

A decomposition from a substitutable many-to-one matching market to a one-to-one matching market*

Pablo Neme[†] Jorge Oviedo[†]

November 4, 2024

Abstract

For a many-to-one market with substitutable preferences on the firm's side, based on the Aizerman-Malishevski decomposition, we define an associated one-to-one market. Given that the usual notion of stability for a one-to-one market does not fit well for this associated one-to-one market, we introduce a new notion of stability. This notion allows us to establish an isomorphism between the set of stable matchings in the many-to-one market and the matchings in the associated one-to-one market that meet this new stability criterion. Furthermore, we present an adaptation of the well-known deferred acceptance algorithm to compute a matching that satisfies this new notion of stability for the associated one-to-one market.

JEL classification: C78, D47.

Keywords: Many-to-one matchings, Aizermann-Malishevski decomposition, one-to-one matchings, deferred acceptance algorithm, stable set

1 Introduction

Many-to-one markets have been extensively studied in the literature, starting with the college admissions problem and extending to the assignment of medical interns to hospitals, as well as applications in labor markets. A fundamental characteristic of these markets is that institutions, which we will refer to as firms, hold preferences over subsets of agents, whom we will refer to as workers.

*We acknowledge the financial support of UNSL through grants 03-2016 and 03-1323, and from Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) through grant PIP 112-200801-00655, and from Agencia Nacional de Promoción Científica y Tecnológica through grant PICT 2017-2355.

[†]Instituto de Matemática Aplicada San Luis (UNSL-CONICET) and Departamento de Matemática, Universidad Nacional de San Luis, San Luis, Argentina. Emails: pabloneme08@gmail.com (P. Neme) and joviedo@unsl.edu.ar (J. Oviedo).

In this paper, we introduce a decomposition of a many-to-one market in which firms have *substitutable* preferences over subsets of workers, transforming it into an associated one-to-one market. We construct this decomposition based on the Aizerman-Malishevski decomposition of substitutable preferences (see [Aizerman and Malishevski, 1981](#)). [Crawford and Kelso \(1982\)](#) introduced the notion of substitutability in preferences, which is the weakest condition needed to guarantee the existence of stable matchings. A firm is said to have substitutable preferences if it continues to desire a worker even when other workers become unavailable.

The Aizerman-Malishevski decomposition is well known in the choice literature (e.g., [Moulin, 1985](#)) and states that any path-independent choice rule can be represented as the union of choices derived from preference relations over individuals. [Chambers and Yenmez \(2017\)](#) apply the Aizerman-Malishevski decomposition to study path-independent choice rules in a matching context. They use this decomposition to develop a deferred acceptance algorithm for many-to-many matching markets with contracts and to analyze its properties. The Aizerman-Malishevski decomposition models a firm as the union of several “copies” of itself, facilitating the representation of firms’ preferences in substitutable cases. A key feature of this decomposition is its capacity to transform a substitutable preference over a subset of workers into multiple, distinct linear preferences over workers. The fact that copies of the same firm have different linear preferences leads to a situation where the usual notion of stability for one-to-one markets does not apply adequately.¹ In [Section 2](#), we present [Example 1](#), which illustrates a many-to-one market where firms have substitutable preferences, along with its corresponding Aizerman-Malishevski decomposition. In this example, we observe that in the many-to-one market, there are four stable matchings, while in the associated one-to-one market, there is only one stable matching under the usual notion of stability. This discrepancy implies that no relation (e.g., no isomorphism) can be established between the stable matchings in the many-to-one market and those in the associated one-to-one market. For this reason, we adapt the notion of stability and introduce *stability** for one-to-one matchings. This new notion resembles the classical notion of stability (i.e., it is individually rational and free of blocking pairs) but is adapted to capture the substitutable preferences inherent in the original market. Building on this new notion, we can establish an isomorphism between the set of stable matchings in the substitutable many-to-one market and the set of *stable** matchings in the associated one-to-one market.

To highlight the difference between the usual notion and *stability**, we introduce two additional conditions not typically required in one-to-one settings. First, for individual rationality* in this associated market, we impose an envy-free condition among firm-copies: no firm-copy matched to a worker should prefer another worker matched to a different copy

¹Recall that, in traditional one-to-one matching markets, stability requires *individual rationality* and the absence of *blocking pairs*. A matching is individually rational if each agent is assigned to an “acceptable” partner. A matching contains a blocking pair if both agents within that pair mutually prefer each other over their current partners in the matching.

of the same firm. Second, in the presence of blocking pairs, we require that the worker in the blocking pair is preferred by the relevant firm-copy over all workers matched to other copies of that same firm.

A common approach to prove the non-emptiness of a stable set is constructive. In the seminal paper by [Gale and Shapley \(1962\)](#), an algorithm is presented—the well-known deferred acceptance algorithm—which constructs a stable matching for traditional one-to-one markets. Although our isomorphism demonstrates that the set of stable* one-to-one matchings is non-empty, we adapt the deferred acceptance algorithm to the related one-to-one market, taking into account the specific characteristics of stability*. Although one-to-one markets are traditionally symmetric, our related one-to-one market is not. This is because it originates from the decomposition of a many-to-one substitutable market. Thus, we present two adapted versions of the deferred acceptance algorithm: one where firm-copies propose and another where workers propose. We show that, whether the firm-copies or the workers are the proposers, the algorithm returns a stable* matching. In this way, independently of the isomorphism, we establish that the set of stable* matchings for the related one-to-one market is non-empty.

The idea of decomposing a many-to-one market into a one-to-one market has been previously studied under a more restrictive preference structure. When firms' preferences are assumed to be *responsive* to an individual ranking of workers—a more restrictive structure than substitutable preferences—[Roth and Sotomayor \(1990\)](#) demonstrate that each firm can be decomposed into identical units (or copies) according to its capacity (quotas q). A key distinction in this decomposition is that each of these copies shares the same individual preferences over workers (derived from responsive preferences), and workers rank all copies of a given firm above those of any other firm, preserving the same order of preferences across all copies. This decomposition transforms a many-to-one market with responsive preferences into a corresponding one-to-one market. Furthermore, [Roth and Sotomayor \(1990\)](#) establish an isomorphism between the stable matchings of a many-to-one market and those of an associated one-to-one market.

The paper is organized as follows. In Section 2, we present the many-to-one market, the Aizerman-Malishevski decomposition, and the associated one-to-one market. For this associated one-to-one market, in Section 3, we present an adapted deferred acceptance algorithm and an isomorphism with the many-to-one market. Finally, in Section 4 some final remarks are presented.

2 Preliminaries

This section contains two subsections. In Subsection 2.1, we present the many-to-one market with substitutable preferences, along with its respective notions of matching, individual rationality, and stability. In Subsection 2.2, we introduce the Aizerman-Malishevski decomposition of substitutable preferences into linear orders, and we show a decomposition of

the many-to-one market into an associated one-to-one market with its respective notions of matching, individual rationality*, and stability*.

