SU(2) STRUCTURES IN FOUR DIMENSIONS AND PLEBANSKI FORMALISM FOR GR

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Abstract. An SU(2) structure in four dimensions can be described as a triple of 2-forms $\Sigma^i \in \Lambda^2(M), i = 1, 2, 3$ satisfying $\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$. Such a triple defines a Riemannian signature metric on M. An SU(2) structure is said to be integrable if the holonomy of this Riemannian metric is contained in SU(2). It is well-known that this is the case if and only if the 2-forms are closed $d\Sigma^i = 0$. The main purpose of the paper is to analyse the second order in derivatives diffeomorphism invariant action functionals that can be constructed for an SU(2) structure. The main result is that there is a unique such action functional if one imposes an additional requirement that the action is also SU(2) invariant, with SU(2) acting on the triple Σ^i as in its vector representation. This action functional has a very simple expression in terms of the intrinsic torsion of the SU(2) structure. We show that its critical points are SU(2) structures whose associated metric is Einstein. The action we describe has also a first order in derivatives version, and we show how this is related to what in the physics literature is known as Plebanski formalism for GR.

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1. Introduction

A G-structure on a smooth manifold M is a reduction of the principal $\mathrm{GL}(n,\mathbb{R})$ bundle of frames on M to a G-subbundle. Most interesting geometric structures on M can be rephrased in the language of G-structures. For example, a Riemannian metric on M is an $\mathrm{O}(n)$ structure, an almost complex structure $J : TM \to TM, J^2 = -\mathbb{I}$ on a manifold of dimension $2n$ is a $\mathrm{GL}(n,\mathbb{C})$ structure.

Many interesting examples of G-structures come from spinors and can be called "spinorial".[1] This is the case when $G = \mathrm{Stab}_\psi \subset \mathrm{Spin}(n)$ is a subgroup that stabilises a spinor $\psi \in S$. When n is even, the spinor representation S of $\mathrm{Spin}(n)$ splits into two irreducible components $S = S_+ \oplus S_-$, and in this case it is natural to consider a semi-spinor $\psi \in S_+$. Given that

$$\dim(\mathrm{GL}(n,\mathbb{R})/G) = \dim(\mathrm{GL}(n,\mathbb{R})/O(n)) + \dim(\mathrm{Spin}(n)/G), \tag{1.1}$$

we see that the $\mathrm{GL}(n,\mathbb{R})$ orbit of G-structures of the same type can be viewed as the space of Riemannian metrics together with a spinor of algebraic type that has G as the stabiliser. Thus, heuristically, we can say

$$\{\text{spinorial G-structure}\} = \{\text{metric}\} + \{\text{spinor}\}. \tag{1.2}$$

What is very interesting about the spinorial G-structures is that they can be expected to be encodable into a collection of differential forms on M. Thus, given a spinor ψ in S or in S_+, a certain collection of differential forms arises as spinor bilinears constructed from ψ with ψ, or possibly with ψ and $\hat{\psi}$, where $\hat{\psi}$ is an appropriate $\mathrm{Spin}(n)$ invariant complex conjugation. Indeed, the tensor product of the spin representation S with itself contains the spaces of all degree differential forms on M

$$S \otimes S = \Lambda^\bullet(M). \tag{1.3}$$

As is seen from a large collection of examples, taking a sufficient number of differential forms obtained as spinor bilinears constructed from ψ and $\hat{\psi}$ is sufficient to reproduce both the metric on M, and the spinor ψ (the latter always mod $\mathbb{Z}_2$ sign ambiguity).

Some of the known examples of such an encoding are as follows.

- **$\{0\}$-structures in 3D.** In this case $\mathrm{Spin}(3) = \mathrm{SU}(2)$, and the spinor is a 2-component spinor $S \sim \mathbb{C}^2$. Taking a unit such spinor $\langle \hat{\psi}, \psi \rangle = 1$, and constructing a real one-form $e^3 \in \Lambda^1$ via $e^3(X) := \langle \hat{\psi}, X\psi \rangle$, and a complex-valued one-form $e(X) := \langle \psi, X\psi \rangle$, where a vector $X \in TM$ acts on ψ by the Clifford multiplication, we get an orthonormal co-frame $e^{1,2,3}$, where $e = e^1 + \mathrm{i}\, e^2$. In turn, starting with a co-frame $e^{1,2,3}$ one recovers the metric via $ds^2 = (e^1)^2 + (e^2)^2 + (e^3)^2$.
- **$\mathrm{SU}(2)$-structures in 4D.** In this case $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Taking a unit $\langle \hat{\psi}, \psi \rangle = 1$ spinor $\psi \in S_+$, its stabiliser is the other $\mathrm{SU}(2)$ that does not act on it. The construction of all possible differential forms returns one real 2-form $\omega(X,Y) = \langle \hat{\psi}, XY\psi \rangle$, and a complex 2-form $\Omega(X,Y) = \langle \psi, XY\psi \rangle$. These objects satisfy $\Omega\Omega = 0, \Omega\omega = 0, (1/2)\Omega\bar{\Omega} = \omega^2$. Alternatively, decomposing $\Omega := \Sigma^1 + \mathrm{i}\,\Sigma^2$ and renaming $\omega := \Sigma^3$ we get a triple of 2-forms satisfying $\Sigma^i\Sigma^j \sim \delta^{ij}, i,j = 1,2,3$. In turn, taking such a triple of 2-forms as the basic geometric data, one can recover the Riemannian metric g_Σ (together with an orientation of M) via

$$g_\Sigma(X,Y)\mathrm{vol}_\Sigma = \frac{1}{6}\epsilon^{ijk} i_X \Sigma^i i_Y \Sigma^j \Sigma^k. \tag{1.4}$$

  Here vol_Σ is the volume form of the metric g_Σ.

[1] But it should be emphasised that there are also interesting G-structures that are not spinorial. For example, the almost complex structure is not spinorial in the sense we use.

- SU(3)**-structures in 6D.** We now have $\mathrm{Spin}(6) = \mathrm{SU}(4)$, and $S_+ = \mathbb{C}^4$. The stabiliser of a semi-spinor $\psi \in S_+$ is SU(3). Taking a unit spinor $\langle \hat{\psi}, \psi \rangle = 1$, the construction of differential forms returns a real 2-form $\omega(X,Y) = \langle \hat{\psi}, XY\psi \rangle$, and a complex 3-form $\Omega(X,Y,Z) = \langle \psi, XYZ\psi \rangle$. This 3-form is decomposable, and also satisfies $\Omega\omega = 0$, as well as $\Omega\bar{\Omega} = (4\mathrm{i}/3)\omega^3$. In this case it is sufficient to take as the basic geometric data the non-degenerate 2-form ω, as well as a real 3-form $\mathrm{Re}(\Omega)$. The metric on M, as well as an almost complex structure in which $\Omega \in \Lambda^{3,0}$, are then recovered from these data by a procedure explained in e.g. [1].
- G_2**-structures in 7D.** In this case there are real spinors $\hat{\psi} = \psi$. Taking a real unit spinor $\psi \in S, \langle \psi, \psi \rangle = 1$, the stabiliser $\mathrm{Stab}_\psi = \mathrm{G}_2$. The construction of possible differential forms returns the 3-form $C(X,Y,Z) = \langle \psi, XYZ\psi \rangle$. There is also a 4-form, and a 7-form produced, but these are not independent and are determined by C. The 3-form C produced by this construction is non-degenerate in a suitable sense. Moreover, the GL(7) orbit of 3-forms of this type is open in Λ^3. Taking a non-degenerate $C \in \Lambda^3$ as the basic geometric data, the metric (and orientation) of M is recovered via

$$g_C(X,Y)\mathrm{vol}_C = \frac{1}{6} i_X C i_Y C C. \tag{1.5}$$

- Spin(7)**-structures in 8D.** In this dimension we again have real semi-spinors $\hat{\psi} = \psi$. Taking a unit real semi-spinor $\psi \in S_+$, the stabiliser is $\mathrm{Stab}_\psi = \mathrm{Spin}(7)$. The construction of differential forms returns a 4-form $\Phi(X,Y,Z,W) = \langle \psi, XYZW\psi \rangle$. This is a 4-form of a special algebraic type, whose GL(8) stabiliser is Spin(7). Taking this 4-form as the basic geometric data, the metric g_Φ is recovered by the procedure explained in [2].

Other examples are possible, but we refrain from enlarging our list.

The purpose of this article is to analyse the case of SU(2) structures in dimension four, in the same spirit as was done recently for the case of G_2 in [3]. More precisely, our aim is to analyse the structure of the intrinsic torsion of an SU(2) structure in 4D, and derive a formula for the (part of the) Riemann curvature tensor as determined by the intrinsic torsion. This links to the classical decomposition of the Riemann curvature tensor in 4D into its self-dual and anti-self-dual components. Our other aim is to analyse the possible (second order in derivatives) action functionals that can be written for SU(2) structures, as well as the corresponding Euler-Lagrange equations.

The geometry of SU(2) structures in dimension four is known. At the same time, there appears to be no reference that treats this geometry from the perspective of the intrinsic torsion. One of the aims of this paper is to fill in this gap, and show that the story in 4D completely parallels the other better understood cases, in particular the case of G_2 structures as treated in [3], and possibly even more closely the geometry of Spin(7) structures as discussed recently in [4] and [5]. The geometry of SU(2) structures in dimension four is also the subject of an ongoing work by other authors, see [6].

