

# Coronas and Callias type operators in coarse geometry

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## Abstract

We interpret the coarse symbol and index class of a Callias type Dirac operator  $\mathcal{D} + \Psi$  on a manifold  $M$  as a pairing between the coarse symbol and index classes associated to  $\mathcal{D}$  and  $K$ -theory classes of the coarse corona of  $M$  or  $M$  itself determined by  $\Psi$ . Local positivity of  $\mathcal{D}$  and local invertibility of  $\Psi$  are incorporated in terms of support conditions on the  $K$ -theoretic level.

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# 1 Introduction

The purpose of this paper is to embed the index theory of Callias type Dirac operators on complete Riemannian manifolds into the coarse homotopy theory developed in [BE20b, BE20a]. Our main innovation is Theorem 4.32 which expresses the index and symbol of Callias type operators in terms of the coarse corona and symbol pairings introduced Definition 3.35 and Definition 3.40.

One of the achievements of the present paper is to provide a  $K$ -theoretic interpretation of Callias type Dirac operators beyond the Fredholm index case. As an application of the theory we discuss a version of the partitioned manifold index theorem [Roe88] and its multipartitioned generalization [Sie12, Zei17, SZ18]. New aspects are that we allow multipartitions along non-compact submanifolds, work with the refined symbol classes in local  $K$ -homology, and that we take the support conditions into account.

In the present paper we work in the setup of coarse homotopy theory developed in [BE20b, BE20a] which is based on the categories **BC** and **UBC** of bornological coarse and uniform bornological coarse spaces and the  $KU$ -module-valued coarse  $K$ -homology functor

$$K\mathcal{X} : \mathbf{BC} \rightarrow \mathbf{Mod}(KU)$$

from [BE23]. In order to keep this paper selfcontained, we review the relevant definitions and results of this theory. The wish to present the setup in a coherent way and to give at least complete definitions accounts for most of the length of this paper.

**Coarse index theory with supports.** As observed by J. Roe [Roe93] the basic global  $K$ -theoretic invariant of a generalized Dirac operator  $\mathcal{D}$  on a complete Riemannian manifold  $M$

is its coarse index class  $\text{ind}\mathcal{X}(\mathcal{D})$  in the coarse  $K$ -homology  $K\mathcal{X}(M)$  (see Definition 2.56 and Definition 4.7). This index class detects the invertibility of  $\mathcal{D}$  on the natural  $L^2$ -space, but also retains some coarsely local information, e.g., on which parts of  $M$  the zero order term in the Weizenboeck formula for  $\mathcal{D}$  is uniformly positive [Roe16].

Classically, the local information on  $\mathcal{D}$  is homotopy theoretically encoded by the analytic symbol class  $\sigma^{\text{an}}(\mathcal{D})$  in the locally finite analytic  $K$ -homology  $K^{\text{an}}(M) \simeq \text{KK}(C_0(M), \mathbb{C})$  of  $M$  (see Definition 3.29 and Definition 4.15, but note that in the present paper we prefer to work with  $E$ -theory). The analytic symbol class can, e.g., be paired with compactly supported  $K$ -theory classes of  $M$ . If one realizes  $KK$ -theory classes by Kasparov modules [Kas88], then the results of these pairings can be interpreted in terms of indices of Fredholm operators constructed from  $\mathcal{D}$ , see Example 1.1. But (somewhat surprisingly) one can also recover the coarse index  $\text{ind}\mathcal{X}(\mathcal{D})$  from the symbol class  $\sigma^{\text{an}}(\mathcal{D})$ , see Remark 4.16.

In order to capture the symbol of  $\mathcal{D}$  within the coarse  $K$ -homology calculus, in [Buna, BE17] we proposed to consider a refined symbol class  $\sigma(\mathcal{D})$  in the local  $K$ -homology  $K^{\mathcal{X}}(M)$  (see Definition 3.25 and Definition 4.14). There is a natural transformation  $a : K^{\mathcal{X}} \rightarrow K\mathcal{X}$  (see (3.11)) of local homology theories called the index map such that

$$a_M(\sigma(\mathcal{D})) = \text{ind}\mathcal{X}(\mathcal{D}) . \quad (1.1)$$

The classical symbol class in analytic  $K$ -homology can be recovered in terms of the natural Paschke duality transformation of local homology theories  $p : K^{\mathcal{X}} \rightarrow K^{\text{an}}$  (see (3.16)) by

$$p_M(\sigma(\mathcal{D})) = \sigma^{\text{an}}(\mathcal{D}) .$$

The advantage of considering the symbol  $\sigma(\mathcal{D})$  instead of  $\sigma^{\text{an}}(\mathcal{D})$  is that one can incorporate local positivity of  $\mathcal{D}$  geometrically in terms of additional support conditions [Buna, BE17]. In coarse geometry, the right notion of a subset is a big family  $\mathcal{Y}$ , i.e., a filtered family of subsets which is closed under forming coarse thickenings. Given such a big family  $\mathcal{Y}$  on  $M$  one can define the local  $K$ -homology  $K_{\mathcal{Y}}^{\mathcal{X}}(M)$  (see Definition 3.26) with support on  $\mathcal{Y}$  and the coarse  $K$ -homology  $K\mathcal{X}(\mathcal{Y}) := \text{colim}_{Y \in \mathcal{Y}} K\mathcal{X}(Y)$  of  $\mathcal{Y}$ . The index map then refines to a map

$$a_{M,\mathcal{Y}} : K_{\mathcal{Y}}^{\mathcal{X}}(M) \rightarrow K\mathcal{X}(\mathcal{Y}),$$

see (3.12).

Assume now that there exists a member  $Y$  of  $\mathcal{Y}$  such that  $\mathcal{D}^2$  is positive on  $M \setminus Y$  in the sense that the quadratic form determined by  $\mathcal{D}^2$  on smooth sections which are compactly supported on  $M \setminus \bar{Y}$  is positive. In Definition 4.14, we define a refined symbol class  $\sigma_{\mathcal{Y}}(\mathcal{D})$  in  $K_{\mathcal{Y}}^{\mathcal{X}}(M)$  such that

$$a_{M,\mathcal{Y}}(\sigma_{\mathcal{Y}}(\mathcal{D})) = \text{ind}\mathcal{X}(\mathcal{D}, \text{ on } \mathcal{Y})$$

in  $K\mathcal{X}(\mathcal{Y})$ .

For example, if  $\mathcal{D}$  is positive on all of  $M$ , then  $\mathcal{D}$  is invertible and  $\text{ind}\mathcal{X}(\mathcal{D}) = 0$ . Taking

$\mathcal{Y} = \{\emptyset\}$ , the class  $\sigma_{\{\emptyset\}}(\mathcal{D})$  in  $K_{\{\emptyset\}}^{\mathcal{X}}(M)$  (called the  $\rho$ -invariant) captures the reason for the equality  $\text{ind}_{\mathcal{X}}(\mathcal{D}) = 0$  in a homotopy theoretic manner.

The local  $K$ -homology  $K^{\mathcal{X}}$  (with supports) and the index map  $a$  are instances of general constructions which can be performed with arbitrary coarse homology theories, see Definition 2.36 and Definition 2.40. In the present paper we just applied them to the coarse  $K$ -homology theory  $K^{\mathcal{X}}$ . These constructions are functorial with respect to natural transformations of coarse homology theories, an observation which waits to be exploited in the future. Furthermore, the index map is a natural transformation of local homology theories (say on the spectrum level) and therefore automatically compatible with Mayer-Vietoris boundary maps. In contrast, the construction of the classical analytic index map

$$a_M^{\text{an}} : K_*^{\text{an}}(M) \rightarrow K^{\mathcal{X}}_*(M)$$

(see, e.g., [HR95, Sec. 5 & 6]) is based on complicated analytic arguments using specific models of  $K_*^{\text{an}}(M)$  and a group-level Paschke duality isomorphism which is not obviously liftable to the spectrum level in a functorial manner.

**Callias type operators.** By Definition 4.20 a Callias type operator is a generalized Dirac operator of the form  $\mathcal{D} + \Psi$  obtained from a generalized Dirac operator  $\mathcal{D}$  by adding a potential  $\Psi$  such that  $\mathcal{D}\Psi + \Psi\mathcal{D}$  is zero order (see Definition 4.19). The relevant index theoretic information on  $\mathcal{D} + \Psi$  is concentrated on the subset of  $M$  on which the potential is not positive in the sense that  $\Psi^2$  dominates  $\mathcal{D}\Psi + \Psi\mathcal{D}$  (see Definition 4.21).

The classical condition is that the potential is positive outside of a compact subset of  $M$  which turns the Callias type operator into a Fredholm operator. Partial positivity as considered in the present paper was already studied in [GHM22].

We consider a big family  $\mathcal{Y}$  such that the potential is positive away from  $\mathcal{Y}$ . Then  $\mathcal{D} + \Psi$  is positive away from  $\mathcal{Y}$ . Since this positivity is caused by the zero order term of  $\mathcal{D} + \Psi$  in Definition 4.24 we can define a refined symbol class with support  $\sigma(\mathcal{D} + \Psi, \text{on } \mathcal{Y})$  in  $K^{\mathcal{X}}(\mathcal{Y}) := \text{colim}_{Y \in \mathcal{Y}} K^{\mathcal{X}}(Y)$ .

If  $\mathcal{D}$  was already positive away from a big family  $\mathcal{Z}$  and  $\Psi$  is asymptotically constant away from  $\mathcal{Z}$ , then  $\mathcal{D} + \Psi$  is positive away from  $\mathcal{Y} \cap \mathcal{Z}$  with index  $\text{ind}_{\mathcal{X}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y} \cap \mathcal{Z})$  in  $K^{\mathcal{X}}(\mathcal{Y} \cap \mathcal{Z})$ , and we define a refined symbol class  $\sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y})$  in  $K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y})$ .

We want to interpret the symbol class  $\sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y})$  and the coarse index class

$$\text{ind}_{\mathcal{X}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y} \cap \mathcal{Z}) = a_{M, \mathcal{Z}}(\sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y}))$$

in terms of pairings between the symbol class  $\sigma_{\mathcal{Z}}(\mathcal{D})$  or the index class  $\text{ind}_{\mathcal{X}}(\mathcal{D}, \text{on } \mathcal{Z})$  with suitable  $K$ -theory classes determined by the potential  $\Psi$ .

The big family  $\mathcal{Y}$  gives rise to the uniform  $\mathcal{Y}$ -corona  $\partial_u^{\mathcal{Y}} M$ , a compact topological space with the potential to capture the behaviour of  $\Psi$  at  $\infty$  (see Definition 3.16). As part of

the coarse  $K$ -homology calculus we introduce the coarse corona pairing Definition 3.35

$$-\cap^{\mathcal{X}} - : K(\partial^{\mathcal{Y}}M) \times K\mathcal{X}(\mathcal{Z}) \rightarrow \Sigma K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z})$$

and the coarse symbol pairing Definition 3.40

$$-\cap^{\mathcal{X}\sigma} - : K(\mathcal{Y}) \times K_{\mathcal{Z}}^{\mathcal{X}}(M) \rightarrow K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}) ,$$

involving the topological  $K$ -theory of the coarse corona  $\partial^{\mathcal{Y}}X$  (Definition 3.2) and of  $\mathcal{Y}$ . Both pairings are components of extra-natural transformations, see Remark 3.36.

Under the assumptions made above, the potential  $\Psi$  gives naturally rise to a class  $[\Psi]$  in  $K(\mathcal{Y})$ . Our main result is Theorem 4.32, which states that under suitable conditions on  $\mathcal{D}$  and  $\Psi$ , one has

$$\sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y}) = [\Psi] \cap^{\mathcal{X}\sigma} \sigma_{\mathcal{Z}}(\mathcal{D})$$

in  $K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y})$ . It provides the interpretation of the symbol class of the Callias type operator in terms of the coarse  $K$ -homology calculus.

We now consider the boundary map

$$\partial : K(\partial_u^{\mathcal{Y}}M) \rightarrow \Sigma K(\mathcal{Y})$$

from (3.10). We assume that there exists a class  $p$  in  $K(\partial_u^{\mathcal{Y}}M)$  with  $\partial p = [\Psi]$ . Then by the compatibility of  $\partial$  with the index map shown in Proposition 3.43 we get

$$\text{ind}\mathcal{X}(\mathcal{D} + \Psi, \text{on } \mathcal{Y} \cap \mathcal{Z}) = c^*p \cap^{\mathcal{X}} \text{ind}\mathcal{X}(\mathcal{D}, \text{on } \mathcal{Z}) \quad (1.2)$$

in  $K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z})$ , where  $c$  is the comparison map (3.9). As a consequence of (1.2), the assumption  $[\Psi] = \partial p$  implies the non-obvious fact that the coarse index  $\text{ind}\mathcal{X}(\mathcal{D} + \Psi, \text{on } \mathcal{Y} \cap \mathcal{Z})$  only depends on  $\mathcal{D}$  via its coarse index  $\text{ind}\mathcal{X}(\mathcal{D}, \text{on } \mathcal{Z})$ , instead of the finer information contained in its symbol class  $\sigma_{\mathcal{Z}}(\mathcal{D})$ .

**Example 1.1.** We assume that  $M$  is connected and that  $\mathcal{Y} = \mathcal{B}$  is the family of bounded subsets. Then we have  $K\mathcal{X}(\mathcal{B}) \simeq KU$  and under this identification  $\text{ind}\mathcal{X}(\mathcal{D} + \Psi, \text{on } \mathcal{B})$  is the Fredholm index of  $\mathcal{D} + \Psi$ . The class  $[\Psi]$  in  $K_{\mathcal{B}}(M)$  is a compactly supported  $K$ -theory class of  $M$  and the Fredholm index of  $\mathcal{D} + \Psi$  is the result of the pairing of  $\sigma^{\text{an}}(\mathcal{D})$  with the class  $[\Psi]$ .

In this case  $\partial^{\mathcal{B}}M$  is the Higson corona  $\partial_h M$  of  $M$ . If  $[\Psi] = \partial p$  for some class  $p$  in  $K(\partial_h M)$ , then the interpretation of the Fredholm index of  $\mathcal{D} + \Psi$  in terms of a pairing between the  $K$ -theory  $K(\partial_h M)$  of the Higson corona and symbol  $\sigma^{\text{an}}(\mathcal{D})$  is known from [Bun95]. ■

**Structure of the paper.** In the first sections we will recall the main definitions and constructions of the coarse homotopy calculus,  $K$ -theory of  $C^*$ -algebras and categories,

and coarse index theory of Dirac operators. Prerequisites for reading this paper are basic homotopy theoretic language (e.g., elements of the theory of stable  $\infty$ -categories), basic  $C^*$ -algebra theory (continuous function calculus, maximal tensor product, Gelfand duality) and some global analysis (Riemannian manifold and generalized Dirac operators).

In Section 2.1 we recall from [BE20b, BE20a] the definitions of the categories  $\mathbf{BC}$  of bornological coarse spaces and  $\mathbf{UBC}$  of uniform bornological coarse spaces and the cone functor  $\mathcal{O}^\infty : \mathbf{UBC} \rightarrow \mathbf{BC}$ .

In Section 2.2 we recall from [BE20b] the notion of a coarse homology theory in general and some basic calculations.

In Section 2.3 we recall from [BE20a] and [BE20b] the notions of a local and locally finite homology theory. We explain the construction of the local homology theory associated to a coarse homology theory and the index map. We further introduce the local homology theory with supports associated to a coarse homology theory.

In Section 2.4 we give a descriptive presentation  $E$ - and  $K$ -theory via universal properties for (graded)  $C^*$ -algebras and categories in the style of [Bun23, BD24].

Section 2.5 is devoted to the construction of the coarse  $K$ -homology functor  $K\mathcal{X} : \mathbf{BC} \rightarrow \mathbf{Mod}(KU)$  in terms of the  $C^*$ -categories of controlled Hilbert spaces taken from [BE20b] and [BE23].

In Section 3.1 and Section 3.2 we introduce various  $C^*$ -algebras of functions associated to a big family by putting support conditions on the functions themselves or their variation. We introduce the corona associated to a pair of big families and show a crucial technical commutator estimate in Lemma 3.9.

In Section 3.3 we define the analytic locally finite  $K$ -homology functor  $K^{\text{an}}(-) := E(C_0(-), \mathbb{C})$  and verify that it is a locally finite local homology theory. We then recall from [QR10, BELa] the Paschke transformation  $p_M : K^{\mathcal{X}}(M) \rightarrow K^{\text{an}}(M)$  and provide conditions on  $M$  ensuring that  $p_M$  is an equivalence. We also explain the factorization of the index map over a map  $K^{\text{an}}(M) \rightarrow K\mathcal{X}(M)$  which allows to recover the coarse index of  $\not{D}$  from its analytic symbol in Remark 4.16.

In Sections 3.4 and 3.5, we construct the coarse corona pairing and the coarse symbol pairing and state and verify their formal functorial properties.

In Section 4.1 we recall from [BE17] the details of the construction of coarse index  $\text{ind}\mathcal{X}(\not{D}, \text{on } \mathcal{Y})$  in  $K\mathcal{X}(\mathcal{Y})$  of a Dirac operator which is positive away from a big family  $\mathcal{Y}$ . We state the suspension theorem (identifying the index operators on products  $\mathbb{R} \otimes M$ ) and the coarse relative index theorem expressing the locality of the coarse index.

In Section 4.2 we first recall from [Buna] how to capture symbols of Dirac operators which

are positive away from  $\mathcal{Y}$  as classes  $\sigma_{\mathcal{Y}}(\mathcal{D})$  in  $K_{\mathcal{Y}}^{\mathcal{X}}(M)$ .

The main result of the present paper is Theorem 4.32 stated in Section 4.3 which identifies the symbol of the Callias type operator  $\mathcal{D} + \Psi$  with support on  $\mathcal{Y}$  as the coarse symbol pairing  $[\Psi] \cap^{\mathcal{X}, \sigma} \sigma(\mathcal{D})$ .

In Section 4.5 we discuss the coarse version of the Dirac-goes-to-Dirac principle.

This principle is then used in Section 4.6 in the discussion of the relation between our pairings and Mayer-Vietoris boundary maps leading to a version of the partitioned manifold theorem.

In Section 4.7 we derive a version of the multipartitioned manifold index theorem by an iterated application of the partitioned manifold theorem.

Finally, in Section 4.8 we argue that the slant products considered in [EWZ22] can be derived by specializing our pairings, and that their basic properties can be derived from general facts about these pairings.

Most of the material discussed in this paper has an immediate generalization to the equivariant case for discrete group actions. We decided not to formulate everything in this generality in order to keep the complexity of the paper reasonable.

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## 2 Coarse cohomology theories

### 2.1 Uniform bornological coarse spaces and cones

In this section we recall the notions of bornological coarse spaces, uniform bornological coarse spaces, topological bornological spaces and the cone construction [BE20b, BEKW20]. We just mention that coarse geometry has been invented by J. Roe [Roe90, Roe03] and refer to the references above for further links to the literature on coarse geometry and attributions.

Let  $X$  be a set.

**Definition 2.1.**

1. A bornology on  $X$  is a subset  $\mathcal{B}$  of the power set of  $X$  which covers  $X$  and is closed

under forming subsets and finite unions.

2. A coarse structure on  $X$  is a subset  $\mathcal{C}$  of the power set of  $X \times X$  which is closed under forming subsets, finite unions, compositions  $(U, U') \mapsto U \circ U'$  (in the sense of relations), the flip of  $X \times X$ , and which contains the diagonal  $\mathbf{diag}(X)$  of  $X$ .
3. A uniform structure on  $X$  is a subset  $\mathcal{U}$  of the power set of  $X \times X$  of subsets  $U$  with  $\mathbf{diag}(X) \subseteq U$  which is closed under forming supersets, finite intersections, compositions, the flip, and which has the property that for any  $U$  in  $\mathcal{U}$  there exists a  $V$  in  $\mathcal{U}$  with  $V \circ V \subseteq U$ .

For a subset  $Z$  of  $X$  and a subset  $U$  (called an entourage) of  $X \times X$  we let

$$U[Z] := \{x \in X \mid \exists z \in Z : (x, z) \in U\} \quad (2.1)$$

denote the  $U$ -thickening of  $Z$ .

Let  $X$  be a set with a coarse structure  $\mathcal{C}$ . The correct notion of a subset in coarse geometry is the notion of a big family.

**Definition 2.2.** A big family is a nonempty subset  $\mathcal{Y}$  of the power set of  $X$  which is closed under taking subsets, finite unions and  $U$ -thickenings for all  $U$  in  $\mathcal{C}$ .

We consider a big family  $\mathcal{Y}$  as a partially ordered set with respect to the inclusion relation. Because  $\mathcal{Y}$  is closed under taking finite unions, this partially ordered set is filtered.

**Remark 2.3.** In the literature on bornological coarse spaces with the basic reference [BE20b], a big family on  $X$  is a family of subsets  $(Y_i)_{i \in I}$  indexed by a filtered poset  $I$  such that  $i \leq j$  implies  $Y_i \subseteq Y_j$  and such that for every  $i$  in  $I$  and  $U$  in  $\mathcal{C}$  there exists  $i'$  in  $I$  with  $U[Y_i] \subseteq Y_{i'}$ . In this convention the notion introduced in Definition 2.2 is the special case of a selfindexing big family which is in addition complete (this is the closedness under taking subsets).

Every big family  $(Y_i)_{i \in I}$  in the sense of [BE20b] generates a big family  $\mathcal{Y} := \{Y \subseteq X \mid \exists i \in I : Y \subseteq Y_i\}$  as defined in Definition 2.2. The canonical map  $I \rightarrow \mathcal{Y}$ ,  $i \mapsto Y_i$  of partially ordered sets is cofinal.

One can define an equivalence relation on the collection of big families in the sense of [BE20b] such that  $(Y_i)_{i \in I} \sim (Y'_{i'})_{i' \in I'}$  if and only if the associated big families in the sense of Definition 2.2 satisfy  $\mathcal{Y} = \mathcal{Y}'$ .

Since the constructions with big families occurring in the present paper only depend on the equivalence classes it suffices to work with complete selfindexing big families as introduced in Definition 2.2. ■



**Example 2.4.** If  $Y$  is a subset of  $X$ , then

$$\{Y\} := \{Z \subseteq X \mid \exists U \in \mathcal{C} : Z \subseteq U[Y]\}. \quad (2.2)$$

is called the big family generated by  $Y$ . If  $\mathcal{Y}$  and  $\mathcal{Z}$  are big families in  $X$ , then we define the big families

$$\mathcal{Y} \cap \mathcal{Z} := \{Y \cap Z \mid Y \in \mathcal{Y} \ \& \ Z \in \mathcal{Z}\} \quad (2.3)$$

and

$$\mathcal{Y} \cup \mathcal{Z} := \{W \subseteq X \mid \exists Y \in \mathcal{Y} \ \& \ Z \in \mathcal{Z} : W \subseteq Y \cup Z\} \quad (2.4)$$

in  $X$ .

If  $\mathcal{Y}$  is a big family in  $X$  and  $\mathcal{Y}'$  is a big family in  $X'$ , then we define the big family

$$\mathcal{Y} \times \mathcal{Y}' := \{Z \subseteq X \times X' \mid \exists Y \in \mathcal{Y}, Y' \in \mathcal{Y}' : Z \subseteq Y \times Y'\} \quad (2.5)$$

in  $X \times X'$ . In order to simplify the notation we will set  $X \times \mathcal{Y}' := \{X\} \times \mathcal{Y}'$ .

If  $\mathcal{Y}$  is a big family in  $X$  and  $Z \subseteq X$  is a subset equipped with the induced coarse structure, then we define the induced big family

$$Z \cap \mathcal{Y} := \{Z \cap Y \mid Y \in \mathcal{Y}\} \quad (2.6)$$

in  $Z$ . ■

A big family on a topological space  $X$  is a family of subsets  $\mathcal{Y}$  which is closed under taking subsets and finite unions and has property that for every  $Y$  in  $\mathcal{Y}$  there exists  $Y'$  in  $\mathcal{Y}$  which contains an open neighbourhood of  $\bar{Y}$ .

We consider the following compatibility relations between bornologies, coarse structures and uniform structures.

**Definition 2.5.**

1. A uniform structure  $\mathcal{U}$  is compatible with a coarse structure  $\mathcal{C}$  if  $\mathcal{C} \cap \mathcal{U} \neq \emptyset$ .
2. A bornology is  $\mathcal{B}$  compatible with a coarse structure  $\mathcal{C}$  if it is a big family.
3. A bornology  $\mathcal{B}$  is compatible with a topology on  $X$  if it is a big family (in the topological sense).

Note that a compatible bornology (in the coarse or topological sense) is simply a big family which covers the whole space.

We consider the following conditions on maps  $f : X \rightarrow Y$  between sets equipped with (some) of the above structures.

**Definition 2.6.**

1.  $f$  is proper if  $f^{-1}(\mathcal{B}_Y) \subseteq \mathcal{B}_X$ .
2.  $f$  is bornological, if  $f(\mathcal{B}_X) \subseteq \mathcal{B}_Y$ .
3.  $f$  is controlled if  $f(\mathcal{C}_X) \subseteq \mathcal{C}_Y$ .
4.  $f$  is uniform if  $f^{-1}(\mathcal{U}_Y) \subseteq \mathcal{U}_X$ .

