+j))) \\
&= \frac{1}{\rho(\alpha)} \int_{\widehat{\Omega}_*} d\hat{\mu}_{\alpha}(\eta) \sum_{j=0}^{s_{\infty}(\Psi_1(\eta),1)-1} (s_1(\eta,j+1) - s_1(\eta,j))\, \tau_j f(\Psi_1(\eta)) \\
&= \frac{1}{\rho(\alpha)} \int_{\mathcal{E}^{\mathbb{Z}}} \prod_{i\in\mathbb{Z}} d\nu_{\alpha}(\varepsilon_i) \sum_{j=0}^{|\tilde{\Psi}_1(\varepsilon_0)|-1} (2\zeta_1(\iota(\varepsilon_0),j)+1)\, \tau_j f\left(I\left(\left(\tilde{\Psi}_1(\varepsilon_i)\right)_{i\in\mathbb{Z}}\right)\right) \\
&= \frac{1}{\rho(\alpha)} \int_{\mathcal{E}^{\mathbb{Z}\setminus\{0\}}} \prod_{i\in\mathbb{Z}\setminus\{0\}} d\nu_{\theta\alpha}(\varepsilon_i) \\
&\quad \times \int_{\mathcal{E}} d\nu_{\alpha}(\varepsilon_0) \sum_{j=0}^{|\tilde{\Psi}_1(\varepsilon_0)|-1} (2\zeta_1(\iota(\varepsilon_0),j)+1)\, \tau_j f\left(I\left((\varepsilon_i)_{i\in\mathbb{Z}\setminus\{0\}} \cup \tilde{\Psi}_1(\varepsilon_0)\right)\right) \\
&= \frac{1}{\rho(\alpha)} \left(\frac{2\alpha_1}{1-\alpha_1}+1\right) \int_{\mathcal{E}^{\mathbb{Z}}} \prod_{i\in\mathbb{Z}} d\nu_{\theta\alpha}(\varepsilon_i) \sum_{j=0}^{|\varepsilon_0|-1} \tau_j f(I((\varepsilon_i)_{i\in\mathbb{Z}})) \\
&= \frac{\rho(\theta\alpha)}{\rho(\alpha)} \left(\frac{2\alpha_1}{1-\alpha_1}+1\right) \int_{\Omega} d\mu_{\theta\alpha}(\eta)\, f(\eta),
\end{aligned}$$

where at the fourth equality we use the relation

$$s_1(\eta,i+1) - s_1(\eta,i) = 2\zeta_1(\eta,i)+1.$$

On the other hand, since

$$\sum_{j=0}^{|\tilde{\Psi}_1(\varepsilon_0)|-1} (s_1(\iota(\varepsilon_0),j+1) - s_1(\iota(\varepsilon_0),j)) = |\varepsilon|,$$

by integrating both sides with respect to ν_{α}, we obtain

$$\rho(\theta\alpha)\left(\frac{2\alpha_1}{1-\alpha_1}+1\right) = \rho(\alpha).$$

Therefore (5.10) is proven.

Finally we show (5.11). Since

$$\int_{\Omega} d\mu_{\alpha}(\eta | s_{\infty}(\eta,0)=0)\, f(\eta) = \int_{\widehat{\Omega}_*} d\hat{\mu}_{\alpha}(\eta)\, f(\eta),$$

by using (5.4) and (5.5), we have (5.11). □

By combining (5.9) and Theorem 5.2, we have the distribution of $\Psi_k(\eta)$ under ϕ_q.

Corollary 5.1. *Suppose that $q \in \mathcal{Q}^{+}$. Then, for any $k \in \mathbb{N}$ and local function $f:\{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, we have*

$$\int_{\Omega} d\phi_q(\eta) f(\Psi_k(\eta)) = \int_{\Omega} d\phi_{\tilde{\theta}^k q}(\eta) f(\eta),$$

and

$$\int_{\Omega} d\phi_q(\eta | s_\infty(\eta,0)=0) f(\Psi_k(\eta)) = \int_{\Omega} d\phi_{\tilde{\theta}^k q}(\eta | s_\infty(\eta,0)=0) f(\eta).$$