At the same time, it is important to emphasise that there are several aspects of the story SU(2) that do not have an analog in the case of other G-structures. This, in particular, makes our treatment very different to that in [6]. For any G-structure we have $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, where $\mathfrak{g}$ is the Lie algebra of G. However, the exceptional phenomenon that occurs only in 4D[2] is that $\mathfrak{g}^\perp$ is also a Lie algebra, namely $\mathfrak{g}^\perp = \mathfrak{su}(2)$. Because of this, even though the GL(4) stabiliser of a triple of 2-forms Σ^i satisfying $\Sigma^i \Sigma^j \sim \delta^{ij}$ is SU(2), the other SU(2) leaves its remnant in the structure of the theory. Indeed, the object Σ^i viewed as a map $\Sigma : \mathbb{R}^3 \to \Lambda^3$ can be interpreted as an SU(2) equivariant map

$$\Sigma : \mathbb{R}^3 \to \Lambda^+. \tag{1.6}$$

Equipped with this interpretation, to be explained in details below, one realises that the SU(2) whose Lie algebra is $\mathfrak{g}^\perp$ stabilises Σ, viewed as the map (1.6). Thus, when appropriately

[2]This is also the case for any $\{0\}$-structure.

interpreted, the structure group in the case of an SU(2) structure in dimension four is actually full Spin(4). What this implies is that all the tensors need to be decomposed into irreducible representations of Spin(4) rather than those of just SU(2). These aspects of the story do not have an analog in the case of the other G-structures. The requirement that the Σ is an SU(2) equivariant map puts additional constraints on the type of objects that can be considered. This is the main difference with the treatment in [6]. With this requirement imposed, the construction becomes much more rigid. In particular, the decomposition of spaces of various differential forms into the irreducible components contains fewer summands. There is also much less freedom in possible diffeomorphism invariant actions that can be written for an SU(2) structure.

When $G \subset \mathrm{O}(n)$, a G-structure is called integrable if the holonomy of the Levi-Civita connection for the metric defined by the G-structure is contained in G. It is well-known that an SU(2) structure is integrable if and only if the triple of 2-forms Σ^i is closed $d\Sigma^i = 0$. This gives a natural set of first-order PDE's for the SU(2)-structure Σ^i. What is less known is that there is also a natural set of second-order PDE's that can be imposed on an SU(2)-structure, and that this PDE's are satisfied if and only if the metric defined by Σ^i is Einstein.

Given that this is one of the main statements of this paper, we would like to explain it already in the Introduction. This is best done as a series of propositions. Let $E = \mathbb{R}^3$. We view an SU(2) structure as an E-valued 2-form, thus $\Sigma \in E \otimes \Lambda^2$.

Theorem 1.1. The intrinsic torsion of an SU(2) structure, which measures its non-integrability, can be described as an E-valued 1-form A^i_μ. We have

$$\nabla_\mu \Sigma^i_{\rho\sigma} = -\epsilon^{ijk} A^j_\mu \Sigma^k_{\rho\sigma}. \tag{1.7}$$

To understand the next statement, we note that there is a natural action of diffeomorphisms on SU(2) structures. In addition, there is also a natural action of SU(2), which acts on $E = \mathbb{R}^3$ as its spin one representation. At the infinitesimal level, both actions are described by the following formulas

$$\delta_X \Sigma^i = d i_X \Sigma^i + i_X d\Sigma^i, \qquad \delta_\phi \Sigma^i = [\phi, \Sigma]^i. \tag{1.8}$$

Here $X \in TM$ is a vector field, and $\phi \in E$ is a section of a vector bundle over M with E as the fibre. The vector space $E = \mathbb{R}^3$ is naturally a Lie algebra, with the Lie bracket given by the cross-product, and $[\cdot,\cdot]$ in the above formula stands for this Lie bracket.

Theorem 1.2. There is a unique (up to an overall multiple) action functional $S[\Sigma]$ that is second order in derivatives of Σ and that is both diffeomorphism and SU(2) invariant, i.e. invariant under both transformations in (1.8). It is given by

$$S[\Sigma] = -\frac{1}{2}\int \Sigma^i [A, A]^i. \tag{1.9}$$

Here A is the intrinsic torsion of the SU(2) structure Σ^i, which is an E-valued 1-form, as determined by (1.7). The wedge product of differential forms is implied. The coefficient in front of the action is chosen for future convenience and will be explained in the main text.

The next proposition describes the critical points of this action functional.

Theorem 1.3. The critical points of (1.9) are SU(2) structures whose associated metric is Einstein.

There is also a first order in derivatives version of (1.9), which is a functional of both Σ and an additional set of E-valued 1-form fields, which, after one imposes their Euler-Lagrange equations, become identified with the intrinsic torsion. As we shall see, this first order version of the action principle is what is known in the physics literature as the Plebanski formalism for General Relativity, see [7].

It is clear from our analysis, as well as the analysis in [3], that many of the steps here have analogs for any other (spinorial) G-structure. It is then very interesting whether there are also analogs of the Plebanski functional for an arbitrary (spinorial) G-structure, and whether there are some "best" second-order PDE's that can be imposed on such a G-structure, the later ideally being the conditions for a critical point of the action. Some steps in this direction in the case of Spin(7) structures in dimension eight were taken in [4], but there are many other settings that are also interesting to analyse, in particular those from the list of examples given above. Another interesting and important question is to analyse the geometric flows that arise as the gradient flows of the natural action functionals. In the case of SU(2) structures this is the gradient flow of the action functional (1.9). We emphasise that our functional (1.9) is not the same as the energy functional considered in [6]. Some information about its gradient flow can be easily extracted from the results in the main text, but we leave the complete treatment to a separate work.

The organisation of this paper is as follows. We start by describing SU(2) structures in more detail, and show how an SU(2) structure on M defines a metric. There is some new material in this section, in particular we give a new formula (2.15) for the metric in terms of an SU(2) structure. We discuss how an SU(2) structure can be viewed as an an SU(2) equivariant map in Section 3. We then proceed to a description of how spaces of E-valued 1- and 2-forms on M split into their irreducible components with respect to the action of $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Some of the material here is new, in particular the decomposition of $E\otimes\Lambda^2$ using the operator J_2. We then discuss, in Section 5, the notion of the intrinsic torsion of an SU(2) structure. We also characterise how (a part of the) Riemann curvature is determined by the intrinsic torsion, and how the Einstein condition can be imposed in this language. We analyse and construct diffeomorphism and SU(2)-invariant functionals that are second order in derivatives and quadratic in perturbations of an SU(2) structure in Section 6. The key result of this section is that there is a unique diffeomorphism and SU(2)-invariant action functional. We construct non-linear action functionals in Section 7. This section contains proofs of all the statements described in the Introduction.

2. SU(2) structures in four dimensions

Definition 2.1. A triple of 2-forms $\Sigma^i\in\Lambda^2(M), i=1,2,3$ satisfying $\Sigma^i\Sigma^j\sim\delta^{ij}$ will be called an SU(2) structure on a 4-dimensional manifold M.

The meaning of the equation $\Sigma^i\Sigma^j\sim\delta^{ij}$ needs to be clarified further. First, this paper uses the convention that the product of differential forms is always the wedge product, with the wedge product symbol omitted. Second, the proportional to symbol means equal up to multiplication by an arbitrary (non-zero) top form on M. The top form needed on the right-hand side of this relation to make it into an equation can be obtained by taking $i=j$ and summing up over the indices. One gets

$$\Sigma^i\Sigma^j=\frac{\delta^{ij}}{3}(\sum_k\Sigma^k\Sigma^k). \tag{2.1}$$

Alternatively, this equation can be interpreted as saying that $\Sigma^1\Sigma^1=\Sigma^2\Sigma^2=\Sigma^3\Sigma^3\neq 0$, while all products $\Sigma^i\Sigma^j$ with $i\neq j$ are equal to zero.

We note that there are 5 relations in $\Sigma^i\Sigma^j\sim\delta^{ij}$, and so the dimension (per point) of the space of SU(2) structures on M is $18-5=13$. This is the same as the dimension of the coset space $\mathrm{GL}(4,\mathbb{R})/\mathrm{SU}(2)$. The fact the $\mathrm{GL}(4,\mathbb{R})$ stabiliser of an SU(2) structure on M is one of the two SU(2)'s in $\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathbb{Z}_2=\mathrm{SO}(4)\subset\mathrm{GL}(4,\mathbb{R})$ follows from the following statement.

