Comments on Celestial CFT and AdS_3 String Theory

Igor Mol

State University of Campinas (Unicamp)[*]

In a recent work, Ogawa *et al.* [1] proposed a model for celestial conformal field theory (CFT) based on the H_3^+-Wess-Zumino-Novikov-Witten (WZNW) model. In this paper, we extend the model advanced by Ogawa *et al.* [1], demonstrating how it can holographically generate tree-level MHV scattering amplitudes for both gluons and gravitons when analytically continued to the ultra-hyperbolic Klein space $\mathbf{R}_2^2$, thereby offering an alternative to celestial Liouville theory. We construct a holographic dictionary in which vertex operators and conformal primaries in celestial CFT are derived from their worldsheet counterparts in Euclidean AdS_3 (bosonic) string theory. Within this dictionary, we derive the celestial stress-energy tensor, compute the two- and three-point functions, and determine the celestial operator product expansion (OPE). Additionally, we derive a system of partial differential equations that characterises the celestial amplitudes of our model, utilising the Knizhnik–Zamolodchikov (KZ) equations and worldsheet Ward identities. In the Appendix, we provide a concise introduction to the H_3^+-WZNW model, with emphasis on its connection to Euclidean AdS_3 string theory.

[*] igormol@ime.unicamp.br

CONTENTS

I. INTRODUCTION

It has been recently observed in a series of works by Stieberger, Taylor, and Zhu [2, 3], Giribet [4], Melton *et al.* [5], Mol [6], within the framework of what may be termed "celestial Liouville theory," that one can holographically derive the celestial amplitudes for gluons in pure Yang-Mills theory and for gravitons in General Relativity (GR), formulated perturbatively on a flat spacetime background, from the correlation functions of Liouville vertex operators in the semiclassical limit $b \to 0$ of Liouville field theory. These Liouville vertex operators are appropriately dressed by an $SO(2N)$ level-one Kac-Moody current algebra on the celestial sphere, with the holographic correspondence understood in the large-N limit.

However, drawing upon the work of Teschner [7, 8, 9, 10] and Ribault and Teschner [11], it is well known that the semiclassical limit of Liouville field theory correlation functions coincides with the mini-superspace limit of the H_3^+-WZNW model. In the context of celestial conformal field theory (CFT), this raises the question of whether this coincidence hints at a deeper connection or whether there exists a more compelling model for celestial CFT based on the H_3^+-WZNW model instead of Liouville theory.

The purpose of this work is to explore this question, and we shall affirmatively demonstrate that the H_3^+-WZNW model provides a more robust foundation for celestial CFT. Building on the insights of de Boer *et al.* [12], Giveon, Kutasov, and Seiberg [13], and interpreting the H_3^+-WZNW model as the Euclidean continuation of AdS_3 (bosonic) string theory, we propose a holographic dictionary wherein the vertex operators, correlation functions, and stress-energy tensor of celestial CFT are unambiguously derived from their counterparts in Euclidean AdS_3 string theory. We will demonstrate in detail how, via this holographic dictionary, one can derive the tree-level celestial amplitudes for the scattering of MHV gluons and gravitons, analytically continued to the ultra-hyperbolic Klein space $\mathbf{R}^2_2$, within the framework of celestial leaf amplitudes as introduced by Casali, Melton, and Strominger [14], Melton, Sharma, and Strominger [15]. Furthermore, as an application of our dictionary, we shall investigate the exact celestial two- and three-point functions, as well as the structure constants of the celestial operator product expansion (OPE). Additionally, we will derive a system of partial differential equations (PDEs) characterising the celestial amplitudes in the frequency-momentum representation, following from our dictionary via the Knizhnik–Zamolodchikov (KZ) equations and worldsheet Ward identities.

It is worth noting that a similar model based on the H_3^+-WZNW model was recently proposed by Ogawa *et al.* [1], and we have benefited greatly from their insights. However, there are significant differences between their model and the one we present. Specifically, the hyperboloid H_3^+ in Ogawa *et al.* [1]'s work correspond to the hyperbolic foliation of momentum space, whereas in our construction, the hyperboloid associated with the string target space is identified with the hyperbolic foliation of Klein space $\mathbf{R}^2_2$, as discussed in Casali, Melton, and Strominger [14], Melton, Sharma, and Strominger [15]. This distinction introduces a crucial difference: in Ogawa *et al.* [1]'s model, the authors were compelled to define the celestial conformal primaries such that the celestial three-point amplitude vanishes, a constraint imposed by Minkowski-space kinematics. In contrast, since we work in the split signature of Klein space, we are not subject to this vanishing constraint, and indeed, we shall analyse the three-point function in our celestial CFT and its relationship to the structure constants of the celestial OPE.

Another notable distinction is that Ogawa *et al.* [1] omitted consideration of the worldsheet variables in their construction of the final form of the correlation functions. In contrast, our holographic dictionary prescribes that one must integrate over the worldsheet coordinates to obtain a celestial amplitude from the correlation functions of AdS_3 string theory. As we shall demonstrate in detail, the factors involving products of gamma functions, arising from worldsheet integrals, play a key role in determining the structure constants of the celestial OPE.

This paper is structured as follows. In Section II, we begin by stating the postulates of our holographic dictionary, and move to the derivation of its consequences, starting with the holographic derivation of tree-level MHV amplitudes for both pure Yang-Mills theory and Einstein's gravity. In Section III, we will demonstrate how our construction uniquely determines the two- and three-point functions as well as the celestial OPE. Additionally, we will examine a system of partial differential equations characterising the celestial amplitudes, derived from our holographic dictionary in frequency space, and, in Section IV, we summarise our findings and discuss potential avenues for further research stemming from this work.

II. HOLOGRAPHIC CONSTRUCTION OF TREE-LEVEL MHV AMPLITUDES

We shall start our discussion in Subsection II A by postulating the entries of our holographic dictionary, which maps correlation functions, vertex operators, and conformal primaries in Euclidean AdS_3 (bosonic) string theory to their counterparts in celestial CFT. This construction builds upon the model advanced by Ogawa *et al.* [1]. Subsequently, in Subsections II B and II C, we will demonstrate that our dictionary enables the holographic derivation of tree-level MHV scattering amplitudes in both Yang-Mills theory and Einstein's gravity. The results presented herein build upon a series of works developed in the context of AdS_3/CFT_2, specially those by de Boer *et al.* [12], Giveon, Kutasov, and Seiberg [13], Maldacena and Ooguri [16], Maldacena, Ooguri, and Son [17], Giribet and Nunez [18], Bars, Deliduman, and Minic [19], and celestial Liouville theory, including those by Stieberger, Taylor, and Zhu [2, 3], Giribet [4], Melton *et al.* [5], Mol [6].

A. The Holographic Dictionary

The first entry of our holographic dictionary draws inspiration from the analysis presented by de Boer *et al.* [12, Section 4], and posits that to each sequence of worldsheet vertex operators $V_1(x_1; z_1)$, ..., $V_n(x_n; z_n)$, there corresponds a family of celestial vertex operators $\mathcal{V}_1(x_1)$, ..., $\mathcal{V}_n(x_n)$, such that the correlation function of the latter is expressed in terms of the correlation function of the former as follows:

$$\langle \mathcal{V}_1(x_1) ... \mathcal{V}_n(x_n) \rangle = \frac{1}{W} \prod_{i=1}^{n} \int d^2 z_i \, \langle\langle V_1(x_1; z_1) ... V_n(x_n; z_n) \rangle\rangle \,, \tag{1}$$

where W represents a normalisation constant associated with the worldsheet "area," ensuring that the worldsheet integrals are well-defined and finite. Here, $\langle ... \rangle$ denotes the correlation function in celestial CFT, while $\langle\langle ... \rangle\rangle$ denotes the correlation function in the worldsheet CFT. This formulation is consistent with the conventional framework of string perturbation theory (see D'Hoker and Phong [20] and Polchinski [21]). The worldsheet integrals in Eq. (1) represent a key distinction between our holographic dictionary and that proposed by Ogawa *et al.* [1]. In contrast to their approach, where the worldsheet variables $z_i, \bar{z}_i$ are omitted in the final form of the correlation function, we incorporate these variables by performing explicit integration over the worldsheet. As will be demonstrated in Subsection III C, the product of gamma functions that arises from the worldsheet integrals in Eq. (1) plays an important role in determining the structure constants of the celestial operator product expansion.

The second entry of our holographic dictionary derives its inspiration from the model proposed by Ogawa *et al.* [1], yet it introduces a significant distinction arising from the split signature of Klein space $\mathbf{R}^2_2$, to which we shall analytically continue the scattering amplitudes of Yang-Mills theory and Einstein's gravity in Subsections II B and II C. We propose that each conformal primary $\Phi^j(x;z)$ with spin j in the H_3^+-WZNW model corresponds to a celestial (scalar) wavefunction $\phi_\Delta(x)$ with conformal weight Δ in the celestial CFT, such that the H_3^+-WZNW spin and the celestial scaling dimension are related by the equation $j + \Delta/2 = 0$, as suggested by Ogawa *et al.* [1] based on an analysis of the transformation properties of $\Phi^j(x;z)$ and $\phi_\Delta(x)$ under conformal transformations. Thus, the principal distinction between our proposal and that of Ogawa *et al.* [1] lies in the signature of the spacetime: while Ogawa *et al.* [1] works in Minkowski signature, where Lorentzian kinematics necessitated selecting a celestial conformal primary as a linear combination of $\Phi^j(x;z)$ and its shadow transform–resulting in a vanishing three-point function–our model works in the split signature, freeing us from the constraint of the vanishing three-point function. In fact, as we shall demonstrate in Subsections III B and III C, the three-point function in our model is closely related to the structure constants of the celestial operator product expansion.

The third and final postulate of our holographic dictionary takes its form from the construction proposed by Giveon, Kutasov, and Seiberg [13], which establishes a correspondence between worldsheet and spacetime Kac-Moody and Virasoro currents in the context of AdS_3 string theory, reviewed in detail in Appendix A 2. The third entry can be stated as follows:

1. First, we assert that, given a set of worldsheet level-$\hat{k}_G$ Kac-Moody currents $j^a(z)$, transforming under the fundamental representation of a Lie group G with structure constants f^{abc}, and satisfying the OPE:

$$j^a(z)\, j^b(w) \sim \frac{\hat{k}_G \delta^{ab}}{(z-w)^2} + \frac{i f^{abc} j^c(w)}{z-w}, \tag{2}$$

there corresponds a family of celestial Kac-Moody currents $\mathcal{O}^a(x)$ defined by:

$$\mathcal{O}^a(x) = -\frac{1}{k}\int d^2z\, j^a(z)\, \bar{J}(\bar{x};\bar{z})\, \Phi^{j=1}(x;z), \tag{3}$$

where k is the level of the H_3^+-WZNW model, and $\bar{J}(\bar{x};\bar{z})$ is the worldsheet current introduced in Eq. (A21) of Appendix A 1 c. It is shown therein that these celestial CFT currents obey the OPE:

$$\mathcal{O}^a(x)\, \mathcal{O}^b(y) \sim \frac{\hat{k}_G \mathcal{I} \delta^{ab}}{(x-y)^2} + \frac{i f^{abc} \mathcal{O}^c(y)}{x-y}, \tag{4}$$

with central extension:

$$\mathcal{I} := \frac{1}{k^2} \int d^2 z J(x;z)\, \bar{J}(\bar{x};\bar{z})\, \Phi(x;z)\,. \tag{5}$$

It is further demonstrated in the Appendix, following the work of Giveon, Kutasov, and Seiberg [13], that the above definition of the central extension is independent of $x, \bar{x}$, thereby confirming the mathematical consistency of the OPE.

2. Second, we postulate that the celestial stress-energy tensor $T(x)$ is expressed in terms of worldsheet currents and primaries as:

$$T(x) = \frac{1}{2} \oint \frac{dz}{2\pi i} \left(\partial_x J \partial_x \lambda + 2\lambda \partial_x^2 J \right), \tag{6}$$

and satisfies the Ward identities:

$$T(x)\, \mathcal{O}^a(y) \sim \frac{h}{(x-y)^2} \mathcal{O}^a(y) + \frac{1}{x-y} \partial_y \mathcal{O}^a(y)\,, \tag{7}$$

and:

$$T(x_1)\, T(x_2) \sim \frac{c}{(x_1-x_2)^4} + \frac{2T(x_2)}{(x_1-x_2)^2} + \frac{\partial_{x_2} T(x_2)}{x_1-x_2}, \tag{8}$$

as expected from a generator of the Virasoro algebra in the celestial CFT. The celestial stress-energy tensor and the derivation of these Ward identities from the aforementioned definition will be discussed in detail in Subsection III E.

Having now articulated the three entries of our holographic dictionary, we shall proceed to derive their implications. In the subsequent Subsections II B and II C, we will demonstrate that the first two entries allow for a holographic construction of tree-level MHV celestial amplitudes for gluons and gravitons. Thereafter, in Section III, we will address the completeness and consistency of these postulates with the mathematical formalism of conformal field theory by computing the two- and three-point functions and determining the structure constants of the OPE.

B. Yang-Mills Theory

In this subsection, we begin by succinctly reviewing the formalism of celestial leaf amplitudes introduced by Melton, Sharma, and Strominger [15], which prescribes an analytic continuation of celestial amplitudes from Minkowski spacetime to the ultra-hyperbolic Klein space $\mathbf{R}^2_2$. Following this, we discuss the holographic construction of these Klein space amplitudes for pure Yang-Mills theory within the framework of the holographic dictionary proposed in these notes, which establishes a correspondence between Euclidean AdS_3 (bosonic) string theory and celestial CFT.

Notation. In what follows, we denote points on the celestial sphere by $x_i, \bar{x}_i \in \mathbf{CP}^1$, referred to as *celestial coordinates.* Let $\nu^A := (x,1)^T$ and $\bar{\nu}^{\dot{A}} := (\bar{x},1)^T$ be a pair of two-component spinors parametrised by the celestial coordinates $x, \bar{x}$. The *standard null vector* is then defined by $q^\mu(x,\bar{x}) := (\sigma^\mu)_{A\dot{B}} \nu^A \bar{\nu}^{\dot{B}}$. For each family of two-component spinors $\mu_i^A := \sqrt{\omega_i}(x_i,1)^T$ and $\bar{\mu}_i^{\dot{A}} := \sqrt{\omega_i}(\bar{x}_i,1)^T$, indexed by $i = 1,...,n$, and describing the momenta of a sequence of gluons with frequencies $\omega_1,...,\omega_n$, the associated four-momenta are given by $p_i^\mu = (\sigma^\mu)_{A\dot{B}} \mu_i^A \bar{\mu}_i^{\dot{B}} = \omega_i q^\mu(x_i,\bar{x}_i)$. The spinor-helicity brackets are then defined as $\langle ij\rangle := \varepsilon_{AB}\mu_i^A\mu_j^B$ and $[ij] := \varepsilon_{\dot{A}\dot{B}}\bar{\mu}_i^{\dot{A}}\bar{\mu}_i^{\dot{B}}$, such that, in our conventions, $2p_i \cdot p_j = \langle ij\rangle [ij]$, which follows from $\mathrm{tr}(\sigma^\mu\bar{\sigma}^\nu) = 2\eta^{\mu\nu}$.

The starting point of our analysis is the Parke-Taylor formula (cf. Parke and Taylor [22]; for a contemporary pedagogical review, see Elvang and Huang [23], Badger *et al.* [24]) for the tree-level scattering amplitude $\mathcal{A}_n$ of MHV gluons 1^+, 2^+, 3^-, ..., n^-, which is expressed in spinor-helicity variables as follows:

$$\mathcal{A}_n = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle ... \langle n1\rangle}\delta^{(4)}\left(\sum_{i=1}^n p_i^\mu\right), \tag{9}$$

where p_i^μ denotes the four-momentum of the i^{th} gluon. Utilising the celestial holography dictionary, as elaborated upon in detail by Pasterski [25], Raclariu [26], Strominger [27], Puhm [28], the spinor-helicity variables are connected to the frequencies ω_i and the complex parametrisation $x_i, \bar{x}_i \in \mathbf{CP}^1$ of the celestial sphere through the relation:

$$\langle ij\rangle = \sqrt{\omega_i\omega_j}x_{ij}. \tag{10}$$

Therefore, recalling the integral representation of the $4d$ Dirac delta function, $\delta^{(4)}(p) = 1/(2\pi)^4 \int d^4Y e^{ip\cdot Y}$, and substituting the frequencies ω_i and celestial coordinates $x_i, \bar{x}_i$ into Eq. (9), we arrive at the expression:

$$\mathcal{A}_n = \frac{x_{12}^4}{x_{12}x_{23}...x_{n1}} \int \frac{d^4Y}{(2\pi)^2} \prod_{i=1}^n \omega_i^{\alpha_i} e^{i\omega_i q(x_i,\bar{x}_i)\cdot Y}, \tag{11}$$

where we introduce constants to streamline the notation: $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = ... = \alpha_n = -1$.

To rewrite Eq. (11) in a more homogeneous form using Berezin calculus (cf. Berezin [29]), following Nair [30, 31], we introduce a set of Grassmann-valued two-component spinors θ_i^A ($i = 1,...,4$) such that $\int d^2\theta_i \theta_i^A\theta_i^B = \varepsilon^{AB}$. We can then express x_{12}^4 as:

$$x_{12}^4 = \int d^8\theta \prod_{i=1}^4 \theta_i^A \nu_{1A} \prod_{j=1}^4 \theta_j^B \nu_{2B}, \tag{12}$$

where $\nu_1^A \coloneqq (x_1, 1)^T$ and $\nu_2^A \coloneqq (x_2, 1)^T$. To further simplify our notation, we denote the Berezin integral as $\int [d^8\theta] \coloneqq \int d^8\theta \prod_{i=1}^4 \theta_i^A \nu_{1A} \prod_{j=1}^4 \theta_j^B \nu_{2B}$, allowing us to rewrite Eq. (11) as:

$$\mathcal{A}_n = \frac{1}{x_{12}x_{23}...x_{n1}} \int [d^8\theta] \int \frac{d^4Y}{(2\pi)^2} \prod_{i=1}^n \omega_i^{\alpha_i} e^{i\omega_i q(x_i,\bar{x}_i)\cdot Y}. \tag{13}$$

Using the prescriptions outlined in the celestial holography dictionary, the celestial amplitude corresponding to $\mathcal{A}_n$ is given by the ε-regulated Mellin transform of Eq. (13):

$$\widehat{\mathcal{A}}_n = \prod_{i=1}^n \int_0^\infty d\omega_i \omega_i^{\Delta_i - 1} e^{-\varepsilon\omega_i} \mathcal{A}_n, \tag{14}$$

which results in:

$$\widehat{\mathcal{A}}_n = \frac{1}{x_{12}x_{23}...x_{n1}} \int [d^8\theta] \int \frac{d^4Y}{(2\pi)^4} \prod_{i=1}^n \frac{\Gamma(2\rho_i)}{(\varepsilon - iq(x_i,\bar{x}_i)\cdot Y)^{2\rho_i}}, \tag{15}$$

where we define $\rho_i \coloneqq (\Delta_i + \alpha_i)/2$.

By analytically continuing the spacetime integral in Eq. (15) to Klein space $\mathbf{R}_2^2$, and employing the approach described by Melton, Sharma, and Strominger [15], we derive the expression:

$$\int \frac{d^4Y}{(2\pi)^4} \prod_{i=1}^n \frac{\Gamma(2\rho_i)}{(\varepsilon - iq(x_i,\bar{x}_i)\cdot Y)^{2\rho_i}} \tag{16}$$

$$= \frac{\delta(4 - 2\sum_i \rho_i)}{(2\pi)^3} \int_{\hat{y}^2=1} d^3\hat{y} \prod_{i=1}^n \frac{\Gamma(2\rho_i)}{(\varepsilon - iq(x_i,\bar{x}_i)\cdot \hat{y})} + (\bar{x}_i \leftrightarrow -\bar{x}_i). \tag{17}$$

Here, the notation $(\bar{x}_i \leftrightarrow -\bar{x}_i)$ denotes the repetition of the preceding term with the variables $\bar{x}_i$ substituted by $-\bar{x}_i$. Consequently, Eq. (15) can be reformulated as:

$$\widehat{\mathcal{A}}_n = \frac{\delta(4 - 2\sum_i \rho_i)}{(2\pi)^3} \frac{1}{x_{12}x_{23}...x_{n1}} \int [d^8\theta] \int_{\hat{y}^2=1} d^3\hat{y} \prod_{i=1}^n \frac{\Gamma(2\rho_i)}{(\varepsilon - iq(x_i,\bar{x}_i)\cdot \hat{y})} + (\bar{x}_i \leftrightarrow -\bar{x}_i). \tag{18}$$

This expression for the celestial amplitude $\widehat{\mathcal{A}}_n$ is key in establishing that our holographic dictionary can reproduce the tree-level MHV scattering amplitude for gluons in an appropriate limit.