2.1 Many-to-one matching market

Let $\mathcal{M} = (\Phi, \mathcal{W}, P)$ be a **many-to-one matching market**, where $\Phi = \{\varphi_1, \dots, \varphi_n\}$ denotes the set of n firms and $\mathcal{W} = \{w_1, \dots, w_k\}$ the set of k workers. Each firm $\varphi \in \Phi$ has a strict, transitive, and complete preference relation P_φ over $2^{\mathcal{W}}$, and each worker $w \in \mathcal{W}$ has a strict, transitive, and complete preference relation P_w over $\Phi \cup \{\emptyset\}$. Let R_w denote the weak preference relation associated with P_w , meaning that if $\varphi_i R_w \varphi_j$ imply that either $\varphi_i P_w \varphi_j$ or $\varphi_i = \varphi_j$. The collection of these preference relations forms the **preference profile** $\mathbf{P} = ((P_\varphi)_{\varphi \in \Phi}, (P_w)_{w \in \mathcal{W}})$. Given a subset of workers $W \subseteq \mathcal{W}$, let $C_\varphi(W)$ denote firm φ 's choice function induced by preference relation P_φ ; that is, $C_\varphi(W) = \{W' \subseteq \mathcal{W} : W' = \max_{W'' \subseteq \mathcal{W}} P_\varphi\}$. Firm φ 's preferences are **substitutable** if for each $W \subseteq \mathcal{W}$ we have that $w \in C_\varphi(W)$ implies that $w \in C_\varphi(W \setminus \{w'\})$ for each $w' \in W \setminus \{w\}$. Note that, if preference P_φ is substitutable, then the induced choice function $C_\varphi(\cdot)$ is not only substitutable but also satisfies **consistency**. We say that a choice function $C_\varphi(\cdot)$ satisfies consistency if $C_\varphi(X) \subseteq Y \subseteq X \implies C_\varphi(X) = C_\varphi(Y)$. Moreover, if the choice function $C_\varphi(\cdot)$ satisfies substitutability and consistency, then it also satisfies **path-independence** (see [Alkan, 2002](#)). Path-independence property states that given a firm φ and two subsets W and W' of workers,

$$C_\varphi(W \cup W') = C_\varphi(C_\varphi(W) \cup W'). \quad (1)$$

Throughout the paper, we assume that firms in the many-to-one matching market have substitutable preferences.

A many-to-one **matching** is a mapping $\mu : \Phi \cup \mathcal{W} \rightarrow 2^{\Phi \cup \mathcal{W}}$ that satisfies the following conditions: (i) $\mu(w) \in \Phi \cup \emptyset$ for each $w \in \mathcal{W}$, (ii) $\mu(\varphi) \in 2^{\mathcal{W}}$ for each $\varphi \in \Phi$, and (iii) $\mu(w) = \varphi$ if and only if $w \in \mu(\varphi)$. A matching μ is **blocked** by a worker w if $\emptyset P_w \mu(w)$. Similarly, μ is blocked by a firm φ if $\mu(\varphi) \neq C_\varphi(\mu(\varphi))$. We say that a matching is **individually rational** if it is not blocked by any individual agent. A matching μ is blocked by a worker-firm pair (w, φ) if $w \notin \mu(\varphi)$, $w \in C_\varphi(\mu(\varphi) \cup \{w\})$, and $\varphi P_w \mu(w)$. Finally, we say that a matching μ is **stable** if it is not blocked by any individual agent or by any worker-firm pair. We denote by $S(\mathcal{M})$ to the **set of all stable matchings** for the matching market \mathcal{M} .

2.2 An associated one-to-one market

In this subsection, we present the *Aizerman-Malishevski decomposition* (see [Chambers and Yenmez, 2017](#)). This decomposition allows us to establish a connection between a many-to-one market with substitutable preferences and a one-to-one market. It captures the essence of the substitutability condition on firms' preferences by splitting each firm into multiple copies and making that each copy of a firm behaves independently of the others, meaning that the copies have different linear preferences over workers.

Aizermann-Malishevski decomposition: If P_φ is a substitutable preference relation on $2^{\mathcal{W}}$, then there is a finite sequence of preference relations $\{P_{\varphi_j}\}_{j \in J_\varphi}$ on \mathcal{W} such that, for all $S \subseteq \mathcal{W}$:

$$C_\varphi(S) = \bigcup_{j \in J_\varphi} \max_S P_{\varphi_j}$$

Next, using the Aizerman-Malishevski decomposition, we introduce a **related one-to-one market**, denoted by M , and provide a notion of stability suitable for this newly associated market. This decomposition allows us to establish a link with the many-to-one market with substitutable preferences. Let

$$\mathcal{F} = \{f_{11}, \dots, f_{1J_1}, \dots, f_{n1}, \dots, f_{nJ_n}\}$$

be the set of firm-copies, where f_{ij} denotes the firm φ_i 's j -th copy. These copies are taken from the Aizermann-Malishevski decomposition, so that $C_{\varphi_i}(S) = \bigcup_{j \in J_i} \max_S P_{ij}$.² The set of workers is $\mathcal{W} = \{w_1, \dots, w_k\}$. Now, each worker w has a strict, transitive, and complete preference relation \bar{P}_w on the set $\mathcal{F} \cup \{\emptyset\}$ such that:

- (1) $\varphi_i P_w \varphi_{i'}$ implies that $f_{ij} \bar{P}_w f_{i'j'}$ for each $j \in J_i$ and each $j' \in J_{i'}$.
- (2) $f_{ij} \bar{P}_w f_{i'j'}$ if and only if $j < j'$, for each $i \in I = \{1, \dots, n\}$ and each $j, j' \in J_i$.

Condition (1) ensures that workers' preferences (\bar{P}_w) over firm-copies reflect exactly the preference relation (P_w) over firms. In turn, condition (2) ensures that \bar{P}_w is a linear order. We denote the associated one-to-one matching market as $M = (\mathcal{F}, \mathcal{W}, (\bar{P}_w)_{w \in \mathcal{W}}, (P_{ij})_{i \in I, j \in J_i})$.

The following example illustrates how we can decompose a many-to-one market with substitutable preferences for firms into an associated one-to-one market using the Aizermann-Malishevski decomposition.

Example 1 Consider a many-to-one matching market (Φ, \mathcal{W}, P) , where $\Phi = \{\varphi_1, \varphi_2\}$, $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$, and the preference profile is presented in the following table:³

$$\begin{array}{ll} P_{\varphi_1} : w_1 w_2, w_1 w_3, w_2 w_4, w_3 w_4, w_1, w_2, w_3, w_4 & P_{w_1} = P_{w_2} : \varphi_2, \varphi_1 \\ P_{\varphi_2} : w_3 w_4, w_1 w_3, w_2 w_4, w_1 w_2, w_4, w_3, w_2, w_1 & P_{w_3} = P_{w_4} : \varphi_1, \varphi_2 \end{array}$$

²Henceforth, to easy notation, we write P_{ij} instead of $P_{\varphi_{ij}}$.

³To ease notation, in the firms' preference over $2^{\mathcal{W}}$, we enlist only the acceptable subsets and we omit the curly brackets, i.e. we write $P_\varphi : w_1 w_2, w_1, w_2$ instead of writing $P_\varphi : \{w_1 w_2\}, \{w_1\}, \{w_2\}, \emptyset$. For the workers' preferences, we also enlist only acceptable firms.

The associated Aizermann-Malishevski decomposition of the substitutable preferences into linear preferences is presented in the following table:

$P_{11} : w_1, w_2, w_3, w_4$	$P_{21} : w_3, w_4, w_2, w_1$
$P_{12} : w_1, w_2, w_4, w_3$	$P_{22} : w_3, w_4, w_1, w_2$
$P_{13} : w_1, w_4, w_2, w_3$	$P_{23} : w_3, w_2, w_4, w_1$
$P_{14} : w_2, w_1, w_3, w_4$	$P_{24} : w_4, w_3, w_2, w_1$
$P_{15} : w_2, w_1, w_4, w_3$	$P_{25} : w_4, w_3, w_1, w_2$
$P_{16} : w_2, w_3, w_1, w_4$	$P_{26} : w_4, w_1, w_3, w_2$

Now, the preferences of workers over firm-copies are presented in the following table:

$$\begin{aligned}\bar{P}_{w_1} = \bar{P}_{w_2} &: f_{21}, f_{22}, f_{23}, f_{24}, f_{25}, f_{26}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16} \\ \bar{P}_{w_3} = \bar{P}_{w_4} &: f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16}, f_{21}, f_{22}, f_{23}, f_{24}, f_{25}, f_{26}\end{aligned}$$

Thus, the one-to-one associated market is denoted by $(\mathcal{F}, \mathcal{W}, P_{\mathcal{F}}, \bar{P}_{\mathcal{W}})$, where

$$\mathcal{F} = \{f_{11}, \dots, f_{16}, f_{21}, \dots, f_{26}\}$$

and $P_{\mathcal{F}}$ is the preference profile derived of the Aizermann-Malishevski decomposition. \diamond

A matching in the associated one-to-one matching market is a mapping $\lambda : \mathcal{F} \cup \mathcal{W} \rightarrow \mathcal{F} \cup \mathcal{W} \cup \{\emptyset\}$ such that $\lambda(f) \in \mathcal{W} \cup \{\emptyset\}$, $\lambda(w) \in \mathcal{F} \cup \{\emptyset\}$ and $\lambda(f) = w \iff \lambda(w) = f$.