Theorem 2.1. An SU(2) structure on M determines a Riemannian metric on M together with two orientations $\mathrm{vol}_\Sigma,\mathrm{vol}'_\Sigma$ on M. The metric together with one of the orientations is determined according to the following formula

$$g_\Sigma(X,Y)\mathrm{vol}'_\Sigma=-\frac{1}{6}\epsilon^{ijk}i_X\Sigma^i i_Y\Sigma^j\Sigma^k. \tag{2.2}$$

The orientation vol'_Σ is such that the signature of the metric defined this way is mostly plus. The other orientation is defined via

$$\mathrm{vol}_\Sigma=\frac{1}{6}\Sigma^i\Sigma^i. \tag{2.3}$$

The 2-forms Σ^i are self-dual with respect to the Hodge star operator corresponding to g_Σ, in the orientation vol_Σ. There exists a choice of a co-frame $e^{1,2,3,4}$ such that when orientations coincide $\mathrm{vol}_\Sigma = \mathrm{vol}'_\Sigma$ we have the following canonical expression for Σ^i

$$\Sigma^1 = e^{41} - e^{23}, \quad \Sigma^2 = e^{42} - e^{31}, \quad \Sigma^3 = e^{43} - e^{12}. \tag{2.4}$$

When the two orientations are opposite of each other $\mathrm{vol}_\Sigma = -\mathrm{vol}'_\Sigma$ we have instead

$$\Sigma^1 = e^{41} - e^{23}, \quad \Sigma^2 = e^{42} - e^{31}, \quad \Sigma^3 = -e^{43} + e^{12}. \tag{2.5}$$

Proof. Let us form $\Omega = \Sigma^1 + \mathrm{i}\,\Sigma^2$. The algebraic relations satisfied by Σ^i imply that $\Omega\Omega = 0$, which then means that it is decomposable. Let us denote by u, v some complex one-forms such that $\Omega = uv$. Because $\Omega\bar\Omega \neq 0$, it is clear that $u, v, \bar u, \bar v$ span $\Lambda^1(M)$. The real 2-form Σ^3 satisfies $\Sigma^3\Omega = \Sigma^3\bar\Omega = 0$, and is therefore of the form

$$\Sigma^3 = \frac{1}{2\mathrm{i}}\alpha u\bar u + \frac{1}{2\mathrm{i}}\beta v\bar v + \frac{1}{2\mathrm{i}}\gamma u\bar v + \frac{1}{2\mathrm{i}}\bar\gamma v\bar u, \quad \alpha, \beta \in \mathbb{R}, \gamma \in \mathbb{C}. \tag{2.6}$$

Alternatively

$$\Sigma^3 = \frac{1}{2\mathrm{i}} \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \bar\gamma & \beta \end{pmatrix} \begin{pmatrix} \bar u \\ \bar v \end{pmatrix}. \tag{2.7}$$

Moreover

$$\Sigma^3\Sigma^3 = \frac{1}{2}(\alpha\beta - |\gamma|^2) uv\bar u\bar v. \tag{2.8}$$

But we must have $\Sigma^3\Sigma^3 = (1/2)\Omega\bar\Omega$, and so $\alpha\beta - |\gamma|^2 = 1$. At the same time, the one-forms u, v are defined modulo

$$\begin{pmatrix} u & v \end{pmatrix} \to \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C}, ad - bc = 1. \tag{2.9}$$

The Hermitian unimodular matrix that appears in (2.7) can always be brought by a $\mathrm{SL}(2, C)$ transformation into the form $\pm\mathbb{I}$. This means that there is a choice of u, v such that

$$\Omega = \Sigma^1 + \mathrm{i}\,\Sigma^2 = uv, \qquad \Sigma^3 = \pm\left(\frac{1}{2\mathrm{i}}u\bar u + \frac{1}{2\mathrm{i}}v\bar v\right), \tag{2.10}$$

where both signs in the expression for Σ^3 are possible. The 1-forms u, v are defined modulo an SU(2) transformation, which leaves Σ^3, Ω invariant. This shows that the triple of 2-forms Σ^i satisfying $\Sigma^i\Sigma^j \sim \delta^{ij}$ defines the Riemannian signature metric

$$g_\Sigma = |u|^2 + |v|^2, \tag{2.11}$$

as well as the canonical orientation of M given by $\Omega\bar\Omega = uv\bar u\bar v$.

We now choose a real basis of co-vectors

$$u = e^4 - \mathrm{i}\,e^3, \qquad v = e^1 + \mathrm{i}\,e^2, \tag{2.12}$$

so that Σ^i take the following form

$$\Sigma^1 = e^{41} - e^{23}, \quad \Sigma^2 = e^{42} - e^{31}, \quad \Sigma^3 = \pm(e^{43} - e^{12}), \tag{2.13}$$

the metric is

$$g_\Sigma = \sum_{I=1}^{4} (e^I)^2, \tag{2.14}$$

and the orientation determined by Σ^i is e^{1234}. It is clear that Σ^i are self-dual 2-forms with respect to the Hodge star corresponding to the metric g_Σ, in the orientation e^{1234}. The formula (2.2) can now be checked by a verification. Changing the sign of Σ^3 changes the sign of the orientation vol'_Σ, and so indeed the canonical expression for Σ^3 is decided by whether the two orientations defined by Σ agree or disagree. □

Definition 2.2. We will say that an SU(2) structure is orientation preserving if $\mathrm{vol}_\Sigma = \mathrm{vol}'_\Sigma$, and orientation changing if $\mathrm{vol}_\Sigma = -\mathrm{vol}'_\Sigma$.

An alternative explicit expression for the metric, in the spirit of the 8D formula in [2], is as follows.

Theorem 2.2. The expression

$$\mp\frac{1}{2}\frac{(\epsilon^{ijk}i_X\Sigma^i i_X\Sigma^j i_X\Sigma^k)(e_1,e_2,e_3)}{(i_X\Sigma^i\Sigma^i)(e_1,e_2,e_3)}, \tag{2.15}$$

where $e_{1,2,3}$ is an arbitrary triple of vectors that together with X span TM, is independent of the choice of e_i. It is homogeneity degree two in X, and equals the metric pairing $g_\Sigma(X,X)$. The minus sign in this formula applies when Σ^i is orientation preserving, and plus sign in the orientation changing case.

Proof. The homogeneity degree in X is obvious. To prove independence of e^i, let us take some other triple of vectors related to X, e^i as

$$(e_i)' = \kappa_i X + \lambda_i{}^j e_j. \tag{2.16}$$

From $i_X\Sigma^i i_X\Sigma^i = 0$ it follows that the denominator is independent of κ_i. The numerator is independent of κ_i because the insertion of two factors of X into Σ^i vanishes. To demonstrate independence of $\lambda_i{}^j$ we note

$$(\epsilon^{ijk}i_X\Sigma^i i_X\Sigma^j i_X\Sigma^k)((e^1)',(e^2)',(e^3)') = \det(\lambda)(\epsilon^{ijk}i_X\Sigma^i i_X\Sigma^j i_X\Sigma^k)(e^1,e^2,e^3). \tag{2.17}$$

Similarly

$$(i_X\Sigma^i\Sigma^j)((e^1)',(e^2)',(e^3)') = \det(\lambda)(i_X\Sigma^i\Sigma^j)(e^1,e^2,e^3). \tag{2.18}$$

The statement of independence of choice of e^i follows. The statement that this expression computes the metric pairing $g_\Sigma(X,X)$ follows by computing it for one of the frame vectors when Σ^i is given by its canonical expressions (2.4),(2.5) in the co-frame basis. □

The statement that the $\mathrm{GL}(4,\mathbb{R})$ stabiliser of a triple Σ^i satisfying $\Sigma^i\Sigma^j \sim \delta^{ij}$ is SU(2) now follows from the facts established above. First, given that Σ^i define a Riemannian signature metric, their $\mathrm{GL}(4,\mathbb{R})$ stabiliser is contained in O(4). Further, we have explicitly determined this stabiliser to be the SU(2) that mixes the complex one-forms u, v and leaves $\Sigma^{1,2,3}$ invariant. The other SU(2) acts on $\Sigma^{1,2,3}$ non-trivially, by mixing them. This fact suggests an alternative viewpoint on SU(2)-structures, to which we now turn.

3. An SU(2) structure as an equivariant map

It turns out to be very convenient to introduce a vector space $E \sim \mathbb{R}^3$, equipped with the usual inner product on $\mathbb{R}^3$. We use the inner product to identify E with its dual E^*, and view $\Sigma^{1,2,3}$ as components of an E-valued 2-form $\Sigma \in \Lambda^2(M)\otimes E$. Alternatively, we can view Σ as a map

$$\Sigma : E \to \Lambda^2(M). \tag{3.1}$$

By construction of the metric g_Σ, the image of the map Σ is identified with the space Λ^+ of self-dual 2-forms for g_Σ. There is a natural wedge-product conformal metric on the space $\Lambda^2(M)$, of signature $(3,3)$, defined via

$$(B_1,B_2)_\wedge = B_1\wedge B_2/v_g, \tag{3.2}$$

where v_g is an arbitrary volume form on M. Restricted to the space $\Lambda^+(M)$ (for any Riemannian signature metric on M), the wedge product metric is definite. The defining conditions $\Sigma^i\wedge\Sigma^j \sim \delta^{ij}$ can then be rephrased as follows.

Definition 3.1. (Alternative definition of an SU(2)-structure). An SU(2)-structure is a map $\Sigma : E \to \Lambda^2(M)$ from a 3-dimensional vector space E equipped with an inner product δ, such that the pull-back of the wedge-product metric on Λ^2 to E coincides with the inner product on E: $\Sigma^*((\cdot,\cdot)_\wedge) = \delta$.

To rephrase this in yet a different way, an SU(2)-structure is a map $\Sigma : E \to \Lambda^2(M)$ that is an isometry on its image.