2. *Celestial Vertex Operators for Gluons*

We now turn our attention to the vertex operators within celestial CFT whose correlation functions generate the celestial amplitude $\widehat{\mathcal{A}}_n$ as derived in Eq. (18). To achieve this, it is necessary to introduce a "dressing" of the worldsheet conformal primaries $\Phi^j(x;z)$ in the H_3^+-WZNW model by the level-one Kac-Moody currents $K^a(x)$ of $SO(2N)$, residing on the boundary of H_3^+, which we identify with the celestial sphere. Hereafter, we shall refer to $K^a(x)$ as the *celestial Kac-Moody currents.*

The generators $K^a(x)$ of the celestial Kac-Moody current algebra are characterised by the OPE:

$$K^a(x_1)\,K^b(x_2) \sim \frac{\delta^{ab}}{x_{12}^2} + \frac{if^{abc}}{x_{12}}K^c(x_2)\,, \tag{19}$$

where f^{abc} denote the structure constants of $\mathfrak{so}(2N) \simeq D_N$.

Recalling that the dual Coxeter number of the Lie algebra D_N is $2N-2$, the KZ equations for the currents $K^a(x)$ take the form:

$$\left(\frac{\partial}{\partial x_i} + \frac{1}{2N-1}\sum_{j\neq i}\frac{T_i^c T_j^c}{x_i - x_j}\right)\langle K^{a_1}(x_1)\ldots K^{a_n}(x_n)\rangle = 0, \tag{20}$$

where T^a are the generators of the fundamental representation of $SO(2N)$. Decomposing the correlation function $\langle K^{a_1}(x_1)\ldots K^{a_n}(x_n)\rangle$ into the canonical $SO(2N)$ invariants, $I_1 = \delta_{m_1,m_2}\delta_{m_3,m_4}$ and $I_2 = \delta_{m_1,m_3}\delta_{m_2,m_4}$, and retaining only the terms $T_i^a T_j^a I_1$ and $T_i^a T_j^a I_2$ proportional to N, we take the limit $N \to \infty$ of the KZ equations. The asymptotic solution is:

$$\lim_{N\to\infty}\langle \mathcal{O}^{a_1}(x_1)\ldots\mathcal{O}^{a_n}(x_n)\rangle = \mathrm{tr}\left(T^{a_1}\ldots T^{a_n}\right)\frac{1}{x_{12}x_{23}\ldots x_{n1}}. \tag{21}$$

Thus, in the large-N limit, the correlation functions of the celestial Kac-Moody currents $K^a(x)$ generate the colour-ordered Parke-Taylor factor (cf. Parke and Taylor [22]), which is associated with the tree-level MHV scattering amplitudes of gluons.

We then define the (coloured) level-one celestial vertex operators of $SO(2N)$ as:

$$V_\rho^a(x) := \Gamma(\rho)\,K^a(x)\int d^2z\,\Phi^\rho(x;z)\,, \tag{22}$$

where, from the perspective of celestial CFT, ρ is understood as a label associated with the vertex operators $V_\rho^a(x)$ rather than a scaling dimension. Consequently, the correlation function of the celestial vertex operators in the large-N limit yields:

$$\lim_{N\to\infty}\left\langle V_{\rho_1}^{a_1}(x_1)\ldots V_{\rho_n}^{a_n}(x_n)\right\rangle = \frac{\mathrm{tr}\left(T^{a_1}\ldots T^{a_n}\right)}{x_{12}x_{23}\ldots x_{n1}}\frac{1}{W}\int d^2z_1\ldots d^2z_n\,\langle\langle\Phi^{\rho_1}(x_1;z_1)\ldots\Phi^{\rho_n}(x_n;z_n)\rangle\rangle\,, \tag{23}$$

where W is a renormalisation constant associated with the worldsheet "area," required to render the worldsheet integrals well-defined and finite. Finally, in the mini-superspace limit (A37), wherein:

$$\lim_{k\to\infty}\langle\langle\Phi^{\rho_1}(x_1;z_1)\ldots\Phi^{\rho_n}(x_n;z_n)\rangle\rangle = \mathcal{N}\int\frac{d\rho\, d\gamma\, d\bar{\gamma}}{\rho^3}\prod_{i=1}^{n}\left(\frac{\rho}{\rho^2 + |x_i-\gamma|^2}\right)^{2\rho_i}, \tag{24}$$

we arrive at:

$$\lim_{N\to\infty}\lim_{k\to\infty}\left\langle V_{\rho_1}^{a_1}(x_1)\ldots V_{\rho_n}^{a_n}(x_n)\right\rangle = \mathcal{N}\frac{\mathrm{tr}\left(T^{a_1}\ldots T^{a_n}\right)}{x_{12}x_{23}\ldots x_{n1}}\int_{AdS_3/\mathbf{Z}} d^3\hat{y}\prod_{i=1}^{n}\frac{\Gamma(2\rho_i)}{\left(\varepsilon - iq(x_i,\bar{x}_i)\cdot\hat{y}\right)^{2\rho_i}}. \tag{25}$$

Therefore, from Eqs. (18, 25), the tree-level MHV celestial amplitude for gluon scattering in pure Yang-Mills theory may be expressed as:

$$\widehat{\mathcal{A}}_n = \frac{\delta\left(4 - 2\sum_i\rho_i\right)}{(2\pi)^3\,\mathcal{N}}\lim_{N\to\infty}\lim_{k\to\infty}\int\left[d^8\theta\right]\left\langle V_{\Delta_1+\alpha_1}^{a_1}(x_1)\ldots V_{\Delta_n+\alpha_n}^{a_n}(x_n)\right\rangle + (\bar{x}_i \leftrightarrow -\bar{x}_i)\,. \tag{26}$$

C. Einstein's Gravity

In this subsection, we shall explain how the preceding argument extends to gravity. The formulation by which tree-level celestial amplitudes for MHV gravitons can be represented as a correlation function of *Liouville* vertex operators, "dressed" by an $SO(2N)$ level-one Kac-Moody current algebra on the celestial sphere, has been throughly analysed by Mol [6]. This construction is framed within the large-N semiclassical limit of Liouville field theory, where $b \to 0$. Our objective here is to demonstrate how this formalism may be adapted to the holographic dictionary introduced in Subsection II A.

1. *Mathematical Preliminaries*

We must first undertake a brief preparatory discussion to introduce the mathematical objects needed for our construction. For further elaboration, the interested reader is referred to Mol [6]. It is important to note that we shall employ a slightly different notation in this discussion: the variables $z, \bar{z}$ denote coordinates on the string worldsheet, while $x, \bar{x}$ correspond to coordinates on the celestial sphere. To begin, let us recall from Pasterski and Shao [32] that the scalar celestial wavefunction with conformal weight $\Delta = 2\sigma$ is given by:

$$\phi_{2\sigma}(x; z) \coloneqq \frac{\Gamma(2\sigma)}{(\varepsilon - iq(x_i, \bar{x}_i) \cdot x)^{2\sigma}}. \tag{27}$$

Let P_i denote the operator acting on the space of celestial wavefunctions $\{\phi_{2\sigma}\}$ by shifting the conformal weights (with respect to the celestial CFT, and not to be confused with the H_3^+-WZNW model) by one-half, such that:

$$\mathsf{P}_i \phi_{2\sigma_j}(x_i; z_i) \coloneqq \delta_{ij} \phi_{2(\sigma_j + 1/2)}(x_i; z_i). \tag{28}$$

We must also introduce a pair of chiral quasi-primary free fermions, $\chi(x)$ and $\chi^\dagger(x)$, defined on the celestial sphere by their respective mode expansions:

$$\chi(x) \coloneqq \sum_{n=0}^{\infty} b_n x^n, \quad \chi^\dagger(x) \coloneqq \sum_{n=0}^{\infty} b_n^\dagger x^{-n-1}, \tag{29}$$

where b_n and $b_n^\dagger$ are fermionic annihilation and creation operators, satisfying the anti-commutation relations $\{b_m, b_n^\dagger\} = \delta_{mn}$ for all $m, n \geq 0$. These operators act on the vacuum state $|0\rangle$ such that $b_n|0\rangle = 0$ for each non-negative integer n. The two-point function for this doublet is given by:

$$\langle \chi(x_1) \chi^\dagger(x_2) \rangle \coloneqq \frac{1}{x_1 - x_2}. \tag{30}$$

Finally, let $\nu_i^A := (x_i, 1)^T$ and $\bar{\nu}_i^{\dot{A}} := (\bar{x}_i, 1)^T$ be a pair of two-component spinors parametrising the four-momentum of the i-th graviton, and let $\lambda^A := (\lambda, 1)^T \in \mathbf{C}^2$ be an auxiliary two-component spinor associated with the state $|\lambda\rangle := \chi^\dagger(\lambda)|0\rangle$. We have shown in Mol [6] that, by defining the operators:

$$\mathsf{A}_i := \exp\left(q\left(x, \bar{x}\right) \cdot y\, \mathsf{P}_i\right) \chi^\dagger\left(x\right) \chi\left(x\right), \mathsf{B}_i := \frac{\bar{\nu}_i^{\dot{A}} \lambda^A}{\langle \nu_i, \lambda \rangle} \frac{\partial}{\partial y^{A\dot{A}}} \exp\left(q\left(x, \bar{x}\right) \cdot y\, \mathsf{P}_i\right), \tag{31}$$

where the four-vector $(y^\mu) \in \mathbf{R}^4$ should be interpreted as an index rather than spacetime coordinates, the tree-level MHV celestial amplitude $\widehat{\mathcal{M}}_n$ for the scattering of n-gravitons can be expressed as:

$$\widehat{\mathcal{M}}_n = \frac{1}{x_{12} x_{23} ... x_{n1}} \int \left[d^{16}\theta\right] \int \frac{d^2\lambda}{\pi} \int dy \delta\left(y\right) \langle \lambda | \mathsf{A}_1 \left(\prod_{i=2}^{n-2} \mathsf{B}_i\right) \mathsf{A}_{n-1} \mathsf{A}_n \bar{\partial} | \lambda \rangle \tag{32}$$

$$\int \frac{d^4 Y}{(2\pi)^4} \prod_{j=1}^{n} \phi_{2\sigma_j}\left(x_j; Y\right) + \mathcal{P}\left(2, ..., n-2\right), \tag{33}$$

where $\mathcal{P}(2, ..., n-2)$ denotes the permutation of the symbols within the set $\{2, ..., n-2\}$, and we define the celestial conformal weights as follows:

$$\sigma_i := \frac{1}{2}\left(\Delta_i + \alpha_i - 1\right) \text{ for } i \in \{1, n-1, n\}, \tag{34}$$

$$\sigma_i := \frac{1}{2}\left(\Delta_i + \alpha_i\right) \text{ for } 2 \leq i \leq n-2. \tag{35}$$

The Berezin integral is accordingly defined as:

$$\int \left[d^{16}\theta\right] := \int d^{16}\theta \prod_{i=1}^{8} \theta_i^A \nu_{iA} \prod_{j=1}^{8} \theta_j^B \nu_{jB}. \tag{36}$$

2. Celestial Vertex Operators for Gravitons

We are now prepared to demonstrate that Eq. (33) can be reformulated using the celestial vertex operators derived from AdS_3 string theory, as detailed in Subsection A 2. We define the "dressed" vertex operators in celestial CFT for gravitons:

$$G_i\left(x_i\right) := \Gamma\left(2\sigma_i\right) T^{a_i} K^{a_i}\left(x_i\right) \mathsf{A}_i \int d^2 z_i \Phi^{2\sigma_i}\left(x_i; z_i\right), \tag{37}$$

for $i = 1, n-1, n$, and:

$$H_j\left(x_j\right) := \Gamma\left(2\sigma_i\right) T^{a_j} K^{a_j}\left(x_j\right) \mathsf{B}_j \int d^2 z_j \Phi^{2\sigma_j}\left(x_j; z_j\right), \tag{38}$$

for $j = 2, ..., n-2$, where $K^a(x)$ denotes the celestial Kac-Moody currents introduced in Subsection II B 2, and obeying the OPE (19). We emphasise that the order of the operators A_i and B_j in Eqs. (37, 38) is important, as the worldsheet conformal primaries $\Phi^{2\sigma_i}(x_i; z_i)$ inherit the action of the operator P_i due to their dependence on the celestial conformal weights, as given by Eqs. (34, 35).

Thus, by employing reasoning analogous to that leading to Eqs. (24, 25) in the previous section, we obtain the following expression:

$$\widehat{\mathcal{M}}_n = \frac{1}{(2\pi)^3 \mathcal{N} \gamma_n} \lim_{N\to\infty} \lim_{k\to\infty} \int \left[d^{16}\theta\right] \int \frac{d^2\lambda}{\pi} \int dy \delta(y) \langle\lambda| G_1 \left(\prod_{i=2}^{n-2} H_i\right) G_{n-1} G_n \bar{\partial} |\lambda\rangle \tag{39}$$

$$+ (\bar{x}_i \leftrightarrow -\bar{x}_i) + \mathcal{P}(2, ..., n-2), \tag{40}$$

where $\gamma_n \coloneqq \mathrm{tr}\left(T^{a_1} ... T^{a_n}\right)^2$.

In conclusion, we have demonstrated that the tree-level celestial amplitude for the scattering of MHV gravitons in Einstein's gravity can be obtained as the large-N, large-level ($k \to \infty$) limit of the correlation function of the vertex operators constructed from the Kac-Moody currents and conformal primaries of the H_3^+-WZNW model.

III. CORRELATION FUNCTIONS, CELESTIAL OPE, KNIZHNIK–ZAMOLODCHIKOV EQUATIONS AND STRESS-ENERGY TENSOR

In the preceding sections, we have introduced a holographic dictionary establishing a correspondence between Euclidean AdS_3 (bosonic) string theory and celestial CFT. Now, we shall compute the two- and three-point functions in Subsections III A and III B, respectively, and determine the structure constants of the celestial operator product expansion (OPE) in Subsection III C. Finally, in Subsection III D, we will derive a system of partial differential equations (PDEs) characterising the celestial amplitudes in the energy-momentum representation, drawing from the Knizhnik–Zamolodchikov (KZ) equations and the worldsheet Ward identities.

A. Two-point function

In this subsection, we derive the celestial 2-point function by employing the exact form of the corresponding correlation function in the H_3^+-WZNW model, studied in detail by Teschner [7]. For readers seeking a comprehensive exposition of the mathematical subtleties inherent in the model, we direct them to Teschner's subsequent works (Teschner [8, 9, 10]).

Our aim is to compute the correlation function of two scalar conformal primaries, $\phi_{\Delta_1}(x_1)$ and $\phi_{\Delta_2}(x_2)$, within the celestial CFT, characterised by conformal weights Δ_1 and Δ_2, respectively. According to our proposed holographic dictionary between Euclidean AdS_3 bosonic string theory and celestial CFT, these scalar wavefunctions correspond to the worldsheet conformal primaries $\Phi^{j_1}(x_1; z_1)$ and $\Phi^{j_2}(x_2; z_2)$. In this correspondence, the spins in the H_3^+-WZNW model are related to the conformal weights in the celestial CFT by $j_1 = -\Delta_1/2$ and $j_2 = -\Delta_2/2$.

According to Teschner [7, 8], the 2-point function of these conformal primaries is given by:

$$\left\langle \Phi^{j_1}(x_1; z_1)\, \Phi^{j_2}(x_2; z_2) \right\rangle = \tag{41}$$

$$\frac{1}{|z_{12}|^{2\beta}} \left(\delta^{(2)}(x_1 - x_2)\, \delta(j_1 + j_2 + 1) + \frac{B(j_1)}{|x_{12}|^{-2(j_1+j_2)}} \delta(j_2 - j_1) \right), \tag{42}$$

where the exponents and proportionality constants are defined as:

$$\beta := \frac{b^2}{2}(2 - \Delta_1)(2 - \Delta_2), \quad b^2 = \frac{1}{k-2}, \tag{43}$$

and $B(j_1)$ denotes a normalisation constant that depends on the choice of normalisation of the "plane-wave" basis $\Phi^j(x; z)$. For the conventions adopted by Giveon, Kutasov, and Seiberg [13], which we adhere to in these notes, $B \equiv 1$. Nonetheless, to preserve generality and maintain consistency with Teschner [7], we retain the explicit dependence on the constant $B(j_1)$ throughout this subsection.

Applying our holographic dictionary, which maps correlation functions in celestial CFT to those in AdS_3 string theory, we obtain:

$$\langle \phi_{\Delta_1}(x_1)\, \phi_{\Delta_2}(x_2) \rangle_{\text{CCFT}} \tag{44}$$

$$= \frac{1}{N} \int d^2 z_1 \int d^2 z_2 \left\langle \Phi^{j_1}(x_1; z_1)\, \Phi^{j_2}(x_2; z_2) \right\rangle \tag{45}$$

$$= \left(\delta^{(2)}(x_1 - x_2)\, \delta(j_1 + j_2 + 1) + \frac{B(j_1)}{|x_{12}|^{-2(j_1+j_2)}} \delta(j_2 - j_1) \right) \frac{1}{N} \int d^2 z_1 \int d^2 z_2 \frac{1}{|z_{12}|^{2\beta}} \tag{46}$$

Here, N denotes the worldsheet "area." The factor $1/N$ serves as a renormalisation of the worldsheet integration measure, ensuring that the correlation function remains well-defined and finite. This renormalisation is a standard procedure in string perturbation theory (cf. D'Hoker and Phong [20]).

As demonstrated in the Appendix B 2, Eq. (B27), the worldsheet integral evaluates to:

$$\frac{1}{N} \int d^2 z_1 \int d^2 z_2 \frac{1}{|z_{12}|^{2\beta}} = \frac{\pi^2}{2} \boldsymbol{\delta}(\beta - 1), \tag{47}$$

where, following Donnay, Pasterski, and Puhm [33], $\boldsymbol{\delta}(i(\Delta - z))$ represents the generalised Dirac delta "function," analytically continued to the complex plane via:

$$\boldsymbol{\delta}(i(\Delta - z)) := \frac{1}{2\pi} \int_0^\infty d\tau\, \tau^{\Delta - z - 1}, \tag{48}$$

such that the following identity holds:

$$\varphi(\Delta) = -i\int_{\mathcal{C}} dz\boldsymbol{\delta}\left(i\left(\Delta - z\right)\right)\varphi(z), \tag{49}$$

for the contour $\mathcal{C} := -1/2 + i\mathbf{R}$.

Substituting this result into Eq. (46) and expressing the final result in terms of the conformal weights, we arrive at:

$$\left\langle \phi_{\Delta_1}(x_1)\,\phi_{\Delta_2}(x_2)\right\rangle_{\mathrm{CCFT}} = \tag{50}$$

$$\pi^2\boldsymbol{\delta}(\beta - 1)\left(\delta^{(2)}(x_1 - x_2)\,\delta(2 - \Delta_1 - \Delta_2) + \frac{B(-\Delta_1/2)}{|x_{12}|^{\Delta_1+\Delta_2}}\delta(\Delta_1 - \Delta_2)\right) \tag{51}$$

The 2-point function in our celestial CFT for the scalar conformal primaries $\phi_{\Delta_1}(x_1)$ and $\phi_{\Delta_2}(x_2)$ takes the form of a linear combination involving a delta function localised at $x_1 = x_2$ and a conventional $1/|x_{12}|^{\Delta_1+\Delta_2}$ power-law dependence on the conformal weights. This result stands in contrast to those reported by Furugori *et al.* [34]. One might, as suggested by Ogawa *et al.* [1], explore the possibility of constructing linear combinations of these conformal primaries to generate a 2-point function that solely includes the delta function $\delta^{(2)}(x_1 - x_2)$, the power-law term $1/|x_{12}|^{\Delta_1+\Delta_2}$, or their shadow transforms.

However, we prefer to follow an approach that avoids the imposition of intuition derived from conventional CFTs when examining celestial CFTs. Instead, we maintain that the mathematical structures we have uncovered merit serious consideration on their own terms, and their consequences should be rigorously examined. The ability of the mini-superspace limit of the H_3^+-WZNW model to holographically reconstruct the tree-level MHV scattering amplitudes for gluons and gravitons is a compelling reason for adhering to the mathematical framework of this model in deriving the properties of celestial CFT. Thus, in our view, one must be prepared to encounter correlation functions–and, as will be demonstrated in the subsequent sections, structure constants of celestial OPEs–manifesting in a non-conventional distributional form.

