SLICES OF STABLE POLYNOMIALS AND CONNECTIONS TO THE GRACE-WALSH-SZEGő THEOREM

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Abstract. Univariate polynomials are called stable with respect to a circular region $\mathcal{A}$, if all of their roots are in $\mathcal{A}$. We consider the special case where $\mathcal{A}$ is a half-plane and investigate affine slices of the set of stable polynomials. In this setup, we show that an affine slice of codimension k always contains a stable polynomial that possesses at most $2(k+2)$ distinct roots on the boundary and at most $(k+2)$ distinct roots in the interior of $\mathcal{A}$. This result also extends to affine slices of weakly Hurwitz polynomials. Subsequently, we apply these results to symmetric polynomials and varieties. Here we show that it is necessary and sufficient for a variety described by polynomials in few multiaffine polynomials to contain points in $\mathcal{A}^n$ with few distinct coordinates for its intersection with $\mathcal{A}^n$ being non-empty. This is at the same time a generalization of the degree principle to stable polynomials and a result similar to Grace-Walsh-Szegő's coincidence theorem on multiaffine symmetric polynomials.

1. Introduction

The study of univariate polynomials whose roots are restricted to a subset of $\mathbb{C}$ is a central topic in mathematics. For instance, a univariate real polynomial is called *hyperbolic* if it is real rooted. Recall that a *circular region* $\mathcal{A}$ is a subset of the complex plane that is bounded by either a circle or a line, and is either open or closed. A univariate complex polynomial is said to be $\mathcal{A}$-*stable* if all its roots lie in $\mathcal{A}$. Here, we consider the case where $\mathcal{A}$ is a half-plane. Since the roots of real polynomials come in conjugated pairs, hyperbolic polynomials are thus exactly real stable polynomials relative to the upper half-plane. Well-known examples of stable polynomials are *Hurwitz stable* polynomials, which are real open left half-plane stable polynomials, and *Schur stable* polynomials, which are unit disk stable polynomials. In particular, stable polynomials have been extensively leveraged to gain insights into combinatorial objects (see e.g. [4, 6, 8, 11]), and Hurwitz polynomials are at the heart of control theory and are used for asymptotic stability for linear continuous-time systems (see e.g., [17] or [7, p. 75]).

Studying the roots of univariate polynomials is deeply related to studying multivariate symmetric polynomials by *Vieta's formula*

$$\prod_{i=1}^{n}(T-x_i) = T^n - e_1(x)T^{n-1} + e_2(x)T^{n-2} + \ldots + (-1)^n e_n(x)$$

where $e_k = \sum_{1\le i_1<\ldots<i_k\le n} X_{i_1}\cdots X_{i_k}$ denotes the k-th *elementary symmetric polynomial.* In the paper we associate points $z \in \mathbb{C}^n$ with monic polynomials

$$f_z = T^n - z_1T^{n-1} + z_2T^{n-2} - \ldots + (-1)^n z_n.$$

In particular, monic hyperbolic polynomials are described by the image of $\mathbb{R}^n$ under the *Vieta map*, i.e., the image under the evaluation of the n elementary symmetric polynomials. Similarly to this hyperbolic picture, monic $\mathcal{A}$-stable polynomials can be identified with the image of $\mathcal{A}^n$ under the Vieta map.

Sets of hyperbolic polynomials obtained by fixing the first k coefficients have been considered by various authors, beginning with the work of Arnold [2, 9, 14, 18] and recently [16, 20]. In the domain of the Vieta map, such sets are called *Vandermonde varieties*, whereas the corresponding sets in the image of the Vieta map are called *hyperbolic slices.* More generally, this notion has been introduced in [21] to sets of hyperbolic polynomials

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that are cut out by a $(n-k)$-dimensional affine subspace. A remarkable property of such hyperbolic slices concerns their local extreme points: It turns out that these local extreme points of linear functionals can be characterized as polynomials with at most k distinct roots. Similarly to this hyperbolic situation, we study affine slices of the set of upper half-plane stable polynomials defined by k linear combinations of coefficients and show in Theorem 2.4 that the local extreme points of such *stable slices* have at most k non-real roots and at most $2k$ distinct real roots.
One of our main motivations for this result is provided by a natural connection to the classical *Grace-Walsh-Szegő's coincidence theorem*. This beautiful result states that for a symmetric multiaffine polynomial $f \in \mathbb{C}[X_1,\dots,X_n]$ evaluated on a circular region $\mathcal{A} \subset \mathbb{C}$ there exists for all $(\zeta_1,\dots,\zeta_n) \in \mathcal{A}^n$ some $\zeta \in \mathcal{A}$ with the property that $f(\zeta_1,\dots,\zeta_n) = f(\zeta,\dots,\zeta)$, under the assumption that the degree of f is n or $\mathcal{A}$ is convex. The coincidence theorem has several applications in stability testing since it allows reduction of the question of verifying multivariate stability to univariate polynomials. However, the assumptions of the theorem are relatively strict. It was proven by Brändén and Wagner [5] that no analogous result can be applied to any multiaffine polynomials invariant under a fixed proper permutation subgroup of S_n. We use our results on stable slices and the connection with symmetric polynomials to prove in Theorem 4.6 and Corollary 4.11 a similar result to Grace-Walsh-Szegő's theorem for multivariate polynomials for functions which can be expressed as polynomials in few multiaffine symmetric polynomials when $\mathcal{A}$ is a half-plane. We show that for any point $\zeta \in \mathcal{A}^n$, we can find a point with few distinct coordinates and the same evaluation. Furthermore, in a similar spirit, we prove a *double-degree principle* for stable varieties in Corollary 4.8 and also a *half-degree principle* for the upper half-plane in Theorem 4.13. Our results on stable slices do not transfer directly to *Hurwitz slices* since the coefficients of those polynomials are real. However, we prove that if we fix k linear combinations of coefficients of a weakly Hurwitz polynomial, then there is a weakly Hurwitz polynomial satisfying the same relations and having only k roots with negative real part and $2k$ distinct roots with real part equal to zero (see Theorem 3.3).

Structure of the article. In Section 2 we study stable slices of univariate polynomials and show in particular that local extreme points of stable slices correspond to polynomials with few distinct roots (Theorem 2.4). In Section 3 we study Hurwitz slices and their boundary by root multiplicities. In Section 4 we apply our results from Section 2 to multivariate symmetric polynomials and formulate a double-degree principle for stable polynomials and our result similar to Grace-Walsh-Szegő's coincidence theorem (Theorem 4.6, Corollaries 4.8 and 4.11). Moreover, we briefly discuss the generalization of Grace-Walsh-Szegő's coincidence theorem to permutation subgroups of S_n as considered in [5]. Finally, we formulate open questions.

2. Stable slices

Throughout the article we denote by $\mathbb{C}[T]$ and $\mathbb{R}[T]$ the rings of univariate complex and real polynomials and $k \le n$ be fixed positive integers. For a complex number x we write $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ for its real and imaginary parts. Furthermore, we commonly identify the set of monic univariate polynomials with $\mathbb{C}^n$ via the bijection

$$(z_1,\dots,z_n) \longmapsto T^n - z_1T^{n-1} + z_2T^{n-2} - \dots + (-1)^n z_n.$$

In this section, we study univariate *stable polynomials*, i.e. polynomials that have all their roots lying in a half-plane. In particular, we are interested in intersections of the set of stable polynomials with affine subspaces of $\mathbb{C}^n$. As multiplication with units in $\mathbb{C}$ does not change the roots of a polynomial, we restrict to monic stable polynomials. We denote the closed upper half-plane by $\mathbb{H}_+$, i.e.

$$\mathbb{H}_+ = \{x \in \mathbb{C} \mid \mathrm{Im}(x) \ge 0\}.$$

Definition 2.1. Let $\mathbb{H}$ be a closed half-plane.