We can now explain in what sense the map $\Sigma : E \to \Lambda^2(M)$ is SU(2) equivariant. The vector space E is naturally a Lie algebra $\mathfrak{su}(2)$, with the Lie bracket given by the cross-product. Similarly, the vector space $\Lambda^+\Big|_p, p \in M$ is naturally identified with an $\mathfrak{su}(2)$ subalgebra of $\mathfrak{so}(4) = \Lambda^2(M)\Big|_p$. The map $\Sigma : E \to \Lambda^2(M)$, restricted to its image $\Lambda^+ \subset \Lambda^2(M)$, is a homomorphism of Lie algebras. This can be used to identify $E \sim \Lambda^+\Big|_p \sim \mathfrak{su}(2)$.

The upshot of this discussion is that a choice of an SU(2)-structure Σ does not need to break O(4) to an SU(2) subgroup that stabilises it. The other SU(2) is still there, and serves as the group with respect to which Σ is equivariant. What this means is that all tensors should be decomposed not just in representations of a single SU(2), but in representations of both SU(2)'s. This phenomenon does not have an analog in the case of other G-structures. This aspect of the story does not feature in the treatment in [6], which considers objects that are not SU(2) covariant. In contrast, we imposes the requirement of SU(2) covariance, which severely restricts that types of objects that are allowed. This becomes clear in the next section, which describes the decomposition of the space of differential forms with values in E. This decomposition is simpler than that in [6], because the second SU(2) is now at play as well.

4. Decomposition of E-valued differential forms

For any G-structure, the intrinsic torsion of a G-structure is an object that measures the failure of this G-structure to be integrable. From general principles, it follows that the intrinsic torsion is an object that takes values in $\Lambda^1(M) \otimes \mathfrak{g}^\perp$, see for example a discussion in the subsection 4.2 of [8]. At the same time, the intrinsic torsion should be determinable from the covariant derivative (computed with respect to the Levi-Civita connection) of the tensors that define the G-structure.

In our case, $\mathfrak{g}^\perp = \mathfrak{su}(2) = E$, and so the intrinsic torsion must be an object with values in $\Lambda^1(M) \otimes E$. The tensor that defines an SU(2) structure takes values in $\Lambda^2(M) \otimes E$, and so does its covariant derivative $\nabla_X \Sigma$ in any direction $X \in TM$. For this reason, we need to understand the decomposition of the spaces $\Lambda^1(M) \otimes E, \Lambda^2(M) \otimes E$, into irreducible representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

Irreducible representations of SU(2) are the spin $k/2$ representations that we denote by S^k. They are of dimension $\dim(S^k) = k+1$. As we have previously discussed, there are two different SU(2)'s in the game. One SU(2) is the group with respect to which the 2-forms Σ^i are invariant. We will choose to denote this copy of SU(2) by $\mathrm{SU}_-(2)$, and the corresponding representations by S^k_-. The other SU(2) is one that acts non-trivially on Σ^i by mixing them, with the map $\Sigma : E \to \Lambda^2$ being equivariant with respect to this copy of SU(2). We will denote it by $\mathrm{SU}_+(2)$, and the corresponding representations by S^k_+. We then have

$$\Lambda^1(M) = S_+ \otimes S_-, \qquad \Lambda^2(M) = S^2_+ \oplus S^2_-, \qquad E = S^2_+, \tag{4.1}$$

and the decomposition of $\Lambda^1(M) \otimes E, \Lambda^2(M) \otimes E$ into irreducibles is

$$\begin{aligned} \Lambda^1(M) \otimes E &= (S^3_+ \otimes S_-) \oplus (S_+ \otimes S_-), \\ \Lambda^2(M) \otimes E &= S^4_+ \oplus S^2_+ \oplus C^\infty(M) \oplus (S^2_+ \otimes S^2_-). \end{aligned} \tag{4.2}$$

4.1. Algebra of Σ's.

To obtain explicit formulas for the irreducible parts of E-valued differential forms, we need some identities satisfied by the 2-forms Σ^i. We will be using the index notation, similar to e.g. [3], which is most suited for the type of calculations that need to be done.

First, one of the two indices of these differential forms can be raised with the metric (that they define), to convert these objects into those in $\mathrm{End}(TM)$. We then have a triple of such endomorphisms of the tangent bundle, satisfying the algebra of the imaginary quaternions

$$\Sigma^i_\mu{}^\alpha \Sigma^j_\alpha{}^\nu = -\delta^{ij}\delta_\mu{}^\nu + \epsilon^{ijk}\Sigma^k_\mu{}^\nu. \tag{4.3}$$

There are also useful relations

$$\Sigma^i_{\mu\nu}\Sigma^i_{\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}, \tag{4.4}$$

$$\epsilon^{ijk}\Sigma^j_{\mu\nu}\Sigma^k_{\rho\sigma} = -2\Sigma^i_{[\mu|\rho|}g_{\nu]\sigma} + 2\Sigma^i_{[\mu|\sigma|}g_{\nu]\rho}. \tag{4.5}$$

4.2. **Decomposition of $\Lambda^1(M)\otimes E$.** Given an SU(2) structure Σ^i, we can define the following operator acting on $\Lambda^1(M)\otimes E$

$$\Lambda^1(M)\otimes E \ni A^i_\mu \to J_\Sigma(A)^i_\mu := \epsilon^{ijk}\Sigma^{j}_{\mu}{}^{\alpha}A^k_\alpha. \tag{4.6}$$

A simple calculation using (4.3) shows that

$$J^2_\Sigma = 2\mathbb{I} + J_\Sigma. \tag{4.7}$$

This means that the eigenvalues of J_Σ are $2, -1$. The eigenspaces of J_Σ are precisely the irreducibles appearing in the first line in (4.2). It is also easy to check that objects of the form

$$\xi^\alpha\Sigma^i_{\alpha\mu} \in \Lambda^1(M)\otimes E \tag{4.8}$$

are eigenvectors of eigenvalue 2. We then have the following characterisation

$$(\Lambda^1(M)\otimes E)_4 = (S_+\otimes S_-) = \{\xi^\alpha\Sigma^i_{\alpha\mu}, \xi\in TM\}. \tag{4.9}$$

The space

$$(\Lambda^1(M)\otimes E)_8 = (S^3_+\otimes S_-) \tag{4.10}$$

can then be characterised as the orthogonal complement of (4.9) in $\Lambda^1(M)\otimes E$.

4.3. GL(4) **orbit of Σ in $\Lambda^2(M)\otimes E$.** To characterise some of the spaces appearing in the decomposition of $\Lambda^2(M)\otimes E$ we first consider the GL(4) orbit of the 2-forms Σ^i. Thus, we consider E-valued 2-forms of the form $h_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}$. Decomposing $h_{\mu\nu}\in \mathrm{GL}(4)$ into its symmetric and anti-symmetric parts, and noting that the anti-symmetric part is valued in $\Lambda^2(M) = S^2_+\oplus S^2_-$, we get the following list of irreducibles appearing

$$h_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]} \in S^2_+\oplus C^\infty(M)\oplus(S^2_+\otimes S^2_-)\subset\Lambda^2(M)\otimes E, \tag{4.11}$$

which is all spaces in the second line of (4.2) apart from S^4_+. These irreducibles in $\Lambda^2(M)\otimes E$ can then be characterised as the images of the map $h_{\mu\nu}\to h_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}$.

One can also act on the index i of Σ^i 2-forms with a GL(3) transformation, i.e., consider the orbit of E-valued 2-forms of the form $h^{ij}\Sigma^j_{\mu\nu}$. Decomposing the matrix h^{ij} into symmetric and anti-symmetric parts, one finds the following list of irreducibles

$$h^{ij}\Sigma^j_{\mu\nu}\in S^4_+\oplus S^2_+\oplus C^\infty(M). \tag{4.12}$$

In the opposite direction, given an object $B^i_{\mu\nu}\in\Lambda^2(M)\otimes E$, its irreducible parts can be extracted as follows

$$\begin{aligned} B^{(i}_{\alpha\beta}\Sigma^{j)\alpha\beta} - \frac{1}{3}\delta^{ij}B^k_{\alpha\beta}\Sigma^{k\alpha\beta} &\in S^4_+, \\ \epsilon^{ijk}B^j_{\alpha\beta}\Sigma^{k\alpha\beta} &\in S^2_+, \\ B^k_{\alpha\beta}\Sigma^{k\alpha\beta} &\in C^\infty(M), \\ B^i_{(\mu|\alpha|}\Sigma^{i\alpha}{}_{\nu)} &\in (S^2_+\otimes S^2_-). \end{aligned} \tag{4.13}$$

4.4. **Decomposition of** $\Lambda^2(M)\otimes E$**.** We can also describe the irreducible subspaces of $\Lambda^2(M)\otimes E$ as eigenspaces of a certain operator in E-valued 2-forms, similar to how we used J_Σ to decompose $\Lambda^1\otimes E$. Let us introduce the following operator

$$J_2:\Lambda^2\otimes E\to\Lambda^2\otimes E,\qquad J_2(B)^i_{\mu\nu}=\epsilon^{ijk}\Sigma^{j\ \alpha}_{[\mu}B^k_{|\alpha|\nu]},\qquad B^i_{\mu\nu}\in\Lambda^2\otimes E. \tag{4.14}$$

A computation gives

$$\begin{aligned} J_2^2(B)^i_{\mu\nu} &= \frac{1}{2}B^i_{\mu\nu}+\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}B^i_{\alpha\beta}+\frac{1}{2}J_2(B)^i_{\mu\nu}+\frac{1}{2}\Sigma^{i\ \alpha}_{[\mu}\Sigma^{j\ \beta}_{\nu]}B^j_{\alpha\beta}, \\ J_2^3(B)^i_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}B^i_{\alpha\beta}+2J_2(B)^i_{\mu\nu}+\Sigma^{i\ \alpha}_{[\mu}\Sigma^{j\ \beta}_{\nu]}B^j_{\alpha\beta}, \\ J_2^4(B)^i_{\mu\nu} &= \frac{1}{2}B^i_{\mu\nu}+\frac{3}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}B^i_{\alpha\beta}+\frac{5}{2}J_2(B)^i_{\mu\nu}+\frac{5}{2}\Sigma^{i\ \alpha}_{[\mu}\Sigma^{j\ \beta}_{\nu]}B^j_{\alpha\beta}. \end{aligned} \tag{4.15}$$

This implies

$$J_2^4-2J_2^3-J_2^2+2J_2=0\qquad\text{or}\qquad J_2(J_2-2)(J_2-1)(J_2+1)=0, \tag{4.16}$$

which implies that the eigenvalues of J_2 are $2,1,-1,0$.

