A Computer Search of New OBZCPs of Lengths up to 49

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***Abstract*—This paper aims to search for new optimal and sub-optimal Odd Binary Z-Complimentary Pairs (OBZCPs) for lengths up to 49. As an alternative to the celebrated binary Golay complementary pairs, optimal OBZCPs are the best almost-complementary sequence pairs having odd lengths. We introduce a computer search algorithm with time complexity $O(2^N)$, where N denotes the sequence length and then show optimal results for all $27 \leq N \leq 33$ and $N = 37, 41, 49$. For those sequence lengths (i.e., $N = 35, 39, 43, 45, 47$) with no optimal pairs, we show OBZCPs with largest zero-correlation zone (ZCZ) widths (i.e., Z-optimal). Finally, based on the Pursley–Sarwate criterion (PSC), we present a table of OBZCPs with smallest combined auto-correlation and cross-correlation.**

***Index Terms*—Aperiodic correlation, Golay complementary pair (GCP), zero-correlation zone (ZCZ), Z-complementary pair (ZCP), odd-length binary ZCP (OBZCP), Pursley-Sarwate criterion.**

I. Introduction

A. Background

Complementary pairs of sequences are useful in coding theory, wireless communication, radar sensing, and signal processing. The general design objective is to find two equal-length sequences whose maximum aperiodic auto-correlation function (AACF) sums are as small as possible (ideally zero). Pioneered by Marcel J. E. Golay [1] in 1951, binary *complementary pairs* were first studied in his design of infrared multislit spectrometry, a detector which isolates the desired radiation with a fixed single wavelength from background radiation with many different wavelengths. Formally, a pair of sequences is called a Golay complementary pair (GCP) [2] if their AACF sums are zero for all the non-zero time-shifts. Since it is generally difficult to find a single unimodular sequence with zero AACF sidelobes[1], GCP provides a solution by allowing two sequences to work in a collaborative way. Some representative applications of the GCPs (and their extensions/variants) include: peak-to-mean envelope power ratio (PMEPR) reduction of multicarrier signals [3]–[6], Doppler resilient radar waveforms [7]–[9], channel estimation [10]–[13], inter-cell interference rejection [14], multicarrier code-division multiple access [15], [16], etc.

Despite their wide applications, however, binary GCPs are limited to even-lengths only. More specifically, the existing binary GCPs are only known to have sequence lengths of $2^\alpha 10^\beta 26^\gamma$ (where α, β, γ are non-negative integers). For the odd-length case, optimal binary almost–complementary pairs are studied in [17]. Their optimization criteria is to maximize the zero-correlation zone (ZCZ) width of an odd-length binary pair, whilst minimizing its out-of-zone AACF sums. Such *optimal* binary pairs exhibit the correlation properties closest to GCPs and are called *optimal* odd-length binary Z-complementary pairs (OBZCPs). It is found that each optimal OBZCP has maximum ZCZ width of $(N+1)/2$, and minimum out-of-zone AACF sum magnitude of 2, where N denotes the sequence length (odd) [17].

By applying insertion to certain binary Golay-Davis-Jedwab (GDJ) complementary pairs [4], *optimal* OBZCPs[2] of lengths $2^m + 1$, where m is a positive integer, are constructed in [17]. Subsequently, *optimal* OBZCPs of *generic* lengths $2^\alpha 10^\beta 26^\gamma + 1$ (where α, β, γ are non-negative integers and $\alpha \geq 1$) are obtained in [18], thanks to certain structural properties of binary GCPs obtained from Turyn's method [19].

B. Contributions

Searching for *optimal* OBZCPs is of interest for understanding their deeper structural properties and for providing higher level of flexibility in practical applications. Besides the aforementioned systematic constructions, it is equally important to look for short OBZCPs because they may be used to generate longer OBZCPs through certain recursive operations. For example, in 2003, Borwein and Ferguson carried out an exhaustive computer search to search new GCPs of lengths up to 100 [21]. Although some OBZCPs are reported in [17], their lengths are capped to 25 only.

The main objective of this paper is thus to look for new optimal (or sub-optimal) OBZCPs of lengths $27 \leq N \leq 49$. By leveraging the High Performance Computing (HPC) facility[3] at the University of Essex, a number of new OBZCPs are found by a fine-grained computer search algorithm involving mapping, grouping, and Gray code counting. Besides the OBZCP optimality criteria on maximum ZCZ width and minimum out-of-zone AACF sums, we also use the Pursley–Sarwate criterion [22], [23] for selection of the best binary odd-length pairs with lowest combined auto-correlation and cross-correlation. For reproducibility of these results, our source code can be found at https://github.com/peterkazakov/obzcp.