(i) We denote by

$$\mathcal{S}_{\mathbb{H}} := \left\{ z \in \mathbb{C}^n \;\middle|\; T^n - z_1T^{n-1} + \dots + (-1)^n z_n \text{ has all roots in } \mathbb{H} \right\}$$

the set of monic $\mathbb{H}$-*stable polynomials* of degree n. If $\mathbb{H} = \mathbb{H}_+$ is the upper half-plane, we write $\mathcal{S}$ for $\mathcal{S}_{\mathbb{H}_+}$.

(ii) We denote the set of points with at most k distinct coordinates on the boundary of $\mathbb{H}$ and at most m coordinates in the interior of $\mathbb{H}$ by

$$\mathbb{H}_{k,m} := \{x \in \mathbb{H}^n \mid |\{x_1 \dots, x_n\} \cap \operatorname{bd} \mathbb{H}| \le k \text{ and } |\{i \in \{1, \dots, n\} \mid x_i \in \operatorname{int} \mathbb{H}\}| \le m\}.$$

Furthermore, the set of all polynomials in $\mathcal{S}_{\mathbb{H}}$ with all roots in $\mathbb{H}_{k,m}$ is denoted by

$$\mathcal{S}_{\mathbb{H}}^{k,m} := \left\{z \in \mathcal{S}_{\mathbb{H}} \;\middle|\; T^n - z_1 T^{n-1} + \dots + (-1)^n z_n \text{ has roots } (x_1, \dots, x_n) \in \mathbb{H}_{k,m}\right\}.$$

If $\mathbb{H} = \mathbb{H}_+$ is the upper half-plane, we write $\mathcal{S}^{k,m}$ for $\mathcal{S}_{\mathbb{H}_+}^{k,m}$.

(iii) For $a = (a_1, \dots, a_k) \in \mathbb{C}^k$ and a surjective linear map $L : \mathbb{C}^n \to \mathbb{C}^k$ we define the *affine slice*

$$\mathcal{S}_{\mathbb{H}} \cap L^{-1}(a) = \{z \in \mathcal{S}_{\mathbb{H}} \mid L(z) = a\}.$$

A set of the form $\mathcal{S}_{\mathbb{H}} \cap L^{-1}(a)$ is called a $\mathbb{H}$-*stable slice*.

Remark 2.2. Observe that the set $\mathcal{S}_{\mathbb{H}}$ can be identified with a semi-algebraic set in $\mathbb{R}^{2n}$. In contrast to the set of hyperbolic polynomials, where an explicit description of the set of hyperbolic polynomials in terms of the coefficients can be obtained via Sturm's Theorem, it seems in general complicated to give an explicit description of $\mathcal{S}_{\mathbb{H}}$. However, in the case of polynomials with real coefficients, this is possible and we will present this case in Section 3.

Our assumption that the linear map $L : \mathbb{C}^n \to \mathbb{C}^k$ is surjective in Definition 2.1 (iii) is only for convenience (see Remark 2.5). Moreover, it suffices to study stable slices of a fixed half-plane. This follows since translations and rotations are linear isomorphisms. Let $\phi : \mathbb{H} \to \mathbb{G}$ be a linear bijection between half-planes and let $\psi = \phi^{-1}$ be its inverse. Then $f_z \in \mathcal{S}_{\mathbb{H}}$ if and only if $f_z \circ \psi \in \mathcal{S}_{\mathbb{G}}$. In particular, we can restrict to $\mathbb{H}_+$-stable slices.

Definition 2.3. Let $A \subset \mathbb{C}^n$ and let $z \in A$. We say that z is a *local extreme point* of A if there is a neighborhood U of z such that z is an extreme point of $\operatorname{conv}(A \cap U)$.

While the set of extreme points of a set A is the set of global minima of linear functions on A, the set of local extreme points of A is the set of local minima of linear functions.
The following theorem which is a generalization of [20, Theorem 4.2] and [21, Theorem 2.8], is our main result on stable slices characterizing local extreme points. As a corollary, we obtain a result for arbitrary stable slices in Corollary 2.10.

Theorem 2.4. The local extreme points of an $\mathbb{H}_+$-stable slice $\mathcal{S} \cap L^{-1}(a)$ correspond to polynomials that have at most k roots in $\mathbb{H}_+ \setminus \mathbb{R}$ and at most $2k$ distinct real roots.

In other words, any local extreme point of the $\mathbb{H}_+$–stable slice $\mathcal{S} \cap L^{-1}(a)$ is contained in the set $\mathcal{S}^{2k,k}$. In the proof, we investigate the multiplicity of the roots of polynomials in the stable slice.

Proof. Let $z \in \mathcal{S} \cap L^{-1}(a)$ be a local extreme point, i.e., there is a neighborhood U of z such that z is an extreme point of $\operatorname{conv}(\mathcal{S} \cap L^{-1}(a) \cap U)$. Consider $f := T^n - z_1 T^{n-1} + \dots + (-1)^n z_n$ and factor $f = p \cdot r$, where p has only roots in $\mathbb{H}_+ \setminus \mathbb{R}$ and r has only real roots.

(1) We show first that p has at most k roots, i.e., $\deg p \le k$. We assume that $\deg p := m > k$ and want to find a contradiction. Write $r = T^{n-m} + r_1 T^{n-m-1} + \dots + r_{n-m}$ and define $r_0 := 1$ and consider the linear map

$$\begin{array}{rccc} \chi : & \mathbb{C}^m & \longrightarrow & \mathbb{C}^n \\ & y & \longmapsto & \left(\sum_{i+j=1} r_i y_j, \dots, \sum_{i+j=n} r_i y_j\right) \end{array},$$

where in each sum $0 \le i \le n - m$ and $1 \le j \le m$. Since $m > k$, there is $b \in \ker(L \circ \chi) \setminus \{0\}$. We define $h := b_1 T^{m-1} + \dots + b_m$ and $g := h \cdot r = c_1 T^{n-1} + \ldots + c_n \neq 0$, where $c = \chi(b)$ by construction and therefore $c \in \ker L$. Now, because p has only roots in $\mathbb{H}_+ \setminus \mathbb{R}$, $p \pm \varepsilon h$ is stable for $\varepsilon > 0$ small enough, since the roots depend continuously on the coefficients [10]. Hence

$$(p \pm \varepsilon h) \cdot r = f \pm \varepsilon h \cdot r = f \pm \varepsilon g$$

is stable for all $\varepsilon > 0$ small enough, i.e., $z \pm \varepsilon c \in \mathcal{S} \cap L^{-1}(a)$. If we choose $\varepsilon > 0$ small enough we can ensure also that $z \pm \varepsilon c \in U$. But then

$$z = \frac{z + \varepsilon c + z - \varepsilon c}{2},$$

a contradiction to z being an extreme point of $\operatorname{conv}(\mathcal{S} \cap L^{-1}(a) \cap U)$.

(2) Now we show that r has at most $2k$ distinct roots. We assume r has distinct roots $x_1, \ldots, x_m$ where $m > 2k$ and want to find a contradiction. We factor f as follows:

$$f = \underbrace{\prod_{i=1}^{m} (T - x_i)}_{=:q} \cdot s,$$

where s is of degree $n - m$. Write $s = T^{n-m} + s_1 T^{n-m-1} + \cdots + s_{n-m}$ and define $s_0 := 1$ and consider the linear map

$$\begin{array}{cccc} \chi : & \mathbb{R}^m & \longrightarrow & \mathbb{C}^n \\ & y & \longmapsto & \left(\sum_{i+j=1} s_i y_j, \ldots, \sum_{i+j=n} s_i y_j \right) \end{array}.$$

Since $m > 2k$, there is $b \in \ker(L \circ \chi) \setminus \{0\}$. We define $h := b_1 T^{m-1} + \cdots + b_m$ and $g := h \cdot s = c_1 T^{n-1} + \ldots + c_n \neq 0$, where $c = \chi(b)$ by construction and therefore $c \in \ker L$. Now, because q has only simple roots in $\mathbb{R}$, $q \pm \varepsilon h$ is hyperbolic and therefore stable for $\varepsilon > 0$ small enough, since the roots depend continuously on the coefficients and complex roots come as conjugated pairs (see e.g. [10]). Hence

$$(q \pm \varepsilon h) \cdot s = f \pm \varepsilon \cdot g$$

is stable for all $\varepsilon > 0$ small enough, i.e., $z \pm \varepsilon c \in \mathcal{S} \cap L^{-1}(a)$. If we choose $\varepsilon > 0$ small enough we can ensure also that $z \pm \varepsilon c \in U$. But then

$$z = \frac{z + \varepsilon c + z - \varepsilon c}{2},$$

a contradiction to z being an extreme point of $\operatorname{conv}(\mathcal{S} \cap L^{-1}(a) \cap U)$.