**Example 2.7.** Let  $X$  and  $X'$  be sets with coarse structures and  $f : X' \rightarrow X$  be a controlled map. If  $\mathcal{Y}$  is a big family on  $X$ , then generalizing (2.6) we define the induced big family

$$f^{-1}(\mathcal{Y}) := \{Z' \subseteq X' \mid \exists Y \in \mathcal{Y} : Z' \subseteq f^{-1}(Y)\}$$

in  $X'$ . For a subset  $Y$  of  $X$  we have  $\{f^{-1}(Y)\} \subseteq f^{-1}(\{Y\})$ , but this inclusion is in general not an equality. ■

We will consider the following categories:

**Definition 2.8.**

1. **BC** (*bornological coarse spaces*): Sets with coarse structures and compatible bornologies and controlled and proper maps.
2. **UBC** (*uniform bornological coarse spaces*): Sets with coarse structures, compatible bornologies and uniform structures and controlled, uniform and proper maps.
3. **TB** (*topological bornological spaces*): Topological spaces with compatible bornological structures and proper continuous maps.

Recall that a uniform structure  $\mathcal{U}$  on a set  $X$  induces a topology on  $X$  generated by the subsets  $U[x]$  for all  $x$  in  $X$  and  $U$  in  $\mathcal{U}$ . We have obvious forgetful functors

$$\iota : \mathbf{UBC} \rightarrow \mathbf{BC} , \quad \tau : \mathbf{UBC} \rightarrow \mathbf{TB}$$

which we will often omit from the notation.

**Example 2.9.** A (generalized, i.e., we allow  $d(x, y) = \infty$ ) metric space  $(X, d)$  presents a uniform bornological coarse space with the following structures:

1.  $\mathcal{B}$  consists of the subsets which can be covered by finitely many metric balls.

2.  $\mathcal{C}$  consists of all subsets which are contained a metric entourage

$$U_r := \{(x, y) \in X \times X \mid d(x, y) \leq r\} \quad (2.7)$$

for some  $r$  in  $[0, \infty)$ .

3.  $\mathcal{U}$  consists of subsets containing a metric entourage  $U_r$  for some  $r \in (0, \infty)$  (note that 0 is excluded here).

A Lipschitz continuous map between metric spaces is uniform and controlled. ■

**Example 2.10.** Let  $\mathbf{Top}^{lc}$  denote the category of locally compact topological spaces and proper continuous maps. We have a fully faithful functor  $\mathbf{Top}^{lc} \rightarrow \mathbf{TB}$  which equips a locally compact space with the bornology of relatively compact subsets.

We have a functor  $\mathbf{Top}_{mc} \rightarrow \mathbf{UBC}$  from metrizable compact topological spaces to uniform bornological coarse spaces which equips a metrizable compact topological space with the maximal coarse and bornological structures and the canonical uniform structure (induced by any choice of metric on  $X$ ). ■

The category  $\mathbf{BC}$  has a symmetric monoidal structure  $\otimes$  such that  $X \otimes Y$  has the following description:

1. The underlying set of  $X \otimes Y$  is  $X \times Y$ .
2. An entourage of  $X \times Y$  is coarse for  $X \otimes Y$  if it is contained a product of entourages of  $X$  and  $Y$ .
3. A subset of  $X \times Y$  is bounded if it is contained in a product of bounded subsets of  $X$  and  $Y$ .

Similarly the category  $\mathbf{UBC}$  has a symmetric monoidal structure  $\otimes$  such that the underlying bornological coarse space of  $X \otimes Y$  is as above and the uniform structure consists of entourages which contain products of uniform entourages of  $X$  and  $Y$ . Finally, also  $\mathbf{TB}$  has a symmetric monoidal structure given by the cartesian structure on the underlying topological spaces and the product bornology as above.

**Definition 2.11.** *The cone functor*

$$\mathcal{O}^\infty : \mathbf{UBC} \rightarrow \mathbf{BC} \quad (2.8)$$

*is defined as follows:*

1. *objects:* Let  $X$  be in  $\mathbf{UBC}$ :

- a) The underlying set of  $\mathcal{O}^\infty(X)$  is  $\mathbb{R} \times X$ .
- b) The underlying bornology of  $\mathcal{O}^\infty(X)$  is the one of  $\mathbb{R} \otimes X$  (where  $\mathbb{R}$  is considered as a metric space).
- c) The coarse structure of  $\mathcal{O}^\infty(X)$  consists of those entourages  $U$  of the coarse product  $\mathbb{R} \otimes X$  which have that property that for every uniform entourage  $V$  of  $X$  there exists  $r$  in  $\mathbb{R}$  such that  $((x, t), (x', t')) \in U$  with  $\min(t, t') \geq r$  implies  $(x, x') \in V$ .

2. morphisms: A map  $f : X \rightarrow Y$  in **UBC** induces a map  $\mathcal{O}^\infty(f) : \mathcal{O}^\infty(X) \rightarrow \mathcal{O}^\infty(Y)$  given by  $(t, x) \mapsto (t, f(x))$ .

The cone boundary

$$\partial^{\text{cone}} : \mathcal{O}^\infty(X) \rightarrow \iota(\mathbb{R} \otimes X) \quad (2.9)$$

is given by the identity of underlying sets. Observe that in general the product uniform structure on  $\mathbb{R} \otimes X$  is not compatible with the coarse structure of  $\mathcal{O}^\infty(X)$ .

The following notions will be relevant in the discussion of Mayer-Vietoris sequences for coarse or local homology theories. Let  $X$  be a bornological coarse space and  $Y, Z$  be two subsets.

**Definition 2.12.** The pair  $(Y, Z)$  is said to be coarsely excisive if for every coarse entourage  $U$  of  $X$  there exists a coarse entourage  $V$  of  $X$  such that

$$U[Y] \cap U[Z] \subseteq V[Y \cap Z] .$$

Equivalently, the pair  $(Y, Z)$  is coarsely excisive iff  $\{Y\} \cap \{Z\}$  is equivalent to  $\{Y \cap Z\}$ .

Assume now that  $X$  is a uniform space with uniform structure  $\mathcal{U}$ .

**Definition 2.13.** We say that  $(Y, Z)$  is uniformly excisive [BE20a, Def. 3.3] if there exists  $U$  in  $\mathcal{U}$  and a function  $\kappa : \{V \subseteq X \times X \mid V \subseteq U\} \rightarrow \mathcal{P}(X \times X)$  (the power set of  $X \times X$ ) such that

1.  $V \subseteq V' \subseteq U$  implies  $\kappa(V) \subseteq \kappa(V')$ ,
2. for every  $V$  in  $\mathcal{U}$  there exists  $W$  in  $\mathcal{U}$  with  $W \subseteq U$  and  $\kappa(W) \subseteq V$ ,
3. for every  $W$  in  $\mathcal{U}$  with  $W \subseteq U$  we have

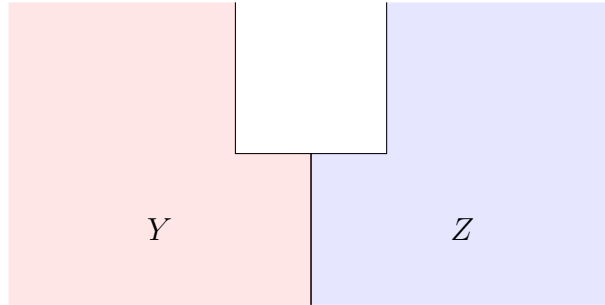
$$W[Y] \cap W[Z] \subseteq \kappa(W)[Y \cap Z] .$$

If  $X$  is a uniform bornological coarse space, then by [BEKW20, Lem. 9.26] the cone functor sends pairs  $(Y, Z)$  of subsets which are coarsely and uniformly excisive to coarsely excisive pairs.

**Example 2.14.** A typical example of coarsely (or uniformly) excisive decomposition is the decomposition  $((-\infty, 0] \times X, [0, \infty) \times X)$  of  $\mathbb{R} \otimes X$  for a coarse (or uniform) space  $X$ .

If  $X$  is presented by a path metric space, then any closed decomposition  $(Y, Z)$  is coarsely and uniformly excisive.

In contrast, the picture below shows a subset of the plane together with a non-coarsely excisive pair of subsets.



Note that this subset is not a path metric space with the restricted metric. ■

## 2.2 Coarse homology theories

In this section we recall from [BE20b, BEKW20] the notion of a coarse homology theory with values in a cocomplete stable  $\infty$ -category  $\mathbf{D}$  (e.g., the category of spectra  $\mathbf{Sp}$  or the category of  $KU$ -modules  $\mathbf{Mod}(KU)$ ).

Before we can state the axioms we need to recall some further notions from coarse geometry and introduce some notation.

Let  $X, Y$  be coarse spaces and  $f, g : X \rightarrow Y$  be two controlled maps.

**Definition 2.15.** We say that  $f$  and  $g$  are close if the map

$$f \sqcup g : \{0, 1\} \otimes X \rightarrow Y, \quad (0, x) \mapsto f(x), \quad (1, x) \mapsto g(x)$$

is controlled.

Here  $\{0, 1\}$  has the maximal coarse structure.

**Definition 2.16.** A bornological coarse space  $X$  is called *flasque* if it admits a selfmap  $f : X \rightarrow X$  (called a *witness of flasqueness*) with the following properties:

1.  $f$  is close to  $\text{id}_X$
2.  $f$  is non-expanding in the sense that for every coarse entourage  $U$  of  $X$  also  $\bigcup_{n \in \mathbb{N}} (f^n \times f^n)(U)$  is a coarse entourage
3.  $f$  is shifting in the sense that for every bounded subset  $B$  there exists  $n$  in  $\mathbb{N}$  with  $B \cap f^n(X) = \emptyset$ .

**Example 2.17.** If  $X$  is a bornological coarse space and  $[0, \infty)$  has the bornological coarse structure induced by the metric, then  $[0, \infty) \otimes X$  is flasque with witness of flasqueness  $f : X \rightarrow X$  given by  $f(t, x) := (t + 1, x)$ . ■

We consider a functor

$$E : \mathbf{BC} \rightarrow \mathbf{D} .$$

If  $\mathcal{Y}$  is a big family on  $X$  in  $\mathbf{BC}$  (see Section 2.1), then we define

$$E(\mathcal{Y}) := \text{colim}_{Y \in \mathcal{Y}} E(Y) , \tag{2.10}$$

where we equip the members of  $\mathcal{Y}$  with the bornological coarse structures induced from  $X$ .

**Definition 2.18.** The functor  $E : \mathbf{BC} \rightarrow \mathbf{D}$  is called a *coarse homology theory* if it has the following properties:

1. *Coarse invariance:* The projection  $\{0, 1\} \otimes X \rightarrow X$  induces an equivalence  $E(\{0, 1\} \otimes X) \rightarrow E(X)$  for every  $X$  in  $\mathbf{BC}$ .
2. *Excision:* For every  $X$  in  $\mathbf{BC}$  each and each pair  $(\mathcal{Y}, Z)$  of a big family  $\mathcal{Y}$  and a subset  $Z$  of  $X$  such that  $X = Y \cup Z$  for some  $Y$  in  $\mathcal{Y}$  (i.e., a complementary pair), we have a push-out square

$$\begin{array}{ccc} E(\mathcal{Y} \cap Z) & \longrightarrow & E(Z) \\ \downarrow & & \downarrow \\ E(\mathcal{Y}) & \longrightarrow & E(X) \end{array} .$$

3. *Vanishing on flasques:*  $E(X) \simeq 0$  if  $X$  is flasque.

4. *U-continuity: For every  $X$  in  $\mathbf{BC}$  the canonical morphism*

$$\operatorname{colim}_{U \in \mathcal{C}} E(X_U) \rightarrow E(X)$$

*is an equivalence, where  $X_U$  is the bornological coarse space derived from  $X$  by replacing the coarse structure by the smaller coarse structure generated by  $U$ .*

**Remark 2.19.** If  $Y$  is a subset of  $X$ , then we have a map  $E(Y) \rightarrow E(\{Y\})$ . If  $E$  is coarsely invariant, then this map is an equivalence. ■

**Remark 2.20.** Coarse invariance of  $E$  is equivalent to the condition that for every two maps  $f, g : X \rightarrow Y$  in  $\mathbf{BC}$  which are close to each other we have  $E(f) \simeq E(g)$ . ■

**Remark 2.21.** For a more symmetric version of excision let  $\mathcal{Z}$  be a big family which is complementary to  $\mathcal{Y}$  in the sense that there exists a member  $Z$  of  $\mathcal{Z}$  such that  $(\mathcal{Y}, Z)$  is a complementary pair. If  $E$  is a coarse homology theory, then we also have a push-out square

$$\begin{array}{ccc} E(\mathcal{Y} \cap \mathcal{Z}) & \longrightarrow & E(\mathcal{Z}) \\ \downarrow & & \downarrow \\ E(\mathcal{Y}) & \longrightarrow & E(X) \end{array} \quad (2.11)$$

If  $(Y, Z)$  is a coarsely excisive decomposition of  $X$ , then by [BE20b, Lem. 3.41] have a push-out square

$$\begin{array}{ccc} E(Y \cap Z) & \longrightarrow & E(Z) \\ \downarrow & & \downarrow \\ E(Y) & \longrightarrow & E(X) \end{array} .$$

■

**Remark 2.22.** In [BE20b] we have constructed a universal coarse homology theory

$$\mathbf{Yo}^s : \mathbf{BC} \rightarrow \mathbf{Sp}\mathcal{X} \quad (2.12)$$

to a certain presentable stable infinity category  $\mathbf{Sp}\mathcal{X}$  such that for every cocomplete stable  $\infty$ -category  $\mathbf{D}$  the restriction

$$(\mathbf{Yo}^s)^* : \mathbf{Fun}^{\operatorname{colim}}(\mathbf{Sp}\mathcal{X}, \mathbf{D}) \rightarrow \mathbf{Fun}(\mathbf{BC}, \mathbf{D})$$

is an equivalence onto the full subcategory of  $\mathbf{Fun}(\mathbf{BC}, \mathbf{D})$  of coarse homology theories. In particular, every coarse homology theory  $E : \mathbf{BC} \rightarrow \mathbf{D}$  has an essentially unique

colimit-preserving factorization

$$\begin{array}{ccc}
 \mathbf{BC} & \xrightarrow{E} & \mathbf{D} \\
 & \searrow \text{Yo}^s & \swarrow \text{dotted } E \\
 & & \mathbf{Sp}\mathcal{X}
 \end{array}$$

indicated by the dotted arrow and denoted by the same symbol. Since the  $\mathbf{Sp}\mathcal{X}$  plays a role analogous to the category of spectra in classical homotopy theory it is called the category of coarse spectra.  $\blacksquare$

**Example 2.23.** A basic consequence of the axioms is a canonical equivalence

$$\Sigma E(-) \simeq E(\mathbb{R} \otimes -) : \mathbf{BC} \rightarrow \mathbf{D} . \quad (2.13)$$

To this end we consider the coarsely excisive decomposition of  $\mathbb{R} \otimes X$  from Example 2.14 and get by specializing (2.21) a push-out square

$$\begin{array}{ccc}
 E(X) & \longrightarrow & E([0, \infty) \otimes X) \\
 \downarrow & & \downarrow \\
 E((-\infty, 0] \otimes X) & \longrightarrow & E(\mathbb{R} \otimes X)
 \end{array}$$

We now use that  $[0, \infty) \otimes X$  and  $(-\infty, 0] \otimes X$  are flasque by Example 2.17 in order to see that the lower left and upper right corners of this square vanish. This provides an equivalence of the lower right corner of the square with the suspension of the upper left corner.  $\blacksquare$

A coarse homology theory can have the following additional properties:

1. Continuity: For every  $X$  in  $\mathbf{BC}$  the canonical map

$$\operatorname{colim}_{Z \in \text{LF}(X)} E(Z) \rightarrow E(X)$$

is an equivalence, where  $\text{LF}(X)$  is the poset of locally finite subsets  $Z$  of  $X$ . Here  $Z$  is locally finite if the induced bornology of  $Z$  consists of the finite subsets.

2. Additivity: For any family  $(X_i)_{i \in I}$  in  $\mathbf{BC}$  the natural map  $E(\bigsqcup_{i \in I}^{\text{free}} X_i) \rightarrow \prod_{i \in I} E(X_i)$  (defined using excision and assuming that the product in  $\mathbf{D}$  exists) is an equivalence, where the free union  $\bigsqcup_{i \in I}^{\text{free}} X_i$  is the bornological coarse space with underlying set  $\bigsqcup_{i \in I} X_i$  with the coarse structure generated by entourages  $\bigcup_{i \in I} U_i$  for families  $(U_i)_{i \in I}$  in  $\prod_{i \in I} \mathcal{C}_{X_i}$ , and whose bounded subsets are subsets  $B$  such that  $B \cap X_i$  is bounded for all  $i$ , and non-empty for at most finitely many  $i$  in  $I$ .
3. Strongness:  $E(X) \simeq 0$  for weakly flasque bornological coarse spaces, where  $X$  is called weakly flasque if it admits a non-expanding and shifting (see Remark 2.22.3) selfmap  $f : X \rightarrow X$  such that  $\text{Yo}^s(f) \simeq \text{Yo}^s(\text{id}_X)$ , see (2.12).



## 2.3 Local and locally finite homology theories

In this section we recall the notions of a local and a locally finite homology theory from [BE20a, BE20b]. We further recall the construction via the cone functor of the local homology theory associated to a coarse homology theory and the implementation of additional support conditions.

We start with the notion of local finiteness which applies to functors defined on spaces equipped with a bornology. We consider a functor  $E : \mathbf{UBC} \rightarrow \mathbf{D}$  or  $E : \mathbf{TB} \rightarrow \mathbf{D}$  and assume that  $\mathbf{D}$  is stable, complete and cocomplete.

**Definition 2.24.** *We say that  $E$  is locally finite if the canonical map*

$$E(X) \rightarrow \mathbf{1}\lim_{B \in \mathcal{B}} \text{Cofib}(E(X \setminus B) \rightarrow E(X)) \quad (2.14)$$

*is an equivalence for every  $X$  with bornology  $\mathcal{B}$ .*

Equivalently,  $E$  is locally finite if

$$\mathbf{1}\lim_{B \in \mathcal{B}} E(X \setminus B) \simeq 0$$

for every  $X$ . The inclusions of locally finite functors into all functors are the right-adjoints of left Bousfield localizations

$$\begin{aligned} (-)^{\text{lf}} : \mathbf{Fun}(\mathbf{UBC}, \mathbf{D}) &\rightleftarrows \mathbf{Fun}^{\text{lf}}(\mathbf{UBC}, \mathbf{D}) : \text{incl} , \\ (-)^{\text{lf}} : \mathbf{Fun}(\mathbf{TB}, \mathbf{D}) &\rightleftarrows \mathbf{Fun}^{\text{lf}}(\mathbf{TB}, \mathbf{D}) : \text{incl} \end{aligned} \quad (2.15)$$

whose units  $E \rightarrow E^{\text{lf}}$  have the components (2.14), see [BE20b, Sec. 7.1.2].

**Remark 2.25.** In this remark we recall the characterization of homology theories on the category  $\mathbf{Top}$  of topological spaces. We let  $\ell : \mathbf{Top} \rightarrow \mathbf{Spc}$  be the Dwyer-Kan localization at the weak homotopy equivalences. This is one of the presentations of the category  $\mathbf{Spc}$  of spaces (anima). Let  $E : \mathbf{Top} \rightarrow \mathbf{D}$  be a functor to a cocomplete stable  $\infty$ -category. The functor  $E$  is a homology theory on  $\mathbf{Top}$  if it sends weak homotopy equivalences to equivalences and has the property that the induced factorization  $\mathbf{Spc} \rightarrow \mathbf{D}$

$$\begin{array}{ccc} \mathbf{Top} & \xrightarrow{E} & \mathbf{D} \\ & \searrow \ell & \nearrow \text{dotted} \\ & \mathbf{Spc} & \end{array}$$

preserves colimits.

By the universal property of  $\mathbf{Spc}$  the evaluation at  $*$  is an equivalence

$$\mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}, \mathbf{D}) \xrightarrow{\mathrm{ev}_*} \mathbf{D}$$

so that the category of  $\mathbf{D}$ -valued homology theories is equivalent to  $\mathbf{D}$  itself. For  $D$  in  $\mathbf{D}$  we write  $D(-)$  for the corresponding homology theory  $X \mapsto \ell(X) \otimes D$  (written using the tensor structure of  $\mathbf{D}$  over  $\mathbf{Spc}$ ).

A homology theory on  $\mathbf{Top}$  in particular has the following properties:

1. homotopy invariance: The map  $E([0, 1] \times X) \rightarrow E(X)$  is an equivalence for every  $X$  in  $\mathbf{Top}$ .
2. excisiveness: For every pair of big families  $(\mathcal{Y}, \mathcal{Z})$  of  $X$  (in the topological sense) covering  $X$  (i.e., there exists members  $Y$  and  $Z$  with  $Y \cup Z = X$ ) we have a push-out square

$$\begin{array}{ccc} E(\mathcal{Y} \cap \mathcal{Z}) & \longrightarrow & E(\mathcal{Y}) \\ \downarrow & & \downarrow \\ E(\mathcal{Z}) & \longrightarrow & E(X) \end{array} .$$

Note that these two properties do not suffice to characterize homology theories. The formulation of excision with big families avoids to talk about the usual additional point-set theoretical assumptions on decompositions of  $X$  into two subsets. But if, e.g.,  $(Y, Z)$  is a closed decomposition of a metric space  $X$  such that the inclusions  $Y \rightarrow U_r[Y]$  and  $Z \rightarrow U_r[Z]$  are homotopy equivalences for all sufficiently small  $r$ , then we have a push-out

$$\begin{array}{ccc} E(Y \cap Z) & \longrightarrow & E(Y) \\ \downarrow & & \downarrow \\ E(Z) & \longrightarrow & E(X) \end{array}$$

provided  $E$  is homotopy invariant and excisive. ■

Let  $E : \mathbf{TB} \rightarrow \mathbf{D}$  be a functor to a cocomplete and complete stable  $\infty$ -category.

**Definition 2.26** ([BE20b, 7.27]). *The functor  $E$  is a locally finite homology theory if it is*

1. *homotopy invariant*
2. *excisive*
3. *locally finite*.

**Remark 2.27.** By [BE20b, 7.1.4] the composition

$$\mathbf{Fun}(\mathbf{Top}, \mathbf{D}) \rightarrow \mathbf{Fun}(\mathbf{TB}, \mathbf{D}) \xrightarrow{(-)^{\text{lf}}} \mathbf{Fun}^{\text{lf}}(\mathbf{TB}, \mathbf{D})$$

of the restriction along the forgetful functor  $\mathbf{TB} \rightarrow \mathbf{Top}$  and the left-adjoint in (2.15) sends homology theories  $E$  to locally finite homology theories  $E^{\text{lf}}$ . The original homology theory can be recovered from  $E^{\text{lf}}$  as the homology theory represented by the spectrum  $E^{\text{lf}}(*)$ .

On the other hand, if  $F : \mathbf{TB} \rightarrow \mathbf{D}$  is a locally finite homology theory, then the spectrum  $F(*)$  represents a cohomology theory  $F(*)(-)$  and we have a natural transformation of functors  $F(*)^{\text{lf}}(-) \rightarrow F(-) : \mathbf{TB} \rightarrow \mathbf{D}$ , which attempts to recover  $F$  from its value on the point. By [BE20b, Thm. 7.43] the map  $F(*)^{\text{lf}}(X) \rightarrow F(X)$  is an equivalence for a large class of topological bornological spaces, e.g, for locally finite simplicial complexes with the topology and bornology induced by the spherical path metric [BE20b, Ex. 7.42]. ■

**Remark 2.28.** We say that  $E : \mathbf{TB} \rightarrow \mathbf{D}$  is open (or closed) excisive if  $E$  sends open (or closed) decompositions to push-out squares. A locally finite and open (or closed) excisive functor  $E : \mathbf{TB} \rightarrow \mathbf{D}$  has an additional contravariant functoriality for topological bornological inclusions  $Y \rightarrow X$  of open (closed) subsets. Indeed, by open (or closed) excision for every closed (or open)  $B$  in  $\mathcal{B} \cap Y$  we have

$$\text{Cofib}(E(Y \setminus B) \rightarrow E(Y)) \xrightarrow{\simeq} \text{Cofib}(E(X \setminus B) \rightarrow E(X)) .$$

In the limit in (2.14) we can restrict to closed (open) bounded subsets contained in  $Y$ . By local finiteness get we a map

$$\begin{aligned} E(X) &\simeq \varprojlim_{B \in \mathcal{B}} \text{Cofib}(E(X \setminus B) \rightarrow E(X)) \rightarrow \varprojlim_{B \in \mathcal{B} \cap Y} \text{Cofib}(E(X \setminus B) \rightarrow E(X)) \\ &\simeq \varprojlim_{B \in \mathcal{B} \cap Y} \text{Cofib}(E(Y \setminus B) \rightarrow E(Y)) \simeq E(Y) . \end{aligned} \quad \blacksquare$$

We now consider functors  $E : \mathbf{UBC} \rightarrow \mathbf{D}$  to a stable cocomplete  $\infty$ -category  $\mathbf{D}$ . Homotopy invariance is defined as above for  $\mathbf{TB}$ . We will consider the following additional properties:

1. excisiveness: For every coarsely and uniformly excisive decomposition  $(Y, Z)$  of  $X$  we have a push-out square

$$\begin{array}{ccc} E(Y \cap Z) & \longrightarrow & E(Y) \\ \downarrow & & \downarrow \\ E(Z) & \longrightarrow & E(X) \end{array} .$$

2. vanishing on flasques:  $E([0, \infty) \otimes X) \simeq 0$  for all  $X$  in  $\mathbf{UBC}$ .