Given that one of our goals in this paper is to show an isomorphism between the set of stable matchings in the many-to-one substitutable market and a one-to-one market, it is necessary to adapt the notions of individual rationality and stability for the associated one-to-one market. The following example illustrates this need:

Example 1 (continued) *The stable matchings of the many-to-one substitutable market are:*

$$\begin{aligned}\mu_{\Phi} &= \begin{pmatrix} \varphi_1 & \varphi_2 \\ w_1 w_2 & w_3 w_4 \end{pmatrix} \mu_1 = \begin{pmatrix} \varphi_1 & \varphi_2 \\ w_1 w_3 & w_2 w_4 \end{pmatrix} \\ \mu_2 &= \begin{pmatrix} \varphi_1 & \varphi_2 \\ w_2 w_4 & w_1 w_3 \end{pmatrix} \mu_{\mathcal{W}} = \begin{pmatrix} \varphi_1 & \varphi_2 \\ w_3 w_4 & w_1 w_2 \end{pmatrix}\end{aligned}$$

Let λ be a matching in the related one-to-one market. Consider the standard notion of individual rationality, i.e., $\lambda(f_{ij})R_{ij}\emptyset$ and $\lambda(w)\bar{R}_w\emptyset$, and the standard notion of stability, i.e., there is no blocking pair (f_{ij}, w) such that $w \neq \lambda(f_{ij})$, $wP_{ij}\lambda(f_{ij})$, and $f_{ij}\bar{P}_w\lambda(w)$.

We can observe that the only one-to-one matching in the related market that satisfies this notion of individual rationality and that has no blocking pairs with respect to this blocking notion is the following:

$$\lambda = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_3 & w_4 & \emptyset & \emptyset & \emptyset & \emptyset & w_2 & w_1 & \emptyset & \emptyset & \emptyset & \emptyset \end{pmatrix}.$$

Note that if we consider any one-to-one matching that assigns workers w_1 and w_2 with any firm-copy $f_{1,\cdot}$, and workers w_3 and w_4 with any firm-copy $f_{2,\cdot}$, it will have blocking pairs: any unmatched firm-copy $f_{1,\cdot}$ with workers w_3 or w_4 , and similarly, any unmatched firm-copy $f_{2,\cdot}$ with workers w_1 or w_2 . This situation clearly implies that, with these notions of individual rationality and stability, it is not possible to define an isomorphism between a set with four elements (the set of stable matchings in the many-to-one substitutable market) and a set with only one element (the set of stable matchings under these notions). \diamond

To conclude this section, we present the necessary definitions to establish a proper notion of “individual rationality” and “stability”, in order to show in the following section that the sets of “stable matchings” in both markets are isomorphic.

Definition 1 A matching λ is **blocked*** by a worker w if $\emptyset \bar{P}_w \lambda(w)$. On the other hand, a matching λ is **blocked*** by a firm-copy f_{ij} if either

- (1) $\emptyset P_{ij} \lambda(f_{ij})$, or
- (2) $\lambda(f_{ij}) \neq \emptyset$ and there is $j' \in J_i$ such that $\lambda(f_{ij'}) P_{ij} \lambda(f_{ij})$.

Note that the absence of blocking* firm-copies implies the negation of condition (2). So, the negation of this condition can be interpreted as an envy-free-like property among the matched copies of a given firm, i.e., no copy that is matched wants to switch partners with another copy. Thus, a one-to-one matching that is not blocked* by any agent we say that it is **Individually rational***.

Definition 2 A matching λ is said to be **blocked*** by a firm-copy and worker pair (f_{ij}, w) if

1. $f_{ij} \bar{P}_w \lambda(w)$; and
2. $w P_{ij} \lambda(f_{ij'})$ for each $j' \in J_i$.

Now, we are in a position to formally define the notion of stability* for the associated one-to-one market.

Definition 3 A matching λ is **stable*** if it is not blocked by any individual agent or any (firm-copy)-worker pair.

Let $S^*(M)$ denote the **set of stable* matchings** of the associated one-to-one matching market.

Note that it is important to highlight that condition (2) in Definition 2 is stronger than the usual notion of pairwise block, since it has to account for the fact that some other copy of the same firm may be matched to a better worker. This is particularly relevant in establishing the equivalence (in an isomorphic sense) between the set of stable matchings in the many-to-one matching market and the set of stable* matchings in the associated one-to-one market.

In the following remark, we summarize the possible scenarios in which a matching in the associated one-to-one market is not stable*.

Remark 1 If $\lambda \notin S^*(M)$, then (at least) one of the following must hold:

- (i) There is $w \in \mathcal{W}$ such that $\emptyset \bar{P}_w \lambda(w)$.
- (ii) There is $f_{ij} \in \mathcal{F}$ such that $\emptyset P_{ij} \lambda(f_{ij})$.
- (iii) There are $f_{ij}, f_{ij'} \in \mathcal{F}$ such that $\lambda(f_{ij'}) P_{ij} \lambda(f_{ij}) P_{ij} \emptyset$.
- (iv) There is a pair $(f_{ij}, w) \in \mathcal{F} \times \mathcal{W}$ such that $f_{ij} \bar{P}_w \lambda(w)$ and $w P_{ij} \lambda(f_{ij'})$ for each j' .

3 Results on the set of stable* matchings

In this section, we present the main results of the paper. In Subsection 3.1, we adapt the well-known deferred acceptance algorithm, initially introduced in the seminal work of Gale and Shapley (1962), to our one-to-one market framework and the notion of stable*. We show that in both cases, whether firms-copies propose or workers propose, the output of the adapted deferred acceptance algorithm is a stable* matching. Thus, we demonstrate that the set of stable* matchings is non-empty.

In Subsection 3.2, we establish an isomorphism between the set of stable matchings in a many-to-one market with substitutable preferences and the set of stable* matchings in the associated one-to-one market.

3.1 Adapted deferred acceptance algorithm

In this subsection, we introduce two variants of the deferred acceptance algorithm that compute a Stable* matching: one in which firm-copies propose and the other in which workers propose. These adaptations take into account the specific characteristics of the associated one-to-one market.

The algorithm presented in Table 1 is the adapted firm-copies-proposing deferred acceptance algorithm for the market $(P_{\mathcal{F}}, \bar{P}_{\mathcal{W}})$. While this algorithm follows the same principle as the one introduced by Gale and Shapley (1962), our adaptation includes an important feature: a firm-copy can only make an offer to a worker if no other copy of the same firm is already matched with a more preferred worker. This restriction is essential to prevent envy between firm-copies. Let O_w^k be the set of firms making a job offer to worker w at stage k .