□

Remark 2.5. The assumption that L is surjective is only for convenience. In particular, if L is not surjective one obtains the same result as in Theorem 2.4, where k can be replaced by $\operatorname{rank} L$.

We point out that the converse of Theorem 2.4 is not true, i.e. not every point $z \in \mathcal{S}^{2k,k} \cap L^{-1}(a)$ is a local extreme point.

Example 2.6. Let $n = 3$, $k = 1$ and

$$\begin{array}{cccc} L : & \mathbb{C}^3 & \longrightarrow & \mathbb{C} \\ & (z_1, z_2, z_3) & \longmapsto & z_3 \end{array}.$$

Then $(i, 0, 0) \in \mathcal{S}^{2k,k} \cap L^{-1}(0)$, but

$$(i, 0, 0) = \frac{((1 - \varepsilon)i, 0, 0) + ((1 + \varepsilon)i, 0, 0)}{2}$$

is not a local extreme point since $((1-\varepsilon)i, 0, 0), ((1+\varepsilon)i, 0, 0) \in \mathcal{S} \cap L^{-1}(0)$ for all $\varepsilon \in [0, 1)$.

We consider the set of stable polynomials of degree n with fixed first coefficients which is an instance of a stable slice.

Definition 2.7. For an integer $k \geq 1$ and a point $a = (a_1, \ldots, a_k) \in \mathbb{C}^k$ we define $\mathcal{S}(a) = \mathcal{S} \cap \{z \in \mathbb{C}^n | z_1 = a_1, \ldots, z_k = a_k\}$ as the set of all monic $\mathbb{H}_+$-stable polynomials of degree n whose first k non-trivial coefficients are determined by the point a.

With our previous notation we have $\mathcal{S}(a) = S \cap L^{-1}(a)$ where $L : \mathbb{C}^n \to \mathbb{C}^k$ denotes the projection to the first k coordinates.

Lemma 2.8. For an integer $k \geq 2$ the stable slice $\mathcal{S}(a)$ is compact.

Proof. As the empty set is compact we can assume that there is $z \in \mathcal{S}(a)$. Furthermore we denote by $x = (x_1, \dots, x_n) \in \mathbb{H}_+$ the roots of the polynomial

$$f_z := T^n - z_1 T^{n-1} + \dots + (-1)^n z_n.$$

Then, if e_1 and e_2 denote the first and second elementary symmetric polynomial

$$\sum_{i=1}^n x_i = e_1(x) = a_1$$

and hence the imaginary part of the $x_i's$ is contained in $[0, \mathrm{Im}(a_1)]$. Furthermore

$$\sum_{i=1}^n x_i^2 = e_1(x)^2 - 2e_2(x) = a_1^2 - 2a_2$$

and hence

$$\sum_{i=1}^n \mathrm{Re}(x_i)^2 = \sum_{i=1}^n \mathrm{Re}(x_i^2) + \mathrm{Im}(x_i)^2 \le \sum_{i=1}^n \mathrm{Re}(x_i^2) + \mathrm{Im}(a_1)^2 = \mathrm{Re}\left(\sum_{i=1}^n x_i^2\right) + n\,\mathrm{Im}(a_1)^2 \ .$$

Since $\sum_{i=1}^n x_i^2 = a_1^2 - 2a_2$ we have

$$\sum_{i=1}^n \mathrm{Re}(x_i)^2 \le \mathrm{Re}(a_1^2 - 2a_2) + n\,\mathrm{Im}(a_1)^2 \ .$$

This shows that also the real part of the x_i's is bounded. Thus the set $\mathcal{S}(a)$ is bounded. Furthermore, as the roots of a polynomial depend continuously on the coefficients it is clear that $\mathcal{S}(a)$ is closed and therefore compact. □

Remark 2.9. For a surjective linear map $L: \mathbb{C}^n \to \mathbb{C}^k$ and a point $a \in \mathbb{C}^k$ the set $\mathcal{S} \cap L^{-1}(a)$ can be unbounded. Then we consider the linear map $\widetilde{L}: \mathbb{C}^n \to \mathbb{C}^{k+2}$, where $\widetilde{L}(z) = (L(z), z_1, z_2)$. The set $\mathcal{S} \cap \widetilde{L}^{-1}(b)$ is compact for any point $b \in \mathbb{C}^{k+2}$, by a similar argument as in the proof of Lemma 2.8. Moreover, if one or both of the first two unit vectors are in the row span of a matrix representation of L, then we can consider $\widehat{L}(z) = (L(z), z_i)$ for $i \in \{1, 2\}$ instead of L or the original stable slice was already compact.

We are now ready to present our main result on general half-plane stable slices.

Corollary 2.10. Let $\mathbb{H}$ be a closed half-plane. Any non-empty $\mathbb{H}$-stable slice $\mathcal{S}_{\mathbb{H}} \cap L^{-1}(a) \neq \emptyset$ contains a point that corresponds to a polynomial with at most $k+2$ roots in the interior of $\mathbb{H}$ and at most $2(k+2)$ distinct roots in the boundary of $\mathbb{H}$, i.e.

$$\mathcal{S}_{\mathbb{H}}^{2(k+2),k+2} \cap L^{-1}(a) \neq \emptyset.$$

Proof. Since $\mathbb{H}$ can be bijectively mapped to $\mathbb{H}_+$ under a linear isomorphism it suffices to show the theorem for $\mathbb{H} = \mathbb{H}_+$. Now the claim follows from Theorem 2.4, Lemma 2.8 and Remark 2.9. □

Corollary 2.10 says that stable slices do always contain a point with few distinct zeros. Moreover, we can characterize the maximal number of distinct roots on the boundary of the half-plane and the number of distinct roots in the interior. Note that the result is independent of the degree n of the univariate polynomials. Thus it is of more interest if n is large. In particular, we observe a stabilization in the structure of local extreme points of stable slices if the number of variables is at least $3k$.

Remark 2.11. In the case that L is the projection to the first $k < n$ coordinates, we can replace $2k$ by k in Theorem 2.4. This is, since $(0, \dots, 0, 1) \in \ker(L \circ \chi)$ and we can choose $h := 1$ in the proof in this case. Moreover, if $k \ge 2$ the considered stable slice is compact in this case by Lemma 2.8. So we can replace $\mathcal{S}_{\mathbb{H}}^{2(k+2),k+2} \cap L^{-1}(a)$ by $\mathcal{S}_{\mathbb{H}}^{k,k} \cap L^{-1}(a)$ in Corollary 2.10.

One could hope that every stable slice contains also points that correspond to polynomials with k distinct roots in $\mathbb{H}_+$, analogous to the case of compact hyperbolic slices, mentioned in [21, Theorem 2.8]. The next example shows that this is not true in general even when L is the projection to the first k coordinates.

