ON THE IWASAWA THEORY OF CAYLEY GRAPHS

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Abstract. This paper explores Iwasawa theory from a graph theoretic perspective, focusing on the algebraic and combinatorial properties of Cayley graphs. Using representation theory, we analyze Iwasawa-theoretic invariants within $\mathbb{Z}_\ell$-towers of Cayley graphs, revealing connections between graph theory, number theory, and group theory. Key results include the factorization of associated Iwasawa polynomials and the decomposition of μ- and λ-invariants. Additionally, we apply these insights to complete graphs, establishing conditions under which these invariants vanish.

1. Introduction

1.1. **Background and motivation.** Fix a prime number ℓ throughout. Let's consider a number field K, and let $\mathbb{Z}_\ell$ denote the ring of ℓ-adic integers. An infinite abelian extension K_∞/K is said to be a $\mathbb{Z}_\ell$-extension if $\mathrm{Gal}(K_\infty/K)$ is isomorphic to $\mathbb{Z}_\ell$ as a topological group. For each integer $n \geq 0$, set K_n/K to be the extension contained in K_∞ of degree ℓ^n. Classical Iwasawa theory studies $\mathbb{Z}_\ell$-extensions of number fields and the asymptotic behavior of certain arithmetic invariants for the fields K_n. In the late 1950s, Iwasawa [Iwa59] investigated the growth of class groups over these $\mathbb{Z}_\ell$-extensions of K, which laid the foundation for Iwasawa theory. Denote by $\mathrm{Cl}(K_n)$ the class group of K_n and $h_{K_n} := \# \mathrm{Cl}(K_n)$ its class number.

Theorem (Iwasawa). *Let K be a number field, K_∞ be a $\mathbb{Z}_\ell$-extension of K and for $n \in \mathbb{Z}_{\geq 0}$ denote by ℓ^{e_n} the exact power of ℓ that divides h_{K_n}. Then there exist invariants $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$, depending on ℓ and independent of n, such that $e_n = \lambda n + \mu \ell^n + \nu$ for $n \gg 0$.*

A natural example of a $\mathbb{Z}_\ell$-extension of K is the cyclotomic $\mathbb{Z}_\ell$-extension, denoted by K_{cyc}. For $K_\infty = K_{\mathrm{cyc}}$, we simply denote the Iwasawa invariants by $\mu_\ell(K)$, $\lambda_\ell(K)$, and $\nu_\ell(K)$. Iwasawa famously conjectured that $\mu_\ell(K) = 0$ for all number fields K. This conjecture has been proved for abelian number fields by Ferrero and Washington [FW79]. In case of general number fields, this is still an open problem.

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The Iwasawa theory of graphs was introduced by Vallières [Val21] and Gonet [Gon21, Gon22]. They explored $\mathbb{Z}_\ell$-towers of *multigraphs* and demonstrated parallels to Iwasawa's theorem regarding the asymptotic variation of graph complexities along these towers, cf. Theorem 3.7 for further details. The $\mathbb{Z}_\ell$-towers of multigraphs exhibit associated μ, λ, and ν-invariants, as well as a graph-theoretic interpretation of the *Iwasawa polynomial*. Interest in Iwasawa theory of graphs has recently gained momentum, cf. for instance, the following works [MV23, MV24, DV23, KM22, RV22, DLRV24, LM24].

In this article, we take a closer look at Cayley graphs associated to finite abelian groups. There exists a captivating relationship between the algebraic properties of the Artin-Ihara L-functions associated with these graphs and the representation theory of their underlying groups. We analyze Iwasawa-theoretic invariants associated to Cayley graphs and show that there is an analogous relationship with representation theory. This leads to intriguing connections between graph theory, number theory, and group theory.

1.2. **Main results.** Let us describe our main results in greater detail. Let G be a finite abelian group G and $S \subseteq G$ be a subset such that

- S generates G,
- $S = S^{-1}$,
- $1 \notin S$.

Set $\widehat{G}$ to denote the group of characters $\operatorname{Hom}\left(G, \mathbb{C}^\times\right)$. The Cayley graph $\operatorname{Cay}(G,S)$ of the group G with respect to the generating set S is a graph where:

- The vertex set V consists of elements of G, i.e., $V = \{v_g \mid g \in G\}$.
- There is an edge from vertex g_1 to vertex g_2 (denoted as $e(g_1, g_2)$) if and only if $g_1 g_2^{-1} \in S$.

The eigenvalues of the adjacency matrix of a Cayley graph are closely related to the irreducible representations of the group G. We mention here that these Cayley graphs are related to a sightly different construction, namely Cayley–Serre graphs. These graphs are obtained as voltage assignments on bouquet graphs. For further details, we refer to [Val21, p.445]. When G is abelian, we show that Iwasawa polynomials associated to a Cayley graph can be factored in a natural way, where each character $\psi \in \widehat{G}$ gives rise to a factor. Let $\beta : S \to \mathbb{Z}_\ell$ be a function such that the following conditions are satisfied

(1) the image of β generates $\mathbb{Z}_\ell$,
(2) $\beta(s^{-1}) = -\beta(s)$ and $\beta(1_G) = 0$,
(3) the image of β lies in $\mathbb{Z}$,
(4) there is a tuple $(h_1, \dots, h_m) \in S^m$ such that $h_1 h_2 \dots h_m \in S$ and

$$\beta(h_1 h_2 \dots h_m) \not\equiv \sum_{i=1}^{m} \beta(h_i) \pmod{\ell}. \tag{1.1}$$

Then associated to such a function is a $\mathbb{Z}_\ell$-tower of connected graphs

$$X \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_k \leftarrow \ldots \tag{1.2}$$

over $X = \mathrm{Cay}(\mathrm{G}, \mathrm{S})$, this is made precise in subsection 3.1. Let $f_X(T)$ be the associated Iwasawa polynomial (see Definition 3.5).

Theorem A (Theorem 4.5)**.** *There are explicit polynomials $P_\psi(T)$ (see (4.2)) such that up to multiplication by a unit in $\mathbb{Z}_\ell[\![T]\!]$, the Iwasawa polynomial $f_X(T)$ is equal to the product $\prod_{\psi \in \widehat{G}} P_\psi(T)$.*

Moreover, the μ- and λ-invariants associated to the $\mathbb{Z}_\ell$-tower (1.2) decompose into a sum of μ- and λ-invariants associated to $P_\psi(T)$. In greater detail, $P_\psi(T)$ is a polynomial with coefficients in a valuation ring $\mathcal{O}$ with uniformizer ϖ. Write $P_\psi(T) = \varpi^{\mu_\psi} Q_\psi(T) u_\psi(T)$, where $Q_\psi(T)$ is a distinguished polynomial and $u_\psi(T)$ is a unit in $\mathbb{Z}_\ell[\![T]\!]$. Set λ_ψ to denote the degree of $Q_\psi(T)$. Set $e \in \mathbb{Z}_{\geq 1}$ to denote the *ramification index*, defined by the relationship $(\ell) = (\varpi^e)$.

Theorem B (Theorem 4.7)**.** *Let $\kappa_\ell(X_n)$ denote the ℓ-primary part of the complexity of X_n. Then, for large enough values of n, we have that $\kappa_\ell(X_n) = \ell^{e_n}$, where*

$$e_n = \mu \ell^n + n\lambda + \nu,$$

where

$$\mu = \frac{1}{e} \sum_\psi \mu_\psi \text{ and } \lambda = \sum_\psi \lambda_\psi - 1,$$

for the ramification constant $e \in \mathbb{Z}_{\geq 0}$ defined above.

Next, we show that the Iwasawa invariants associated to the factor polynomials $P_\psi(T)$ can be suitably calculated. Under some very explicit combinatorial criteria, it is shown that μ_ψ and λ_ψ vanish. More specifically, we refer to Lemma 4.8, Lemma 4.9 and Proposition 4.10.

Finally, our findings on Cayley graphs are utilized to examine the Iwasawa theory of complete graphs. Consider a positive integer n and let K_n represent the complete graph with no self-loops on n vertices. Denote by C_n the cyclic group of order n, and define $C'_n = C_n \setminus \{1\}$. The complete graph K_n is then the Cayley graph associated to the pair (C_n, C'_n). We factor the Iwasawa polynomial, and obtain the following result on the vanishing of the μ-invariant.

