

Applicability criteria of proper charge neutrality and special relativistic MHD models extended by two-fluid effects

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(Dated: 14 October 2024)

The applicability of relativistic magnetohydrodynamics (RMHD) and its generalization to two-fluid models (including the Hall and inertial effects) is systematically investigated by using the method of dominant balance in the two-fluid equations. Although proper charge neutrality or quasi-neutrality is the key assumption for all MHD models, this condition is difficult to be met when both relativistic and inertial effects are taken into account. The range of application for each MHD model is illustrated in the space of dimensionless scale parameters. Moreover, the number of field variables of relativistic Hall MHD (RHMHD) is shown to be greater than that of RMHD and Hall MHD. Nevertheless, the RHMHD equations may be solved at a lower computational cost than RMHD, since root-finding algorithm, which is the most time-consuming part of the RMHD code, is no longer required to compute the primitive variables.

I. INTRODUCTION

Magnetized plasmas subject to relativistic effects are common in various high-energy celestial bodies, such as pulsar, black hole magnetosphere, corona of accretion disk, jet from active galactic nucleus and gamma-ray burst. Relativistic magnetohydrodynamics (RMHD) has been used in theoretical and numerical studies as a model to analyze the macroscopic motion of relativistic magnetized plasmas. Since RMHD ignores the microscopic scales of such as inertial length and gyro-radius, it is well known that the magnetic field is frozen in the plasma motion in the collisionless limit. Therefore, RMHD is inappropriate for dealing with magnetic reconnection^{1,2} at least in the microscopic region where magnetic field lines reconnect. In many cases, magnetic reconnection is a key process in which magnetic energy is efficiently converted into kinetic and thermal energy. In addition, RMHD becomes invalid in the limit of low plasma density or weak magnetic field. The Vlasov-Maxwell equations, on the other hand, are based on first principles and have been solved by Particle-In-Cell (PIC) simulations in recent years. However, this direct approach is the most computationally expensive and these kinetic models are difficult to solve analytically. Thus, there is a demand for intermediate models which bridge the gap between RMHD and kinetic ones. In this study, we focus on extended RMHD models that include the two-fluid effects, which are expected to be more widely applicable than RMHD while maintaining a moderate computational cost.

In non-relativistic electron-ion plasmas, a model^{3,4} including the two-fluid effects (i.e., the Hall effect and the electron inertia effect) is called extended MHD (XMHD) in the recent literature⁵. The XMHD equations are derived from the two-fluid equations by imposing the quasi-neutrality (QN) condition, which approximately eliminates microscopic motions such as plasma oscillation and cyclotron oscillation. XMHD is also shown to have a Hamiltonian structure which conserves canonical vorticities (instead of magnetic flux)⁶⁻⁹. Due to the electron-inertia effect, magnetic reconnection can occur even in the collisionless limit¹⁰⁻¹². Furthermore, the Hall ef-

fect is well-known for significantly enhancing the reconnection speed, according to the Global Environment Modeling (GEM) Reconnection Challenge¹³. Since the electron-inertia effect manifests itself on an even smaller scale than the Hall effect, Hall MHD¹⁴ is often used as well, neglecting only the electron-inertia effect. In the case of electron-positron plasmas, the Hall effect vanishes and only the inertial effect remains, so this model is called inertial MHD (IMHD)⁵.

It is natural to assume that there are also some MHD models that include the relativistic effects alongside the two-fluid effects. Such an extension of RMHD in electron-positron plasma was explored early on the literature^{15,16}. Additionally, the extension of generalized Ohm's law was attempted and applied to pulsar magnetosphere¹⁷Note1. Koide¹⁹ derived a generalized RMHD model from the relativistic two-fluid equations by imposing the proper charge neutrality (PCN) condition, which will be referred to as relativistic extended MHD (RXMHD) in this paper. A variational principle of RXMHD was later proposed by Kawazura et al.²⁰. The general relativistic version of RXMHD was presented by Koide²¹ and by Comisso and Asenjo²² using a covariant form. RXMHD was applied to relativistic collisionless magnetic reconnection²³. Relativistic Hall MHD (RHMHD) is similarly obtained by neglecting electron-inertia, and its properties have been studied by Kawazura^{24,25}. However, for the QN condition to hold in non-relativistic MHD, the flow velocity must be sufficiently slower than the speed of light. Moreover, as will be clarified in this paper, the PCN condition in RMHD actually holds in the limit of neglecting the two-fluid effects. Therefore, hybrid models which include both the relativistic and two-fluid effects may violate the charge neutrality condition, requiring careful consideration of the applicability of RXMHD (and RHMHD). In fact, all models bearing the name "MHD" assume either QN or PCN a priori. Once these neutrality conditions (i.e., single-fluid approximation) fail, we should solve the two-fluid equations or kinetic models directly.

Note1 Generalized relativistic Ohm's law is also proposed by¹⁸ in a different way.

In this study, starting from the relativistic two-fluid equations, we systematically reproduce various MHD models (including RXMHD) using the method of dominant balance²⁶ and theoretically illustrate their scopes of application. Since there are too many dimensionless parameters in the original two-fluid equations, we will not explore all cases but focus only on the realm of MHD where the MHD balances hold; the MHD terms are not negligible but dominant. Specifically, we consider a situation in which the Lorentz force ($\mathbf{J} \times \mathbf{B}$ term) is dominant in the equation of motion for the center-of-mass velocity of the two fluids. If the pressure term or the electric force is far more dominant than the Lorentz force, the MHD model is unlikely to be applicable²⁷. Therefore, in order to make nonessential parameters invisible and highlight only the dominant terms, the plasma pressure and the external electric field will be ignored from the beginning. The MHD models are finally classified in terms of three dimensionless parameters corresponding to the scales of the plasma density, the flow velocity, and the external magnetic field. Furthermore, the dimensionless parameters can be reduced to two if the flow velocity is assumed to be on the same order of the Alfvén velocity. The applicability of the various MHD models will be visualized in this parameter space, supposing that a dimensionless coefficient before each term is considered negligible if it is less than, say, 10^{-4} . For these relativistic and two-fluidic MHD models, we will write them in the form of a dynamical system $\partial_t u = F(u)$ and identify the number of the time-evolving field variables u . We will show that RHMHD has more variables than HMHD and RMHD. In the case of RMHD, the right-hand side $F(u)$ is notorious for being an implicit function of u , which requires extra computational cost²⁸. RHMHD will be shown to resolve this problem of RMHD, although the number of variables increases.

II. BASIC EQUATIONS

A. Two-fluid equations

We denote the Minkowski spacetime of the reference frame by

$$x^\mu := (ct, x, y, z) = (x^0, x^i), \quad x_\mu := (-ct, x, y, z) = (x_0, x_i), \quad (1)$$

where c is the speed of light and the Minkowski metric tensor is $\text{diag}(-1, 1, 1, 1)$. The partial derivatives will be shortly denoted by $\partial_\mu = \partial/\partial x^\mu$ and $\partial^\mu = \partial/\partial x_\mu$. The proper four-velocity is defined as

$$U^\mu := (\gamma, \gamma \mathbf{v}/c), \quad U_\mu := (-\gamma, \gamma \mathbf{v}/c), \quad (2)$$

where \mathbf{v} is the reference-frame three-velocity (called simply "velocity"), and $\gamma := 1/\sqrt{1 - |\mathbf{v}|^2/c^2}$ is the Lorentz factor. In this paper, Greek indices ($\mu = 0, 1, 2, 3$) denote the time-space (4D) components, while Roman indices ($v^i = v_i$, $i = 1, 2, 3$) or bold faces (\mathbf{v}) denote the spatial (3D) components. The Einstein summation convention will be used in what follows.

Momentarily Co-moving Reference Frame (MCRF) refers to the inertial frame co-moving with particles. Physical quantities of relativistic fluid are said to be "proper" when they are observed in the frame co-moving with the velocity \mathbf{v} . Therefore, the proper number density is given by $N = n/\gamma$, when n is the number density in the reference-frame.

In this study, we start with the special-relativistic fluid equations for both positively and negatively charged gases, where dissipation due to collision is neglected for simplicity. The equations of motion, the continuity equations and Maxwell's equations are written as

$$\partial_\nu (h_\pm N_\pm U_\pm^\mu U_\pm^\nu) = -\partial^\mu p_\pm \pm ec N_\pm U_\pm^\nu F^\mu{}_\nu, \quad (3)$$

$$\partial_\nu (N_\pm U_\pm^\nu) = 0, \quad (4)$$

$$\partial_\nu {}^*F^{\mu\nu} = 0, \quad (5)$$

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu, \quad (6)$$

where the subscripts plus (+) and minus (−) indicate that they are the quantities for positively and negatively charged particles, respectively. Moreover, h_\pm is the entropy per unit particle, p_\pm is the pressure, $F^{\mu\nu}$ is the electromagnetic field tensor, ${}^*F^{\mu\nu}$ is the Hodge dual tensor of $F^{\mu\nu}$, and $J^\mu := ec(N_+ U_+^\mu - N_- U_-^\mu)$ is the four-current. The governing equations, (3) to (6), are called the two-fluid equations. We use the SI unit system; μ_0 is the vacuum magnetic permeability, and e is the elementary charge.

Maxwell's equations are also expressed in 3 + 1 form as

$$\partial_j E^j/c = \mu_0 ec \tilde{n}, \quad (7)$$

$$\partial_j B^j/c = 0, \quad (8)$$

$$\epsilon^i{}_{jk} \partial^j E^k/c = -\partial_0 B^i, \quad (9)$$

$$\epsilon^i{}_{jk} \partial^j B^k = \mu_0 J^i - \partial_0 E^i/c, \quad (10)$$

where E^i is the electric field, B^i is the magnetic field, $e\tilde{n} = e(n_+ - n_-)$ is the charge density, and ϵ_{ijk} is the Levi-Civita symbol.