5.3. **Two-sided Markov distribution case.** If $(\eta(x))_{x\in\mathbb{Z}}$ is a two-sided stationary Markov chain, then we can show that $(\Psi_k(\eta)(x))_{x\in\mathbb{Z}}$ is also a two-sided stationary Markov chain. Before describing the precise statement, we prepare some notations and recall some facts on Markov chains on $\{0,1\}$. In the following discussion, we denote the transition matrix of a given two-sided stationary Markov chain $(\eta(x))_{x\in\mathbb{Z}}$ by $P^{(\eta)} = (p^{(\eta)}(r,s))_{r,s=0,1}$, where

$$p^{(\eta)}(r,s) := \mathbb{P}(\eta(1)=s | \eta(0)=r),$$

and assume that $p^{(\eta)}(0,1)+p^{(\eta)}(1,1)<1$. If we define $a^{(\eta)}, b^{(\eta)}$ as

$$a^{(\eta)} := p^{(\eta)}(0,1)p^{(\eta)}(1,0),$$
$$b^{(\eta)} := p^{(\eta)}(0,0)p^{(\eta)}(1,1),$$

then $a^{(\eta)}, b^{(\eta)}$ satisfies $0<a^{(\eta)}<1$, $0 \le b^{(\eta)}<1$ and $\sqrt{a^{(\eta)}}+\sqrt{b^{(\eta)}}<1$. In addition, $p^{(\eta)}(0,0)$ and $p^{(\eta)}(1,0)$ are expressed in terms of $a^{(\eta)}, b^{(\eta)}$ as

$$p^{(\eta)}(0,0) = \frac{1-a+b+\sqrt{(1-(a+b))^2-4ab}}{2},$$
$$p^{(\eta)}(1,0) = \frac{1+a-b+\sqrt{(1-(a+b))^2-4ab}}{2}.$$

Conversely, for a given a,b such that $0<a<1$, $0\le b<1$ and $\sqrt{a}+\sqrt{b}<1$, one can construct a transition matrix with condition $p(0,1)+p(1,1)<1$. Actually, we can show the following.

Lemma 5.4. *We define*

$$\mathcal{P} := \left\{ (p(i,j))_{i,j=0,1} \subset [0,1]^4 \ ; \ \sum_{j=0,1} p(i,j) = 1, \ i=0,1, \ p(0,1)+p(1,1)<1 \right\},$$

and

$$\mathcal{Q} := \left\{ (a,b) \subset [0,1]^2 \ ; \ a>0, \ \sqrt{a}+\sqrt{b}<1 \right\}.$$

In addition, we define a map $Q:\mathcal{P}\to\mathcal{Q}$ as

$$Q\left((p(i,j))_{i,j=0,1}\right) := (p(0,1)p(1,0), p(0,0)p(1,1)).$$

Then, Q is a bijection between $\mathcal{P}$ and $\mathcal{Q}$.

Proof of Lemma 5.4. From the definition, it is clear that Q is injective. To show that Q is surjective, we define a map $P:\mathcal{Q}\to\mathbb{R}^4$ as

$$P(a,b) = (p(a,b;i,j))_{i,j=0,1} := \begin{pmatrix} \frac{1-a+b+\sqrt{(1-(a+b))^2-4ab}}{2} & \frac{1+a-b-\sqrt{(1-(a+b))^2-4ab}}{2} \\ \frac{1+a-b+\sqrt{(1-(a+b))^2-4ab}}{2} & \frac{1-a+b-\sqrt{(1-(a+b))^2-4ab}}{2} \end{pmatrix}.$$

By a direct computation, one can show that $P(\mathcal{Q}) \subset \mathcal{P}$, and $Q \circ P(a,b) = (a,b)$ for any $(a,b) \in \mathcal{Q}$. Hence Q gives a bijection between $\mathcal{P}$ and $\mathcal{Q}$. $\square$

Remark 5.1. *We note that $P = Q^{-1}$. In addition, if we define $F(a,b)$ as*

$$F(a,b) := \begin{cases} \dfrac{1-(a+b)-\sqrt{(1-(a+b))^2-4ab}}{2b} & b > 0, \\ a & b = 0, \end{cases}$$

then $P(a,b)$ can be represented as

$$P(a,b) = \begin{pmatrix} \frac{1}{1+F(a,b)} & \frac{F(a,b)}{1+F(a,b)} \\ \frac{a(1+F(a,b))}{F(a,b)} & b(1+F(a,b)) \end{pmatrix}.$$

We note that $F(ab,b)$ coincides with the generating function of the Narayana numbers.