3.  $u$ -continuity:  $\operatorname{colim}_{U \in \mathcal{C} \cap \mathcal{U}} E(X_U) \xrightarrow{\simeq} E(X)$ , where  $X_U$  is obtained from  $X$  by replacing the coarse structure by the smaller coarse structure generated by  $U$ .

**Definition 2.29.** *The functor  $E$  is called a local homology theory if it has the following properties:*

1. *homotopy invariance*
2. *excisiveness*
3. *vanishing on flasques*
4.  *$u$ -continuity.*

We have then following constructions of local homology theories:

1. If  $E : \mathbf{UBC} \rightarrow \mathbf{D}$  is a local homology theory which is coarsening invariant [BE20a, Def. 11.8] and  $\mathbf{D}$  is complete, then  $E^{\text{lf}}$  is again a (coarsening invariant, see [BE20a, Lem. 11.10]) local homology theory which is in addition locally finite. The assumption on coarsening invariance is needed in order to ensure that  $E^{\text{lf}}$  is  $u$ -continuous. Without this assumption we would have the problem of commuting the limit defining  $E^{\text{lf}}$  with the colimit appearing in the  $u$ -continuity condition.
2. If  $E : \mathbf{TB} \rightarrow \mathbf{D}$  is a locally finite homology theory, then  $E\tau : \mathbf{UBC} \rightarrow \mathbf{TB}$  is a local homology theory [BE20a, Lem. 3.16].
3. If  $E : \mathbf{BC} \rightarrow \mathbf{D}$  is a coarse homology theory, then  $E\iota : \mathbf{UBC} \rightarrow \mathbf{D}$  is a local homology theory [BE20a, Lem 3.13].
4. If  $E : \mathbf{BC} \rightarrow \mathbf{D}$  is a coarse homology theory which is in addition strong, then  $E\mathcal{O}^\infty := E \circ \mathcal{O}^\infty : \mathbf{UBC} \rightarrow \mathbf{D}$  is a local homology theory [BE20a, Lem. 9.6]. It is coarsening invariant by [BE20a, Ex. 11.9].

**Example 2.30.** The cone boundary (2.9) together with (2.13) induces a natural transformation of local homology theories

$$a : \Sigma^{-1}E\mathcal{O}^\infty \rightarrow E\iota : \mathbf{UBC} \rightarrow \mathbf{D} . \quad (2.16)$$

which we call the index map. ■

Note that a local homology theory  $E : \mathbf{UBC} \rightarrow \mathbf{D}$  with complete target is not necessarily locally finite. By [BE20a, Sec. 11] we have comparison transformations

$$E(*)^{\text{lf}}(-) \rightarrow E^{\text{lf}} \leftarrow E : \mathbf{UBC} \rightarrow \mathbf{D} , \quad (2.17)$$

where  $E(*)^{\text{lf}}(-)$  denotes the locally finite homology theory represented by the spectrum  $E(*)$ .

**Example 2.31.** Let  $E : \mathbf{BC} \rightarrow \mathbf{D}$  be a strong and additive coarse homology theory with complete target. If  $X$  in  $\mathbf{UBC}$  is presented by a locally finite finite-dimensional simplicial complex with the spherical path metric, then by [BE20a, Prop. 11.23] transformations from (2.17) induce equivalences

$$\Sigma E(*)^{\text{lf}}(X) \xrightarrow{\simeq} (E\mathcal{O}^\infty)^{\text{lf}}(X) \xleftarrow{\simeq} E\mathcal{O}^\infty(X) .$$

Here we used the equivalence  $E\mathcal{O}^\infty(*) \simeq \Sigma E(*)$ . ■

**Example 2.32.** Let  $X$  in  $\mathbf{UBC}$  be a uniform bornological coarse space whose bornology is the bornology of relatively compact subsets of the underlying topological space. Note that  $X$  is then necessarily locally compact.

We assume that  $E : \mathbf{BC} \rightarrow \mathbf{D}$  is a strong coarse homology theory with complete target  $\mathbf{D}$ . By the coarsening invariance of  $E\mathcal{O}^\infty$  [BEKW20, Prop. 9.33], [BE20a, Ex. 11.9] we know that  $E\mathcal{O}^\infty(X)$ , and therefore also  $(E\mathcal{O}^\infty)^{\text{lf}}(X)$  do not depend on the choice of  $\mathcal{C}$ . Since the restrictions the uniform structures as above to a compact subset of  $X$  is determined by the topology of  $X$  alone we conclude that  $(E\mathcal{O}^\infty)^{\text{lf}}(X)$  does not depend on the uniform structure as well. So  $(E\mathcal{O}^\infty)^{\text{lf}}(X)$  is an invariant of the locally compact topological space  $X$ . By Example 2.31, if  $X$  is a locally finite finite-dimensional simplicial complex and  $E$  is in addition additive, then  $(E\mathcal{O}^\infty)^{\text{lf}}(X) \simeq \Sigma E(*)^{\text{lf}}(X)$  where the right-hand side clearly only depends on the locally compact space  $X$ .

Assume now that  $X$  in  $\mathbf{UBC}$  has the additional structure of a finite-dimensional, locally finite simplicial complex whose simplices are uniformly equicontinuous and coarsely bounded. Let  $X_{\text{simp}}$  be  $X$  equipped with the uniform and coarse structure induced by the spherical path metric of the simplices. In both cases we assume that the bornologies consist of relatively compact subsets. Then the identity of the underlying set is a morphism  $X_{\text{simp}} \rightarrow X$  in  $\mathbf{UBC}$ . We have a commutative diagram

$$\begin{array}{ccccc} E\mathcal{O}^\infty(X_{\text{simp}}) & \xrightarrow{\simeq} & (E\mathcal{O}^\infty)^{\text{lf}}(X_{\text{simp}}) & \xleftarrow{\simeq} & \Sigma E(*)^{\text{lf}}(X_{\text{simp}}) \\ \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ E\mathcal{O}^\infty(X) & \longrightarrow & (E\mathcal{O}^\infty)^{\text{lf}}(X) & \xleftarrow{\simeq} & \Sigma E(*)^{\text{lf}}(X) \end{array}$$

which presents  $\Sigma E(*)^{\text{lf}}(X)$  as a retract of  $E\mathcal{O}^\infty(X)$ . ■

**Remark 2.33.** The functor

$$\Sigma E(*)^{\text{lf}}(-) : \mathbf{UBC} \rightarrow \mathbf{D}$$

has the additional contravariant functoriality for inclusions of open subsets. We do not know whether  $E\mathcal{O}^\infty(-)$  has such a contravariant functoriality. ■

**Example 2.34.** The consideration from Example 2.32 applies in particular to a complete finite-dimensional Riemannian manifold  $M$  presenting an object of **UBC**. Assume that  $E$  is strong and additive with complete target  $\mathbf{D}$ . Then  $\Sigma E(*)^{\text{lf}}(M)$  is a retract of  $E\mathcal{O}^\infty(M)$ . We get a factorization of the index map over a locally finite version  $a_M^{\text{lf}}$

$$\begin{array}{ccccc}
 & & \curvearrowright & & \\
 \Sigma^{-1}E\mathcal{O}^\infty(M) & \longrightarrow & E(*)^{\text{lf}}(M) & \xleftarrow{\simeq} & \Sigma^{-1}E\mathcal{O}^\infty(M_{\text{simp}}) \\
 & \searrow^{a_M} & \downarrow^{a_M^{\text{lf}}} & \swarrow^{a_{M_{\text{simp}}}} & \\
 & & E(M) & & 
 \end{array}$$

of the index map. ■

We now take advantage of the geometric construction of local homology theories via the cone functor in order to add support conditions. Let  $X$  be in **UBC** and  $\mathcal{Y}$  and  $\mathcal{Z}$  be two big families in  $X$ .

**Definition 2.35.** We define the big families

$$\mathcal{O}^\infty(\mathcal{Y}) := \mathbb{R} \times \mathcal{Y}, \quad \mathcal{O}^-(\mathcal{Y}) := \{\mathbb{R}^-\} \times \mathcal{Y}$$

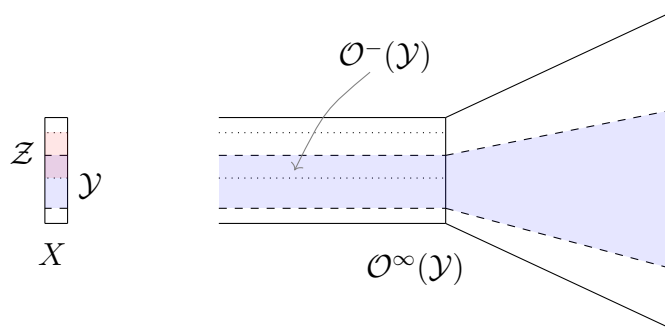
and

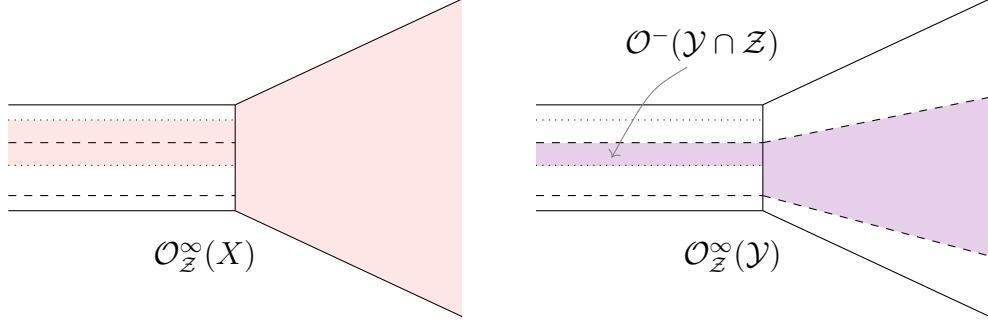
$$\mathcal{O}_Z^\infty(X) := \{\mathbb{R}^-\} \times \mathcal{Z} \cup \{\mathbb{R}^+\} \times X, \quad \mathcal{O}_Z^\infty(\mathcal{Y}) := \mathcal{O}_Z^\infty(X) \cap \mathcal{O}^\infty(\mathcal{Y})$$

in  $\mathcal{O}^\infty(X)$ .

Recall Example 2.4 for the notations in the above definition.

Note that all members of  $\mathcal{O}^-(\mathcal{Y})$  are flasque.





We let  $\mathbf{UBC}^{(2)}$  be the category of pairs  $(X, \mathcal{Z})$  of  $X$  in  $\mathbf{UBC}$  and a big family  $\mathcal{Z}$ . A morphism  $f : (X, \mathcal{Z}) \rightarrow (X', \mathcal{Z}')$  is a morphism  $f : X \rightarrow X'$  in  $\mathbf{UBC}$  such that  $f(\mathcal{Z}) \subseteq \mathcal{Z}'$ . We define the category  $\mathbf{BC}^{(2)}$  of pairs  $(X, \mathcal{Z})$  of bornological coarse spaces with a big family similarly. We have a forgetful functor  $\iota : \mathbf{UBC}^{(2)} \rightarrow \mathbf{BC}^{(2)}$  and the functor

$$\mathcal{O}_-^\infty : \mathbf{UBC}^{(2)} \rightarrow \mathbf{BC}^{(2)}, \quad (X, \mathcal{Z}) \mapsto (\mathcal{O}^\infty(X), \mathcal{O}_\mathcal{Z}^\infty(X)).$$

If  $E : \mathbf{BC} \rightarrow \mathbf{D}$  is a functor to a cocomplete stable  $\infty$ -category, then we get the functor

$$E\mathcal{O}_-^\infty : \mathbf{UBC}^{(2)} \rightarrow \mathbf{D}, \quad (X, \mathcal{Z}) \mapsto E(\mathcal{O}_\mathcal{Z}^\infty(X)). \quad (2.18)$$

It comes with a natural transformation  $E\mathcal{O}_-^\infty \rightarrow E\mathcal{O}^\infty$  induced by the maps  $\mathcal{O}_\mathcal{Z}^\infty(X) \rightarrow \mathcal{O}^\infty(X)$  for every  $(X, \mathcal{Z})$  in  $\mathbf{UBC}^{(2)}$ , where we consider  $E\mathcal{O}^\infty$  as a functor on  $\mathbf{UBC}^{(2)}$  using the forgetful functor  $(X, \mathcal{Z}) \mapsto X$ .

We now assume that  $E : \mathbf{BC} \rightarrow \mathbf{D}$  is a strong coarse homology theory. As discussed above, the induced functor  $E\mathcal{O}^\infty : \mathbf{UBC} \rightarrow \mathbf{D}$  is a local homology theory.

**Definition 2.36.** *We call the functor  $E\mathcal{O}_-^\infty : \mathbf{UBC}^{(2)} \rightarrow \mathbf{D}$  from (2.18) the associated local homology theory with support.*

**Remark 2.37.** Let  $\mathcal{Z}$  be a big family in  $X$ . For a subset  $Y$  or a big family  $\mathcal{Y}$  on  $X$ , in order to simplify the notation, we will write  $E\mathcal{O}_\mathcal{Z}^\infty(Y) := E\mathcal{O}_{Y \cap \mathcal{Z}}^\infty(Y)$  or  $E\mathcal{O}_\mathcal{Z}^\infty(\mathcal{Y}) := \operatorname{colim}_{Y \in \mathcal{Y}} E\mathcal{O}_{\mathcal{Z} \cap Y}^\infty(Y)$ . This notation is consistent since  $E\mathcal{O}_\mathcal{Z}^\infty(\mathcal{Y}) \simeq E(\mathcal{O}_\mathcal{Z}^\infty(\mathcal{Y}))$ , where  $\mathcal{O}_\mathcal{Z}^\infty(\mathcal{Y})$  is defined in Definition 2.35.  $\blacksquare$

**Lemma 2.38.** *We have a functorial fibre sequence*

$$E\iota(\mathcal{Z}) \rightarrow E\mathcal{O}_{\{\emptyset\}}^\infty(X) \rightarrow E\mathcal{O}_\mathcal{Z}^\infty(X). \quad (2.19)$$

*Proof.* This is the Mayer-Vietoris sequence for the decomposition  $(\{\mathbb{R}^-\} \times \mathcal{Z}, \{\mathbb{R}^+\} \times X)$  of  $\mathcal{O}_\mathcal{Z}^\infty(X)$  together with the flasqueness of the members of  $\{\mathbb{R}^-\} \times \mathcal{Z}$ .  $\square$

The following lemma says that  $E\mathcal{O}_-^\infty(-)$  has the analogues for pairs of the properties of a local homology theory.

**Lemma 2.39.** *The functor  $E\mathcal{O}_-^\infty(-) : \mathbf{UBC}^{(2)} \rightarrow \mathbf{D}$  has the following properties:*

1. *homotopy invariant: For  $(X, \mathcal{Z})$  in  $\mathbf{UBC}$  the projection  $([0, 1] \otimes X, [0, 1] \times \mathcal{Z}) \rightarrow (X, \mathcal{Z})$  induces an equivalence  $E\mathcal{O}_{[0,1] \times \mathcal{Z}}^\infty([0, 1] \otimes X) \rightarrow E\mathcal{O}_\mathcal{Z}^\infty(X)$ .*
2. *excisiveness: If  $(X, \mathcal{Z})$  is in  $\mathbf{UBC}^{(2)}$  and  $(A, B)$  is a coarsely and uniformly excisive decomposition of  $X$ , then the square*

$$\begin{array}{ccc} E\mathcal{O}_\mathcal{Z}^\infty(A \cap B) & \longrightarrow & E\mathcal{O}_\mathcal{Z}^\infty(B) \\ \downarrow & & \downarrow \\ E\mathcal{O}_\mathcal{Z}^\infty(A) & \longrightarrow & E\mathcal{O}_\mathcal{Z}^\infty(X) \end{array}$$

*is a push-out square.*

3. *vanishing on flasques: For every  $(X, \mathcal{Z})$  in  $\mathbf{UBC}^{(2)}$  we have  $E\mathcal{O}_{\mathbb{R} \times \mathcal{Z}}^\infty(\mathbb{R} \otimes X) \simeq 0$ .*
4. *u-continuity: For every  $(X, \mathcal{Z})$  in  $\mathbf{UBC}^{(2)}$  we have  $\operatorname{colim}_{U \in \mathcal{C}} E\mathcal{O}_\mathcal{Z}^\infty(X_U) \simeq E\mathcal{O}_\mathcal{Z}^\infty(X)$ .*

*Proof.* One option is to redo the argument of [BE20a, Lem. 9.6] with the additional support condition. Alternatively one could employ the fibre sequence (2.19) and the fact that  $E\iota$  and  $E\mathcal{O}$  are known to be local homology theories (see [BE20a, Lem 3.13] for  $E\iota$  and [BE20a, Lem. 9.6] for  $E\mathcal{O}$ ).  $\square$

We now assume that  $E : \mathbf{BC} \rightarrow \mathbf{D}$  is a strong coarse homology theory. We incorporate the support conditions into the index map (2.16) as follows. We define the functor

$$E\iota' : \mathbf{UBC}^{(2)} \rightarrow \mathbf{D}, \quad (X, \mathcal{Z}) \mapsto E\iota(\mathcal{Z}).$$

We added the superscript  $'$  in order to make clear that this functor takes the big family instead of the space.

**Definition 2.40.** *The index map*

$$a : \Sigma^{-1}E\mathcal{O}_-^\infty \rightarrow E\iota' : \mathbf{UBC}^{(2)} \rightarrow \mathbf{D}$$

*with support conditions is the natural transformation of functors  $\mathbf{UBC}^{(2)} \rightarrow \mathbf{D}$  with components*

$$a_{X, \mathcal{Z}} : \Sigma^{-1}E\mathcal{O}_\mathcal{Z}^\infty(X) \rightarrow E\iota(\mathcal{Z}) \tag{2.20}$$

*given by*

$$\Sigma^{-1}E\mathcal{O}_\mathcal{Z}^\infty(X) \xrightarrow{\partial^{\operatorname{cone}}} \Sigma^{-1}E(\{\mathbb{R}^-\} \otimes \mathcal{Z} \cup \{\mathbb{R}^+\} \otimes X) \xrightarrow{\partial^{MV}} E(\mathcal{Z}).$$



Here we consider  $\{\mathbb{R}^-\} \otimes \mathcal{Z} \cup \{\mathbb{R}^+\} \otimes X$  as a big family of  $\mathbb{R} \otimes X$ ,  $\partial^{MV}$  is the Mayer-Vietoris boundary associated to the decomposition into  $(\{\mathbb{R}^-\} \otimes \mathcal{Z}, \{\mathbb{R}^+\} \otimes X)$ , and we use that the inclusion  $X \rightarrow \mathbb{R} \times X$ ,  $x \mapsto (0, x)$  induces an equivalence  $E(\mathcal{Z}) \simeq E(\{0\} \times \mathcal{Z})$ .

**Remark 2.41.** The index map is the boundary map of the fibre sequence (2.19). In particular, we have a fibre sequence

$$\Sigma^{-1}E\mathcal{O}_{\{\emptyset\}}^\infty(X) \rightarrow \Sigma^{-1}E\mathcal{O}_{\mathcal{Z}}^\infty(X) \xrightarrow{a_{X,\mathcal{Z}}} E(\mathcal{Z}). \quad (2.21)$$

■

## 2.4 $C^*$ -algebras and categories and their $K$ -theory

In this section we recall some basic constructions from non-commutative homotopy theory. In particular we recall the  $E$ - and  $K$ -theory for graded  $C^*$ -algebra. Instead of providing explicit cycle-by-relation constructions we will give a complete description in terms of universal properties. All needed classes will be derived from explicit homomorphisms between suitable  $C^*$ -algebras (see Remark 2.48 and Remark 2.49), and the boundary operators in long exact sequences come from general homotopy-theoretic principles (see Remark 2.42).

Let  $G$  be a countable discrete group. By  $GC^*\mathbf{Alg}^{\text{nu}}$  we denote the category of not necessarily unital  $C^*$ -algebras with a  $G$ -action by automorphisms. The left action of  $G$  on itself turns the Hilbert space  $L^2(G)$  into a  $G$ -Hilbert space. We let  $K_G$  denote the algebra of compact operators on  $L^2(G) \otimes \ell^2$  with the induced  $G$ -action. In the present paper we will need the cases of a trivial group and the two-element group  $G = C_2$ . The category  $GC^*\mathbf{Alg}^{\text{nu}}$  has a symmetric monoidal structure  $\otimes$  given by the maximal tensor product.

We will consider the following properties of a functor  $F : GC^*\mathbf{Alg}^{\text{nu}} \rightarrow \mathbf{D}$  to some cocomplete stable  $\infty$ -category  $\mathbf{D}$ :

1. homotopy invariance: For every  $A$  in  $C^*\mathbf{Alg}^{\text{nu}}$  the inclusion  $A \rightarrow A \otimes C([0, 1])$  (induced by the inclusion  $\mathbb{C} \rightarrow C([0, 1])$  as constant functions) induces an equivalence  $F(A) \rightarrow F(A \otimes C([0, 1]))$ .
2.  $K_G$ -stability:  $F$  sends  $K_G$ -equivalences to an equivalences. Thereby  $f : A \rightarrow B$  is a  $K_G$ -equivalence if  $f \otimes \text{id}_{K_G} : A \otimes K_G \rightarrow B \otimes K_G$  is a homotopy equivalence, or equivalently, is sent to an equivalence by every homotopy invariant functor.
3. exactness:  $F(0) \simeq 0$  and  $F$  sends every exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  in  $GC^*\mathbf{Alg}^{\text{nu}}$  to a fibre sequence  $F(A) \rightarrow F(B) \rightarrow F(C)$  in  $\mathbf{D}$ .
4. s-finitary: For every  $A$  in  $GC^*\mathbf{Alg}^{\text{nu}}$  the natural morphism induces an equivalence

$\operatorname{colim}_{A' \subseteq_{\text{sep}} A} F(A') \simeq F(A)$ , where  $A' \subseteq_{\text{sep}} A$  is the poset of  $G$ -invariant separable subalgebras of  $A$ .

5. sum-preserving: For any family  $(A_i)_{i \in I}$  in  $GC^* \mathbf{Alg}^{\text{nu}}$  the canonical map induced by the inclusions  $A_j \rightarrow \bigoplus_{i \in I} A_i$  for all  $j$  in  $I$  induces an equivalence  $\bigoplus_{i \in I} F(A_i) \simeq F(\bigoplus_{i \in I} A_i)$ .

**Remark 2.42.** Note that exactness is a property the functor can have or not. If  $F$  is exact, then the boundary map  $\partial : F(C) \rightarrow \Sigma F(A)$  is implicitly encoded into the functor.

Consider for example the  $K$ -theory functor  $K : C^* \mathbf{Alg}^{\text{nu}} \rightarrow \mathbf{Mod}(KU)$  described further below. This functor turns out to be exact and therefore sends an exact sequence of  $C^*$ -algebras to a fibre sequence of  $KU$ -modules. Upon taking homotopy groups we get a long exact sequence

$$K_*(A) \rightarrow K_*(B) \rightarrow K_*(X) \xrightarrow{\partial} K_{*-1}(A)$$

of abelian groups. Classically one considers the graded group-valued functor  $K_*$  only. In order to even state the exactness of the latter one must provide the boundary as an additional datum. In the cycle-by-relation picture it is given by an explicit formula involving unitaries and projections, and this approach requires various verifications of well-definedness, exactness, and naturality. Here different authors use different formulas which have to be compared. Eventually, all these constructions, at least up to sign, provide the boundary map coming from homotopy theory. We refer to the discussion of this question in [BLP23]. ■

Following [Bun23] (for  $G = e$ ) and [BD24] (for general  $G$ ) we adopt the following definition:

**Definition 2.43** ([BD24]). *The equivariant  $E$ -theory functor*

$$e^G : GC^* \mathbf{Alg}^{\text{nu}} \rightarrow \mathbf{E}^G$$

*is an initial homotopy invariant,  $K_G$ -stable, exact, sum-preserving and  $s$ -finitary functor to a cocomplete stable  $\infty$ -category.*

In other words, the equivariant  $E$ -theory functor is uniquely characterized by the universal property that  $\mathbf{E}^G$  is cocomplete and stable, and that

$$(e^G)^* : \mathbf{Fun}^{\operatorname{colim}}(\mathbf{E}^G, \mathbf{D}) \rightarrow \mathbf{Fun}^{h, K_G, \text{ex}, \text{sfin}, \oplus}(GC^* \mathbf{Alg}^{\text{nu}}, \mathbf{D})$$

is an equivalence for any cocomplete stable  $\infty$ -category  $\mathbf{D}$ , where the target is the full subcategory of  $\mathbf{Fun}(GC^* \mathbf{Alg}^{\text{nu}}, \mathbf{D})$  of functors having the list of properties indicated by the superscripts. In the case of  $G = \{e\}$  we will omit the superscript  $G$ .