Example 2.12. We consider $\mathcal{S} \cap L^{-1}(a)$, where

$$a := (-23i, -463, 8461i) \quad \text{and} \quad \begin{array}{rccc} L: & \mathbb{C}^4 & \longrightarrow & \mathbb{C}^3 \\ & (z_1, z_2, z_3, z_4) & \longmapsto & (z_1, z_2, z_3) \end{array}$$

is the projection to the first 3 coordinates. Then $\mathcal{S} \cap L^{-1}(a)$ is non-empty, since

$$(-23i, -463, 8461i, 8020) \in \mathcal{S} \cap L^{-1}(a).$$

The coefficient vector corresponds to a polynomial with roots $-20 + i, i, 20 + i$ and $20i$. Furthermore, $\mathcal{S} \cap L^{-1}(a)$ contains no point corresponding to a polynomial with at most 3 distinct roots.

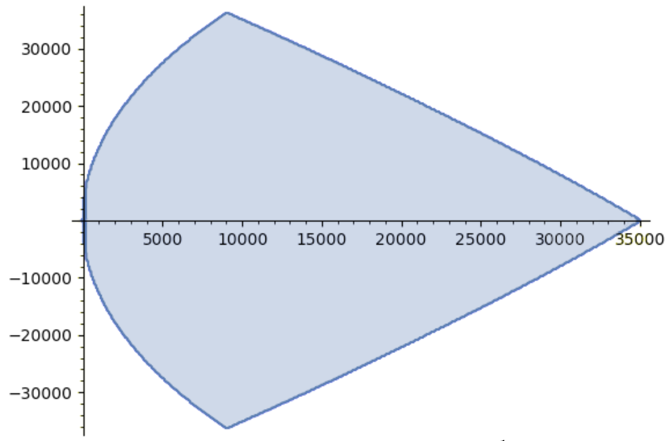


FIGURE 1. The stable slice $\mathcal{S} \cap L^{-1}(a)$

3. Hurwitz slices

In this section we consider *Hurwitz polynomials*, i.e. real univariate polynomials with all roots in the left half-plane. Moreover, polynomials with all roots having nonpositive real part are called *weakly Hurwitz*. We show in Theorem 3.3 that the local extreme points of affine slices of the set of monic Hurwitz polynomials have few distinct roots and study a partial order on the set of monic Hurwitz polynomials in Subsection 3.2.

Like for stable polynomials we identify monic weakly Hurwitz polynomials with their coefficients. Any monic weakly Hurwitz polynomial has nonnegative coefficients.

Similarly to hyperbolic polynomials, monic Hurwitz polynomials can be characterized as poynomials with a positive definite *finite Hurwitz matrix* [12] (see also [22, Section 9.3]). While the finite Hurwitz matrix of any weakly Hurwitz polynomial is positive semidefinite, its converse is not true [3]. Kemperman [13] showed that weakly Hurwitz polynomials can be characterized in a similar way by their *infinite Hurwitz matrix* (see also [1, Thm. 4.9] for another characterization).

3.1. Hurwitz slices and their local extreme points.

In contrast to the study of stable polynomials in Section 2 where we considered surjective linear maps $\mathbb{C}^n \to \mathbb{C}^k$ over the field $\mathbb{C}$, we restrict to real linear maps over $\mathbb{R}$. However, since the roots of weakly Hurwitz polynomials can be complex, we cannot directly apply any result about hyperbolic polynomials.

Definition 3.1. We write $\mathbb{H}_{\text{left}}$ for the left half-plane in $\mathbb{C}$, i.e.

$$\mathbb{H}_{\text{left}} := \{x \in \mathbb{C}^n \mid \operatorname{Re}(x) \leq 0\}$$

The set of monic *weakly Hurwitz* polynomials is defined by

$$\mathcal{HW} := \mathcal{S}_{\mathbb{H}_{\text{left}}} \cap \mathbb{R}^n := \{z \in \mathbb{R}^n \mid f_z \text{ has all roots in } \mathbb{H}_{\text{left}}\}.$$

Moreover, for a linear map $L: \mathbb{R}^n \to \mathbb{R}^k$ we call the set $\mathcal{HW} \cap L^{-1}(a)$ a *Hurwitz slice*.

We have the following connection between Hurwitz polynomials and stable polynomials.

Remark 3.2. The set of monic weakly Hurwitz polynomials $\mathcal{HW}$ can be embedded in $\mathcal{S}$ in the following way: If $f(T) \in \mathcal{HW}$ is Hurwitz then the monic polynomial

$$\tilde{f}(T) = (-i)^n \cdot f(i \cdot T) = T^n + \sum_{k=1}^{n} i^k z_k T^{n-k}$$

is upper half-plane stable with coefficients alternating from the sets $\mathbb{R}$ or $i \cdot \mathbb{R}$. The map $\tilde{} : \mathcal{HW} \to \mathcal{S}$ is linear, injective, not surjective, and its inverse is $g(T) \mapsto i^n g(-i \cdot T)$.

For instance, the polynomial

$$f = (T+2)(T+1+i)(T+1-i) = T^3 + 4T^2 + 6T + 4$$

is Hurwitz and

$$\tilde{f} = (-i)^3 f(iT) = T^3 - 4iT^2 - 6T + 4i$$

is $\mathbb{H}_+$-stable with alternating real and purely complex coefficients.
We get the same results about multiplicities of the roots of local extreme points of Hurwitz slices as for stable slices in Theorem 2.4.

Theorem 3.3. Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be a surjective linear map. The local extreme points of a Hurwitz slice $\mathcal{HW} \cap L^{-1}(a)$ correspond to polynomials that have at most k roots with negative real part and at most $2k$ distinct roots with real part equal to zero.

The result is the same as in Theorem 2.4 and the proof follows the same strategy.
The proof of the theorem is similarly to the proof of Theorem 2.4, but one has to be a bit careful because we are dealing with real coefficients only.

Proof. Let $z \in \mathcal{HW} \cap L^{-1}(a)$ be a local extreme point, i.e., there is a neighborhood U of z such that z is an extreme point of $\operatorname{conv}(\mathcal{HW} \cap L^{-1}(a) \cap U)$. Consider $f := f_z = T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n$ and factor $f = p \cdot r$, where p has only roots with negative real part and r has only roots with real part equal to zero. Note that since f has real coefficients, the roots of f come in complex conjugated pairs, so p and r have also real coefficients.

(1) In order to show that p has at most k roots, we assume that $\deg p := m > k$ and want to derive a contradiction. Write $r = T^{n-m} + r_1 T^{n-m-1} + \cdots + r_{n-m}$, define $r_0 := 1$ and consider the linear map

$$\begin{array}{rccc} \chi : & \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\ & y & \longmapsto & \left(\sum_{i+j=1} r_i y_j, \ldots, \sum_{i+j=n} r_i y_j\right) \end{array},$$

where in each sum $0 \le i \le n-m$ and $1 \le j \le m$. Similarly to part (1) in the proof of Theorem 2.4 one verifies, by considering some $0 \ne b \in \ker(L \circ \chi)$, that z cannot be a local extreme point of $\mathcal{HW} \cap L^{-1}(a)$.