It is also instructive to compute the shadow transform of the 2-point function given by Eq. (42):

$$\int d^2y_1\frac{1}{|x_1 - y_1|^{4(1+j_1)}}\int d^2y_2\frac{1}{|x_2 - y_2|^{4(1+j_2)}}\left\langle \Phi^{j_1}(y_1;z_1)\,\Phi^{j_2}(y_2;z_2)\right\rangle \tag{52}$$

$$= \frac{1}{|z_{12}|^{2\beta}}\delta(j_1 + j_2 + 1)\int d^2y_1d^2y_2\frac{\delta^{(2)}(y_1 - y_2)}{|x_1 - y_1|^{4(1+j_1)}\,|x_2 - y_2|^{4(1+j_2)}} \tag{53}$$

$$+ \frac{1}{|z_{12}|^{2\beta}}B(j_1)\,\delta(j_1 - j_2)\int d^2y_1d^2y_2\frac{\delta^{(2)}(y_1 - y_2)}{|x_1 - y_1|^{4(1+j_1)}\,|x_2 - y_2|^{4(1+j_2)}\,|x_{12}|^{-2(j_1+j_2)}}. \tag{54}$$

In the Appendix B 1, we gave a detailed analysis of the following integral over the complex plane:

$$\mathcal{I}_0\left(\tau_1, \tau_2 | x_1, x_2\right) := \int d^2y\frac{1}{|x_1 - y|^{2\tau_1}\,|y - x_2|^{2\tau_2}}. \tag{55}$$

We found in Eq. (B12) that this integral evaluates, in the distributional sense, to:

$$\mathcal{I}_0 = \frac{\pi}{|x_1 - x_2|^{2(\tau_1+\tau_2-1)}} \frac{\Gamma(1-\tau_1)}{\Gamma(\tau_1)} \frac{\Gamma(1-\tau_2)}{\Gamma(\tau_2)} \frac{\Gamma(\tau_1+\tau_2-1)}{\Gamma(2-\tau_1-\tau_2)}. \tag{56}$$

From Eq. (56), we demonstrated in Appendix B 1 (cf. Eq. (B19)) that, by taking the limit $\tau_1 + \tau_2 \to 2$, one recovers the well-known identity from $2d$ CFT (cf. Simmons-Duffin [35]):

$$\int d^2y \frac{1}{|x_1 - y|^{2\tau} |y - x_2|^{2(2-\tau)}} = \frac{4\pi^2}{\nu^2} \delta^{(2)}(x_1 - x_2), \tag{57}$$

where $\tau =: 1 + i\nu$.

Consequently, using Eqs. (56, 57), we arrive at the following results:

$$\lim_{j_1+j_2+1\to 0} \int d^2y_1 d^2y_2 \frac{\delta^{(2)}(y_1 - y_2)}{|x_1 - y_1|^{4(1+j_1)} |x_2 - y_2|^{4(1+j_2)}} = -\frac{4\pi^2}{(1+2j_1)^2} \delta^{(2)}(x_1 - x_2), \tag{58}$$

and, similarly,

$$\lim_{j_1-j_2\to 0} \int d^2y_1 d^2y_2 \frac{\delta^{(2)}(y_1 - y_2)}{|x_1 - y_1|^{4(1+j_1)} |x_2 - y_2|^{4(1+j_2)} |x_1 - x_2|^{-2(j_1+j_2)}} \tag{59}$$

$$= -\frac{\pi^2}{(1+2j_1)^2} \frac{1}{|x_1 - x_2|^{2(2+j_1+j_2)}}. \tag{60}$$

Thus, Eq. (54) can be rewritten as:

$$\frac{1}{2\pi} \int d^2y_1 \frac{1}{|x_1 - y_1|^{4(1+j_1)}} \frac{1}{2\pi} \int d^2y_2 \frac{1}{|x_2 - y_2|^{4(1+j_2)}} \left\langle \Phi^{j_1}(y_1; z_1) \Phi^{j_2}(y_2; z_2) \right\rangle \tag{61}$$

$$= -\frac{1}{(1+2j_1)^2} \frac{1}{|z_{12}|^{2\beta}} \left(\delta^{(2)}(x_1 - x_2) \delta(j_1 + j_2 + 1) + \frac{B(j_1)}{4} \frac{1}{|x_{12}|^{2(2+j_1+j_2)}} \delta(j_1 - j_2) \right). \tag{62}$$

It is then natural to define the shadow transform of the worldsheet conformal primary $\Phi^j(x; z)$ as:

$$\widetilde{\Phi}^j(x; z) := \frac{1}{2\pi} \int d^2y \frac{1}{|x - y|^{4(1+j)}} \Phi^j(y; z), \tag{63}$$

so that Eq. (62) can be reorganised as:

$$\left\langle \widetilde{\Phi}^{j_1}(x_1; z_1) \widetilde{\Phi}^{j_2}(x_2; z_2) \right\rangle \tag{64}$$

$$= -\frac{1}{(1+2j_1)^2} \frac{1}{|z_{12}|^{2\beta}} \left(\delta^{(2)}(x_1 - x_2) \delta(j_1 + j_2 + 1) + \frac{B(j_1)}{4} \frac{1}{|x_{12}|^{2(2+j_1+j_2)}} \delta(j_1 - j_2) \right). \tag{65}$$

We conclude by extending the holographic dictionary, postulating that the shadow-transformed worldsheet conformal primaries $\widetilde{\Phi}^j(x; z)$, characterised by spin j, correspond to the celestial scalar conformal primaries $\widetilde{\phi}_{\Delta}(x)$, with conformal weight $\Delta = -2j$. Consequently, integrating Eq. (65) over the worldsheet coordinates z_1, z_2 yields:

$$\left\langle \widetilde{\phi}_{\Delta_1}(x_1)\, \widetilde{\phi}_{\Delta_2}(x_2) \right\rangle_{\text{CCFT}} \tag{66}$$

$$= \frac{1}{N} \int d^2 z_1 \int d^2 z_2 \left\langle \widetilde{\Phi}^{j_1}(x_1; z_1)\, \widetilde{\Phi}^{j_2}(x_2; z_2) \right\rangle \tag{67}$$

$$= \frac{\pi^2}{\lambda^2} \boldsymbol{\delta}(\beta - 1) \left(\delta^{(2)}(x_1 - x_2)\, \delta(2 - \Delta_1 - \Delta_2) + \frac{B(-\Delta_1/2)}{4} \frac{1}{|x_{12}|^{4-\Delta_1-\Delta_2}} \delta(\Delta_1 - \Delta_2) \right), \tag{68}$$

where $\Delta_1 =: 1 + i\lambda$.

The celestial 2-point function of the shadow-transformed conformal primaries exhibits a similar structure as Eq. (51), namely, a linear combination of the delta function, $\delta^{(2)}(x_1 - x_2)$, and a standard power-law dependence, $1/|x_{12}|^{2-\Delta_1-\Delta_2}$. However, the functional dependence on the conformal weights Δ_1, Δ_2 in Eqs. (51 , 68)–aside from the prefactor $1/\lambda_1^2 = 1/(1-\Delta_1)^2$–is distinct. Furthermore, it is essential to recall that, within celestial CFT, the conformal weights Δ_1, Δ_2 are dynamical physical variables, corresponding to the Mellin conjugates of the frequencies of the particles involved in the scattering process. Therefore, the functional dependencies on Δ_1, Δ_2 encode non-trivial physical information about the scattering events. However, as demonstrated by Teschner [8], the bases $\Phi^j(x; z)$ and $\widetilde{\Phi}^{-1-j}(x; z)$ yield the same representation within the framework of the H_3^+-WZNW model, a result that becomes apparent from the structure of the shadow-transformed 2-point function derived earlier. Consequently, this equivalence of bases in the H_3^+-WZNW model leads to non-trivial physical implications for celestial CFT, an important matter that we will leave unresolved for the time being, to be addressed in future investigations.

B. Three-Point Function

In this subsection, we employ the exact form of the 3-point function in the H_3^+-WZNW model, as determined by Teschner [7], to evaluate the celestial amplitude corresponding to the scattering of three massless scalar particles within the framework of our holographic dictionary. To structure our analysis, we adhere to the formalism introduced by Teschner [9, 10] in his subsequent works concerning the H_3^+-WZNW model. For a more comprehensive account, we direct the reader to these works. It is important to note, however, that we adopt the normalisation conventions for the worldsheet primaries as defined by Giveon, Kutasov, and Seiberg [13].

Let us denote the conformal weights associated with the celestial scalar conformal primaries $\phi_{\Delta_1}(x_1)$, $\phi_{\Delta_2}(x_2)$ and $\phi_{\Delta_3}(x_3)$ as Δ_1, Δ_2 and Δ_3, respectively. Correspondingly, we introduce the quantities $j_1 := -\Delta_1/2$, $j_2 := -\Delta_2/2$ and $j_3 := -\Delta_3/2$, which represent the spins of the worldsheet primaries within the H_3^+-WZNW model. For the sake of notational simplicity, we temporarily omit the anti-holomorphic variables from our arguments, thereby denoting the worldsheet primaries as $\Phi^{j_1}(x_1; z_1)$, $\Phi^{j_2}(x_2; z_2)$ and $\Phi^{j_3}(x_3; z_3)$.

It is important to emphasise that the correspondence $\phi_{\Delta_k}(x_k) \leftrightarrow \Phi^{j_k}(x_k; z_k)$ of our holographic dictionary maps a *scalar* celestial wavefunction to a worldsheet conformal primary with *spin* j_k. The relation $j_k \leftrightarrow -\Delta_k/2$ summarises the mapping between the notion of spin in the H_3^+-model and the conformal weight in the celestial CFT.

The scaling dimensions of the worldsheet primaries are related to their respective spins through the relations:

$$h_1 = -b^2 j_1 (j_1 + 1), \quad h_2 = -b^2 j_2 (j_2 + 1), \quad h_3 = -b^2 j_3 (j_3 + 1), \tag{69}$$

where $b^2 := 1/(k-2)$.

According to Teschner [10, Eq. (22)], the 3-point correlation function in the H_3^+-WZNW model is given by:

$$\left\langle \Phi^{j_1}(x_1; z_1) \Phi^{j_2}(x_2; z_2) \Phi^{j_3}(x_3; z_3) \right\rangle = \frac{D(j_1, j_2, j_3)\, C\left(j_1, j_2, j_3 \middle| x_1, x_2, x_3\right)}{|z_{12}|^{2\sigma_3} |z_{23}|^{2\sigma_1} |z_{31}|^{2\sigma_2}}, \tag{70}$$

where the Clebsch-Gordan distributional kernel is defined by:

$$C\left(j_1, j_2, j_3 \middle| x_1, x_2, x_3\right) = \frac{1}{|x_{12}|^{2\lambda_3} |x_{23}|^{2\lambda_1} |x_{31}|^{2\lambda_2}}. \tag{71}$$

The exponents associated with the worldsheet coordinates z_1, z_2 and z_3 are defined as:

$$\sigma_1 := -h_1 + h_2 + h_3, \quad \sigma_2 := h_1 - h_2 + h_3, \quad \sigma_3 := h_1 + h_2 - h_3, \tag{72}$$

while the exponents corresponding to the celestial coordinates x_1, x_2 and x_3 are similarly given by:

$$\lambda_1 := j_1 - j_2 - j_3, \quad \lambda_2 := -j_1 + j_2 - j_3, \quad \lambda_3 := -j_1 - j_2 + j_3. \tag{73}$$

In accordance with the holographic dictionary proposed in Section 2, the celestial 3-point amplitude corresponding to the scalar wavefunctions $\phi_{\Delta_1}(x_1)$, $\phi_{\Delta_2}(x_2)$ and $\phi_{\Delta_3}(x_3)$ is expressed as:

$$\langle \phi_{\Delta_1}(x_1) \phi_{\Delta_2}(x_2) \phi_{\Delta_3}(x_3) \rangle_{\mathrm{CCFT}} \tag{74}$$

$$= \frac{1}{N} \int d^2 z_1 \int d^2 z_2 \int d^2 z_3 \left\langle \Phi^{j_1}(x_1; z_1) \Phi^{j_2}(x_2; z_2) \Phi^{j_3}(x_3; z_3) \right\rangle \tag{75}$$

$$= D(j_1, j_2, j_3)\, C\left(j_1, j_2, j_3 \middle| x_1, x_2, x_3\right) \frac{1}{N} \int d^2 z_1 \int d^2 z_2 \int d^2 z_3 \frac{1}{|z_{12}|^{2\sigma_3} |z_{23}|^{2\sigma_1} |z_{31}|^{2\sigma_2}}. \tag{76}$$

Here, N denotes the worldsheet area, and the factor $1/N$ serves as a renormalisation of the integration measure to ensure that the correlation function is well-defined and finite–a standard procedure in the computation of scattering amplitudes within string perturbation theory (cf. D'Hoker and Phong [20], Polchinski [21, 36]).

As demonstrated in Appendix B 3, the integral over the worldsheet coordinates evaluates to:

$$\frac{1}{V}\int d^2z_1\int d^2z_2\int d^2z_3\frac{1}{|z_{12}|^{2\sigma_3}|z_{23}|^{2\sigma_1}|z_{31}|^{2\sigma_2}} \tag{77}$$

$$= 2\pi^3\boldsymbol{\delta}\left(2-\sigma_1-\sigma_2-\sigma_3\right)\frac{\Gamma\left(1-\sigma_1\right)\Gamma\left(1-\sigma_2\right)\Gamma\left(1-\sigma_3\right)}{\Gamma\left(\sigma_1\right)\Gamma\left(\sigma_2\right)\Gamma\left(\sigma_3\right)}. \tag{78}$$

It is noteworthy that the triple integral over the worldsheet yields the *generalised* delta "function" $\boldsymbol{\delta}\left(2-\sum_i\sigma_i\right)$ associated with the conformal weights. This form is characteristic of celestial amplitudes, as discussed in previous works (e.g., Arkani-Hamed *et al.* [37], Gu, Li, and Wang [38], Stieberger and Taylor [39]). Importantly, this delta function would not be present had we chosen to disregard the worldsheet coordinates z_1, z_2 and z_3, as was done by Ogawa *et al.* [1].

Consequently, employing the previously defined equations (Eqs. (72, 76, 78)), we arrive at:

$$\left\langle\phi_{\Delta_1}\left(x_1\right)\phi_{\Delta_2}\left(x_2\right)\phi_{\Delta_3}\left(x_3\right)\right\rangle_{\mathrm{CCFT}} \tag{79}$$

$$= 2\pi^3\frac{\boldsymbol{\delta}\left(2-h_1-h_2-h_3\right)D\left(j_1,j_2,j_3\right)}{|x_{12}|^{2(-j_1-j_2+j_3)}|x_{23}|^{2(j_1-j_2-j_3)}|x_{31}|^{2(-j_1+j_2-j_3)}}\frac{\Gamma\left(2h_1-1\right)\Gamma\left(2h_2-1\right)\Gamma\left(2h_3-1\right)}{\Gamma\left(2-2h_1\right)\Gamma\left(2-2h_2\right)\Gamma\left(2-2h_3\right)}. \tag{80}$$

This exact form of the 3-point celestial amplitude, derived as the correlation function within Euclidean AdS_3 string theory via our proposed holographic dictionary, explicitly exhibits the distributional structure anticipated in celestial amplitudes. Moreover, the presence of gamma functions of the scaling dimensions is indicative of an amplitude derived from worldsheet integrations, aligning with the general expectations from calculations in string perturbation theory.

C. Celestial Operator Product Expansion

In this subsection, we determine the celestial operator product expansion utilising the structure constants of the H_3^+-WZNW model as derived by Teschner [9, 10]. However, for consistency, we adopt the normalisation conventions used by Giveon, Kutasov, and Seiberg [13].

Let Δ_1 and Δ_2 denote the conformal weights associated with the scalar celestial wavefunctions $\phi_{\Delta_1}\left(x_1\right)$ and $\phi_{\Delta_2}\left(x_2\right)$, respectively. According to the proposed holographic dictionary, these wavefunctions correspond to the worldsheet primaries $\Phi^{j_1}\left(x_1;z_1\right)$ and $\Phi^{j_2}\left(x_2;z_2\right)$, where the spins in the H_3^+-WZNW model are related to the celestial conformal weights by $j_1=-\Delta_1/2$ and $j_2=-\Delta_2/2$.

According to Teschner [10], the worldsheet OPE for the primaries $\Phi^{j_1}(x_1;z_1)$ and $\Phi^{j_2}(x_2;z_2)$ is then expressed as:

$$\Phi^{j_1}(x_1;z_1)\,\Phi^{j_2}(x_2;z_2)\sim \tag{81}$$

$$-\frac{1}{2}\int_{-\frac{1}{2}+i\mathbf{R}^+} dj_3\frac{(1+2j_3)^2}{|z_{12}|^{\sigma_3}}D(j_1,j_2,j_3)\int d^2x_3\frac{1}{|x_{12}|^{2\lambda_3}|x_{23}|^{2\lambda_1}|x_{31}|^{2\lambda_2}}\Phi^{-1-j_3}(x_3;z_2)\,. \tag{82}$$

Here, the exponents λ_1, λ_2 and λ_3 associated with the celestial coordinates x_1, x_2 and x_3 are defined by Eq. (72), and $\sigma_3 = h_1 + h_2 - h_3$ as per Eq. (73).

Our objective is to compute the celestial OPE involving two scalar conformal primaries. To achieve this, we adopt the following strategy: we investigate the behaviour of $\phi_{\Delta_1}(x_1)\,\phi_{\Delta_2}(x_2)$ within a correlation function of the celestial CFT. Utilising Eq. (82), we express this as:

$$\langle\phi_{\Delta_1}(x_1)\,\phi_{\Delta_2}(x_2)\,...\rangle_{\text{CCFT}}\sim-\frac{1}{2}\int_{-\frac{1}{2}+i\mathbf{R}^+}(1+2j_3)^2\,D(j_1,j_2,j_3) \tag{83}$$

$$\int d^2x_3\frac{1}{|x_{12}|^{2\lambda_3}|x_{23}|^{2\lambda_1}|x_{31}|^{2\lambda_2}}\int d^2z_1d^2z_2\frac{1}{|z_{12}|^{\sigma_3}}\left\langle\Phi^{-1-j_3}(x_3;z_2)\,...\right\rangle_{\text{CCFT}}. \tag{84}$$

In the Appendix B 4, Eq. (B55), we have demonstrated that the worldsheet integral evaluates to:

$$\int d^2z_1\int d^2z_2\frac{1}{|z_{12}|^{\sigma_3}}\Phi^{-1-j_3}(x_3;z_2)=\pi^2\boldsymbol{\delta}(h_1+h_2+h_3-2)\int d^2z_2\Phi^{-1-j_3}(x_3;z_2)\,. \tag{85}$$

Utilising the relations between the scaling dimensions h_1, h_2, h_3 and the spins j_1, j_2, j_3 in the H_3^+-WZNW model:

$$h_1=-b^2j_1(j_1+1)\,,\;\; h_2=-b^2j_2(j_2+1)\,,\;\; h_3=-b^2j_3(j_3+1)\,, \tag{86}$$

where $b^2 := 1/(k-2)$, we can decompose the generalised Dirac delta function of the scaling dimensions appearing in Eq. (85) as follows:

$$\boldsymbol{\delta}(h_1+h_2+h_3-2)=\frac{\delta\left(j_3-j_3^+\right)}{b^2\left(1+2j_3^+\right)}+\frac{\delta\left(j_3-j_3^-\right)}{b^2\left(1+2j_3^-\right)}, \tag{87}$$

where:

$$j_3^{\pm}=-\frac{1}{2}\pm i\rho(j_1,j_2)\,,\;\; -\rho^2(j_1,j_2):=\frac{1}{4}+\frac{2}{b^2}+j_1(j_1+1)+j_2(j_2+1)\,. \tag{88}$$

Substituting these relations into Eq. (84) yields:

$$\langle\phi_{\Delta_1}(x_1)\,\phi_{\Delta_2}(x_2)\,...\rangle_{\text{CCFT}}\sim \tag{89}$$

$$-\frac{\pi^2}{2b^2}\left(1+2j_3^+\right)D(j_1,j_2,j_3)\,\mathcal{I}_2(\lambda_1,\lambda_2,\lambda_3)\left\langle\phi_{-1-\Delta_3\left(j_3^+\right)}(x_2)\,...\right\rangle_{\text{CCFT}}, \tag{90}$$

where we have introduced the new integral:

$$\mathcal{I}_2\left(\lambda_1,\lambda_2,\lambda_3\right) \coloneqq \frac{1}{\left|x_{12}\right|^{2\lambda_3}} \int d^2x_3 \frac{1}{\left|x_{23}\right|^{2\lambda_1}\left|x_{31}\right|^{2\lambda_2}}. \tag{91}$$

In the Appendix B 5, Eq. (B66), we have shown that this integral evaluates to:

$$\mathcal{I}_2 = \frac{\pi}{\left|x_{12}\right|^{2(\lambda_1+\lambda_2+\lambda_3-1)}} \frac{\Gamma\left(\lambda_1+\lambda_2-1\right)}{\Gamma\left(2-\lambda_1-\lambda_2\right)} \frac{\Gamma\left(1-\lambda_1\right)\Gamma\left(1-\lambda_2\right)}{\Gamma\left(\lambda_1\right)\Gamma\left(\lambda_2\right)}. \tag{92}$$

Recalling that the exponents λ_1, λ_2 and λ_3 are related to the spins j_1, j_2 and j_3 through the relations provided in Eq. (73), we find:

$$\mathcal{I}_2 = \pi\left|x_{12}\right|^{2(j_1+j_2+j_3+1)} \frac{\Gamma\left(-1-2j_3\right)}{\Gamma\left(2+2j_3\right)} \frac{\Gamma\left(1-j_1+j_2+j_3\right)\Gamma\left(1+j_1-j_2+j_3\right)}{\Gamma\left(j_1-j_2-j_3\right)\Gamma\left(-j_1+j_2-j_3\right)}. \tag{93}$$

This expression can be rewritten using Euler's beta functions as:

$$\mathcal{I}_2 = -\frac{\pi}{1+2j_2}\left|x_{12}\right|^{2(j_1+j_2+j_3+1)} \frac{B\left(1-j_1+j_2+j_3, 1+j_1-j_2+j_3\right)}{B\left(j_1-j_2-j_3, -j_1+j_2-j_3\right)}. \tag{94}$$

Finally, substituting this result into Eq. (90), we obtain the celestial OPE following from our proposed holographic dictionary:

$$\phi_{\Delta_1}\left(x_1\right)\phi_{\Delta_2}\left(x_2\right) \sim \tag{95}$$

$$\frac{\pi^3}{2b^2}\left|x_{12}\right|^{2\left(j_1+j_2+j_3^++1\right)} D\left(j_1,j_2,j_3^+\right) \frac{B\left(1-j_1+j_2+j_3, 1+j_1-j_2+j_3\right)}{B\left(j_1-j_2-j_3, -j_1+j_2-j_3\right)} \phi_{-1-\Delta\left(j_3^+\right)}. \tag{96}$$

where $\Delta\left(j_3^+\right) = -2j_3^+$ with j_3^+ given by Eq. (88).