Since for any  $A$  in  $GC^*\mathbf{Alg}^{\text{nu}}$  precomposition with  $A \otimes - : GC^*\mathbf{Alg}^{\text{nu}} \rightarrow GC^*\mathbf{Alg}^{\text{nu}}$  preserves homotopy invariant,  $K_G$ -stable, exact, sum-preserving and  $s$ -finitary functors the category  $E^G$  has furthermore a uniquely determined symmetric monoidal structure such that  $e^G$  extends to a symmetric monoidal functor. It turns out that  $E^G$  is a presentably symmetric monoidal  $\aleph_1$ -presentable stable  $\infty$ -category [BD24].

The tensor unit of  $E$  is  $e(\mathbb{C})$ . One can check [Bun23] that the commutative ring spectrum

$$KU := \mathbf{map}_E(e(\mathbb{C}), e(\mathbb{C}))$$

in  $\mathbf{CAlg}(\mathbf{Sp})$  is equivalent to the usual complex  $K$ -theory spectrum. The symmetric monoidal functor  $\text{Res}_G : C^*\mathbf{Alg}^{\text{nu}} \rightarrow GC^*\mathbf{Alg}^{\text{nu}}$  equipping  $C^*$ -algebras with the trivial  $G$ -action descends to a symmetric monoidal functor  $\text{Res}_G : E \rightarrow E^G$ . Consequently,  $E^G$  is enriched over the category  $\mathbf{Mod}(KU)$  of  $KU$ -modules. For  $A, B$  in  $GC^*\mathbf{Alg}^{\text{nu}}$  we will write

$$E^G(A, B) := \mathbf{map}_{E^G}(e^G(A), e^G(B))$$

for the bivariant  $E^G$ -theory  $KU$ -module of  $A, B$ . Considering the mapping spectra in  $E^G$  as  $KU$ -modules encodes Bott-periodicity in a natural way.

**Remark 2.44.** As shown in [BD24], for separable  $A, B$  in  $GC^*\mathbf{Alg}^{\text{nu}}$  there is a canonical isomorphism of groups between  $\pi_0 E^G(A, B)$  and the classical equivariant  $E$ -theory groups as defined, e.g., in [GHT00] using homotopy classes of asymptotic morphisms. Furthermore, the restriction of  $e^G$  to separable  $G$ - $C^*$ -algebras yields a functor  $GC^*\mathbf{Alg}_{\text{sep}}^{\text{nu}} \rightarrow (E^G)^{\aleph_1}$ , where  $(E^G)^{\aleph_1}$  denotes the full subcategory of  $\aleph_1$ -compact objects in  $E^G$ . Upon going to the homotopy category this functor is equivalent to the equivariant  $E$ -theory functor constructed in [GHT00]. This justifies to consider the functor characterized in Definition 2.43 as the correct  $\infty$ -categorical enhancement of classical  $E$ -theory.  $\blacksquare$

We next describe the extension of  $E$ -theory to graded  $C^*$ -algebras. Our approach is an  $\infty$ -categorical version of [Haa99]. The category  $(C^*\mathbf{Alg}^{\text{nu}})^{\text{gr}}$  of graded  $C^*$ -algebras is the same as the category  $C_2C^*\mathbf{Alg}^{\text{nu}}$  of  $C_2$ - $C^*$ -algebras, but we equip it with the graded version  $\hat{\otimes}$  of the maximal tensor product which involves Koszul sign rules. One again checks that for any  $A$  in  $C_2C^*\mathbf{Alg}^{\text{nu}}$  precomposition by the functor  $A \hat{\otimes} - : C_2C^*\mathbf{Alg}^{\text{nu}} \rightarrow C_2C^*\mathbf{Alg}^{\text{nu}}$  preserves homotopy invariant,  $K_{C_2}$ -stable, exact, sum-preserving and  $s$ -finitary functors. Consequently, the category  $E^{C_2}$  has an essentially unique symmetric monoidal structure  $\hat{\otimes}$  such that  $e^{C_2}$  has an essentially unique symmetric monoidal refinement with respect to the graded tensor product  $\hat{\otimes}$  on  $C_2C^*\mathbf{Alg}^{\text{nu}} = (C^*\mathbf{Alg}^{\text{nu}})^{\text{gr}}$ .

We now consider the graded  $C^*$ -algebra  $\mathcal{S} := C_0(\mathbb{R})$  with the grading involution sending  $f(x)$  to  $f(-x)$ . If  $A$  is a graded  $C^*$ -algebra and  $\Phi$  is a selfadjoint odd unbounded multiplier of  $A$  such that  $(i + \Psi)^{-1} \in A$ , then using function calculus we can define a morphism of graded  $C^*$ -algebras

$$\Psi_* : \mathcal{S} \rightarrow A, \quad f \mapsto f(\Psi). \quad (2.22)$$

We can recover  $\Psi$  by  $\Psi = \Psi_*(X)$ , where  $X$  is the multiplier of  $\mathcal{S}$  generating the identity morphism (given by  $(Xf)(x) := xf(x)$ ), and where we implicitly used the extension of  $\Psi_*$  to some unbounded multipliers. The algebra  $\mathcal{S}$  has a structure of a coalgebra in  $(C^*\mathbf{Alg}^{\text{nu}})^{\text{gr}}$  with counit  $\epsilon : \mathcal{S} \rightarrow \mathbb{C}$  given by  $f \mapsto f(0)$  (induced by the multiplier 0 on  $\mathbb{C}$ ) and the coproduct

$$\Delta : \mathcal{S} \rightarrow \mathcal{S} \hat{\otimes} \mathcal{S}, \quad (2.23)$$

induced by the multiplier  $X \hat{\otimes} 1 + 1 \hat{\otimes} X$  (see, e.g., [HG04] for a nice description). Since  $e^{C_2}$  is symmetric monoidal,  $e^{C_2}(\mathcal{S})$  becomes a coalgebra in  $E^{C_2}$ .

**Definition 2.45.** *We denote by*

$$E^{\text{gr}} := \mathbf{Comod}_{E^{C_2}}(e^{C_2}(\mathcal{S}))$$

*the presentably symmetric monoidal stable  $\infty$ -category of comodules over the coalgebra  $e^{C_2}(\mathcal{S})$  in  $E^{C_2}$ . We further define the lax symmetric monoidal functor*

$$e^{\text{gr}} := e^{C_2}(\mathcal{S}) \hat{\otimes} e^{C_2}(-) : (C^*\mathbf{Alg}^{\text{nu}})^{\text{gr}} \rightarrow E^{\text{gr}} .$$

Thus by definition, the functor  $e^{\text{gr}}$  sends a graded  $C^*$ -algebra  $A$  to the cofree comodule  $e^{C_2}(\mathcal{S}) \hat{\otimes} e^{C_2}(A)$  over  $e^{C_2}(\mathcal{S})$ . The functor  $e^{\text{gr}}$  is homotopy invariant,  $K_{C_2}$ -stable, exact, sum-preserving and s-finitary. We do not know whether this functor satisfies a universal property.

**Remark 2.46.** In order to see that our definition reproduces the classical  $E$ -theory groups for graded  $C^*$ -algebras  $A, B$  we observe (using the universal property of the cofree comodule functor) that

$$E^{\text{gr}}(A, B) \simeq E^{C_2}(\mathcal{S} \hat{\otimes} A, B) . \quad (2.24)$$

If  $A$  and  $B$  are separable, then it follows from [Haa99] and the fact [BD24] that the group  $\pi_0 E^{C_2}(\mathcal{S} \hat{\otimes} A, B)$  is the classical  $C_2$ -equivariant  $E$ -theory group that  $\pi_0 E^{\text{gr}}(A, B)$  is classical graded  $E$ -theory group of  $A$  and  $B$ .  $\blacksquare$

Let  $\text{triv} : C^*\mathbf{Alg}^{\text{nu}} \rightarrow (C^*\mathbf{Alg}^{\text{nu}})^{\text{gr}}$  be the functor (the same as  $\text{Res}_{C_2}$ ) which equips a  $C^*$ -algebra with the trivial grading. Then we have a commutative diagram

$$\begin{array}{ccc} C^*\mathbf{Alg}^{\text{nu}} & \xrightarrow{e} & E \\ \downarrow \text{triv} & & \downarrow e^{C_2}(\mathcal{S}) \hat{\otimes} \text{triv}(-) \\ (C^*\mathbf{Alg}^{\text{nu}})^{\text{gr}} & \xrightarrow{e^{\text{gr}}} & E^{\text{gr}} \end{array} . \quad (2.25)$$

We argue that the right vertical arrow is fully faithful. In other words, the  $E$ -theory for graded  $C^*$ -algebras extends the  $E$ -theory for ungraded ones. To this end, for  $A, B$  in  $E$ ,

we consider the diagram

$$\begin{array}{ccccc}
& & \xrightarrow{\simeq} & \mathbf{map}_{\mathbf{E}^{\text{gr}}}(e^{C_2}(\mathcal{S}) \hat{\otimes} \text{triv}(A), e^{C_2}(\mathcal{S}) \hat{\otimes} \text{triv}(B)) & \\
& \swarrow & & \downarrow \epsilon_*, (2.24) & \\
\mathbf{map}_{\mathbf{E}}(A, B) & \xrightarrow{\text{Res}_{C_2}} & \mathbf{map}_{\mathbf{E}^{C_2}}(\text{Res}_{C_2}(A), \text{Res}_{C_2}(B)) & \xrightarrow{\epsilon^*} & \mathbf{map}_{\mathbf{E}^{C_2}}(e^{C_2}(\mathcal{S}) \hat{\otimes} \text{triv}(A), \text{triv}(B)) \\
\parallel & & \downarrow \simeq & & \downarrow \simeq \\
\mathbf{map}_{\mathbf{E}}(A, B) & \xrightarrow{\pi^*} & \mathbf{map}_{\mathbf{E}}((e^{C_2}(\mathbb{C}) \rtimes C_2) \otimes A, B) & \xrightarrow{(\epsilon \rtimes C_2)^*} & \mathbf{map}_{\mathbf{E}}((e^{C_2}(\mathcal{S}) \rtimes C_2) \otimes A, B) , \\
& \searrow & \xrightarrow{\simeq} & & \nearrow
\end{array}$$

where the two unnamed vertical equivalences are given by the dual Green-Julg adjunction (see e.g. [BD24, Prop. 3.61.2]). One can further check using the explicit description of the unit of the dual Green-Julg adjunction that the map made by  $\pi^*$  is induced by the homomorphism  $\pi : \mathbb{C} \rtimes C_2 \rightarrow \mathbb{C}$  induced by the trivial representation of  $C_2$ . The dotted arrow is an equivalence since by [BE17, Lem. 7.3] the composition

$$e^{C_2}(\mathcal{S}) \rtimes C_2 \xrightarrow{\epsilon \rtimes C_2} e^{C_2}(\mathbb{C}) \rtimes C_2 \xrightarrow{\pi} e(\mathbb{C})$$

is an equivalence. We can conclude that the dashed arrow is an equivalence as desired.

We use the graded  $E$ -theory in order to define the  $K$ -theory functor for graded  $C^*$ -algebras.

**Definition 2.47.** *We define the lax symmetric monoidal  $K$ -theory functor for graded  $C^*$ -algebras as the composition*

$$K^{\text{gr}} : (C^* \mathbf{Alg}^{\text{nu}})^{\text{gr}} \xrightarrow{e^{\text{gr}}} \mathbf{E}^{\text{gr}} \xrightarrow{\mathbf{map}_{\mathbf{E}^{\text{gr}}}(e^{\text{gr}}(\mathbb{C}), -)} \mathbf{Mod}(KU) .$$

We further define the lax symmetric monoidal  $K$ -theory functor for  $C^*$ -algebras by

$$K : C^* \mathbf{Alg}^{\text{nu}} \xrightarrow{\text{triv}} (C^* \mathbf{Alg}^{\text{nu}})^{\text{gr}} \xrightarrow{K^{\text{gr}}} \mathbf{Mod}(KU) .$$

Since the right vertical arrow in (2.25) is fully faithful we have  $K(-) \simeq \mathbf{E}(\mathbb{C}, -)$  which shows that our definition reproduces the classical definition of the  $K$ -theory functor for  $C^*$ -algebras.

**Remark 2.48.** In this remark we recall the model-free description of the  $K$ -theory classes associated to a projection or a unitary in an ungraded algebra.

Let  $p$  be a self-adjoint projection in a  $C^*$ -algebra  $Q$ . It gives rise to a homomorphism  $\pi : \mathbb{C} \rightarrow Q$  sending 1 to  $p$ . The morphism  $e(\pi) : e(\mathbb{C}) \rightarrow e(Q)$  in  $\pi_0 \mathbf{E}(\mathbb{C}, Q) \cong K_0(Q)$  is the  $K$ -theory class  $[p]$  associated to  $p$ .

Let  $I$  be a  $C^*$ -algebra and  $I^+$  denote its unitalization. A normalized unitary  $u$  in  $I$  is a unitary  $u$  in  $I^+$  which is sent to 1 by the canonical homomorphism  $I^+ \rightarrow \mathbb{C}$ . If  $u$  is a normalized unitary in  $I$ , then by function calculus it gives rise to a homomorphism

$$\mu : C(S^1, \{1\}) \rightarrow I, \quad f \mapsto f(u),$$

where  $C(S^1, \{1\}) := \ker(C(S^1) \xrightarrow{f \mapsto f(1)} \mathbb{C})$ . The composition  $\Omega e(\mathbb{C}) \simeq e(C(S^1, \{1\})) \xrightarrow{e(\mu)} e(I)$  represents the class  $[u]$  in  $\pi_0 \Sigma E(\mathbb{C}, I) \cong K_{-1}(I)$  associated to  $u$ .

Assume that

$$0 \rightarrow I \rightarrow A \rightarrow Q \rightarrow 0 \tag{2.26}$$

is an exact sequence of  $C^*$ -algebras such that  $A \rightarrow Q$  is a unit-preserving morphism between unital algebras. Let  $p$  be a projection in  $Q$ . Then we can find a lift  $\tilde{p}$  in  $A$  which is selfadjoint and has spectrum in the unit interval  $[0, 1]$ . We get a normalized unitary  $u := e^{2\pi i \tilde{p}}$  in  $I$  where we identify  $I^+$  with the subalgebra  $I + 1_A \mathbb{C}$  of  $A$ . Using the relation of the triangulated structure of  $\text{hoE}$  with Puppe sequences one can check that

$$\partial[p] = [u], \tag{2.27}$$

where  $\partial$  is the boundary operator  $K_0(Q) \rightarrow K_{-1}(I)$  associated to the exact sequence (2.26). ■

**Remark 2.49.** Recall that

$$K^{\text{gr}}(A) \simeq E^{\text{gr}}(\mathbb{C}, A) \simeq E^{C_2}(\mathcal{S}, A).$$

In particular, every homomorphism  $\mathcal{S} \rightarrow A$  of graded  $C^*$ -algebras (e.g., the one in (2.22)) represents a class in  $K_0^{\text{gr}}(A)$ . If this homomorphism is  $\Psi_* : \mathcal{S} \rightarrow A$ ,  $\Psi_*(f) = f(\Psi)$ , for an unbounded multiplier  $\Psi$  of  $A$ , then we write  $[\Psi]$  for the corresponding class in  $K_0^{\text{gr}}(A)$ .

We will use the following variation on this construction: Let  $a$  be in  $(0, \infty)$ . Then pull-back along the map

$$\kappa : (-a, a) \rightarrow \mathbb{R}, \quad t \mapsto \frac{t}{\sqrt{a - |t|^2}} \tag{2.28}$$

induces an isomorphism  $\kappa^* : \mathcal{S} \rightarrow C_0((-a, a))$  of graded  $C^*$ -algebras. It is a homotopy inverse of the extension by zero map  $\epsilon : C_0((-a, a)) \rightarrow \mathcal{S}$ .

Assume that  $j : B \rightarrow A$  is the inclusion of a graded subalgebra and  $\Psi$  is an unbounded multiplier of  $A$  as above such that  $\epsilon(f)(\Psi) \in B$  for all  $f$  in  $C_0((-a, a))$ . Then we get a class  $[\Psi_B] \in K_0^{\text{gr}}(B)$  represented by the homomorphism

$$\Psi_B(f) = (\kappa^* f)(\Psi) : \mathcal{S} \rightarrow B.$$

It satisfies  $j_*[\Psi_B] = [\Psi]$ . ■

**Remark 2.50.** In this remark we provide an explicit formula for the cup product of  $K$ -theory classes. Let  $A, B$  be graded  $C^*$ -algebras,  $[a]$  be a class in  $K_0^{\text{gr}}(A)$  represented by  $a : \mathcal{S} \rightarrow A$ , and  $[b]$  be a class in  $K_0^{\text{gr}}(B)$  represented by  $b : \mathcal{S} \rightarrow B$ . Then the product  $[a] \cup [b]$  in  $K_0^{\text{gr}}(A \hat{\otimes} B)$  is by definition the image of  $([a], [b])$  under the structure map

$$K^{\text{gr}}(A) \times K^{\text{gr}}(B) \rightarrow K^{\text{gr}}(A \hat{\otimes} B)$$

of the symmetric monoidal structure of  $K^{\text{gr}}$ . Explicitly,  $[a] \cup [b]$  is represented by the composition

$$\mathcal{S} \xrightarrow{\Delta} \mathcal{S} \hat{\otimes} \mathcal{S} \xrightarrow{a \hat{\otimes} b} A \hat{\otimes} B ,$$

where  $\Delta$  is the coproduct (2.23) of  $\mathcal{S}$ . ■

**Example 2.51.** If  $V$  is a finite-dimensional graded Hilbert space with an action of  $\mathbf{Cl}^n$  such that the generators act by anti-selfadjoint operators, then  $\text{End}_{\mathbf{Cl}^n}(V)$  is a graded  $C^*$ -algebra. The evaluation gives an isomorphism of  $\mathbf{Cl}^n$ -modules

$$\mathbf{Cl}^n \hat{\otimes} \text{Hom}_{\mathbf{Cl}^n}(\mathbf{Cl}^n, V) \xrightarrow{\cong} V , \quad (2.29)$$

where  $\mathbf{Cl}^n$  acts by right-multiplication in itself. It induces an isomorphism of graded  $C^*$ -algebras

$$\mathbf{Cl}^n \hat{\otimes} \text{End}_{\mathbb{C}}(\text{Hom}_{\mathbf{Cl}^n}(\mathbf{Cl}^n, V)) \cong \text{End}_{\mathbf{Cl}^n}(V) , \quad (2.30)$$

where the first factor  $\mathbf{Cl}^n$  in (2.30) acts on the first factor  $\mathbf{Cl}^n$  in (2.29) by left multiplication. Since  $e^{\text{gr}}(\text{End}_{\mathbb{C}}(\text{Hom}_{\mathbf{Cl}^n}(\mathbf{Cl}^n, V))) \simeq e^{\text{gr}}(\mathbb{C})$  by Morita invariance we therefore have an equivalence  $e^{\text{gr}}(\text{End}_{\mathbf{Cl}^n}(V)) \simeq e^{\text{gr}}(\mathbf{Cl}^n)$ .

Let  $X : \mathbb{R}^n \rightarrow \mathbf{Cl}^n$  be the canonical linear map which we consider as an unbounded multiplier on  $C_0(\mathbb{R}^n, \mathbf{Cl}^n)$ . Then  $(iX)_* : \mathcal{S} \rightarrow C_0(\mathbb{R}^n, \mathbf{Cl}^n)$  represents an invertible class (Kasparov's Bott element)  $\beta$  in  $E^{\text{gr}}(\mathbb{C}, C_0(\mathbb{R}^n, \mathbf{Cl}^n))$  and hence an equivalence

$$e^{\text{gr}}(\mathbb{C}) \xrightarrow{\beta} e^{\text{gr}}(C_0(\mathbb{R}^n, \mathbf{Cl}^n)) \simeq \Omega^n e^{\text{gr}}(\mathbf{Cl}^n) \simeq \Omega^n e^{\text{gr}}(\text{End}_{\mathbf{Cl}^n}(V)) .$$

Upon tensoring with  $e^{\text{gr}}(A)$  and applying  $K^{\text{gr}}$  its  $n$ -fold desuspension  $\Sigma^n e^{\text{gr}}(\mathbb{C}) \xrightarrow{\beta} e^{\text{gr}}(\text{End}_{\mathbf{Cl}^n}(V))$  induces an equivalence

$$\Sigma^n K^{\text{gr}}(A) \simeq K^{\text{gr}}(A \hat{\otimes} \text{End}_{\mathbf{Cl}^n}(V)) \quad (2.31)$$

for any graded  $C^*$ -algebra  $A$ . In particular,

$$K_*^{\text{gr}}(A \hat{\otimes} \text{End}_{\mathbf{Cl}^n}(V)) \cong K_{*-n}^{\text{gr}}(A) . \quad (2.32)$$

■

**Example 2.52.** We consider the exact sequence (2.26) and a normalized unitary  $u = e^{2\pi i \tilde{p}}$  in  $I$  for some selfadjoint  $\tilde{p}$  in  $A$  with spectrum in  $[0, 1]$  whose image in  $Q$  is a projection.

Taking  $V = \mathbb{C}1^1$  as a right  $\mathbb{C}1^1$ -module we have  $\text{End}_{\mathbb{C}1^1}(\mathbb{C}1^1) \cong \mathbb{C}1^1$  and (2.32) gives an isomorphism

$$K_{-1}(I) \cong K_{-1}^{\text{gr}}(I) \cong K_0^{\text{gr}}(I \hat{\otimes} \mathbb{C}1^1). \quad (2.33)$$

One can check using the explicit description of Kasparov's Bott element  $\beta$  and (2.28) that the image of  $[u]$  in  $K_0^{\text{gr}}(I \hat{\otimes} \mathbb{C}1^1)$  under (2.33) is represented by the map

$$\mathcal{S} \rightarrow I \hat{\otimes} \mathbb{C}1^1, \quad f \mapsto f \left( \frac{2\tilde{p} - 1}{\sqrt{1 - (2\tilde{p} - 1)^2}} \hat{\otimes} i\sigma \right). \quad (2.34)$$

■

We now extend the  $E$ -theory and  $K$ -theory functors from  $C^*$ -algebras to  $C^*$ -categories. We start with the characterization of  $C^*$ -categories following [Bunb]. A  $\mathbb{C}$ -linear  $*$ -category  $\mathbf{C}$  is a category enriched in complex vector spaces with an involution  $*$  :  $\mathbf{C}^{\text{op}} \rightarrow \mathbf{C}$  which fixes objects and acts complex anti-linearly on morphism vector spaces. A morphism between  $\mathbb{C}$ -linear  $*$ -categories is a functor which is compatible with the enrichment in  $\mathbb{C}$ -vector spaces and the involutions.

A  $C^*$ -algebra can be considered as a  $\mathbb{C}$ -linear  $*$ -category with a single object. We consider a morphism  $f$  in  $\mathbf{C}$ . We define the maximal norm of  $f$  by  $\|f\|_{\text{max}} := \sup_{\rho} \|\rho(f)\|$ , where  $\rho$  runs over all functors  $\rho : \mathbf{C} \rightarrow A$  of  $\mathbb{C}$ -linear  $*$ -categories from  $\mathbf{C}$  to  $C^*$ -algebras  $A$ . We say that  $\mathbf{C}$  is a pre- $C^*$ -category if  $\|f\|_{\text{max}} < \infty$  for every morphism  $f$  in  $\mathbf{C}$ . Finally,  $\mathbf{C}$  is a  $C^*$ -category if it is a pre- $C^*$ -category such that its morphism spaces are complete in the norm  $\| - \|_{\text{max}}$ . A morphism between  $C^*$ -categories is just a morphism of  $\mathbb{C}$ -linear  $*$ -categories.

**Remark 2.53.** A morphism  $\mathbf{C} \rightarrow \mathbf{D}$  of  $C^*$ -categories is an ideal inclusion if it induces a bijection on objects and inclusions on the level of morphisms so that the image satisfies the obvious generalization of the conditions for an ideal to categories. For an ideal inclusion we can form the quotient  $\mathbf{D}/\mathbf{C}$  which has the same objects as  $\mathbf{D}$  and whose morphism spaces are the quotients of the morphism spaces of  $\mathbf{D}$  by the images of the corresponding morphism spaces of  $\mathbf{C}$ . ■

A  $G$ - $C^*$ -category is a  $C^*$ -category with an action of  $G$  by automorphisms. We let  $GC^*\mathbf{Cat}^{\text{nu}}$  denote the category of  $G$ - $C^*$ -categories and equivariant morphisms. By [Joa03] we have an adjunction

$$A^f : GC^*\mathbf{Cat}^{\text{nu}} \rightleftarrows GC^*\mathbf{Alg}^{\text{nu}} : \text{incl} \quad (2.35)$$

where the inclusion views a  $G$ - $C^*$ -algebra as a  $G$ - $C^*$ -category with a single object.

A graded  $C^*$ -category is a  $C^*$ -category with a strict  $C_2$ -action fixing objects. We let  $(C^*\mathbf{Cat}^{\text{nu}})^{\text{gr}}$  be the full subcategory of  $C_2C^*\mathbf{Cat}^{\text{nu}}$  of graded  $C^*$ -categories. The adjunction (2.35) restricts to an adjunction

$$A^{f,\text{gr}} : (C^*\mathbf{Cat}^{\text{nu}})^{\text{gr}} \rightleftarrows (C^*\mathbf{Alg}^{\text{nu}})^{\text{gr}} : \text{incl}.$$



We use this functor in order to extend the  $E$ -theory to graded  $C^*$ -categories.