In this paper, the four potential $A^\mu = (\phi/c, A^i)$ is also introduced to express the electromagnetic field and we employ the Coulomb gauge $\partial_j A^j = 0$. Maxwell's equations are then transformed into

$$-\partial_j \partial_j \phi = \mu_0 ec^2 \tilde{n}, \quad (11)$$

$$\frac{1}{c^2} \partial_t \partial_t A^i + \partial_i \partial_j A^j - \partial_j \partial_j A^i = -\frac{1}{c^2} \partial_i (\partial_t \phi) + \mu_0 J^i. \quad (12)$$

B. Transformation into MHD variables

Let us rewrite Eq. (3) and Eq. (4) in terms of MHD variables without any approximation. It is insightful to introduce four-dimensional center-of-mass flux (divided by c) by

$$Q^\mu = \left(n, n \frac{v^i}{c} \right) := \frac{m_+ N_+ U_+^\mu + m_- N_- U_-^\mu}{m_+ + m_-}, \quad (13)$$

where m_\pm is the mass of particle, and four-dimensional current (with the same dimension as Q^μ) by

$$K^\mu = \frac{J^\mu}{ec} = \left(\tilde{n}, n \frac{u^i}{c} \right) := N_+ U_+^\mu - N_- U_-^\mu. \quad (14)$$

The MHD equations are sometimes called the single-fluid model, assuming that the two species of charged fluid move together approximately; $\tilde{n} \ll n$ and $|\mathbf{u}| \ll |\mathbf{v}|$.

By denoting (3) for the positive and negative species by $(3)_+$ and $(3)_-$ respectively, the equation of center-of-mass motion is obtained from the sum $(3)_+ + (3)_-$ as follows

$$mc^2 \partial_\nu [f Q^\mu Q^\nu + \mu^2 \tilde{f} (Q^\mu K^\nu + K^\mu Q^\nu) + \mu^2 f' K^\mu K^\nu] = -\partial^\mu p + ecK^\nu F^\mu{}_\nu. \quad (15)$$

On the other hand, generalized Ohm's law is obtained from $[m_-(3)_+ - m_+(3)_-]/(m_+ + m_-)$,

$$mc^2 \partial_\nu [\mu^2 \tilde{f} Q^\mu Q^\nu + \mu^2 f' (Q^\mu K^\nu + K^\mu Q^\nu) + \mu^2 (-\tilde{\mu} f' + \mu^2 \tilde{f}) K^\mu K^\nu] = \frac{\mu}{2} \partial^\mu p + \frac{1}{2} \partial^\mu \tilde{p} + ec(Q^\nu - \tilde{\mu} K^\nu) F^\mu{}_\nu, \quad (16)$$

where the following abbreviations are used

$$m := m_+ + m_-, \quad \mu^2 := \frac{m_+ m_-}{m^2}, \quad \tilde{\mu} := \frac{m_+ - m_-}{m}, \quad (17)$$

$$p := p_+ + p_-, \quad \tilde{p} := p_+ - p_-, \quad (18)$$

$$f_\pm := \frac{h_\pm}{N_\pm m_\pm c^2}, \quad (19)$$

$$\tilde{f} := f_+ - f_-, \quad (20)$$

$$f := \frac{1}{m} (m_+ f_+ + m_- f_-), \quad (21)$$

$$f' := \frac{1}{m} (m_- f_+ + m_+ f_-) = f - \tilde{\mu} \tilde{f}. \quad (22)$$

Both the classical and relativistic MHD equations are derived by neglecting \tilde{f} owing to $\tilde{f} \ll f$ (which then leads to $f' = f$). Therefore, the orders of f and \tilde{f} are important for the validity of the MHD approximation.

Similarly, we obtain the conservation law of mass density

$$\partial_\nu Q^\nu = 0 \quad (23)$$

from $[m_+(4)_+ + m_-(4)_-]/m$, and the conservation law of charge density

$$\partial_\nu K^\nu = 0 \quad (24)$$

from $(4)_+ - (4)_-$.

C. Assumption of cold plasma

The validity of the MHD approximation primarily relies on $\tilde{n} \ll n$, $|\mathbf{u}| \ll |\mathbf{v}|$ and $\tilde{f} \ll f$ being sufficiently fulfilled. To focus on this topic, we neglect the pressure terms (i.e., the cold plasma approximation) in what follows because they simply appear as additional terms and make the governing equations lengthy. Therefore, $p = \tilde{p} = 0$ and $h_\pm = m_\pm c^2$ are assumed. Then, (20) and (21) are reduced to

$$f = \frac{1}{m} \left(\frac{m_+}{N_+} + \frac{m_-}{N_-} \right), \quad \tilde{f} = \frac{1}{N_+} - \frac{1}{N_-}. \quad (25)$$

Using the relations,

$$N_\pm = \sqrt{-N_\pm N_\pm U_\pm^\alpha U_{\pm\alpha}} = \sqrt{-\left(Q^\alpha Q_\alpha + \frac{m_\pm^2}{m^2} K^\alpha K_\alpha \right) \mp \frac{m_\pm}{m} (Q^\alpha K_\alpha + K^\alpha Q_\alpha)}, \quad (26)$$

we can express f and \tilde{f} in terms of Q^μ and K^μ (in a very complicated way). In Maxwell's equations, the electromagnetic field $F^{\mu\nu}$ is generated by J^μ , which is ecK^μ . Therefore, the two-fluid equations are fully expressed by the MHD variables, Q^μ , K^μ and $F^{\mu\nu}$.

Let us clarify the number of field variables in the two-fluid equations. From the definitions given above, the equations (15), (16), (23) and (24) clearly describe the time evolution of the 8 variables Q^μ and K^μ (which correspond to n , v^i , \tilde{n} and u^i). Maxwell's equations provide the time evolution of the 6 variables E^i and B^i (which is $F^{\mu\nu}$), but they must be solved under the two constraints (7) and (8) (which include no time derivative). In fact, we can eliminate the variable \tilde{n} because \tilde{n} is uniquely determined by E^i via (7), and the charge conservation law (24) is automatically satisfied by (7) and (10). Therefore, in the cold plasma approximation, the two-fluid equations constitute a dynamical system of 13 field variables under 1 constraint in total. In a sense, the *degree of freedom* is $13 - 1 = 12$. Even when A^i is used instead of B^i , the Coulomb gauge $\partial_i A^i = 0$ is imposed instead of $\partial_i B^i = 0$ and the *degree of freedom* is the same. Reducing the number of field variables is one of the major purposes of the following MHD approximation.

III. DOMINANT BALANCE

To derive reduced models from the two-fluid equations systematically, we first normalize all terms in the equations and consider the dominant balances that are suitable for magnetized plasma.

A. Normalization

We normalize all the equations by introducing 8 representative scales (with subscript \star) as follows

$$\hat{n} = \frac{n}{n_\star}, \quad \hat{v}^i = \frac{v^i}{v_\star}, \quad \hat{\tilde{n}} = \frac{\tilde{n}}{\tilde{n}_\star}, \quad \hat{u}^i = \frac{u^i}{u_\star}, \quad \hat{B}^i = \frac{B^i}{B_\star}, \quad \hat{\phi}^i = \frac{\phi}{\phi_\star}, \quad \hat{x}^i = \frac{x^i}{L_\star}, \quad \hat{t} = \frac{t}{T_\star}, \quad (27)$$

where we have introduced the common scale for all three-dimensional components of vector fields (i.e., $v^1 \sim v^2 \sim v^3$) for simplicity. Note that L_\star and T_\star are the representative spatial and temporal scales, respectively, of plasma dynamics that we are interested in.

The conservation law of mass (23) is written in terms of the normalized quantities (with the hat symbol) as

$$\frac{L_\star}{v_\star T_\star} \hat{\partial}_0 \hat{Q}^0 + \hat{\partial}_i \hat{Q}^i = 0. \quad (28)$$

Except when we consider the special cases (such as steady solution or incompressible limit), the two terms on the left hand side balance each other. First of all, we assume this balance as usual,

$$\text{Balance 1: } L_\star = v_\star T_\star. \quad (29)$$

Because this balance is merely a relation among scale parameters, it should be actually interpreted as $L_\star \sim v_\star T_\star$ or $O(L_\star) = O(v_\star T_\star)$. But, the equality “=” will be used in this paper to reduce the number of the scale parameters by imposing this balance.

Next, consider the Poisson equation (11) which is normalized to

$$-\frac{\hat{\phi}_\star}{L_\star^2} \hat{\partial}_j \hat{\partial}_j \hat{\phi} = \mu_0 e c^2 \tilde{n}_\star \hat{n}. \quad (30)$$

We assume that there is no externally-applied electrostatic potential (e.g., $\hat{\phi} \rightarrow 0$ at infinity). Then, $\hat{\phi}$ is generated only by the charge density of plasma itself via this equation, and it is natural to assume the balance between the left and right hand sides,

$$\text{Balance 2: } \hat{\phi}_\star = \mu_0 e c^2 \tilde{n}_\star L_\star^2. \quad (31)$$

Since the two balances (29) and (31) are assumed among the eight representative scales, let us define 5 dimensionless parameters for later use as follows

$$\varepsilon := \frac{1}{L_\star} \sqrt{\frac{m}{\mu_0 n_\star e^2}}, \quad (32)$$

$$\sigma := B_\star^2 / (\mu_0 m n_\star c^2), \quad (33)$$

$$\beta_\star := v_\star / c, \quad (34)$$

$$\alpha := \tilde{n}_\star / n_\star, \quad (35)$$

$$\varepsilon_m := u_\star / v_\star, \quad (36)$$

where ε denotes the normalized inertial length and σ is called the magnetization parameter. As we have mentioned earlier, the smallness of α and ε_m will be essential for the MHD approximation.