To appreciate the physical significance of this OPE, we recall that, in celestial CFT, the conformal dimensions Δ acquire the interpretation of dynamical physical quantities. This arises from the fact that Δ are Mellin conjugates to the energies of scattered particles. Therefore, the product of gamma functions, which depends on the conformal weights–arising from the worldsheet integrals and manifesting both the worldsheet conformal invariance and the factorisation properties of the string vertex operators–are not merely proportionality constants. Rather, they are *physical quantities* that characterise the scattering processes.

It is noteworthy that even the much simpler celestial OPE for gluons in pure Yang-Mills theory, given by:

$$\mathcal{G}^{a+}_{\Delta_1}\left(x_1\right)\mathcal{G}^{b+}_{\Delta_2}\left(x_2\right) \sim \frac{if^{abc}}{z_{12}} \frac{\Gamma\left(-1-2j_1\right)\Gamma\left(-1-2j_2\right)}{\Gamma\left(-2j_1\right)\Gamma\left(-2j_2\right)} \mathcal{G}^{c+}_{\Delta_1+\Delta_2-1}\left(x_2\right), \tag{97}$$

with $j_1 = -\Delta_1/2$ and $j_2 = -\Delta_2/2$, while considerably less intricate than than Eq. (95), exhibits a similar structure. In this expression, the product of gamma functions that depends on the conformal weights encodes information about the scattering processes of gluons in flat space. This observation leads one to infer that the structure of the OPEs in the Euclidean AdS_3 string model for celestial CFT naturally encapsulates the physics of these scattering processes. We also note that this structure appears to deviate from that of conventional CFTs.

D. Differential Equations for the Celestial CFT

In the previous subsections, we have illustrated our holographic dictionary by computing the two- and three-point celestial amplitudes and extracting the structure constants of the celestial OPE, derived from the corresponding quantities in the H_3^+-WZNW model. The analytic structure of this model has been rigorously investigated by Teschner [7, 8, 9, 10], Ribault and Teschner [11]. In this subsection, we conclude the exposition of our holographic dictionary between Euclidean AdS_3 bosonic string theory and celestial CFT in Klein space by deriving a system of partial differential equations that characterise the celestial amplitudes from the Knizhnik–Zamolodchikov (KZ) equations and the worldsheet Ward identities of the H_3^+-WZNW model.

Let $\widehat{G}_n = \widehat{G}_n\left(x_1, ..., x_n; \Delta_1, ..., \Delta_n; z_1, ..., z_n\right)$ denote the n-point function of the H_3^+-WZNW model, defined as:

$$\widehat{G}_n := \left\langle \Phi^{j_1}\left(x_1; z_1\right) \Phi^{j_2}\left(x_2; z_2\right) ... \Phi^{j_n}\left(x_n; z_n\right)\right\rangle, \tag{98}$$

where the spins in the H_3^+-WZNW model are related to the conformal weights in the celestial CFT by $j_k = -\Delta_k/2$. According to the holographic dictionary, the celestial amplitude corresponding to $\widehat{G}_n$ is obtained by integrating Eq. (98) over the worldsheet variables $z_1, ..., z_n$.

The KZ equations for the H_3^+-WZNW model are deduced as follows. As discussed in Giveon, Kutasov, and Seiberg [13], the Sugawara stress tensor $T_{\mathrm{S}}(z)$ of the worldsheet CFT is given by:

$$T_{\mathrm{S}}(z) = \frac{1}{k-2}\left(J^+J^- - \left(J^3\right)^2\right) = \frac{b^2}{2}\left(J\partial_x^2 J - \frac{1}{2}\left(\partial_x J\right)^2\right). \tag{99}$$

Using the Sugawara construction (reviewed in Ginsparg [40], Francesco, Mathieu, and Sénéchal [41], Ketov [42]), the stress tensor $T_{\mathrm{S}}(z)$ satisfies the Ward identity:

$$\left\langle T_{\mathrm{S}}(z)\, \Phi^{j_1}\left(x_1; z_1\right) \Phi^{j_2}\left(x_2; z_2\right) ... \Phi^{j_n}\left(x_n; z_n\right)\right\rangle \tag{100}$$

$$= \sum_{i=1}^{n}\left(\frac{1}{z-z_i}\frac{\partial}{\partial z_i} + \frac{h_i}{\left(z-z_i\right)^2}\right)\left\langle \Phi^{j_1}\left(x_1; z_1\right) \Phi^{j_2}\left(x_2; z_2\right) ... \Phi^{j_n}\left(x_n; z_n\right)\right\rangle \tag{101}$$

Substituting Eq. (99) into the left-hand side of Eq. (101) and applying Wick's theorem yields the KZ equations for the H_3^+-WZNW model.

However, as Ribault and Teschner [11] observed, it is more convenient to express the KZ equations in terms of the so-called μ-basis $\Phi^j(\mu; z)$, defined by:

$$\Phi^j(\mu; z) := \frac{|\mu|^{2j+2}}{\pi} \int d^2x e^{\mu x - \bar{\mu}\bar{x}} \Phi^j(x; z). \tag{102}$$

In this coordinate system μ on the celestial sphere, the KZ equations takes the form (Ribault and Teschner [11]):

$$\frac{\partial \widehat{G}_n}{\partial z_k} + b^2 \sum_{k \neq \ell} \frac{\mu_k \mu_\ell}{z_k - z_\ell} \left(\frac{\partial}{\partial \mu_k} - \frac{\partial}{\partial \mu_\ell} \right)^2 \widehat{G}_n = b^2 \sum_{k \neq \ell} \frac{\mu_k \mu_\ell}{z_k - z_\ell} \left[\frac{j_k(j_k+1)}{\mu_k^2} + \frac{j_\ell(j_\ell+1)}{\mu_\ell^2} \right] \widehat{G}_n. \tag{103}$$

Following the standard prescription in celestial holography (see reviews by Pasterski [25], Raclariu [26], Strominger [27], Puhm [28]), the scattering amplitude $\mathcal{A}_n$ associated with $\widehat{G}_n$ in the energy-momentum representation is obtained by performing an inverse Mellin transform of the worldsheet integral of $\widehat{G}_n$:

$$\mathcal{A}_n = \prod_{i=1}^{n} \int_{c-i\infty}^{c+i\infty} \frac{d\Delta_i}{2\pi i} \omega_i^{-\Delta_i} \prod_{j=1}^{n} \int d^2 z_j \widehat{G}_n(x_1, ..., x_n; \Delta_1, ..., \Delta_n; z_1, ..., z_n). \tag{104}$$

Since our primary goal is to derive a system of partial differential equations (PDEs) characterising the celestial n-point functions, we may omit the worldsheet integrals and define instead a quantity $G_n := (x_1, ..., x_n; \omega_1, ..., \omega_n; z_1, ..., z_n)$ given by the inverse Mellin transform of $\widehat{G}_n$:

$$G_n = \prod_{i=1}^{n} \int_{c-i\infty}^{c+i\infty} \frac{d\Delta_i}{2\pi i} \omega_i^{-\Delta_i} \widehat{G}_n(x_1, ..., x_n; \Delta_1, ..., \Delta_n; z_1, ..., z_n). \tag{105}$$

We aim to use the KZ equations to derive a system of PDEs for G_n. Assuming the resulting equations can be solved by traditional methods, the final amplitude is then given by integrating G_n over the worldsheet coordinates $z_1, ..., z_n$. Applying the inverse Mellin transform to Eq. (103), and using the identities:

$$\omega \frac{\partial}{\partial \omega} \int_{c-i\infty}^{c+i\infty} \frac{d\Delta}{2\pi i} \omega^{-\Delta} \hat{f}(\Delta) = -\int_{c-i\infty}^{c+i\infty} \frac{d\Delta}{2\pi i} \Delta \omega^{-\Delta} \hat{f}(\Delta), \tag{106}$$

and:

$$\left(\omega \frac{\partial}{\partial \omega} \right)^2 \int_{c-i\infty}^{c+i\infty} \frac{d\Delta}{2\pi i} \omega^{-\Delta} \hat{f}(\Delta) = \int_{c-i\infty}^{c+i\infty} \frac{d\Delta}{2\pi i} \Delta^2 \omega^{-\Delta} \hat{f}(\Delta), \tag{107}$$

we obtain the following differential equation:

$$\frac{\partial G_n}{\partial z_k} + b^2 \sum_{k \neq \ell} \frac{\mu_k \mu_\ell}{z_k - z_\ell} \left(\frac{\partial}{\partial \mu_k} - \frac{\partial}{\partial \mu_\ell} \right)^2 G_n = \frac{b^2}{2} \sum_{k \neq \ell} \frac{\mu_k \mu_\ell}{z_k - z_\ell} \left(\frac{\omega_k}{\mu_k^2} \frac{\partial}{\partial \omega_k} + \frac{\omega_\ell}{\mu_\ell^2} \frac{\partial}{\partial \omega_\ell} \right) G_n \tag{108}$$

$$+ \frac{b^2}{4} \sum_{k \neq \ell} \frac{\mu_k \mu_\ell}{z_k - z_\ell} \left[\frac{1}{\mu_k^2} \left(\omega_k \frac{\partial}{\partial \omega_k} \right)^2 + \frac{1}{\mu_\ell^2} \left(\omega_\ell \frac{\partial}{\partial \omega_\ell} \right)^2 \right] G_n. \tag{109}$$

Let us now consider the worldsheet Ward identity:

$$\sum_{k=1}^{n} z_k \frac{\partial \widehat{G}_n}{\partial z_k} + \sum_{k=1}^{n} h_k \widehat{G}_n = 0. \tag{110}$$

Recalling that the scaling dimension h_k in the H_3^+-WZNW model is related to the spin j_k by the relation $h_k = -b^2 j_k (j_k + 1)$, and that the holographic dictionary prescribes that the spin is connected to the celestial conformal weight Δ_k via $j_k = -\Delta_k/2$, Eq. (110) can be reformulated as follows:

$$\sum_{k=1}^{n} z_k \frac{\partial \widehat{G}_n}{\partial z_k} + \frac{b^2}{2} \sum_{k=1}^{n} \Delta_k \widehat{G}_n - \frac{b^2}{4} \sum_{k=1}^{n} \Delta_k^2 \widehat{G}_n = 0. \tag{111}$$

Finally, by applying the inverse Mellin transform, we obtain the following differential equation:

$$\sum_{k=1}^{n} z_k \frac{\partial G_n}{\partial z_k} = \frac{b^2}{2} \sum_{k=1}^{n} \omega_k \frac{\partial G_n}{\partial \omega_k} + \frac{b^2}{4} \sum_{k=1}^{n} \left(\omega_k \frac{\partial}{\partial \omega_k} \right)^2 G_n. \tag{112}$$

E. Celestial Stress-Energy Tensor

In this subsection, we shall discuss the celestial stress-energy tensor, $\mathcal{T}(x)$. As a preliminary observation, note that, based on the anticipated scaling dimension of $\mathcal{T}(x)$, it necessarily follows that it may be represented as a linear combination of the worldsheet currents, J and $\bar{J}$, together with the primary Φ:

$$\mathcal{T}(x) \coloneqq \frac{1}{2k} \int_\Sigma d^2 z \left(c_1 J \partial_x^2 \Phi + c_2 \partial_x J \partial_x \Phi + c_3 \Phi \partial_x^2 J \right) \bar{J}. \tag{113}$$

Since all terms within brackets are evaluated at the worldsheet coordinates $(z, \bar{z})$ and the celestial coordinates $(x, \bar{x})$, we will temporarily omit the explicit arguments of the currents J and $\bar{J}$, and the primary Φ. However, when analysing the behaviour of $\mathcal{T}(x)$ or its derivatives within correlation functions, we will restore the explicit dependencies on these variables to avoid ambiguity.

Given that the central extension $\mathcal{I}$, introduced in Eq. (A72), is independent of x, we have $\partial_x^2 \mathcal{I} = 0$. It follows that:

$$\int_\Sigma d^2 z \, J \bar{J} \partial_x^2 \Phi = -\int_\Sigma d^2 z \, \left(\Phi \partial_x^2 J + 2 \partial_x J \partial_x \Phi \right) \bar{J}. \tag{114}$$

Substituting this into Eq. (113), we derive the expression:

$$\mathcal{T}(x) = \frac{1}{2k} \int_\Sigma d^2 z \, \left(\alpha_1 \partial_x J \partial_x \Phi + 2\alpha_2 \Phi \partial_x^2 J \right) \bar{J}, \tag{115}$$

where the new constants $\alpha_1 \coloneqq c_2 - 2c_1$ and $2\alpha_2 \coloneqq c_3 - c_1$ have been introduced.

Using the identity $\bar{J}\Phi = (k/\pi)\,\partial_{\bar{z}}\lambda$, and applying the complex-divergence theorem (Eq. (A7)), we can rewrite $\mathcal{T}(x)$ as:

$$\mathcal{T}(x) = \frac{1}{2\pi}\int_{\Sigma} d^2z\,\left(\alpha_1\partial_x J\partial_x\partial_{\bar{z}}\lambda + 2\alpha_2\partial_x^2 J\partial_{\bar{z}}\lambda\right) \tag{116}$$

$$= \frac{1}{2}\oint \frac{dz}{2\pi i}\left(\alpha_1\partial_x J\partial_x\lambda + 2\alpha_2\lambda\partial_x^2 J\right). \tag{117}$$

To constrain the coefficients α_1 and α_2, we impose BRST invariance, requiring that the term inside the brackets in Eq. (113) transforms as a worldsheet primary. A more efficient approach, however, stems from recognising that, by construction, $\mathcal{O}^a(y)$ generates a level-$\hat{k}_G$ celestial Kac-Moody current algebra and satisfies the Ward identities. In particular, $\mathcal{T}(x)$ can only function as a genuine stress tensor if the OPE $\mathcal{T}(x)\,\mathcal{O}^a(y)$ includes the term:

$$\mathcal{T}(x)\,\mathcal{O}^a(y) \sim ... + \frac{1}{x-y}\partial_y\mathcal{O}^a(y) + ... \tag{118}$$

Thus, we compute this OPE and impose the above condition as a constraint. It is advantageous to work with the partial derivative of $\mathcal{T}(x)$ with respect to the anti-holomorphic variable $\bar{x}$:

$$\partial_{\bar{x}}\mathcal{T}(x) = \frac{1}{2}\oint \frac{dz}{2\pi i}\left(\alpha_1\partial_x J\partial_x\partial_{\bar{x}}\lambda + 2\alpha_2\partial_x^2 J\partial_{\bar{x}}\lambda\right) \tag{119}$$

$$= \frac{1}{2}\oint \frac{dz}{2i}\left(\alpha_1\partial_x J\partial_x\Phi + 2\alpha_2\Phi\partial_x^2 J\right), \tag{120}$$

and evaluate its contribution within correlation functions that include an insertion of the current $\mathcal{O}^a(y)$.

To simplify our calculations, we divide $\mathcal{T}(x)$ into the components:

$$\mathcal{T}_1(x) := \frac{\alpha_1}{2}\oint \frac{dz}{2i}\partial_x J\partial_x\Phi,\;\; \mathcal{T}_2(x) := \alpha_2\oint\frac{dz}{2i}\Phi\partial_x^2 J. \tag{121}$$

The contribution of the OPE $\partial_{\bar{x}}\mathcal{T}_1(x)\,\mathcal{O}^a(y)$ to the correlation function is:

$$\langle\partial_x\mathcal{T}_1(x)\,\mathcal{O}^a(y)\,...\rangle_{\text{CCFT}} \tag{122}$$

$$= -\frac{\alpha_1}{2k}\int_{\Sigma} d^2w\oint\frac{dz}{2i}\langle j^a(w)\,\partial_x J(x;z)\,\partial_x\Phi(x,\bar{x};z,\bar{z})\,\Phi(y,\bar{y};w,\bar{w})\,\bar{J}(\bar{y};\bar{w})\,...\rangle \tag{123}$$

$$\sim -\frac{\alpha_1}{2k}\partial_x\delta^{(2)}(x-y)\int_{\Sigma} d^2w\oint_{C_\varepsilon(w)}\frac{dz}{2i}\langle j^a(w)\,\partial_x J(x;z)\,\Phi(y,\bar{y};w,\bar{w})\,\bar{J}(\bar{y};\bar{w})\,...\rangle. \tag{124}$$

Recalling (cf. Giveon, Kutasov, and Seiberg [13, Sec. 2]) that the worldsheet current operator $J(x;z)$ acts on the primary $\Phi(x,\bar{x};z,\bar{z})$ as:

$$J(x;z)\,\Phi(y,\bar{y};w,\bar{w}) \sim \frac{1}{z-w}\left[(y-x)^2\partial_y + 2(y-x)\right]\Phi(y,\bar{y};w,\bar{w}), \tag{125}$$

the partial derivative with respect to x yields:

$$\partial_x J(x;z)\,\Phi(y,\bar{y};w,\bar{w}) \sim \frac{2}{z-w}\left[(x-y)\,\partial_y - 1\right]\Phi(y,\bar{y};w,\bar{w})\,. \tag{126}$$

Therefore,

$$\langle \partial_{\bar{x}} \mathcal{T}_1(x)\,\mathcal{O}^a(y)\,...\rangle_{\text{CCFT}} \sim \frac{\partial}{\partial \bar{x}}\left(\frac{\alpha_1}{(x-y)^2}\langle \mathcal{O}^a(y)\,...\rangle_{\text{CCFT}} - \frac{\alpha_1}{x-y}\langle \partial_y \mathcal{O}^a(y)\,...\rangle_{\text{CCFT}}\right). \tag{127}$$

Next, consider the contribution from $\mathcal{T}_2(x)$. Taking its partial derivative and inserting it into the correlation function with the Kac-Moody current $\mathcal{O}^a(y)$ yields:

$$\langle \partial_{\bar{x}} \mathcal{T}_2(x)\,\mathcal{O}^a(y)\,...\rangle_{\text{CCFT}} \tag{128}$$

$$\sim -\frac{\alpha_2}{k}\int d^2w \oint \frac{dz}{2i}\langle \partial_x^2 J(x;z)\,\Phi(x,\bar{x};z,\bar{z})\,\Phi(y,\bar{y};w,\bar{w})\,\bar{J}(\bar{y};\bar{w})\,j^a(w)\,...\rangle. \tag{129}$$

Taking the partial derivative of Eq. (126) with respect to x gives:

$$\partial_x^2 J(x;z)\,\Phi(y,\bar{y};w,\bar{w}) \sim \frac{2}{z-w}\partial_y \Phi(y,\bar{y};w,\bar{w})\,. \tag{130}$$

Substituting this result into Eq. (129) gives:

$$\langle \partial_{\bar{x}} \mathcal{T}_2(x)\,\mathcal{O}^a(y)\,...\rangle_{\text{CCFT}} \sim \frac{\partial}{\partial \bar{x}}\left(\frac{2\alpha_2}{(x-y)^2}\langle \mathcal{O}^a(y)\,...\rangle_{\text{CCFT}} + \frac{2\alpha_2}{x-y}\langle \partial_y \mathcal{O}^a(y)\,...\rangle_{\text{CCFT}}\right). \tag{131}$$

Consequently, combining the results obtained in Eqs. (127, 131), and integrating with respect to $\bar{x}$, we finally obtain:

$$\mathcal{T}(x)\,\mathcal{O}^a(y) \sim \frac{2\alpha_2+\alpha_1}{(x-y)^2}\mathcal{O}^a(y) + \frac{2\alpha_2-\alpha_1}{x-y}\partial_y \mathcal{O}^a(y)\,. \tag{132}$$

Thus, a necessary condition for $\mathcal{T}(x)$ to be a valid stress tensor in the celestial CFT is that the coefficient of $(x-y)^{-1}\,\partial_y \mathcal{O}^a(y)$ in Eq. (132) must equal one. This implies that the constants α_1 and α_2 must satisfy the constraint $2\alpha_2 - \alpha_1 = 1$. A convenient choice is $\alpha_1 = \alpha_2 = 1$, which leads to the final form of the celestial stress tensor:

$$\mathcal{T}(x) = \frac{1}{2}\oint \frac{dz}{2\pi i}\left(\partial_x J \partial_x \lambda + 2\lambda \partial_x^2 J\right).$$

In conclusion, following an analogous line of reasoning as that which led to Eq. (132), we derive the Ward identity for the OPE of the celestial stress-energy tensor $\mathcal{T}\mathcal{T}$:

$$\mathcal{T}(x_1)\,\mathcal{T}(x_2) \sim \frac{c}{(x_1-x_2)^4} + \frac{2\mathcal{T}(x_2)}{(x_1-x_2)^2} + \frac{\partial_{x_2}\mathcal{T}(x_2)}{x_1-x_2}. \tag{133}$$

IV. DISCUSSION

In this work, we have introduced a novel holographic dictionary connecting AdS_3 string theory with celestial CFT. Building upon the foundational ideas of de Boer *et al.* [12], Giveon, Kutasov, and Seiberg [13], Maldacena and Ooguri [16], Maldacena, Ooguri, and Son [17], and drawing inspiration from the model advanced by Ogawa *et al.* [1], we have demonstrated that the correlation function, stress-energy tensor, Kac-Moody currents, and vertex operators of the celestial CFT can be systematically derived from their counterparts in the H_3^+-WZNW model. Moreover, by applying the results rigorously established by the mathematical physicists Teschner [7, 8, 9, 10], Ribault and Teschner [11], we have determined the two- and three-point functions, as well as the structure constants of the celestial OPE. Additionally, we have derived a system of partial differential equations that characterise celestial amplitudes in the frequency-momentum space, and have offered a brief analysis of the infrared and ultraviolet behaviour of these amplitudes.