Theorem C (Theorem 4.12)**.** *For $\beta : C'_n \to \mathbb{Z}_\ell$, and $n \in \mathbb{Z}_{\geq 1}$ let μ_n (resp. λ_n) be the μ-invariant (resp. λ-invariant) associated to the tower over K_n. Suppose that $\ell \nmid n$ and $\sum_{s \in C'_n} \bar{\beta}(s)^2 \neq 0$. Then, $\mu_n = 0$ and $\lambda_n = 1$.*

In our final section, we illustrate our results through two concrete examples. In fact, we are able to draw the graphs in towers, which makes them come to life, cf. (5.1) and (5.2). We note that an undirected Cayley graph $X = \mathrm{Cay}(G, S)$ can be interpreted as a Galois cover of a bouquet. The formula $f_X(T) = \prod_{\psi \in \widehat{G}} P_\psi(T)$ in Theorem 4.5 has parallels with [RV22, (4.5) on p. 20]. The latter formula, due to the second author and Daniel Vallieres, emerges in the context of proving an analogue of Kida's formula for cyclic groups within the graph-theoretic framework. This coincidence indicates that broader generalizations of our work are worth exploring. Such generalizations may encompass more general types of graphs, higher-dimensional analogues, or new classes of groups G.

1.3. **Organization.** The article is structured into five sections. Section 2 covers foundational concepts and establishes relevant notation, including the formal definition of a *multigraph*, discussion on the Galois theory of graph covers, and introduction of Artin-Ihara L-functions associated with graphs. Section 3 focuses on establishing the Iwasawa theory of graphs, which involves a specific combinatorial framework for parameterizing connected $\mathbb{Z}_\ell$-towers over a graph. This section details the Iwasawa polynomial and its connection to the asymptotic complexity growth of graphs within $\mathbb{Z}_\ell$-towers, encapsulated by the Iwasawa μ- and λ-invariants. In section 4, the main results of the article are proven. These results are illustrated through an example presented in section 5.

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2. Preliminaries

In this section, we recall some preliminary notions and set up relevant notation. The notation we use is consistent with [Val21, MV23, MV24, RV22, DLRV24]. In order to be consistent with previous work, we discuss generalities for multigraphs. The Cayley graphs we consider in this article are undirected graphs, with no self loops.

2.1. **Galois theory of covers.** We fix a prime number ℓ and set $\mathbb{Z}_\ell$ to denote the ℓ-adic integers. Let X be a finite multigraph; recall that this means that X can be described as a quadruple (V_X, E_X^+, i, ι), where

- $V_X = \{v_1, \ldots, v_{g_X}\}$ is a finite set of *vertices*,
- E_X^+ is a collection of *edges* between vertices.
- The set of edges is equipped with an *incidence function*
$$i : E_X^+ \to V_X \times V_X.$$
Here, the interpretation is that an edge e starts at v_i and ends at v_j if $i(e) = (v_i, v_j)$,
- $\iota : E_X^+ \to E_X^+$ is the *inversion map*.

In addition, it is required that the following compatibility relations hold

(1) ι^2 is the identity on E_X^+,

(2) $\iota(e) \neq e$ for all $e \in E_X^+$,
(3) $i(\iota(e)) = \tau(i(e))$,

where $\tau : V_X \times V_X \to V_X \times V_X$ is the map defined by $\tau(e_1, e_2) := (e_2, e_1)$. We write $e \sim e'$ if $e' = \iota(e)$, and denote by E_X the set of equivalence classes for this relation. We think of E_X^+ as edges with orientation (or directed edges) and E_X as undirected edges. The map $\pi : E_X^+ \to E_X$ is the map that assigns to e its equivalence class. When using the word "edge" we shall mean an element in E_X^+. We shall henceforth simply use the term "*graph*" to refer to a multigraph. An edge from v_i to v_i is referred to as a *loop*. We also define the incidence matrix $A_X = (a_{i,j})$ of the graph X, where $a_{i,j}$ is the number of edges from v_i to v_j. We define source and target maps $o, t : E_X^+ \to V_X$ to be the compositions of i with the projections to the first and second factor of $V_X \times V_X$ respectively. Observe that the source map o assigns to each directed edge its starting vertex, while the target map t assigns to each edge its ending vertex. In particular, e is a loop precisely if $o(e) = t(e)$. Note that (V_X, E_X) is an undirected graph. For $v \in V_X$, let $E_{X,v}^+ := \{e \in E_X^+ \mid o(e) = v\}$, i.e., the set of directed edges emanating from v. The *degree* of v is defined as the number of edges emanating from v, i.e., $\deg(v) := \# E_{X,v}^+$. The betti numbers of X are defined as follows

$$b_i(X) := \operatorname{rank}_{\mathbb{Z}} H_i(X, \mathbb{Z}).$$

The *Euler characteristic* is defined as follows $\chi(X) := b_0(X) - b_1(X)$. When X is connected, $b_0(X) = 1$ and $b_1(X) = \# E_X - \# V_X + 1$. We have that $\chi(X) = \# V_X - \# E_X$.

Assumption 2.1. *It will be assumed throughout that all our multigraphs are connected with no vertices having degree equal to* 1*. Moreover, we assume that* $\chi(X) \neq 0$*, i.e., the graph is not a cycle graph.*

The divisor group $\operatorname{Div}(X)$ consists of formal sums of the form $D = \sum_{v \in V_X} n_v v$, where $n_v \in \mathbb{Z}$ for all v. It is the free abelian group on the vertices V_X of X. The degree of D is the sum $\deg(D) := \sum_v n_v \in \mathbb{Z}$, which defines a homomorphism

$$\deg : \operatorname{Div}(X) \to \mathbb{Z},$$

the kernel of which is denoted $\operatorname{Div}^0(X)$. Let $\mathcal{M}(X)$ be the abelian group of $\mathbb{Z}$-valued functions on V_X. We note that $\mathcal{M}(X)$ can be freely generated by the characteristic functions χ_v defined by

$$\chi_v(v') := \begin{cases} 1 \text{ if } v' = v; \\ 0 \text{ otherwise.} \end{cases}$$

Set $\operatorname{div}(\chi_v) := \sum_{w \in V_X} \rho_w(v) w$, where

$$\rho_w(v) := \begin{cases} \operatorname{val}_X(v) - 2 \cdot \text{ number of loops at } v & \text{if } w = v; \\ - \text{ number of edges from } w \text{ to } v & \text{if } w \neq v. \end{cases}$$

With this notation in hand, we extend $\operatorname{div}(\chi_v)$ to a map $\operatorname{div} : \mathcal{M}(X) \to \operatorname{Div}(X)$ as follows. For $f \in \mathcal{M}(X)$, one has that

$$\operatorname{div}(f) = -\sum_{v} m_v(f) \cdot v,$$

where

$$m_v(f) = \sum_{e \in E^+_{X,v}} \left(f(t(e)) - f(o(e)) \right).$$

Note that $\operatorname{div}(f)$ has degree 0. We let $\operatorname{Pr}(X)$ be the image of div, and set $\operatorname{Pic}^0(X) := \operatorname{Div}^0(X)/\operatorname{Pr}(X)$. Set κ_X to be the cardinality of $\operatorname{Pic}^0(X)$. This quantity is analogous to the notion of the *class number of a number ring* and referred to in the literature as the *complexity* of X. For a more comprehensive account, see [CP18].

We come to the notion of a Galois cover of graphs. First, we introduce the notion of a *morphism* $f : Y \to X$ between graphs. This consists of a pair (f_V, f_E), where $f_V : V_Y \to V_X$ and $f_E : E_Y^+ \to E_X^+$ are functions satisfying the following compatibility relations:

(a) $f_V(o(e)) = o(f_E(e))$,
(b) $f_V(t(e)) = t(f_E(e))$,
(c) $\iota\left(f_E(e)\right) = f_E\left(\iota(e)\right)$.

We use f to denote f_V or f_E, depending on the context.

Definition 2.2. *Let X and Y be two graphs and $f : Y \to X$ be a graph morphism. If f satisfies the following conditions:*

(a) $f : V_Y \to V_X$ is surjective,
(b) for all $w \in V_Y$, the restriction $f|_{E^+_{Y,w}}$ induces a bijection

$$f|_{E^+_{Y,w}} : E^+_{Y,w} \xrightarrow{\approx} E^+_{X,f(w)},$$

then f is said to be a cover*. The cover $f : Y \to X$ is called* Galois *if the following two conditions are satisfied.*

(1) The graphs X and Y are connected.
(2) The group $\operatorname{Aut}_f(Y/X) := \{\sigma \in \operatorname{Aut}(Y) : f \circ \sigma = f\}$ acts transitively on the fiber $f^{-1}(v)$ for all $v \in V_X$.

We denote a Galois cover $f : Y \to X$ also by Y/X and suppress the role of the covering map. Moreover, we set $\operatorname{Gal}(Y/X) := \operatorname{Aut}_f(Y/X)$.

2.2. **Artin–Ihara L-functions.** Before introducing the Iwasawa theory of graphs, we discuss the role of Artin-Ihara L-functions. These are essentially graph-theoretic analogs of Artin L-functions. The standard reference for the content of this subsection is [Ter11].