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[1]Although a Huffman sequence has zero AACF sidelobes except for the end time-shift, its sequence elements may have different magnitudes.

[2]In this paper, without any specific announcement, we are only interested in Type-I OBZCPs each having ZCZ around the in-phase time-shift. There are also Type-II OBZCPs where the ZCZs are centered around the end-shift positions, leading to zero correlation sidelobes away from the in-phase time-shift, but these are not our focus.

[3]https://www.essex.ac.uk/staff/it-services/hpc.

This paper is organized as follows. In Section II, we give some preliminaries and important equations/facts on OBZCPs. In Section III, we introduce the general considerations and programming techniques of the proposed algorithm. In Section IV, new OBZCPs found by computer search are presented. A list of OBZCPs with smallest smallest combined auto-correlation and cross-correlation are given in Section V. Finally, this paper is concluded in Section VI.

II. Preliminaries

Throughout this paper, we are interested in binary sequence pairs whose elements are drawn from the set of $\mathbb{Z}_2 = \{0, 1\}$. A length-N vector is called a binary sequence if it is over $\mathbb{Z}_2^N$. For convenience, whenever necessary, binary sequences may also be shown over $\{1, -1\}^N$. For $\mathbf{a} = (a_0, a_1, \cdots, a_{N-1})$ over $\mathbb{Z}_2^N$, let $\mathbf{a}(z)$ be the associated polynomial of z as follows,

$$\mathbf{a}(z) = \sum_{\tau=0}^{N-1} (-1)^{a_\tau} z^\tau. \tag{1}$$

For two binary sequences $\mathbf{a}$ and $\mathbf{b}$ over $\mathbb{Z}_2^N$, define

$$\rho_{\mathbf{a},\mathbf{b}}(\tau) = \begin{cases} \sum_{i=0}^{N-1-\tau} (-1)^{a_i + b_{i+\tau}}, & 0 \le \tau \le N-1; \\ \sum_{i=0}^{N-1-\tau} (-1)^{a_{i+\tau} + b_i}, & -(N-1) \le \tau \le -1; \\ 0, & |\tau| \ge N. \end{cases} \tag{2}$$

When $\mathbf{a} \neq \mathbf{b}$, $\rho_{\mathbf{a},\mathbf{b}}(\tau)$ is called the aperiodic cross-correlation function of $\mathbf{a}$ and $\mathbf{b}$; otherwise, it is called the AACF. For simplicity, the AACF of $\mathbf{a}$ will be sometimes written as $\rho_{\mathbf{a}}(\tau)$.

A. Binary Z-complementary pairs (ZCPs)

***Definition* 1.** *Let $\mathbf{a}$ and $\mathbf{b}$ be over $\mathbb{Z}_2^N$. $(\mathbf{a}, \mathbf{b})$ is said to be a binary ZCP with ZCZ width of Z if and only if [20]*

$$\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) = 0, \quad \text{for any } 1 \le \tau \le Z-1. \tag{3}$$

In this case, $\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)$ for $Z \le \tau \le N-1$, is called the out-of-zone aperiodic auto-correlation sum of $\mathbf{a}$ and $\mathbf{b}$ at time-shift τ. When $Z = N$, a ZCP reduces to a Golay complementary pair (GCP) [2]. An OBZCP refers to a binary ZCP with odd-length.

Fact 1. *Each OBZCP $(\mathbf{a}, \mathbf{b})$ has the maximum ZCZ of width $(N+1)/2$ [24], i.e., $Z \le (N+1)/2$, where N denotes the sequence length. An OBZCP is said to be Z-optimal if $Z = (N+1)/2$.*

Fact 2. *The magnitude of each out-of-zone aperiodic auto-correlation sum for a Z-optimal OBZCP $(\mathbf{a}, \mathbf{b})$ is lower bounded by 2 [17], i.e.,*

$$\left|\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)\right| \ge 2, \quad \text{for any } (N+1)/2 \le \tau \le N-1.$$

A Z-optimal OBZCP is said to be optimal if $\left|\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)\right| = 2$ holds for all $(N+1)/2 \le \tau \le N-1$.

Fact 3. *For an optimal (or a Z-optimal) OBZCP $(\mathbf{a}, \mathbf{b})$, the following equations are satisfied [17]*

$$\begin{cases} a_0 + a_{N-1} + b_0 + b_{N-1} & \equiv 0 \ (mod\ 2), \\ a_r + a_{N-1-r} + b_r + b_{N-1-r} & \equiv 1 \ (mod\ 2), \end{cases} \tag{4}$$

where $1 \le r \le (N-3)/2$.