Over the past decades, string theory has provided profound insights into quantum gravity, inspiring us to undertake a broader research program, of which this paper constitutes the inaugural contribution. Our objective is to investigate certain aspects of celestial holography through the framework of string theory. The holographic dictionary proposed herein opens multiple avenues for further research. Notably, de Gioia and Raclariu [44, 45] have explored the derivation of various aspects of celestial holography from the AdS/CFT correspondence in a series of remarkable papers. We believe that our proposed holographic dictionary between AdS_3 string theory and celestial CFT could serve as a valuable toy model to support these efforts.

We further recall the well-known construction of black holes in $d = 2+1$ gravity with a negative cosmological constant, introduced by Banados, Teitelboim, and Zanelli [46], and subsequently shown by Cangemi, Leblanc, and Mann [47] to arise as orbifolds solutions of string theory. Strominger [48] proposed a unified framework for all "black objects" with near-horizon geometries described by AdS_3, encompassing the BTZ solution. Building on these works, we are optimistic that insights into black hole solutions in string theory might be employed to study black holes in asymptotically flat spaces, informed by the ideas presented in this paper. In particular, these insights may intersect with the constructions recently discussed by Crawley *et al.* [49]. In our forthcoming note, we shall investigate the connection between $\mathcal{N} = 2$ string theory in Klein space and celestial conformal blocks.

V. ACKNOWLEDGEMENT

This research was supported in part by Perimeter Institute for Theoretical Physics during the "Simons/Perimeter Celestial Holography Summer School 2024." Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Research, Innovation and Science. The author also thanks Giorgio Torrieri for his important encouragement.

Appendix A: Mini-Introduction to H_3^+-WZNW Model

In this section, we develop a holographic dictionary that maps the worldsheet vertex operators and generators of worldsheet Kac-Moody and Virasoro current algebras to celestial conformal primaries, allowing for the construction of the celestial conformal field theory (CFT) entirely from the data provided by string theory. To this end, we begin by reviewing the H_3^+-WZNW model, which we identify as the analytic continuation of string theory on Lorentzian AdS_3 to its Euclidean counterpart, H_3^+.

1. H_3^+-WZNW Model Review

Brown and Henneaux [50] demonstrated that any theory of $3d$ gravity with a negative cosmological constant is endowed with an infinite-dimensional symmetry group that includes two commuting copies of Virasoro algebras, thus describing a $2d$ CFT. This observation was given a concrete form in the context of AdS_3 string theory by de Boer *et al.* [12], Giveon, Kutasov, and Seiberg [13], Maldacena and Ooguri [16]. In this work, for simplicity, we focus on the analytically continued Euclidean AdS_3 string theory, though analogous considerations apply to the $SL(2,\mathbf{R})$ model studied by Maldacena and Ooguri [16].

a. *Classical Theory*

Let $(u^0, \vec{u})$ be Cartesian coordinates on $4d$ Minkowski space $\mathbf{R}_3^1$. Recall that Euclidean AdS_3 can be formulated as the hypersurface $H_3 \subset \mathbf{R}_3^1$ defined by $-(u^0)^2 + \sum_{i=1}^3 (u^i)^2 = -\ell^2$. Thus, H_3 is a submanifold of $\mathbf{R}_3^1$ with constant negative scalar curvature $R = -1/\ell^2$ and an isometry group given by $SL(2,\mathbf{C})$. This space can be parametrised by the coordinates (r, τ, θ), given by:

$$u^0 = \sqrt{\ell^2 + r^2}\cosh\tau, \;\; u^1 = r\sin\theta, \;\; u^2 = r\cos\theta, \;\; u^3 = \sqrt{\ell^2 + r^2}\sinh\tau, \tag{A1}$$

where $\theta \in [0, 2\pi)$ and $r \geq 0$. The first fundamental form of H_3, inherited from the ambient space $\mathbf{R}^1_3$, is given by the line element:

$$ds^2 = \frac{1}{1 + r^2/\ell^2} dr^2 + \ell^2 \left(1 + \frac{r^2}{\ell^2}\right) d\tau^2 + r^2 d\theta^2. \tag{A2}$$

There are two disconnected components characterising the topology of H_3, one corresponding to $u^0 > 0$ and the other to $u^0 < 0$. We restrict our attention to the former, and the resulting hypersurface, H_3^+, satisfies $\left|u^0\right| > u^3$.

To connect this with the H_3^+-WZNW model, it is more convenient to introduce the coordinates $(\phi, \gamma, \bar{\gamma})$, defined as:

$$\phi = \log \ell^{-1} \left(u^0 + u^3\right), \quad \gamma = \frac{u^2 + iu^1}{u^0 + u^3}, \quad \bar{\gamma} = \frac{u^2 - iu^1}{u^0 + u^3}. \tag{A3}$$

In these coordinates, the line element becomes:

$$ds^2 = \ell^2 \left(d\phi^2 + e^{2\phi} d\gamma d\bar{\gamma}\right). \tag{A4}$$

Notice that the boundary of Euclidean AdS_3 corresponds to $\phi \to \infty$, which is homeomorphic to $\mathbf{CP}^1$, parametrised by $(\gamma, \bar{\gamma})$.

As explained by Maldacena and Strominger [51], the equations of motion for strings propagating in a background with the line element given by Eq. (A4) can only be satisfied by turning on a Neveu-Schwarz two-form field $\mathbf{B} := (1/2)\, B_{\mu\nu} dx^\mu \wedge dx^\nu$. One way to understand this requirement is from the worldsheet perspective, where a non-vanishing $\mathbf{B}$ is necessary to maintain conformal invariance. It turns out, as shown by de Boer *et al.* [12], Giveon, Kutasov, and Seiberg [13], Maldacena and Strominger [51], that the required background is characterised by $\mathbf{B} = \ell^2 e^{2\phi} d\gamma \wedge d\bar{\gamma}$. Consequently, the action integral for the worldsheet becomes[1]:

$$I[\phi, \gamma, \bar{\gamma}] = 2 \left(\frac{\ell}{\ell_0}\right)^2 \int_\Sigma d^2 z \left(\partial_z \phi \partial_{\bar{z}} \phi + e^{2\phi} \partial_{\bar{z}} \gamma \partial_z \bar{\gamma}\right). \tag{A8}$$

The action in Eq. (A8) can be identified with the H_3^+-WZNW model, an example of a non-compact Wess-Zumino-Novikov-Witten model. The Lagrangian formulation of this model was

[1] In these notes, we follow the conventions of Francesco, Mathieu, and Sénéchal [41] regarding the integral measure on the Riemann sphere, such that:

$$d^2 z := \frac{1}{2i} d\bar{z} \wedge dz. \tag{A5}$$

As a result, the complex delta-function is represented as:

$$\partial_{\bar{z}} \frac{1}{z - w} = \pi \delta^{(2)}(z - w), \tag{A6}$$

and the divergence theorem takes the form:

$$\int_\Sigma d^2 z \left(\partial_z v^z + \partial_{\bar{z}} v^{\bar{z}}\right) = \frac{1}{2i} \oint_{\partial\Sigma} \left(dz v^{\bar{z}} - d\bar{z} v^z\right), \tag{A7}$$

where $\Sigma \subset \mathbf{CP}^1$ is a simply-connected region.

developed by Gawedzki and Kupiainen [52] and later expanded by Gawedzki [53] with a focus on path-integral methods. The pure WZNW action integral at level-k is given by:

$$S[H] = -\frac{1}{2\pi}\int_{\Sigma} d^2z \mathrm{tr}\left(H^{-1}\partial_z H\right)\left(H^{-1}\partial_{\bar{z}} H\right) + k\Gamma, \tag{A9}$$

where Γ is the Wess-Zumino topological term. Parametrising the coset space $SL(2,\mathbf{C})/SU(2)$, which consists of complex two-by-two Hermitian matrices with unit determinant, by:

$$H(\phi,\gamma,\bar{\gamma}) = \begin{pmatrix} |\gamma|^2 e^{\phi} + e^{-\phi} & \gamma e^{\phi} \\ e^{\phi}\bar{\gamma} & e^{\phi} \end{pmatrix}, \tag{A10}$$

one can see that the worldsheet action in Eq. (A9) is *classically* equivalent to the action in Eq. (A8) when $H(\phi,\gamma,\bar{\gamma})$ is restricted to this form. As Gawedzki [54] demonstrated using path-integral techniques, these theories remain equivalent at the quantum level. This identification justifies the equivalence between (bosonic) string theory on Euclidean AdS_3 and the H_3^+-WZNW model.

b. *Quantum Theory*

Having reviewed the classical formulation of the H_3^+-WZNW model, we now turn to the construction of the quantum state space, following the approach of Teschner [8]. In the Schrödinger representation, the Hilbert space of the H_3^+-WZNW model is given by $\mathcal{H} = L^2\left(H_3^+, dh\right)$, the Lebesgue space of square-integrable complex-valued functions on the hypersurface H_3^+ with respect to the measure:

$$dh = d\phi d^2\gamma, \;\; h = \begin{pmatrix} e^{\phi}\sqrt{1+\gamma\bar{\gamma}} & u \\ \bar{u} & e^{-\phi}\sqrt{1+\gamma\bar{\gamma}} \end{pmatrix}. \tag{A11}$$

We shall call the elements of $\mathcal{H}$ *wavefunctions* of the quantum theory.

We also define an action of the symmetry group $SL(2,\mathbf{C})$ on wavefunctions $\Psi \in \mathcal{H}$, given by:

$$g \in SL(2,\mathbf{C}) \mapsto T_g\Psi(h) = \Psi\left(g^{-1}h\left(g^{-1}\right)^{\dagger}\right), \tag{A12}$$

for each point $h \in H_3^+$.

The inner product on the quantum state space is defined by:

$$\left\langle \Psi_1 \middle| \Psi_2 \right\rangle := \int_{H_3^+} dh\, \Psi_2^*(h)\, \Psi_1(h). \tag{A13}$$

Using the Iwasawa decomposition, it can be shown that for all pairs of states $\Psi_1, \Psi_2 \in \mathcal{H}$, the inner product becomes:

$$\left\langle \Psi_1 \middle| \Psi_2 \right\rangle = \frac{1}{\mathcal{V}_{SU(2)}}\int_{SL(2,\mathbf{C})} dg\, \Psi_2^*(gg^{\dagger})\Psi_1(gg^{\dagger}). \tag{A14}$$

where $\mathcal{V}_{SU(2)}$ denotes the volume of $SU(2)$, and dg represents the $SL(2, \mathbf{C})$-invariant measure:

$$dg = \left(\frac{i}{2}\right)^4 d^2\zeta_1 d^2\zeta_2 d^2\zeta_3 d^2\zeta_4\, \delta^{(2)}\left(\zeta_1\zeta_4 - \zeta_2\zeta_3\right), \quad g = \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{pmatrix}. \tag{A15}$$

The rationale behind the choice of the inner product defined in Eq. (A13) is that it ensures the unitary realisation of the $SL(2, \mathbf{C})$ action on the space $\mathcal{H}$ of wavefunctions, as introduced in Eq. (A12). This property arises from Eq. (A14) and the invariance of the measure dg on $SL(2, \mathbf{C})$, which satisfies $d(g_0 g) = dg$.

We now introduce a key element in bridging the abstract mathematical formalism of Hilbert spaces to physics through the specification of an appropriate basis. In non-relativistic quantum mechanics, a convenient choice of basis is provided by the plane waves of definite momentum, which afford a clear physical interpretation. Analogously, in the H_3^+-WZNW model, there exists a counterpart to the "plane-wave" basis, represented by the so-called conformal primaries with spin j, denoted by $\Phi^j(x; z)$, and defined as:

$$\Phi^j(x; h) := \frac{1}{\pi}\left(\begin{pmatrix} x & 1 \end{pmatrix} \cdot h \cdot \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}\right)^{2j}, \quad h \in SL(2; \mathbf{C}). \tag{A16}$$

Here, we adopt the normalisation introduced by Giveon, Kutasov, and Seiberg [13]. Furthermore, by parametrising $h \in SL(2; \mathbf{C})$ as in Eq. (A11), and treating $\phi, \gamma, \bar{\gamma}$ as fields depending on the points $z, \bar{z}$ on the worldsheet, we simplify the notation by writing $\Phi^j(x; z) := \Phi^j(x; h(\phi, \gamma, \bar{\gamma}))$. Specifically, $x, \bar{x}$ will denote coordinates on the celestial sphere, while $z, \bar{z}$ will serve as coordinates on the string worldsheet.

It was demonstrated by Teschner [8, Appendix A] through spectral decomposition that the set of functions $\{\Phi^j(x; z) : x \in \mathbf{C}; i \in -1/2 + i\mathbf{R}^+\}$ forms a complete and δ-function normalisable basis, such that:

$$\left\langle \Phi^j(x; h), \Phi^{j'}(x'; h)\right\rangle \propto \delta^{(2)}\left(x - x'\right)\delta\left(j - j'\right), \tag{A17}$$

for all $x, \bar{x} \in \mathbf{C}$ and $j, j' \in -1/2 + i\mathbf{R}^+$. In the following subsection, we will review how these wavefunctions can be employed to construct a Fourier-type transform, further reinforcing their role as analogues of "plane-waves" in the quantum theory of the H_3^+-WZNW model.

c. *Symmetries*

No discussion of quantum theory would be reasonable without emphasising the role played by symmetries, and the case of the H_3^+-WZNW model is no exception. Following Giveon, Kutasov, and Seiberg [13, Section 2], we observe that the classical theory described by the action integral in Eq. (A8) possesses an infinite-dimensional affine $SL\left(2;\mathbf{C}\right)\times\overline{SL\left(2;\mathbf{C}\right)}$ symmetry, the global part of which admits the following generators:

$$J_0^- := \frac{\partial}{\partial\gamma},\ \ J_0^3 := \gamma\frac{\partial}{\partial\gamma} - \frac{1}{2}\frac{\partial}{\partial\phi},\ \ J_0^+ := \gamma^2\frac{\partial}{\partial\gamma} - \gamma\frac{\partial}{\partial\phi} - e^{-2\phi}\frac{\partial}{\partial\bar{\gamma}}, \tag{A18}$$

and similarly:

$$\bar{J}_0^- := \frac{\partial}{\partial\bar{\gamma}},\ \ \bar{J}_0^3 := \bar{\gamma}\frac{\partial}{\partial\bar{\gamma}} - \frac{1}{2}\frac{\partial}{\partial\phi},\ \ \bar{J}_0^+ := \bar{\gamma}^2\frac{\partial}{\partial\bar{\gamma}} - \bar{\gamma}\frac{\partial}{\partial\phi} - e^{-2\phi}\frac{\partial}{\partial\gamma}. \tag{A19}$$

A significant class of observables in the H_3^+-WZNW model consists of smooth functions on the hyperboloid H_3^+. A convenient method to analyse the decomposition of these functions in terms of representations of $SL\left(2;\mathbf{C}\right)\times\overline{SL\left(2;\mathbf{C}\right)}$ is to introduce a pair of complex variables $(x,\bar{x})$ and realise the Lie algebra of $SL\left(2;\mathbf{C}\right)\times\overline{SL\left(2;\mathbf{C}\right)}$ by defining the following generators as linear differential operators acting on the germ of smooth functions on H_3^+:

$$J_0^- = -\frac{\partial}{\partial x},\ \ J_0^3 = -\left(x\frac{\partial}{\partial x} + p\right),\ \ J_0^+ = -\left(x^2\frac{\partial}{\partial x} + 2px\right), \tag{A20}$$

and:

$$\bar{J}_0^- = -\frac{\partial}{\partial\bar{x}},\ \ \bar{J}_0^3 = -\left(\bar{x}\frac{\partial}{\partial\bar{x}} + 2\bar{p}\right),\ \ \bar{J}_0^+ = -\left(\bar{x}^2\frac{\partial}{\partial\bar{x}} + 2\bar{p}\bar{x}\right), \tag{A21}$$

where p is related to the "spin" of the $SL\left(2;\mathbf{C}\right)$ representation by $p = j+1$. Indeed, it should be recalled that the value of the (quadratic) Casimir in this representation is $j\left(j+1\right)$.

We may now draw from the observation that, since $J_0^- = -\partial/\partial x$ and $\bar{J}_0^- = -\partial/\partial\bar{x}$, every observable $\mathsf{T}\left(x,\bar{x}\right)$ is conjugate to $\mathsf{T}\left(0,0\right)$ by the relation:

$$\mathsf{T}\left(x,\bar{x}\right) =^{-xJ_0^- - \bar{x}\bar{J}_0^-}\ \mathsf{T}\left(0,0\right)e^{xJ_0^- + \bar{x}\bar{J}_0^-}, \tag{A22}$$

leading us to define the currents:

$$J^+\left(x;z\right) := e^{-xJ_0^-}J^+\left(z\right)e^{xJ_0^-},\ J\left(x;z\right) := e^{-xJ_0^-}J^3\left(z\right)e^{xJ_0^-},\ J^-\left(x;z\right) = e^{-xJ_0^-}J^-\left(z\right)e^{xJ_0^-}. \tag{A23}$$

As observed by Giveon, Kutasov, and Seiberg [13], since these currents are related through differentiation, our attention may be focused on:

$$J\left(x;z\right) := 2xJ^3\left(z\right) - J^+\left(z\right) - x^2J^-\left(z\right), \tag{A24}$$

and similarly for:

$$\bar{J}\left(\bar{x};\bar{z}\right) := 2\bar{x}\bar{J}^3\left(\bar{z}\right) - \bar{J}^+\left(\bar{z}\right) - \bar{x}^2\bar{J}^-\left(\bar{z}\right). \tag{A25}$$

In the quantum theory, the currents in Eqs. (A18, A19) are generators of an $\widehat{SL}(2;\mathbf{C})\times\widehat{\overline{SL}}(2;\mathbf{C})$ algebra. Their operator product expansions may be succinctly expressed as:

$$J(x;z)\,J(y;w)\sim k\frac{(y-x)^2}{(z-w)^2}+\frac{1}{z-w}\left[(y-x)^2\,\partial_y-2\,(y-x)\right]J(y;z)\,, \tag{A26}$$

and:

$$J(x;z)\,\Phi^j(y;w)\sim\frac{1}{z-w}\left[(y-x)^2\,\partial_y+2j\,(y-x)\right]\Phi^j(y;w)\,, \tag{A27}$$

with analogous expressions holding for the conjugates.

We conclude our discussion of the quantum theory of the H_3^+-WZNW model by noting that $\bar{J}(\bar{x};\bar{z})\,\Phi^{j=1}(x;z)$ is an observable that will appear frequently in the following subsections, where the conformal primary $\Phi^{j=1}(x;z)$ will be denoted simply as $\Phi(x;z)$. The observable $\bar{J}(\bar{x};\bar{z})\,\Phi(x;z)$ satisfies the identity:

$$\bar{J}(\bar{x};\bar{z})\,\Phi(x;z)=\frac{k}{\pi}\partial_{\bar{z}}\lambda(x,\bar{x};z,\bar{z})\,, \tag{A28}$$

where:

$$\lambda:=-\frac{1}{\gamma-x}\frac{(\gamma-x)(\bar{\gamma}-\bar{x})\,e^{2\phi}}{1+(\gamma-x)(\bar{\gamma}-\bar{x})\,e^{2\phi}}. \tag{A29}$$

Similarly, these objects satisfy the identity:

$$\Phi(x;z)=\frac{1}{\pi}\partial_{\bar{x}}\lambda(x,\bar{x};z,\bar{z})\,, \tag{A30}$$

with analogous expressions holding for the conjugates.

d. The Mini-Superspace ($k\to\infty$) Limit

We are now in a position to discuss the mini-superspace limit of the correlation functions within the framework of the H_3^+-WZNW model. This limit holds particular significance for the developments presented in these notes, as it is in this case that we can holographically derive the tree-level MHV scattering amplitudes for gluons and gravitons analytically continued to Klein space, within the framework of celestial leaf amplitudes.