The algorithm will eventually stop in a finite number of stages, as firms begin by making offers to the top of their preference lists and continue to make offers (if authorized) to less preferred workers as their previous offers are rejected. Given that the set of firms and workers is finite, there will eventually be no more rejections, leading to the cessation of the algorithm. Before presenting that the output of the algorithm is a stable* matching, we illustrate the procedure for Example 1.

Algorithm:

Input A preference profile $(\bar{P}_{\mathcal{W}}, P_{\mathcal{F}})$.

Output A matching $\lambda_{\mathcal{F}}$.

DEFINE: $\lambda^0(w) = \emptyset$ and $\lambda^0(f) = \emptyset$ for each $w \in \mathcal{W}$ and each $f \in \mathcal{F}$.

Stage 1 (a) Each firm-copy f makes a job offer to the best acceptable worker at P_f .

(b) Each worker w selects, with respect to \bar{P}_w , the best firm-copy among those in $O_w^1 \cup \lambda^0(w)$, say \tilde{f} .

Set $R_w^1 = O_w^1 \setminus \{\tilde{f}\}$ for each $w \in \mathcal{W}$. Define

$$\lambda^1(f_{ij}) = \begin{cases} \emptyset & \text{if } f_{ij} \in \cup_{w \in \mathcal{W}} R_w^1, \\ w & \text{if } f_{ij} = \tilde{f}. \end{cases}$$

Stage k (a) Each firm-copy $f_{ij} \in \cup_{\tilde{w} \in \mathcal{W}} R_{\tilde{w}}^{k-1}$ is *authorized* to make an offer to the best acceptable worker w with respect to $P_{f_{ij}}$ who has not previously rejected it, if there are no $f_{ij'}$ and $w' \neq w$ such that $\lambda^{k-1}(w') = f_{ij'}$ and $w' P_{f_{ij}} w$. Otherwise, f_{ij} is *unauthorized* to make an offer at this stage.

(b) Each worker w selects, with respect to \bar{P}_w , the best firm-copy among those in $O_w^k \cup \lambda^{k-1}(w)$, say \tilde{f} .

Set $R_w^k = O_w^k \cup \lambda^{k-1}(w) \setminus \{\tilde{f}\}$ for each $w \in \mathcal{W}$. Define

$$\lambda^k(f_{ij}) = \begin{cases} \emptyset & \text{if } f_{ij} \in \cup_{w \in \mathcal{W}} R_w^k, \\ \emptyset & \text{if } f_{ij} \text{ is unauthorized to make an offer,} \\ w & \text{if } f_{ij} = \tilde{f}, \text{ and} \\ \lambda^{k-1}(f_{ij}) & \text{otherwise.} \end{cases}$$

If $\cup_{w \in \mathcal{W}} R_w^k = \emptyset$:

STOP and let $\lambda_{\mathcal{F}} = \lambda^k$.

ELSE:

CONTINUE TO STAGE $k + 1$

Table 1: Adapted Firm-copies-proposal Deferred Acceptance algorithm

Example 1 (continued) Considering the associated one-to-one market previously presented, we applied the adapted firm-copies-proposing deferred acceptance algorithm as follows:

Stage 1: Firm-copies offers are presented in the following table:

f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}	f_{21}	f_{22}	f_{23}	f_{24}	f_{25}	f_{26}
w_1	w_1	w_1	w_2	w_2	w_2	w_3	w_3	w_3	w_4	w_4	w_4

So, the sets of offers that workers receive in Stage 1 are:

$$\begin{aligned} O_{w_1}^1 &= \{f_{11}, f_{12}, f_{13}\} \\ O_{w_2}^1 &= \{f_{13}, f_{14}, f_{15}\} \\ O_{w_3}^1 &= \{f_{21}, f_{22}, f_{23}\} \\ O_{w_4}^1 &= \{f_{24}, f_{25}, f_{26}\} \end{aligned}$$

Workers, according to their preferences \bar{P} , accept an offer and the resulting rejection sets are:

$$\begin{aligned} R_{w_1}^1 &= \{f_{12}, f_{13}\} \\ R_{w_2}^1 &= \{f_{15}, f_{16}\} \\ R_{w_3}^1 &= \{f_{22}, f_{23}\} \\ R_{w_4}^1 &= \{f_{25}, f_{26}\} \end{aligned}$$

Thus,

$$\lambda^1 = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_1 & \emptyset & \emptyset & w_2 & \emptyset & \emptyset & w_3 & \emptyset & \emptyset & w_4 & \emptyset & \emptyset \end{pmatrix}.$$

Stage 2 (and henceforth): Each firm-copy $f_{ij} \in \bigcup_{w \in \mathcal{W}} R_w^1$ is no longer authorized to make an offer to any $w \in \mathcal{W}$. To see this consider, for instance, $f_{12} \in R_{w_1}^1$. We have that $w_1 = \lambda(f_{11})P_{12}w_k$, where $k \in \{2, 4\}$. Therefore, the output of the adapted firm-copies-proposing deferred acceptance algorithm is:

$$\lambda_{\mathcal{F}} = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_1 & \emptyset & \emptyset & w_2 & \emptyset & \emptyset & w_3 & \emptyset & \emptyset & w_4 & \emptyset & \emptyset \end{pmatrix}.$$

◇

The following theorem proves the output of the adapted firm-copies-proposing deferred acceptance algorithm is a stable* matching.

Theorem 1 *Let $\lambda_{\mathcal{F}}$ be the output of the firm-copies-proposing deferred acceptance algorithm. Then, $\lambda_{\mathcal{F}}$ is a stable* matching.*

Proof. Assume that $\lambda_{\mathcal{F}} \notin S^*(M)$. Then, by Remark 1 there are four possible situations:

- (i) **There is $w \in \mathcal{W}$ such that $\emptyset \bar{P}_w \lambda_{\mathcal{F}}(w)$.** Given that at each stage k , w selects \tilde{f} as the best firm-copy among those in $O_w^k \cup \lambda^{k-1}(w)$ with respect to \bar{P}_w , and in stage 1 $\lambda^0(w) = \emptyset$, we have that $\lambda_{\mathcal{F}}(w) \bar{R}_w \emptyset$. Then, this situation does not occur.
- (ii) **There is $f_{ij} \in \mathcal{F}$ such that $\emptyset P_{ij} \lambda_{\mathcal{F}}(f_{ij})$.** Given that at each stage f_{ij} makes an offer, if it is authorized, to an acceptable worker with respect to P_{ij} , we have that $\lambda_{\mathcal{F}}(f_{ij}) R_{ij} \emptyset$. Then, this situation does not occur.
- (iii) **There are $f_{ij}, f_{i'j'} \in \mathcal{F}$ such that $\lambda_{\mathcal{F}}(f_{i'j'}) P_{ij} \lambda_{\mathcal{F}}(f_{ij}) P_{ij} \emptyset$.** Assume that there are $w, w' \in \mathcal{W}$ such that $w = \lambda_{\mathcal{F}}(f_{ij})$ and $w' = \lambda_{\mathcal{F}}(f_{i'j'})$. Assume also that there are two stages t, t'

such that t' is the stage where $f_{ij'}$ is assigned to w' ($\lambda^{t'}(f_{ij'}) = w'$ and $\lambda^{t'-1}(f_{ij'}) \neq w'$) and such that t is the stage where f_{ij} is assigned to w ($\lambda^t(f_{ij}) = w$ and $\lambda^{t-1}(f_{ij}) \neq w$). We have to consider two subcases.

$t' < t$: Since $w' P_{ij} w$, we have that f_{ij} is unauthorized to make an offer to w , so it is not possible that $\lambda_{\mathcal{F}}(f_{ij}) = w$.