(2) Now we show that r has at most $2k$ distinct roots. We assume that all the distinct roots of r are $x_1, \ldots, x_m$ where $m > 2k$ and we want to find a contradiction. We factor f as follows:

$$f = \underbrace{\prod_{i=1}^{m} (T - x_i)}_{=:q} \cdot s,$$

where s is of degree $n-m$. Note that f and therefore q and s have real coefficients. Write $s = T^{n-m} + s_1 T^{n-m-1} + \cdots + s_{n-m}$ and define $s_0 := 1$ and consider the linear map

$$\begin{array}{rccc} \chi : & \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\ & y & \longmapsto & \left(\sum_{i+j=1} s_i y_j, \ldots, \sum_{i+j=n} s_i y_j\right) \end{array}.$$

Since $m > 2k$, there is $b \in \ker(L \circ \chi) \setminus \{0\}$ with $b_{2i-1} = 0$ for all $i \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor\}$. We define $h := b_1 T^{m-1} + \cdots + b_m$ and $g := h \cdot s = c_1 T^{n-1} + \ldots + c_n \ne 0$, where $c = \chi(b)$ by construction and therefore $c \in \ker L$. Note that q corresponds to a hyperbolic polynomial $\tilde{q}$ via the embedding stated in Remark 3.2 where the degree is m instead of n. The same transformation maps h to a hyperbolic polynomial $\tilde{h}$. Now, because $\tilde{q}$ has only distinct roots, $\tilde{q} \pm \varepsilon \tilde{h}$ is hyperbolic for $\varepsilon > 0$ small enough since the roots depend continuously on the coefficients and complex roots come as conjugated pairs (see e.g. [10]). Moreover, we have $\tilde{q} \pm \varepsilon\tilde{h} = T^m + w_2T^{m-2} + w_4T^{m-4} + \ldots$ for some real numbers w_{2i}. Thus, $\tilde{q} \pm \varepsilon\tilde{h}$ lies in the image of the map $\tilde{}$ and we can apply the inverse of the transformation $\tilde{}$ from Remark 3.2 which is also linear. So $q \pm \varepsilon h$ is weakly Hurwitz for $\varepsilon > 0$ small enough. Hence

$$(q \pm \varepsilon h) \cdot s = f \pm \varepsilon \cdot g$$

is Hurwitz for all $\varepsilon > 0$ small enough, i.e., $z \pm \varepsilon c \in \mathcal{HW} \cap L^{-1}(a)$. If we choose $\varepsilon > 0$ small enough we can ensure also that $z \pm \varepsilon c \in U$. But then

$$z = \frac{z + \varepsilon c + z - \varepsilon c}{2},$$

a contradiction to z being an extreme point of $\operatorname{conv}(\mathcal{HW} \cap L^{-1}(a) \cap U)$.

□

In the case where L is the projection to the first k coordinates, one can again replace $2k$ by k in the proof of Theorem 3.3. Furthermore, since closed subsets of compact sets are compact, we get from Lemma 2.8 and Remark 3.2 also that $\mathcal{HW} \cap L^{-1}(a)$ is compact if L is the projection to the first k coordinates. More generally, Remark 2.9 translates in the same way.

3.2. **Geometry and combinatorics of Hurwitz slices.** In this subsection, we briefly discuss the interplay of the geometry and combinatorics of the set of weakly Hurwitz polynomials and Hurwitz slices. This is inspired by the rich geometry and combinatorics of linear slices of the set of monic univariate hyperbolic polynomials and should be seen as a starting point for further investigations.

The boundary of the set $\mathcal{HW}$ consists of polynomials of the form $f = p \cdot q$ where $p, q \in \mathbb{R}[T]$ are monic, $\deg(p) + \deg(q) = n$, p is Hurwitz of even degree $r < n$ and for any root z of q, we have $\operatorname{Re}(z) = 0$. In a neighborhood of p, one can perturb all coefficients but the leading coefficient of p and the obtained polynomial is again a monic Hurwitz polynomial. All imaginary roots $\pm ib_1, \ldots, \pm ib_r$ of q come in complex conjugated pairs. We assume $0 \leq |b_1| < \ldots < |b_r|$ and we have

$$q = T^s \prod_{i=1}^{l} ((T - ib_i)(T + ib_i))^{\mu_i} = T^s \prod_{i=1}^{l} (T^2 + b_i^2)^{\mu_i}$$

with $s \in \{0, 1\}$. Note that s is uniquely determined by the degree of q. We have $s = 0$ if $\deg q$ is even and $s = 1$ otherwise. The real polynomial

$$q_e = T^s \prod_{i=1}^{l} (T + b_i^2)^{\mu_i}$$

has only real roots $0 \geq -b_1^2 > \ldots > -b_l^2$ with multiplicities $s, \mu_1, \ldots, \mu_l$. For a monic polynomial $f \in \mathbb{R}[T]$ whose roots are all of the form ib with $b \in \mathbb{R}$, we call f_e its associated *even polynomial.* We have a $1:1$ correspondence between monic polynomials $q \in \mathbb{R}[T]$ for which all of its roots have real part 0 and hyperbolic polynomials with only nonpositive roots. A *composition of* n is a sequence of positive integers summing up to n. Let $\mu = (\mu_1, \ldots, \mu_l)$ be the composition of n given by the multiplicities of the ordered roots of q_e. We call the tuple (s, μ) the *root multiplicity* of the even polynomial q_e.

For a weakly Hurwitz polynomial $f = p \cdot q$ we call the triple (r, s, μ) the *multiplicity* of f which we denote by $\operatorname{mult}(f)$. For instance, we have $\operatorname{mult}(T^5) = (0, 1, (2))$ and $(T+1)(T-i)(T+i)(T-2i)(T+2i)$ has multiplicity $(1, 0, (1, 1))$. Moreover, we define the set

$$\mathcal{HW}_{(r,s,\mu)} = \{f \in \mathcal{HW} | \operatorname{mult}(f) = (r, s, \mu)\}$$

of monic Hurwitz polynomials of degree n with r roots in the interior and the roots on the boundary are encoded by the root multiplicity (s, μ). For different multiplicity triples, the associated sets are disjoint. Note that $\mathcal{HW}_{(r,s,\mu)} \neq \varnothing$ if and only if $r + s + 2\sum_{i=1}^{l} \mu_i = n$, since one can find for any multiplicity (r, s, μ) a monic Hurwitz polynomial with $\operatorname{mult}(f) = (r, s, \mu)$. From the definition of $\mathcal{HW}_{(r,s,\mu)}$ we can say which sets $\mathcal{HW}_{(r',s',\mu')}$ are contained in $\operatorname{cl} \mathcal{HW}_{(r,s,\mu)}$. To do so, we define a partial order.

Definition 3.4. Let $\mathcal{C}$ be the set of all triples (r, s, μ) where $r \leq n$ is a positive integer and, if $n-r$ is even then $s = 0$ and $\mu = (\mu_1, \dots, \mu_l)$ is a composition of $\frac{n-r}{2}$, and otherwise $s = 1$ and $\mu = (\mu_1, \dots, \mu_l)$ is a composition of $\frac{n-r-1}{2}$. We define the partial order $\trianglelefteq$ on $\mathcal{C}$ as the transitive and reflexive closure of the following relations. We say $(r, s, \mu) \trianglelefteq (r, s, \lambda)$ if μ can be obtained from λ by replacing some of the commas in the composition λ by the plus operation. We define $(r-1, 1, \mu) \trianglelefteq (r, 0, \mu)$ and $(r-1, 0, (1, \mu_1, \dots, \mu_l)) \trianglelefteq (r, 1, \mu)$.