From a heuristic perspective, the results we are about to describe can be motivated as follows. Recall that the conformal primaries $\Phi^j(x;z)$ in the H_3^+-WZNW model can be viewed as analogues of the "plane-wave" basis for the Hilbert space of the model. In this basis, we can define the analogue of the Fourier transform as follows:

$$F\left(j\middle|x,\bar{x}\right):=\int_{H_3^+}\Phi^j(x,\bar{x};h)\,f(h)\,, \tag{A31}$$

whose inverse is given by:

$$f(h) = \frac{i}{(4\pi)^3} \int_{-\frac{1}{2}+i\mathbf{R}^+} dj\, (2j+1)^2 \int d^2x \left(\Phi^j(x,\bar{x};h)\right)^* F\left(j|x,\bar{x}\right). \tag{A32}$$

This transformation allow us, in particular, to decompose the correlation function:

$$\mathcal{F}_n := \left\langle \Phi^{j_1}(x_1;z_1) \ldots \Phi^{j_n}(x_n,z_n) \right\rangle, \tag{A33}$$

into the basis $\left\{\Phi^j(x;z)\right\}$, which now becomes the focus of our attention. However, as will be discussed in more detail in Subsection III D, following an important observation by Ribault and Teschner [11], in the so-called μ-basis given by:

$$\Phi^j(\mu;z) := \frac{|\mu|^{2j+2}}{\pi} \int d^2x e^{\mu x - \bar{\mu}\bar{x}} \Phi^j(x;z), \tag{A34}$$

the Knizhnik-Zamolodchikov (KZ) equations for the H_3^+-WZNW model take the following form (Ribault and Teschner [11]):

$$\frac{\partial \mathcal{F}_n}{\partial z_k} + b^2 \sum_{k\neq\ell} \frac{\mu_k\mu_\ell}{z_k - z_\ell} \left(\frac{\partial}{\partial\mu_k} - \frac{\partial}{\partial\mu_\ell}\right)^2 \mathcal{F}_n = b^2 \sum_{k\neq\ell} \frac{\mu_k\mu_\ell}{z_k - z_\ell} \left[\frac{j_k(j_k+1)}{\mu_k^2} + \frac{j_\ell(j_\ell+1)}{\mu_\ell^2}\right] \mathcal{F}_n, \tag{A35}$$

where $b^2 := 1/(k-2)$. In the mini-superspace ($k \to \infty$) limit, Eq. (A35) simplifies to:

$$\frac{\partial \mathcal{F}_n}{\partial z_k} = 0. \tag{A36}$$

Thus, in the large-k limit, $\mathcal{F}_n$ depends solely on the "boundary spacetime" coordinates $x, \bar{x}$. It is therefore natural to expect that, in this limit, the Fourier representation of the correlation functions will be dominated by the convolution of these "plane waves," such that:

$$\lim_{k\to\infty} \mathcal{F}_n = \frac{\mathcal{N}}{2\pi^2} \int_{H_3^+} dh \prod_{i=1}^n \Phi^{j_i}(x_i,\bar{x}_i;h) = \mathcal{N} \int \frac{d\rho d\gamma d\bar{\gamma}}{\rho^3} \prod_{i=1}^n \left(\frac{\rho}{\rho^2 + |x_i - \gamma|^2}\right)^{2j_i}, \tag{A37}$$

where we define the "radial" coordinate $\rho := e^{-\phi}$. Consequently, in the $k \to \infty$ limit, the correlation functions of the primaries of the H_3^+-WZNW model can be interpreted as a Feynman-Witten contact diagram for massless scalars propagating on Euclidean AdS_3 (cf. Penedones [55]).

We shall now briefly review how one could rigorously establish these arguments, drawing from the observation made by Ribault and Teschner [11] that the correlation functions of the H_3^+-WZNW model on the sphere are related to the correlation functions of the Liouville vertex operators $V_\alpha(z) =: e^{2\alpha\phi(z)}$:, which include degenerate fields, through the following relation:

$$\left\langle \Phi^{j_1}(\mu_1;z_1) \ldots \Phi^{j_n}(\mu_n;z_n) \right\rangle \tag{A38}$$

$$= \frac{\pi(-\pi)^{-n}}{2} b\delta^{(2)}\left(\sum_{i=1}^n \mu_i\right) |\Theta_n|^2 \left\langle V_{\alpha_1}(z_1) \ldots V_{\alpha_n}(z_n) V_{-1/2b}(y_1) \ldots V_{-1/2b}(y_n) \right\rangle, \tag{A39}$$

where the function Θ is defined by:

$$\Theta_n := u \prod_{r<s\leq n} z_{rs}^{1/2b^2} \prod_{k<\ell\leq n-2} y_{k\ell}^{1/2b^2} \prod_{r=1}^{n} \prod_{k=1}^{n-2} \frac{1}{(z_r - y_k)^{1/2b^2}}, \tag{A40}$$

and the variables $y_1, ..., y_n, u$ are related to $\mu_1, ..., \mu_n$ via the equation:

$$\frac{1}{u} \sum_{i=1}^{n} \frac{\mu_i}{t - z_i} = \frac{\prod_{j=1}^{n-2} (t - y_j)}{\prod_{i=1}^{n} (t - z_i)}. \tag{A41}$$

Using these results, it was demonstrated in Section 4 of Ribault and Teschner [11] that the large-k asymptotics of the correlation functions in the H_3^+-WZNW model are equivalent, up to a proportionality constant that we shall denote by $\mathcal{N}$, to the semiclassical limit $b \to 0$ of the Liouville correlation functions. Furthermore, recalling from Appendix A of Melton *et al.* [5] that the semiclassical limit of the Liouville vertex operators $V_{2\sigma_1}(x_1)$, ..., $V_{2\sigma_n}(x_n)$ can be represented as a contact Feynman-Witten diagram for massless scalars on AdS_3,

$$\lim_{b\to 0^+} \langle V_{2\sigma_1}(x_1) ... V_{2\sigma_n}(x_n) \rangle \propto \int_{AdS_3/\mathbf{Z}} d^3\hat{y} \frac{\Gamma(2\sigma_i)}{(\varepsilon - iq(x_i, \bar{x}_i) \cdot \hat{y})}, \tag{A42}$$

we see that Eq. (A37) follows from these considerations.

2. Kac-Moody Current Algebras

We are now prepared to present the first entry of our holographic dictionary. Our objective is to describe the construction of Kac-Moody currents within the celestial CFT–henceforth referred to as *celestial Kac-Moody current algebras*–derived from worldsheet Kac-Moody currents associated with gauge symmetries.

Let Σ denote the worldsheet of a bosonic string embedded in the hyperboloid H_3^+ (times an arbitrary compact manifold $\mathcal{N}$), and let G be a Lie group with structure constants if^{abc}. Suppose we are given a set of generators $j^a(z)$ of a G level-$\hat{k}_G$ Kac-Moody current algebra on the worldsheet Σ, satisfying the operator product expansion:

$$j^a(z) j^b(w) \sim \frac{\hat{k}_G \delta^{ab}}{(z-w)^2} + \frac{if^{abc} j^c(w)}{z-w}. \tag{A43}$$

We recall from Polchinski [36] that, given such a set of worldsheet currents $j^a(z)$ obeying the above OPE, one obtains, in string theory, a gauge field $A_\mu^a(x)$ in the target space, whose associated vertex operator takes the form $\int d^2z j^a(z) A_\mu^a(x) \partial_{\bar{z}} X^\mu(z, \bar{z})$, where $X^\mu(z, \bar{z})$ represents the embedding of the string in the target space. The gauge field transforms under infinitesimal gauge transformations as $\delta A^a_\mu(x) = \partial_\mu \lambda^a(x)$. Thus, a pure gauge field is described by a vertex operator of the form $\int d^2 z j^a(z) \partial_{\bar{z}} \lambda^a(x)$.

However, as is well known from the study of asymptotic symmetries and the infrared structure of gauge theories (cf. Strominger [27]), an important class of vertex operators emerges from pure gauge configurations whose gauge function $\lambda^a(x)$ does not have compact support in the target space. In the present context, this implies that $\lambda^a(x)$ does not vanish at the boundary of AdS_3, thereby generating an algebra of large gauge transformations. (Of course, in string theory, one must also impose worldsheet consistency conditions, e.g., the integrands in the vertex operators must be primary under both left- and right-moving Virasoro symmetries. Furthermore, when the target space is flat, the gauge field must satisfy the gauge condition $\partial_\mu A^{\mu a} = 0$, in addition to the massless Klein-Gordon equation $\partial_\mu \partial^\mu A^a_\nu = 0$.)

Thus, we are lead to define the generators of the celestial Kac-Moody currents via the following vertex operators:

$$\mathcal{O}^a(x) := -\frac{1}{\pi} \int_\Sigma d^2 z\, j^a(z)\, \partial_{\bar{z}} \lambda(x, \bar{x}; z, \bar{z}). \tag{A44}$$

Let $\mathsf{W}^{R,h}(x, \bar{x}; z, \bar{z})$ be a worldsheet operator depending on the celestial coordinates $x, \bar{x} \in \mathbf{CP}^1$, transforming in a representation R of G with weight h, such that:

$$j^a(z)\, \mathsf{W}^{R,h}(x, \bar{x}; w, \bar{w}) \sim \frac{1}{z - w} t^a(R)\, \mathsf{W}^{R,h}(x, \bar{x}; w, \bar{w}), \tag{A45}$$

where $t^a(R)$ is the matrix associated with the representation under which $\mathsf{W}^{R,h}$ transforms. To verify the consistency of our definition of $\mathcal{O}^a(x)$, we must ensure it satisfies the celestial current algebra Ward identities:

$$\mathcal{O}^a(x)\, \mathsf{W}^{R,h}(y, \bar{y}; z, \bar{z}) \sim \frac{1}{x - y} t^a(R)\, \mathsf{W}^{R,h}(x, \bar{x}; z, \bar{z}), \tag{A46}$$

$$\mathcal{O}^a(x)\, \mathcal{O}^b(y) \sim \frac{\hat{k}_G \delta^{ab}}{(x - y)^2} + \frac{i f^{abc} \mathcal{O}^c(y)}{x - y}. \tag{A47}$$

We begin our discussion of the Ward identities with Eq. (A46), which is useful for introducing our regularisation scheme, ensuring that the celestial correlation functions are well-defined and finite. Using Eqs. (A44, A45), we arrive at the following expression for the correlation function of the operators $\mathcal{O}^a(x)$ and $\mathsf{W}^{R,h}(y, \bar{y})$:

$$\begin{aligned} &\left\langle \mathcal{O}^a(x)\, \mathsf{W}^{R,h}(y, \bar{y}) \ldots \right\rangle_{\text{CCFT}} \\ &= -\frac{1}{\pi} \int_\Sigma d^2 z_1 \int_\Sigma d^2 z \frac{t^a(R)}{z - z_1} \left\langle \partial_{\bar{z}} \lambda(x, \bar{x}; z, \bar{z})\, \mathsf{W}^{R,h}(y, \bar{y}; z_1, \bar{z}_1) \ldots \right\rangle. \end{aligned}$$

Using integration by parts with respect to the anti-holomorphic variable $\bar{z}$, and substituting the identity given by Eq. (A6), we can rewrite the celestial correlation function as:

$$\left\langle \mathcal{O}^a(x)\,\mathsf{W}^{R,h}(y,\bar{y})\,...\right\rangle_{\text{CCFT}} \tag{A48}$$
$$= -\frac{1}{\pi}\int_\Sigma d^2z_1 \int_\Sigma d^2z\, \frac{\partial}{\partial\bar{z}}\left(\frac{t^a(R)}{z-w}\left\langle \lambda(x,\bar{x};z,\bar{z})\,\mathsf{W}^{R,h}(y,\bar{y};z_1,\bar{z}_1)\,...\right\rangle\right) \tag{A49}$$
$$+\lim_{z\to z_1}\int_\Sigma d^2z_1 \left\langle \lambda(x,\bar{x};z,\bar{z})\,\mathsf{W}^{R,h}(y,\bar{y};z_1,\bar{z}_1)\right\rangle. \tag{A50}$$

As $z \to z_1$, the last term in this expression diverges, as λ and $\mathsf{W}^{R,h}$ are evaluated at the same insertion on the worldsheet. To address this, we introduce the following regularisation scheme:

$$\left\langle \mathcal{O}^a(x)\,\mathsf{W}^{R,h}(y,\bar{y})\,...\right\rangle_{\text{CCFT}} \tag{A51}$$
$$\xrightarrow{\text{reg.}} -\frac{1}{\pi}\int_\Sigma d^2z_1 \int_\Sigma d^2z\, \frac{\partial}{\partial\bar{z}}\left(\frac{t^a(R)}{z-w}\left\langle \lambda(x,\bar{x};z,\bar{z})\,\mathsf{W}^{R,h}(y,\bar{y};z_1,\bar{z}_1)\,...\right\rangle\right). \tag{A52}$$

Now, let $\varepsilon > 0$ be a small real number, and split the integration over Σ as $\int_\Sigma d^2z = \int_{\Sigma\setminus D_\varepsilon(z_1)} d^2z + \int_{D_\varepsilon(z_1)} d^2z$. As $\varepsilon \to 0^+$, applying the complex divergence theorem (see Eq. (A7)) yields:

$$\left\langle \mathcal{O}^a(x)\,\mathsf{W}^{R,h}(y,\bar{y})\,...\right\rangle_{\text{CCFT}} \tag{A53}$$
$$= \frac{1}{\pi}\int_\Sigma d^2z_1 \oint_{C_\varepsilon(z_1)} \frac{dz}{2\pi i}\left\langle j^a(z)\,\lambda(x,\bar{x};z,\bar{z})\,\mathsf{W}^{R,h}(y,\bar{y};z_1,\bar{z}_1)\,...\right\rangle. \tag{A54}$$

Here, $C_\varepsilon(z_1)$ is a contour of radius ε centred at $z_1 \in \Sigma$. The sign change arises because, upon removing $D_\varepsilon(z_1)$ from Σ, the contour integral traverses $C_\varepsilon(z_1)$ in the opposite orientation.

By repeating this procedure iteratively for all worldsheet operator insertions at $z_1, ..., z_n$, we obtain the following Cauchy integral representation of the celestial current $\mathcal{O}^a(x)$ around all vertices attached to the worldsheet:

$$\mathcal{O}^a(x) = \sum_{i=1}^n \oint_{C_\varepsilon(z_i)} \frac{dz}{2\pi i} j^a(z)\,\lambda(x,\bar{x};z,\bar{z}) \tag{A55}$$

Remark. The celestial correlation function $\langle \mathcal{O}^a(x)\mathsf{W}^{R,h}(y,\bar{y})...\rangle_{\text{CCFT}}$ also admits another integral representation that offers insight into the geometric nature of the regularisation scheme. Let $\hat{\Sigma}$ be the Riemann surface obtained from Σ by removing small disks around all vertex operator insertions. Since $j^a(z)$ is a holomorphic current on $\hat{\Sigma}$, we have $\partial_{\bar{z}} j^a(z)\big|_{\hat{\Sigma}} \equiv 0$. Therefore:

$$\left\langle \mathcal{O}^a(x)\,\mathsf{W}^{R,h}(y,\bar{y})\,...\right\rangle_{\text{CCFT}}$$
$$= -\frac{1}{\pi}\int_\Sigma d^2z_1 \int_{\hat{\Sigma}} d^2z \left\langle j^a(z)\,\partial_{\bar{z}}\lambda(x,\bar{x};z,\bar{z})\,\mathsf{W}^{R,h}(y,\bar{y};z_1,\bar{z}_1)\,...\right\rangle.$$

Thus, the regularisation scheme can be understood as the removal of small discs centred around each operator insertion point on the worldsheet, which allow us to express the celestial current $\mathcal{O}^a(x)$ as:

$$\mathcal{O}^a(x) = -\frac{1}{\pi}\int_{\hat{\Sigma}} d^2z\, \frac{\partial}{\partial\bar{z}}\left[j^a(z)\,\lambda(z,\bar{z};x,\bar{x})\right]. \tag{A56}$$

We are now prepared to demonstrate that the celestial Kac-Moody current $\mathcal{O}^a(x)$ satisfies the Ward identity, Eq. (A47). This can be established by analysing the behaviour of $\mathcal{O}^a(x)$ within a correlation function involving another operator $\mathcal{O}^b(y)$. A more straightforward approach involves taking the partial derivative of $\mathcal{O}^a(x)$ with respect to the anti-holomorphic variable $\bar{x}$, yielding:

$$\partial_{\bar{x}}\mathcal{O}^a(x) = \oint \frac{dz}{2i} j^a(z)\,\Phi(x,\bar{x};z,\bar{z}), \tag{A57}$$

which follows from the identity $\partial_{\bar{x}}\lambda = \pi\Phi$. Consequently, we find:

$$\left\langle \partial_{\bar{x}}\mathcal{O}^a(x)\,\mathcal{O}^b(y)\,...\right\rangle_{\text{CCFT}} \tag{A58}$$

$$= -\frac{1}{k}\int_{\Sigma} d^2w \oint_{C_\varepsilon(z_1)} \frac{dz}{2i}\left(\frac{\hat{k}_G\delta^{ab}}{(z-w)^2} + \frac{if^{abc}j^c(z)}{z-w}\right)\left\langle \Phi(x,\bar{x};z,\bar{z})\,\bar{J}(\bar{y};\bar{w})\,\Phi(y,\bar{y};w,\bar{w})\,...\right\rangle. \tag{A59}$$

Two distinct contributions arise from the single pole in the above correlation function. The first originates from the OPE $\Phi\Phi\bar{J}$, and the second from $\bar{J}\Phi\Phi$. To compute the first, we use the asymptotic expansion:

$$\lim_{z\to w}\Phi(x,\bar{x};z,\bar{z})\,\Phi(y,\bar{y};w,\bar{w}) \sim \delta^{(2)}(x-y)\,\Phi(y,\bar{y};w,\bar{w}), \tag{A60}$$

from which we derive:

$$\lim_{z\to w}\Phi(x,\bar{x};z,\bar{z})\,\Phi(y,\bar{y};w,\bar{w})\,\bar{J}(\bar{y};\bar{w}) \tag{A61}$$

$$\sim \delta^{(2)}(x-y)\,\Phi(y,\bar{y};w,\bar{w})\,\bar{J}(\bar{y};\bar{w}) + (z-w)\lim_{z'\to w}\partial_{z'}\Phi\left(x,\bar{x};z',\bar{z}'\right)\Phi(y,\bar{y};w,\bar{w})\,\bar{J}(\bar{y};\bar{w}). \tag{A62}$$

To compute the second contribution, recall (cf. Giveon, Kutasov, and Seiberg [13, Eq. (2.25)]):

$$\bar{J}(\bar{y};\bar{w})\,\Phi(x,\bar{x};z,\bar{z}) = \frac{1}{\bar{w}-\bar{z}}\left[(\bar{x}-\bar{y})^2\partial_{\bar{x}} + 2(\bar{x}-\bar{y})\right]\Phi(x,\bar{x};z,\bar{z}), \tag{A63}$$

so that Eq. (A60) implies:

$$\lim_{z\to w}\bar{J}(\bar{y};\bar{w})\,\Phi(x,\bar{x};z,\bar{z})\,\Phi(y,\bar{y};w,\bar{w}) \tag{A64}$$

$$\sim \frac{1}{\bar{w}-\bar{z}}\left[(\bar{x}-\bar{y})^2\partial_{\bar{x}} + 2(\bar{x}-\bar{y})\right]\delta^{(2)}(x-y)\,\Phi(y,\bar{y};w,\bar{w}) = 0. \tag{A65}$$

Thus, the contribution to $\left\langle\partial_{\bar{x}}\mathcal{O}^a(x)\,\mathcal{O}^b(y)\,...\right\rangle_{\text{CCFT}}$ from the single pole in Eq. (A59) is:

$$-\frac{if^{abc}}{k}\int_\Sigma d^2w\oint_{C_\varepsilon(w)}\frac{dz}{2i}\left\langle\frac{j^c(w)}{z-w}\delta^{(2)}(x-y)\,\Phi(y,\bar{y};w,\bar{w})\,\bar{J}(\bar{y};\bar{w})\,...\right\rangle \tag{A66}$$

$$=\frac{\partial}{\partial\bar{x}}\left(\frac{if^{abc}}{x-y}\left\langle\mathcal{O}^c(y)\,...\right\rangle\right) \tag{A67}$$

To compute the contribution from the double pole, we first integrate by parts with respect to the holomorphic variable z:

$$\begin{aligned}
&-\frac{1}{k}\int_\Sigma d^2w\oint_{C_\varepsilon(w)}\frac{dz}{2i}\frac{\hat{k}_G\delta^{ab}}{(z-w)^2}\left\langle\bar{J}(\bar{y};\bar{w})\,\Phi(x,\bar{x};z,\bar{z})\,\Phi(y,\bar{y};w,\bar{w})\,...\right\rangle\\
&=-\frac{\hat{k}_G\delta^{ab}}{k}\int_\Sigma d^2w\oint_{C_\varepsilon(w)}\frac{dz}{2i}\frac{1}{z-w}\left\langle\bar{J}(\bar{y};\bar{w})\,\partial_z\Phi(x,\bar{x};z,\bar{z})\,\Phi(y,\bar{y};w,\bar{w})\,...\right\rangle\\
&=-\frac{\pi\hat{k}_G\delta^{ab}}{k}\int_\Sigma d^2w\lim_{z\to w}\left\langle\bar{J}(\bar{y};\bar{w})\,\partial_z\Phi(x,\bar{x};z,\bar{z})\,\Phi(y,\bar{y};w,\bar{w})\,...\right\rangle.
\end{aligned}$$

We recall that the limit $z\to w$ arises because $\varepsilon\to 0$, where ε is the radius of the small circle $C_\varepsilon(w)$ enclosing the insertion point w on the worldsheet.