To characterise the eigenspaces we consider an arbitrary 3×3 matrix $M^{ij}=M^{ij}_s+M^{ij}_a, M^{ij}_s=M^{(ij)}_s, M^{ij}_a=M^{[ij]}_a$ and compute

$$J_2(M^{ij}\Sigma^j_{\mu\nu})=\mathrm{Tr}(M)\Sigma^i_{\mu\nu}-M^{ji}\Sigma^j_{\mu\nu}=\mathrm{Tr}(M)\Sigma^i_{\mu\nu}-M^{ij}_s\Sigma^j_{\mu\nu}+M^{ij}_a\Sigma^j_{\mu\nu}. \tag{4.17}$$

This means that the eigenspace of J_2 of eigenvalue 2 is spanned by multiples of $\Sigma^i_{\mu\nu}$. The eigenspace of eigenvalue 1 is S^2_+ spanned by $M^{ij}_a\Sigma^j_{\mu\nu}$. The eigenspace of eigenvalue -1 is S^4_+ spanned by $M^{ij}_s\Sigma^j_{\mu\nu}$ with $\mathrm{Tr}(M_s)=0$.

We can also apply the operator J_2 to objects of the type $h_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}$. We get

$$J_2(h_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]})=\frac{1}{2}h_\alpha{}^\alpha\Sigma^i_{\mu\nu}. \tag{4.18}$$

This means that the space $S^2_+\otimes S^2_-$ spanned by $h_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}$ with tracefree $h_{\mu\nu}$ is eigenspace of J_2 of eigenvalue 0.

All in all, we get

$$\Lambda^2\otimes E=(\Lambda^2\otimes E)_5\oplus(\Lambda^2\otimes E)_3\oplus(\Lambda^2\otimes E)_1\oplus(\Lambda^2\otimes E)_9. \tag{4.19}$$

The last space here is $(\Lambda^2\otimes E)_9=\Lambda^-\otimes E$.

5. Intrinsic torsion and the associated curvature

5.1. **Intrinsic torsion.** From general principles, it follows that the torsion of a G-structure should be described by a an object valued in $\Lambda^1(M)\otimes\mathfrak{g}^\perp$, which in our case is $\Lambda^1(M)\otimes E$. At the same time, the intrinsic torsion quantifies non-integrability of the G-structure, and thus the failure of the tensors defining this G-structure to be parallel with respect to the Levi-Civita connection. Thus, we expect that $\nabla_\mu\Sigma^i_{\alpha\beta}$ should be expressible via the intrinsic torsion $A^i_\mu\in\Lambda^1(M)\otimes E$. The following proposition is a statement to this effect

Theorem 5.1. There exists a set of objects $A^i_\mu\in\Lambda^1(M)\otimes E$ such that

$$\nabla_\mu\Sigma^i_{\alpha\beta}=-\epsilon^{ijk}A^j_\mu\Sigma^k_{\alpha\beta}. \tag{5.1}$$

Proof. Comparing the right-hand side of the formula (5.1) with the set of objects that appear in (4.13), we see that the statement is that there are no $S^4_+, C^\infty(M), S^2_+\otimes S^2_-$ irreducible components in $X^\mu\nabla_\mu\Sigma^i_{\alpha\beta}\in\Lambda^2(M)\otimes E, \forall X^\mu\in TM$. The $S^4_+, C^\infty(M)$ components are extracted as

$$2\Sigma^{(i|\alpha\beta|}\nabla_\mu\Sigma^{j)}_{\alpha\beta}=\nabla_\mu(\Sigma^{i|\alpha\beta|}\Sigma^j_{\alpha\beta})=4\nabla_\mu\delta^{ij}=0. \tag{5.2}$$

Here we have used the fact that the operation of raising-lowering of the indices commutes with ∇_μ. Similarly, the $S^2_+ \otimes S^2_-$ component is extracted as

$$2\Sigma^i_{(\mu}{}^\alpha \nabla_\rho \Sigma^i_{\nu)\alpha} = \nabla_\rho \Sigma^i_\mu{}^\alpha \Sigma^i_{\nu\alpha} = 3\nabla_\rho g_{\mu\nu} = 0. \tag{5.3}$$

This shows that no undesired components are present in $\nabla_\mu \Sigma^i_{\alpha\beta}$ and that (5.1) holds. □

5.2. **Bianchi identity.** Establishing a version of the formula (5.1) is one of the more laborious parts of the analysis of a non-integrable G-structure. The rest of the analysis is much more algorithmic. We take another covariant derivative and anti-symmetrise to get

$$2R_{\mu\nu[\rho}{}^\alpha \Sigma^i_{|\alpha|\sigma]} = 2\nabla_{[\mu}\nabla_{\nu]}\Sigma^i_{\rho\sigma} = -2\epsilon^{ijk}(\nabla_{[\mu}A^j_{\nu]}\Sigma^k_{\rho\sigma} + A^j_{[\nu}\nabla_{\mu]}\Sigma^k_{\rho\sigma}) = \tag{5.4}$$
$$-2\epsilon^{ijk}(\nabla_{[\mu}A^j_{\nu]}\Sigma^k_{\rho\sigma} + A^j_{[\mu}\epsilon^{klm}A^l_{\nu]}\Sigma^l_{\rho\sigma}) = -\epsilon^{ijk}F^j_{\mu\nu}\Sigma^k_{\rho\sigma},$$

where

$$F^i_{\mu\nu} := 2\nabla_{[\mu}A^i_{\nu]} + \epsilon^{ijk}A^j_\mu A^k_\nu \tag{5.5}$$

is the curvature of the connection A^i_μ. Importantly, we observe that, in the case of SU(2) structures in dimension four, the intrinsic torsion assembles itself into an $\mathfrak{su}(2)$-valued one-form, or an SU(2) connection. This connection gives rise to its curvature 2-form (5.5).

The left-hand side in (5.4) is just the projection of the Riemann tensor that is valued in the symmetric second power $\Lambda^2(M) \otimes_S \Lambda^2(M)$ onto E with respect to the second pair of indices. So, there is no loss of information if we multiply both sides of (5.4) with $\epsilon^{ijk}\Sigma^{j\rho\sigma}$ to get

$$R_{\mu\nu}{}^{\rho\sigma}\Sigma^k_{\rho\sigma} = 2F^k_{\mu\nu}. \tag{5.6}$$

This is the most useful form of the "Bianchi identity" (5.4), using the terminology of [3]. In words, the self-dual part of the Riemann curvature $R_{\mu\nu\rho\sigma}$ with respect to the pair of indices $\rho\sigma$ equals a multiple of the curvature tensor $F^i_{\mu\nu}$, which is also the curvature of the intrinsic torsion A^i_μ. The fact that the intrinsic torsion assembles itself into an SU(2) connection does not have analogs in the case of other G-structures.

5.3. **Ricci tensor.** We can extract the Ricci tensor from (5.6) via

$$\Sigma^i_\mu{}^\alpha R_{\alpha\nu\rho\sigma}\Sigma^i_{\rho\sigma} = (g_{\mu\rho}g^\alpha{}_\sigma - g_{\mu\sigma}g^\alpha{}_\rho + \epsilon_\mu{}^\alpha{}_{\rho\sigma})R_{\alpha\nu\rho\sigma} = -2R_{\mu\nu}, \tag{5.7}$$

where we used (4.4). On the other hand, applying this to the right-hand side of (5.6) we get

$$R_{\mu\nu} = -\Sigma^i_\mu{}^\alpha F^i_{\alpha\nu}. \tag{5.8}$$

Thus, in particular,

$$R = \Sigma^{i\mu\nu}F^i_{\mu\nu}. \tag{5.9}$$

The inverse of the formula for the Ricci curvature is

$$F^i_{\mu\nu} = \Psi^{ij}\Sigma^j_{\mu\nu} - \frac{R}{6}\Sigma^i_{\mu\nu} + R_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}. \tag{5.10}$$

Here Ψ^{ij} is the matrix of components of the chiral half of the Weyl curvature. Using $R_{\mu\nu} = \tilde{R}_{\mu\nu} + \frac{1}{4}Rg_{\mu\nu}$, where $\tilde{R}_{\mu\nu}$ is the tracefree part of the Ricci curvature, we can also rewrite this as

$$F^i_{\mu\nu} = \Psi^{ij}\Sigma^j_{\mu\nu} + \frac{R}{12}\Sigma^i_{\mu\nu} + \tilde{R}_{[\mu}{}^\alpha\Sigma^i_{|\alpha|\nu]}. \tag{5.11}$$

The first two terms here are self-dual as 2-forms, the last is anti-self-dual.