Let X be a graph and $a_1, \ldots, a_k \in E_X^+$ such that for $i < k$, one has that $t(a_i) = o(a_{i+1})$. Then, the sequence $a_1, \ldots, a_k$ gives rise to a *walk* $w = a_1 a_2 \ldots a_k$.

Here, k is the *length* of w and is denoted by $l(w)$. The walk w is said to have a *backtrack* if $\iota(a_i) = a_{i+1}$ for some $i < k$. It is said to have a *tail* if $a_k = \iota(a_1)$. A *cycle* is a walk such that $o(a_1) = t(a_k)$. A cycle $\mathfrak{c}$ is said to be a prime if it has no backtrack or tail, and there is no cycle u and integer $f > 1$ for which $\mathfrak{c} \neq u^f$. In other words, $\mathfrak{c}$ is a prime if one can go around it only once. Let Y/X be a Galois cover with abelian Galois group $G := \mathrm{Gal}(Y/X)$. Set $\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ to be the group of characters of G. Given a character $\psi \in \widehat{G}$, the Artin-Ihara L-function is defined as follows:

$$L_{Y/X}(u, \psi) := \prod_{\mathfrak{c}} \left(1 - \psi\left(\frac{Y/X}{\mathfrak{c}}\right) u^{l(\mathfrak{c})}\right)^{-1}.$$

In the above product, $\mathfrak{c}$ runs over all primes of X and $\left(\frac{Y/X}{\mathfrak{c}}\right) \in G$ refers to the Frobenius automorphism at $\mathfrak{c}$ (cf. [Ter11, Definition 16.1]). When $Y \to X$ is the identity $X \xrightarrow{\mathrm{Id}} X$, we recover the *Ihara zeta function* $\zeta_X(u) := L_{X/X}(u, 1)$.

Given a connected graph X, let $\chi(X)$ denote its *Euler characteristic*, which is defined as follows:

$$\chi(X) = |V_X| - |E_X|.$$

It follows from the Assumption 2.1 that $\chi(X) < 0$.

Let Y/X represent a covering of graphs with the property that it is abelian, having an automorphism group denoted as G. Suppose, for each i from 1 to g_X, w_i is a specific vertex chosen from the fiber of v_i. Considering σ as an element of G, we define the matrix $A(\sigma)$ as a $g_X \times g_X$ matrix, with its entries denoted by $a_{i,j}(\sigma)$. This is determined as follows:

$$a_{i,j}(\sigma) = \begin{cases} \text{Twice the number of loops at the vertex } w_i, & \text{if } i = j \text{ and } \sigma = 1; \\ \text{The number of edges connecting } w_i \text{ to } w_j^\sigma, & \text{otherwise.} \end{cases}$$

For $\psi \in \widehat{G}$ (the character group of G), we define A_ψ as the twisted sum of the $A(\sigma)$ matrices

$$A_\psi = A_{\psi,X} := \sum_{\sigma \in G} \psi(\sigma) \cdot A(\sigma).$$

Let D be the matrix $(d_{i,j})$ with

$$d_{i,j} := \begin{cases} 0 & \text{if } i \neq j; \\ \deg(v_i) & \text{if } i = j. \end{cases}$$

If Y/X serves as an abelian covering of multigraphs and ψ represents a character of $G = \mathrm{Gal}(Y/X)$, then according to the three-term determinant formula from [Ter11, Theorem 18.15] applied to the Artin-Ihara L-function, we have:

$$L_{Y/X}(u,\psi)^{-1} = (1-u^2)^{-\chi(X)} \cdot \det(I - A_\psi u + (D-I)u^2). \tag{2.1}$$

We let $h_X(u,\psi) := \det\left(I - A_\psi u + (D-I)u^2\right)$ and for ease of notation, set $h_X(u) := h_X(u,1)$. The result below gives us an explicit relationship between the derivative of h_X and the complexity of X.

Theorem 2.3. *For a graph X satisfying Assumption 2.1, one has that*

$$h'_X(1) = -2\chi(X)\kappa_X.$$

Proof. For a proof of the result, we refer to [Nor98] or [HMSV24, Theorem 2.11]. □

For abelian covers, the Artin-Ihara L-functions described satisfy a relation as a consequence of the Artin formalism, as the following result shows.

Theorem 2.4. *Let Y/X be an abelian Galois cover of graphs with $G = \mathrm{Gal}(Y/X)$, then one has*

$$\zeta_Y(u) = \zeta_X(u) \times \prod_{\psi \in \widehat{G}, \psi \neq 1} L_{Y/X}(u,\psi),$$

where $1 \in \widehat{G}$ denotes the trivial character.

Proof. For a proof of this result, see [Ter11, Corollary 18.11]. □

The Artin formalism described above gives us a formula for the complexity of Y. This formula is expressed in terms of the complexity of X and a product of special values of the *twisted polynomials* $h_X(u,\psi)$.

Corollary 2.5. *Let X be a graph satisfying Assumption 2.1 and Y/X be an abelian Galois cover of X with $G := \mathrm{Gal}(Y/X)$. Then, the following relationship holds*

$$|G|\kappa_Y = \kappa_X \prod_{\psi \in \widehat{G}, \psi \neq 1} h_X(1,\psi).$$

Proof. The result is an immediate consequence of Theorem 2.3 and Theorem 2.4, see [Val21, p.440] for further details. □

Remark 2.6. *Some remarks are in order.*

1. *It follows from the above relation in particular that $h_X(1,\psi) \neq 0$ for all $\psi \in \widehat{G}$ such that $\psi \neq 1$.*
2. *The special value of the Artin-Ihara L-function at $u = 1$ has been studied by Hammer, Mattman, Sands and Vallieres, cf.* [HMSV24].

3. Iwasawa theory of graphs

The Iwasawa theory of graphs is a branch of mathematics that combines topology, combinatorics, and the Galois theory of covers of graphs. In this section, we summarize the key ideas and set up relevant notion. For a comprehensive account, please see [Val21, Gon21, MV23, MV24].

3.1. **$\mathbb{Z}_\ell$-covers of graphs.** Throughout this section, we choose a section

$$\gamma : E_X \to E_X^+$$

of π and set $\mathcal{S} := \gamma(E_X)$. This set $\mathcal{S}$ is referred to as an *orientation* of X, the understanding here is that for any edge, the orientation prescribes a direction to this edge.

Definition 3.1. *Let $\mathcal{G}$ be a finite group, a* voltage *assignment valued in $\mathcal{G}$ is a function $\alpha : \mathcal{S} \to \mathcal{G}$. We extend α to all of E_X^+ by setting $\alpha(\iota(e)) := \alpha(e)^{-1}$.*

Let $(\mathcal{S}, \alpha)$ be a pair consisting of an orientation $\mathcal{S}$ and a voltage assignment $\alpha : \mathcal{S} \to \mathcal{G}$. Let $(\mathcal{S}', \alpha')$ be another pair. Then we define as equivalence $(\mathcal{S}, \alpha) \sim (\mathcal{S}', \alpha')$ if the extensions of α and α' to E_X^+ coincide. Given a pair $(\mathcal{S}, \alpha)$ and another orientation $\mathcal{S}'$, then it is easy to see that there is a unique voltage assignment $\alpha' : \mathcal{S}' \to \mathcal{G}$ such that $(\mathcal{S}, \alpha) \sim (\mathcal{S}', \alpha')$. Associated to the datum $(\mathcal{G}, \mathcal{S}, \alpha)$ is a graph $X(\mathcal{G}, \mathcal{S}, \alpha)$, which we describe as follows. The set of vertices of $X(\mathcal{G}, \mathcal{S}, \alpha)$ is $V_X \times \mathcal{G}$. On the other hand, the set of edges is identified with the set $E_X^+ \times \mathcal{G}$, where $(e, \sigma) \in E_X^+ \times \mathcal{G}$ is the edge that connects $(o(e), \sigma)$ to $(t(e), \sigma \cdot \alpha(e))$. On the other hand, the inversion map is given by $\overline{(e, \sigma)} = \big(\iota(e), \sigma \cdot \alpha(e)\big)$. It is clear that if $(\mathcal{S}, \alpha) \sim (\mathcal{S}', \alpha')$, then $X(\mathcal{G}, \mathcal{S}, \alpha) = X(\mathcal{G}, \mathcal{S}', \alpha')$.