**Definition 2.54.** *We define the  $E$ -theory of graded  $C^*$ -categories as the composition*

$$e^{\text{gr}} : (C^* \mathbf{Cat}^{\text{nu}})^{\text{gr}} \xrightarrow{A^{f,\text{gr}}} (C^* \mathbf{Alg}^{\text{nu}})^{\text{gr}} \xrightarrow{e^{\text{gr}}} E^{\text{gr}} .$$

We further define the  $K$ -theory of graded  $C^*$ -categories by

$$K^{\text{gr}} : (C^* \mathbf{Cat}^{\text{nu}})^{\text{gr}} \xrightarrow{e^{\text{gr}}} E^{\text{gr}} \xrightarrow{\text{map}_{E^{\text{gr}}}(e^{\text{gr}}(\mathbb{C}), -)} \mathbf{Mod}(KU) .$$

The compositions

$$e : C^* \mathbf{Cat}^{\text{nu}} \xrightarrow{\text{triv}} (C^* \mathbf{Cat}^{\text{nu}})^{\text{gr}} \xrightarrow{e} E^{\text{gr}} \quad (2.36)$$

and

$$K : C^* \mathbf{Cat}^{\text{nu}} \xrightarrow{\text{triv}} (C^* \mathbf{Cat}^{\text{nu}})^{\text{gr}} \xrightarrow{K^{\text{gr}}} \mathbf{Mod}(KU) \quad (2.37)$$

are the usual  $E$ - and  $K$ -theory functors for  $C^*$ -categories.

One can check that  $e^{\text{gr}} : (C^* \mathbf{Cat}^{\text{nu}})^{\text{gr}} \rightarrow E^{\text{gr}}$  again has a symmetric monoidal refinement if we equip  $(C^* \mathbf{Cat}^{\text{nu}})^{\text{gr}}$  with the maximal tensor product. We will use that the  $E$ -theory functor for  $C^*$ -categories from (2.36) is a finitary homological functor in the sense of [BE23, Def. 3.24]. In particular, it preserves filtered colimits, sends unitary equivalences to equivalences, sends exact sequences to fibre sequences, and annihilates flasque  $C^*$ -categories [BE, Def. 11.3]. The arguments are the same as for the case of  $KK$ -theory in [BELb, Sec. 6 & 7].

## 2.5 $X$ -controlled Hilbert spaces and coarse $K$ -homology

For the present paper the main example of a coarse homology theory is coarse  $K$ -theory

$$K\mathcal{X} : \mathbf{BC} \rightarrow \mathbf{Mod}(KU)$$

with values in the stable  $\infty$ -category of  $KU$ -modules which we describe in the following.

Let  $X$  be a set. An  $X$ -controlled Hilbert space is a pair  $(H, \chi)$  of a Hilbert space and finitely additive projection valued measure  $\chi : \mathcal{P}(X) \rightarrow B(H)$ . We say that  $(H, \chi)$  is determined on points if  $\bigoplus_{x \in X} \chi(\{x\})H \cong H$ . If  $X$  has a bornology  $\mathcal{B}$ , then we say that  $(H, \chi)$  is locally finite if  $\chi(B)$  is finite-dimensional for all  $B$  in  $\mathcal{B}$ .

Let  $(H, \chi)$  and  $(H', \chi')$  be two  $X$ -controlled Hilbert spaces. If  $U$  is an entourage of  $X$ , then a bounded operator  $A : H \rightarrow H'$  is  $U$ -controlled if  $\chi'(Z')A\chi(Z) = 0$  for all subsets  $Z, Z'$  of  $X$  with  $Z' \cap U[Z] = \emptyset$ . If  $X$  has a coarse structure  $\mathcal{C}$ , then a controlled operator is an operator which is  $U$ -controlled from some  $U$  in  $\mathcal{C}$ .

If  $f$  in  $\ell^\infty(X)$  is a bounded function on  $X$ , then we can consider it as a  $\text{diag}(X)$ -controlled operator  $\chi(f) : H \rightarrow H$  acting by  $f(x)$  on the summand  $\chi(\{x\})H$  of  $H$ .

We first construct a functor

$$\mathbf{C} : \mathbf{BC} \rightarrow C^* \mathbf{Cat}^{\text{nu}}$$

which associates to every bornological coarse space the Roe category  $\mathbf{C}(X)$ .

**Definition 2.55.**

1. *Objects of  $\mathbf{C}(X)$ :* The objects of  $\mathbf{C}(X)$  are the locally finite  $X$ -controlled Hilbert spaces  $(H, \chi)$  which are determined on points.
2. *Morphisms of  $\mathbf{C}(X)$ :* The morphisms  $A : (H, \chi) \rightarrow (H', \chi')$  are bounded operators which can be approximated in norm by controlled operators.
3. *Involution:* The involution on  $\mathbf{C}(X)$  takes the adjoint operator.
4.  *$\mathbf{C}(f)$ :* For a morphism  $X \rightarrow Y$  of bornological coarse spaces the functor  $\mathbf{C}(f) : \mathbf{C}(X) \rightarrow \mathbf{C}(Y)$  sends  $(H, \chi)$  to  $(H, f_*\chi)$ . It acts by the identity on morphisms.

In the following definition we use the  $K$ -theory functor for  $C^*$ -categories from (2.37).

**Definition 2.56.** We define the coarse  $K$ -homology functor as the composition

$$K\mathcal{X} : \mathbf{BC} \xrightarrow{\mathbf{C}} C^* \mathbf{Cat}^{\text{nu}} \xrightarrow{K} \mathbf{Mod}(KU) .$$

**Theorem 2.57** ([BE20b],[BE23],[BC19]).  $K\mathcal{X}$  is a coarse homology theory which is in addition continuous, additive and strong and has a lax symmetric monoidal refinement.

**Remark 2.58.** Coarse  $K$ -homology as a  $\mathbb{Z}$ -graded group-valued functor for proper metric spaces has been introduced again by Roe, see [HR00] for a text-book account. Upon taking homotopy groups the above construction extends the classical one to general bornological coarse spaces. Note that  $KU \simeq K\mathcal{X}(\ast)$ . ■

The idea of using  $C^*$ -categories of  $X$ -controlled Hilbert spaces appeared already in [HR04a]. But note that the details are different. The  $C^*$ -category  $\mathbf{C}(X)$  from Definition 2.56 is defined for arbitrary bornological coarse spaces and its objects are locally finite, and the control is implemented by a finitely additive projection-valued measure. The  $X$ -controlled Hilbert spaces in [HR04a] are defined only for proper metric spaces and usually are not locally finite, and the control is implemented by an action of the algebra  $C_0(X)$  of continuous functions vanishing at  $\infty$ . ■

**Remark 2.59.** Theorem 2.57 is deduced from [BE23, Thm. 7.3] and the fact that  $K : C^*\mathbf{Cat}^{\text{nu}} \rightarrow \mathbf{Mod}(KU)$  is a finitary homological functor in the sense of [BE23, Def. 3.23]. Note that the  $E$ -theory functor  $e : C^*\mathbf{Cat}^{\text{nu}} \rightarrow E$  itself is a finitary homological functor with values in the stable  $\infty$ -category  $E$ . Therefore the composition

$$e\mathcal{X} : \mathbf{BC} \xrightarrow{\mathbf{C}} C^*\mathbf{Cat}^{\text{nu}} \xrightarrow{e} E$$

is an  $E$ -valued coarse homology theory. We will in particular use coarse invariance, excision, vanishing on flasques and the version of (2.13) for  $e\mathcal{X}$ .  $\blacksquare$

Let  $X$  be a bornological coarse space,  $Y$  be a subset, and  $\mathcal{Y}$  be a big family on  $X$ .

**Definition 2.60.** We let  $\mathbf{C}(Y \subseteq X)$  be the wide subcategory of the Roe category  $\mathbf{C}(X)$  given by operators of the form  $\chi'(Y)A\chi(Y)$  for  $A$  in  $\mathbf{C}(X)$ . We further define  $\mathbf{C}(\mathcal{Y} \subseteq X)$  as the ideal in  $\mathbf{C}(X)$  generated by the wide subcategories  $\mathbf{C}(Y \subseteq X)$  for all members  $Y$  of  $\mathcal{Y}$ .

**Remark 2.61.** Note that  $\mathbf{C}(\mathcal{Y} \subseteq X)$  has the same set of objects as  $\mathbf{C}(X)$  and the inclusion  $\mathbf{C}(\mathcal{Y} \subseteq X) \rightarrow \mathbf{C}(X)$  is an ideal inclusion in the sense explained in Remark 2.53 allowing to form the quotient  $\frac{\mathbf{C}(X)}{\mathbf{C}(\mathcal{Y} \subseteq X)}$ . In contrast, the canonical morphism  $\mathbf{C}(\mathcal{Y}) \rightarrow \mathbf{C}(X)$  is not an ideal and we can not form the corresponding quotient. But this morphism factorizes over a morphism  $\mathbf{C}(\mathcal{Y}) \rightarrow \mathbf{C}(\mathcal{Y} \subseteq X)$  which by Lemma 2.62 below induces an equivalence in  $K$ -theory.  $\blacksquare$

Let  $\mathcal{Y}$  be a big family in a bornological coarse space  $X$ .

**Lemma 2.62.** We have a canonical equivalence

$$K\mathcal{X}(\mathcal{Y}) \simeq K(\mathbf{C}(\mathcal{Y} \subseteq X)) . \quad (2.38)$$

*Proof.* By definition

$$K\mathcal{X}(\mathcal{Y}) \simeq \operatorname{colim}_{Y \in \mathcal{Y}} K\mathcal{X}(Y) \simeq \operatorname{colim}_{Y \in \mathcal{Y}} K(\mathbf{C}(Y)) .$$

By [BE23, Lem. 6.10] for every  $Y$  in  $\mathcal{Y}$  we have a unitary equivalence  $\mathbf{C}(Y) \rightarrow \mathbf{C}(Y \subseteq X)$ . Since  $K$  sends unitary equivalences of  $C^*$ -categories to equivalences we get

$$\operatorname{colim}_{Y \in \mathcal{Y}} K(\mathbf{C}(Y)) \simeq \operatorname{colim}_{Y \in \mathcal{Y}} K(\mathbf{C}(Y \subseteq X)) .$$

Since  $\operatorname{colim}_{Y \in \mathcal{Y}} \mathbf{C}(Y \subseteq X) \cong \mathbf{C}(\mathcal{Y} \subseteq X)$  by definition and  $K$  preserves filtered colimits we get

$$\operatorname{colim}_{Y \in \mathcal{Y}} K(\mathbf{C}(Y \subseteq X)) \simeq K(\mathbf{C}(\mathcal{Y} \subseteq X)) .$$

Combining these equivalence we get the equivalence (2.38).  $\square$

**Remark 2.63.** In the argument for Lemma 2.62 we can replace  $K$  by  $e$ . We then get a canonical equivalence

$$e\mathcal{X}(\mathcal{Y}) \simeq e(\mathbf{C}(\mathcal{Y} \subseteq \mathcal{X})) . \quad \blacksquare$$

## 3 Coarse coronas

### 3.1 Coarse coronas and a commutator estimate

Let  $X$  be a set,  $U$  be an entourage on  $X$ , and  $W$  be a subset of  $X$ . For a function  $f : X \rightarrow \mathbb{C}$  we define the  $U$ -variation of  $f$  on  $W$  by

$$\mathrm{Var}_U(f, W) := \sup_{(x,y) \in U \cap (W \times W)} |f(x) - f(y)| .$$

For a set  $X$  we let  $\ell^\infty(X)$  denote the  $C^*$ -algebra of all bounded functions  $X \rightarrow \mathbb{C}$  with the supremum norm  $\|f\| := \sup_{x \in X} |f(x)|$ .

Let  $\mathcal{Y}$  be a filtered family of subsets in  $X$ .

#### Definition 3.1.

1. The  $C^*$ -algebra  $\ell^\infty(\mathcal{Y})$  of functions vanishing away from  $\mathcal{Y}$  is defined as the sub- $C^*$ -algebra of  $\ell^\infty(X)$  of functions  $f$  satisfying

$$\lim_{Y \in \mathcal{Y}} \|f_{M \setminus Y}\| = 0 .$$

2. For a coarse space  $X$  with coarse structure  $\mathcal{C}$  we define the algebra of bounded functions with vanishing variation away from  $\mathcal{Y}$  as

$$\ell_{\mathcal{Y}}^\infty(X) := \{f \in \ell^\infty(X) \mid \forall U \in \mathcal{C} : \lim_{Y \in \mathcal{Y}} \mathrm{Var}_U(f, X \setminus Y) = 0\} .$$

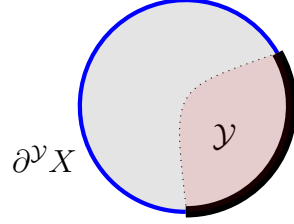
From now on we assume that  $X$  is a coarse space. We have an exact sequence of  $C^*$ -algebras

$$0 \rightarrow \ell^\infty(\mathcal{Y}) \rightarrow \ell_{\mathcal{Y}}^\infty(X) \rightarrow C(\partial^{\mathcal{Y}} X) \rightarrow 0 \quad (3.1)$$

defining the unital quotient  $C^*$ -algebra  $C(\partial^{\mathcal{Y}} X)$ .

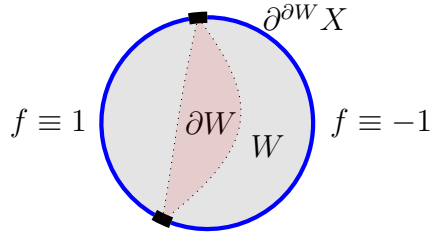
**Definition 3.2.** The coarse  $\mathcal{Y}$ -corona of  $X$  is the compact topological space  $\partial^{\mathcal{Y}} X$  defined as the Gelfand dual of  $C(\partial^{\mathcal{Y}} X)$ .

**Remark 3.3.** We have the following cartoon picture of the coarse  $\mathcal{Y}$ -corona:



The grey interior is the space  $X$ , with the boundary depicting its  $\mathcal{Y}$ -completion, i.e., the Gelfand dual  $\overline{X}^{\mathcal{Y}}$  of the algebra  $\ell_{\mathcal{Y}}^{\infty}(X)$ . The boundary  $\overline{X}^{\mathcal{Y}} \setminus X$  of this compactification consists of two parts: The boundary  $\partial^{\mathcal{Y}} X$  (depicted blue in the picture), and its complement in  $\overline{X}^{\mathcal{Y}} \setminus X$ , which comes from the Gelfand dual of  $\ell^{\infty}(X)$ . ■

**Example 3.4.** Let  $W$  be a subset of  $X$ . The coarse boundary of  $W$  is defined as the big family  $\partial W$  of  $X$  consisting of the subsets  $Y$  of  $X$  such that there exists a coarse entourage  $U$  of  $X$  with  $Y \subseteq U[W] \cap U[X \setminus W]$ .



Denote by  $f$  the function such that  $f|_W = 1$  and  $f|_{X \setminus W} = -1$ . Then  $f \in \ell_{\partial W}^{\infty}(X)$ . Its class  $[f]$  in  $C(\partial^{\partial W} X)$  defines a map  $\partial^{\partial W} X \rightarrow \{-1, 1\}$  which decomposes this corona into two disjoint components. ■

**Remark 3.5** (A continuous description of the coarse corona). Assume that  $X$  is a paracompact topological space and that the coarse structure on  $X$  is compatible with the topology in the sense that there exists an open coarse entourage. For example, the coarse and topological structures could be both induced by a metric. We then define the  $C^*$ -algebra

$$C_{\mathcal{Y}}(X) := C(X) \cap \ell_{\mathcal{Y}}^{\infty}(X)$$

of continuous functions with bounded variation away from  $\mathcal{Y}$ , as well as the algebra

$$C(\mathcal{Y}) := \operatorname{colim}_{Y \in \mathcal{Y}} \ker(C_b(X) \rightarrow C_b(X \setminus Y)) \quad (3.2)$$

of continuous functions that vanish away from  $\mathcal{Y}$ .

**Lemma 3.6.** *We have  $C(\mathcal{Y}) = \ell^{\infty}(\mathcal{Y}) \cap C(X)$  and the canonical inclusion is an isomorphism*

$$\frac{C_{\mathcal{Y}}(X)}{C(\mathcal{Y})} \xrightarrow{\cong} \frac{\ell_{\mathcal{Y}}^{\infty}(X)}{\ell^{\infty}(\mathcal{Y})}.$$

*Proof.* It is clear from the definitions that  $C(\mathcal{Y}) \subseteq \ell^\infty(\mathcal{Y}) \cap C_{\mathcal{Y}}(X)$ . We show the converse inclusion. By assumption we can choose an open coarse entourage  $U$  on  $X$ . For every  $Y$  in  $\mathcal{Y}$ , we let  $\rho_Y$  be a continuous function on  $X$  taking values in  $[0, 1]$ , which is supported on  $U[Y]$  and constant equal to one on  $Y$ . Here we use that by paracompactness of  $X$ , the open covering  $(U[Y], X \setminus \bar{Y})$  admits a subordinated partition of unity.

Then we have the equality  $f = \lim_{Y \in \mathcal{Y}} \rho_Y f$  and  $\rho_Y f \in \ker(C_b(X) \rightarrow C_b(X \setminus U[Y]))$ . Since  $\mathcal{Y}$  is a big family we have  $U[Y] \in \mathcal{Y}$  for every  $U$  in  $\mathcal{C}$  and we can conclude from (3.2) that  $f \in C(\mathcal{Y})$ .

For the second assertion, we have to prove that for any function  $f$  in  $\ell_{\mathcal{Y}}^\infty(X)$  there exists  $\tilde{f}$  in  $C_{\mathcal{Y}}(X)$  such that  $f - \tilde{f} \in \ell^\infty(\mathcal{Y})$ . We start with the open covering  $(U[\{x\}])_{x \in X}$  of  $X$ . Since the topology of  $X$  is paracompact, it admits a locally finite open refinement  $(O_j)_{j \in J}$  and a subordinate partition of unity  $(\chi_j)_{j \in J}$ . Note that  $O_j$  is  $U$ -bounded for every  $j$  in  $J$ .

For each  $j \in J$  we pick a point  $x_j$  in  $O_j$ . Given a function  $f$  in  $\ell_{\mathcal{Y}}^\infty(X)$ , we define the continuous bounded function

$$\tilde{f}(y) := \sum_{j \in J} f(x_j) \chi_j(y). \quad (3.3)$$

We claim that  $f - \tilde{f} \in \ell^\infty(\mathcal{Y})$ . Let  $\epsilon$  be in  $(0, \infty)$ . Then by the variation condition on  $f$ , there exists a member  $Y$  of  $\mathcal{Y}$  such that  $\text{Var}_U(f, X \setminus Y) \leq \epsilon$ . This implies for every  $y$  in  $X \setminus U[Y]$  that

$$|f(y) - \tilde{f}(y)| \leq \sum_{j \in J} |f(y) - f(x_j)| \chi_j(y) \leq \epsilon \sum_{j \in J} \chi_j(y) = \epsilon,$$

where we used that the sum is actually taken only over those  $j$  in  $J$  such that  $y \in U[x_j] \neq \emptyset$  which implies that  $x_j, y \in X \setminus Y$ .

Since  $f \in \ell_{\mathcal{Y}}^\infty(X)$  and  $f - \tilde{f} \in \ell^\infty(\mathcal{Y})$  we see that  $\tilde{f} \in \ell_{\mathcal{Y}}^\infty(X)$ . By the first assertion,  $\tilde{f} \in C_{\mathcal{Y}}(X)$  as desired.  $\square$

Recall that the coarse  $\mathcal{Y}$ -corona of  $X$  is by Definition 3.2 given by

$$\partial^{\mathcal{Y}} X := \text{spec} \left( \frac{\ell_{\mathcal{Y}}^\infty(X)}{\ell^\infty(\mathcal{Y})} \right).$$

As a consequence of the second assertion of Lemma 3.6, if  $X$  is a paracompact topological space such that the coarse structure contains an open entourage, then we get the alternative description

$$\partial^{\mathcal{Y}} X \cong \text{spec} \left( \frac{C_{\mathcal{Y}}(X)}{C(\mathcal{Y})} \right) \quad (3.4)$$

in terms of continuous functions on  $X$ .

In the case that  $X$  is a proper metric space and that  $\mathcal{Y} = \mathcal{B}$  is the collection of bounded subsets of  $X$ , the right-hand side of (3.4) was considered in [Roe93, §5.1] and is known as the Higson corona  $\partial_h X$ . ■

**Example 3.7.** Let  $\mathcal{Y}, \mathcal{Y}', \mathcal{Z}, \mathcal{Z}'$  be big families on a coarse space  $X$ . Then we have a homomorphism

$$\ell_{\mathcal{Y}}^{\infty}(\mathcal{Z}) \otimes \ell_{\mathcal{Y}'}^{\infty}(\mathcal{Z}') \rightarrow \ell_{\mathcal{Y} \cup \mathcal{Y}'}^{\infty}(\mathcal{Z} \cap \mathcal{Z}'), \quad (f \otimes g) \mapsto fg. \quad (3.5)$$

For  $\mathcal{Z} = \mathcal{Z}' = \{X\}$  this induces a well-defined homomorphism

$$C(\partial^{\mathcal{Y}} X) \otimes C(\partial^{\mathcal{Y}'} X) \rightarrow C(\partial^{\mathcal{Y} \cup \mathcal{Y}'} X),$$

which corresponds by Gelfand duality to a map

$$\partial^{\mathcal{Y} \cup \mathcal{Y}'} X \rightarrow \partial^{\mathcal{Y}} X \times \partial^{\mathcal{Y}'} X.$$

■

**Remark 3.8.** Let  $\varphi : X \rightarrow X'$  be a morphism of coarse spaces. Then for  $f$  in  $\ell^{\infty}(X')$ , a subset  $Y'$  of  $X'$  and a coarse entourage  $U$  of  $X$ , we have

$$\text{Var}_U(\varphi^* f, X \setminus \varphi^{-1}(Y')) \leq \text{Var}_{(\varphi \times \varphi)(U)}(f, X' \setminus Y').$$

Therefore, if  $\mathcal{Y}, \mathcal{Y}'$  are big families on  $X$ , respectively  $X'$  such that  $\varphi^{-1}(\mathcal{Y}') \subseteq \mathcal{Y}$ , we get a \*-homomorphism  $\varphi^* : \ell_{\mathcal{Y}'}^{\infty}(X') \rightarrow \ell_{\mathcal{Y}}^{\infty}(X)$ . Clearly, this homomorphism also sends  $\ell^{\infty}(\mathcal{Y}')$  to  $\ell^{\infty}(\mathcal{Y})$ . Therefore, by Gelfand, duality,  $\varphi$  extends to a continuous map

$$\varphi : \partial^{\mathcal{Y}} X \rightarrow \partial^{\mathcal{Y}'} X'.$$

In particular, if  $X$  is a coarse space with two big families  $\mathcal{Y}, \mathcal{Y}'$  such that  $\mathcal{Y}' \subseteq \mathcal{Y}$ , then we may apply the above observation to the identity map of  $X$ , which yields a continuous surjective map  $\partial^{\mathcal{Y}} X \rightarrow \partial^{\mathcal{Y}'} X$ . ■

Let  $X$  be a coarse space and  $\mathcal{Y}$  be a big family on  $X$ . Let  $(H, \chi), (H', \chi')$  be  $X$ -controlled Hilbert spaces which are determined on points and  $A : H \rightarrow H'$  be a bounded operator. The argument for the following commutator estimate is taken from [QR10].