5.4. **Einstein condition.** As is well-known, the Riemann curvature viewed as a symmetric endomorphism of $\Lambda^2(M)$, decomposed into its self-dual and anti-self-dual blocks, reproduces the decomposition into Ricci and Weyl parts of the curvature. This is most usefully captured by the following matrix representation

$$\text{Riemann} = \begin{pmatrix} W^+ + R & Rc^0 \\ Rc^0 & W^- + R \end{pmatrix}. \tag{5.12}$$

Here $W^\pm$ are the two chiral halves of the Weyl curvature, and Rc^0 is the tracefree part of the Ricci tensor. The trace part is denoted by R and is the scalar curvature. The first row of this matrix is the self-dual part of Riemann with respect to the second pair of indices, and the second row is the anti-self-dual part. Similarly, the first (second) column is the self-dual (anti-self-dual) part of Reimann with respect to the first pair of indices. We thus see that the curvature $F^i_{\mu\nu}$ of the intrinsic torsion encodes precisely the first row of the matrix (5.12), and thus the self-dual part W^+ if the Weyl curvature, as well as all of the Ricci curvature.

It is now clear that the Einstein condition can be encoded as one on the curvature $F^i_{\mu\nu}$. The condition that $F^i_{\mu\nu}$ is self-dual as a 2-form is equivalent to the condition that the tracefree part Rc^0 of Ricci vanishes

$$F^i \in \Lambda^+ \Leftrightarrow Rc^0 = 0. \tag{5.13}$$

The scalar curvature can then be set to any desired value by imposing a condition on the self-dual part of F^i. All in all, Einstein equations are most usefully stated as the condition

$$F^i_{\mu\nu} = \left(\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}\right)\Sigma^j_{\mu\nu}. \tag{5.14}$$

Here Ψ^{ij} is an arbitrary symmetric tracefree 3×3 matrix, which encodes the W^+ part of the curvature, and is not constrained by the Einstein equations. The constant Λ is a multiple of the scalar curvature. The equation (5.14) is equivalent to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ Einstein condition.

6. Linearised analysis

The purpose of this section is to consider perturbations of SU(2) structures, and construct the most general diffeomorphism invariant Lagrangian for such perturbations. This linearised story provides a very good intuition for the non-linear story in the next section.

6.1. **Perturbation of a** SU(2) **structure.** The tangent space to the GL(4) orbit of Σ^i contains irreducible representations $(\Lambda^2 \otimes E)_{3+1+9}$. We can parametrise these spaces as

$$(\Lambda^2 \otimes E)_{1+9} \ni 2h_{[\mu}{}^{\alpha}\Sigma^i_{|\alpha|\nu]}, \qquad (\Lambda^2 \otimes E)_3 \ni 2\epsilon^{ijk}\Sigma^j_{\mu\nu}\xi^k, \tag{6.1}$$

with $h_{\mu\nu}$ being a symmetric tensor and $\xi^i \in E$. The role of the numerical factors chosen is to simplify some formulas that follow. This means that perturbations of $\Sigma^i_{\mu\nu}$, which we denote by $\delta\Sigma^i_{\mu\nu} := \sigma^i_{\mu\nu}$ can be parametrised as

$$\sigma^i_{\mu\nu} = 2h_{[\mu}{}^{\alpha}\Sigma^i_{|\alpha|\nu]} + 2\epsilon^{ijk}\Sigma^j_{\mu\nu}\xi^k. \tag{6.2}$$

The inverse is given by

$$h_{\mu\nu} = -\frac{1}{2}\sigma^i_{(\mu}{}^{\alpha}\Sigma^i_{|\alpha|\nu)} - \frac{1}{12}\eta_{\mu\nu}\Sigma^{i\rho\sigma}\sigma^i_{\rho\sigma}, \tag{6.3}$$
$$\xi^i = -\frac{1}{16}\epsilon^{ijk}\Sigma^{j\mu\nu}\sigma^k_{\mu\nu}.$$

6.2. **Transformation properties under diffeomorphisms.** Let us consider a background of a constant triple of 2-forms Σ^i. The diffeomorphisms act $\delta_X\Sigma^i = \mathcal{L}_X\Sigma^i = i_X d\Sigma^i + di_X\Sigma^i$. In the case of a constant triple of 2-forms we get $\delta_X\sigma^i = di_X\Sigma^i$. In index notation

$$\delta_X\sigma^i_{\mu\nu} = 2\partial_{[\mu}X^\alpha\Sigma^i_{|\alpha|\nu]}. \tag{6.4}$$

This means that

$$\delta_X h_{\mu\nu} = \partial_{(\mu}X_{\nu)}, \qquad \delta_X\xi^i = \frac{1}{4}\Sigma^{i\mu\nu}\partial_\mu X_\nu. \tag{6.5}$$

6.3. **Transformation properties under** SU(2)**.** In addition to diffeomorphisms, we can also consider how quantities transform under the SU(2) transformations that rotate Σ^i. The infinitesimal version of these transformations is

$$\delta_\phi\sigma^i_{\mu\nu} = 2\epsilon^{ijk}\Sigma^j_{\mu\nu}\phi^k. \tag{6.6}$$

Under these transformations

$$\delta_\phi h_{\mu\nu} = 0, \qquad \delta_\phi\xi^i = \phi^i. \tag{6.7}$$

6.4. **Second order action functional.** We now determine the most general diffeomorphism invariant action functional that can be written in terms of fields $h_{\mu\nu}$ and ξ^i, subject to the transformation properties (6.5). We first write the general linear combination of all possible terms. The types of terms are dictated by simple representation theory. First, we can write the most general linear combination of terms that can be constructed solely from $h_{\mu\nu}$. This is standard and independent of the dimension. We write this as

$$\frac{\rho}{2}(\partial_\mu h_{\nu\rho})^2 + \frac{\alpha}{2}(\partial_\mu h)^2 - \beta h\partial^\mu\partial^\nu h_{\mu\nu} - \gamma(\partial^\mu h_{\mu\nu})^2. \tag{6.8}$$

We then need to determine all possible terms involving two copies of ξ^i, as well as $h\xi$ terms. The field $h_{\mu\nu}$ lives in $S^2_+\otimes S^2_-$ as well as $C^\infty(M)$. The field ξ^i is in S^2_+. We have the following tensor products

$$\begin{aligned}(S^2_+\otimes S^2_-)\otimes S^2_+ &= (S^4_+\otimes S^2_-)\oplus(S^2_+\otimes S^2_-)\oplus S^2_-, \\ S^2_+\otimes_s S^2_+ &= S^4_+\oplus C^\infty(M),\end{aligned} \tag{6.9}$$

where $\otimes_s$ means symmetrised tensor product. We need to combine these irreducible pieces with those arising from the symmetrised product of two partial derivatives, which is in $S^2_+\otimes S^2_-\oplus C^\infty$. This makes it clear that the only term that can be constructed from two copies of ξ^i is $(\partial_\mu\xi^i)^2$. There is also just a single term that can be constructed from $h_{\mu\nu}$ and ξ^i, which is

$$(\partial^\mu h_{\mu\nu})(\partial^\alpha\xi^i)\Sigma^{i\ \nu}_{\alpha}. \tag{6.10}$$

This gives the following most general Lagrangian

$$\mathcal{L} = \frac{\rho}{2}(\partial_\mu h_{\nu\rho})^2 + \frac{\alpha}{2}(\partial_\mu h)^2 - \beta h\partial^\mu\partial^\nu h_{\mu\nu} - \gamma(\partial^\mu h_{\mu\nu})^2 + \frac{\lambda}{2}(\partial_\mu\xi^i)^2 + \mu(\partial^\mu h_{\mu\nu})(\partial^\alpha\xi^i)\Sigma^{i\ \nu}_{\alpha}.$$

6.5. **Diffeomorphism invariant Lagrangian.** We now calculate the effect of a diffeomorphism on $\mathcal{L}$, integrating by parts when necessary. We use the symbol $\approx$ to denote equality modulo integration by parts. We have

$$\begin{aligned}\delta_X\mathcal{L} \approx (\rho-\gamma+\frac{\mu}{4})\partial^2(\partial^\mu h_{\mu\nu})X^\nu - (\alpha+\beta)\partial^2 h(\partial X) + (-\beta+\gamma+\frac{\mu}{4})(\partial X)(\partial^\mu\partial^\nu h_{\mu\nu}) \\ -(\frac{\lambda}{4}+\frac{\mu}{2})\partial^2\xi^i\Sigma^{i\mu\nu}\partial_\mu X_\nu.\end{aligned}$$

Equating the coefficients in front of the independent terms to zero, and parametrising the solution by ρ, μ we have

$$\beta = -\alpha = \rho + \frac{\mu}{2}, \quad \gamma = \rho + \frac{\mu}{4}, \qquad \lambda = -2\mu. \tag{6.11}$$

The most general diffeomorphism invariant Lagrangian is then

$$\mathcal{L} = \rho\mathcal{L}_{GR} + \mu\mathcal{L}', \tag{6.12}$$

with

$$\mathcal{L}_{GR} = \frac{1}{2}(\partial_\mu h_{\nu\rho})^2 - \frac{1}{2}(\partial_\mu h)^2 - h\partial^\mu\partial^\nu h_{\mu\nu} - (\partial^\mu h_{\mu\nu})^2, \tag{6.13}$$

$$\mathcal{L}' = -\frac{1}{4}(\partial_\mu h)^2 - \frac{1}{2}h\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{4}(\partial^\mu h_{\mu\nu})^2 - (\partial_\mu \xi^i)^2 + (\partial^\mu h_{\mu\nu})(\partial^\alpha \xi^i)\Sigma^{i\ \nu}_{\alpha}.$$

We note that this is a very similar story to what happens in the case of Spin(7) structures in eight dimensions, see [4]. In that context, as here, the most general diffeomoprhism invariant Lagrangian is also given by a linear combination of two terms.