This operation is functorial and results in Galois covers of X. Consider a voltage assignment $\alpha : \mathcal{S} \to \mathcal{G}$ and a group homomorphism $f : \mathcal{G} \to \mathcal{G}_1$. Then, there is a natural morphism of multigraphs denoted $f_* : X(\mathcal{G}, \mathcal{S}, \alpha) \to X(\mathcal{G}_1, \mathcal{S}, f \circ \alpha)$. This morphism is defined on vertices and edges as follows

$$f_*(v, \sigma) = (v, f(\sigma)) \text{ and } f_*(e, \sigma) = (e, f(\sigma)).$$

If both $X(\mathcal{G}, \mathcal{S}, \alpha)$ and $X(\mathcal{G}_1, \mathcal{S}, f \circ \alpha)$ are connected, and f is surjective, then f_* is a cover according to Definition 2.2. Moreover, it is a Galois cover, with the group of covering transformations being isomorphic to $\ker(f)$. In particular, if $f : \mathcal{G} \to \{1\}$ represents the group morphism into the trivial group and both X and $X(\mathcal{G}, \mathcal{S}, \alpha)$ are connected, then a Galois cover, denoted $f_* : X(\mathcal{G}, \mathcal{S}, \alpha) \to X$, is obtained, with the group of covering transformations isomorphic to $\mathcal{G}$.

We now recall the notion of a $\mathbb{Z}_\ell$-tower of multigraphs.

Definition 3.2. *Let ℓ be a rational prime, and let X be a connected graph. A $\mathbb{Z}_\ell$-tower over X consists of a series of covers of connected graphs*

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n \leftarrow \ldots$$

such that for every positive integer n, the cover X_n/X obtained by composing the covers is Galois with Galois group $\mathrm{Gal}(X_n/X)$ isomorphic to $\mathbb{Z}/\ell^n\mathbb{Z}$.

We call a function $\alpha : \mathcal{S} \to \mathbb{Z}_\ell$ a $\mathbb{Z}_\ell$-valued voltage assignment. Let $\alpha_{/n}$ denote the mod-ℓ^n reduction of α. This yields a $\mathbb{Z}_\ell$-tower over X:

$$X \leftarrow X(\mathbb{Z}/\ell\mathbb{Z}, \mathcal{S}, \alpha_{/1}) \leftarrow X(\mathbb{Z}/\ell^2\mathbb{Z}, \mathcal{S}, \alpha_{/2}) \leftarrow \ldots \leftarrow X(\mathbb{Z}/\ell^k\mathbb{Z}, \mathcal{S}, \alpha_{/k}) \leftarrow \ldots.$$

It turns out that all $\mathbb{Z}_\ell$-towers arise from voltage assignments, cf. [RV22, section 2.3] for details. Setting $t := \#\mathcal{S}$, we choose an ordering $\mathcal{S} = \{s_1, \dots, s_t\}$ and represent α as a vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t) \in \mathbb{Z}_\ell^t$, where α_i denotes $\alpha(s_i)$. To relate growth patterns in Picard groups to Iwasawa invariants, it is necessary to impose certain conditions on the multigraphs $X(\mathbb{Z}/\ell^n\mathbb{Z}, \mathcal{S}, \alpha_{/n})$.

Assumption 3.3. *We assume that the derived multigraphs $X(\mathbb{Z}/\ell^n\mathbb{Z}, S, \alpha_{/n})$ are connected for all $n \geq 0$.*

We describe an explicit condition that $X(\mathcal{G}, \mathcal{S}, \alpha)$ is connected. If $w = a_1 a_2 \dots a_n$ is a walk in X, then we define

$$\alpha(w) = \alpha(e_1) \cdot \dots \cdot \alpha(e_n) \in \mathcal{G}.$$

Recall that $\alpha : \mathcal{S} \to \mathcal{G}$ satisfies the condition that $\alpha(\iota(e)) = \alpha(e)^{-1}$. This implies that if c_1 and c_2 are homotopically equivalent, then $\alpha(c_1) = \alpha(c_2)$. Choose a vertex $v_0 \in V_X$, and let $\pi_1(X, v_0)$ be the fundamental group of X with base-point v_0. We deduce that α induces a group homomorphism

$$\rho_\alpha : \pi_1(X, v_0) \to \mathcal{G}, \tag{3.1}$$

defined by $\rho_\alpha([\gamma]) = \alpha(\gamma)$.

Theorem 3.4. *Assume that X is a connected graph. Then, $X(\mathcal{G}, \mathcal{S}, \alpha)$ is connected if and only if ρ_α is surjective.*

Proof. For a proof of this result, cf. [RV22, Theorem 2.11]. □

3.2. **The Iwasawa polynomial.** Let X be a connected graph. The matrix $D_X = (d_{i,j})$ is the $g_X \times g_X$ matrix for which

$$d_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

The difference matrix $Q_X := D_X - A_X$ is referred to as the *Laplacian matrix*.

The rings $\mathbb{Z}_\ell[x]$ and $\mathbb{Z}_\ell[\![T]\!]$ denote the polynomial ring and formal power series ring with coefficients in $\mathbb{Z}_\ell$, respectively. The ring $\mathbb{Z}_\ell[x; \mathbb{Z}_\ell]$ consists of expressions in the form $f(x) = \sum_a c_a x^a$, where $a \in \mathbb{Z}_\ell$ and the coefficients c_a belong to $\mathbb{Z}_\ell$. Let $\alpha : \mathcal{S} \to \mathbb{Z}_\ell$ be a function and extend α to E_X^+ such that $\alpha(\iota(e)) = -\alpha(e)$. The matrix $M(x)$ associated to the pair (X, α) has entries in $\mathbb{Z}_\ell[x; \mathbb{Z}_\ell]$ and is defined by

$$M(x) = M_{X,\alpha}(x) := D_X - \left(\sum_{\substack{e \in E_X^+ \\ i(e) = (v_i, v_j)}} x^{\alpha(e)} \right)_{(i,j)}. \tag{3.2}$$

We also introduce the notation $\binom{b}{n}$ for $\frac{b(b-1)\dots(b-n+1)}{n!}$, and define $(1+T)^b$ as the formal power series $\sum_{n=0}^{\infty} \binom{b}{n} T^n$.

Definition 3.5. *With respect to notation above, the* Iwasawa polynomial *associated to the $\mathbb{Z}_\ell$-tower defined by α is defined as follows*

$$f_{X,\alpha}(T) := \det M(1+T) \in \mathbb{Z}_\ell[\![T]\!].$$

As defined above, the $f_{X,\alpha}(T)$ is not in general a polynomial, however, becomes one upon multiplication by a suitably chosen unit in $\mathbb{Z}_\ell[\![T]\!]$. This unit can be taken to be a suitably large power of $(1+T)$.

Let ζ_{ℓ^n} be a primitive ℓ^n-th root of unity. For the tower
(3.3)

$$X \leftarrow X(\mathbb{Z}/\ell\mathbb{Z}, \mathcal{S}, \alpha_{/1}) \leftarrow X(\mathbb{Z}/\ell^2\mathbb{Z}, \mathcal{S}, \alpha_{/2}) \leftarrow \ldots \leftarrow X(\mathbb{Z}/\ell^n\mathbb{Z}, \mathcal{S}, \alpha_{/n}) \leftarrow \ldots$$

related to α, for any positive integer n, it follows from [MV24, Corollary 5.6] that

$$f_{X,\alpha}(1-\zeta_{\ell^n}) = h_X(1, \psi_n), \tag{3.4}$$

where ψ_n denotes the character of $\mathbb{Z}/\ell^n\mathbb{Z}$, defined by $\psi_n(\bar{1}) = \zeta_{\ell^n}$.

Lemma 3.6. *The Iwasawa polynomial $f_{X,\alpha}(T)$ is divisible by T.*

Proof. We find that $f_{X,\alpha}(0) = \det\big(M(1+0)\big) = \det(Q_X)$. It is easy to see that Q_X is a singular matrix, indeed, $u := (1,1,\ldots,1)^t$ is in its null-space. Therefore, $f_{X,\alpha}(0) = 0$, in other words, T divides $f_{X,\alpha}(T)$. □

Note that $f_{X,\alpha}(T)$ is a Laurent series in $(1+T)$. In light of the above result, we write $f_{X,\alpha}(T) = (1+T)^{-m} T g_{X,\alpha}(T)$, where $m \in \mathbb{Z}_{\geq 0}$ is the smallest number such that $g_{X,\alpha}(T)$ above is a polynomial. Of significance are the Iwasawa invariants that are associated to the Iwasawa polynomial. We recall that a polynomial $g(T) \in \mathbb{Z}_\ell[T]$ is a *distinguished polynomial* if it is a monic polynomial and all the non-leading coefficients of $g(T)$ are divisible by ℓ. It follows from the ℓ-adic Weierstrass Preparation theorem, that there is a factorization of the Iwasawa polynomial

$$g_{X,\alpha}(T) = \ell^\mu P(T) u(T),$$

where $P(T)$ is a distinguished polynomial and $u(T)$ is a unit in $\mathbb{Z}_\ell[\![T]\!]$. Since $u(T)$ is a unit in $\mathbb{Z}_\ell[\![T]\!]$, the constant term $u(0)$ is a unit in $\mathbb{Z}_\ell$. In particular, we have that $\deg g \geq 1$. The invariants $\mu_\ell(X,\alpha) := \mu$ and $\lambda_\ell(X,\alpha) := \deg P(T)$ are the μ- and λ-invariant associated to the tower (3.3).