$t' \geq t$: Note that since w is matched with f_{ij} before than w' and the fact that $w' P_{ij} w$ implies that w' reject f_{ij} in an earlier step than t , say \tilde{t} . In this case, we claim that there is a firm-copy $f_{\tilde{i}k}$ that is not a copy of f_{ij} such that $w' = \lambda^{\tilde{t}}(f_{\tilde{i}k})$ and $f_{\tilde{i}k} \bar{P}_{w'} f_{i,j}$. Since w' reject f_{ij} at stage \tilde{t} , such a firm-copy exists that is temporarily matched to w' at stage \tilde{t} , and w' prefers this firm-copy over f_{ij} . Moreover, since $\tilde{t} \leq t$, we have that $f_{\tilde{i}k}$ is not a copy of f_{ij} , otherwise f_{ij} will not be authorized to make an offer to w . Thus, in this case, the claim holds. Note that this contradicts the definition of workers' preferences \bar{P} item (1), since $f_{ij'} \bar{P}_{w'} f_{\tilde{i}k} \bar{P}_{w'} f_{i,j}$.

Then, by the two cases analyzed, this situation does not occur.

- (iv) **There is a pair $(f_{ij}, w) \in \mathcal{F} \times \mathcal{W}$ such that $f_{ij} \bar{P}_w \lambda_{\mathcal{F}}(w)$ and $w P_{ij} \lambda_{\mathcal{F}}(f_{ij'})$ for each $j' \in J_i$.** If $w P_{ij} \lambda_{\mathcal{F}}(f_{ij'})$ for each $j' \in J_i$, then each $f_{ij'}$ has made an offer to w in a previous stage and it was rejected because w was temporarily matched with a better firm-copy that is not a copy of $f_{ij'}$, otherwise $f_{ij'}$ is unauthorized to make an offer. Then, $\lambda_{\mathcal{F}}(w) \bar{P}_w f_{ij'}$ for each $j' \in J_i$. Then, this situation does not occur.

Therefore, since none of the four cases hold, we have that $\lambda_{\mathcal{F}} \in S^*(M)$.

□

The algorithm presented in Table 2 is the adapted worker-proposing deferred acceptance algorithm for the market $(\bar{P}_{\mathcal{W}}, P_{\mathcal{F}})$. This adaptation also ensures that the output is envy-free among firm-copies. We say that a worker makes a *valid* offer to a firm-copy if the acceptance of the worker's proposal by that firm-copy does not generate envy among the other copies.

Let O_f^k be the set of workers making a job offer to firm f at stage k .

Before proving that the output of the algorithm is a stable* matching, we first illustrate the procedure with Example 1.

Example 1 (continued) Considering the associated one-to-one market previously presented, we applied the adapted worker-proposing deferred acceptance algorithm as follows:

Stage 1: The workers' offers are presented in the following table:

w_1	w_2	w_3	w_4
f_{21}	f_{21}	f_{11}	f_{11}

So, the sets of offers that firm-copies receive in stage 1 are: $O_{f_{1,1}}^1 = \{w_3, w_4\}$ and $O_{f_{21}}^1 = \{w_1, w_2\}$. Firm-copies, according to their preferences P , accept the offers, and the resulting rejection sets are

Algorithm:

Input A preference profile $(\bar{P}_{\mathcal{W}}, P_{\mathcal{F}})$.

Output A matching $\lambda_{\mathcal{W}}$.

DEFINE: $\lambda^0(w) = \emptyset$ and $\lambda^0(f_{ij}) = \emptyset$ for each $w \in \mathcal{W}$ and each $f_{ij} \in \mathcal{F}$.

Stage 1 (a) Each worker w makes an offer to the best acceptable firm-copy at \bar{P}_w .

(b) Each firm-copy f_{ij} selects, with respect to $P_{f_{ij}}$, the best worker among those in $O_{f_{ij}}^1 \cup \lambda^0(f_{ij})$, say $\tilde{w}_{f_{ij}}$.

Set $R_{f_{ij}}^1 = O_{f_{ij}}^1 \setminus \{\tilde{w}_{f_{ij}}\}$ for each $f_{ij} \in \mathcal{F}$. Define

$$\lambda^1(w) = \begin{cases} \emptyset & \text{if } w \in \cup_{f_{ij} \in \mathcal{F}} R_{f_{ij}}^1, \\ f_{ij} & \text{if } w = \tilde{w}_{f_{ij}}. \end{cases}$$

Stage k (a) Each worker $w \in \cup_{f_{ij} \in \mathcal{F}} R_{f_{ij}}^{k-1}$ make an new offer to the best

acceptable firm-copy f_{ij} with respect to \bar{P}_w who has not previously rejected it.

(b) For each firm-copy f_{ij} , define the set of "Valid Offers"

$$\tilde{O}_{f_{ij}}^k = O_{f_{ij}}^k \setminus \{w \in O_{f_{ij}}^k : \exists w' \in \mathcal{W} \text{ with } w' = \lambda^{k-1}(f_{ij'}) P_{f_{ij'}} w \text{ for some } j' \in J_i\}.$$

Each firm f_{ij} selects the best worker, with respect to $P_{f_{ij}}$

among those in $\tilde{O}_{f_{ij}}^k \cup \lambda^{k-1}(f_{ij})$, say \tilde{w}^{ij} .

Set $R_{f_{ij}}^k = O_{f_{ij}}^k \cup \lambda^{k-1}(f_{ij}) \setminus \{\tilde{w}^{ij}\}$ for each $f_{ij} \in \mathcal{F}$. Define

$$\lambda^k(w) = \begin{cases} \emptyset & \text{if } w \in \cup_{f_{ij} \in \mathcal{F}} R_{f_{ij}}^k, \\ f_{ij} & \text{if } w = \tilde{w}^{ij}, \text{ and} \\ \lambda^{k-1}(w) & \text{otherwise.} \end{cases}$$

If $\cup_{f_{ij} \in \mathcal{F}} R_{f_{ij}}^k = \emptyset$:

STOP and let $\lambda_{\mathcal{W}} = \lambda^k$.

ELSE:

CONTINUE TO STAGE $k + 1$

Table 2: Adapted Worker-proposal Deferred Acceptance algorithm

$R_{f_{11}}^1 = w_4$ and $R_{f_{21}}^1 = w_2$. Thus,

$$\lambda^1 = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_3 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & w_2 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \end{pmatrix}.$$

Stage 2: Each worker $w_2, w_4 \in R_{f_{11}}^1 \cup R_{f_{21}}^1$ submits a new offer. Worker w_2 offers to firm-copy f_{22} , while w_4 offers to firm-copy $f_{1,2}$. Since both offers are valid, and firm-copies f_{12} and f_{22} accept them

without generating any further rejections, the algorithm stops. Thus,

$$\lambda_{\mathcal{W}} = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_3 & w_4 & \emptyset & \emptyset & \emptyset & \emptyset & w_2 & w_1 & \emptyset & \emptyset & \emptyset & \emptyset \end{pmatrix}.$$

◇

The following theorem shows that the output of the adapted worker-proposing deferred acceptance algorithm is a stable* matching. The proof follows the same spirit as the proof of Theorem 1, adapted for the case where workers are the ones making the offers.