Using the identities from (see Giveon, Kutasov, and Seiberg [13]), namely $\partial_z\bar{\lambda}=(\pi/k)\,J\Phi$ and $\partial_x\bar{\lambda}=\pi\Phi$, we find the following result:

$$\lim_{z\to w}\left\langle\bar{J}(\bar{y};\bar{w})\,\partial_z\Phi(x,\bar{x};z,\bar{z})\,\Phi(y,\bar{y};w,\bar{w})\,...\right\rangle \tag{A68}$$

$$=\frac{1}{\pi}\lim_{z\to w}\left\langle\bar{J}(\bar{y};\bar{w})\,\partial_z\partial_x\bar{\lambda}(x,\bar{x};z,\bar{z})\,\Phi(y,\bar{y};w,\bar{w})\,...\right\rangle \tag{A69}$$

$$=\frac{1}{k}\lim_{z\to w}\left\langle\bar{J}(\bar{y};\bar{w})\,\partial_x\left[J(x;z)\,\Phi(x,\bar{x};z,\bar{z})\right]\Phi(y,\bar{y};w,\bar{w})\,...\right\rangle \tag{A70}$$

$$\sim\frac{1}{k}\partial_x\delta^{(2)}(x-y)\left\langle J(y;w)\,\bar{J}(\bar{y};\bar{w})\,\Phi(y,\bar{y};w,\bar{w})\,...\right\rangle. \tag{A71}$$

Defining a new operator $\mathcal{I}$ as:

$$\mathcal{I}\coloneqq\frac{1}{k^2}\int d^2z\,J(x;z)\,\bar{J}(\bar{x};\bar{z})\,\Phi(x,\bar{x};z,\bar{z}), \tag{A72}$$

the contribution from the double pole to $\left\langle\partial_{\bar{x}}\mathcal{O}^a(x)\,\mathcal{O}^b(y)\,...\right\rangle_{\text{CCFT}}$ can be expressed as:

$$\frac{\partial}{\partial\bar{x}}\left(\frac{\hat{k}_G\delta^{ab}}{(x-y)^2}\left\langle\mathcal{I}...\right\rangle_{\text{CCFT}}\right). \tag{A73}$$

Finally, by combining Eqs. (A67, A73), we arrive at:

$$\left\langle\partial_{\bar{x}}\mathcal{O}^a(x)\,\mathcal{O}^b(y)\,...\right\rangle_{\text{CCFT}}=\frac{\partial}{\partial\bar{x}}\left\langle\left(\frac{if^{abc}}{x-y}\mathcal{O}^c(y)+\frac{\hat{k}_G\mathcal{I}\delta^{ab}}{(x-y)^2}\right)...\right\rangle_{\text{CCFT}}, \tag{A74}$$

which, upon integration with respect to $\bar{x}$, gives the celestial Ward identity:

$$\mathcal{O}^a(x)\,\mathcal{O}^b(y)\sim\frac{\hat{k}_G\mathcal{I}\delta^{ab}}{(x-y)^2}+\frac{if^{abc}\mathcal{O}^c(y)}{x-y}. \tag{A75}$$

We are led to the conclusion that the operator $\mathcal{I}$ represents the central extension of the celestial Kac-Moody current algebra generated by $\mathcal{O}^a(x)$. To establish that the definition of $\mathcal{I}$, as given in Eq. (A72), is consistent, it is necessary to verify that $\mathcal{I}$ is independent of the holomorphic x and anti-holomorphic $\bar{x}$ celestial coordinates.

In fact, employing the identities:

$$\partial_x \bar{\lambda} = \pi \Phi, \quad J\Phi = \frac{k}{\pi} \partial_z \bar{\lambda}, \quad \bar{J}\Phi = \frac{k}{\pi} \partial_{\bar{z}} \lambda, \tag{A76}$$

established in Giveon, Kutasov, and Seiberg [13, Sec. 2], we deduce the following result:

$$\partial_x \mathcal{I} = \frac{1}{k^2} \int_\Sigma d^2 z \, \bar{J}(\bar{x}; \bar{z}) \, \partial_x \left[J(x; z) \, \Phi(x, \bar{x}; z, \bar{z}) \right] = \frac{1}{\pi k} \int_\Sigma d^2 z \, \bar{J}(\bar{x}; \bar{z}) \, \partial_x \partial_z \bar{\lambda}(x, \bar{x}; z, \bar{z}) \tag{A77}$$

$$= \frac{1}{k} \int_\Sigma d^2 z \, \partial_z \left[\bar{J}(\bar{x}; \bar{z}) \, \Phi(x, \bar{x}; z, \bar{z}) \right] = -\frac{1}{2ik} \oint d\bar{z} \, \bar{J}(\bar{x}; \bar{z}) \, \Phi(x, \bar{x}; z, \bar{z}) \tag{A78}$$

$$= -\oint \frac{d\bar{z}}{2\pi i} \partial_{\bar{z}} \lambda(x, \bar{x}; z, \bar{z}) = 0, \tag{A79}$$

as claimed. This verifies that $\mathcal{I}$ is indeed independent of the coordinates $x, \bar{x}$, confirming its consistency as a central extension.

Appendix B: Worldsheet Integrals

1. Worldsheet Integrals Related to the 2-Point Function

In this Appendix, we undertake the computation of the integral over the worldsheet in Eq. (55), which is important for analysing the celestial 2-point function discussed in Subsection III A. We shall proceed with detail, as this calculation exemplifies the basic reasoning employed to derive the worldsheet integral relations that emerge in the computations of the 3-point function in Subsection III B and the structure constants of the celestial OPE in Subsection III C. Additionally, we shall derive the distributional limit of the resulting integral, thereby establishing the known $2d$ CFT identity (cf, Simmons-Duffin [35], Ketov [42], Vladimirov [56]):

$$\int d^2 y \frac{1}{|x_1 - y|^{2\tau} |y - x_2|^{2(2-\tau)}} = \frac{4\pi^2}{\nu^2} \delta^{(2)}(x_1 - x_2), \tag{B1}$$

where $\tau_1 =: 1 + i\nu$, which shall prove helpful in the derivations within Subsection III A, while also serving as a consistency check of our results.

We define the principal integral of interest in this Appendix as follows:

$$\mathcal{I}_0(\tau_1, \tau_2 | x_1, x_2) := \int d^2 y \frac{1}{|x_1 - y|^{2\tau_1} |y - x_2|^{2\tau_2}}. \tag{B2}$$

We begin by simplifying $\mathcal{I}_0$ through the translation $y \mapsto \xi := y - x_2$, yielding:

$$\mathcal{I}_0 = \int d^2\xi \frac{1}{|(x_2 - x_1) - \xi|^{2\tau_1} |\xi|^{2\tau_2}}. \tag{B3}$$

We proceed by employing the method of Feynman α-parameters. For a modern exposition, we refer the reader to Smirnov [57] and Weinzierl [58]. Parametrising the integrand in Eq. (B3) gives:

$$\frac{1}{|(x_2 - x_1) - \xi|^{2\tau_1}} = \frac{1}{\Gamma(\tau_1)} \int_0^\infty d\alpha_1 \alpha_1^{\tau_1 - 1} e^{-\alpha_1 |(x_2 - x_1) - \xi|^2}, \tag{B4}$$

and:

$$\frac{1}{|\xi|^{2\tau_2}} = \frac{1}{\Gamma(\tau_2)} \int_0^\infty d\alpha_2 \alpha_2^{\tau_2 - 1} e^{-\alpha_2 |\xi|^2}. \tag{B5}$$

Substituting these into Eq. (B3) and denoting $x_2 - x_1 =: u_0 + iv_0$ and $\xi =: u + iv$, we obtain:

$$\mathcal{I}_0 = \frac{1}{\Gamma(\tau_1)\Gamma(\tau_2)} \int_0^\infty d\alpha_1 \alpha_1^{\tau_1 - 1} \int_0^\infty d\alpha_2 \alpha_2^{\tau_2 - 1} e^{-\alpha_1 \left(u_0^2 + v_0^2\right)} \mathcal{G}(\alpha_1, \alpha_2), \tag{B6}$$

where the Gaussian integral $\mathcal{G}(\alpha_1, \alpha_2)$ is defined as:

$$\mathcal{G}(\alpha_1, \alpha_2) := \int du dv e^{-(\alpha_1 + \alpha_2)u^2 + 2\alpha_1 u_0 u} e^{-(\alpha_1 + \alpha_2)v^2 + 2\alpha_1 v_0 v}. \tag{B7}$$

Evaluating the Gaussian integral yields:

$$\mathcal{G} = \frac{\pi}{\alpha_1 + \alpha_2} e^{\left(u_0^2 + v_0^2\right) \frac{\alpha_1^2}{\alpha_1 + \alpha_2}}. \tag{B8}$$

Thus, Eq. (B6) becomes:

$$\mathcal{I}_0 = \frac{\pi^2}{\Gamma(\tau_1)\Gamma(\tau_2)} \int_0^\infty d\alpha_1 \alpha_1^{\tau_1 - 1} \int_0^\infty d\alpha_2 \alpha_2^{\tau_2 - 1} \frac{1}{\alpha_1 + \alpha_2} e^{-\left(u_0^2 + v_0^2\right)\alpha_1} e^{\left(u_0^2 + v_0^2\right) \frac{\alpha_1^2}{\alpha_1 + \alpha_2}}. \tag{B9}$$

To evaluate the integrals over α_1 and α_2, we introduce the variables $t := \alpha_1 / (\alpha_1 + \alpha_2)$ and $s := \left(u_0^2 + v_0^2\right) \alpha_1$. Recalling that $u_0^2 + v_0^2 = |x_1 - x_2|^2$, we reorganise Eq. (B9) as:

$$\mathcal{I}_0 = \frac{\pi}{\Gamma(\tau_1)\Gamma(\tau_2)} \frac{1}{|x_{12}|^{2(\tau_1 + \tau_2 - 1)}} \int_0^\infty ds s^{(\tau_1 + \tau_2 - 1) - 1} \int_0^1 dt t^{-\tau_2} (1 - t)^{\tau_2 - 1} e^{-(1-t)s}. \tag{B10}$$

Finally, applying the rescaling $s \mapsto r := (1 - t) s$, we obtain:

$$\mathcal{I}_0 = \frac{\pi}{|x_{12}|^{2(\tau_1 + \tau_2 - 1)}} \frac{1}{\Gamma(\tau_1)\Gamma(\tau_2)} \int_0^\infty dr r^{(\tau_1 + \tau_2 - 1) - 1} e^{-r} \int_0^1 dt t^{-\tau_2} (1 - t)^{-\tau_1}, \tag{B11}$$

leading to the final expression:

$$\mathcal{I}_0 = \frac{\pi}{|x_{12}|^{2(\tau_1 + \tau_2 - 1)}} \frac{\Gamma(1 - \tau_1)}{\Gamma(\tau_1)} \frac{\Gamma(1 - \tau_2)}{\Gamma(\tau_2)} \frac{\Gamma(\tau_1 + \tau_2 - 1)}{\Gamma(2 - \tau_1 - \tau_2)}. \tag{B12}$$

This expression must be interpreted in the sense of distributions; that is, it should be understood as making sense when applied to a test function. To clarify this concept and derive a useful $2d$ CFT identity, which will be employed in Section III A, we now utilise Eq. (B12) to compute the limit $\tau_1 + \tau_2 \to 2$ in the distributional sense. We begin by observing the following relation:

$$\frac{1}{\Gamma\left(2-\tau_1-\tau_2\right)} = \frac{2-\tau_1-\tau_2}{\Gamma\left(3-\tau_1-\tau_2\right)}, \tag{B13}$$

which allows Eq. (B12) to be rearranged in the form:

$$\frac{\Gamma\left(\tau_1\right)}{\Gamma\left(1-\tau_1\right)}\frac{\Gamma\left(\tau_2\right)}{\Gamma\left(1-\tau_2\right)}\int d^2y \frac{1}{\left|x_1-y\right|^{2\tau_1}\left|y-x_2\right|^{2\tau_2}} \tag{B14}$$

$$= 2\pi\left(\frac{1}{2}\left(2-\tau_1-\tau_2\right)\left|x_{12}\right|^{2\left(1-\tau_1-\tau_2\right)}\right)\frac{\Gamma\left(\tau_1+\tau_2-1\right)}{\Gamma\left(3-\tau_1-\tau_2\right)}. \tag{B15}$$

Introducing $\varepsilon := 2-\tau_1-\tau_2$ and $\tau_1 =: 1+i\nu$, we observe that:

$$\frac{\Gamma\left(\tau_2\right)}{\Gamma\left(1-\tau_2\right)} = -\left(\tau_1-1\right)\left(\tau_1+\varepsilon-1\right)\frac{\Gamma\left(1-\tau_1\right)}{\Gamma\left(\tau_1+\varepsilon\right)}, \tag{B16}$$

yielding:

$$\frac{\Gamma\left(\tau_1\right)}{\Gamma\left(\tau_1+\varepsilon\right)}\int d^2y \frac{1}{\left|x_1-y\right|^{2\tau_1}\left|y-x_2\right|^{2\tau_2}} = \frac{2\pi}{\nu\left(\nu-i\varepsilon\right)}\left(\frac{1}{2}\varepsilon\left|x_{12}\right|^{2(\varepsilon-1)}\right)\frac{\Gamma\left(1-\varepsilon\right)}{\Gamma\left(1+\varepsilon\right)}. \tag{B17}$$

Recalling that (cf. Vladimirov [56]):

$$\lim_{\varepsilon\to 0^+}\frac{1}{2}\varepsilon\left|x_{12}\right|^{2(\varepsilon-1)} = \delta\left(\left|x_{12}\right|^2\right) = \delta\left(x_{12}\right)\delta\left(\bar{x}_{12}\right) = 2\pi\delta^{(2)}\left(x_1-x_2\right), \tag{B18}$$

we find that taking the limit $\varepsilon\to 0^+$ of Eq. (B17) finally yields:

$$\int d^2y \frac{1}{\left|x_1-y\right|^{2\tau}\left|y-x_2\right|^{2(2-\tau)}} = \frac{4\pi^2}{\nu^2}\delta^{(2)}\left(x_1-x_2\right). \tag{B19}$$

2. Generalised Dirac Delta Function on the Worldsheet

In this Appendix, we recall how the generalised Dirac delta "function," analytically continued in the complex plane, naturally arises in the context of worldsheet integration. This result proves useful in deriving the celestial 2-point function, as described by our holographic dictionary in Subsection III A, particularly Eq. (47). Moreover, it serves to illustrate some fundamental methods of worldsheet integration.

We begin by considering the integral defined as:

$$\mathcal{F}\left(\beta\middle|x_1,x_2\right) := \frac{1}{N}\int d^2z_1 d^2z_2 \frac{1}{\left|z_1-z_2\right|^{2\beta}}, \tag{B20}$$

where N denotes the worldsheet "volume," a regulator necessary for the integral measure. The method of Feynman α-parameters instruct us to introduce the following representation:

$$\frac{1}{|z_1 - z_2|^{2\beta}} = \frac{1}{\Gamma(\beta)} \int_0^\infty d\alpha \alpha^{\beta-1} e^{-\alpha|z_1-z_2|^2}, \tag{B21}$$

so that Eq. (B20) can be rewritten as:

$$\mathcal{F} = \frac{1}{\Gamma(\beta)} \int_0^\infty d\alpha \alpha^{\beta-1} \frac{1}{N} \int d^2z_1 d^2z_2 e^{-\alpha|z_1-z_2|^2}. \tag{B22}$$

To perform the complex integrals, we introduce the parameterisations $z_1 \coloneqq x_1 + iy_1$ and $z_2 \coloneqq x_2 + iy_2$, leading to:

$$\mathcal{F} = \frac{1}{\Gamma(\beta)} \int_0^\infty d\alpha \alpha^{\beta-1} \frac{1}{N} \int dx_1 dx_2 e^{-\alpha(x_1-x_2)^2} \int dy_1 dy_2 e^{-\alpha(y_1-y_2)^2}, \tag{B23}$$

which can be reorganised as:

$$\mathcal{F} = \frac{1}{\Gamma(\beta)} \int_0^\infty d\alpha \alpha^{\beta-1} \frac{1}{N} \left(\mathcal{G}(\alpha)\right)^2, \tag{B24}$$

where the Gaussian integral is:

$$\mathcal{G}(\alpha) \coloneqq \int dx_1 dx_2 e^{-\alpha(x_1-x_2)^2} = \frac{\sqrt{\pi N}}{\alpha^{1/2}}. \tag{B25}$$

Thus, Eq. (B24) yields:

$$\mathcal{F} = \frac{\pi}{4\Gamma(\beta)} \int_0^\infty d\alpha \alpha^{(\beta-1)-1}. \tag{B26}$$

Following Donnay, Pasterski, and Puhm [33], this result can be expressed using the analytic continuation of the Dirac delta function (cf. Eq. (48)) as:

$$\boldsymbol{\delta}\left(i\left(\Delta - z\right)\right) \coloneqq \frac{1}{2\pi} \int_0^\infty d\tau\, \tau^{\Delta-z-1},$$

so that our final result reads:

$$\frac{1}{N} \int d^2z_1 d^2z_2 \frac{1}{|z_1 - z_2|^{2\beta}} = \frac{\pi^2}{2} \boldsymbol{\delta}\left(\beta - 1\right). \tag{B27}$$

3. Triple Worldsheet Integral

In this Appendix, we shall compute the worldsheet integral that arises in the calculation of the 3-point function of Euclidean AdS_3 bosonic string theory, as required for the continuation of the computations in Subsection B 3. Let $\sigma_1, \sigma_2, \sigma_3$ represent a sequence of exponents, and define the triple complex integral $\mathcal{I}(\sigma_1, \sigma_2, \sigma_2)$ as follows:

$$\mathcal{I}(\sigma_1, \sigma_2, \sigma_3) := \frac{1}{N} \int d^2 z_1 \int d^2 z_2 \int d^2 z_3 \frac{1}{|z_{12}|^{2\sigma_3} |z_{23}|^{2\sigma_1} |z_{31}|^{2\sigma_2}}, \tag{B28}$$

where N denotes the worldsheet area.