6.6. **Lagrangian that is also SU(2) gauge invariant.** The Lagrangian (6.12) is diffeomoprhism invariant (modulo integration by parts). It is also invariant under global (i.e. rigid) SO(3) rotations acting on E. However, it is clear that it is possible to demand also local SO(3) invariance. From (6.7) we see that these transformations act only on ξ^i. It is clear that the Lagrangian $\mathcal{L}'$ is not invariant under such local transformations. Therefore, only $\mathcal{L}_{GR}$ is both diffeomorphism and SU(2) gauge invariant. It is therefore to be expected that there exists a unique non-linear Lagrangian for $\Sigma^i_{\mu\nu}$, which is second order in derivatives, and both diffeomorphism and SU(2) gauge invariant. We can also expect this non-linear action to have critical points that are Einstein metrics. This is exactly what happens, as we shall now verify.

7. Action functionals

In preparation to the construction of the action, we will first show that the intrinsic torsion is completely determined by the exterior derivative $d\Sigma^i$.

7.1. **Torsion in terms of $d\Sigma^i$.** In (5.1) we have related the torsion A^i_μ to the covariant derivative $\nabla_\mu \Sigma^i_{\alpha\beta}$ of the 2-forms Σ^i. We now explain that the knowledge of the exterior derivative is sufficient

Theorem 7.1. The intrinsic torsion is determined by the exterior derivatives of the 2-forms Σ^i. Specifically, we have

$$A = \frac{1}{4}(J_\Sigma - \mathbb{I})({}^*d\Sigma^i), \tag{7.1}$$

where J_Σ is the operator (4.6) and ${}^*d\Sigma^i$ is the Hodge dual of the 3-form $d\Sigma^i$.

Proof. We project the equation (5.1) to the space of 3-forms, anti-symmetrising over all 3 indices. We have

$$\partial_{[\nu}\Sigma^i_{\alpha\beta]} = -\epsilon^{ijk}A^j_{[\nu}\Sigma^k_{\alpha\beta]}. \tag{7.2}$$

We can write this in index-free differential form notations as

$$d\Sigma^i + \epsilon^{ijk}A^j\Sigma^k = 0. \tag{7.3}$$

To solve this, we multiply with the ϵ tensor and use the self-duality of $\Sigma^i_{\mu\nu}$

$$\epsilon^{\mu\nu\alpha\beta}\partial_\nu\Sigma^i_{\alpha\beta} = -2\epsilon^{ijk}A^j_\nu\Sigma^{k\mu\nu} = 2(J_\Sigma(A))^{i\mu}, \tag{7.4}$$

where J_Σ is the operator on $\Lambda^1(M)\otimes E$ that was introduced in (4.6). We can write this in an index-free way as

$${}^*d\Sigma^i = 2J_\Sigma(A). \tag{7.5}$$

The J_Σ operator is invertible, with the inverse given by

$$J_\Sigma^{-1} = \frac{1}{2}(J_\Sigma - \mathbb{I}). \tag{7.6}$$

This establishes (7.1). □

We have an immediate well-known corollary.

Corollary 7.1. An SU(2) structure is integrable if and only if $d\Sigma^i = 0$.

7.2. **Calculation of A.** We can obtain an explicit expression for the intrinsic torsion in terms of $d\Sigma^i$. We have

$$(*d\Sigma)^i_\mu = \epsilon_\mu{}^{\alpha\beta\gamma}\partial_\alpha\Sigma^i_{\beta\gamma}, \tag{7.7}$$

and

$$A^i_\mu = \frac{1}{4}(J_\Sigma - \mathbb{I})(*d\Sigma)^i_\mu = -\frac{1}{4}\epsilon_\mu{}^{\alpha\beta\gamma}\partial_\alpha\Sigma^i_{\beta\gamma} + \frac{1}{4}\epsilon^{ijk}\Sigma^{j\,\alpha}_\mu\epsilon_\alpha{}^{\beta\gamma\delta}\partial_\beta\Sigma^k_{\gamma\delta}. \tag{7.8}$$

We can can simplify the last term using

$$\epsilon^{\mu\nu\rho\sigma}\Sigma^i_{\alpha\sigma} = 3\delta^{[\rho}_\alpha\Sigma^{i\mu\nu]}. \tag{7.9}$$

This gives

$$A^i_\mu = -\frac{1}{4}\epsilon_\mu{}^{\alpha\beta\gamma}\partial_\alpha\Sigma^i_{\beta\gamma} - \frac{1}{4}\epsilon^{ijk}\Sigma^{j\alpha\beta}\partial_\mu\Sigma^k_{\alpha\beta} - \frac{1}{2}\epsilon^{ijk}\Sigma^{j\alpha\beta}\partial_\beta\Sigma^k_{\mu\alpha}. \tag{7.10}$$

This expression is useful for the action functionals described below.

7.3. **Bianchi identity.** A useful consequence of (7.3) is obtained by taking its exterior derivative. We get

$$\epsilon^{ijk}dA^j\Sigma^k - \epsilon^{ijk}A^j d\Sigma^k = 0. \tag{7.11}$$

We now substitute $d\Sigma^k$ from (7.3) as $d\Sigma^k = -\epsilon^{klm}A^l\Sigma^m$. We then use $A^jA^l = (1/2)\epsilon^{jls}\epsilon^{spq}A^pA^q$ to rewrite

$$\epsilon^{ijk}A^j\epsilon^{klm}A^l\Sigma^m = \epsilon^{ijk}(\frac{1}{2}\epsilon^{jlm}A^lA^m)\Sigma^k. \tag{7.12}$$

All in all, we get

$$\epsilon^{ijk}F^j\Sigma^k = 0, \tag{7.13}$$

where F^i is the curvature

$$F^i = dA^i + \frac{1}{2}\epsilon^{ijk}A^jA^k. \tag{7.14}$$

We note that (7.13) can be interpreted as the statement that there is no S^2_+ component in the decomposition of the $F \in E\otimes\Lambda^2$ into its irreducible components.

7.4. **Transformation properties under diffeomorphisms.** Under diffeomorphisms $\delta_X\Sigma^i = di_X\Sigma^i + i_Xd\Sigma^i$. We now assume that the intrinsic torsion solves (7.3) and determine how it transforms under diffeomorphisms. Taking the variation of (7.3) we have

$$d(i_Xd\Sigma^i) + \epsilon^{ijk}\delta_XA^j\Sigma^k + \epsilon^{ijk}A^j(di_X\Sigma^i + i_Xd\Sigma^i) = 0. \tag{7.15}$$

We can also insert the vector field X into (7.3) to get

$$i_Xd\Sigma^i + \epsilon^{ijk}(i_XA^j)\Sigma^k - \epsilon^{ijk}A^ji_X\Sigma^k = 0. \tag{7.16}$$

Substituting $i_Xd\Sigma^i$ from here into (7.15) we have

$$\begin{aligned} d(\epsilon^{ijk}A^ji_X\Sigma^k - \epsilon^{ijk}(i_XA^j)\Sigma^k) + \epsilon^{ijk}\delta_XA^j\Sigma^k + \epsilon^{ijk}A^jdi_X\Sigma^k \\ +\epsilon^{ijk}A^j(\epsilon^{klm}A^li_X\Sigma^m - \epsilon^{klm}(i_XA^l)\Sigma^m) = 0. \end{aligned} \tag{7.17}$$

The terms in the first line become

$$\epsilon^{ijk}dA^ji_X\Sigma^k - \epsilon^{ijk}d(i_XA^j)\Sigma^k + \epsilon^{ijk}(i_XA^j)\epsilon^{klm}A^l\Sigma^m + \epsilon^{ijk}\delta_XA^j\Sigma^k, \tag{7.18}$$

where we have used (7.3) again. The first term in the second line can also be simplified. We again use $A^jA^l = (1/2)\epsilon^{jls}\epsilon^{spq}A^pA^q$ to get

$$\epsilon^{ijk}A^j\epsilon^{klm}A^li_X\Sigma^m = \epsilon^{ijk}(\frac{1}{2}\epsilon^{jlm}A^lA^m)i_X\Sigma^k. \tag{7.19}$$