Theorem 2 *Let $\lambda_{\mathcal{W}}$ be the output of the worker-proposing deferred acceptance algorithm. Then, $\lambda_{\mathcal{W}}$ is a stable* matching.*

Proof. Assume that $\lambda_{\mathcal{W}} \notin S^*(M)$. Then, by Remark 1 there are four possible situations:

- (i) **There is $f_{ij} \in \mathcal{F}$ such that $\emptyset P_{ij} \lambda_{\mathcal{W}}(f_{ij})$.** Given that at each stage k , f_{ij} selects \tilde{w}^{ij} as the best worker among those in $\tilde{O}_{f_{ij}}^k \cup \lambda^{k-1}(w)$ with respect to P_{ij} , and in stage 1 $\lambda^0(f_{ij}) = \emptyset$, we have that $\lambda_{\mathcal{W}}(f_{ij}) R_{ij} \emptyset$. Then, this situation does not occur.
- (ii) **There is $w \in \mathcal{W}$ such that $\emptyset \bar{P}_w \lambda(w)$.** Given that at each stage w makes an offer, to an acceptable firm with respect to \bar{P}_w , we have that $\lambda(w)_{\mathcal{W}} \bar{R}_w \emptyset$. Then, this situation does not occur.
- (iii) **There are $f_{ij}, f_{ij'} \in \mathcal{F}$ such that $\lambda_{\mathcal{W}}(f_{ij'}) P_{ij} \lambda_{\mathcal{W}}(f_{ij}) P_{ij} \emptyset$.** Assume that there are $w, w' \in \mathcal{W}$ such that $w = \lambda_{\mathcal{W}}(f_{ij})$ and $w' = \lambda_{\mathcal{W}}(f_{ij'})$. Assume also that there are two stages t, t' such that t' is the stage where $f_{ij'}$ is assigned to w' ($\lambda^{t'}(f_{ij'}) = w'$ and $\lambda^{t'-1}(f_{ij'}) \neq w'$) and such that t is the stage where f_{ij} is assigned to w ($\lambda^t(f_{ij}) = w$ and $\lambda^{t-1}(f_{ij}) \neq w$). There are two subcases to consider.
 - $t' < t$: Since $w' P_{ij} w$, we have that w is not a valid offer for f_{ij} , so it is not possible that $\lambda_{\mathcal{W}}(f_{ij}) = w$.
 - $t' \geq t$: Note that since w is matched with f_{ij} before that w' and the fact that $w' P_{ij} w$ implies that w' reject f_{ij} in an earlier step than t , say \tilde{t} . In this case, we claim that there is a firm $f_{\tilde{ik}}$ that is not a copy of f_{ij} such that $w' = \lambda^{\tilde{t}}(f_{\tilde{ik}})$ and $f_{\tilde{ik}} \bar{P}_w f_{ij}$. Since w' reject f_{ij} at stage \tilde{t} , such a firm-copy exists and is temporarily matched to w' at stage \tilde{t} , and w' prefers this firm-copy over f_{ij} . Moreover, since $\tilde{t} \leq t$, we have that $f_{\tilde{ik}}$ is not a copy of f_{ij} , otherwise w is not a valid offer for f_{ij} . Thus, in this case, the claim holds. Note that this contradicts the definition of workers' preferences \bar{P} item (1), since $f_{ij'} \bar{P}_{w'} f_{\tilde{ik}} \bar{P}_{w'} f_{ij}$. Then, by the two cases analyzed, this situation does not occur.
- (iv) **There is a pair $(f_{ij}, w) \in \mathcal{F} \times \mathcal{W}$ such that $f_{ij} \bar{P}_w \lambda_{\mathcal{W}}(w)$ and $w P_{ij} \lambda_{\mathcal{W}}(f_{ij'})$ for each $j' \in J_i$.** There are two subcases to consider.

If w makes a non-valid offer to f_{ij} at some step: then, there is a stage t and a worker w' such that $w' = \lambda^t(f_{ij}) P_{ij} w$, contradicting that $w P_{ij} \lambda_{\mathcal{W}}(f_{ij'})$ for each $j' \in J_i$.

If w makes a valid offer to f_{ij} at some step: Then, w was rejected by f_{ij} at some stage k . Thus, there is $w' \in \mathcal{W}$ such that $w' = \lambda^k(f_{ij})P_{ij}w$, contradicting that $wP_{ij}\lambda_{\mathcal{W}}(f_{ij'})$ for each $j' \in J_i$.

Therefore, since none of the four cases hold, we have that $\lambda_{\mathcal{W}} \in S^*(M)$. \square

3.2 The Isomorphism between markets

In this subsection, we prove that there is an isomorphism between the set of stable matchings in a many-to-one market with substitutable preferences and the set of stable* matchings of the associated one-to-one market, which is derived from the Aizermann-Malishevski decomposition.

Theorem 3 *There is an isomorphism between $S^*(M)$ and $S(\mathcal{M})$.*

Proof. Given $\lambda \in S^*(M)$, define $T(\lambda) = \mu$ as

$$\mu(\varphi_i) = \bigcup_{j \in J_i} \lambda(f_{ij})$$

for each $\varphi_i \in \Phi$. Moreover, if $w \in \mu(\varphi_i)$, then $\mu(w) = \varphi_i$.

Similarly, if $\mu \in S(\mathcal{M})$, define $T^{-1}(\mu) = \lambda$ as $\lambda(w) = f_{ij}$ for $w = \max_{\mu(\varphi_i)} P_{ij}$ such that there is no $j' < j$ with $w = \max_{\mu(\varphi_i)} P_{ij'}$. Moreover, if $\lambda(w) = f_{ij}$, then $\lambda(f_{ij}) = w$.

Formally we need to show that,

(i) Given $\lambda \in S^*(M)$, then $T(\lambda) = \mu \in S(\mathcal{M})$. We prove this implication by proving the following two claims.

Claim 1 $\mu = T(\lambda)$ is a many-to-one matching.

By definition of T , for each $w \in \mathcal{W}$, $w \in \mu(\varphi_i)$ implies that there is $j \in J_i$ such that $w = \lambda(f_{ij})$. Assume μ is not a matching, then there is $\varphi_{i'}$ with $i' \neq i$ such that $w \in \mu(\varphi_{i'})$. So, there is $j' \in J_{i'}$ such that $w = \lambda(f_{i'j'})$. Thus, $w = \lambda(f_{ij}) = \lambda(f_{i'j'})$ which contradicts λ being a one-to-one matching.

The construction of T ensures the bilateral nature of the assignment.

Claim 2 $\mu = T(\lambda) \in S^*(M)$.

By way of contradiction, assume $\mu \notin S(\mathcal{M})$. Then, we have three cases to analyze:

(a) If $\mu(w) = \varphi_i$, then there is $j \in J_i$ such that $w = \lambda(f_{ij})$. Assume that $\varnothing P_w \varphi_i$, then by the construction of the one-to-one associated market (particularly condition (1) on workers' preferences) it follows that $\varnothing \bar{P}_w f_{ij}$, contradicting that $\lambda \in S^*(M)$.

- (b) Let $w \in \mu(\varphi_i)$. Assume that $w \notin C_{\varphi_i}(\mu(\varphi_i))$. Since $\mu(\varphi_i) = \bigcup_j \lambda(f_{ij})$ and $C_{\varphi}(\mu(\varphi_i)) = \bigcup_j \max_{\mu(\varphi)} P_{ij}$, then there is $j' \in J$ such that $\lambda(f_{ij'}) = w$ and $\max_{\mu(\varphi_i)} P_{ij'} \neq w$. Then, λ is blocked by $f_{ij'}$ since it satisfies condition (2) of Definition 1, contradicting that $\lambda \in S^*(M)$.
- (c) Assume that there is a blocking pair to μ , i.e., there is a pair (φ_i, w) such that $\varphi_i P_w \mu(w) = \varphi_{i'}$ for some i' , and $w \in C_{\varphi}(\mu(\varphi_i) \cup \{w\})$. Then, by the Aizermann-Malishevski decomposition, there is ij' such that $w = \max_{\mu(\varphi_i) \cup \{w\}} P_{ij'}$ which implies that $w P_{ij'} \lambda(f_{ij''})$ for each $j'' \in J_i \setminus \{j'\}$. By the construction of the associated one-to-one market, if $\varphi_i P_w \varphi_{i'}$, then $f_{ij'} \bar{P}_w f_{i'j}$ for each $j \in J_{i'}$ (in particular for $\lambda(w) = f_{i'j}$ with $j \in J_{i'}$). Hence, $(w, f_{ij'})$ is a blocking* pair to λ , contradicting that $\lambda \in S^*(M)$.