B. Pursley–Sarwate Criterion [22], [23]

For a binary sequence pair $(\mathbf{a}, \mathbf{b})$ of length N, the cross-correlation demerit factor between $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\text{CDF}(\mathbf{a}, \mathbf{b}) = \frac{\sum_{\tau=1-N}^{N-1} |\rho_{\mathbf{a},\mathbf{b}}(\tau)|^2}{N^2}. \tag{5}$$

For a binary sequence $\mathbf{a}$, the auto-correlation demerit factor of $\mathbf{a}$ is defined by

$$\text{ADF}(\mathbf{a}) = -1 + \text{CDF}(\mathbf{a}, \mathbf{a}). \tag{6}$$

Furthermore, let us define the Pursley–Sarwate criterion of binary pair $(\mathbf{a}, \mathbf{b})$ as follows:

$$\text{PSC}(\mathbf{a}, \mathbf{b}) = \sqrt{\text{ADF}(\mathbf{a}) \cdot \text{ADF}(\mathbf{b})} + \text{CDF}(\mathbf{a}, \mathbf{b}). \tag{7}$$

According to Pursley and Sarwate [22], we have $\text{PSC}(\mathbf{a}, \mathbf{b}) \ge 1$. Katz and Moore pointed out in [23] that binary GCPs of lengths $2^\alpha 10^\beta 26^\gamma$ (where α, β, γ are non-negative integers) satisfy the Pursley–Sarwate lower bound with equality, i.e., $\text{PSC}(\mathbf{a}, \mathbf{b}) = 1$.

In this paper, we also use Pursley–Sarwate criterion as a sieve to select OBZCPs. Specifically, we look for *optimal* OBZCPs or *Z-optimal* OBZCPs with PSC values closest to 1.

III. Proposed Algorithm

A. General Considerations

In this section we describe our main considerations for our proposed algorithm with the time complexity of $O(2^N)$. Exhaustive computer search of all pairs is infeasible due to the high complexity of $O(2^{2N})$. We present a novel algorithm based on a two-step approach using the algebraic constructions and software data structures within a reasonable memory constraint.

In order to limit duplicated calculations, the first consideration is to exclude equivalent pairs. Similar to the definition in [20], two pairs $(\mathbf{a} = (a_0, \ldots, a_{N-1}), \mathbf{b} = (b_0, \ldots, b_{N-1}))$ are said to be equivalent, if one can be obtained by the other with one of the following operations:

- Interchange: if $(\mathbf{a}, \mathbf{b})$ is a solution, then so is $(\mathbf{b}, \mathbf{a})$;
- Negation: if $(\mathbf{a}, \mathbf{b})$ is a solution, then so is $(\text{neg}(\mathbf{a}), \mathbf{b})$, where $\text{neg}(\mathbf{a}) = (1 - a_0, 1 - a_1, \ldots, 1 - a_{N-1})$;
- Reversal: if $(\mathbf{a}, \mathbf{b})$ is a solution, then so is $(\text{rvr}(\mathbf{a}), \mathbf{b})$, where $\text{rvr}(\mathbf{a}) = (a_{N-1}, \ldots, a_1, a_0)$.

Naturally, a series of these operations above lead to an equivalent solution. For instance $(\text{neg}(\mathbf{b}), \text{rvr}(\text{neg}(\mathbf{a})))$ is also equivalent to $(\mathbf{a}, \mathbf{b})$.

To proceed, without loss of generality, we assume that $a_{N-1} = b_{N-1} = 1$. We then run as separate cases for the all possible combinations of the middle bits $a_{(N-1)/2}$, $b_{(N-1)/2}$ and $a_0 = b_0$ due to (4).

To visualize, we consider the following OBZCP structure in our search algorithm:

$$\begin{bmatrix} a_0 & a_1 & \dots & a_{(N-3)/2} & a_{(N-1)/2} & a_{(N+1)/2} & \dots & a_{N-2} & 1 \\ a_0 & b_1 & \dots & b_{(N-3)/2} & b_{(N-1)/2} & b_{(N+1)/2} & \dots & b_{N-2} & 1 \end{bmatrix}.$$

Next, we leverage the following three important data processing strategies in our proposed algorithm:

- Mapping;
- Grouping;
- Gray counting and updates.