Using $\hat{A}^i = A^i / (B_\star L_\star)$, Ampere-Maxwell's law (12) is normalized as

$$\beta_\star^2 \hat{\partial}_0 \hat{\partial}_0 \hat{A}^i + \hat{\partial}_i \hat{\partial}_j \hat{A}^j - \hat{\partial}_j \hat{\partial}_j \hat{A}^i = \frac{\beta_\star}{\varepsilon \sqrt{\sigma}} \left(-\alpha \hat{\partial}_i \hat{\partial}_0 \hat{\phi} + \varepsilon_m \hat{n} \hat{u}^i \right). \quad (37)$$

The right hand side is regarded as the source terms which generate magnetic field and, hence, can not be much larger than the left hand side. In contrast to the Poisson equation (11), we allow for externally-applied magnetic field, which can exist ($A^i \neq 0$) even when the right hand side is small or zero (that

is vacuum magnetic field). Thus, we should consider only the following regime;

$$\text{Balance 3: } \frac{\beta_\star}{\varepsilon \sqrt{\sigma}} \max(\alpha, \varepsilon_m) \leq 1. \quad (38)$$

Again, this inequality “ \leq ” actually means “ \lesssim ” because this is a relation among the scale parameters.

Next, to estimate the orders of f and \tilde{f} , let us normalize N_+ and N_- as follows

$$\begin{aligned} \hat{N}_\pm^2 &= \frac{N_\pm^2}{n_\star^2} = \hat{n}^2 (1 - \beta_\star^2 |\hat{v}|^2) \pm 2 \frac{m_\mp}{m} \hat{n} (\alpha \hat{n} - \beta_\star^2 \varepsilon_m \hat{v} \cdot \hat{n} \hat{u}) \\ &\quad + \frac{m_\mp^2}{m^2} (\alpha^2 \hat{n}^2 - \beta_\star^2 \varepsilon_m^2 |\hat{n} \hat{u}|^2). \end{aligned} \quad (39)$$

The first term on the right hand side is $O(1)$. Since we are interested in the case of $\alpha, \varepsilon_m \ll 1$ and the inequalities $m_\mp/m \leq 1$ and $\beta_\star < 1$ always hold, the second and third terms on the right hand side are of small order; $\max(\alpha, \beta_\star^2 \varepsilon_m) \ll 1$. As a loose assumption, we consider the situation where

$$\text{Balance 4: } \eta := \max(\alpha, \beta_\star^2 \varepsilon_m) \leq 1, \quad (40)$$

holds. Namely, we give up applying the MHD approximation when $\eta \gg 1$. Assuming (40), we obtain the estimates, $n_\star f = O(1)$ and $n_\star \tilde{f} = O(\eta)$, and hence normalize them by $\hat{f} = n_\star f$ and $\hat{\tilde{f}} = n_\star \tilde{f} / \eta$. More explicitly, when $\eta \ll 1$, the leading-order terms are calculated by series expansion as follows

$$\hat{f} = \frac{\gamma}{\hat{n}} + \frac{\gamma^3}{2\hat{n}} \mu^2 (3\gamma^2 \Lambda_a^2 - \Lambda_b) + O(\alpha^3, \beta_\star^2 \varepsilon_m^3), \quad (41)$$

$$\eta \hat{\tilde{f}} = -\frac{\gamma^2}{\hat{n}} \Lambda_a + O(\alpha^2, \beta_\star^2 \varepsilon_m^2), \quad (42)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \beta_\star^2 |\hat{v}|^2}}, \quad (43)$$

$$\Lambda_a := \alpha \frac{\hat{n}}{\hat{n}} - \beta_\star^2 \varepsilon_m \hat{v} \cdot \hat{u} = O(\alpha, \beta_\star^2 \varepsilon_m), \quad (44)$$

$$\Lambda_b := \alpha^2 \frac{\hat{n}^2}{\hat{n}^2} - \beta_\star^2 \varepsilon_m^2 |\hat{u}|^2 = O(\alpha^2, \beta_\star^2 \varepsilon_m^2). \quad (45)$$

Here, we emphasize that the first order terms in α and ε_m are vacant in the series expansion of \hat{f} , which turns out to be important later.

It should be also remarked that we exclude the strongly-relativistic situation such as $\beta_\star |\hat{v}| = |\mathbf{v}|/c = 0.9999$, in which the Lorentz factor γ becomes much greater than 1 and our estimate $\hat{f} = O(1)$ is no longer valid. This means a breakdown of the assumed balance^{Note1} and strongly relativistic flow regions must be treated separately using a different normalization. For example, we suggest that all the equations should be

^{Note1} If $x = O(1)$, the function $f(x) = 1/\sqrt{1-x^2}$ is estimated as $O(1)$ in scale analysis. But, only the neighborhood of $x = 1$ should be treated separately as an exceptional case due to singularity. For example, the method of matched asymptotic expansion is necessary for this kind of problems.

Lorenz-transformed to the inertia frame moving with the flow speed $0.9999c$ so that the Lorentz factor becomes $\gamma = O(1)$.

Now, the equations (15), (16), (23) and (24) are normalized as follows

$$\begin{aligned} & \hat{\partial}_0 \left[\hat{f} \hat{Q}^i \hat{Q}^0 + \mu^2 \eta \hat{f} \left(\alpha \hat{Q}^i \hat{K}^0 + \varepsilon_m \hat{K}^i \hat{Q}^0 \right) + \mu^2 \alpha \varepsilon_m \hat{f}' \hat{K}^i \hat{K}^0 \right] \\ & + \hat{\partial}_j \left[\hat{f} \hat{Q}^i \hat{Q}^j + \mu^2 \eta \varepsilon_m \hat{f} \left(\hat{Q}^i \hat{K}^j + \hat{Q}^j \hat{K}^i \right) + \mu^2 \varepsilon_m^2 \hat{f}' \hat{K}^i \hat{K}^j \right] \\ & - \frac{\alpha^2}{\beta_*^2 \varepsilon^2} \hat{K}^0 \hat{F}^i{}_0 - \frac{\varepsilon_m}{\beta_* \varepsilon} \sqrt{\sigma} \hat{K}^j \hat{F}^i{}_j = 0, \end{aligned} \quad (46)$$

$$\begin{aligned} & \hat{\partial}_0 \left[\mu^2 \eta \varepsilon_m \hat{f} \hat{Q}^i \hat{Q}^0 + \mu^2 \varepsilon_m \hat{f}' \left(\alpha \hat{Q}^i \hat{K}^0 + \varepsilon_m \hat{K}^i \hat{Q}^0 \right) \right. \\ & \quad \left. - \mu^2 \varepsilon_m^2 \alpha \left(\hat{\mu} \hat{f}' - \mu^2 \eta \hat{f} \right) \hat{K}^i \hat{K}^0 \right] \\ & + \hat{\partial}_j \left[\mu^2 \eta \varepsilon_m \hat{f} \hat{Q}^i \hat{Q}^j + \mu^2 \varepsilon_m^2 \hat{f}' \left(\hat{Q}^i \hat{K}^j + \hat{Q}^j \hat{K}^i \right) \right. \\ & \quad \left. - \mu^2 \varepsilon_m^3 \left(\hat{\mu} \hat{f}' - \mu^2 \eta \hat{f} \right) \hat{K}^i \hat{K}^j \right] \\ & - \frac{\alpha \varepsilon_m}{\beta_*^2 \varepsilon^2} \left(\hat{Q}^0 - \hat{\mu} \alpha \hat{K}^0 \right) \hat{F}^i{}_0 - \frac{\varepsilon_m}{\beta_* \varepsilon} \sqrt{\sigma} \left(\hat{Q}^j - \hat{\mu} \varepsilon_m \hat{K}^j \right) \hat{F}^i{}_j = 0, \end{aligned} \quad (47)$$

$$\hat{\partial}_0 \hat{Q}^0 + \hat{\partial}_i \hat{Q}^i = 0, \quad (48)$$

$$\frac{\alpha}{\varepsilon_m} \hat{\partial}_0 \hat{K}^0 + \hat{\partial}_i \hat{K}^i = 0, \quad (49)$$

where the normalized electric field \hat{E}^i is give by

$$\hat{E}^i = \hat{F}^i{}_0 = \frac{c L_*}{\phi_*} F^i{}_0 = -\hat{\partial}_i \hat{\phi} - \frac{\beta_* \varepsilon \sqrt{\sigma}}{\alpha} \hat{\partial}_0 \hat{A}^i. \quad (50)$$

B. Imposition of MHD balance

The MHD approximation is understood as the reduction to a single-fluid model, satisfying

$$\alpha \ll 1 \quad \text{and} \quad \varepsilon_m \ll 1. \quad (51)$$

If they were not satisfied, we would have to solve the two-fluid equations as they are. However, in the limit of $\alpha, \varepsilon_m \rightarrow 0$ (then $\eta \rightarrow 0$), many terms in (46) are negligible and ultimately (46) becomes the equation of motion for neutral fluid. Since this simple limit is not interesting, we assume that the electromagnetic ($\mathbf{J} \times \mathbf{B}$) force, which is $\hat{K}^j \hat{F}^i{}_j$ in (46), is not negligible but dominant. Namely, the flow \mathbf{v} is dominantly accelerated by this term due to

$$\text{Balance 5: } \frac{\varepsilon_m}{\beta_* \varepsilon} \sqrt{\sigma} = 1. \quad (52)$$

The another meaning of this balance can be understood by defining a representative cyclotron frequency as

$$\omega_{c*} = \frac{e B_*}{m} = \frac{c}{L_*} \frac{\sqrt{\sigma}}{\varepsilon}. \quad (53)$$

Since we are considering magnetized plasmas, this frequency is supposed to be much faster than the time scale of flow dynamics,

$$\frac{1/\omega_{c*}}{T_*} = \frac{\beta_* \varepsilon}{\sqrt{\sigma}} \ll 1. \quad (54)$$

The balance (52) indicates that ε_m is indeed small and of the same order as ω_{c*}^{-1}/T_* . It is well-known that the two-fluid equations generally encompass ion's and electron's cyclotron motions. By taking the limit of $\varepsilon_m \rightarrow 0$ while keeping the balance (52), we can eliminate these fast motions from the flow dynamics. We also remark that the $\mathbf{v} \times \mathbf{B}$ term (which is $\hat{Q}^j \hat{F}^i{}_j$) in Ohm's law (47) becomes of order 1 due to (52).