**Lemma 3.9.** *If  $f$  is in  $\ell_{\mathcal{Y}}^{\infty}(X)$  and  $A$  is  $U$ -controlled for some coarse entourage  $U$ , then*

$$\lim_{Y \in \mathcal{Y}} \|\chi'(X \setminus Y)(\chi'(f)A - A\chi(f))\chi(X \setminus Y)\| = 0.$$

*Proof.* Let  $\epsilon$  in  $(0, \infty)$  be given and set  $\eta := \epsilon/4\|A\|$ . We then choose  $Y$  in  $\mathcal{Y}$  such that  $\text{Var}_U(f, X \setminus Y') \leq \eta$  for each  $Y'$  in  $\mathcal{Y}$  with  $Y \subseteq Y'$ . We define the partition  $(S_k)_{k \in \mathbb{Z}}$  of  $X \setminus Y$  by

$$S_k := \{x \in X \setminus Y \mid (k-1)\eta \leq f(x) < k\eta\}.$$

Since  $f$  is bounded, only finitely many of these sets are non-empty. If  $k, l$  are in  $\mathbb{Z}$ , then  $x \in S_k$  and  $y \in S_l$  implies  $|f(x) - f(y)| \geq (|k - l| - 1)\eta$ . Since the  $U$ -variation of  $f$  on  $X \setminus Y = \bigcup_{k \in \mathbb{Z}} S_k$  is bounded by  $\eta$ , the condition  $|k - l| \geq 2$  implies that  $S_k \cap U[S_l] = U[S_k] \cap S_l = \emptyset$ . Since  $A$  is  $U$ -controlled we can conclude that  $\chi'(S_k)A\chi(S_l) = 0$

We set

$$\tilde{f} := Y \cdot f + \eta \sum_{k \in \mathbb{Z}} k \cdot S_k,$$

where for notational simplicity, we identify subsets of  $X$  with the corresponding indicator function. Then by construction,  $\|\tilde{f} - f\| \leq \eta$  and hence

$$\|(\chi'(f)A - A\chi(f)) - (\chi'(\tilde{f})A - A\chi(\tilde{f}))\| \leq 2\eta\|A\| = \frac{\epsilon}{2}. \quad (3.6)$$

Since  $A$  is  $U$ -controlled, we have

$$\begin{aligned} & \chi'(X \setminus U[Y])(\chi'(\tilde{f})A - A\chi(\tilde{f}))\chi(X \setminus U[Y]) \\ &= \eta \sum_{k \in \mathbb{Z}} k \cdot \chi'(X \setminus U[Y])(\chi'(S_k)A - A\chi(S_k))\chi(X \setminus U[Y]). \end{aligned} \quad (3.7)$$

Inserting the identities  $\chi(X \setminus Y) = \sum_{k \in \mathbb{Z}} \chi(S_k)$  and  $\chi'(X \setminus Y) = \sum_{k \in \mathbb{Z}} \chi'(S_k)$  and using that  $\chi'(S_k)A\chi(S_l) = 0$  whenever  $|k - l| \geq 2$ , we get

$$\sum_{k \in \mathbb{Z}} k \chi'(X \setminus Y)(\chi'(S_k)A - A\chi(S_k))\chi(X \setminus Y) = \sum_{k \in \mathbb{Z}} (\chi'(S_k)A\chi(S_{k-1}) - \chi'(S_k)A\chi(S_{k+1})).$$

The right-hand side is an operator with norm bounded by  $2\|A\|$ . Using  $\chi(X \setminus U[Y]) = \chi(X \setminus U[Y])\chi(X \setminus Y)$  and plugging the above equality into (3.7), we get

$$\|\chi'(X \setminus U[Y])(\chi'(\tilde{f})A - A\chi(\tilde{f}))\chi(X \setminus U[Y])\| \leq 2\eta\|A\| = \frac{\epsilon}{2}.$$

Combining this with (3.6), we see that

$$\|\chi'(X \setminus Y')(\chi'(f)A - A\chi(f))\chi(X \setminus Y')\| \leq \epsilon$$

for all  $Y'$  in  $\mathcal{Y}$  with  $U[Y] \subseteq Y'$ . □

Recall that  $X$  is a coarse space with a big family  $\mathcal{Y}$  and that  $(H, \chi)$  and  $(H', \chi')$  are  $X$ -controlled Hilbert spaces which are determined on points. Let now  $A : H \rightarrow H'$  be a bounded operator which can be approximated in norm by controlled bounded operators, and  $f$  be in  $\ell_{\mathcal{Y}}^{\infty}(X)$ .

**Corollary 3.10.** *We have  $\lim_{Y \in \mathcal{Y}} \|\chi'(X \setminus Y)(\chi'(f)A - A\chi(f))\chi(X \setminus Y)\| = 0$ .*

Let  $X$  be a bornological coarse space,  $\mathcal{Y}$  be a big family in  $X$  and recall the Definition 2.60 of the ideal  $\mathbf{C}(\mathcal{Y} \subseteq X)$ .

**Corollary 3.11.** *For a morphism  $A : (H, \chi) \rightarrow (H', \chi')$  in  $\mathbf{C}(X)$  and a function  $f$  in  $\ell_{\mathcal{Y}}^{\infty}(X)$  we have  $\chi'(f)A - A\chi(f) \in \mathbf{C}(\mathcal{Y} \subseteq X)$ .*



## 3.2 Uniform coronas and the comparison map

In this section, we define the uniform version of the  $\mathcal{Y}$ -corona introduced in the previous section. We consider a uniform space  $X$  with uniform structure  $\mathcal{U}$ . Recall that a function  $f : X \rightarrow \mathbb{C}$  is called uniformly continuous if

$$\lim_{U \in \mathcal{U}^{\text{op}}} \text{Var}_U(f, X) = 0 .$$

We let  $C_u(X)$  denote the  $C^*$ -algebra of uniformly continuous and bounded functions on  $X$  with the supremum norm. Note that  $C_u(X)$  is a closed subalgebra of the  $C^*$ -algebra  $C_b(X)$  of bounded continuous functions.

Let  $\mathcal{Y}$  be a filtered family of subsets of  $X$ .

**Definition 3.12.** *We define the  $C^*$ -algebra*

$$C_u(\mathcal{Y}) := \text{colim}_{Y \in \mathcal{Y}} \ker(C_u(X) \rightarrow C_u(X \setminus Y)) ,$$

where the colimit is taken in  $C^*$ -algebras.

**Remark 3.13.**  $C_u(\mathcal{Y})$  is a subalgebra of  $C_u(X)$  in a canonical way and consists of uniformly continuous functions which are asymptotically small away from  $\mathcal{Y}$ . We have an inclusion  $C_u(\mathcal{Y}) \subseteq \ell^\infty(\mathcal{Y}) \cap C_u(X)$ . This inclusion is an equality if  $\mathcal{U}$  is induced by a metric on  $X$ . However, for general, non-metric, uniform spaces  $X$  we do not expect that this inclusion is an equality: If  $f$  is in  $\ell^\infty(\mathcal{Y}) \cap C_u(X)$ , then it is not clear how to approximate it by uniformly continuous functions supported on a suitable member  $Y$  of  $\mathcal{Y}$ . ■

**Example 3.14.** If  $X$  is a proper metric space and  $\mathcal{B}$  is the bornology generated by bounded subsets of  $X$ , then we have an equality  $C_0(X) = C_u(\mathcal{B}) = \ell^\infty(\mathcal{B}) \cap C_u(X)$ . ■

Let  $X$  be a set with compatible coarse and uniform structures  $\mathcal{C}$  and  $\mathcal{U}$ . Let  $\mathcal{Y}$  be a big family on  $X$ . Recall the Definition 3.1.2 of  $\ell_{\mathcal{Y}}^\infty(X)$ .

**Definition 3.15.** *We define the uniform corona algebra of  $X$  by*

$$C_{u,\mathcal{Y}}(X) := \ell_{\mathcal{Y}}^\infty(X) \cap C_u(X) .$$

We have an exact sequence of  $C^*$ -algebras

$$0 \rightarrow C_u(\mathcal{Y}) \rightarrow C_{u,\mathcal{Y}}(X) \rightarrow C(\partial_u^{\mathcal{Y}} X) \rightarrow 0 \tag{3.8}$$

defining the unital quotient  $C^*$ -algebra  $C(\partial_u^{\mathcal{Y}} X)$ .

**Definition 3.16.** *The uniform  $\mathcal{Y}$ -corona of  $X$  is the compact topological space  $\partial_u^{\mathcal{Y}}X$  defined as the Gelfand dual of  $C(\partial_u^{\mathcal{Y}}X)$ .*

**Example 3.17.** If  $X$  comes from a proper metric space (as in Example 2.9) and  $\mathcal{B}$  is the big family of bounded subsets of  $X$ , then any continuous function with bounded variation away from  $\mathcal{B}$  is automatically uniformly continuous. In view of Example 3.14 and Remark 3.5 the uniform and the coarse corona coincide,

$$\partial_u^{\mathcal{B}}X = \partial^{\mathcal{B}}X,$$

and both agree with the Higson corona of  $X$ . ■

**Remark 3.18.** If  $\mathcal{Y}, \mathcal{Y}'$  are big families such that  $\mathcal{Y} \subseteq \mathcal{Y}'$ , then

$$C_{u,\mathcal{Y}}(X) \subseteq C_{u,\mathcal{Y}'}(X) .$$

The algebra  $C_{u,\emptyset}(X)$  consists of functions which are constant on coarse components of  $X$ . Furthermore  $C_{u,\{X\}}(X) = C_u(X)$  and  $\partial_u^{\{X\}}X$  contains  $X$  as a subspace. ■

**Remark 3.19.** Consider the cone  $\mathcal{O}^\infty(X)$  of  $X$  from Definition 2.11. One of the main observations that will be exploited below is that pullback along the canonical projection  $\text{pr} : \mathcal{O}^\infty(X) \rightarrow X$  yields a homomorphism

$$\text{pr}^* : C_u(X) \rightarrow \ell_{\mathcal{O}^-(X)}^\infty(\mathcal{O}^\infty(X)) ,$$

which follows directly from the definition of the coarse structure of  $\mathcal{O}^-(X)$ . More generally, if  $\mathcal{Z}$  and  $\mathcal{Y}$  are two big families on  $X$ , then the above homomorphism restricts to a map

$$C_{u,\mathcal{Z}}(\mathcal{Y}) := C_{u,\mathcal{Z}}(X) \cap C_u(\mathcal{Y}) \rightarrow \ell_{\mathcal{O}_{\mathcal{Z} \cap \mathcal{Y}}^-(X)}^\infty(\mathcal{O}^\infty(\mathcal{Y})) ,$$

using the big families in  $\mathcal{O}^\infty(X)$  defined in (2.35). ■

By Remark 3.13, the canonical inclusion  $C_{u,\mathcal{Y}}(X) \hookrightarrow \ell_{\mathcal{Y}}^\infty(X)$  sends  $C_u(\mathcal{Y})$  to  $\ell^\infty(\mathcal{Y})$ . Hence by Gelfand duality, it induces a comparison map

$$c : \partial^{\mathcal{Y}}X \rightarrow \partial_u^{\mathcal{Y}}X . \tag{3.9}$$

This map is a homeomorphism in many cases, as Proposition 3.22 below shows. We consider the following version of a bounded geometry condition on a set  $X$  with compatible uniform and coarse structures.

**Assumption 3.20.** *We assume that  $X$  admits a partition of unity  $(\chi_j)_{j \in J}$  such that:*

1. The family  $(\chi_j)_{j \in J}$  is jointly uniformly continuous in the sense that for every  $\epsilon$  in  $(0, \infty)$  there exists a uniform entourage  $V$  of  $X$  such that for all  $(x, y)$  in  $V$  and  $j$  in  $J$  we have  $|\chi_j(x) - \chi_j(y)| < \epsilon$ .
2. The covering  $(\text{supp}(\chi_j))_{j \in J}$  is uniformly locally finite, i.e.,  $\sup_{x \in X} |\{j \in J \mid x \in \text{supp} \chi_j\}| < \infty$
3. The family  $(\text{supp}(\chi_j))_{j \in J}$  of subsets is coarsely bounded, i.e., there exists a coarse entourage  $U$  such that  $\text{supp}(\chi_j) \times \text{supp}(\chi_j) \subseteq U$  of  $X$  for all  $j$  in  $J$ . ■

**Example 3.21.** If  $(X, d)$  is a metric space of bounded geometry, then it satisfies Assumption 3.20. Indeed, we can find a suitable family  $(x_j)_{j \in J}$  of points in  $X$  such that  $(U_2[\{x_j\}])_{j \in J}$  is uniformly locally finite and  $(U_1[\{x_j\}])_{j \in J}$  is an open covering of  $X$ . In order to define the required partition of unity we set

$$\tilde{\chi}_j(x) := \begin{cases} 1 - d(x, U_1[\{x_j\}]) & d(x, U_1[\{x_j\}]) \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad \chi_j(x) := \frac{\tilde{\chi}_j(x)}{\sum_{k \in J} \tilde{\chi}_k(x)} \quad \blacksquare$$

**Proposition 3.22.** *If  $X$  is a set with compatible uniform and coarse structures satisfying Assumption 3.20, then (3.9) is a homeomorphism.*

*Proof.* Without loss of generality we can assume that  $\chi_j \neq 0$  for all  $j$  in  $J$  and pick a point  $x_j$  in  $\text{supp}(\chi_j)$ .

Let  $f$  be in  $\ell_{\mathcal{Y}}^{\infty}(X)$  and set

$$\tilde{f}(x) := \sum_{j \in J} f(x_j) \chi_j(x).$$

As in Remark 3.5 and using Assumption 3.20.3 we see that  $f$  is continuous, has vanishing variation away from  $\mathcal{Y}$ , and satisfies  $f - \tilde{f} \in \ell^{\infty}(\mathcal{Y})$ . It remains to show that  $\tilde{f}$  is uniformly continuous. By Assumption 3.20.2 the number  $N := \sup_{x \in X} |\{j \in J \mid x \in \text{supp} \chi_j\}|$  is finite. Let  $\epsilon$  be in  $(0, \infty)$  and choose by Assumption 3.20.1 a uniform entourage  $V$  such that  $|\chi_j(x) - \chi_j(y)| \leq \frac{\epsilon}{2N\|f\|_{\infty}}$  for all  $(x, y)$  in  $V$  and  $j$  in  $J$ . Then for all  $(x, y)$  in  $V$  we have

$$|\tilde{f}(x) - \tilde{f}(y)| \leq \sum_{j \in J} |f(x_j)| |\chi_j(x) - \chi_j(y)| \leq \|f\|_{\infty} \sum_{\substack{j \in J \\ x \in \text{supp}(\chi_j) \text{ or } y \in \text{supp}(\chi_j)}} \frac{\epsilon}{2N\|f\|_{\infty}} \leq \epsilon.$$

This shows that  $\tilde{f}$  uniformly continuous. □

Recall that  $X$  is a set with compatible uniform and coarse structures and that  $\mathcal{Y}$  is a big family in  $X$ .

**Definition 3.23.** We define the  $K$ -theory of  $X$  with support in  $\mathcal{Y}$  by

$$K(\mathcal{Y}) := K(C_u(\mathcal{Y})) .$$

**Remark 3.24.** Let  $X$  and  $X'$  be uniform spaces with filtered families  $\mathcal{Y}$  and  $\mathcal{Y}'$  of subsets of  $X$ , respectively  $X'$ . If  $f : X \rightarrow X'$  is a uniformly continuous map such that  $f^{-1}(\mathcal{Y}') \subseteq \mathcal{Y}$ , then the pullback homomorphism  $f^* : C_u(X') \rightarrow C_u(X)$  sends  $C_u(\mathcal{Y}')$  to  $C_u(\mathcal{Y})$ , hence  $f$  induces a map

$$f^* : K(\mathcal{Y}') \rightarrow K(\mathcal{Y}) . \quad \blacksquare$$

The exact sequence (3.8) induces a boundary map

$$\partial : K(\partial_u^{\mathcal{Y}} X) \rightarrow \Sigma K(\mathcal{Y}) \quad (3.10)$$

and the comparison map (3.9) induces a pullback map

$$c^* : K(\partial_u^{\mathcal{Y}} X) \rightarrow K(\partial^{\mathcal{Y}} X) .$$

These maps involve the topological  $K$ -theory of the uniform and coarse  $\mathcal{Y}$ -corona of  $X$ , which is defined as the  $K$ -theory of the  $C^*$ -algebras of continuous functions on these spaces.

### 3.3 Analytic locally finite $K$ -homology and Paschke duality

Recall Definition 2.56 of the strong coarse  $K$ -homology theory  $K\mathcal{X} : \mathbf{BC} \rightarrow \mathbf{Mod}(KU)$  (see also Theorem 2.57) and the cone functor  $\mathcal{O}^\infty : \mathbf{UBC} \rightarrow \mathbf{BC}$  introduced in Definition 2.11. As explained in Section 2.3, using the cone functor, we can associate to any strong coarse homology theory a local homology theory.

**Definition 3.25.** We define the local  $K$ -homology functor for uniform bornological coarse spaces as the composition

$$K^{\mathcal{X}} = \Sigma^{-1} K\mathcal{X}\mathcal{O}^\infty : \mathbf{UBC} \xrightarrow{\mathcal{O}^\infty} \mathbf{BC} \xrightarrow{\Sigma^{-1}K\mathcal{X}} \mathbf{Mod}(KU) .$$

By specializing (2.16) we get the index map, a natural transformation of functors

$$a : K^{\mathcal{X}} \rightarrow K\mathcal{X} \circ \iota : \mathbf{UBC} \rightarrow \mathbf{Mod}(KU) . \quad (3.11)$$

Note that if we write the evaluation of  $K\mathcal{X} \circ \iota$  on objects of  $\mathbf{UBC}$ , then we will omit the symbol  $\iota$ .

We now specialize the construction from (2.18).

**Definition 3.26.** We define the local  $K$ -homology with support by

$$K_-^{\mathcal{X}} : \mathbf{UBC}^{(2)} \rightarrow \mathbf{Mod}(KU) , \quad (X, \mathcal{Z}) \mapsto K_{\mathcal{Z}}^{\mathcal{X}}(X) := \Sigma^{-1} K \mathcal{X}(\mathcal{O}_{\mathcal{Z}}^{\infty}(X)) .$$

The index map (2.20) specializes to a natural transformation

$$a_{X, \mathcal{Z}} : K_{\mathcal{Z}}^{\mathcal{X}}(X) \rightarrow K \mathcal{X}(\mathcal{Z}) . \quad (3.12)$$

**Remark 3.27.** Note that  $K_{\{X\}}^{\mathcal{X}}(X) \simeq K^{\mathcal{X}}(X)$  and that we have a fibre sequence

$$K_{\{\emptyset\}}^{\mathcal{X}}(X) \rightarrow K^{\mathcal{X}}(X) \xrightarrow{a_X} K \mathcal{X}(X) , \quad (3.13)$$

which is a special case of (2.21). Thus the spectrum  $K_{\{\emptyset\}}^{\mathcal{X}}(X)$  captures the secondary invariants of the reasons for the vanishing of the index of classes in  $K^{\mathcal{X}}(X)$ .

The fibre sequence (3.13) is our analogue of the Higson-Roe surgery sequence [HR04b, Def. 1.5] and  $K_{\{\emptyset\}}^{\mathcal{X}}(X)$  is the analogue of the analytic structure set. It follows from Lemma 2.39 that  $K_{\{\emptyset\}}^{\mathcal{X}}(-) : \mathbf{UBC} \rightarrow \mathbf{Mod}(KU)$  is a local homology theory, so it is in particular excisive and homotopy invariant. The analogue of excisiveness for the analytic structure set of Higson-Roe has been shown in [Sie12].

Our order to compare our constructions with [HR04b] precisely one would have to work with the equivariant generalization

$$K_{\{\emptyset\}}^{G, \mathcal{X}}(X) \rightarrow K^{G, \mathcal{X}}(X) \xrightarrow{a_X^G} K \mathcal{X}^G(X) \quad (3.14)$$

for  $G$ -uniform bornological coarse spaces  $X$ . If  $X$  is homotopy equivalent in  $G\mathbf{UBC}$  to a finite-dimensional locally finite simplicial complex with a free proper action of  $G$ , then it follows from the equivariant Paschke duality equivalence considered in [BELa] that the long exact sequence obtained from (3.14) by taking homotopy groups is equivalent to the sequence defined in [HR04b, Def. 1.5]. ■

We consider a uniform bornological coarse space  $X$  with bornology  $\mathcal{B}$  and recall the Definition 3.12 of  $C_u(\mathcal{B})$ . In this section it is useful to change notation and consider the functor

$$C_0 : \mathbf{UBC} \rightarrow (C^* \mathbf{Alg}^{\text{nu}})^{\text{op}} , \quad X \mapsto C_0(X) := C_u(\mathcal{B})$$

which on morphisms is given by the pull-back.

**Remark 3.28.** If  $X$  is the uniform bornological coarse space associated to a proper metric space, then  $C_0(X)$  is the usual algebra of continuous functions on  $X$  vanishing at  $\infty$ . This justifies the notation. ■

**Definition 3.29.** We define the analytic locally finite  $K$ -homology as the composition

$$K^{\text{an}} : \mathbf{UBC} \xrightarrow{C_0} (C^* \mathbf{Alg}^{\text{nu}})^{\text{op}} \xrightarrow{E(C_0(-), \mathbb{C})} \mathbf{Mod}(KU) .$$

The following justifies the name, compare Definition 2.26 and Definition 2.29.

**Proposition 3.30.** The functor  $K^{\text{an}}$  is a locally finite local homology theory.

*Proof.* For any  $X$  in  $\mathbf{UBC}$  the map  $C_0(X) \rightarrow C_0([0, 1] \otimes X)$  is a homotopy equivalence of  $C^*$ -algebras. By homotopy invariance of the  $E$ -theory functor  $e : C^* \mathbf{Alg}^{\text{nu}} \rightarrow \mathbf{E}$  we see that  $K^{\text{an}}([0, 1] \otimes X) \rightarrow K^{\text{an}}(X)$  is an equivalence. This shows homotopy invariance of  $K^{\text{an}}$ .

Assume that  $(Y, Z)$  is a uniformly and coarsely excisive decomposition of  $X$  in  $\mathbf{UBC}$ . Then we have a map of exact sequences of commutative  $C^*$ -algebras

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & C_0(X) & \longrightarrow & C_0(Y) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & C_0(Z) & \longrightarrow & C_0(Z \cap Y) \longrightarrow 0 \end{array}$$

where the maps in the right square are the restrictions along the inclusions of subsets and  $A, B$  are defined as the kernels. We show that the left vertical map is an isomorphism as indicated. Assume that  $f$  is in  $A$  and sent to zero in  $B$ . Then  $f|_Z = 0$  and  $f|_Y = 0$  so that  $f = 0$  since  $Y \cup Z = X$ . Assume now that  $f$  is in  $B$ . Then  $f|_{Z \cap Y} = 0$ . As the  $C^*$ -algebra  $C_0(Z)$  admits an approximate unit, there exists a function  $h$  in  $C_0(Z)$  such that  $\|fh - f\| \leq \epsilon$ . By definition of  $C_0(Z)$  we can in addition assume that there exists a bounded subset  $B$  of  $Z$  such that  $h|_{Y \setminus B} = 0$ . Let  $\tilde{f}$  be the extension by zero of  $fh$  to  $X$ . Then  $\tilde{f}|_{X \setminus B} = 0$  and  $\tilde{f}$  is uniformly continuous since  $(Y, Z)$  is uniformly excisive. Let  $\hat{f}$  be the extension by zero of  $f$  to  $X$ . Then  $\|\hat{f} - \tilde{f}\| \leq \epsilon$ . Since  $\epsilon$  was arbitrary we conclude that  $\hat{f} \in A$  and that it is a preimage of  $f$ .

We now use the exactness of the  $E$ -theory functor  $E(-, \mathbb{C})$  in order to get a map of fibre sequences

$$\begin{array}{ccccc} K^{\text{an}}(Z \cap Y) & \longrightarrow & K^{\text{an}}(Z) & \longrightarrow & E(B, \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \cong \\ K^{\text{an}}(Y) & \longrightarrow & K^{\text{an}}(X) & \longrightarrow & E(A, \mathbb{C}) \end{array}$$

Since the right vertical map is an equivalence we conclude that the left square is a push-out. This shows that  $K^{\text{an}}$  is excisive.

The algebra  $C_0([0, \infty) \otimes X)$  is contractible by the homotopy  $C_0([0, \infty) \otimes X) \rightarrow C_0([0, 1] \times [0, \infty) \otimes X)$  sending  $f(t, x)$  to  $(s, t, x) \mapsto f(st, x)$ . This implies  $K^{\text{an}}([0, \infty) \otimes X) \simeq 0$  and therefore  $K^{\text{an}}$  vanishes on flasques.

Since  $K^{\text{an}}(X)$  does not depend on the coarse structure of  $X$  at all the functor  $K^{\text{an}}$  is trivially  $u$ -continuous.

This completes the verification that  $K^{\text{an}}$  is a local homology theory. We now show that  $K^{\text{an}}$  is locally finite. For  $B$  in  $\mathcal{B}$  we consider the exact sequence

$$0 \rightarrow C_B(X) \rightarrow C_0(X) \rightarrow C_0(X \setminus B) \rightarrow 0 ,$$

where  $C_B(X)$  is defined as the kernel of the restriction map along  $X \setminus B \rightarrow X$ . By the exactness of the  $E$ -theory functor  $E(-, \mathbb{C})$  we conclude that

$$\text{Cofib}(K^{\text{an}}(X \setminus B) \rightarrow K^{\text{an}}(X)) \simeq E(C_B(X), \mathbb{C}) .$$

Since the  $E$ -theory functor preserves filtered colimits we get

$$\begin{aligned} (K^{\text{an}})^{\text{lf}}(X) &\simeq \mathbf{lim}_{B \in \mathcal{B}^{\text{op}}} \text{Cofib}(K^{\text{an}}(X \setminus B) \rightarrow K^{\text{an}}(X)) \\ &\simeq \mathbf{lim}_{B \in \mathcal{B}^{\text{op}}} E(C_B(X), \mathbb{C}) \simeq E(\mathbf{colim}_{B \in \mathcal{B}} C_B(X), \mathbb{C}) . \end{aligned}$$

Under this equivalence the canonical map  $K^{\text{an}}(X) \rightarrow (K^{\text{an}})^{\text{lf}}(X)$  is induced by the map of  $C^*$ -algebras

$$\mathbf{colim}_{B \in \mathcal{B}} C_B(X) \rightarrow C_0(X) . \quad (3.15)$$

In order to show that  $K^{\text{an}}$  is locally finite it therefore suffices to show that (3.15) is an isomorphism of  $C^*$ -algebras. By Definition 3.12, the domain of this map is given by

$$\mathbf{colim}_{B \in \mathcal{B}} \ker \left( \mathbf{colim}_{B' \in \mathcal{B}} \ker(C_u(X) \rightarrow C_u(X \setminus B')) \rightarrow \mathbf{colim}_{B' \in \mathcal{B}} \ker(C_u(X \setminus B) \rightarrow C_u(X \setminus (B \cup B'))) \right) .$$

Since  $\mathcal{B}$  is filtered and  $\ker$  is a finite limit we can interchange  $\mathbf{colim}_{B' \in \mathcal{B}}$  with  $\ker$  and get

$$\mathbf{colim}_{(B', B) \in \mathcal{B} \times \mathcal{B}} \ker \left( \ker(C_u(X) \rightarrow C_u(X \setminus B')) \rightarrow \ker(C_u(X \setminus B) \rightarrow C_u(X \setminus (B \cup B'))) \right) .$$

If  $B' \subseteq B$ , then the map that the outer kernel is taken of is the zero map, so the argument of the colimit is just  $\ker(C_u(X) \rightarrow C_u(X \setminus B'))$ . Since the diagonal  $\mathcal{B} \rightarrow \mathcal{B} \times \mathcal{B}$  is cofinal, the domain of (3.15) is isomorphic to  $\mathbf{colim}_{B \in \mathcal{B}} \ker(C_u(X) \rightarrow C_u(X \setminus B))$  which is precisely the definition of the codomain.  $\square$

There is a Paschke duality transformation

$$p : K^{\mathcal{X}} \rightarrow K^{\text{an}} \quad (3.16)$$

whose components we will describe following [BELa]. The essential idea for the construction is taken from [QR10].