(ii) Given $\mu \in S(\mathcal{M})$, then $T^{-1}(\mu) = \lambda \in S^*(M)$. We prove this implication by proving two more claims.

Claim 3 λ is a one-to-one matching.

By way of contradiction, assume λ is not a matching. So, there is $w \in \mathcal{W}$ such that $\lambda(w) = f_{ij}$ and $\lambda(w) = f_{i'j}$. If $i \neq i'$, then $w \in \mu(\varphi_i)$ and $w \in \mu(\varphi_{i'})$, which would contradict that μ is a many-to-one matching. If $i = i'$ and $j \neq j'$, then either $j < j'$ or vice-versa, making λ contradict the construction of $T^{-1}(\mu)$.

The construction of T^{-1} ensures the bilateral nature of the assignment.

Claim 4 $\lambda \in S^*(M)$.

By way of contradiction, assume $\lambda \notin S^*(M)$.

- (a) Assume that λ is blocked* by a worker w . If $\lambda(w) = f_{ij}$ then $w \in \mu(\varphi_i)$. Since $\mu \in S(\mathcal{M})$, then $\varphi_i P_w \emptyset$. By construction of the preferences in the associated one-to-one market, it follows that $f_{ij} \bar{P}_w \emptyset$ for each $j \in J_i$, a contradiction.
- (b) Assume that λ is blocked* by a firm-copy. Since $\mu \in S(\mathcal{M})$ then $C_{\varphi_i}(\mu(\varphi_i)) = \mu(\varphi_i) = \bigcup_{j \in J_i} \max_{\mu(\varphi_i)} P_{ij}$. By construction of T^{-1} if $\lambda(f_{ij'}) = w$, then $\max_{\mu(\varphi_i)} P_{ij'} = w P_{ij'} \lambda(f_{ij''})$ for each $j'' \in J_i \setminus \{j'\}$ and $w P_{ij'} \emptyset$, a contradiction.
- (c) Assume that there are f_{ij} and $f_{ij'}$ such that $\lambda(f_{ij'}) P_{ij} \lambda(f_{ij}) P_{ij} \emptyset$. Since $\mu \in S(\mathcal{M})$, and $T^{-1}(\mu) = \lambda$, we have that for each $w \in \mu(\varphi_i)$ there is P_{ij} such that $w = \max_{\mu(\varphi_i)} P_{ij}$ and $w = \lambda(f_{ij})$. Then, for each f_{ij} such that $\lambda(f_{ij}) P_{ij} \emptyset$ there is no $f_{ij'}$ such that $\lambda(f_{ij'}) P_{ij} \lambda(f_{ij})$, a contradiction.
- (d) Assume that (w, f_{ij}) form a blocking* pair, i.e., $w P_{ij} \lambda(f_{ij'})$ for each $j' \in J_i$ and $f_{ij} \bar{P}_w \lambda(w)$.⁴ By construction of T^{-1} if $\lambda(f_{ij}) \neq \emptyset$ then $\lambda(f_{ij}) = \max_{\mu(\varphi_i)} P_{ij}$. It

⁴W.l.o.g. assume $j = \min\{j'' \in J_i \text{ such that } w P_{ij''} \lambda(f_{ij'}) \text{ for each } j' \in J_i\}$. Also assume that some firm-copy is matched, otherwise stability and stability* coincide.

follows that $wP_{ij} \left[\max_{\mu(\varphi_i)} P_{ij} \right]$, which in turn implies that $w = \max_{\mu(\varphi_i) \cup \{w\}} P_{ij}$. Then, we have that $w \in C_{\varphi_i}(\mu(\varphi_i) \cup \{w\}) = \bigcup_{j' \in J_i} \max_{\mu(\varphi_i) \cup \{w\}} P_{ij'}$. By construction of the associated one-to-one market, $f_{ij} \bar{P}_w f_{i'j'}$ implies $\varphi_i P_w \varphi_{i'}$. Then, (w, φ_i) is a blocking pair to μ , contradicting that $\mu \in S(\mathcal{M})$.

By (i) and (ii), T is an isomorphism between the set of stable many-to-one matchings and the set of stable* one-to-one matchings. \square

The following example illustrates the isomorphism between the two markets of Example 1.

Example 1 (continued) Recall that the stable matchings of the many-to-one substitutable market are:

$$\begin{aligned} \mu_\Phi &= \begin{pmatrix} \varphi_1 & \varphi_2 \\ w_1 w_2 & w_3 w_4 \end{pmatrix} \mu_1 = \begin{pmatrix} \varphi_1 & \varphi_2 \\ w_1 w_3 & w_2 w_4 \end{pmatrix} \\ \mu_2 &= \begin{pmatrix} \varphi_1 & \varphi_2 \\ w_2 w_4 & w_1 w_3 \end{pmatrix} \mu_{\mathcal{W}} = \begin{pmatrix} \varphi_1 & \varphi_2 \\ w_3 w_4 & w_1 w_2 \end{pmatrix} \end{aligned}$$

The stable* matchings of the associated one-to-one market are:

$$\begin{aligned} \lambda_{\mathcal{F}} &= \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_1 & \emptyset & \emptyset & w_2 & \emptyset & \emptyset & w_3 & \emptyset & \emptyset & w_4 & \emptyset & \emptyset \end{pmatrix} \\ \lambda_1 &= \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_2 & \emptyset & w_4 & \emptyset & \emptyset & \emptyset & w_3 & \emptyset & \emptyset & \emptyset & \emptyset & w_1 \end{pmatrix} \\ \lambda_2 &= \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_1 & \emptyset & \emptyset & \emptyset & \emptyset & w_3 & w_4 & \emptyset & w_2 & \emptyset & \emptyset & \emptyset \end{pmatrix} \\ \lambda_{\mathcal{W}} &= \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_3 & w_4 & \emptyset & \emptyset & \emptyset & \emptyset & w_2 & w_1 & \emptyset & \emptyset & \emptyset & \emptyset \end{pmatrix} \end{aligned}$$

It easy to see that $T(\lambda_{\mathcal{F}}) = \mu_\Phi$, $T(\lambda_1) = \mu_1$, $T(\lambda_2) = \mu_2$, and $T(\lambda_{\mathcal{W}}) = \mu_{\mathcal{W}}$. \diamond

4 Final remarks

In this paper, we present a method to decompose a many-to-one market with substitutable preferences into a one-to-one market using the Aizerman-Malishevski decomposition. We define a notion of stability* for an associated one-to-one market that is adjusted to show that the set of stable matchings in a many-to-one market and the set of stable* matchings in the associated one-to-one market are isomorphic. Furthermore, we present an adaptation of the deferred acceptance algorithm (originally introduced by Gale and Shapley, 1962) for our associated one-to-one market. Regardless of which side proposes –be it the firm-copies

or the workers– a stable* matching is always obtained. Although the fact that both markets are isomorphic already indicates that the set of stable* matchings in the related one-to-one market is non-empty, the algorithm provides an alternative proof of this fact.