On the other hand, the terms involving the electric field $\hat{F}^i{}_0$ in (46) and (47) are, respectively, written as

$$\frac{\alpha^2}{\beta_*^2 \varepsilon^2} \hat{K}^0 \hat{F}^i{}_0 = \frac{1}{\sigma} \frac{\alpha^2}{\varepsilon_m^2} \hat{K}^0 \left[-\hat{\partial}_i \hat{\phi} - \sigma \frac{\varepsilon_m}{\alpha} \hat{\partial}_0 \hat{A}^i \right], \quad (55)$$

and

$$\begin{aligned} & \frac{\alpha \varepsilon_m}{\beta_*^2 \varepsilon^2} \left(\hat{Q}^0 - \hat{\mu} \alpha \hat{K}^0 \right) \hat{F}^i{}_0 \\ & = \frac{1}{\sigma} \frac{\alpha}{\varepsilon_m} \left(\hat{Q}^0 - \hat{\mu} \alpha \hat{K}^0 \right) \left[-\hat{\partial}_i \hat{\phi} - \sigma \frac{\varepsilon_m}{\alpha} \hat{\partial}_0 \hat{A}^i \right], \end{aligned} \quad (56)$$

using the balance (52). In the limit of $\varepsilon_m \rightarrow 0$ or $\sigma \rightarrow 0$, only the electrostatic force term ($\hat{\partial}_i \hat{\phi}$) gets too large $1/(\sigma \varepsilon_m) \rightarrow \infty$ to balance with other terms. This implies that the existence of very fast plasma oscillation breaks down the assumed balance totally. To maintain the balance consistently, the charge separation α must be small enough that *all terms* in (55) and (56) are equal or less than the order 1, which requires $\alpha \leq \sqrt{\sigma} \varepsilon_m$, $\alpha \leq \sigma \varepsilon_m$ and $\alpha \leq \varepsilon_m$. To consider the most general situation satisfying all of them, we assume

$$\text{Balance 6: } \alpha = \varepsilon_m \varepsilon_\sigma, \quad (57)$$

where ε_σ is the abbreviation of

$$\varepsilon_\sigma := \min(\sigma, 1). \quad (58)$$

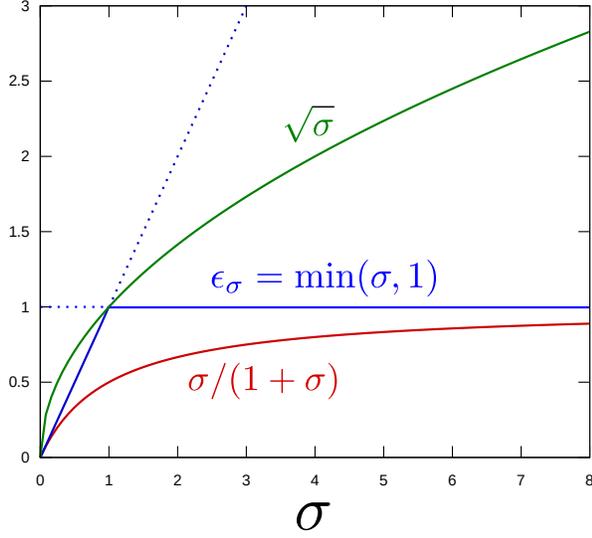
This is the last balance that we impose to derive MHD models. The magnitude of α is now determined by other scale parameters. The meaning of this balance is again understood by introducing a representative plasma frequency as

$$\omega_{p*} := c \sqrt{\frac{e^2 \mu_0 n_*}{m}} = \frac{1}{L_*} \frac{c}{\varepsilon}. \quad (59)$$

Since $\varepsilon_\sigma \leq \sqrt{\sigma}$ holds mathematically (see Fig. 1), the balance (57) leads to

$$\alpha \leq \frac{1/\omega_{p*}}{T_*} = \beta_* \varepsilon = \varepsilon_m \sqrt{\sigma}. \quad (60)$$

Therefore, when the plasma frequency is much faster than the time scale of the flow dynamics ($\omega_{p*}^{-1}/T_* \rightarrow 0$), the balance 6 requires α to be small ($\alpha \rightarrow 0$), which diminishes the fast plasma oscillation. Note that ω_{p*}^{-1}/T_* is not always a small

FIG. 1: Plots of ϵ_σ , $\sqrt{\sigma}$ and $\sigma/(1+\sigma)$

number when σ is much greater than 1. The balance 6 requires smallness of α more strictly than the condition $\alpha \leq \omega_{p*}^{-1}/T_*$ when $\sigma \geq 1$.

At this point, we summarize the situation where all the balances 1, 2, ..., 6 are imposed together. Given the balances 5 and 6, the balance 3 can be reduced to

$$\text{Balance 3': } \frac{\beta_*^2}{\sigma} \leq 1, \quad (61)$$

Therefore, the situation can be divided into the two cases, $\beta_*^2 \leq \sigma \leq 1$ or $1 \leq \sigma$. In either case, the balance 4 is simply rewritten as

$$\text{Balance 4': } \eta = \epsilon_m \epsilon_\sigma \leq 1. \quad (62)$$

By omitting the hat symbol $\hat{}$ in what follows, the normalized equations are summarized as follows

$$\begin{aligned} & \partial_0 \left[f Q^i Q^0 + \epsilon_I^2 \epsilon_\sigma \tilde{f} (\epsilon_\sigma Q^i K^0 + K^i Q^0) + \epsilon_\sigma \epsilon_I^2 f' K^i K^0 \right] \\ & + \partial_j \left[f Q^i Q^j + \epsilon_I^2 \epsilon_\sigma \tilde{f} (Q^i K^j + Q^j K^i) + \epsilon_I^2 f' K^i K^j \right] \\ & - \epsilon_\sigma K^0 \left[-\frac{\epsilon_\sigma}{\sigma} \partial_i \phi - \partial_0 A^i \right] - K^j F^i_j = 0, \end{aligned} \quad (63)$$

$$\begin{aligned} & \epsilon_I^2 \partial_0 \left[\epsilon_\sigma \tilde{f} Q^i Q^0 + f' (\epsilon_\sigma Q^i K^0 + K^i Q^0) \right. \\ & \quad \left. - \epsilon_\sigma (\epsilon_H f' - \epsilon_\sigma \epsilon_I^2 \tilde{f}) K^i K^0 \right] \\ & + \epsilon_I^2 \partial_j \left[\epsilon_\sigma \tilde{f} Q^i Q^j + f' (Q^i K^j + Q^j K^i) \right. \\ & \quad \left. - (\epsilon_H f' - \epsilon_\sigma \epsilon_I^2 \tilde{f}) K^i K^j \right] \\ & - (Q^0 - \epsilon_H \epsilon_\sigma K^0) \left[-\frac{\epsilon_\sigma}{\sigma} \partial_i \phi - \partial_0 A^i \right] \\ & - (Q^j - \epsilon_H K^j) F^i_j = 0, \end{aligned} \quad (64)$$

$$\partial_0 Q^0 + \partial_i Q^i = 0, \quad (65)$$

$$\epsilon_\sigma \partial_0 K^0 + \partial_i K^i = 0, \quad (66)$$

$$-\partial_j \partial_j \phi = \tilde{n}, \quad (67)$$

$$\beta_*^2 \partial_0 \partial_0 A^i + \partial_i \partial_j A^j - \partial_j \partial_j A^i = \frac{\beta_*^2}{\sigma} (-\epsilon_\sigma \partial_i \partial_0 \phi + n u^i), \quad (68)$$

where

$$f' = f - \epsilon_H \epsilon_\sigma \tilde{f}. \quad (69)$$

We have introduced

$$\epsilon_H := \tilde{\mu} \epsilon_m, \quad (70)$$

$$\epsilon_I := \mu \epsilon_m, \quad (71)$$

because ϵ_m appears only in these forms. By noting $\Lambda_a = O(\epsilon_m \epsilon_\sigma)$ and $\Lambda_b = O(\epsilon_m^2 \epsilon_\sigma)$, the estimate (41) becomes

$$f = \frac{1}{n \sqrt{1 - \beta_*^2 |v|^2}} + O(\epsilon_\sigma \epsilon_I^2). \quad (72)$$

Due to the balances 5 and 6, the number of non-dimensional parameters (namely, the dimension of the parameter space) has been reduced to three; ϵ_m , σ , β_* . However, the governing equations are still equivalent to the two-fluid equations. In the following sections, we derive reduced models by taking the specific limit of ϵ_m , σ , β_* .

IV. REDUCTION TO VARIOUS MHD MODELS

A. Vacuum limit $\beta_*^2/\sigma \rightarrow 0$

The limit $\beta_*^2/\sigma \rightarrow 0$ corresponds to vacuum state since the right hand side of the Ampere-Maxwell equation (68) vanishes. This limit may be uninteresting because the plasma current is too small to disturb the vacuum magnetic field. For example, the limit of large magnetization parameter $\sigma \rightarrow \infty$ inevitably results in this vacuum state (due to $\beta_* \leq 1$). According to the dispersion relation of RMHD, the relativistic Alfvén velocity v_{A*} is well-known as

$$v_{A*} = \frac{B_* c}{\sqrt{\mu_0 m m_* c^2 + B_*^2}} = \sqrt{\frac{\sigma}{1 + \sigma}} c. \quad (73)$$

When $\sigma \rightarrow \infty$ (i.e., strong magnetic field or low density limit), the displacement current becomes dominant and the Alfvén wave turns into the electromagnetic wave in vacuum.

The interaction between plasma motion and electromagnetic field is most active in the situation $\beta_*^2/\sigma \simeq 1$. Therefore, the Alfvén ordering $\beta_*^2/\sigma = 1$ is conventionally employed in (non-relativistic) MHD focusing on only this situation, which is admissible as far as $\sigma \leq 1$. It is also interesting to note that the scale of ϵ_σ is similar to $v_{A*}^2/c^2 = \sigma/(1 + \sigma)$.