Recall that the bornology  $\mathcal{B}$  of  $X$  is a big family.

In the following we use the maximal tensor product of  $C^*$ -categories in the special case that the first factor is a  $C^*$ -algebra. Then the set of objects of the tensor product is the set of objects of the right factor  $\mathbf{C}(\mathcal{O}^\infty(X))$ , and the morphisms of the tensor product are generated by tensor products of algebra elements and morphisms. Further note that the set of objects of the quotient on the right-hand side of (3.31) is also canonically identified with the set of objects of  $\mathbf{C}(\mathcal{O}^\infty(X))$ , see Remark 2.61.

**Lemma 3.31.** *We have a morphism*

$$\mu : C_0(X) \otimes \mathbf{C}(\mathcal{O}^\infty(X)) \rightarrow \frac{\mathbf{C}(\mathcal{O}^\infty(\mathcal{B}) \subseteq \mathcal{O}^\infty(X))}{\mathbf{C}(\mathcal{O}^-(\mathcal{B}) \subseteq \mathcal{O}^\infty(X))} \quad (3.17)$$

given as follows:

1. *objects: It is the identity on objects.*
2. *morphisms: It sends the morphism  $f \otimes A : (H, \chi) \rightarrow (H', \chi')$  to  $[\chi'(\mathbf{pr}^* f)A] : (H, \chi) \rightarrow (H', \chi')$ , where  $\mathbf{pr} : \mathbb{R} \times X \rightarrow X$  is the projection.*

*Proof.* This is a special case of the more general Lemma 3.39 below, which reduces to Lemma 3.31 upon setting  $\mathcal{Y} = \mathcal{B}$  and  $\mathcal{Z} = \{X\}$ .  $\square$

The Paschke morphism

$$p_X : K^{\mathcal{X}}(X) \rightarrow K^{\text{an}}(X) \quad (3.18)$$

is defined as the composition

$$\begin{aligned} K^{\mathcal{X}}(X) &\simeq \Sigma^{-1}K(\mathbf{C}(\mathcal{O}^\infty(X))) \simeq \Sigma^{-1}\mathbf{E}(\mathbb{C}, \mathbf{C}(\mathcal{O}^\infty(X))) \\ &\xrightarrow{C_0(X) \otimes -} \Sigma^{-1}\mathbf{E}(C_0(X), C_0(X) \otimes \mathbf{C}(\mathcal{O}^\infty(X))) \\ &\xrightarrow{\mu_*} \Sigma^{-1}\mathbf{E}\left(C_0(X), \frac{\mathbf{C}(\mathcal{O}^\infty(\mathcal{B}) \subseteq \mathcal{O}^\infty(X))}{\mathbf{C}(\mathcal{O}^-(\mathcal{B}) \subseteq \mathcal{O}^\infty(X))}\right) \\ &\stackrel{!}{\simeq} \Sigma^{-1}\mathbf{E}(C_0(X), \mathbf{C}(\mathcal{O}^\infty(\mathcal{B}) \subseteq \mathcal{O}^\infty(X))) \\ &\stackrel{!!}{\rightarrow} \Sigma^{-1}\mathbf{E}(C_0(X), \mathbf{C}(\mathcal{O}^\infty(*))) \stackrel{!!!}{\simeq} \mathbf{E}(C_0(X), \mathbb{C}) \simeq K^{\text{an}}(X), \end{aligned}$$

where for ! we use that the projection

$$e(\mathbf{C}(\mathcal{O}^\infty(\mathcal{B}) \subseteq \mathcal{O}^\infty(X))) \rightarrow e\left(\frac{\mathbf{C}(\mathcal{O}^\infty(\mathcal{B}) \subseteq \mathcal{O}^\infty(X))}{\mathbf{C}(\mathcal{O}^-(\mathcal{B}) \subseteq \mathcal{O}^\infty(X))}\right)$$

induces an equivalence since the functor  $e$  is exact and we have  $e(\mathbf{C}(\mathcal{O}^-(\mathcal{B}) \subseteq \mathcal{O}^\infty(X))) \simeq 0$ . For the latter equivalence, we use (see Lemma 2.62 and Remark 2.63) that

$$e(\mathbf{C}(\mathcal{O}^-(\mathcal{B}) \subseteq \mathcal{O}^\infty(X))) \simeq \operatorname{colim}_{n \in \mathbb{N}, B \in \mathcal{B}} e(\mathbf{C}((-\infty, n] \times B)),$$



where  $(-\infty, n] \times B$  has the bornological coarse structure induced from  $\mathcal{O}^\infty(X)$ . These subspaces are flasque and therefore  $\mathbf{C}((-\infty, n] \times B)$  is flasque as a  $C^*$ -category and hence annihilated by  $e$ . The morphism  $!!$  is induced by the projections  $B \rightarrow *$  for the members  $B$  of  $\mathcal{B}$ , and  $!!!$  employs the equivalence

$$e(\mathbf{C}(\mathcal{O}^\infty(*))) \stackrel{e(\mathbf{C}(\partial^{\text{cone}}))}{\simeq} e(\mathbf{C}(\mathbb{R})) \stackrel{\text{Remark 2.59}, (2.13)}{\simeq} \Sigma e(\mathbf{C}(*)) \simeq \Sigma e(\mathbb{C}) .$$

We refer to [BELa] for more details and a better description of the Paschke morphism which exhibits  $p_X$  as a component of a natural transformation. In [BELa] it is shown that the Paschke morphism  $p_X$  is an equivalence under very general conditions on  $X$ .

**Theorem 3.32** ([BELa]). *If  $X$  is in UBC homotopy equivalent to a countable, finite-dimensional, locally finite simplicial complex with the spherical path metric, then  $p_X$  in (3.18) is an equivalence.*

*Proof.* In order to convince the reader that the statement is true we sketch the proof and refer to [BELa] for more details. We use (an import from [BELa]) the non-trivial fact that  $p_X$  is the component of a natural transformation  $p : K^{\mathcal{X}} \rightarrow K^{\text{an}}$ .

By the homotopy invariance of  $K^{\mathcal{X}}$  and  $K^{\text{an}}$  we can then assume that  $X$  itself is a locally finite finite-dimensional simplicial complex with the spherical path metric.

We then argue by induction on the dimension. For the induction step we use that gluing the disjoint union of the  $n$ -simplices into the  $n - 1$ -skeleton provides a coarsely and uniformly excisive decomposition, and that the disjoint union of  $n$ -simplices is homotopy equivalent to a discrete space. In these two steps it is important that the simplices are equicontinuously parameterized, a fact guaranteed by the choice of the metric. We then use that both functors  $K^{\mathcal{X}}$  and  $K^{\text{an}}$  are excisive and homotopy invariant.

In order to start the induction we must show the assertion for discrete spaces. If  $X$  is discrete, then  $K^{\mathcal{X}}(X) = \prod_{x \in X} KU$  since  $K^{\mathcal{X}}$  is additive, and also  $K^{\text{an}}(X) \cong \prod_{x \in X} KU$  since  $K^{\text{an}}$  is locally finite. Under these identifications the map  $p_X$  becomes the identity.  $\square$

**Remark 3.33.** We note that the Paschke morphism  $p_X$  in (3.18) is similar but not the same as the Paschke isomorphism as discussed, e.g., in [HR00]. The maps have different domains.

The Paschke morphism from [HR00] is an equivalence for any locally compact metric space. We do not know whether we can remove the additional assumption on  $X$  in Theorem 3.32 for the Paschke morphism with domain  $K^{\mathcal{X}}(X)$ . In fact, we do not expect that the functor  $K^{\mathcal{X}}$  is locally finite.  $\blacksquare$

### 3.4 The coarse corona pairing

In this and the next section we construct the coarse corona pairing  $\cap^{\mathcal{X}}$  and the coarse symbol pairing  $\cap^{\mathcal{X}\sigma}$  mentioned in the introduction. The principal ideas going into their constructions are surely well known and variants in different setups have been considered previously, see, e.g., [QR10, Wul21].

We consider a bornological coarse space  $X$  with two big families  $\mathcal{Y}$  and  $\mathcal{Z}$ .

In order to understand the following assertion note that the sets of objects of the  $C^*$ -categories in the domain and target of the functor in (3.19) are canonically identified.

**Lemma 3.34.** *We have a morphism of  $C^*$ -categories*

$$\nu : C(\partial^{\mathcal{Y}}X) \otimes \mathbf{C}(\mathcal{Z} \subseteq X) \rightarrow \frac{\mathbf{C}(\mathcal{Z} \subseteq X)}{\mathbf{C}(\mathcal{Y} \cap \mathcal{Z} \subseteq X)}. \quad (3.19)$$

given as follows:

1. *objects: It acts as identity on objects.*
2. *morphisms: It sends the morphism  $[f] \otimes A : (H, \chi) \rightarrow (H', \chi')$  to  $[\chi'(f)A] : (H, \chi) \rightarrow (H', \chi')$ .*

*Proof.* Let  $f$  be in  $\ell_{\mathcal{Y}}^{\infty}(X)$  and  $A$  be in  $\mathbf{C}(\mathcal{Z} \subseteq X)$ . First note that if in addition  $f \in \ell^{\infty}(\mathcal{Y})$ , then  $\chi'(f)A \in \mathbf{C}(\mathcal{Y} \cap \mathcal{Z} \subseteq X)$ . Hence the class  $[\chi'(f)A]$  only depends on the class  $[f]$  in  $\ell_{\mathcal{Y}}^{\infty}(X)/\ell^{\infty}(\mathcal{Y}) = C(\partial^{\mathcal{Y}}X)$ , so the map  $\nu$  is well-defined on elementary tensors. Furthermore, by Corollary 3.11 we have  $\chi'(f)A - A\chi(f) \in \mathbf{C}(\mathcal{Y} \subseteq X)$ . Since the individual terms belong to  $\mathbf{C}(\mathcal{Z} \subseteq X)$  we conclude that actually  $\chi'(f)A - A\chi(f)$  belongs to  $\mathbf{C}(\mathcal{Y} \cap \mathcal{Z} \subseteq X)$ .

The arguments above show that  $\nu$  is a well-defined functor (see the proof of Lemma 3.39 for an analogous argument with more details) defined on the algebraic tensor product in the domain. Since its target is a  $C^*$ -category,  $\nu$  uniquely extends to a functor defined on the maximal tensor product, by the universal property of the latter.  $\square$

Upon applying  $K$ -theory to  $\nu$  from Lemma 3.34 and using the symmetric monoidal structure of  $K$  and Lemma 2.62 we get a pairing

$$K(\partial^{\mathcal{Y}}X) \times K\mathcal{X}(\mathcal{Z}) \rightarrow K\mathcal{X}(\mathcal{Z}, \mathcal{Y} \cap \mathcal{Z}). \quad (3.20)$$

**Definition 3.35.** *We define the coarse corona pairing*

$$-\cap^{\mathcal{X}}- : K(\partial^{\mathcal{Y}}X) \times K\mathcal{X}(\mathcal{Z}) \rightarrow \Sigma K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z}) \quad (3.21)$$

as the composition

$$K(\partial^{\mathcal{Y}} X) \times K\mathcal{X}(\mathcal{Z}) \xrightarrow{\nu} K\mathcal{X}(\mathcal{Z}, \mathcal{Y} \cap \mathcal{Z}) \xrightarrow{\partial} \Sigma K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z}). \quad (3.22)$$

of (3.20) with the boundary map for the pair  $(\mathcal{Z}, \mathcal{Y} \cap \mathcal{Z})$ .

**Remark 3.36.** We now explain the naturality properties of the coarse corona pairing. To begin with, we consider the category  $\mathbf{BC}^{(3)}$ , whose objects are triples  $(X, \mathcal{Y}, \mathcal{Z})$  of a space  $X$  together with two big families  $\mathcal{Y}, \mathcal{Z}$  on  $X$ , and whose morphisms  $f : (X, \mathcal{Y}, \mathcal{Z}) \rightarrow (X', \mathcal{Y}', \mathcal{Z}')$  are morphisms in  $\mathbf{BC}$  such that  $f(\mathcal{Y}) \subseteq \mathcal{Y}'$  and  $f(\mathcal{Z}) \subseteq \mathcal{Z}'$ . We let  $\mathbf{BC}_{\text{tw}}^{(3)}$  be the following variation on the twisted arrow category of  $\mathbf{BC}^{(3)}$ : Its objects consist of an object  $(X, \mathcal{Y}, \mathcal{Z})$  of  $\mathbf{BC}^{(3)}$ , a pair  $(\tilde{X}, \tilde{\mathcal{Y}})$  of a coarse space  $\tilde{X}$  and a big family  $\tilde{\mathcal{Y}}$  on  $X$ , as well as a controlled map  $f : X \rightarrow \tilde{X}$  such that  $f^{-1}(\tilde{\mathcal{Y}}) \subseteq \mathcal{Y}$ . Morphisms of  $\mathbf{BC}_{\text{tw}}^{(3)}$  are diagrams

$$\begin{array}{ccc} (\tilde{X}, \tilde{\mathcal{Y}}) & \xleftarrow{f} & (X, \mathcal{Y}, \mathcal{Z}) \\ g \uparrow & & \downarrow h \\ (\tilde{X}', \tilde{\mathcal{Y}}') & \xleftarrow{f'} & (X', \mathcal{Y}', \mathcal{Z}') \end{array} \quad (3.23)$$

such that the underlying maps commute, where  $h$  is a morphism in  $\mathbf{BC}^{(3)}$ , and where  $g$  is a controlled map such that  $g^{-1}(\tilde{\mathcal{Y}}) \subseteq \mathcal{Y}'$ . Here the morphism direction is downward.

There are two functors

$$L, R : \mathbf{BC}_{\text{tw}}^{(3)} \longrightarrow \mathbf{Mod}(KU)$$

whose action on objects is given by

$$\begin{aligned} L((\tilde{X}, \tilde{\mathcal{Y}}) \xleftarrow{f} (X, \mathcal{Y}, \mathcal{Z})) &= K(\partial^{\tilde{\mathcal{Y}}} \tilde{X}) \times K\mathcal{X}(\mathcal{Z}) \\ R((\tilde{X}, \tilde{\mathcal{Y}}) \xleftarrow{f} (X, \mathcal{Y}, \mathcal{Z})) &= K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z}). \end{aligned}$$

Naturality of the coarse corona pairing is the assertion that there is a natural transformation  $L \rightarrow R$  whose component at  $f : (X, \mathcal{Y}, \mathcal{Z}) \rightarrow (\tilde{X}, \tilde{\mathcal{Y}})$  is the composition

$$\cap^{\mathcal{X}} \circ (f^* \times \text{id}) : K(\partial^{\tilde{\mathcal{Y}}} \tilde{X}) \times K\mathcal{X}(\mathcal{Z}) \rightarrow \Sigma K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z}).$$

This natural transformation arises from applying the  $K$ -theory functor to a similar (1-categorical) natural transformation at the level of  $C^*$ -categories, involving the morphism  $\nu$  from Lemma 3.34.

In particular, the natural transformation assigns to the morphism (3.23) in  $\mathbf{BC}_{\text{tw}}^{(3)}$  the commutative diagram

$$\begin{array}{ccccc} K(\partial^{\tilde{\mathcal{Y}}} \tilde{X}) \times K\mathcal{X}(\mathcal{Z}) & \xrightarrow{f^* \times \text{id}} & K(\partial^{\mathcal{Y}} X) \times K\mathcal{X}(\mathcal{Z}) & \xrightarrow{\cap^{\mathcal{X}}} & \Sigma K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z}) \\ g^* \times h_* \downarrow & & & & \downarrow h_* \\ K(\partial^{\tilde{\mathcal{Y}}'} \tilde{X}') \times K\mathcal{X}(\mathcal{Z}') & \xrightarrow{f'^* \times \text{id}} & K(\partial^{\mathcal{Y}'} X') \times K\mathcal{X}(\mathcal{Z}') & \xrightarrow{\cap^{\mathcal{X}}} & \Sigma K\mathcal{X}(\mathcal{Y}' \cap \mathcal{Z}') \end{array} \quad (3.24)$$

We summarize this by saying that  $\cap^{\mathcal{X}}$  is an extra-natural transformation. ■

**Remark 3.37.** particular case of Remark 3.36 is the following: If  $f : (X, \mathcal{Y}, \mathcal{Z}) \rightarrow (X', \mathcal{Y}', \mathcal{Z}')$  is a morphism in  $\mathbf{BC}^{(3)}$  satisfying the additional condition that  $f^{-1}(\mathcal{Y}') \subseteq \mathcal{Y}$ , then we have the commutative diagram

$$\begin{array}{ccc}
K(\partial^{\mathcal{Y}} X) \times K\mathcal{X}(\mathcal{Z}) & \xrightarrow{\cap^{\mathcal{X}}} & \Sigma K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z}) \\
\uparrow f^* \times \text{id} & & \downarrow f_* \\
K(\partial^{\mathcal{Y}'} X') \times K\mathcal{X}(\mathcal{Z}) & & \\
\downarrow \text{id} \times f_* & & \\
K(\partial^{\mathcal{Y}'} X') \times K\mathcal{X}(\mathcal{Z}') & \xrightarrow{\cap^{\mathcal{X}}} & \Sigma K\mathcal{X}(\mathcal{Y}' \cap \mathcal{Z}') .
\end{array}$$

This is the special case of (3.24) for  $g = f' = \text{id}$  and  $h = f$ . ■

Let  $X$  be a bornological coarse space and let  $\mathcal{W}, \mathcal{W}'$  be two big families on  $X$  such that  $X = W \cup W'$  for some members  $W$  of  $\mathcal{W}$  and  $W'$  of  $\mathcal{W}'$ . We then have the following compatibility of the coarse corona pairing with the Mayer-Vietoris boundary map corresponding to this decomposition:

**Lemma 3.38.** *The following diagram commutes:*

$$\begin{array}{ccc}
K(\partial^{\mathcal{Y}} X) \times \Sigma^{-1} K\mathcal{X}(\mathcal{Z}) & \xrightarrow{\cap^{\mathcal{X}}} & K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z}) \\
\text{id} \times \partial^{\text{MV}} \downarrow & & \downarrow \partial^{\text{MV}} \\
K(\partial^{\mathcal{Y}} X) \times K\mathcal{X}(\mathcal{Z} \cap \mathcal{W} \cap \mathcal{W}') & \xrightarrow{\cap^{\mathcal{X}}} & \Sigma K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z} \cap \mathcal{W} \cap \mathcal{W}')
\end{array} \tag{3.25}$$

*Proof.* Consider the commutative diagram

$$\begin{array}{ccc}
K(\partial^{\mathcal{Y}} X) \times K\mathcal{X}(\mathcal{W} \cap \mathcal{W}' \cap \mathcal{Z}) & \xrightarrow{\quad\quad\quad} & K(\partial^{\mathcal{Y}} X) \times K\mathcal{X}(\mathcal{W}' \cap \mathcal{Z}) \\
\downarrow & \searrow \cap^{\mathcal{X}} & \downarrow \cap^{\mathcal{X}} \\
& \Sigma K\mathcal{X}(\mathcal{W} \cap \mathcal{W}' \cap \mathcal{Z} \cap \mathcal{Y}) & \xrightarrow{\quad\quad\quad} & \Sigma K\mathcal{X}(\mathcal{W}' \cap \mathcal{Z} \cap \mathcal{Y}) \\
& \downarrow & & \downarrow \\
& \Sigma K\mathcal{X}(\mathcal{W}' \cap \mathcal{Z} \cap \mathcal{Y}) & \xrightarrow{\quad\quad\quad} & \Sigma K\mathcal{X}(\mathcal{Z} \cap \mathcal{Y}) \\
& \nearrow \cap^{\mathcal{X}} & & \nwarrow \cap^{\mathcal{X}} \\
K(\partial^{\mathcal{Y}} X) \times K\mathcal{X}(\mathcal{W}' \cap \mathcal{Z}) & \xrightarrow{\quad\quad\quad} & K(\partial^{\mathcal{Y}} X) \times K\mathcal{X}(\mathcal{Z}) ,
\end{array}$$

where both the interior and the exterior square is a pushout square, each giving rise to a Mayer-Vietoris boundary map. The four trapezoids commute by Remark 3.37, which in each case is applied to a map of triples that is the identity on  $X$ . We therefore obtain a map of pushout diagrams, which yields the desired commutative diagram (3.25). □

### 3.5 The coarse symbol pairing

We consider a uniform bornological coarse space  $X$  with big families  $\mathcal{Y}$  and  $\mathcal{Z}$  and let  $\mathcal{O}^-(\mathcal{Y})$  and  $\mathcal{O}_{\mathcal{Z}}^\infty(X)$  be the big families in  $\mathcal{O}^\infty(X)$  defined in (2.35).

**Lemma 3.39.** *We have a morphism of  $C^*$ -categories*

$$\mu : C_u(\mathcal{Y}) \otimes \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(X) \subseteq \mathcal{O}^\infty(X)) \rightarrow \frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(X))}{\mathbf{C}(\mathcal{O}^-(\mathcal{Z} \cap \mathcal{Y}) \subseteq \mathcal{O}^\infty(X))} \quad (3.26)$$

given as follows:

1. *objects: It acts as identity on objects.*
2. *morphisms: It sends the morphism  $f \otimes A : (H, \chi) \rightarrow (H', \chi')$  to  $[\chi'(\mathbf{pr}^*f)A] : (H, \chi) \rightarrow (H', \chi')$ , where  $\mathbf{pr} : \mathbb{R} \times X \rightarrow X$  is the projection.*

*Proof.* Let  $f$  be in  $C_u(\mathcal{Y})$  and  $A : (H, \chi) \rightarrow (H', \chi')$  be a morphism in  $\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(X) \subseteq \mathcal{O}^\infty(X))$ . We first show that  $\chi'(\mathbf{pr}^*f)A$  is a morphism of  $\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(X))$ . By assumption,  $f$  can be approximated by functions which are supported on members of  $\mathcal{Y}$ . This implies that  $\mathbf{pr}^*f \in \ell^\infty(\mathcal{O}^\infty(\mathcal{Y}))$ . Similarly,  $A$  can be approximated by morphisms supported on members of  $\mathcal{O}_{\mathcal{Z}}^\infty(X)$ . Consequently, the product  $\chi'(\mathbf{pr}^*f)A$  can be approximated by morphisms supported on members of  $\mathcal{O}^\infty(\mathcal{Y}) \cap \mathcal{O}_{\mathcal{Z}}^\infty(X) = \mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y})$ . We conclude that  $\chi'(\mathbf{pr}^*f)A \in \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(X))$  as desired.

We next show that  $\mu$  is compatible with the composition. Given another morphism  $B : (H', \chi') \rightarrow (H'', \chi'')$  in  $\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(X))$  and a function  $g$  in  $C_u(\mathcal{Y})$ , we have

$$\mu(g \otimes B)\mu(f \otimes A) - [\mu(gf \otimes BA)] = [\chi''(\mathbf{pr}^*g)(B\chi'(\mathbf{pr}^*f) - \chi''(\mathbf{pr}^*f)B)A]$$

We therefore have to show that  $B\chi'(\mathbf{pr}^*f) - \chi''(\mathbf{pr}^*f)B$  belongs to the ideal  $\mathbf{C}(\mathcal{O}^-(\mathcal{Z} \cap \mathcal{Y}) \subseteq \mathcal{O}^\infty(X))$ .

To this end, we claim that the uniform continuity of  $f$  implies that  $\mathbf{pr}^*f \in \ell_{\mathcal{O}^-(X)}^\infty(\mathcal{O}^\infty(X))$ . Indeed, for each  $\varepsilon$  in  $(0, \infty)$ , there exists a uniform entourage  $U$  of  $X$  such that  $|f(x) - f(y)| \leq \varepsilon$  whenever  $(x, y) \in U$ . Let  $V$  be a coarse entourage of  $\mathcal{O}^\infty(X)$ . Then by Definition 2.11.1c, there exists  $r$  in  $\mathbb{R}$  such that  $((x, t), (x', t')) \in V$  and  $\min(t, t') \geq r$  imply  $(x, x') \in U$ . Therefore  $\text{Var}_V(\mathbf{pr}^*f, \mathcal{O}^\infty(X) \setminus W) \leq \varepsilon$  for any member  $W := (-\infty, s] \times X$  of  $\mathcal{O}^-(X)$  with  $s \geq r$ . This shows that  $\mathbf{pr}^*f \in \ell_{\mathcal{O}^-(X)}^\infty(X)$ , as claimed.