This means that (7.17) can be rewritten as

$$\epsilon^{ijk}F^ji_X\Sigma^k + \epsilon^{ijk}(\delta_XA^j - d(i_XA^j))\Sigma^k + \epsilon^{ijk}(i_XA^j)\epsilon^{klm}A^l\Sigma^m - \epsilon^{ilk}A^l\epsilon^{kjm}(i_XA^j)\Sigma^m = 0,$$

where we changed the names of the dummy indices suggestively. The last two terms can be simplified using the identity

$$\epsilon^{ijk}\epsilon^{klm} + \epsilon^{ilk}\epsilon^{kmj} + \epsilon^{imk}\epsilon^{kjl} = 0. \tag{7.20}$$

This gives

$$\epsilon^{ijk}F^j i_X \Sigma^k + \epsilon^{ijk}(\delta_X A^j - d(i_X A^j) - \epsilon^{jpq}A^p(i_X A^q))\Sigma^k = 0. \tag{7.21}$$

We can finally insert X into (7.13) to rewrite the first term here as $-\epsilon^{ijk}i_X F^j \Sigma^k$. Overall, this produces terms that are all of the type of operator J_Σ acting on an E-valued 1-form. The operator J_Σ is invertible, which allows us to write

$$\delta_X A^i = d(i_X A^i) + \epsilon^{ijk}A^j(i_X A^k) + i_X F^i. \tag{7.22}$$

The first two terms here assemble into the covariant derivative of $i_X A^i$ computed using the connection A^i. The last term is the insertion of X into the curvature F^i. This is of course as expected, because using the formula for F^i and noting a cancelation this can be rewritten as

$$\delta_X A^i = d i_X A^i + i_X dA^i. \tag{7.23}$$

This confirms that the torsion transforms covariantly under diffeomorphisms, and gives a very useful formula (7.22).

7.5. **Transformation properties under** SU(2) **gauge transformations.** Let us also determine how the torsion transforms under the local SU(2) gauge transformations $\delta_\phi \Sigma^i = \epsilon^{ijk}\phi^k\Sigma^k$. Taking the variation of (7.3) we have

$$d(\epsilon^{ijk}\phi^j\Sigma^k) + \epsilon^{ijk}\delta_\phi A^j \Sigma^k + \epsilon^{ijk}A^j\epsilon^{klm}\phi^l\Sigma^m = 0. \tag{7.24}$$

The first term gives a contribution containing $d\phi^i$, as well as one with $d\Sigma^i$. The latter can be transformed using (7.3). This gives

$$\epsilon^{ijk}(\delta_\phi A^j + d\phi^j)\Sigma^k + \epsilon^{ijk}A^j\epsilon^{klm}\phi^l\Sigma^m - \epsilon^{ijk}\phi^j\epsilon^{klm}A^l\Sigma^m = 0. \tag{7.25}$$

The last two terms can again be transformed using (7.20). This puts all terms in the same form of J_Σ acting on an E-valued 1-form. Because J_Σ is invertible we get

$$\delta_\phi A^i = -d\phi^i - \epsilon^{ijk}A^j\phi^k, \tag{7.26}$$

which is the usual gauge transformation with parameter $-\phi^i$. This implies that the curvature F^i transforms covariantly

$$\delta_\phi F^i = \epsilon^{ijk}\phi^j F^k. \tag{7.27}$$

7.6. **Diffeomorphism invariant action.** We have confirmed that the torsion transforms covariantly under diffeomorphisms. This means that any action that is schematically of the type $\int A^2$ is diffeomorphism invariant. Now, the representation theoretic decomposition (4.9), (4.10) of $A \in \Lambda^1 \otimes E$ shows that there are two irreducible components of the intrinsic torsion. This means that there are only two quadratic invariants that can be constructed from A. One can always take as a basis of such invariants the quantities $(A^i_\mu)^2$ and $A^i_\mu J_\Sigma(A)^{i\mu}$. It can be confirmed that the linearisation of this general diffeomorphism invariant action coincides with the linearised action (6.12). This establishes that the most general diffeomorphism invariant action for Σ, which is second order in derivatives and is also invariant under global SO(3) rotations of Σ, is given by a linear combination of $\int (A^i_\mu)^2$ and $\int A^i_\mu J_\Sigma(A)^{i\mu}$.

7.7. **Diffeomorphism and** SU(2) **invariant action.** Let us now impose the requirement that the action is both diffeomorphism and SU(2) gauge invariant. At the linearised level, we have seen that this has the effect that only one of the two diffeomorphism invariant terms survives, and one gets linearised Einstein-Hilbert action. It is clear that from the two terms A^2 and $AJ_\Sigma(A)$ the first one is not gauge invariant. Using physics terminology, this term is a mass term for the connection, which cannot be gauge invariant. Let us discuss the other term. We claim that it is both diffeomorphism and SU(2) invariant. To see this, it is best to rewrite it using some integration by parts identities. Consider $\int \Sigma^i dA^i$. Integrating by parts we have

$$\int \Sigma^i dA^i \approx -\int d\Sigma^i A^i = -\int \epsilon^{ijk} A^i A^j \Sigma^k. \tag{7.28}$$

In the last equality we have used (7.3). The quantity on the right-hand side is a multiple of $AJ_\Sigma(A)$. This means that

$$\int \Sigma^i F^i = \int \Sigma^i (dA^i + \frac{1}{2}\epsilon^{ijk} A^j A^k) \approx -\frac{1}{2}\int \Sigma^i \epsilon^{ijk} A^j A^k. \tag{7.29}$$

The integrand on the left is built from objects that transform covariantly under local SU(2) gauge transformations and is invariant under them. The integral is then both diffeomorphism and gauge invariant. This means that this is also the case for the object on the left-hand side. This establishes that there is unique action for SU(2) structures in dimension four that is both diffeomorphism and SU(2) gauge invariant. It is of the schematic type torsion squared, and is given by

$$S[\Sigma] = -\frac{1}{2}\int_M \Sigma^i \epsilon^{ijk} A^j(\Sigma) A^k(\Sigma), \tag{7.30}$$

where we now indicated that the connection (intrinsic torsion) is determined by Σ. This is the action described in the Introduction, see (1.9). Our discussion above makes it clear that the action (7.30) is "the best" second order in derivatives action that can be written for SU(2) structures. It is the best action because it is the unique action that in addition to diffeomorphism invariance also possesses SU(2) gauge invariance. As we shall see below, it is also best in the sense that its critical points are Einstein. One can substitute the expression for $A(\Sigma)$ given by (7.10) to obtain a second order in derivatives action for Σ.

7.8. **Plebanski action and Einstein condition.** We can now discuss the Plebanski action, which is a first-order in derivatives version of (7.30). The idea is to write an action that is a functional of both Σ^i and an independent E-valued one-form field A^i, such that the Euler-Lagrange equations for A^i coincide with (7.3). A moment of reflect shows that this action is $\int \Sigma^i F^i$. This action is then to be supplemented by the constraint terms that guarantee that Σ^i satisfy their algebraic constraints. One is also free to add to this action the volume term with an arbitrary coefficient. This produces the action known in the literature as the Plebanski action [7]. It is given by

$$S[\Sigma, A, \Psi] = \int_M \Sigma^i (dA^i + \frac{1}{2}\epsilon^{ijk} A^j A^k) - \frac{1}{2}\left(\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}\right)\Sigma^i \Sigma^j. \tag{7.31}$$

Here Ψ^{ij} is an arbitrary traceless symmetric 3×3 matrix, whose components serve as Lagrange multipliers to impose the constraints $\Sigma^i\Sigma^j \sim \delta^{ij}$. Indeed, the variation with respect to the field Ψ^{ij} gives $\Sigma^i\Sigma^j \sim \delta^{ij}$, which are the algebraic conditions that need to be satisfied by an SU(2) structure defining 2-forms Σ^i. It is also not difficult to see that its Euler-Lagrange equation arising by varying with respect to A^i is precisely (7.3), and the Euler-Lagrange equation arising by varying with respect to Σ^i is precisely (5.14).

This establishes the following

Theorem 7.2. The critical points of (7.30), or equivalently of (7.31) are SU(2) structures whose associated metric is Einstein.

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References

[1] Lucio Bedulli, Luigi Vezzoni, "The Ricci tensor of SU(3)-manifolds," J. Geom. Phys. 57 (2007), n. 4, 1125-1146, arXiv:math/0606786 [math.DG].
[2] Spiro Karigiannis, "Flows of Spin(7)-structures," arXiv:0709.4594 [math.DG].
[3] Shubham Dwivedi, Panagiotis Gianniotis, Spiro Karigiannis, "Flows of G2-structures, II: Curvature, torsion, symbols, and functionals," arXiv:2311.05516 [math.DG].
[4] K. Krasnov, "Dynamics of Cayley Forms," [arXiv:2403.16661 [math.DG]].
[5] S. Dwivedi, "A gradient flow of Spin(7)-structures," arXiv:2404.00870 [math.DG].
[6] U. Fowdar, "On flows of SU(2) structures," talk at "Special Holonomy and Geometric Structures on Complex Manifolds" meeting, March 2024, link to the talk.
[7] J. F. Plebanski, "On the separation of Einsteinian substructures," J. Math. Phys. **18** (1977), 2511-2520 doi:10.1063/1.523215
[8] R. Bryant, "Some remarks on G_2-structures," arXiv:math/0305124 [math.DG].

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