B. Single-fluid limit $\varepsilon_m \rightarrow 0$

In the limit of $\varepsilon_m \rightarrow 0$ (then $\varepsilon_H, \varepsilon_I \rightarrow 0$), a lot of terms can be neglected in (63) and (64) as follows

$$\begin{aligned} & \partial_0(\gamma n v^i) + \partial_j(\gamma n v^i v^j) \\ & - \varepsilon_\sigma \tilde{n} \left(-\frac{\varepsilon_\sigma}{\sigma} \partial_i \phi - \partial_0 A^i \right) - n u^j F^i_j = 0, \end{aligned} \quad (74)$$

$$-n \left(-\frac{\varepsilon_\sigma}{\sigma} \partial_i \phi - \partial_0 A^i \right) - m v^j F^i_j = 0. \quad (75)$$

These are the well-known RMHD equations, where the expression of f has been simplified into

$$f = \frac{\gamma}{n} = \frac{1}{n \sqrt{1 - \beta_*^2 |v|^2}}. \quad (76)$$

and \tilde{f} is completely neglected as if the ‘‘proper charge neutrality’’ $N_+ = N_-$ holds. In fact, there exists small-order charge separation $\tilde{n} \neq 0$ (with $\nabla \cdot \mathbf{J} \neq 0$) and the associated electrostatic force takes part in the dominant balance. We will discuss more about the RMHD equations in Sec. VI.

C. Non-relativistic limit $\sigma \rightarrow 0$

Here, we consider the limit of small σ . Because of the balance 3', the limit of $\sigma \rightarrow 0$ forces $\beta_*^2 \rightarrow 0$ as well (and $\varepsilon_\sigma \rightarrow 0$). Therefore, let us take these non-relativistic limits while keeping

$$\frac{\beta_*^2}{\sigma} \rightarrow \frac{v_*^2}{v_{A*}^2} = \text{const.} (\leq 1). \quad (77)$$

Then, the equations (63), (64) and (68) are reduced to the extended MHD (XMHD) equations,

$$\partial_0(n v^i) + \partial_j(n v^i v^j + \varepsilon_I^2 n u^i u^j) - n u^j F^i_j = 0, \quad (78)$$

$$\begin{aligned} & \varepsilon_I^2 \partial_0(n u^i) + \varepsilon_I^2 \partial_j [n(v^i u^j + v^j u^i) - \varepsilon_H n u^i u^j] \\ & - n(-\partial_i \phi - \partial_0 A^i) - n(v^j - \varepsilon_H u^j) F^i_j = 0, \end{aligned} \quad (79)$$

$$\partial_i \partial_j A^j - \partial_j \partial_i A^i = \frac{v_*^2}{v_{A*}^2} n u^i. \quad (80)$$

where $f = 1/n$ has been substituted. Since $v_*^2/v_{A*}^2 \rightarrow 0$ is again the uninteresting vacuum limit, it is conventional to use the Alfvén ordering $v_* = v_{A*}$.

Note that the displacement current in (68) has been neglected and hence $\nabla \times \mathbf{B} = (v_*^2/v_{A*}^2) \mathbf{J}$ is a constraint; we can eliminate $n u^i$ (or \mathbf{J}) using this relation. The generalized Ohm's law (79) is regarded as the evolution equation for A^i , where ϕ is determined such that the Coulomb gauge $\partial_i A^i = 0$ holds. Therefore, the XMHD equations constitute a dynamical system of 7 fields (n, v^i, A^i) with one constraint $\partial_i A^i = 0$. Since $\nabla \cdot \mathbf{J} = \partial_i(n u^i) = 0$, the so-called ‘‘quasi-neutrality condition’’ holds as if $\tilde{n} = 0$. In fact, small charge separation $\tilde{n} = -\partial_j \partial_j \phi \neq 0$ exists although it no longer appears explicitly in the XMHD equations. For small $\sigma \ll 1$, the balance

6 ($\alpha = \varepsilon_m \sigma$) requests α to be further smaller than ε_m . The charge conservation law is indeed reduced to $\nabla \cdot \mathbf{J} = 0$ in the limit $\sigma \rightarrow 0$.

The electron-inertia effect is manifested by the terms with ε_I^2 , which is the second-order of ε_m . If we neglect only ε_I^2 , the Hall MHD (HMHD) equations are reproduced. If we neglect ε_H too or simply take the limit of $\varepsilon_m \rightarrow 0$, the MHD equations are finally obtained.

D. Relativistic Hall MHD model

Now, we are positioned to search for the other MHD models which include both the relativistic and two-fluid effects. To derive a reduced model, we still need to assume the smallness of ε_m but should not neglect it completely. An approximation that comes to mind immediately is to neglect $O(\varepsilon_m^2)$, namely, the electron-inertia effect ε_I^2 only. The resultant equations deserve to be called relativistic Hall MHD (RHMHD). It is remarkable that f in (72) includes no additional term due to the Hall effect, $O(\varepsilon_H)$ or $O(\varepsilon_m)$. By neglecting ε_I^2 , the proper charge neutrality $f = \gamma/n$ still holds approximately and \tilde{f} vanishes.

Therefore, in RHMHD, (63) and (64) are reduced to

$$\begin{aligned} & \partial_0(\gamma n v^i) + \partial_j(\gamma n v^i v^j) \\ & - \varepsilon_\sigma \tilde{n} \left(-\frac{\varepsilon_\sigma}{\sigma} \partial_i \phi - \partial_0 A^i \right) - n u^j F^i_j = 0, \end{aligned} \quad (81)$$

$$\begin{aligned} & - (n - \varepsilon_H \varepsilon_\sigma \tilde{n}) \left(-\frac{\varepsilon_\sigma}{\sigma} \partial_i \phi - \partial_0 A^i \right) \\ & - n(v^j - \varepsilon_H u^j) F^i_j = 0. \end{aligned} \quad (82)$$

The terms including ε_H are the difference from RMHD.

If we further assume the smallness of $\sigma \ll 1$ additionally, we can also neglect the term of $O(\varepsilon_H \varepsilon_\sigma)$ in Ohm's law. Although it is just a minor reduction, let us call it weakly-relativistic Hall MHD (W-RHMHD).

E. Weakly-relativistic XMHD model

If one wants to allow for both the electron-inertia and relativistic effects, it is difficult to derive a reduced model from the two-fluid equations. One option is to neglect $O(\varepsilon_I^2 \sigma)$ by assuming the sufficient smallness of both ε_m and $\sigma (\ll 1)$. Then, the proper charge neutrality holds again and we can derive similar equations to XMHD.

$$\begin{aligned} & \partial_0(\gamma n v^i) + \partial_j(\gamma n v^i v^j + \varepsilon_I^2 \gamma n u^i u^j) \\ & - \sigma \tilde{n} (-\partial_i \phi - \partial_0 A^i) - n u^j F^i_j = 0, \end{aligned} \quad (83)$$

$$\begin{aligned} & \varepsilon_I^2 \partial_0(\gamma n u^i) + \varepsilon_I^2 \partial_j [\gamma n (v^i u^j + v^j u^i) - \varepsilon_H \gamma n u^i u^j] \\ & - (n - \varepsilon_H \sigma \tilde{n}) (-\partial_i \phi - \partial_0 A^i) - n(v^j - \varepsilon_H u^j) F^i_j = 0. \end{aligned} \quad (84)$$

We call this model weakly-relativistic XMHD because it is valid only when σ and β_*^2 are sufficiently smaller than 1.

V. RANGE OF APPLICATION

Under the balances 1 to 6, there remain three non-dimensional parameters $(\varepsilon_m, \sigma, \beta_*)$, which are related to the three physical scales (n_*, v_*, B_*) of plasma (for fixed length scale L_*). More rigorously speaking, $\tilde{\mu}$ and μ are two additional parameters which appear only in the forms, $\varepsilon_H = \tilde{\mu}\varepsilon_m$ and $\varepsilon_I = \mu\varepsilon_m$. Because of the inequalities $\tilde{\mu} \leq 1$ and $\mu \leq 1$, they do not alter the balances 1 to 6 but possibly make ε_H and ε_I further smaller than ε_m . As the two typical examples, we will consider electron-ion (Hydrogen) plasma $m_-/m_+ \simeq 10^{-3}$ (for which $\mu \simeq 10^{-3/2} = 0.0316$ and $\tilde{\mu} \simeq 1$) and electron-positron plasma $m_-/m_+ = 1$ (for which $\mu = 0.5$ and $\tilde{\mu} = 0$).

Now, we carefully consider the magnitude of ε_m , which obviously measures the impact of the two-fluid effect as we have seen in the previous section. According to the balance 5 and 3', it depends on the other parameters as follows

$$\varepsilon_m = \frac{\beta_*}{\sqrt{\sigma}} \varepsilon = \frac{v_*}{L_* \omega_{c*}} \quad \text{with} \quad \frac{\beta_*}{\sqrt{\sigma}} \leq 1. \quad (85)$$

Here, we obtain $\varepsilon = d_*/L_*$ by newly introducing a representative inertial length (or skin depth) $d_* = c/\omega_{p*}$. More specifically, the inertial lengths for positively (+) and negatively (−) charged gases are given by

$$d_{\pm} = \sqrt{\frac{m_{\pm}}{m_+ + m_-}} d_*. \quad (86)$$

Indeed, d_+ and d_- respectively correspond to ion's and electron's inertial lengths for electron-ion plasma.

In the case of non-relativistic limit ($\sigma \rightarrow 0$ and $\beta_* \rightarrow 0$) with application of the Alfvén ordering $\beta_*^2/\sigma \rightarrow 1$ ($v_* = v_{A*}$), we simply obtain $\varepsilon_m = \varepsilon$. Therefore,

$$\varepsilon_H \simeq d_+/L_* \quad \text{and} \quad \varepsilon_I \simeq d_-/L_* (\simeq \sqrt{m_-/m_+} \varepsilon_H), \quad (87)$$

for electron-ion plasma, and

$$\varepsilon_H = 0 \quad \text{and} \quad \varepsilon_I = \sqrt{2} d_+/L_* = \sqrt{2} d_-/L_*, \quad (88)$$

for electron-positron plasma. In this way, the Hall and electron-inertia effects are associated with the small-scales d_+ and d_- , where the density n_* is important because d_* is proportional to $1/\sqrt{n_*}$ only. For sufficiently dense plasma $n_* \rightarrow \infty$, the two-fluid effect becomes negligible $\varepsilon \rightarrow 0$. However, this is a consequence of applying the Alfvén ordering. Namely, v_* and B_* are not fixed independently but varied along with n_* .