Theorem 3.7 (Gonet [Gon21, Gon22], Vallieres [Val21], McGown–Vallieres [MV23, MV24])**.** *Let X be a graph and α a $\mathbb{Z}_\ell$-valued voltage assignment such that Assumptions 2.1 and 3.3 are satisfied. For $n \in \mathbb{Z}_{\geq 1}$, set $X_n := X(\mathbb{Z}/\ell^n\mathbb{Z}, \mathcal{S}, \alpha_{/n})$ and $\kappa_\ell(X_n)$ denote the ℓ-primary part of the complexity of X_n. Let $\mu := \mu_\ell(X,\alpha)$ and $\lambda := \lambda_\ell(X,\alpha)$. Then, there exists $n_0 > 0$ and $\nu \in \mathbb{Z}$ such that for all $n \geq n_0$,*

$$\kappa_\ell(X_n) = \ell^{(\ell^n \mu + n\lambda + \nu)}.$$

Proof. The result above is [MV24, Theorem 6.1]. □

Remark 3.8. *Vallieres proved a restricted version of the above result for certain $\mathbb{Z}_\ell$-towers consisting of Cayley–Serre multigraphs* [Val21, Theorem 5.6]*. This result was subsequently generalized by McGown and Vallieres. The result was proven independently via a different method by Gonet.*

4. Cayley graphs and Iwasawa theory

4.1. **A factorization of the Iwasawa polynomial.** Let G be a finite abelian group and S be a subset of G such that

- S generates G,
- $S = S^{-1}$,
- $1 \notin S$.

We shall set $r := \#S$. Associated with the pair (G, S) is a Cayley graph, denoted by $X := \mathrm{Cay}(G, S)$. This Cayley graph is defined by taking the vertex set $V = \{v_g \mid g \in G\}$ as the elements of G and connecting two vertices g_1 and g_2 by an edge $e(g_1, g_2)$ if $g_1 g_2^{-1} \in S$. Since $S = S^{-1}$, it follows that there is an edge e starting at g_1 and ending at g_2, precisely if there is an edge $\iota(e)$ starting at g_2 and ending at g_1. Since $1_G \notin S$, it follows that $\mathrm{Cay}(G, S)$ has no loops. Throughout, we assume that the Assumption 2.1 is satisfied.

Definition 4.1. *We shall consider voltage assignments that arise from functions on S. Let ℓ be a prime number and $\beta : S \to \mathbb{Z}_\ell$ be a function such that:*

(1) the image of β generates $\mathbb{Z}_\ell$ (as a $\mathbb{Z}_\ell$-module),
(2) $\beta(s^{-1}) = -\beta(s)$ and $\beta(1_G) = 0$,
(3) the image of β lies in $\mathbb{Z}$,
(4) there exists $m > 0$ and a tuple $(h_1, \dots, h_m) \in S^m$ such that $h_1 h_2 \dots h_m \in S$ and

$$\beta(h_1 h_2 \dots h_m) \not\equiv \sum_{i=1}^{m} \beta(h_i) \pmod{\ell}. \tag{4.1}$$

We define a $\mathbb{Z}_\ell$-valued voltage assignment $\alpha = \alpha_\beta : E_X^+ \to \mathbb{Z}_\ell$ *by $\alpha(e) := \beta(g_1 g_2^{-1})$ where e is the edge joining v_{g_1} to v_{g_2}.*

Remark 4.2. *The condition (1) above is a consequence of (4). It follows from (2) that $\alpha(\iota(e)) = \beta(g_2 g_1^{-1}) = -\beta(g_1 g_2^{-1}) = -\alpha(e)$. Condition (3) simplifies some of our calculations, see for instance in the definition of m_β in the formula for $P_\psi(T)$ in* (4.2)*. Finally, (4) is used in the Proposition below to establish connectedness of the graphs in towers.*

Proposition 4.3. *With respect to notation above, the graphs $X(\mathbb{Z}/\ell^k, S, \alpha_{/k})$ are connected for all $k \geq 0$ (i.e. the Assumption 3.3 is satisfied).*

Proof. We deduce the result from Theorem 3.4 by showing that the homomorphism $\rho_\alpha^k : \pi_1(X, v_0) \to \mathbb{Z}/\ell^k\mathbb{Z}$ is surjective for all $k \geq 1$. Fix a tuple $(h_1, \dots, h_m)$ such that

$$\beta(h_1h_2\dots h_m)\not\equiv\sum_{i=1}^m\beta(h_i)\pmod{\ell}.$$

We take v_0 to be the vertex v_{1_G}. Consider the sequence of elements

$$a_0:=1_G, a_1:=h_1, a_2:=h_1h_2,\dots,a_m:=h_1h_2\dots h_m.$$

The sequence of elements gives rise to a loop γ starting and ending at $v_0=v_{a_0}$ traversing the vertices $v_{a_1},\dots,v_{a_m}$. For $i=1,\dots,m$, let e_i be the edge joining a_{i-1} to a_i and e_{m+1} be the edge joining a_m back to a_0. We find that

$$\alpha(e_i)=\begin{cases}-\beta(h_i) & \text{if } i<m+1;\\ \beta(h_1\dots h_m) & \text{if } i=m+1.\end{cases}$$

Therefore, $\rho_\alpha^k(\gamma)=-\sum_{i=1}^m\beta(h_i)+\beta(h_1\dots h_m)$ and thus, $\rho_\alpha^k(\gamma)\in\left(\mathbb{Z}/\ell^k\mathbb{Z}\right)^\times$. Therefore, the map ρ_α^k is surjective for all $k\geq 1$. □

The condition (4.1) can be specialized to a neater condition as follows.

Proposition 4.4. *Let $\beta: S\to\mathbb{Z}_\ell$ be a function satisfying the conditions (1)-(3) of Definition 4.1. Assume moreover that there exists $h\in S$ with order $M>1$ such that the following conditions are satisfied*

(1) $(M,\ell)=1$,

(2) $\beta(h)\not\equiv 0 \mod \ell$.

Then, the congruence condition (4.1) is satisfied.

Proof. Taking $h_1=h_2=\dots=h_{M-1}=h$, the congruence condition becomes

$$\beta(h^{M-1})\not\equiv(M-1)\beta(h)\pmod{\ell}.$$

Note that $h^{M-1}=h^{-1}$ and hence is contained in S. The condition becomes $-\beta(h)\not\equiv(M-1)\beta(h)\pmod{\ell}$, i.e., $\ell\nmid M\beta(h)$. Since $(M,\ell)=1$, and $\beta(h)\not\equiv 0\pmod{\ell}$, the result follows. □

We choose an ordering and write $G=\{g_1,\dots,g_n\}$ and set $v_i:=v_{g_i}$. For $g\in G$, set

$$\delta_S(g):=\begin{cases}1 & \text{if } g\in S;\\ 0 & \text{if } g\notin S.\end{cases}$$

Recall that a voltage assignment $\alpha: E_X^+\to\mathbb{Z}_\ell$ gives rise to a $\mathbb{Z}_\ell$-tower over X and that

$$f_{X,\alpha}(T)=\det\left(\mathrm{M}_{X,\alpha}(1+T)\right)=\det\left(d_{i,j}-\delta_S(g_ig_j^{-1})(1+T)^{\beta(g_ig_j^{-1})}\right)_{i,j}$$

is the associated Iwasawa polynomial.

Let $\bar{\mathbb{Q}}$ (resp. $\bar{\mathbb{Q}}_\ell$) be a choice of algebraic closure of $\mathbb{Q}$ (resp $\mathbb{Q}_\ell$). Choose an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_\ell$ and thus view any algebraic number as an element in $\bar{\mathbb{Q}}_\ell$. Let K be the field generated by $|G|$-th roots of unity in $\bar{\mathbb{Q}}_\ell$. Set $\mathcal{O}$ to denote its valuation ring and ϖ its uniformizer. Thus any character $\psi \in \widehat{G}$ takes values in $\mathcal{O}^\times$. Take k to denote the residue field $\mathcal{O}/(\varpi)$ and set $\bar{\psi} : G \to k^\times$ to denote the mod-ϖ reduction of ψ. Recall that $r := \#S$. For $\psi \in \widehat{G}$, set

$$P_\psi(T) = P_{X,\alpha,\psi}(T) := (1+T)^{m_\beta} r - \sum_{s\in S} (1+T)^{m_\beta - \beta(s)} \psi(s) \in \mathcal{O}[T], \tag{4.2}$$

where $m_\beta := \max\{\beta(s) \mid s \in S\}$. Given elements $f, g \in \mathcal{O}[\![T]\!]$, write $f \sim g$ to mean that $f = ug$ for some unit u of $\mathcal{O}[\![T]\!]$.