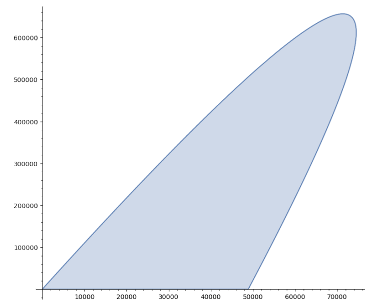
For instance, we have

$$(3,0,(2)) \trianglelefteq (3,0,(1,1)), (2,1,(1,1)) \trianglelefteq (3,0,(1,1)) \text{ and } (2,0,(1,1,1)) \trianglelefteq (3,1,(1,1)).$$

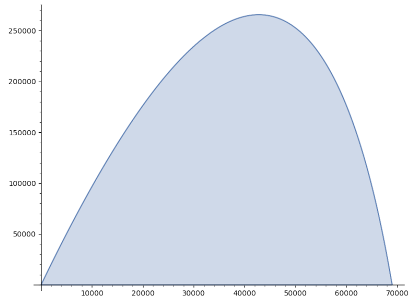
If μ' is a composition that can be obtained from μ by replacing some of the commas in μ plus signs, this means that we can continuously collapse a conjugated pair of roots of a polynomial in $\mathcal{HW}_{(r,s,\mu)}$ to obtain a polynomial in $\mathcal{HW}_{(r,s,\mu')}$. We have

$$\bigcup_{(r',s',\mu') \trianglelefteq (r,s,\mu)} \mathcal{HW}_{(r',s',\mu')} \subset \operatorname{cl} \mathcal{HW}_{(r,s,\mu)}.$$

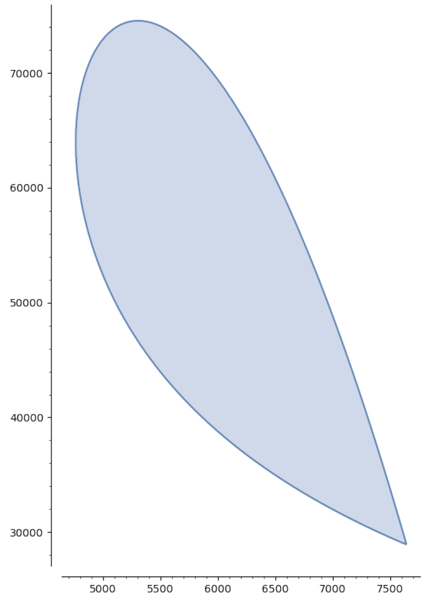
For fixed r the partial order $\trianglelefteq$ is the partial order considered to study the geometry of hyperbolic slices in [15, 16]. There are many open questions about the interplay of the geometry of $\mathcal{HW}$ and the poset $(\mathcal{C}, \trianglelefteq)$. Can one use the understanding of the geometry of hyperbolic slices to understand Hurwitz slices? Is the set $\mathcal{HW}_{(r,s,\mu)}$ contractible? Is the geometry of the set $\mathcal{HW}$ completely described by the poset $(\mathcal{C}, \trianglelefteq)$, i.e. is the set $\mathcal{HW}$ a stratified manifold with a stratification indexed by the poset? Is the partial order $(\mathcal{C}, \trianglelefteq)$ a lattice?



(A) $\mathcal{HW} \cap \{z_1 = -14, z_2 = 546, z_3 = -6064\}$



(B) $\mathcal{HW} \cap \{z_1 = -32, z_2 = 561, z_3 = -5830\}$



(C) $\mathcal{HW} \cap \{z_1 = -14, z_2 = 546, z_4 = 405010\}$

FIGURE 2. Hurwitz slices for $n = 5$ where $(|z_4|, |z_5|)$ resp. $(|z_3|, |z_5|)$ are displayed

In Figure 2 we present three examples of Hurwitz slices for $n = 5$. The multiplicity of any polynomial on the upper arc in (A) is $(3, 0, (1))$ at all points but the two endpoints. At the left endpoint the multiplicity is $(3, 0, (1))$ with a double root at 0 and $(2, 1, (1))$ at the right endpoint with a root at $\approx \pm 21i$. The bottom line corresponds to the multiplicity $(4, 1, (0))$. The same multiplicities are true for the arcs in (B). In (C) any boundary point has multiplicity structure $(5, 0, (0))$.
In general, in a Hurwitz slice not every multiplicity occurs. It is an open question to classify which multiplicities do occur in Hurwitz slices. Is a Hurwitz slice where the first coefficients are fixed always connected? We do not expect connectivity for other slices.

By Theorem 3.3 for $k < \frac{n}{3}$ the Hurwitz slice can at least not be strictly convex. Adm, Garloff and Tyaglov classified [1, Thm. 4.9] the subset of weakly Hurwitz polynomials with r roots in the interior of the left half-plane. They showed that a monic polynomial $f(T) = p_0(T^2) + Tp_1(T^2) \in \mathbb{R}[T]$ is weakly Hurwitz with r roots in the open left half-plane if and only if the first r principal minors of the finite Hurwitz matrix are negative and the remaining ones are 0 and if the polynomial $\gcd(p_0, p_1)$ has only negative roots. Do the roots of $\gcd(p_0, p_1)$ correspond to the root multiplicity (s, μ)? Finally, one could study the combinatorics and geometry of general stable slices.

4. A Grace-Walsh-Szegő like theorem for symmetric polynomials in few multiaffine polynomials

Throughout this section, let $\mathbb{H}$ be a closed half-plane and let $\underline{X} := (X_1, \dots, X_n)$ be a tuple of n variables.
The main result of this section is a statement, similar to the well-known Grace-Walsh-Szegő coincidence theorem, and a generalization of the degree principle in Theorem 4.6, Corollary 4.11 and Corollary 4.8. We refer to [19, p. 107] for background on the coincidence theorem. The main tool in this section will be our results on root multiplicities of local extreme points of stable slices from Section 2. Recall that a multivariate polynomial is called *multiaffine*, if it is linear in every variable.

Theorem 4.1 (Grace-Walsh-Szegő coincidence theorem)**.** Let $\mathcal{A}$ be a closed circular region and let $f \in \mathbb{C}[\underline{X}]$ be a multiaffine symmetric polynomial. If $\deg(f) = n$ or if $\mathcal{A}$ is convex, then for any $(x_1, \dots, x_n) \in \mathcal{A}^n$ there exists a $y \in \mathcal{A}$ with $f(x_1, \dots, x_n) = f(y, \dots, y)$.

We address an extension to the case where the symmetric polynomial f is no longer assumed to be multiaffine but can be written as a polynomial in k multiaffine symmetric polynomials in Theorem 4.6. However, we cannot expect a diagonal point in $\mathcal{A}^n$ any longer.

Definition 4.2. Let $V \subseteq \mathbb{C}^n$ be a variety. We say that V is $\mathbb{H}$-*stable* if $V \cap \mathbb{H}^n = \emptyset$. Moreover, we say that a polynomial $f \in \mathbb{C}[\underline{X}]$ is $\mathbb{H}$-*stable* if the variety $V(f)$ is $\mathbb{H}$-stable.

Remark 4.3. In Definition 4.2 we follow the standard terminology for stability of multivariate polynomials which is in contrast to the definition of stability of univariate polynomials in Definition 2.1. We say that a multivariate polynomial is $\mathbb{H}$-stable if there is no zero in $\mathbb{H}^n$, while any root of a univariate polynomial has to be contained in $\mathbb{H}$ if it is $\mathbb{H}$-stable. Since the complement of $\mathbb{H}$ is an open half-plane $\mathbb{H}^c$ one can see that for univariate polynomials $\mathbb{H}$-stability in Definition 2.1 is the same as $\mathbb{H}^c$-stability in Definition 4.2.

Recall that any n-variate symmetric polynomial can uniquely be written as a polynomial in the first n elementary symmetric polynomials by the fundamental theorem of symmetric functions. We are interested in symmetric polynomials, which can be written as polynomials in few linear combinations of elementary symmetric polynomials, which generalizes the notion of multiaffine symmetric polynomials.

Definition 4.4. Let $f \in \mathbb{C}[\underline{X}]$ be a symmetric polynomial and write f in terms of elementary symmetric polynomials, say $f = g(e_1, \dots, e_n)$ for some $g \in \mathbb{C}[Z_1, \dots, Z_n]$.

1. We say that f is $(l_1, \dots, l_k)$-*sufficient* if $g \in \mathbb{C}[l_1, \dots, l_k]$ where $l_1, \dots, l_k$ are linear forms.
2. Moreover, we say that a symmetric variety $V \subseteq \mathbb{C}^n$ is $(l_1, \dots, l_k)$-*sufficient*, if it can be described by $(l_1, \dots, l_k)$-sufficient polynomials.

Remark 4.5. A polynomial f is $(l_1, \dots, l_k)$-*sufficient* for some linear forms $l_1, \dots, l_k$, if and only if f can be written as a polynomial in k symmetric and multiaffine polynomials. In particular, every symmetric and multiaffine polynomial is l_1-sufficient for some linear form l_1.