In general, it is interesting to note that ε_m does not originally depend on the density n_* but on the ratio v_*/B_* in (85). When σ gets larger than 1, the two-fluid effect ε_m gets smaller than ε by the factor $\beta_*/\sqrt{\sigma}$. This tendency agrees with Kawazura²⁴ in which the ion skin depth is modified to shrink as the magnetic field strength increases relativistically. In the limit of $B_* \rightarrow \infty$ with fixed n_* and v_* , we find that the two-fluid effect becomes negligible and the use of RMHD is justified (although it tends to be almost vacuum plasma, $\beta_*^2/\sigma \rightarrow 0$).

In the case of electron-positron plasma, the Hall effect vanishes identically, $\varepsilon_H = 0$. Then, the XMHD model includes only the electron-inertia effect, which is especially called Inertial MHD (IMHD). Similarly, we can obtain weakly-relativistic IMHD from weakly-relativistic XMHD when $\varepsilon_H = 0$.

Model	Included order	Neglected order
MHD		$O(\sigma), O(\varepsilon_H), O(\varepsilon_I^2)$
HMHD	$O(\varepsilon_H)$	$O(\sigma), O(\varepsilon_I^2)$
IMHD	$O(\varepsilon_I^2)$	$O(\sigma), O(\varepsilon_H)$
XMHD	$O(\varepsilon_H), O(\varepsilon_I^2)$	$O(\sigma)$
RMHD	$O(\sigma)$	$O(\varepsilon_H), O(\varepsilon_I^2)$
RHMHD	$O(\sigma), O(\varepsilon_H)$	$O(\varepsilon_I^2)$
W-RHMHD	$O(\sigma), O(\varepsilon_H)$	$O(\varepsilon_I^2), O(\varepsilon_H \sigma)$
W-RIMHD	$O(\sigma), O(\varepsilon_I^2)$	$O(\varepsilon_H), O(\varepsilon_I^2 \sigma)$
W-RXMHD	$O(\sigma), O(\varepsilon_H), O(\varepsilon_I^2)$	$O(\varepsilon_I^2 \sigma)$

TABLE I: Classification of models (H = Hall, I = Inertial, X = eXtended, R = Relativistic, W-R = Weakly-Relativistic)

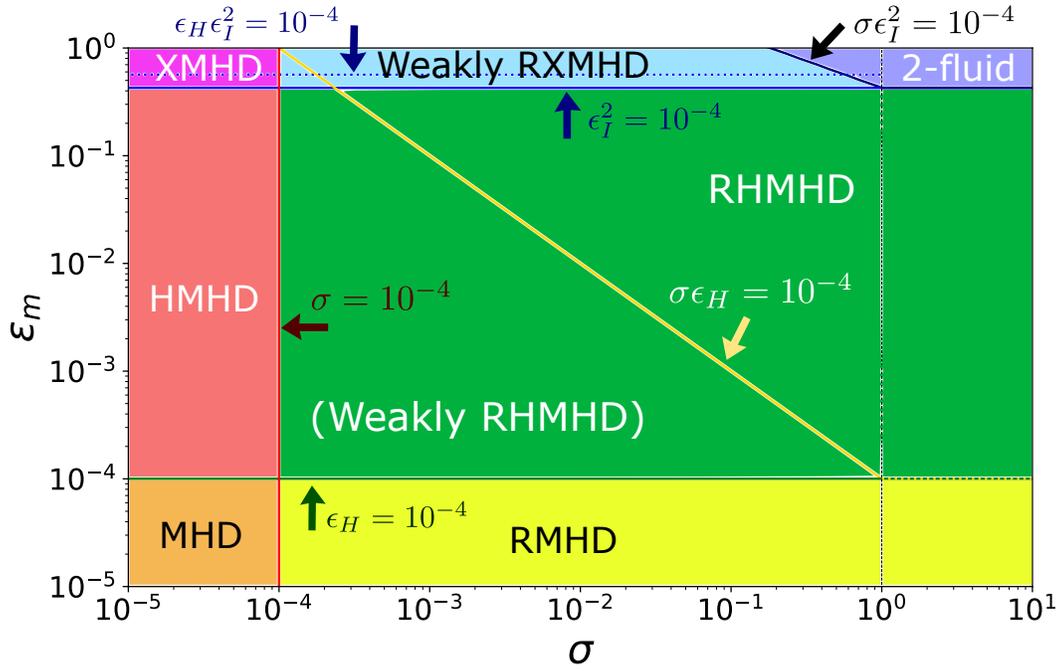
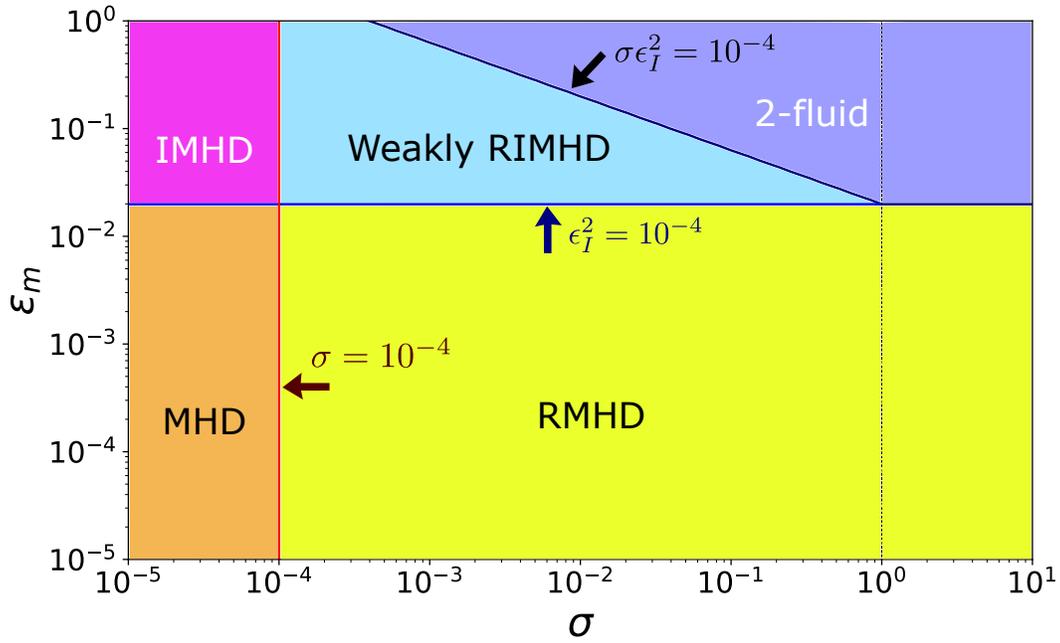
All the models which we have presented so far are summarized in Table I. All these reduced models need to neglect the order of

$$\varepsilon_{\sigma} \varepsilon_I^2 = \frac{m_+ m_-}{m^2} \beta_*^2 \frac{\varepsilon_{\sigma}}{\sigma} \varepsilon^2, \quad (89)$$

in common, which is necessary for $f = \gamma/n$ to hold approximately and to get rid of \tilde{f} . Then, either proper charge neutrality or quasi-neutrality holds. In other words, we have to solve the two-fluid equations directly if this $\varepsilon_{\sigma} \varepsilon_I^2$ is not sufficiently smaller than 1.

Since it is still difficult to imagine the applicable scope of each model, let us assume 10^{-4} as a clear threshold for example. Namely, the non-dimensional parameters (such as σ and ε_H) can be considered negligible if they are below 10^{-4} . Otherwise, they are not neglected. Then, the lines such as $\sigma = 10^{-4}$ and $\varepsilon_H = 10^{-4}$ divide the parameter space $(\varepsilon_m, \sigma, \beta_*)$ into subspaces, in which a certain group of MHD models is applicable. Recall that β_* should satisfy $\beta_*^2 \leq \sigma$ and it appears only in Ampere-Maxwell's equation (68). As in Table I, the models are classified in terms of σ and ε_m (regardless of β_*), which are illustrated in Fig. 2 for electron-ion plasma and in Fig. 3 for electron-positron plasmas. Due to the smallness of mass ratio $m_-/m_+ \simeq 10^{-3}$, the electron-inertia effect is readily neglected ($\varepsilon_I^2 < 10^{-4}$) in the majority of cases for electron-ion plasma. But, we have to keep in mind that electron inertia may be important *locally* at singular point or layer, where the small-scale structure $L_* \sim d_-$ emerges (as in the location where magnetic reconnection occurs).

Although Figs. 2 and 3 look simple enough, let us further illustrate the application ranges in terms of (n_*, v_*, B_*) . Since the vacuum state $\beta_*^2/\sigma \ll 1$ is uninteresting, it is reasonable to fix β_* to the maximal value $\beta_* = \sqrt{\varepsilon_{\sigma}}$. As shown in the plots of Fig. 1, there is in fact no order difference between $\sqrt{\varepsilon_{\sigma}}$ and $v_{A*}/c = \sqrt{\sigma/(1+\sigma)}$ in the scale analysis; $\sqrt{\varepsilon_{\sigma}} \simeq v_{A*}/c$.