Theorem 4.5. *Let G be a finite abelian group and S a subset of G such that*

- *S generates G,*
- *$S = S^{-1}$,*
- *$1 \notin S$.*

Let $X = \mathrm{Cay}(G, S)$ be the associated Cayley graph. Let $\beta : S \to \mathbb{Z}_\ell$ be a function satisfying the conditions (1)–(4) of Definition 4.1 and

$$\alpha = \alpha_\beta : E_X^+ \to \mathbb{Z}_\ell$$

be the associated $\mathbb{Z}_\ell$-valued voltage assignment. Then, we have that

$$f_{X,\alpha}(T) \sim \prod_{\psi \in \widehat{G}} P_\psi(T).$$

Proof. For $\psi \in \widehat{G}$, set $v_\psi = (\psi(g_1), \dots, \psi(g_n))$ observe that

$$\left(M_{X,\alpha}(x) v_\psi\right)_i = r\psi(g_i) - \sum_{j; g_i g_j^{-1} \in S} x^{\beta(g_i g_j^{-1})} \psi(g_j) = r\psi(g_i) - \sum_{s\in S} x^{\beta(s)} \psi(g_i s^{-1}).$$

Therefore,

$$M_{X,\alpha}(x) v_\psi = \left(r - \sum_{s\in S} x^{\beta(s)} \psi(s^{-1}) \right) v_\psi. \tag{4.3}$$

Choose an enumeration of $\widehat{G} = \{\psi_1, \dots, \psi_j, \dots, \psi_n\}$, and let $\mathbf{F} := \left(\psi_j(g_i)\right)_{i,j}$. It follows from the orthogonality relations that $\mathbf{F}$ is invertible with $\mathbf{F}^{-1} = \left(\frac{1}{n}\bar{\psi}_i(g_j)\right)_{i,j}$. The relation (4.3) implies that $v_{\psi_1}, \dots, v_{\psi_n}$ is a basis of eigenvectors and $\mathbf{F}$ is the change of basis matrix. Thus, (4.3) can be rephrased as follows

$$M_{X,\alpha}(x) = \mathbf{F} B_{X,\alpha}(x) \mathbf{F}^{-1},$$

where

$$B_{X,\alpha}(x) = \left(b_{i,j}(x)\right),$$

where

$$b_{i,j}(x) = \begin{cases} 0 & \text{if } i \neq j; \\ \left(r - \sum_{s\in S} x^{\beta(s)} \psi_i(s^{-1})\right) & \text{if } i = j. \end{cases}$$

Therefore,

$$\begin{aligned} f_{X,\alpha}(T) &= \det\left(M_{X,\alpha}(1+T)\right) \\ &= \det\left(D_{X,\alpha}(1+T)\right) \\ &= \prod_{\psi\in\widehat{G}}\left(r - \sum_{s\in S}(1+T)^{-\beta(s)}\psi(s)\right), \\ &= (1+T)^{-m_\beta n}\prod_{\psi\in\widehat{G}} P_\psi(T); \end{aligned}$$

since $(1+T)$ is a unit in $\mathbb{Z}_\ell[\![T]\!]$, the result follows. $\square$

Remark 4.6. *Let $\beta, \beta' : S \to \mathbb{Z}_\ell$ be functions satisfying the conditions of Definition 4.1. Suppose that there is a constant $c \in \mathbb{Z}$ such that $\ell \nmid c$ and $\beta(s) = c\beta'(s)$ for all $s \in S$. Let $\alpha := \alpha_\beta$ and $\alpha' := \alpha_{\beta'}$ be the associated $\mathbb{Z}_\ell$-valued functions. Then, it is easy to see that*

$$f_{X,\alpha'}(T) = f_{X,\alpha}\left((1+T)^c - 1\right).$$

Note that $T \sim (1+T)^c - 1$ and thus it follows from an application of the ℓ-adic Weierstrass preparation theorem that

$$\mu_\ell(X,\alpha) = \mu_\ell(X,\alpha') \text{ and } \lambda_\ell(X,\alpha) = \lambda_\ell(X,\alpha').$$

Thus, as far as the computation of Iwasawa invariants is concerned, we may as well assume that there are no common divisors of the values $\{\beta(s) \mid s \in S\}$ (other than ± 1).

4.2. **The Iwasawa invariants μ_ψ and λ_ψ and their properties.** Write $P_\psi(T) = \varpi^{\mu_\psi} Q_\psi(T) u_\psi(T)$, where $Q_\psi(T)$ is a distinguished polynomial and $u_\psi(T) \in \mathcal{O}[\![T]\!]^\times$. Set λ_ψ to denote the λ-invariant of $P_\psi(T)$, defined as follows $\lambda_\psi := \deg\left(Q_\psi(T)\right)$. Let e denote the ramification index, defined by $(\ell) = (\varpi^e)$ in $\mathcal{O}$. Note that $P_1(0) = r - \sum_{s\in S} 1(s) = r - r = 0$, and thus, T divides $P_1(T)$. In particular, $\lambda_1 \geq 1$.

Theorem 4.7. *With respect to notation above,*

$$\mu_\ell(X,\alpha) = \frac{1}{e}\left(\sum_{\psi\in\widehat{G}} \mu_\psi\right);$$

$$\lambda_\ell(X,\alpha) = \left(\sum_{\psi\in\widehat{G}} \lambda_\psi\right) - 1.$$

In particular, if $\mu_\ell(X,\alpha) = 0$ if and only if $\mu_\psi = 0$ for all $\psi \in \widehat{G}$.

Proof. By Theorem 4.5,

$$Tg_{X,\alpha}(T) = f_{X,\alpha}(T) \sim \prod_{\psi\in\widehat{G}} P_\psi(T),$$

and therefore

$$Tg_{X,\alpha}(T) \sim \varpi^{\sum_{\psi\in\widehat{G}}\mu_\psi} \prod_{\psi\in\widehat{G}} Q_\psi(T).$$

Therefore,

$$\sum_\psi \mu_\psi = e\mu_\ell(X,\alpha),$$

$$\sum_\psi \lambda_\psi = 1 + \lambda_\ell(X,\alpha),$$

from which the result follows. □

For each $j \in [0, 2m_\beta]$, let

$$S_j := \beta^{-1}\{(m_\beta - j)\} = \{s \in S \mid \beta(s) = m_\beta - j\}.$$

From (4.2), we deduce that

$$P_\psi(T) = \sum_{j=0}^{2m_\beta} a_{j,\psi}(1+T)^j,$$

where

$$a_{j,\psi} := \begin{cases} -\sum_{s\in S_j}\psi(s) & \text{if } j \neq m_\beta; \\ r - \sum_{s\in S_j}\psi(s) & \text{if } j = m_\beta. \end{cases} \tag{4.4}$$

Also note that $P_\psi(0) = r - \sum_{s\in S}\psi(s)$. Recall that $\bar{\psi}$ is the mod-ϖ reduction of ψ. Set $\bar{r}$ to denote the mod-ℓ reduction of r.

Lemma 4.8. *Let ψ be a nontrivial character of G, then the following are equivalent*

(1) $\mu_\psi = 0$ and $\lambda_\psi = 0$,
(2) $P_\psi(T)$ is a unit in $\mathcal{O}[\![T]\!]$,
(3) $\sum_{s\in S}\bar{\psi}(s) \neq \bar{r}$.

Proof. The equivalence of the conditions (1) and (2) is an easy consequence of the ℓ-adic Weierstrass preparation theorem. The power series $P_\psi(T)$ is a unit if and only if its constant term $P_\psi(0)$ is a unit in $\mathcal{O}$. Since $P_\psi(0) = r - \sum_{s\in S}\psi(s)$, it follows that (2) and (3) are equivalent. □

Lemma 4.9. *With respect to notation above, $\lambda_1 \geq 2$. Moreover, the following conditions are equivalent*

(1) $\mu_1 = 0$ and $\lambda_1 = 2$,
(2) $P_1''(T)$ is a unit in $\mathcal{O}[\![T]\!]$,
(3) $\sum_{s\in S}\bar{\beta}(s)^2 \neq 0$.

Proof. Note that $P_1(0) = 0$ and hence $P_1(T)$ is divisible by T. Recall that from (4.2), we have that

$$P_1(T) = (1+T)^{m_\beta} r - \sum_{s\in S}(1+T)^{m_\beta - \beta(s)}$$

and thus,

$$P_1'(0) = m_\beta r - \sum_{s\in S}(m_\beta - \beta(s)) = \sum_{s\in S}\beta(s).$$

Since $\beta(s^{-1}) = -\beta(s)$ and $S = S^{-1}$, it follows that $P_1'(0) = 0$. Hence, T^2 divides $P_1(T)$ and $\lambda_1 \geq 2$.