Now we describe the statement on the distribution of $(\Psi_k(\eta)(x))_{x\in\mathbb{Z}}$ when $(\eta(x))_{x\in\mathbb{Z}}$ is a two-sided stationary Markov chain.

Theorem 5.3. *Suppose that $(\eta(x))_{x\in\mathbb{Z}}$ is a two-sided stationary Markov chain on $\{0,1\}$, and $Q^{(\eta)}(0,1) + Q^{(\eta)}(1,1) < 1$. Then, $(\Psi_1(\eta)(x))_{x\in\mathbb{Z}}$ is also a two-sided stationary Markov chain such that*

$$a^{(\Psi_1(\eta))} = \frac{a^{(\eta)}b^{(\eta)}}{\left(1-a^{(\eta)}\right)^2}, \quad b^{(\Psi_1(\eta))} = \frac{b^{(\eta)}}{\left(1-a^{(\eta)}\right)^2}. \tag{5.12}$$

Proof of Theorem 5.3. From [FG, Lemma 3.7, Corollary 4.8], it is known that the distribution of $(\eta(x))_{x\in\mathbb{Z}}$ can be expressed as μ_α, where α is given by $\alpha_k := a^{(\eta)}\left(b^{(\eta)}\right)^{k-1}$ for any $k \in \mathbb{N}$. On the other hand, from Theorem 5.2, we see that the distribution of $(\Psi_k(\eta)(x))_{x\in\mathbb{Z}}$ is $\mu_{\theta\alpha}$, and

$$(\theta\alpha)_k = \frac{a^{(\eta)}\left(b^{(\eta)}\right)^k}{\left(1-a^{(\eta)}\right)^{2k}} = \frac{a^{(\eta)}b^{(\eta)}}{\left(1-a^{(\eta)}\right)^2}\left(\frac{b^{(\eta)}}{\left(1-a^{(\eta)}\right)^2}\right)^{k-1},$$

for any $k \in \mathbb{N}$. Hence, $(\Psi_k(\eta)(x))_{x\in\mathbb{Z}}$ is a two-sided stationary Markov chain on $\{0,1\}$, and $a^{(\Psi_1(\eta))}, b^{(\Psi_1(\eta))}$ is given by (5.12). $\square$

5.4. **Integrated current of k-skipped configuration.** First we recall the notion of *energy* of the BBS configuration. For any $\eta \in \Omega$ with the condition $\sum_{x\in\mathbb{Z}} \eta(x) < \infty$ and $k \in \mathbb{N} \cup \{\infty\}$, we define $E_k(\eta)$ as

$$E_k(\eta) := \sum_{\ell=1}^{k} \sum_{x\in\mathbb{Z}} \eta_\ell^{\uparrow}(x).$$

We note that $E_\infty(\eta)$ is equal to the total number of 1s in η, and $E_k(\eta)$ can be represented as

$$E_k(\eta) = E_\infty(\eta) - E_\infty(\Psi_k(\eta)).$$

In [MSSS], it is shown that $E_k(\eta)$ coincides with the notion of the k-energy defined via the crystal theory formulation [FOY]. When we consider the infinite ball configuration, then $E_k(\eta)$ may become infinite, but still we can consider the current of the energy.

In this subsection, we consider the long-time behavior of the integrated current $C^n(\Psi_k(\eta))$, where $C^n(\eta)$ is defined as

$$\begin{aligned} C^n(\eta) &:= \sum_{m=0}^{n-1} W_\infty(T^m\eta, 0) \\ &:= \sum_{m=0}^{n-1}\sum_{\ell=1}^{\infty} \mathcal{W}_\ell(T^m\eta, 0), \end{aligned}$$

for any $\eta \in \Omega$. We note that the quantity $C^n(\eta)$ can be considered as the integrated current of $E_\infty(\eta)$ at the origin. Hence, the asymptotic behavior of $C^n(\Psi_k(\eta))$ is closely related to that of the integrated current of the k-energy $E_k(\eta)$.