FIG. 2: Electron-Ion Plasma (σ vs. ϵ_m)FIG. 3: Electron-Positron Plasma (σ vs. ϵ_m)

Therefore, we refer to

$$\beta_* = \sqrt{\epsilon_\sigma} \simeq v_{A*}/c \quad (90)$$

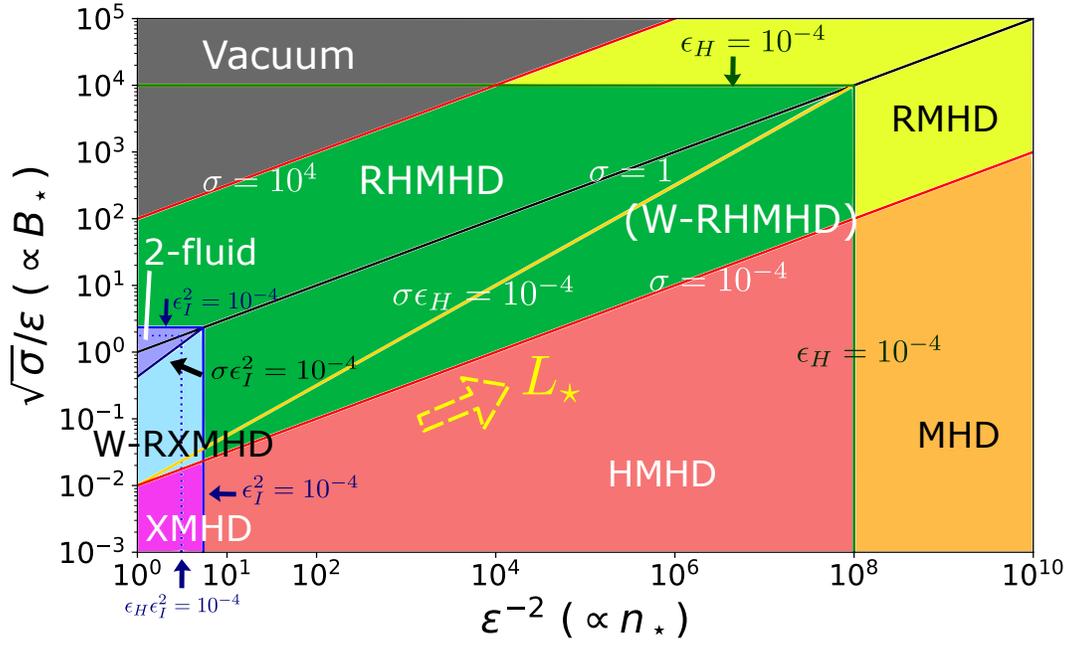
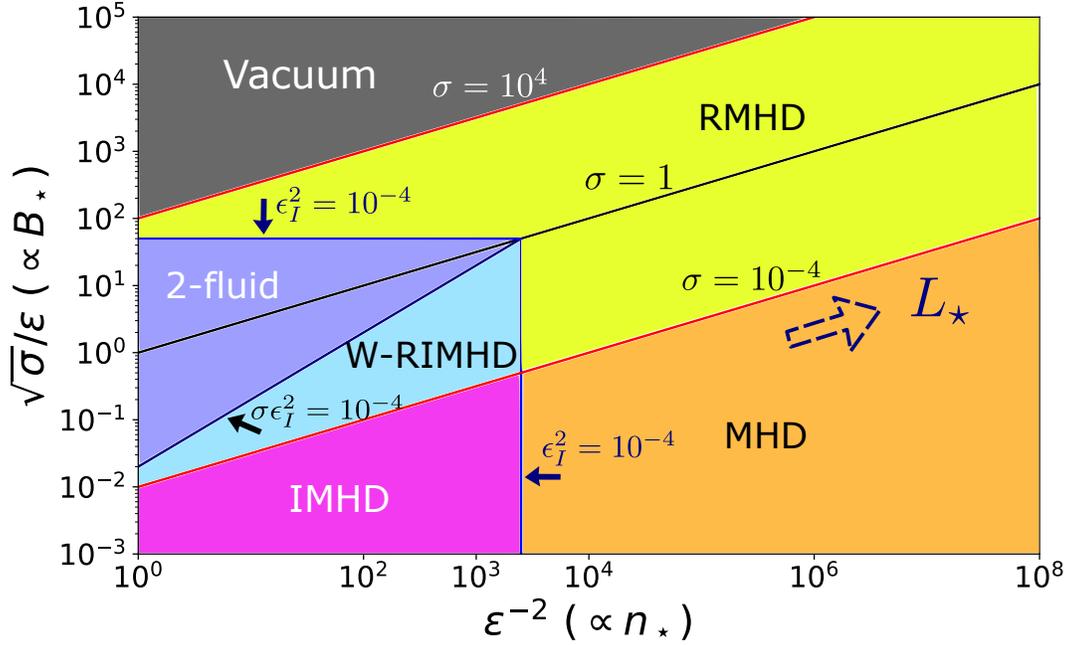
as relativistic Alfvén ordering. By imposing this relativistic Alfvén ordering on β_* , we obtain $\epsilon_m = \epsilon \sqrt{\epsilon_\sigma / \sigma}$ and the two

remaining parameters are chosen as

$$\epsilon^{-2} = \frac{\mu_0 n_* e^2 L_*^2}{m} \propto n_*, \quad (91)$$

$$\sqrt{\sigma} / \epsilon = \frac{e B_* L_*}{mc} \propto B_*, \quad (92)$$

representing the scales of (n_*, B_*) directly. Therefore, we can

FIG. 4: Electron-Ion Plasma (n_* vs. B_* vs. L_*)FIG. 5: Electron-Positron Plasma (n_* vs. B_* vs. L_*)

remap Fig. 2 into Fig. 4 and Fig. 3 into Fig. 5 on the 2d plane ($\epsilon^{-2}, \sqrt{\sigma}/\epsilon$). The region of $\sigma > 10^4$ is filled in gray because it is considered vacuum ($\beta_*^2/\sigma < 10^{-4}$). In the strong magnetic field limit $B_* \rightarrow \infty$, we inevitably enter this vacuum regime but RMHD is still valid and no problem to keep using it. In the dense plasma limit $n_* \rightarrow \infty$, we enter the conventional MHD regime. In this figure, the limit of large scale $L_* \rightarrow \infty$ corresponds to the movement in the direction indi-

cated by the fat arrow, which is parallel to the $\sigma = \text{const.}$ line. In the triangle region indicated by "2-fluid", the charge neutrality approximation $\epsilon_\sigma \epsilon_I^2 < 10^{-4}$ is not satisfied. This region exists on the low density side $\epsilon^{-2} < 10^4 \mu^2$ and for intermediate strength of magnetic field. In the weak magnetic field limit $B_* \rightarrow 0$, we can apply the non-relativistic MHD models (such as XMHD, HMHD, IMHD, MHD). But, when magnetic field is weak such that $\sqrt{\sigma}/\epsilon < \sqrt{10^4 \mu^2}$ holds and in the low

density limit $n_* \rightarrow 0$, we have to solve the two-fluid equations without assuming charge neutrality. The limit of small scale $L_* \rightarrow 0$ also enters the "2-fluid" region eventually.

VI. REMARKS ON RHMHD

In comparison to RMHD, the RHMHD equations just have a few additional terms in Ohm's law due to the Hall effect. But, this difference is quite influential when solving these equations theoretically and numerically.

In general, when a dynamical system $\partial_t u = F(u)$ for u is solved numerically, the recurrence formula such as $u^{n+1} = u^n + F(u^n)\Delta t$ is iterated for time marching $t \rightarrow t + \Delta t$. To execute this iteration, the right hand side $F(u)$ must be calculated uniquely using the dynamical variable u . This is a fundamental requirement for the well-posedness of time-evolving system.

In 3d vector format, the RHMHD equations are composed of the evolution equations (that include time derivative ∂_t of some quantity),

$$\partial_t n = -\nabla \cdot (n\mathbf{v}), \quad (93)$$

$$\partial_t (n\boldsymbol{\gamma}\mathbf{v}) = -\nabla \cdot (n\boldsymbol{\gamma}\mathbf{v}\mathbf{v}) + \varepsilon_\sigma \tilde{n}\mathbf{E} + \mathbf{J} \times \mathbf{B}, \quad (94)$$

$$\beta_*^2 \partial_t \mathbf{E} = \nabla \times \mathbf{B} - \frac{\beta_*^2}{\sigma} \mathbf{J}, \quad (95)$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad (96)$$

and the constraints,

$$n(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \varepsilon_H (\varepsilon_\sigma \tilde{n}\mathbf{E} + \mathbf{J} \times \mathbf{B}) = 0, \quad (97)$$

$$\nabla \cdot \mathbf{E} = \frac{\varepsilon_\sigma}{\sigma} \tilde{n}, \quad (98)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (99)$$

A drastic change from the two-fluid equations is that the time derivative of the current \mathbf{J} no longer exists in (97) due to neglect of electron-inertia $\varepsilon_f^2 \rightarrow 0$. Therefore, to calculate the right hand sides of the evolution equations, we have to determine (or eliminate) \mathbf{J} using the other variables.

From Ohm's law (97), the electric field of the component parallel to the magnetic field must be zero ($\mathbf{E} \cdot \mathbf{B} = 0$). By combining (95) and (96), we obtain

$$\begin{aligned} \partial_t (\mathbf{E} \times \mathbf{B}) = & \nabla \cdot \left(\frac{\mathbf{B}\mathbf{B}}{\beta_*^2} + \mathbf{E}\mathbf{E} \right) - \nabla \cdot \left(\frac{|\mathbf{B}|^2}{2\beta_*^2} + \frac{|\mathbf{E}|^2}{2} \right) \\ & - \frac{1}{\sigma} (\varepsilon_\sigma \tilde{n}\mathbf{E} + \mathbf{J} \times \mathbf{B}). \end{aligned} \quad (100)$$

This evolution equation for $\mathbf{E} \times \mathbf{B}$ can be solved instead of (95) because the electric field has only the perpendicular component and can be reproduced by $\mathbf{E} = \mathbf{B} \times (\mathbf{E} \times \mathbf{B})/|\mathbf{B}|^2$. Moreover, we can easily eliminate $\varepsilon_\sigma \tilde{n}\mathbf{E} + \mathbf{J} \times \mathbf{B}$ in (94) and (100) by using Ohm's law (97). Then, the right hand sides no longer include \mathbf{J} . Therefore, the RHMHD equations are regarded as a dynamical system of 10 fields ($n, M_P, M_{EM}, \mathbf{B}$), where $M_P = n\boldsymbol{\gamma}\mathbf{v}$ and $M_{EM} = \mathbf{E} \times \mathbf{B}$, satisfying two constraints $M_{EM} \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{B} = 0$. The right hand sides

of (93), (94), (96) and (100) are explicitly written in terms of ($n, M_P, M_{EM}, \mathbf{B}$), using

$$\mathbf{v} = \frac{M_P/n}{\sqrt{1 + \beta_*^2 (M_P/n)^2}}, \quad (101)$$

$$\mathbf{E} = \mathbf{B} \times M_{EM}/|\mathbf{B}|^2. \quad (102)$$

In this way, the RHMHD equations are numerically solvable.