We find that

$$\begin{aligned}P_1''(0) =& m_\beta(m_\beta - 1)r - \sum_{s\in S}(m_\beta - \beta(s))(m_\beta - \beta(s) - 1)\\ =& m_\beta(m_\beta - 1)r - \sum_{s\in S}\big(m_\beta(m_\beta-1) - 2\beta(s)m_\beta + \beta(s)^2 + \beta(s)\big)\\ =&(1-2m_\beta)\sum_{s\in S}\beta(s) + \sum_{s\in S}\beta(s)^2 = \sum_{s\in S}\beta(s)^2.\end{aligned}$$

It is an immediate consequence of the ℓ-adic Weierstrass preparation theorem that (1) and (2) are equivalent. We find that $P_1''(T)$ is a unit in $\mathcal{O}[\![T]\!]$ if and only if $P_1''(0)$ is not divisible by ℓ. On the other hand, note that $P_1''(0) = \sum_{s\in S}\beta(s)^2$ and therefore, (2) is equivalent to (3). ☐

Proposition 4.10. *Let $\beta : S \to \mathbb{Z}_\ell$ be a function satisfying the conditions of Definition 4.1. Moreover, assume that there exists $j \neq 0$ such that S_j is a singleton. Then, the following assertions hold*

1. *$\mu_\psi = 0$ for all $\psi \in \widehat{G}$,*
2. *$\mu(X, \alpha) = 0$.*

Proof. It follows from Theorem 4.7 that the assertions (1) and (2) are equivalent. Thus, it suffices to show that $\mu_\psi = 0$ for all $\psi \in \widehat{G}$. From (4.4), we find that

$$\overline{a_{j,\psi}} = \bar{\psi}(s),$$

where $S_j = \{s\}$. Since $\bar{\psi}(s)$ is a root of unity, it follows therefore that $\overline{a_{j,\psi}} \neq 0$. This implies that the mod-ϖ reduction of $P_\psi(T)$ is nontrivial, which in turn implies that $\mu_\psi = 0$. ☐

4.3. **Complete graphs.** In this section, we let n be a positive integer and K_n be the complete graph on n vertices. Let C_n be the cyclic group of order n and and $C_n' := C_n \setminus \{1\}$. Then, we find that $K_n = \mathrm{Cay}(C_n, C_n')$, we apply the results of the previous subsection to study the μ- and λ-invariants of K_n. Let $\beta : C_n' \to \mathbb{Z}_\ell$ be a function satisfying the conditions of Definition 4.1. Let μ_n and λ_n be the Iwasawa invariants associated to (K_n, α). Let $\mu_{n,\psi}$ and $\lambda_{n,\psi}$ be the Iwasawa invariants μ_ψ and λ_ψ introduced in the previous section.

Lemma 4.11. *Assume that $\ell \nmid n$. Then, for all nontrivial characters ψ,*

$$\mu_{n,\psi} = \lambda_{n,\psi} = 0.$$

Proof. Recall that Lemma 4.8 asserts that

$$\mu_{n,\psi} = \lambda_{n,\psi} = 0$$

if

$$\bar{r} \neq \sum_{s \in C'_n} \bar{\psi}(s).$$

Since ψ is nontrivial, $\sum_{s \in C'_n} \bar{\psi}(s) = -1$ and $\bar{r} = \overline{(n-1)}$. Since $\ell \nmid n$, the result follows. ☐

Theorem 4.12. *Suppose that $\ell \nmid n$ and $\sum_{s \in C'_n} \bar{\beta}(s)^2 \neq 0$. Then, $\mu_n = 0$ and $\lambda_n = 1$.*

Proof. It follows from Lemma 4.11 that $\mu_{n,\psi} = \lambda_{n,\psi} = 0$ for all nontrivial $\psi \in \widehat{G}$. On the other hand, $\mu_{n,1} = 0$ and $\lambda_{n,1} = 2$ by Lemma 4.9. Therefore, it follows from Theorem 4.7 that $\mu_n = 0$ and $\lambda_n = 1$. ☐

5. Illustrative examples

In this section, we illustrate our results through two examples.

5.1. **Example 1.** Let $G = \mathbb{Z}/4\mathbb{Z} := \{0, 1, 2, 3\}$, $S = \{1, 2, 3\}$, $X := \mathrm{Cay(G, S)}$ and $\ell := 3$. Note that the condition $S = S^{-1}$ is satisfied. Moreover, the Assumption 2.1 is satisfied and $\chi(X) = -2$. Now, define the function $\beta : S \to \mathbb{Z}_3$ as follows

$$\beta(s) = \begin{cases} 1, & \text{if } s = 1; \\ 0, & \text{if } s = 2; \\ -1, & \text{if } s = 3; \end{cases}$$

and $\alpha : E_X^+ \to \mathbb{Z}_3$ the associated function. We check that the conditions (1)–(4) of Definition 4.1 are all satisfied.

(1) The image of β is $\{-1, 0, 1\}$ and clearly generates $\mathbb{Z}_3$.
(2) The condition $\beta(-s) = -\beta(s)$ is easy to verify.
(3) By definition, the image of β lies in $\mathbb{Z}$.
(4) We apply Proposition 4.4 to verify the condition (4). Indeed setting $h := 1$, this element has order $m = 4$. We note that $(m, \ell) = 1$ and $\beta(h) = 1$. Also, h, h^2 and h^3 are contained in S.

The matrix $M(1+T)$ (see (3.2)) is given by

$$\begin{pmatrix} 3 & -(1+T)^{-1} & -1 & -(1+T) \\ -(1+T) & 3 & -(1+T)^{-1} & -1 \\ -1 & -(1+T) & 3 & -(1+T)^{-1} \\ -(1+T)^{-1} & -1 & -(1+T) & 3 \end{pmatrix}$$

Let $\psi \in \widehat{G}$ be the character which is defined by $\psi(n) := \exp\left(\frac{2\pi \mathbf{i} n}{4}\right) = \mathbf{i}^n$, where $\mathbf{i}$ is a squareroot of -1. Set $P_j(T) := P_{\psi^j}(T)$ for $j = 0, \dots, 3$; note that $m_\beta = 1$. Setting $x := (1+T)$, we find that

$$\begin{aligned} P_j(T) =& -\mathbf{i}^{3j}x^2 + (3 - \mathbf{i}^{2j})x - \mathbf{i}^j \\ =& \begin{cases} -(x-1)^2 & \text{if } j = 0; \\ (\mathbf{i}x^2 + 4x - \mathbf{i}) & \text{if } j = 1; \\ (x+1)^2 & \text{if } j = 2; \\ (-\mathbf{i}x^2 + 4x + \mathbf{i}) & \text{if } j = 3. \end{cases} \end{aligned}$$

On the other hand, a computation of the determinant yields

$$\begin{aligned} f_{X,\alpha}(T) =& \det(M(x)) \\ =& x^{-4} \begin{pmatrix} 3x & -1 & -x & -x^2 \\ -x^2 & 3x & -1 & -x \\ -x & -x^2 & 3x & -1 \\ -1 & -x & -x^2 & 3x \end{pmatrix} \\ =& -x^{-4}(x+1)^2(x-1)^2(x^4 + 14x^2 + 1) \\ =& x^{-4}(x+1)^2(x-1)^2(ix^2 + 4x - i)(-ix^2 + 4x + i) \\ =& x^{-4} \prod_{j=0}^{3} P_j(T). \end{aligned}$$

This illustrates the Theorem 4.5.

Set $K_\psi = \mathbb{Q}_3(\mathbf{i})$, and $\mathcal{O}$ (resp. ϖ) be the valuation ring (resp. unformizer) of K_ψ. We have that $\varpi = (3)$. By Theorem 4.7, we have that

$$\begin{aligned} \mu_3(X, \alpha) =& \left(\sum_{j=0}^{3} \mu_{\psi^j}\right); \\ \lambda_3(X, \alpha) =& \left(\sum_{j=0}^{3} \lambda_{\psi^j}\right) - 1. \end{aligned} \tag{5.1}$$

For $j \in \{1, 2, 3\}$, then Lemma 4.8 asserts that the following are equivalent

(1) $\mu_{\psi^j} = 0$ and $\lambda_{\psi^j} = 0$,
(2) $\mathbf{i}^j + \mathbf{i}^{2j} + \mathbf{i}^{3j} \not\equiv 0 \pmod 3$.