In [CKST], the law of large numbers (LLN), central limit theorem (CLT) and large deviations principle(LDP) of C^n have been proved when $(\eta(x))_{x\in\mathbb{Z}}$ is a two-sided stationary Markov chain on $\{0,1\}$, and $Q^{(\eta)}(0,1) + Q^{(\eta)}(1,1) < 1$. Moreover, under the same assumption on $(\eta(x))_{x\in\mathbb{Z}}$, they show that $\eta \mapsto T\eta$ is ergodic. Before presenting their results, we prepare some notations. For any $p, q \in [0,1)$ such that $p + q < 1$, we define $m_{p,q}$ and $v_{p,q}$ as

$$m_{p,q} := \frac{p(1-p+q)}{(1+p-q)(1-p-q)}, \quad v_{p,q} := \frac{p\left((1-p)(1+q)^2 + 2q(1+p)^2\right)}{(1+p)^3(1-q)^2}.$$

Theorem 5.4 (Theorem 3.34 in [CKST])**.** *Suppose that $(\eta(x))_{x\in\mathbb{Z}}$ is a two-sided stationary Markov chain on $\{0,1\}$, and $Q^{(\eta)}(0,1) + Q^{(\eta)}(1,1) < 1$. Then, we have*

$$\lim_{n\to\infty} \frac{C^n(\eta)}{n} = m_{Q^{(\eta)}(0,1),Q^{(\eta)}(1,1)} \quad a.s.,$$

and

$$\lim_{n\to\infty} \frac{C^n(\eta) - n m_{Q^{(\eta)}(0,1),Q^{(\eta)}(1,1)}}{n^{\frac{1}{2}} v_{Q^{(\eta)}(0,1),Q^{(\eta)}(1,1)}} = N(0,1) \quad \text{in distribution},$$

where $N(0,v)$ is the standard normal distribution. In addition, $n^{-1}C^n(\eta)$ satisfies a large deviation principle with rate function given by [CKST, (3.39)].

By combining Theorem 5.3 and Theorem 5.4, we obtain the LLN, CLT and LDP of $C^n(\Psi_k(\eta))$ as follows :

Corollary 5.2. *Suppose that $(\eta(x))_{x\in\mathbb{Z}}$ is a two-sided stationary Markov chain on $\{0,1\}$, and $Q^{(\eta)}(0,1) + Q^{(\eta)}(1,1) < 1$. Then, for any $k \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \frac{C^n(\Psi_k(\eta))}{n} = m_{Q^{(\Psi_k(\eta))}(0,1),Q^{(\Psi_k(\eta))}(1,1)} \quad a.s.,$$

and

$$\lim_{n\to\infty} \frac{C^n(\Psi_k(\eta)) - n m_{Q^{(\Psi_k(\eta))}(0,1),Q^{(\Psi_k(\eta))}(1,1)}}{n^{\frac{1}{2}} v_{Q^{(\Psi_k(\eta))}(0,1),Q^{(\Psi_k(\eta))}(1,1)}} = N(0,1) \quad \text{in distribution.}$$

In addition, $n^{-1}C^n(\Psi_k(\eta))$ satisfies a large deviations principle with certain rate function.

Acknowledgement

The author thanks to Makiko Sasada for insightful discussion and for pointing out some gaps in Section 5.3. The work of HS has been supported by JST CREST Grant No. JPMJCR1913 and JSPS KAKENHI Grant No. JP21K20332, 24KJ1037, 24K16936 and 24K00528.

References

[CS] D.A. Croydon and M. Sasada : *Generalized Hydrodynamic Limit for the Box-Ball System.* Commun. Math. Phys. 383, 427-463 (2021)

[CS2] D. A. Croydon and M. Sasada : *Invariant measures for the box-ball system based on stationary Markov chains and periodic Gibbs measures.* J. Math. Phys. 60, 083301 (2019)

[CKST] D. A. Croydon, T. Kato, M. Sasada and S. Tsujimoto : *Dynamics of the box-ball system with random initial conditions via Pitman's transformation.* to appear in Mem. Amer. Math. Soc., preprint appears at arXiv:1806.02147, 2018

[FG] P. A. Ferrari and D. Gabrielli : *BBS invariant measures with independent soliton components.* Electron. J. Probab. 25: 1-26 (2020)

[FNRW] P. A. Ferrari, C. Nguyen, L. Rolla, and M. Wang : *Soliton decomposition of the box-ball system.* Forum of Mathematics, Sigma 9 (2021).