For instance, for $n \geq 3$ the polynomial $e_1^2(\underline{X}) + e_2(\underline{X}) + 2e_3(\underline{X})$ is (l_1, l_2) sufficient for $l_1(Z_1, \dots, Z_n) = Z_1$ and $l_2(Z_1, \dots, Z_n) = Z_2 + 2Z_3$. For checking sufficiency of polynomials and more on the notion of sufficiency we refer to Subsection 3.3 in [21].

The following Theorem is our main result of this section and can be seen at the same time as some kind of *degree principle* for checking stability and as a weak form of generalization of the Grace-Walsh-Szegő's coincidence theorem.

Theorem 4.6. Let $V \subseteq \mathbb{C}^n$ be a symmetric $(l_1, \dots, l_k)$-sufficient variety. Then V is $\mathbb{H}$-stable, if and only if $V \cap \mathbb{H}_{2(k+2),k+2} = \varnothing$.

Proof. The forward implication follows from the definition. To prove the converse implication we suppose that V is not $\mathbb{H}$-stable and we want to show that

$$V \cap \mathbb{H}_{2(k+2),k+2} \neq \varnothing.$$

So let $x \in V \cap \mathbb{H}^n$ and consider $z := (e_1(x), \dots, e_n(x)) \in \mathcal{S}_{\mathbb{H}} \cap L^{-1}(a)$, where

$$L : \begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^k \\ y & \longmapsto & (l_1(y), \dots, l_k(y)) \end{array} \quad \text{and} \quad a := L(z) \in \mathbb{C}^k.$$

Then by Corollary 2.10 we find $\tilde{z} \in \mathcal{S}_{\mathbb{H}}^{2(k+2),k+2} \cap L^{-1}(a)$, i.e. there is $\tilde{x} \in \mathbb{H}_{2(k+2),k+2}$ with $\tilde{z} = (e_1(\tilde{x}), \dots, e_n(\tilde{x}))$ and $L(\tilde{z}) = a = L(z)$. This means that $\tilde{x} \in V$, since V is $(l_1, \dots, l_k)$-sufficient. □

The following proposition is a direct consequence of the unique representation of a symmetric polynomial of degree d in terms of the elementary symmetric polynomials and may serve as a motivation for Definition 4.4. We consider new variables $\underline{Z} := (Z_1, \dots, Z_n)$. For a symmetric polynomial in $\mathbb{R}[\underline{X}]$ there is a unique polynomial $g \in \mathbb{R}[\underline{Z}]$ with $f(\underline{X}) = g(e_1(\underline{X}), \dots, e_n(\underline{X}))$.

Proposition 4.7. Let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial of degree d. Then f is $(Z_1, \dots, Z_d)$-sufficient, i.e. f can be written as $f = g(e_1, \dots, e_d)$ for some $g \in \mathbb{C}[Z_1, \dots, Z_d]$. Moreover, g is linear in $Z_{\lfloor \frac{d}{2} \rfloor + 1}, \dots, Z_d$.

Proof. See Proposition 2.3 in [20]. □

From Theorem 4.6 and Proposition 4.7, we obtain immediately the following *double-degree principle.*

Corollary 4.8 (Double-degree principle)**.** Let $f_1, \dots, f_m \in \mathbb{C}[\underline{X}]$ be symmetric polynomials of degree at most d. Then

$$V(f_1, \dots, f_m) \cap \mathbb{H}^n = \varnothing \iff V(f_1, \dots, f_m) \cap \mathbb{H}_{2(d+2),d+2} = \varnothing.$$

Remark 4.9. In the case that $\mathbb{H}$ is a rotation of the upper half-plane we can replace $\mathbb{H}_{2(d+2),d+2}$ by $\mathbb{H}_{d,d}$ in Corollary 4.8. This follows immediately from Remark 2.11 for $d = 2$ and the case $d = 1$ is trivial.

Although one might hope for a stronger degree principle, the next example shows that stability of a variety defined by symmetric polynomials of degree $\leq d$ cannot always be checked by testing points with at most d many distinct coordinates.

Example 4.10. Let $n = 4$ and consider $f_1 := e_1 - 23i$, $f_2 := e_2 - 463i$ and $f_3 := e_3 - 8461i$. Then

$$V(f_1, f_2, f_3) \cap \mathbb{H}_+^4 \neq \varnothing \text{ and } V(f_1, f_2, f_3) \cap \{x \in \mathbb{H}_+^4 \mid |\{x_1, \dots, x_4\}| \leq 3\} = \varnothing,$$

which can either be computed directly using Gröbner basis or concluded by using Example 2.12.

From Remark 4.5 and Theorem 4.6, we get immediately the following variation of Grace-Walsh-Szegő's coincidence Theorem.

Corollary 4.11. Let $f \in \mathbb{C}[\underline{X}]$ be a symmetric polynomial that can be written as a polynomial in k symmetric and multiaffine polynomials. Furthermore, let $x \in \mathbb{H}^n$. Then there is $\tilde{x} \in \mathbb{H}_{2(k+2),k+2}$ with $f(x) = f(\tilde{x})$.

Note that different from Grace-Walsh-Szegő's coincidence theorem we do not require f to be multiaffine. But our result is weaker in the following sense: We do not consider closed inner or outer circle. Moreover, if f is symmetric of degree $d \geq 2$ and multiaffine and $x \in \mathcal{A}^n$, then we can find $\tilde{x} \in \mathbb{H}_{d,d}$ with

$$f(x) = f(\tilde{x}),$$

while one can find $y \in \mathbb{H}$ with Grace-Walsh-Szegő's coincidence Theorem such that

$$f(x) = f(y, \ldots, y).$$

Remark 4.12. The results of this section translate to open half-planes in the following way: Let $\mathbb{G}$ be an open circular region and $x \in \mathbb{G}^n$. Then $x \in \mathbb{H}^n$ for some closed half-plane $\mathbb{H} \subset \mathbb{G}$. So $\mathbb{G}_{2(k+2),k+2}$ can be replaced by $\mathbb{G}_{0,3(k+2)}$ in Theorem 4.6 and Corollary 4.11 and $\mathbb{G}_{2(d+2),d+2}$ can be replaced by $\mathbb{G}_{0,3(d+2)}$ in Corollary 4.8.

If $\mathbb{H} = \mathbb{H}_+$ is the upper half-plane, one can also formulate a generalization of the half-degree principle.

Theorem 4.13 (Half-degree principle for the upper half-plane)**.** Let $f \in \mathbb{C}[\underline{X}]$ be a symmetric polynomial of degree $d \leq n$ and $\lambda, \mu \in \mathbb{R}$. Then

$$\inf_{x \in \mathbb{H}_+^n} \lambda \operatorname{Re}(f(x)) + \mu \operatorname{Im}(f(x)) = \inf_{x \in \mathbb{H}_{+k,k}} \lambda \operatorname{Re}(f(x)) + \mu \operatorname{Im}(f(x)),$$

where $k = \max\{\lfloor \frac{d}{2} \rfloor, 2\}$.