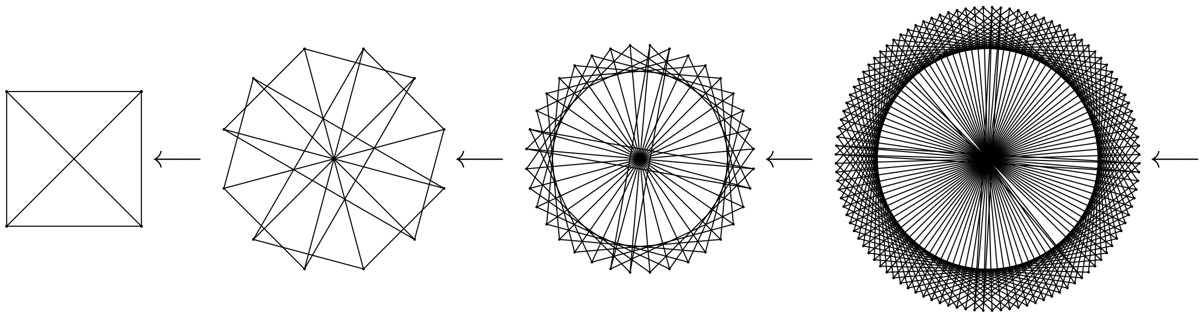
Since $\mathbf{i}^j + \mathbf{i}^{2j} + \mathbf{i}^{3j} = -1$, it follows that for $j \in \{1, 2, 3\}$,

$$\mu_{\psi^j} = 0 \text{ and } \lambda_{\psi^j} = 0.$$

We have that $P_0(T) = -T^2$, hence, $\mu_1 = 0$ and $\lambda_1 = 2$. Thus, from (5.1), we have that

$$\mu_3(X, \alpha) = 0 \text{ and } \lambda_3(X, \alpha) = 1.$$

One can visualize the $\mathbb{Z}_3$-tower as follows:



5.2. **Example 2.** We consider another example where $\mu(X,\alpha) > 0$. Let $G = \mathbb{Z}/6\mathbb{Z} := \{0,1,2,3,4,5\}$, $S = \{1,2,3,4,5\}$, $X = \mathrm{Cay}(\mathrm{G},\mathrm{S})$ and $\ell = 2$. Clearly, the condition $S = S^{-1}$ and the Assumption 2.1 are satisfied. Now, define the function $\beta : S \to \mathbb{Z}_2$ as follows:

$$\beta(s) = \begin{cases} 1, & \text{if } s = 1 \\ 1, & \text{if } s = 2 \\ 0, & \text{if } s = 3 \\ -1, & \text{if } s = 4 \\ -1, & \text{if } s = 5; \end{cases}$$

and $\alpha : E_X^+ \to \mathbb{Z}_2$ the associated function. We check that the conditons (1)–(4) of Definition 4.1 are all satisfied.

(1) The image of β is $\{-1,0,1\}$ and clearly generates $\mathbb{Z}_2$.
(2) The condition $\beta(-s) = -\beta(s)$ is easy to verify.
(3) By definition, the image of β lies in $\mathbb{Z}$.
(4) Consider the tuple $(1,1,1) \in S^3$. Then $\beta(1+1+1) = \beta(3) = 0$, $\beta(1) + \beta(1) + \beta(1) = 3$ and hence $\beta(1+1+1) \not\equiv 3\beta(1) \pmod 2$.

Now, the matrix $M(1+T)$ is given by

$$\begin{pmatrix} 5 & -(1+T)^{-1} & -(1+T)^{-1} & -1 & -(1+T) & -(1+T) \\ -(1+T) & 5 & -(1+T)^{-1} & -(1+T)^{-1} & -1 & -(1+T) \\ -(1+T) & -(1+T) & 5 & -(1+T)^{-1} & -(1+T)^{-1} & -1 \\ -1 & -(1+T) & -(1+T) & 5 & -(1+T)^{-1} & -(1+T)^{-1} \\ -(1+T)^{-1} & -1 & -(1+T) & -(1+T) & 5 & -(1+T)^{-1} \\ -(1+T)^{-1} & -(1+T)^{-1} & -1 & -(1+T) & -(1+T) & 5 \end{pmatrix}$$

Let $\psi \in \widehat{G}$ be the character which is defined by $\psi(n) := \exp\left(\frac{2\pi \mathbf{i} n}{6}\right)$. Let $\omega := \exp\left(\frac{2\pi \mathbf{i}}{6}\right)$. Therefore, $\psi(n) = \omega^n$. Note that $\omega^3 = -1$, $\omega^2 + \omega = \mathbf{i}\sqrt{3}$ and $\omega^2 - \omega = -1$, where $\mathbf{i}$ is the square root of -1.

Set $P_j(T) := P_{\psi^j}(T)$ for $j = 0, \dots, 5$; note that $m_\beta = 1$. Setting $x := (1+T)$, we find that

$$\begin{aligned} P_j(T) =& 5x - (\omega^j + \omega^{2j} + \omega^{3j}x + \omega^{4j}x^2 + \omega^{5j}x^2) \\ =& \begin{cases} -2(x-1)^2 & \text{if } j = 0; \\ 6x - \mathbf{i}\sqrt{3}(1-x^2) & \text{if } j = 1; \\ 4x + x^2 + 1 & \text{if } j = 2; \\ 6x & \text{if } j = 3. \\ 4x + x^2 + 1 & \text{if } j = 4; \\ 6x + \mathbf{i}\sqrt{3}(1-x^2) & \text{if } j = 5; \end{cases} \end{aligned}$$

Now a computation of the determinant gives us

$$\begin{aligned} f_{X,\alpha}(T) =& \det(M(x)) \\ & \begin{pmatrix} 5 & x^{-1} & x^{-1} & -1 & x & x \\ x & 5 & x^{-1} & x^{-1} & -1 & x \\ x & x & 5 & x^{-1} & x^{-1} & -1 \\ -1 & x & x & 5 & x^{-1} & x^{-1} \\ x^{-1} & -1 & x & x & 5 & x^{-1} \\ x^{-1} & x^{-1} & -1 & x & x & 5 \end{pmatrix} \\ =& -36x^{-5}(x-1)^2(x^2+4x+1)^2(x^4+10x^2+1) \\ =& x^{-6}\prod_{j=0}^{5} P_j(T). \end{aligned}$$

This again illustrates the Theorem 4.5.

Set $K_\psi = \mathbb{Q}_2(\omega)$, and $\mathcal{O}$ (resp. π) be the valuation ring (resp. unformizer) of K_ψ. We have that $\pi = (2)$. Again, by Theorem 4.7, we have that

$$\begin{aligned} \mu_2(X,\alpha) &= \left(\sum_{j=0}^{5} \mu_{\psi^j}\right); \\ \lambda_2(X,\alpha) &= \left(\sum_{j=0}^{5} \lambda_{\psi^j}\right) - 1. \end{aligned} \tag{5.2}$$

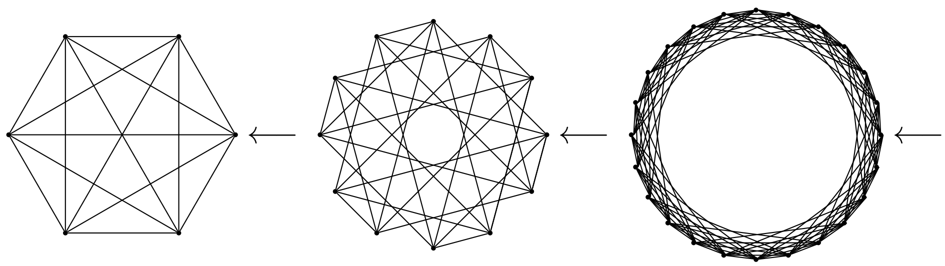
Now, we calculate each of these μ_{ψ^j}'s and λ_{ψ^j}'s.

(1) For $P_0 = -2T^2$, $\mu_{\psi^0} = 1$, $\lambda_{\psi^0} = 2$.
(2) For $P_1 = \mathbf{i}\sqrt{3}(T^2 + 2T(1-\sqrt{3}) - 2\mathbf{i}\sqrt{3})$, $\mu_\psi = 0$, $\lambda_\psi = 2$.
(3) For $P_2 = T^2 + 6T + 6$, $\mu_{\psi^2} = 0$, $\lambda_{\psi^2} = 2$.
(4) For $P_3 = 6(T+1)$, $\mu_{\psi^3} = 1$, $\lambda_{\psi^3} = 0$.
(5) For $P_4 = T^2 + 6T + 6$, $\mu_{\psi^4} = 0$, $\lambda_{\psi^4} = 2$.
(6) For $P_5 = -\mathbf{i}\sqrt{3}(T^2 + 2T(1-\sqrt{3}) - 2\mathbf{i}\sqrt{3})$, $\mu_{\psi^5} = 0$, $\lambda_{\psi^5} = 2$.

Now, using equation (5.2), we get

$$\mu_2(X, \alpha) = 2 \text{ and } \lambda_2(X, \alpha) = 9.$$

One can visualize the $\mathbb{Z}_2$-tower as follows:



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