B. Mapping

The purpose of mapping is to link each sequence $\mathbf{b}$ with a proper sequence $\mathbf{a}$ such that the ZCZ can be achieved. Based on (3), we have

$$\rho_{\mathbf{b}}(\tau) = -\rho_{\mathbf{a}}(\tau) \tag{8}$$

for each $\tau = 1, 2, \dots, (N-3)/2$. Suppose that there are k sequences of $\mathbf{a}$ which satisfy the above equation. We store all the ρ values and their corresponding sequences $\mathbf{a}$ in a map $\{\rho_{\mathbf{a}} \to [\mathbf{a}_1, \dots, \mathbf{a}_k]\}$ where

$$\rho_{\mathbf{a}} = (\rho_{\mathbf{a}}(1), \rho_{\mathbf{a}}(2), \dots, \rho_{\mathbf{a}}((N-3)/2))$$

for all $\mathbf{a} \in \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ that generate the same ρ. We will denote by map$[\rho_{\mathbf{a}}]$ for the whole list of $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$.

Example 1. *For $N = 11$, $\rho_{\mathbf{a}} = (2, -1, 2, 1)$, we have*

$$\begin{aligned} \textit{map}[\rho_{\mathbf{a}}] = \{&(1,0,0,0,0,0,1,0,0,1,1),\\ &(1,0,1,1,0,0,1,1,1,1,1),\\ &(1,1,0,0,1,0,0,0,0,0,1),\\ &(1,1,1,1,1,0,0,1,1,0,1)\}. \end{aligned}$$

For each sequence $\mathbf{b}$, proper software implementation of such map will permit us to find all possible corresponding sequences $\mathbf{a}$ in constant $O(1)$ time.

Theoretically, with this approach we can attack the whole problem. However, this may require us to put maximally all 2^N sequences of $\mathbf{a}$ and their corresponding ρ values in the memory. This is quite significant considering the physical memory limitations. Next, we introduce grouping in order to overcome this limitation.

C. Grouping

In the first step, we group the sequence elements by using the property below:

$$a_r + a_{N-1-r} + b_r + b_{N-1-r} \equiv 1 \pmod 2. \tag{9}$$

For ease of presentation, denote by $N_{\mathbf{a}}$ the integer associated to the binary sequence $\mathbf{a}$ and vice versa.

For any binary vector $\mathbf{c} = (c_1, c_2, \dots, c_{(N-3)/2})$, define a set $\mathbf{C}$ comprising of all the possible length-N binary sequences $\mathbf{a}$, each satisfying

$$c_i = a_i \oplus a_{N-1-i}, i = 1, 2, \dots, (N-3)/2 \tag{10}$$

where $\oplus$ is the binary xor operation. By noting that $a_0, a_{(N-1)/2}, a_{N-1}$ are fixed, each set $\mathbf{C}$ contains $2^{(N-3)/2}$ sequences of $\mathbf{a}$ and we have $2^{(N-3)/2}$ sets for all possible combinations of $\mathbf{c}$.

Example 2. *Consider $N = 7$, $Z = (N+1)/2 = 4$, $\mathbf{c} = (1, 0)$. One can generate the following $\mathbf{C}(\mathbf{c})$:*

$$\begin{aligned} &\{(a_0, 0, 0, a_3, 0, 1, 1), (a_0, 1, 0, a_3, 0, 0, 1),\\ &(a_0, 0, 1, a_3, 1, 1, 1), (a_0, 1, 1, a_3, 1, 0, 1)\}. \end{aligned}$$

D. Gray code counting

For every sequence $\mathbf{b}$, note that $b_{N-1} = 1$, $b_{(N-1)/2}$ is fixed and $b_0 = a_0$ as shown in (4). To proceed, $(b_1, b_2, \dots, b_{(N-3)/2})$ is said to be the lower part of sequence $\mathbf{b}$ and $(b_{(N+1)/2}, b_{(N+3)/2}, \dots, b_{(N-2)})$ its upper part. Our key idea is to use a Gray code to represent the lower part, whereby the upper part is recalculated based on (4).

In the sequel, "codeword" and "sequence" may be used interchangeably. The weight of each codeword is equal to its number of 1's. Comparison between codewords (sequences), e.g., $\mathbf{a} \geq \mathbf{b}$, is performed by comparing their associated integers, i.e., $N_{\mathbf{a}} \geq N_{\mathbf{b}}$. A Gray code is characterized by having the next codeword with only one bit changed. This gives us two advantages:

- We can update only one corresponding bit in $\mathbf{b}$'s upper part, i.e. if b_i is changed, then b_{N-1-i} is changed.
- For such a change, the weight of sequence $\mathbf{b}$ is either increased or decreased by 2 or stays unchanged. Hence, we can track the weight of sequence $\mathbf{b}$ without recalculating it for each modification. By pre-calculating all the admissible weight pairs associated to $(\mathbf{a}, \mathbf{b})$ (see *Property 3* of [17]), we can directly exclude all the non-matching combinations.