[FOY] K. Fukuda, Y. Yamada and M. Okado : *Energy functions in box ball systems.* Int. J. Mod. Phys. A 15(09), 1379 (2000)

[IKT] R. Inoue, A. Kuniba and T. Takagi : *Integrable structure of box–ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry.* J. Phys. A: Math. Theor. 45(7), 073001 (2012)

[KL] A. Kuniba and H. Lyu : *Large Deviations and One-Sided Scaling Limit of Randomized Multicolor Box-Ball System.* J Stat Phys 178, 38–74 (2020)

[KLO] A. Kuniba, H. Lyu and M. Okado : *Randomized box-ball systems, limit shape of rigged configurations and thermodynamic Bethe ansatz.* Nuclear Physics B 937 240–271 (2018)

[KMP] A. Kuniba, G. Misguich and V. Pasquier : *Generalized hydrodynamics in box-ball system.* J. Phys. A: Math. Theor. 53 404001 (2020)

[KMP2] A. Kuniba, G. Misguich and V. Pasquier : *Generalized hydrodynamics in complete box-ball system for $U_q(\hat{sl}_n)$.* Scipost phys. 10, 095 (2021)

[KMP3] A. Kuniba, G. Misguich and V. Pasquier : *Current correlations, Drude weights and large deviations in a box-ball system.* J. Phys. A: Math. Theor. **55** 244006 (2022)

[KNTW] Saburo Kakei, Jonathan J C Nimmo, Satoshi Tsujimoto and Ralph Willox : *Linearization of the box-ball system: an elementary approach.* Journal of Integrable Systems. Volume 3, Issue 1, 2018, xyy002

[KS] A.N. Kirillov, and R. Sakamoto : *Relationships Between Two Approaches: Rigged Configurations and 10-Eliminations.* Lett Math Phys 89, 51-65 (2009)

[KOSTY] A. Kuniba, M. Okado, R. Sakamoto, T. Takagi and Y. Yamada : *Crystal interpretation of Kerov-Kirillov-Reshetikhin bijection.* Nuclear Physics B, **740**, 299–327 (2006)

[KOY] A. Kuniba, M. Okado and Y. Yamada : *Box–ball system with reflecting end.* J. Nonlin. Math. Phys. **12** 475–507 (2005)

[KTT] A. Kuniba, T. Takagi and A. Takenouchi : *Bethe ansatz and inverse scattering transform in a periodic box-ball system.* Nucl. Phys. B 747, 354–397 (2006)

[LLP] L. Levine, H. Lyu and J. Pike : *Double Jump Phase Transition in a Soliton Cellular Automaton.* Int Math Res Notices volume 2022, Issue 1, 665–727 (2020)

[LLPS] J. Lewis, H. Lyu, P. Pylyavskyy, and A. Sen : *Scaling limit of soliton lengths in a multicolor box-ball system.* arXiv:1911.04458

[MIT] Jun Mada, Makoto Idzumi and Tetsuji Tokihiro : *On the initial value problem of a periodic box-ball system.* J. Phys. A: Math. Gen. 39 (2006)

[MSSS] M. Mucciconi, M. Sasada, T. Sasamoto and H. Suda : *Relationships between two linearizations of the box-ball system : Kerov-Kirillov-Reschetikhin bijection and slot configuration.* Forum of Mathematics, Sigma. 2024;12:e55.

[OSS] S. Olla, M. Sasada and H. Suda : *Scaling limits of solitons in the box-ball system.* in preparation.

[T] T. Takagi : *Inverse scattering method for a soliton cellular automaton.* Nuclear Physics **B707**, 577–601.

[TM] D. Takahashi and J. Matsukidaira : *Box and ball system with a carrier and ultra-discrete modified KdV equation.* J. Phys. A **30** L733–L739 (1997).

[TS] D. Takahashi and J. Satsuma : *A soliton celler automaton.* J. Phys. Soc. Japan **59** 3514–3519 (1990)