Proof. Write $f = g(e_1, \ldots, e_d)$ for some $g \in \mathbb{R}[Z_1, \ldots, Z_d]$ and note that g is linear in $Z_{\lfloor \frac{d}{2} \rfloor + 1}, \ldots, Z_d$ by Proposition 4.7. Let now $x \in \mathbb{H}_+^n$ and consider $z := (e_1(x), \ldots, e_n(x)) \in \mathcal{S}(a)$, where $a := (e_1(x), \ldots, e_k(x))$. Since $\mathcal{S}(a)$ is compact and g is linear on $\mathcal{S}(a)$, the minimum of g on $\mathcal{S}(a)$ is taken on an extreme point of the convex hull of $\mathcal{S}(a)$, i.e. on a point $\tilde{z} \in \mathcal{S}^{k,k}$ by Remark 2.11. □

4.1. **A converse to Grace-Walsh-Szegő's coincidence theorem.** In another direction, Brändén and Wagner [5] proved that for the open upper half-plane $\operatorname{int} \mathbb{H}_+$ and a group $G \subsetneq S_n$ acting on $\mathbb{C}[\underline{X}]$ via permutation of the variables, no analogous result to Theorem 4.1 holds. More concretely, if $G \subset S_n$ is a group acting on $\mathbb{C}[\underline{X}]$ by restriction and all G-invariant multiaffine polynomials f satisfy that for all $x \in \operatorname{int} \mathbb{H}_+^n$ there is a $y \in \operatorname{int} \mathbb{H}_+$ with $f(y) = f(x)$, then G must be already the full symmetric group S_n.
By considering *Young subgroups* of S_n we find that a weaker statement still holds.

Definition 4.14. For a group $G \subset S_n$ we denote by $S(G) = \tilde{S}_{i_1} \times \cdots \times \tilde{S}_{i_k} \subset S_n$ a Young subgroup of G, where $\tilde{S}_{i_j}$ is the symmetric group on i_j elements acting on $\mathbb{C}^n$ by permuting the $i_1 + \cdots + i_{j-1} + 1$ to $i_1 + \cdots + i_j$-th coordinates and k be the minimal number of factors needed. Furthermore, we define $k(G) := k$.

In particular, we have $\sum_{j=1}^k i_j = n$ for any group $G \subset S_n$.

Theorem 4.15. Let $\mathbb{H}$ be a half-plane, $f \in \mathbb{C}[\underline{X}]^G$ a G-invariant multiaffine polynomial and $x \in \mathbb{H}^n$. Then there are $y_1, \ldots, y_k \in \mathbb{H}$, such that

$$f(x) = f(\underbrace{y_1, \ldots, y_1}_{i_1\text{-times}}, \ldots, \underbrace{y_k, \ldots, y_k}_{i_k\text{-times}}),$$

where $k := k(G)$.

Proof. Let $S(G) = \tilde{S}_{i_1} \times \cdots \times \tilde{S}_{i_k} \subset S_n$ be as in Definition 4.14 and $x \in \mathbb{H}^n$. The polynomial

$$f_1 := f(X_1, \ldots, X_{i_1}, x_{i_1+1}, \ldots, x_n) \subseteq \mathbb{C}[X_1, \ldots, X_{i_1}]$$

is $\tilde{S}_{i_1}$-invariant and multiaffine, so by Grace-Walsh-Szegő's theorem, there is $y_1 \in \mathbb{H}$, such that

$$f(x) = f_1(x_1, \ldots, x_{i_1}) = f_1(\underbrace{y_1, \ldots, y_1}_{i_1\text{-times}}).$$

By induction, we define the $\tilde{S}_{i_j}$-invariant polynomial

$$f_j = f(\underbrace{y_1,\dots,y_1}_{i_1\text{-times}},\dots,\underbrace{y_{j-1},\dots,y_{j-1}}_{i_{j-1}\text{-times}},X_1,\dots,X_{i_j},x_{i_1+\dots+i_j+1},\dots,x_n)$$

and, by Grace-Walsh-Szegő's theorem, there is a $y_j \in \mathcal{A}$, such that

$$f(x) = f_j(x_{i_1+\dots+i_{j-1}+1},\dots,x_{i_1+\dots+i_j}) = f_j(\underbrace{y_j,\dots,y_j}_{i_j\text{-times}}).$$

□

Using the result of Brändén and Wanger we also find the following converse statement:

Theorem 4.16. Let $G \subset S_n$ and $H = \tilde{S}_{j_1} \times \dots \times \tilde{S}_{j_m} \subset S_n$ be a supergroup of G, where $\tilde{S}_{j_l}$ is the symmetric group on j_l elements acting on $\mathbb{C}^n$ by permuting the $j_1 + \dots + j_{l-1} + 1$ to $j_1 + \dots + j_l$-th coordinates. If for any G-invariant multiaffine polynomial $f \in \mathbb{C}[\underline{X}]^G$ and any $x \in (\operatorname{int}\mathbb{H}_+)^n$, there are $y_1,\dots,y_m \in \operatorname{int}\mathbb{H}_+$, such that

$$f(x) = f(\underbrace{y_1,\dots,y_1}_{j_1\text{-times}},\dots,\underbrace{y_m,\dots,y_m}_{j_m\text{-times}}),$$

then every G-invariant multiaffine polynomial is already H-invariant.

Proof. Let $l \in \{1,\dots,m\}$. For any $\tilde{S}_{j_l}$-invariant j_l-variate multiaffine polynomial $f \in \mathbb{C}[X_{j_1+\dots+j_{l-1}+1},\dots,X_{j_1+\dots+j_l}]$ and any $x \in (\operatorname{int}\mathbb{H}_+)^{j_l}$, there is $y_l \in \operatorname{int}\mathbb{H}_+$, such that

$$f(x) = f(\underbrace{y_l,\dots,y_l}_{j_l\text{-times}}).$$

By the converse to Grace-Walsh-Szegő's coincidence theorem by Brändén and Wagner [5], every G-invariant multiaffine polynomial is already S_{j_l}-invariant. □

5. Conclusion and open questions

In our paper, we restrict to half-plane stable polynomials. However, the notion of stable polynomials can be formulated for any circular region, i.e. any open or closed subset of $\mathbb{C}$ that is bounded by a circle or by a line. It is well known that a Möbius transformation maps circular regions to circular regions and testing stability of a polynomial can always be reduced to testing whether an associated polynomial of possibly smaller degree is $\mathbb{H}_+$-stable. Let $\mathcal{A}$ be a circular region and let $\phi(z) = \frac{az+b}{cz+d}$ be a Möbius transformation mapping $\mathbb{H}_+$ to $\mathcal{A}$. Then a monic polynomial $f \in \mathbb{C}[T]$ is $\mathcal{A}$-stable if and only if the polynomial $(cT+d)^{\deg(f)} f\left(\frac{aT+b}{cT+d}\right)$ is $\mathbb{H}_+$-stable. The roots of the associated polynomial are contained in the image of the roots of f under ϕ^{-1}. However, the obtained polynomial must not necessarily be monic or can have fewer roots. This happens if one of the roots is a pole point of ϕ^{-1}. For instance, if $f = p\cdot(T-1)$ is $\{x \in \mathbb{C} : |x| \le 1\}$-stable and p has only roots different from 1, then

$$(T+i)^{\deg(p)} p\left(\frac{T-i}{T+i}\right)(T+i)\left(\frac{T-i}{T+i}-1\right) = (T+i)^{\deg(p)} p\left(\frac{T-i}{T+i}\right)\cdot(-2i)$$

is a non-monic $\mathbb{H}_+$-stable polynomial of degree $\deg(f)-1$. Thus our proofs of Theorems 2.4 and 4.6 do not transfer to circular regions which are bounded by a circle. Nevertheless, the following questions seem worth to be asked.

Question 5.1. Can Theorem 2.4 and Theorem 4.6 be adapted to arbitrary circular regions? If not, can our variation of the coincidence theorem be extended to a closed domain bounded by a circle?

Question 5.2. Can our double-degree principle in Corollary 4.8 be improved?

Finally, we gave a possible combinatorial encoding for subsets of the set of weakly Hurwitz polynomials. For hyperbolic polynomials, there is the rich interplay between geometry and combinatorics of its roots. Hyperbolic slices with fixed first k coefficients and their strata are known to be contractible. Moreover, Lien [15] showed that in this case, one can reconstruct the compositions of the stratification from the compositions of its 0-dimensional strata, and Schabert and Lien [16] showed that this poset has a structure similar to polytopes, giving the same bounds on its number of i-dimensional strata. We ask if similar results hold for Hurwitz slices with fixed first k coefficients.

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