E. Sketch of the Proposed Algorithm

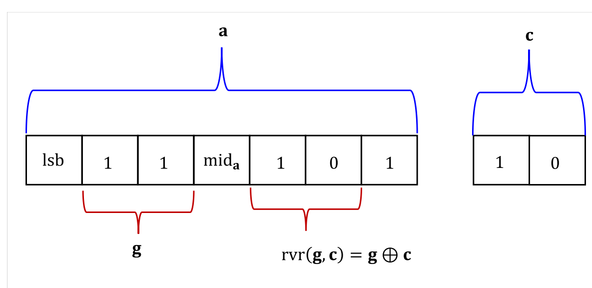

Fig. 1: Structure of sequence $\mathbf{a}$ of length 7.

In the proposed algorithm, $\mathrm{rvr}(\mathbf{g}, \mathbf{c})$ is defined as $\mathrm{rvr}(\mathbf{g} \oplus \mathbf{c})$, which is obtained by binary xor operation of $\mathbf{g}$ and $\mathbf{c}$, followed by reversing the resulting binary vector. Sequence $\mathbf{a}$ is constructed by $(\mathrm{lsb}, \mathbf{g}, \mathrm{mid}_{\mathbf{a}}, \mathrm{rvr}(\mathbf{g}, \mathbf{c}), 1)$, where lsb and $\mathrm{mid}_{\mathbf{a}}$ are single binary elements and $\mathbf{g}$, $\mathrm{rvr}(\mathbf{g}, \mathbf{c})$ are binary vectors of length $(N-3)/2$. For illustration purpose, we show in Figure 1 a length-7 $\mathbf{a}$ which belongs to $\mathbf{C}$ for $\mathbf{g} = (1, 1)$ and $\mathbf{c} = (1, 0)$. In the algorithm, **b_upper** and **b_lower**, two binary vectors of length $(N-3)/2$, are used for construction of $\mathbf{b}$.

Algorithm 1 OBZCPs

Require: $N \geq 5$, N is odd
Require: MAX_ACC ≥ 1, maximum correlation
Require: $\text{mid}_{\mathbf{a}}$, predefined value for $a_{(N-1)/2}$
Require: $\text{mid}_{\mathbf{b}}$, predefined value for $b_{(N-1)/2}$
Require: lsb, least significant bit of $\mathbf{a}$ and $\mathbf{b}$
1: **for** $N_{\mathbf{c}} \leftarrow 0$ to $2^{(N-3)/2} - 1$ **do**
2: **for** $N_{\mathbf{g}} \leftarrow 0$ to $2^{(N-3)/2} - 1$ **do** ▷ Add $\mathbf{a} \in \mathbf{C}$ to the map
3: $\mathbf{a} \leftarrow (\text{lsb}, \mathbf{g}, \text{mid}_{\mathbf{a}}, \text{rvr}(\mathbf{g}, \mathbf{c}), 1)$
4: Add $\mathbf{a}$ to map$[\rho_{\mathbf{a}}]$
5: **end for**
6: $\mathbf{b\_lower} \leftarrow (1, 0, 0, \ldots, 0)$ ▷ Construct initial $\mathbf{b}$
7: Calculate $\mathbf{b\_upper}$ such that Eq.(4) holds for $\mathbf{b} = (\text{lsb}, \mathbf{b\_lower}, \text{mid}_{\mathbf{b}}, \mathbf{b\_upper}, 1)$ and given $\mathbf{c}$. $\mathbf{a}$'s bits are already aggregated with an xor in $\mathbf{C}$.
8: **for** $j \leftarrow 1$ to $2^{(N-3)/2}$ **do** ▷ Iterate over possible $\mathbf{b}$ candidates via Gray code
9: $\rho_{\mathbf{a}}(\tau) \leftarrow -\rho_{\mathbf{b}}(\tau), \tau = 1, \ldots, (N-3)/2$
10: **for** $\mathbf{a} \in$ map$[\rho_{\mathbf{a}}]$ **do**
11: **if** $N_{\mathbf{a}} > N_{\mathbf{b}}$ **then** ▷ Skip duplication
12: Print $(\mathbf{a}, \mathbf{b})$ if out-of-zone correlation $\leq$ MAX_ACC
13: **end if**
14: **end for**
15: Generate next Gray code for $\mathbf{b\_lower}$ part by changing single bit $\mathbf{b}_x$
16: Update $\mathbf{b\_upper}$ part by reverting single bit
17: $\mathbf{b}_{N-1-x}$
18: **end for**
19: **end for**
20: Filter out the equivalent pairs

The proposed algorithm is based on one outer loop to iterate over all the sets and two consecutive inner loops for 1) loading data in the memory map and 2) iterating over the Gray code by changing a single bit on each step and retrieving matching values from the map such that equation (4) holds.