A common result when using the deferred acceptance algorithm across all matching markets is that the output is proposing-side optimal: if, for instance, the firms are the ones making the offers, the output of the algorithm is the firm-optimal stable matching. Unfortunately, this is not valid in our associated one-to-one market. Recall from Example 1 the stable* matchings resulting from the adapted deferred acceptance algorithm when both the firm-copies and the workers propose:

$$\lambda_{\mathcal{F}} = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_1 & \emptyset & \emptyset & w_2 & \emptyset & \emptyset & w_3 & \emptyset & \emptyset & w_4 & \emptyset & \emptyset \end{pmatrix}$$

and

$$\lambda_{\mathcal{W}} = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} & f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ w_3 & w_4 & \emptyset & \emptyset & \emptyset & \emptyset & w_2 & w_1 & \emptyset & \emptyset & \emptyset & \emptyset \end{pmatrix},$$

respectively. If we observe, for instance, the firm f_{12} , we find that

$$w_4 = \lambda_{\mathcal{W}}(f_{12})P_{12}\lambda_{\mathcal{F}}(f_{12}) = \emptyset, \quad (2)$$

indicating that the optimality of $\lambda_{\mathcal{F}}$ fails.

Let us consider the following result on isomorphic lattices: *Given two partially ordered sets $(A, >_A)$ and $(B, >_B)$ and an isomorphism \mathcal{I} between the sets A and B that preserves the order: for $a, b \in A$ such that $a >_A b$, it follows that $\mathcal{I}(a) >_B \mathcal{I}(b)$, if $(A, >_A)$ has a lattice structure, then $(B, >_B)$ also has a lattice structure (see Birkhoff, 1967, for a thorough treatment of lattice theory).*

If we consider the orders $P_{\mathcal{W}}$ and $\bar{P}_{\mathcal{W}}$ induced by workers' preferences in the many-to-one and related one-to-one markets, we can assert that the set of stable* matchings has a lattice structure with respect to the induced order $\bar{P}_{\mathcal{W}}$ since the ordering between matchings is preserved. However, the previous example also shows that the same cannot be stated for the firm-copies side. Despite that μ_{Φ} is unanimously preferred (either considering partial order induced by firms' preferences or considering Blair's partial order (Blair, 1988)) by all firms to $\mu_{\mathcal{W}}$ in the many-to-one market, $\mu_{\Phi} = T(\lambda_{\mathcal{F}})$, and $\mu_{\mathcal{W}} = T(\lambda_{\mathcal{W}})$, (2) shows that the isomorphisms T do not preserve the order.

Several important conclusions from this paper may guide future research. One significant avenue is the establishment of an order among matchings on the firm-copies side, which we think has to be closely related to Blair's partial order (see Blair, 1988, for more details). This new possible order is essential for demonstrating that the set of many-to-many matchings with substitutable preferences forms a lattice structure. Once a new order is established for matchings in the related one-to-one market on the firm-copies side, we can ask for firms preferences in the original many-to-one model not only be substitutable but also satisfy the

well-known “law of aggregated demand” (LAD).⁵ This requirement could open numerous avenues for future research. Below, we outline some of these potential directions:

- (i) It may allow for the definition of pointing functions for the associated one-to-one market, which would enable the calculation of the join and meet between two stable* matchings (see [Martínez et al., 2001](#); [Alkan, 2002](#); [Li, 2014](#), for more details on pointing functions in many-to-many markets).
- (ii) By defining pointing functions in the associated one-to-one market, it may be possible to provide an appropriate definition of side-optimality, ensuring that the output of the adapted deferred acceptance algorithm is side-optimal.
- (iii) [Bonifacio et al. \(2022\)](#) presents an algorithm that computes, using the notion of cycles in preferences, the entire set of stable matchings for a many-to-many market with substitutable preferences that satisfy the LAD (this algorithm is a generalization of the one presented in [Irving and Leather, 1986](#)). This algorithm runs the lattice of the set of stable matchings, from the firm-optimal to the worker-optimal stable matching. We believe that by requiring the LAD on firms’ preferences (in the many-to-many market), it may be possible to identify a cycle structure in the preferences of firm-copies (in the associated one-to-one market), thus it may be possible to develop an algorithm that computes the entire set of stable* matchings.
- (iv) Another important result for matching markets is the well-known “Rural Hospitals Theorem”.⁶ It is coherent that this theorem does not hold in the associated one-to-one market since, in the many-to-one market, without requiring the LAD on firms’ preferences beyond substitutability, the result does not hold. However, as we can observe in [Example 1](#) (the firms’ preferences in the many-to-one market satisfy both substitutability and LAD), although this result does not hold for a firm-copy (see the case of the firm-copy f_{12} in [\(2\)](#)), we can observe that the number of matched copies of a firm is the same in every stable* matching. Therefore, it may be possible to adapt this result to the associated one-to-one market when requiring both substitutability and the LAD in the preferences of firms in the original many-to-one market.

⁵The law of aggregated demand states that when a firm chooses from an expanded set, it selects at least as many workers as before. This property was first studied by [Alkan \(2002\)](#) under the name ‘cardinal monotonicity’. See also [Hatfield and Milgrom \(2005\)](#).

⁶The *Rural Hospital Theorem* is proven in different contexts by many authors (see [McVitie and Wilson, 1971](#); [Roth, 1984, 1985](#); [Martínez et al., 2000](#); [Alkan, 2002](#), among others). The version of this theorem for a many-to-many matching market, where all agents have substitutable choice functions satisfying the LAD, which also applies to our setting, is presented in [Alkan \(2002\)](#). It states that each agent is matched with the same number of partners in every stable matching.

References

- AIZERMAN, M. AND A. MALISHEVSKI (1981): "General theory of best variants choice: Some aspects," *IEEE Transactions on Automatic Control*, 26, 1030–1040.
- ALKAN, A. (2002): "A class of multipartner matching markets with a strong lattice structure," *Economic Theory*, 19, 737–746.
- BIRKHOFF, G. (1967): *Lattice theory*, American Mathematical Society.
- BLAIR, C. (1988): "The Lattice Structure of the Set of Stable Matchings with Multiple Partners," *Mathematics of Operations Research*, 13, 619–628.
- BONIFACIO, A. G., N. JUAREZ, P. NEME, AND J. OVIEDO (2022): "Cycles to compute the full set of many-to-many stable matchings," *Mathematical Social Sciences*, 117, 20–29.
- CHAMBERS, C. P. AND M. B. YENMEZ (2017): "Choice and matching," *American Economic Journal: Microeconomics*, 9, 126–147.
- CRAWFORD, V. P. AND A. KELSO (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483–1504.
- GALE, D. AND L. SHAPLEY (1962): "College admissions and the stability of marriage," *American Mathematical Monthly*, 69, 9–15.
- HATFIELD, J. W. AND P. R. MILGROM (2005): "Matching with Contracts," *American Economic Review*, 95, 913–935.
- IRVING, R. AND P. LEATHER (1986): "The complexity of counting stable marriages," *SIAM Journal of Computing*, 15, 655–667.
- LI, J.-R. (2014): "A new proof of the lattice structure of many-to-many pairwise-stable matchings," *Journal of the Operations Research Society of China*, 2, 369–377.
- MARTÍNEZ, R., J. MASSÓ, A. NEME, AND J. OVIEDO (2000): "Single Agents and the Set of Many-to-One Stable Matchings," *Journal of Economic Theory*, 91, 91–105.
- (2001): "On the lattice structure of the set of stable matchings for a many-to-one model," *Optimization*, 50, 439–457.
- MCVITIE, D. G. AND L. B. WILSON (1971): "Three procedures for the stable marriage problem," *Communications of the ACM*, 14, 491–492.
- MOULIN, H. (1985): "Choice functions over a finite set: A summary," *Social Choice and Welfare*, 2, 147–160.
- ROTH, A. E. (1984): "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," *Journal of Political Economy*, 92, 991.

——— (1985): “The college admissions problem is not equivalent to the marriage problem,”
Journal of Economic Theory, 36, 277–288.

ROTH, A. E. AND M. SOTOMAYOR (1990): *Two-sided matching*, Cambridge University Press.