On the other hand, in the case of RMHD which neglects the Hall effect $\varepsilon_H = 0$, Ohm's law (97) becomes $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ that does not include \mathbf{J} . Therefore, we are forced to eliminate \mathbf{J} by combining (94) and (100) as follows

$$\begin{aligned} \partial_t (n\boldsymbol{\gamma}\mathbf{v} + \sigma \mathbf{E} \times \mathbf{B}) = & -\nabla \cdot (n\boldsymbol{\gamma}\mathbf{v}\mathbf{v}) + \sigma \nabla \cdot \left(\frac{\mathbf{B}\mathbf{B}}{\beta_*^2} + \mathbf{E}\mathbf{E} \right) \\ & - \sigma \nabla \cdot \left(\frac{|\mathbf{B}|^2}{2\beta_*^2} + \frac{|\mathbf{E}|^2}{2} \right), \end{aligned} \quad (103)$$

where $M_{tot} := n\boldsymbol{\gamma}\mathbf{v} + \sigma \mathbf{E} \times \mathbf{B}$ is the total momentum of plasma and electromagnetic field. According to Ohm's law, \mathbf{E} is always replaced by $-\mathbf{v} \times \mathbf{B}$. Thus, the RMHD equations are a dynamical system of 7 fields (n, M_{tot}, \mathbf{B}) with a constraint $\nabla \cdot \mathbf{B} = 0$. However, to calculate the right hand sides of (93), (96) and (103), we need to write \mathbf{v} in terms of (n, M_{tot}, \mathbf{B}). It is well known that this is not analytically feasible and requires the use of a root-finding algorithm (such as the Newton-Raphson method). For hot plasma, the evolution equation for the total energy E_{tot} is also solved simultaneously, and the reconstruction of the primitive variables ($n, \mathbf{v}, p, \mathbf{B}$) from the time-evolving ones ($n, M_{tot}, E_{tot}, \mathbf{B}$) is one of the most computationally expensive part of RMHD simulation.

In the presence of the Hall effect, we can avoid using root-finding algorithm and the time-marching algorithm becomes straightforward while the number of field variables increases from 7 to 10. The RHMHD equations are possibly solved at a lower cost than the RMHD equations.

Finally, in the presence of the electron-inertia effect $\varepsilon_f^2 \neq 0$, Ohm's law is regarded as the evolution equation for $\boldsymbol{\gamma}\mathbf{J}$. The number of field variables is 13 under one constraint $\nabla \cdot \mathbf{B} = 0$, which is essentially the same as the original two-fluid equations. The numbers of field variables and constraints are summarized in Table II. As we have remarked before, it is more natural in XMHD (and IMHD) to solve \mathbf{A} instead of \mathbf{B} under the constraint $\nabla \cdot \mathbf{A} = 0$. Since cold plasma is assumed in this work for simplicity, one more field variable (such as pressure or temperature) would be added when temperature is not negligible.

VII. CONCLUSION

In this paper, we have investigated the applicability of various MHD models to special relativistic plasmas, using the method of dominant balance in the two-fluid equations. To simplify the formulation and consideration, we have assumed cold plasma (the limit of zero temperature and pressure) because electromagnetic force, not pressure, is a dominant force

Model	Field variables	Constraints
MHD	7 ($n, \mathbf{v}, \mathbf{B}$)	1 ($\nabla \cdot \mathbf{B} = 0$)
HMHD	7 ($n, \mathbf{v}, \mathbf{B}$)	1 ($\nabla \cdot \mathbf{B} = 0$)
IMHD	7 ($n, \mathbf{v}, \mathbf{B}$)	1 ($\nabla \cdot \mathbf{B} = 0$)
XMHD	7 ($n, \mathbf{v}, \mathbf{B}$)	1 ($\nabla \cdot \mathbf{B} = 0$)
RMHD	7 ($n, \mathbf{v}, \mathbf{B}$)	1 ($\nabla \cdot \mathbf{B} = 0$)
RHMHD	10 ($n, \mathbf{v}, \mathbf{B}, \mathbf{E}$)	2 ($\nabla \cdot \mathbf{B} = 0, \mathbf{E} \cdot \mathbf{B} = 0$)
W-RHMHD	10 ($n, \mathbf{v}, \mathbf{B}, \mathbf{E}$)	2 ($\nabla \cdot \mathbf{B} = 0, \mathbf{E} \cdot \mathbf{B} = 0$)
W-RIMHD	13 ($n, \mathbf{v}, \mathbf{J}, \mathbf{B}, \mathbf{E}$)	1 ($\nabla \cdot \mathbf{B} = 0$)
W-RXMHD	13 ($n, \mathbf{v}, \mathbf{J}, \mathbf{B}, \mathbf{E}$)	1 ($\nabla \cdot \mathbf{B} = 0$)

TABLE II: Number of field variables for cold plasma (H = Hall, I = Inertial, X = eXtended, R = Relativistic, W-R = Weakly-Relativistic)

in the MHD balance. Although there is no problem in including nondominant pressure effect, the case of relativistic pressure should be investigated as a future topic which might also break down the MHD balance (or the charge neutrality approximation). Similarly, externally-applied electric field is assumed to be absent because it is rarely dominant.

Under these assumptions, the relativistic two-fluid equations are nondimensionalized by eight representative scales, resulting in seven nondimensional parameters. For the electromagnetic force to be a dominant term, the six balances (1 to 6) are imposed as constraints among these parameters. Since the balances 3 and 4 are inequalities, the number of the nondimensional parameters is reduced to three ($\epsilon_m, \sigma, \beta_*$) satisfying an inequality $\beta_*^2/\sigma \leq 1$. The parameter $\epsilon_m = v_*/(L_*\omega_{c*})$ is smaller than 1 if we focus on the flow dynamics slower than the cyclotron frequency ω_{c*} . The RMHD equations are obtained in the limit $\epsilon_m \rightarrow 0$. By taking the mass ratio as an additional parameter, this parameter ϵ_m appears only through either $\epsilon_H = \tilde{\mu}\epsilon_m$ or $\epsilon_I^2 = \mu^2\epsilon_m^2$ in the two-fluid equations. We have shown that the approximation of proper charge neutrality can be justified by neglecting the order of $\epsilon_\sigma\epsilon_I^2$ where $\epsilon_\sigma = \min(\sigma, 1)$. When $\sigma \ll 1$, this approximation naturally corresponds to the quasi-neutrality condition of non-relativistic MHD. All the reduced models, or the generalized MHD models, are derived by neglecting $O(\epsilon_\sigma\epsilon_I^2)$ while allowing for the Hall effect $O(\epsilon_H)$, electron-inertia effect $O(\epsilon_I^2)$ and relativistic effects $O(\beta_*^2)$ and $O(\sigma)$. A special care is therefore needed when both the electron-inertia and relativistic effects are taken into account simultaneously, because their multiplication $O(\epsilon_\sigma\epsilon_I^2)$ is not negligible unless both σ and ϵ_I^2 are much smaller than 1. Only for the weakly relativistic case $\sigma \ll 1$, we can use the W-RXMHD and W-RIMHD models, where proper charge neutrality is still valid. If $\epsilon_\sigma\epsilon_I^2 \ll 1$ is not fulfilled, the two-fluid equations should be solved without any approximation.

The case of $\beta_*^2/\sigma \ll 1$ is often uninteresting because it is almost vacuum (i.e., the kinetic energy is much smaller than the energy of externally-applied magnetic field). On the other hand, the inequality $\beta_*^2/\sigma \leq 1$ indicates that the maximum velocity scale should be $\beta_* = \sqrt{\epsilon_\sigma}$ which is understood as

relativistic Alfvén ordering ($v_* \simeq v_{A*}$). Interesting MHD phenomena are expected in this velocity scale. By focusing on this velocity scale, the number of the nondimensional parameters is further reduced to two (ϵ, σ) which are related to the scales of number density n_* , magnetic field B_* and length L_* . We have illustrated the applicable ranges of the various MHD models in terms of these scales. For a low density case or in a small scale, it is shown that the charge neutrality condition is violated at an intermediate strength of magnetic field around $\sigma \sim 1$.

We have also summarized the number of field variables for each generalized MHD model. The RHMHD model is shown to be a dynamical system of 10 fields ($n, \mathbf{v}, \mathbf{B}, \mathbf{E}$) satisfying two constraints ($\nabla \cdot \mathbf{B} = 0$ and $\mathbf{E} \cdot \mathbf{B} = 0$). This number 10 is different from 7 of the other non-relativistic MHD models and 13 of the original two-fluid model. Moreover, the RHMHD equations describe the time marching of variables ($n, n\gamma\mathbf{v}, \mathbf{E} \times \mathbf{B}, \mathbf{B}$) and the primitive variables ($n, \mathbf{v}, \mathbf{B}, \mathbf{E}$) can be written explicitly by them. Since the RMHD equations requires the root-finding algorithm to calculate ($n, \mathbf{v}, \mathbf{B}$) from ($n, n\gamma\mathbf{v} + \sigma\mathbf{E} \times \mathbf{B}, \mathbf{B}$), the RHMHD model has an advantage in that the time-marching algorithm is simpler and less expensive than RMHD at the expense of increasing the field variables from 7 to 10. Naturally, RHMHD is a higher fidelity model than RMHD since it includes the Hall effect, which is known to be important for magnetic reconnection process in electron-ion plasma. The application of RHMHD is therefore expected to be beneficial both theoretically and numerically for analysing relativistic plasma phenomena.

ACKNOWLEDGMENTS

We thank Y. Kawazura and K. Toma for helpful discussion. This work was supported by JST SPRING, Grant Number JP-MJSP2114, a Scholarship of Tohoku University, Division for Interdisciplinary Advanced Research and Education, and IFS Graduate Student Overseas Presentation Award.

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