It is noted that the time complexity of the outer loops equals to the number of sets that we iterate on, i.e., $O(2^{(N-3)/2})$. The time complexity of the two consecutive inner loops is determined by the maximum of:

- The complexity of the first inner loop which equals to the number of all constructed $\mathbf{a}$'s in each set, i.e., $O(2^{(N-3)/2})$;
- The complexity of the second inner loop which equals to the number of all the $\mathbf{b\_upper}$ values that we iterate on via Gray code, i.e., $O(2^{(N-3)/2})$. During the second inner loop, we construct all the possible $\mathbf{b}$ matching to the set $\mathbf{C}(\mathbf{c})$. It is also noted that the corresponding search in the table between $\mathbf{a}$ and $\mathbf{b}$ has lookup complexity of $O(1)$. Additionally, the most inner loop on line 10 is usually of size $0, 1, 2$, and rarely up to 4 and hence does not impact the overall complexity.

Thus, the total time complexity of the inner loops is $O(2^{(N-3)/2})$.

Based on the above analysis, the overall algorithm has a complexity of $O(2^N)$ since it is executed 8 times with all possible combinations of $a_{(N-1)/2}, b_{(N-1)/2}, a_0$.

IV. Computer Search Results

In this section, we present optimal OBZCPs from computer search, and in case they do not exist, Z-optimal OBZCPs for $27 \leq N \leq 49$, in Tables I to XII.

Optimal OBZCPs up to $N = 25$ are reported in [17] and some optimal pairs for $N = 33, 41$ can be obtained via the constructions in [18]. During this computer search, we have found only two non-equivalent optimal pairs for $N = 37$. We have also found optimal pairs of length $N = 49$, but the search is non-exhaustive as we reached the limit of our computational resources.

Our computer search shows that there are no optimal OBZCPs for $N = 35, 39, 43, 45, 47$ and this motivates us to search Z-optimal pairs with the maximum out-of-zone aperiodic autocorrelation sum having magnitude of 6. In this work, we only present a few Z-optimal pairs in the tables below for $N = 35, 39, 43, 45, 47$.

It is noted that all the OBZCPs are presented in hexadecimal notation. This allows us to map every hexadecimal digit to four binary digits, i.e. ($0 \to 0000$, $1 \to 0001$, ..., $F \to 1111$) and by stripping initial zeros if necessary. For instance `159FE24` represents 101011001111111110001001000.

Since $N = 35$ is the first odd length that does not have optimal pairs, we present on the left-hand-side of Fig. 1 on the magnitude plot of the AACF sums of the pair `(7905A9444, 710C1A3B2)`. On the right-hand-side of Fig. 1, we show the magnitude plot of the optimal pair `(15BCD1FAF3340,10599EA0E984A)` with length 49.

Our algorithm is implemented using the software language "Rust" known as blazingly fast and memory-efficient [25] and includes optimizations such as Rayon parallel computing library, usage of native CPU instructions via compilation flags, such as SIMD, loops unwinding, etc. Our source code can be found at https://github.com/peterkazakov/obzcp. The OBZCP search for lengths up to 41 was carried out over a 2.9 GHz 6-Core Intel i9 computer. The execution time over for $N = 31$ and $N = 33$ are 3 and 14 minutes, respectively. Each subsequent length requires approximately 4 times more minutes, for instance $N = 39$ requires a day of computations.

The other results for $N = 43, 45, 47, 49$ are found through Essex HPC.

V. OBZCPs with Smallest PSC Values

Finally, we leverage the Pursley–Sarwate criterion and present OBZCPs with smallest PSC values for sequence lengths up to 49 in Table XIII.

One can see that the PSC value of each pair is close to 1, indicating that each pair is almost complementary. We also notice a trend that the PSC values for optimal OBZCPs generally decrease for larger sequence lengths with just one exception for $N = 13$.