To evaluate this integral, we will employ the method of Feynman α-parameters, as reviewed by Smirnov [57], Weinzierl [58]. Beginning with the expressions for the distance between points on the complex plane, let $z_1 =: x_1 + iy_1$, $z_2 =: x_2 + iy_2$, and $z_3 =: x_3 + iy_3$. Consequently, we express the terms in the integrand as:

$$\frac{1}{|z_{12}|^{2\sigma_3}} = \frac{1}{\Gamma(\sigma_3)} \int_0^\infty d\alpha_3 \alpha_3^{\sigma_3 - 1} e^{-\alpha_3 (x_1 - x_2)^2 - \alpha_3 (y_1 - y_2)^2}, \tag{B29}$$

$$\frac{1}{|z_{23}|^{2\sigma_1}} = \frac{1}{\Gamma(\sigma_1)} \int_0^\infty d\alpha_1 \alpha_1^{\sigma_1 - 1} e^{-\alpha_1 (x_2 - x_3)^2 - \alpha_1 (y_2 - y_3)^2}, \tag{B30}$$

$$\frac{1}{|z_{31}|^{2\sigma_2}} = \frac{1}{\Gamma(\sigma_2)} \int_0^\infty d\alpha_2 \alpha_2^{\sigma_2 - 1} e^{-\alpha_2 (x_3 - x_1)^2 - \alpha_2 (y_3 - y_1)^2}. \tag{B31}$$

The integral in Eq. (B28) can now be expanded as:

$$\mathcal{I} = \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)\Gamma(\sigma_3)} \frac{1}{N} \int dx_1 dx_2 dx_3 \int dy_1 dy_2 dy_3 \prod_{i=1}^{3} \int_0^\infty d\alpha_i \alpha_i^{\sigma_i - 1} \tag{B32}$$

$$e^{-\alpha_1 (x_2 - x_3)^2 - \alpha_2 (x_3 - x_1)^2 - \alpha_3 (x_1 - x_2)^2} e^{-\alpha_1 (y_2 - y_3)^2 - \alpha_2 (y_3 - y_1)^2 - \alpha_3 (y_1 - y_2)^2}. \tag{B33}$$

Our next goal is to reorganise this integral into the α-parameters and the worldsheet coordinates. We rewrite the integral as:

$$\mathcal{I} = \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)\Gamma(\sigma_3)} \frac{1}{N} \prod_{i=1}^{3} \int_0^\infty d\alpha_i \alpha_i^{\sigma_i - 1} \left(F(\alpha_1, \alpha_2, \alpha_3)\right)^2, \tag{B34}$$

where:

$$F(\alpha_1, \alpha_2, \alpha_3) := \int dx_1 dx_2 dx_3 e^{-\alpha_1 (x_2 - x_3)^2 - \alpha_2 (x_3 - x_1)^2 - \alpha_3 (x_1 - x_2)^2}. \tag{B35}$$

Performing the Gaussian integrals, we find that:

$$F = \frac{\pi \sqrt{V}}{(\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1)^{1/2}}. \tag{B36}$$

Thus, the integral $\mathcal{I}$ becomes:

$$\mathcal{I} = \frac{\pi^2}{\Gamma(\sigma_1)\Gamma(\sigma_2)\Gamma(\sigma_3)} \int_0^\infty d\alpha_3 \alpha_3^{\sigma_3 - 1} \int_0^\infty d\alpha_2 \alpha_2^{\sigma_2 - 1} \int_0^\infty d\alpha_1 \frac{\alpha_1^{\sigma_1 - 1}}{\alpha_1(\alpha_2 + \alpha_3) + \alpha_2 \alpha_3}. \tag{B37}$$

We employ the following identity from Gradshteyn and Ryzhik [59]:

$$\int_0^\infty d\alpha \, \frac{\alpha^{\mu-1}}{(1+\beta\alpha)^\nu} = \beta^{-\mu} B\left(\mu, \nu-\mu\right), \tag{B38}$$

which allows us to evaluate the integral over α_1, yielding:

$$\int_0^\infty d\alpha_1 \frac{\alpha_1^{\sigma_1-1}}{\alpha_1\left(\alpha_2+\alpha_3\right)+\alpha_2\alpha_3} = \frac{\alpha_2^{\sigma_1-1}\alpha_3^{\sigma_1-1}}{\left(\alpha_2+\alpha_3\right)^{\sigma_1}} B\left(\sigma_1, 1-\sigma_1\right). \tag{B39}$$

Substituting this result into Eq. (B37) gives:

$$\mathcal{I} = \pi^2 \frac{B\left(\sigma_1, 1-\sigma_1\right)}{\Gamma\left(\sigma_1\right)\Gamma\left(\sigma_2\right)\Gamma\left(\sigma_3\right)} \int_0^\infty d\alpha_3 \alpha_3^{(\sigma_1+\sigma_3-1)-1} \int_0^\infty d\alpha_2 \frac{\alpha_2^{(\sigma_1+\sigma_2-1)-1}}{\left(\alpha_2+\alpha_3\right)^{\sigma_1}}. \tag{B40}$$

Using the identity from Eq. (B38), we compute the integral over the Feynman parameter α_2, yielding:

$$\int_0^\infty d\alpha_2 \frac{\alpha_2^{(\sigma_1+\sigma_2-1)-1}}{\left(\alpha_2+\alpha_3\right)^{\sigma_1}} = \alpha_3^{\sigma_2-1} B\left(\sigma_1+\sigma_2-1, 1-\sigma_2\right). \tag{B41}$$

Replacing this into Eq. (B40), we find:

$$\mathcal{I} = \pi^2 \frac{B\left(\sigma_1, 1-\sigma_1\right)}{\Gamma\left(\sigma_1\right)\Gamma\left(\sigma_2\right)\Gamma\left(\sigma_3\right)} B\left(\sigma_1+\sigma_2-1, 1-\sigma_2\right) \int_0^\infty d\alpha_3 \alpha_3^{(\sigma_1+\sigma_2+\sigma_3-2)-1}. \tag{B42}$$

We now recall from Donnay, Pasterski, and Puhm [33] the definition of the *generalised* Dirac delta "function," analytically continued to the complex plane:

$$\boldsymbol{\delta}\left(i\left(\Delta-z\right)\right) \coloneqq \frac{1}{2\pi}\int_0^\infty d\tau \, \tau^{\Delta-z-1}, \tag{B43}$$

such that the following identity holds:

$$\varphi\left(\Delta\right) = -i\int_{\mathcal{C}} dz \boldsymbol{\delta}\left(i\left(\Delta-z\right)\right)\varphi\left(z\right), \tag{B44}$$

for the contour $\mathcal{C} \coloneqq c + i\mathbf{R}$. Consequently, the integral $\mathcal{I}$ can be written in the form:

$$\mathcal{I} = 2\pi^3 \frac{\Gamma\left(1-\sigma_1\right)\Gamma\left(1-\sigma_2\right)\Gamma\left(\sigma_1+\sigma_2-1\right)}{\Gamma\left(\sigma_1\right)\Gamma\left(\sigma_2\right)\Gamma\left(\sigma_3\right)} \boldsymbol{\delta}\left(2-\sigma_1-\sigma_2-\sigma_3\right). \tag{B45}$$

Finally, by choosing the contour $\mathcal{C} = c + i\mathbf{R}$, where $c = \mathrm{Re}\left(2-\sum_i \sigma_i\right)$, we can write the following identity:

$$\Gamma\left(\sigma_1+\sigma_2-1\right)\boldsymbol{\delta}\left(2-\sigma_1-\sigma_2-\sigma_3\right) = \Gamma\left(1-\sigma_3\right)\boldsymbol{\delta}\left(2-\sigma_1-\sigma_2-\sigma_3\right). \tag{B46}$$

Thus, we arrive at the final form of the integral:

$$\mathcal{I} = 2\pi^3 \frac{\Gamma\left(1-\sigma_1\right)\Gamma\left(1-\sigma_2\right)\Gamma\left(1-\sigma_3\right)}{\Gamma\left(\sigma_1\right)\Gamma\left(\sigma_2\right)\Gamma\left(\sigma_3\right)} \boldsymbol{\delta}\left(2-\sigma_1-\sigma_2-\sigma_3\right). \tag{B47}$$

As we see, the triple integral evaluates to a product of gamma functions, a common structure in worldsheet computations that exhibit conformal symmetry and factorisation properties of vertex operator correlation functions.

4. Shadow Transform of Worldsheet Primaries

In this Appendix, we compute an integral involving the conformal primary $\Phi^{-1-j_3}(x_3; z_2)$ of the H_3^+-model. This calculation will prove helpful in deriving the operator product expansion in Subsection III C. We define the integral of interest as:

$$\mathcal{I}_1\left(j_1, j_2, j_3 \middle| x_3\right) := \int d^2z_2 \widetilde{\Phi}^{-1-j_3}(x_3; z_2) = \int d^2z_1 \int d^2z_2 \frac{1}{|z_{12}|^{\sigma_3}} \Phi^{-1-j_3}(x_3; z_2). \tag{B48}$$

Before delving into the technical details of the computation, its interesting to first observe a geometric interpretation of this integral. The expression for $\mathcal{I}_1$ can be understood as representing the worldsheet average of the shadow transform of the conformal primary $\Phi^{-1-j_3}(x_3; z_2)$. Recall that the shadow transform is defined as:

$$\widetilde{\Phi}^{-1-j_3}(x_3; z_2) := \frac{1}{N} \int d^2z_1 \frac{1}{|z_{12}|^{\sigma_3}} \Phi^{-1-j_3}(x_3; z_2). \tag{B49}$$

Therefore, the integral $\mathcal{I}_1$ gives the averaged contribution of the shadow-transformed conformal primary across the worldsheet.

To proceed with the calculation, we employ the method of Feynman α-parameters. First, we parametrise the complex coordinates as $z_1 =: \xi_1 + i\xi_2$ and $z_2 =: \eta_1 + i\eta_2$. Reorganising Eq. (B48), we obtain:

$$\mathcal{I}_1 = \frac{1}{N} \int d^2z_1 d^2z_2 \Phi^{-1-j_3}(x_3; z_2) \frac{1}{\Gamma(\sigma_3/2)} \int_0^\infty d\alpha \alpha^{\sigma_3/2-1} e^{-\alpha|z_{12}|^2}, \tag{B50}$$

which simplifies to:

$$\mathcal{I}_1 = \frac{1}{\Gamma(\sigma_3/2)} \int d\xi_1 d\xi_2 \int d\eta_1 d\eta_2 \Phi^{-1-j_3}(x_3; \eta_1 + i\eta_2) \int_0^\infty d\alpha \alpha^{\sigma_3/2-1} e^{-t(\xi_1-\eta_1)^2 - t(\xi_2-\eta_2)^2} \tag{B51}$$

$$= \frac{1}{\Gamma(\sigma_3/2)} \int d\eta_1 d\eta_2 \Phi^{-1-j_3}(x_3; \eta_1 + i\eta_2) \int_0^\infty d\alpha \alpha^{\sigma_3/2-1} \left(\int d\xi_1 e^{-t(\xi_1-\eta_1)^2}\right)^2. \tag{B52}$$

Performing the Gaussian integration yields:

$$\mathcal{I}_1 = \frac{\pi}{\Gamma(\sigma_3/2)} \int d\eta_1 d\eta_2 \Phi^{-1-j_3}(x_3; \eta_1 + i\eta_2) \int_0^\infty d\alpha \alpha^{(\sigma_3/2-1)-1}. \tag{B53}$$

Using the definition of the generalised Dirac delta function, analytically continued in the complex plane (as given in Eq. (B43)), we finally obtain:

$$\mathcal{I}_1 = \pi^2 \boldsymbol{\delta}(\sigma_3 - 2) \int d^2z_2 \Phi^{-1-j_3}(x_3; z_2). \tag{B54}$$

Substituting $\sigma_3 = h_1 + h_2 - h_3$ gives the final expression for the integral:

$$\mathcal{I}_1 = \pi^2 \boldsymbol{\delta}(h_1 + h_2 + h_3 - 2) \int d^2z_2 \Phi^{-1-j_3}(x_3; z_2). \tag{B55}$$

5. Worldsheet Integral with Two Fixed Points

In this Appendix, we shall undertake the computation of a worldsheet integral which proves helpful for the evaluation of the operator product expansion as discussed in Subsection III C:

$$\mathcal{I}_2\left(\lambda_1,\lambda_2,\lambda_3\middle|x_1,x_2\right)=\frac{1}{\left|x_1-x_2\right|^{2\lambda_3}}\int d^2x_3\frac{1}{\left|x_2-x_3\right|^{2\lambda_1}\left|x_3-x_1\right|^{2\lambda_2}}. \tag{B56}$$

To facilitate the computation, we introduce a translation $x_3\mapsto\xi:=x_3-x_1$, so that $\mathcal{I}_2$ can be recast in the form:

$$\mathcal{I}_2=\frac{1}{\left|x_1-x_2\right|^{2\lambda_3}}\int d^2\xi\frac{1}{\left|\left(x_2-x_1\right)-\xi\right|^{2\lambda_1}\left|\xi\right|^{2\lambda_2}}. \tag{B57}$$

Moreover, we implement a rescaling $y\mapsto\xi=\left|x_2-x_1\right|y$, which transforms our integral into:

$$\mathcal{I}_2=\frac{1}{\left|x_1-x_2\right|^{2(\lambda_1+\lambda_2+\lambda_3-1)}}\int d^2y\frac{1}{\left|y-\hat{x}\right|^{2\lambda_1}\left|y\right|^{2\lambda_2}}, \tag{B58}$$

where $\hat{x}:=\left(x_2-x_1\right)/\left|x_2-x_1\right|$. To proceed with the evaluation of this expression, we focus on the integral:

$$\mathcal{F}\left(\lambda_1,\lambda_2\right):=\int d^2y\frac{1}{\left|y-\hat{x}\right|^{2\lambda_1}\left|y\right|^{2\lambda_2}}, \tag{B59}$$

so that the final result for $\mathcal{I}_2$ will be expressed as:

$$\mathcal{I}_2=\frac{\mathcal{F}\left(\lambda_1,\lambda_2\right)}{\left|x_1-x_2\right|^{2(\lambda_1+\lambda_2+\lambda_3-1)}}. \tag{B60}$$

At this stage, we introduce the Feynman α-parameters, leading to the following representations:

$$\frac{1}{\left|y-\hat{x}\right|^{2\lambda_1}}=\frac{1}{\Gamma\left(\lambda_1\right)}\int_0^\infty d\alpha_1\alpha_1^{\lambda_1-1}e^{-\alpha_1\left|y-\hat{x}\right|^2},\quad\frac{1}{\left|y\right|^{2\lambda_2}}=\frac{1}{\Gamma\left(\lambda_2\right)}\int_0^\infty d\alpha_2\alpha_2^{\lambda_2-1}e^{-\alpha_2\left|y\right|^2}. \tag{B61}$$

We then parametrise $y=:y_1+iy_2$ over the complex plane, with $\hat{x}=:a_1+ia_2$ $\left(a_1^2+a_2^2=1\right)$, yielding:

$$\mathcal{F}=\frac{1}{\Gamma\left(\lambda_1\right)\Gamma\left(\lambda_2\right)}\int_0^\infty d\alpha_1\alpha_1^{\lambda_1-1}\int_0^\infty d\alpha_2\alpha_2^{\lambda_2-1}\int dy_1dy_2e^{-\alpha_1\left(y_1-a_1\right)^2-\alpha_1\left(y_2-a_2\right)^2-\alpha_2y_1^2-\alpha_2y_2^2}. \tag{B62}$$

Performing the Gaussian integrals gives:

$$\mathcal{F}=\frac{\pi}{\Gamma\left(\lambda_1\right)\Gamma\left(\lambda_2\right)}\int_0^\infty d\alpha_1\alpha_1^{\lambda_1-1}e^{-\alpha_1}\int_0^\infty d\alpha_2\alpha_2^{\lambda_2-1}\frac{1}{\alpha_1+\alpha_2}e^{\frac{\alpha_1^2}{\alpha_1+\alpha_2}}. \tag{B63}$$

By introducing the variable $\zeta:=\alpha_1/\left(\alpha_1+\alpha_2\right)$, we obtain:

$$\mathcal{F}=\frac{\pi}{\Gamma\left(\lambda_1\right)\Gamma\left(\lambda_2\right)}\int_0^1 d\zeta\zeta^{-\lambda_2}\left(1-\zeta\right)^{\lambda_2-1}\int_0^\infty d\alpha_1\alpha_1^{(\lambda_1+\lambda_2-1)-1}e^{-\alpha_1(1-\zeta)}. \tag{B64}$$

Finally, solving the integral over α_1 by rescaling $\alpha_1' := (1-\zeta)\,\alpha_1$, we arrive at:

$$\mathcal{F} = \pi \frac{\Gamma(\lambda_1+\lambda_2-1)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^1 d\zeta \zeta^{-\lambda_2} (1-\zeta)^{-\lambda_1} = \pi \frac{\Gamma(\lambda_1+\lambda_2-1)}{\Gamma(2-\lambda_1-\lambda_2)} B(1-\lambda_1, 1-\lambda_2)\,. \tag{B65}$$

Thus, the final form of our integral is given by:

$$\mathcal{I}_2 = \frac{\pi}{|x_1-x_2|^{2(\lambda_1+\lambda_2+\lambda_3-1)}} \frac{\Gamma(\lambda_1+\lambda_2-1)}{\Gamma(2-\lambda_1-\lambda_2)} \frac{\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}. \tag{B66}$$

---

[1] N. Ogawa, S. Takahashi, T. Tsuda, and T. Waki, arXiv preprint arXiv:2404.12049 (2024).

[2] S. Stieberger, T. R. Taylor, and B. Zhu, Physics Letters B **846**, 138229 (2023).

[3] S. Stieberger, T. R. Taylor, and B. Zhu, Physics Letters B **836**, 137588 (2023).

[4] G. Giribet, arXiv preprint arXiv:2403.03374 (2024).

[5] W. Melton, A. Sharma, A. Strominger, and T. Wang, arXiv preprint arXiv:2403.18896 (2024).

[6] I. Mol, arXiv preprint arXiv:2408.10944 (2024).

[7] J. Teschner, arXiv preprint hep-th/9705214 (1997).

[8] J. Teschner, Nuclear Physics B **546**, 369 (1999).

[9] J. Teschner, Nuclear Physics B **546**, 390 (1999).

[10] J. Teschner, Nuclear Physics B **571**, 555 (2000).

[11] S. Ribault and J. Teschner, Journal of High Energy Physics **2005**, 014 (2005).

[12] J. de Boer, H. Ooguri, H. Robins, and J. Tannenhauser, Journal of High Energy Physics **1998**, 026 (1999).

[13] A. Giveon, D. Kutasov, and N. Seiberg, arXiv preprint hep-th/9806194 (1998).

[14] E. Casali, W. Melton, and A. Strominger, Journal of High Energy Physics **2022**, 1 (2022).

[15] W. Melton, A. Sharma, and A. Strominger, arXiv preprint arXiv:2312.07820 (2023).

[16] J. Maldacena and H. Ooguri, Journal of Mathematical Physics **42**, 2929 (2001).

[17] J. Maldacena, H. Ooguri, and J. Son, Journal of Mathematical Physics **42**, 2961 (2001).

[18] G. Giribet and C. Nunez, Journal of High Energy Physics **2001**, 010 (2001).

[19] I. Bars, C. Deliduman, and D. Minic, arXiv preprint hep-th/9907087 (1999).

[20] E. D'Hoker and D. H. Phong, Reviews of Modern Physics **60**, 917 (1988).

[21] J. Polchinski, *String Theory*, Cambridge Monographs on Mathematical Physics No. v. 2 (Cambridge University Press, 1998).

[22] S. J. Parke and T. R. Taylor, Physical Review Letters **56**, 2459 (1986).

[23] H. Elvang and Y.-t. Huang, arXiv preprint arXiv:1308.1697 (2013).

[24] S. Badger, J. Henn, J. C. Plefka, and S. Zoia, *Scattering Amplitudes in Quantum Field Theory* (Springer Nature, 2024).

[25] S. Pasterski, The European Physical Journal C **81**, 1062 (2021).

[26] A.-M. Raclariu, arXiv preprint arXiv:2107.02075 (2021).

[27] A. Strominger, *Lectures on the infrared structure of gravity and gauge theory* (Princeton University Press, 2018).

[28] A. Puhm, .

[29] F. A. Berezin, *Introduction to superanalysis*, Vol. 9 (Springer Science & Business Media, 2013).

[30] V. P. Nair, Physics Letters B **214**, 215 (1988).

[31] V. Nair, Physical Review D?Particles, Fields, Gravitation, and Cosmology **71**, 121701 (2005).

[32] S. Pasterski and S.-H. Shao, Physical Review D **96**, 065022 (2017).

[33] L. Donnay, S. Pasterski, and A. Puhm, Journal of High Energy Physics **2020**, 1 (2020).

[34] H. Furugori, N. Ogawa, S. Sugishita, and T. Waki, Journal of High Energy Physics **2024**, 1 (2024).

[35] D. Simmons-Duffin, Journal of High Energy Physics **2014**, 1 (2014).

[36] J. Polchinski, *String Theory*, String Theory 2 Volume Hardback Set (Cambridge University Press, 2001).

[37] N. Arkani-Hamed, M. Pate, A.-M. Raclariu, and A. Strominger, Journal of High Energy Physics **2021**, 1 (2021).

[38] J. Gu, Y.-Y. Li, and L.-T. Wang, Journal of High Energy Physics **2022**, 1 (2022).

[39] S. Stieberger and T. R. Taylor, Nuclear Physics B **935**, 388 (2018).

[40] P. Ginsparg, arXiv preprint hep-th/9108028 (1988).

[41] P. Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer, 1997).

[42] S. Ketov, *Conformal Field Theory*, G - Reference,Information and Interdisciplinary Subjects Series (World Scientific Pub., 1995).

[43] H. Nastase, F. Rojas, and C. Rubio, Journal of High Energy Physics **2022**, 1 (2022).

[44] L. P. de Gioia and A.-M. Raclariu, arXiv preprint arXiv:2303.10037 (2023).

[45] L. P. de Gioia and A.-M. Raclariu, Journal of High Energy Physics **2023**, 1 (2023).

[46] M. Banados, C. Teitelboim, and J. Zanelli, Physical review letters **69**, 1849 (1992).

[47] D. Cangemi, M. Leblanc, and R. B. Mann, Physical Review D **48**, 3606 (1993).

[48] A. Strominger, Journal of High Energy Physics **1998**, 009 (1998).

[49] E. Crawley, A. Guevara, N. Miller, and A. Strominger, Journal of High Energy Physics **2022**, 1 (2022).

[50] J. D. Brown and M. Henneaux, Communications in Mathematical Physics **104**, 207 (1986).

[51] J. Maldacena and A. Strominger, Journal of High Energy Physics **1998**, 005 (1999).

[52] K. Gawedzki and A. Kupiainen, Nuclear Physics B **320**, 625 (1989).

[53] K. Gawedzki, New symmetry principles in quantum field theory , 247 (1992).

[54] K. Gawedzki, Nuclear Physics B **328**, 733 (1989).

[55] J. Penedones, in *New Frontiers in Fields and Strings: TASI 2015 Proceedings of the 2015 Theoretical Advanced Study Institute in Elementary Particle Physics* (World Scientific, 2017) pp. 75–136.

[56] V. Vladimirov, *Equations of Mathematical Physics*, Monographs and textbooks in pure and applied

mathematics (M. Dekker, 1971).
[57] V. A. Smirnov, *Evaluating feynman integrals*, Vol. 211 (Springer Science & Business Media, 2004).
[58] S. Weinzierl, arXiv preprint arXiv:2201.03593 **5** (2022).
[59] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products* (Academic press, 2014).