TABLE I: Optimal OBZCPs for $N = 27$

Column 1	Column 2	Column 3	Column 4
(6AC2984, 42265F0)	(419B094, 4038DAA)	(72C2320, 6581DAA)	(5CB287E, 409159C)
(6A92700, 41D3994)	(77724F1, 71B4ABF)	(623AFC9, 4C14A0D)	(668A84F, 4F0B77B)

TABLE II: Optimal OBZCPs for $N = 29$

Column 1	Column 2	Column 3
(144E4E10, 114693FA)	(17CA7B3A, 14640A7C)	(1FCADB38, 1C64AA7E)
(1835D190, 17CA5190)	(1A9FDB38, 15605B38)	(1F6CC570, 11D1AD20)
(1AEF4E13, 16F06EEB)		

TABLE III: Optimal OBZCPs for $N = 31$

Column 1	Column 2	Column 3
(4ED3AE80, 43856426)	(6C7FC945, 5EB32F1D)	(64AFE6B9, 640ACBC7)

TABLE IV: Optimal OBZCPs for $N = 33$

Column 1	Column 2	Column 3
(1EA6D8C0A, 1EA6C73F4)	(180CBE6AC, 180CA1952)	(1F8B39ED4, 1F8B2612A)
(1E72A06CA, 1A6C05630)		

TABLE V: Z-optimal OBZCPs for $N = 35$

Column 1	Column 2	Column 3
(7905A9444, 710C1A3B2)	(72E2F6394, 5DE937410)	(4AC870914, 40DDE22C2)
(54C870928, 40DDE22C2)	(793B3EE96, 5C6F5043A)	(44EA20C2D, 42D04DDE3)

TABLE VI: Optimal OBZCPs for $N = 37$

Column 1	Column 2
(1D29F4D110, 11273940E8)	(17B506C9C4, 144430A7C2)

TABLE VII: Z-optimal OBZCPs for $N = 39$

Column 1	Column 2	Column 3
(69B1294470, 5AEF8C06E8)	(6CC43E1164, 4A75EC2828)	(70122A279C, 5B235A9E08)
(72577E7298, 6E0592287A)	(59F26A2072, 44AF3882D0)	(78C84D1254, 77E235C7D2)

TABLE VIII: Optimal OBZCPs for $N = 41$

Column 1	Column 2	Column 3
(1A2903A133C, 15D6FCA133C)	(18215995906, 1821586A6F8)	(18A5F99F9AE, 175A069F9AE)
(1945F99F9D6, 16BA069F9D6)	(1F5EC9C62FA, 10A136C62FA)	(195DC396EFC, 16A23C96EFC)
(1F51C9C6DFA, 10AE36C6DFA)	(1A2903C337A, 15D6FCC337A)	(1E14DDB4188, 11EB22B4188)
(1C9EF59CBA0, 13610A9CBA0)	(1A774FA7BB0, 1588B0A7BB0)	(19FA296B9F9, 1605D66B9F9)

TABLE IX: Z-optimal OBZCPs for $N = 43$

Column 1	Column 2	Column 3
(668D847F75A,62EEF0F6BB4),	(74BF732EE12,43637D50C1A),	(7121C0324F5,4AB9D90D7E5)

TABLE X: Z-optimal OBZCPs for $N = 45$

Column 1	Column 2	Column 3
(1C3009533530,15C8AF20693C)	(1ADB163A02A9,118C8F16C00D)	(13589BC02838,11EF69162A6E)

TABLE XI: Z-optimal OBZCPs for $N = 47$

Column 1	Column 2	Column 3
(5D9F943CC229,4BBA9D0B6FE3)	(5375CCE48E1E,4752A49003F4)	(7815ACEB273E,579B7779603A)

TABLE XII: *Optimal* OBZCPs for $N = 49$

Column 1	Column 2
(15BCD1FAF3340,10599EA0E984A)	(1FB67150F994A,1A533E0AE3240)
(1564E7FF86694,152CC3E031B2A)	(156CC7FF8E494,1524E3E03992A)
(15BCD1FFA6614,150CCBE0E984A)	(1FB67155ACC1E,1F066B4AE3240)
(1564E7FAD33C0,107996A031B2A)	(156CC7FADB1C0,1071B6A03992A)

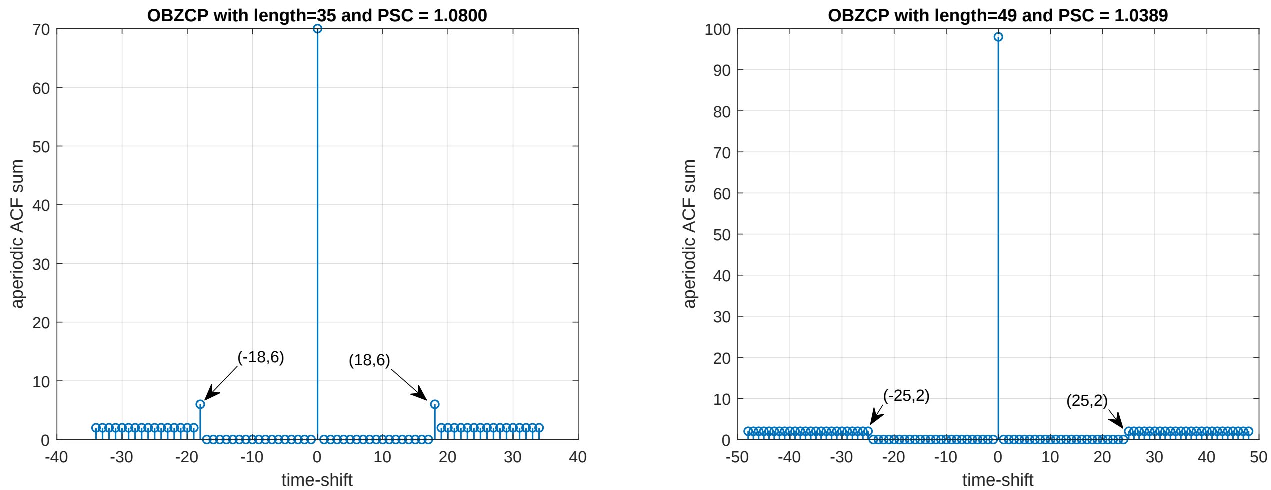

Fig. 2: Plots of AACF sums for OBZCPs, i.e., (7905A9444, 710C1A3B2) and (15BCD1FAF3340,10599EA0E984A), of lengths 35 and 49, respectively.

TABLE XIII: OBZCPs with smallest PSC values

N	PSC (g_1, g_2)	ADF (g_1)	ADF (g_2)	CDF (g_1, g_2)	g_1	g_2
3	1.4444	1.1111	1.1111	0.33333	7	5
5	1.2997	0.48	0.8	0.68	1E	16
7	1.185	0.77551	0.28571	0.71429	5D	4F
9	1.1636	0.79012	0.39506	0.60494	1E8	14C
11	1.149	0.41322	0.67769	0.61983	7C6	5DA
13	1.1005	0.07100	0.30769	0.95266	1F35	1709
15	1.1107	0.38222	0.20444	0.83111	612E	4C0A
17	1.0983	0.38754	0.609	0.61246	1FCA9	1A6E3
19	1.0931	0.40443	0.27147	0.76177	7A284	78CDA
21	1.0819	0.29932	0.46259	0.70975	1B6A87	109883
23	1.0762	0.41966	0.58601	0.58034	78B519	76D1DF
25	1.0740	0.4224	0.5248	0.6032	159FB70	11DA0CA
27	1.0664	0.36488	0.25514	0.76132	5CB287E	409159C
29	1.0608	0.27111	0.39477	0.73365	144E4E10	114693FA
31	1.0571	0.4308	0.30593	0.69407	64AFE6B9	640ACBC7
33	1.0554	0.33792	0.24977	0.76492	1E72A06CA	1A6C05630
35	1.0792	0.19102	0.25633	0.85796	44EA20C2D	42D04DDE3
37	1.0515	0.30095	0.35354	0.72535	1D29F4D110	11273940E8
39	1.0874	0.27745	0.3879	0.75937	78C84D1254	77E235C7D2
41	1.0460	0.38548	0.31886	0.69542	19FA296B9F9	1605D66B9F9
43	1.0964	0.38615	0.33423	0.73716	668D847F75A	62EEF0F6BB4
45	1.0748	0.37728	0.34963	0.7116	1ADB163A02A9	118C8F16C00D
47	1.0972	0.32866	0.41195	0.72929	5D9F943CC229	4BBA9D0B6FE3
49	1.0388	0.33986	0.39983	0.67014	156CC7FF8E494	1524E3E03992A

VI. Conclusions

This paper has introduced a number of new primitive *optimal* and *Z-optimal* OBZCPs for $27 \leq N \leq 49$ which are obtained through a computer search. By using selected algebraic properties of OBZCPs and with certain programming techniques, we have presented an algorithm with time complexity of $O(2^N)$. Next to the known OBZCP constructions for $N = 33$ and $N = 41$, we have found *optimal* sequence pairs for $N = 37$ and $N = 49$. Additionally, a list of OBZCPs with smallest PSC values have been reported.

An interesting future work of this research is to develop systematic *optimal* (or sub-*optimal*) OBZCP constructions by exploiting the obtained *short* sequence pairs. Another ambitious future direction is to carry out an exhaustive computer search for OBZCPs up to length 100 and analyze their deeper structural properties.

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