In view of the claim Corollary 3.11 implies  $\chi''(\mathbf{pr}^*f)B - B\chi'(\mathbf{pr}^*f) \in \mathbf{C}(\mathcal{O}^-(X) \subseteq \mathcal{O}^\infty(X))$ . Furthermore, by the assumptions on  $f$  and  $B$  the individual terms belong to  $\mathbf{C}(\mathcal{O}^\infty(\mathcal{Y}) \cap \mathcal{O}_{\mathcal{Z}}^\infty(X) \subseteq \mathcal{O}^\infty(X))$ . In total,  $B\chi'(\mathbf{pr}^*f) - \chi''(\mathbf{pr}^*f)B$  belongs to the ideal associated to the big family

$$\mathcal{O}^-(X) \cap \mathcal{O}^\infty(\mathcal{Y}) \cap \mathcal{O}_{\mathcal{Z}}^\infty(X) = \mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z})$$

as desired.

We thus get a functor  $\mu$  defined on the algebraic tensor product, which then uniquely extends to the maximal tensor product.  $\square$

We apply the  $K$ -theory functor to the  $*$ -homomorphism  $\mu$  from Lemma 3.39. With a view on Definitions 3.23 and 3.25 and Lemma 2.62, the fact that  $K$  is symmetric monoidal yields a pairing

$$\mu : K(\mathcal{Y}) \times K_{\mathcal{Z}}^{\mathcal{X}}(X) \rightarrow \Sigma^{-1}K\mathcal{X}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y}), \mathcal{O}^{-}(\mathcal{Y} \cap \mathcal{Z})) . \quad (3.27)$$

Note that the members of  $\mathcal{O}^{-}(\mathcal{Y} \cap \mathcal{Z})$  are flasque and consequently, that

$$K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}) \simeq \Sigma^{-1}K\mathcal{X}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y})) \rightarrow \Sigma^{-1}K\mathcal{X}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y}), \mathcal{O}^{-}(\mathcal{Y} \cap \mathcal{Z}))$$

is an equivalence.

**Definition 3.40.** *We define the coarse symbol pairing*

$$-\cap^{\mathcal{X}\sigma} - : K(\mathcal{Y}) \times K_{\mathcal{Z}}^{\mathcal{X}}(X) \rightarrow K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y})$$

as the composition

$$K(\mathcal{Y}) \times K_{\mathcal{Z}}^{\mathcal{X}}(X) \xrightarrow{\mu} \Sigma^{-1}K\mathcal{X}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y}), \mathcal{O}^{-}(\mathcal{Y} \cap \mathcal{Z})) \xleftarrow{\simeq} K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}) . \quad (3.28)$$

The name of this pairing will be justified in Section 4.2 where we show in Theorem 4.32 that it sends symbols of Dirac type operators to symbols (with support) of suitable Callias type operators.

**Remark 3.41.** The categories  $\mathbf{UBC}^{(3)}$  and  $\mathbf{UBC}_{\text{tw}}^{(3)}$  may be defined analogously to the categories  $\mathbf{BC}^{(3)}$  and  $\mathbf{BC}_{\text{tw}}^{(3)}$  of Remark 3.36, using spaces with compatible coarse and uniform structures instead of coarse spaces. One may then say that the coarse symbol pairings  $\cap^{\mathcal{X}\sigma}$  are the components of an extra-natural transformation.

A particular case is the following analog of Remark 3.37: For a morphism  $f : (X, \mathcal{Y}, \mathcal{Z}) \rightarrow (X', \mathcal{Y}', \mathcal{Z}')$  in  $\mathbf{UBC}^{(3)}$  satisfying the additional condition that  $f^{-1}(\mathcal{Y}') \subseteq \mathcal{Y}$ , we have a commutative diagram

$$\begin{array}{ccc} K(\mathcal{Y}) \times K_{\mathcal{Z}}^{\mathcal{X}}(X) & \xrightarrow{\cap^{\mathcal{X}\sigma}} & \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}) \\ \uparrow f^* \times \text{id} & & \downarrow f_* \\ K(\partial^{\mathcal{Y}'} X') \times K_{\mathcal{Z}}^{\mathcal{X}}(X) & & \\ \downarrow \text{id} \times f_* & & \\ K(\mathcal{Y}') \times K_{\mathcal{Z}'}^{\mathcal{X}}(X') & \xrightarrow{\cap^{\mathcal{X}\sigma}} & \Sigma K_{\mathcal{Z}'}^{\mathcal{X}}(\mathcal{Y}') . \end{array}$$

■

**Remark 3.42.** If  $\mathcal{Y}'$  is a further big family, then (3.26) can be generalized to a paring

$$C_u(\mathcal{Y}) \otimes \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y}') \subseteq \mathcal{O}^{\infty}(X)) \rightarrow \frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y} \cap \mathcal{Y}') \subseteq \mathcal{O}^{\infty}(X))}{\mathbf{C}(\mathcal{O}^-(\mathcal{Y} \cap \mathcal{Y}' \cap \mathcal{Z}) \subseteq \mathcal{O}^{\infty}(X))}$$

which yields

$$-\cap^{\mathcal{X}\sigma} - : K(\mathcal{Y}) \times K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}') \rightarrow K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y} \cap \mathcal{Y}') . \quad (3.29)$$

■

Let  $X$  be in **UBC** and let  $\mathcal{Y}, \mathcal{Z}$  be big families on  $X$ . The following proposition describes the compatibility of the pairings introduced above with the index map (3.11). Note that we drop the forgetful functor  $\iota : \mathbf{UBC} \rightarrow \mathbf{BC}$ .

**Proposition 3.43.** *We have a commutative diagram*

$$\begin{array}{ccc} \Sigma K(\mathcal{Y}) \times K_{\mathcal{Z}}^{\mathcal{X}}(X) & \xrightarrow{\cap^{\mathcal{X}\sigma}} & \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}) \\ \partial \times \text{id} \uparrow & & \downarrow a_{\mathcal{Y}, \mathcal{Z}} \\ K(\partial_u^{\mathcal{Y}} X) \times K_{\mathcal{Z}}^{\mathcal{X}}(X) & & \\ \downarrow c^* \times a_{X, \mathcal{Z}} & & \\ K(\partial_u^{\mathcal{Y}} X) \times K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Z}) & \xrightarrow{\cap^{\mathcal{X}}} & \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y} \cap \mathcal{Z}) . \end{array}$$

*Proof.* Consider the diagram of in  $C^*$ -categories with exact rows

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ C_u(\mathcal{Y}) \otimes \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(X) \subseteq \mathcal{O}^{\infty}(X)) & \xrightarrow{\mu} & \frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y}) \subseteq \mathcal{O}^{\infty}(X))}{\mathbf{C}(\mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z}) \subseteq \mathcal{O}^{\infty}(X))} \\ \downarrow & & \downarrow \\ C_{u, \mathcal{Y}}(X) \otimes \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(X) \subseteq \mathcal{O}^{\infty}(X)) & \longrightarrow & \frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(X) \subseteq \mathcal{O}^{\infty}(X))}{\mathbf{C}(\mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z}) \subseteq \mathcal{O}^{\infty}(X))} \\ \downarrow & & \downarrow \\ C(\partial_u^{\mathcal{Y}} X) \otimes \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(X) \subseteq \mathcal{O}^{\infty}(X)) & \longrightarrow & \frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(X) \subseteq \mathcal{O}^{\infty}(X))}{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y}) \subseteq \mathcal{O}^{\infty}(X))} \\ \downarrow & & \downarrow \\ 0 & & 0 , \end{array}$$

where the horizontal maps are all given by the description  $f \otimes A \mapsto [\chi(\mathbf{pr}^* f)A]$ , thus making the diagram commutative. We now apply the  $K$ -theory functor to obtain a map

of fiber sequences of  $KU$ -modules. In particular, inserting Definition 2.56, we get the commutative square

$$\begin{array}{ccc} \Sigma K(\mathcal{Y}) \times K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(X)) & \xrightarrow{\mu} & \Sigma K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y}), \mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z})) \\ \partial \times \text{id} \uparrow & & \uparrow \partial \\ K(\partial_u^{\mathcal{Y}} X) \times K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(X)) & \longrightarrow & K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(X), \mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y})) , \end{array}$$

involving the boundary maps for these fiber sequences. Desuspending this diagram and using Definition 3.25, we may expand the resulting diagram to involve the coarse symbol map

$$\begin{array}{ccccc} & & \cap^{\mathcal{X}\sigma} & & \\ & & \curvearrowright & & \\ \Sigma K(\mathcal{Y}) \times K_{\mathbb{Z}}^{\mathcal{X}}(X) & \xrightarrow{\mu} & K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y}), \mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z})) & \xleftarrow{\cong} & \Sigma K_{\mathbb{Z}}^{\mathcal{X}}(\mathcal{Y}) \\ \partial \times \text{id} \uparrow & & \uparrow \partial & & \parallel \\ K(\partial_u^{\mathcal{Y}} X) \times K_{\mathbb{Z}}^{\mathcal{X}}(X) & \longrightarrow & \Sigma^{-1} K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(X), \mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y})) & \xrightarrow{\partial} & K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y})) . \end{array} \quad (3.30)$$

Here the rightmost square commutes by compatibility of the boundary maps for the map of fiber sequences

$$\begin{array}{ccccc} K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y})) & \longrightarrow & K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(X)) & \longrightarrow & K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(X), \mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y})) \\ \downarrow & & \downarrow & & \parallel \\ K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y}), \mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z})) & \longrightarrow & K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(X), \mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z})) & \longrightarrow & K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(X), \mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y})) . \end{array}$$

The projection map  $\text{pr} : \mathcal{O}^{\infty}(X) \rightarrow X$  is controlled and satisfies  $\text{pr}^{-1}(\mathcal{Y}) = \mathcal{O}^{\infty}(\mathcal{Y})$ , hence extends to a map  $\mathbf{pr} : \partial^{\mathcal{O}^{\infty}(\mathcal{Y})} \mathcal{O}^{\infty}(X) \rightarrow \partial^{\mathcal{Y}} X$  between the corresponding coarse coronas (see Remark 3.8). This allows to write the second row of (3.30) as a coarse corona pairing over the space  $\mathcal{O}^{\infty}(X)$ , for the big families  $\mathcal{O}_{\mathbb{Z}}^{\infty}(X)$  and  $\mathcal{O}^{\infty}(\mathcal{Y})$ :

$$\begin{array}{ccccc} K(\partial_u^{\mathcal{Y}} X) \times K_{\mathbb{Z}}^{\mathcal{X}}(X) & \longrightarrow & \Sigma^{-1} K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(X), \mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y})) & \xrightarrow{\partial} & K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(\mathcal{Y})) \\ c^* \times \text{id} \downarrow & & \nearrow \nu & & \nearrow \\ K(\partial^{\mathcal{Y}} X) \times K_{\mathbb{Z}}^{\mathcal{X}}(X) & & & & \\ \text{pr}^* \times \text{id} \downarrow & & & & \\ K(\partial^{\mathcal{O}^{\infty}(\mathcal{Y})} \mathcal{O}^{\infty}(X)) \times \Sigma^{-1} K\mathcal{X}(\mathcal{O}_{\mathbb{Z}}^{\infty}(X)) & & & \xrightarrow{\cap^{\mathcal{X}}} & \end{array} \quad (3.31)$$

Here the top left triangle commutes by definition of the top left horizontal arrow, while the other triangle is the definition of the coarse corona pairing. The bottom arrow of the



diagram (3.31) may be extended to the following diagram:

$$\begin{array}{ccc}
K(\partial^{\mathcal{O}^\infty(\mathcal{Y})}\mathcal{O}^\infty(X)) \times \Sigma^{-1}K\mathcal{X}(\mathcal{O}_Z^\infty(X)) & \xrightarrow{\cap^{\mathcal{X}}} & K\mathcal{X}(\mathcal{O}_Z^\infty(\mathcal{Y})) = \Sigma K_Z^{\mathcal{X}}(\mathcal{Y}) \\
\text{id} \times \partial^{\text{MV}} \downarrow & & \downarrow \partial^{\text{MV}} \\
K(\partial^{\mathcal{O}^\infty(\mathcal{Y})}\mathcal{O}^\infty(X)) \times K\mathcal{X}(\{0\} \otimes \mathcal{Z}) & \xrightarrow{\cap^{\mathcal{X}}} & \Sigma K\mathcal{X}(\{0\} \otimes (\mathcal{Y} \cap \mathcal{Z})) \\
\text{id} \times i_* \uparrow \simeq & & \uparrow \simeq i_* \\
K(\partial^{\mathcal{O}^\infty(\mathcal{Y})}\mathcal{O}^\infty(X)) \times K\mathcal{X}(\mathcal{Z}) & & \\
i^* \times \text{id} \downarrow & \xrightarrow{\cap^{\mathcal{X}}} & \Sigma K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z}) \\
K(\partial^{\mathcal{Y}}X) \times K\mathcal{X}(\mathcal{Z}) & & 
\end{array} \quad \begin{array}{l} \curvearrowright \\ a_{\mathcal{Y}, \mathcal{Z}} \end{array} \quad (3.32)$$

The upper square is commutative by compatibility of the coarse corona pairing with the Mayer-Vietoris boundary by applying Lemma 3.38 with  $\mathcal{O}^\infty(X)$ ,  $\{\mathbb{R}^-\} \times X$ ,  $\{\mathbb{R}^+\} \times X$ ,  $\mathcal{O}_Z^\infty(\mathcal{Y})$ , and  $\mathcal{O}_Z^\infty(X)$  in place of  $X$ ,  $\mathcal{W}$ ,  $\mathcal{W}'$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$ . Commutativity of the bottom square is extra-naturality of the coarse corona pairing for the inclusion map  $i : X \rightarrow \mathcal{O}^\infty(X)$ , see Remark 3.37. The triangle on the right is essentially the definition of the index map, see Remark 2.41. Finally, since  $\text{pr} \circ i = \text{id}$ , it is clear that the diagram

$$\begin{array}{ccc}
& & K(\partial^{\mathcal{Y}}X) \times K_Z^{\mathcal{X}}(X) \\
& & \text{pr}^* \times \text{id} \downarrow \\
& & K(\partial^{\mathcal{O}^\infty(\mathcal{Y})}\mathcal{O}^\infty(X)) \times \Sigma^{-1}K\mathcal{X}(\mathcal{O}_Z^\infty(X)) \\
& & \text{id} \times \partial^{\text{MV}} \downarrow \\
\text{id} \times a_{X, \mathcal{Z}} \curvearrowright & & K(\partial^{\mathcal{O}^\infty(\mathcal{Y})}\mathcal{O}^\infty(X)) \times K\mathcal{X}(\{0\} \otimes \mathcal{Z}) \\
& & \text{id} \times i_* \uparrow \simeq \\
& & K(\partial^{\mathcal{O}^\infty(\mathcal{Y})}\mathcal{O}^\infty(X)) \times K\mathcal{X}(\mathcal{Z}) \\
& & i^* \times \text{id} \downarrow \\
& & K(\partial^{\mathcal{Y}}X) \times K\mathcal{X}(\mathcal{Z})
\end{array} \quad (3.33)$$

commutes. The assertion of the proposition follows from pasting the diagrams (3.30)–(3.33).  $\square$

**Remark 3.44.** In the case  $X$  is a coarsifying proper metric space and  $\mathcal{Y} = \mathcal{B}$  the commutative diagram (3.43) is essentially equivalent to the statement of [EM06, Thm. 36].  $\blacksquare$

**Remark 3.45.** In this remark we explain why in the construction of the coarse index pairing we can not simply replace the local  $K$ -homology  $K^{\mathcal{X}}$  by its locally finite version

$K^{\mathcal{X},\text{lf}}$  (see Section 2.3) or the locally finite  $K$ -homology  $K^{\text{an}}$  introduced in Definition 3.29. We fix  $X$  in **UBC** with a big family  $\mathcal{Y}$ . For  $B$  in  $\mathcal{B}$  we have a restriction map

$$K(\mathcal{Y}) \rightarrow K(\mathcal{Y} \cap (X \setminus B)) .$$

By naturality of the coarse symbol pairing we can produce a map of diagrams indexed by  $B$  in  $\mathcal{B}^{\text{op}}$

$$K(\mathcal{Y}) \times \text{Cofib}(K^{\mathcal{X}}(X \setminus B) \rightarrow K^{\mathcal{X}}(X)) \rightarrow \text{Cofib}(K^{\mathcal{X}}(\mathcal{Y} \cap (X \setminus B)) \rightarrow K^{\mathcal{X}}(\mathcal{Y})) .$$

But note that taking the limit over  $\mathcal{B}^{\text{op}}$  does **not** produce a map

$$K(\mathcal{Y}) \times K^{\mathcal{X},\text{lf}}(X) \rightarrow K^{\mathcal{X},\text{lf}}(\mathcal{Y}) . \quad (3.34)$$

The problem is that in the construction above we must first take the colimit over  $\mathcal{Y}$  and then the limit over  $\mathcal{B}^{\text{op}}$ , while for the target in (3.34) we would first take the limit and then the colimit.

Similarly it is not clear whether in general the coarse index pairing has a factorization over the Paschke transformation  $p_X : K^{\mathcal{X}}(X) \rightarrow K^{\text{an}}(X)$  from (3.18), i.e., whether we can construct a pairing

$$K(\mathcal{Y}) \times K^{\text{an}}(X) \xrightarrow{-\cap^{\text{an}\sigma}-} K^{\text{an}}(\mathcal{Y})$$

rendering the obvious comparison square commutative. But note that if  $\mathcal{Y}$  consists of bounded subsets, then such a pairing exists. ■

Let  $X$  be in **UBC** and let  $\mathcal{Y}$ ,  $\mathcal{Y}'$  and  $\mathcal{Z}$  be big families on  $X$ . Recall the generalization of the coarse symbol pairing discussed in Remark 3.42.

**Lemma 3.46.** *The following square commutes*

$$\begin{array}{ccc} K(\mathcal{Y}) \times K(\mathcal{Y}') \times K_{\mathcal{Z}}^{\mathcal{X}}(X) & \xrightarrow{(3.5) \times \text{id}} & K(\mathcal{Y} \cap \mathcal{Y}') \times K_{\mathcal{Z}}^{\mathcal{X}}(X) \\ \text{id} \times \cap^{\mathcal{X}\sigma} \downarrow & & \downarrow \cap^{\mathcal{X}\sigma} \\ K(\mathcal{Y}) \times K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}') & \xrightarrow{\cap^{\mathcal{X}\sigma}} & K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y} \cap \mathcal{Y}') \end{array}$$

*Proof.* This follows from unfolding definitions, using (3.29) and the associativity of the symmetric monoidal structure of  $K$ . □

## 4 Applications to Dirac operators

### 4.1 The coarse index of generalized Dirac operators

Let  $M$  be a complete Riemannian manifold. Its underlying metric space represents a uniform bornological coarse space which we will also denote by  $M$ . We consider a

generalized Dirac operator

$$\mathcal{D} : C^\infty(M, E) \rightarrow C^\infty(M, E)$$

of degree  $n$ . Thus  $E$  is a hermitean vector bundle of graded modules over the Clifford algebra  $\mathbf{Cl}^n$  and  $\mathcal{D}$  is an odd first order formally selfadjoint differential operator commuting with the  $\mathbf{Cl}^n$ -action and such that

$$\mathcal{D}^2 = \nabla^* \nabla + R_0 \tag{4.1}$$

for some hermitean connection  $\nabla$  on  $E$  and  $R_0$  in  $C^\infty(M, \mathbf{End}(E))$ . We consider  $\mathcal{D}$  and its powers as symmetric operators on the Hilbert space  $L^2(M, E)$  with domain  $C_c^\infty(M, E)$ . By [?] the unbounded operator  $\mathcal{D}$  and its powers are essentially selfadjoint and we will use the same symbols also to denote their unique selfadjoint extensions.

The Dirac operator gives rise to a coarse index class  $\text{ind} \mathcal{X}(\mathcal{D})$  in  $K \mathcal{X}_{-n}(M)$  whose construction we will describe below in detail. Under additional local positivity assumptions this index class is actually supported on a suitable big family on  $M$ .

**Remark 4.1.** That local positivity implies a restriction of the support of the coarse index has been observed in [Roe16]. In the generality of the present paper (and even equivariantly) for Callias type operators, this has been studied in [GHM22]. ■

The norm in  $L^2(M, E)$  of a section  $\phi$  will be denoted by  $\|\phi\|$ . Let  $Y$  be a subset of  $M$ .

**Definition 4.2.** We say that  $\mathcal{D}$  is positive away from  $Y$  if there exists a number  $a$  in  $(0, \infty)$  such that  $\|\mathcal{D}\phi\|^2 \geq a^2 \|\phi\|^2$  for all  $\phi$  in  $C_c^\infty(M \setminus \bar{Y}, E)$ .

We will say that  $\mathcal{D}$  is positive away from a big family  $\mathcal{Y}$  if it is positive away from some member of  $\mathcal{Y}$ .

**Example 4.3.** We consider the summand  $R_0$  from (4.1). Let  $\sigma(R_0(m))$  denote the spectrum of the selfadjoint endomorphism  $R_0(m)$  of the fibre of  $E$  at  $m$  in  $M$ . If  $\inf_{m \in M \setminus Y} \min(\sigma(R_0(m))) > 0$ , then  $\mathcal{D}$  is positive away from  $Y$ . ■

We assume that  $\mathcal{D}$  is positive away from a big family  $\mathcal{Y}$ . One should expect that  $\mathcal{D}$  is close to being invertible outside  $\mathcal{Y}$  and that the relevant index theoretic information is supported on this big family. Indeed, following insights of [Roe16], one constructs a refined index class  $\text{ind} \mathcal{X}(\mathcal{D}, \text{on } \mathcal{Y})$  in  $K \mathcal{X}_{-n}(\mathcal{Y})$  such that

$$j_* \text{ind} \mathcal{X}(\mathcal{D}, \text{on } \mathcal{Y}) = \text{ind} \mathcal{X}(\mathcal{D}),$$

where  $j_* : K \mathcal{X}(\mathcal{Y}) \rightarrow K \mathcal{X}(M)$  is the canonical morphism. As we need the details of the construction in the subsequent sections we recall the precise definition of this index class from [BE17] in Definition 4.7.

Let  $X$  be a bornological coarse space. A graded  $X$ -controlled Hilbert space determined on points of degree  $n$  is an  $X$ -controlled Hilbert space  $(H, \chi)$  determined on points (see Section 2.5) such that  $H$  is in addition a graded module over  $\mathbf{Cl}^n$ , with the generators of  $\mathbf{Cl}^n$  acting as anti-selfadjoint operators. A bounded operator  $A$  in  $B(H)$  is called locally compact if  $\chi(B)A$  and  $A\chi(B)$  are compact for all bounded subsets  $B$  of  $X$ . Note that if  $(H, \chi)$  is locally finite, then local compactness is automatic.

**Definition 4.4.** *The Roe algebra  $C^*(H, \chi)$  associated to  $(H, \chi)$  is the  $C^*$ -subalgebra of  $B(H)$  generated by  $\mathbf{Cl}^n$ -equivariant controlled and locally compact operators.*

**Example 4.5.** If  $(H, \chi)$  is an object of the Roe category  $\mathbf{C}(X)$  of  $X$ , then it is of degree 0 and  $\text{End}_{\mathbf{C}(X)}((H, \chi)) = C^*(H, \chi)$ . ■

We now come back to the Dirac operator  $\not{D}$  on  $E \rightarrow M$ . We consider the graded Hilbert space  $L^2(M, E)$  with its induced  $\mathbf{Cl}^n$ -action, where the  $L^2$ -scalar product is defined using the hermitean structure of  $E$  and the Riemannian volume measure. In order to turn it into a graded  $M$ -controlled Hilbert space determined on points of degree  $n$ , we choose a partition  $(B_i)_{i \in I}$  of  $M$  by regular Borel subsets with uniformly bounded diameter and base points  $b_i$  in  $B_i$  for every  $i$  in  $I$ . We then define the projection valued measure

$$\chi := \sum_{i \in I} \chi_{B_i} \delta_{b_i} ,$$

where  $\chi_{B_i}$  is the multiplication on  $L^2(M, E)$  by the characteristic function of  $B_i$ . Thus for a subset  $Z$  of  $M$  we have  $\chi(Z) = \sum_{i \in I, b_i \in Z} \chi_{B_i}$ .

By construction, the graded  $M$ -controlled Hilbert space  $(L^2(M, E), \chi)$  is determined on points, but it is not locally finite if  $M$  has positive dimension. We let

$$C^*(M, E) := C^*(L^2(M, E), \chi)$$

denote the associated Roe algebra. Note that it is independent of the choice of the partition  $(B_i)_{i \in I}$ . We let furthermore  $C^*(\mathcal{Y} \subseteq M, E)$  be the subalgebra of  $C^*(M, E)$  generated by operators of the form  $\chi(Y)A\chi(Y)$  for  $A$  in  $C^*(M, E)$  and  $Y$  a member of  $\mathcal{Y}$ . Note that  $C^*(\mathcal{Y} \subseteq M, E)$  is an ideal and again does not depend on the choice of the partition.

The following lemma is [Roe16, Lem. 2.3].

**Lemma 4.6.** *If  $\not{D}$  is positive away from  $\mathcal{Y}$ , then there exists a  $\epsilon$  in  $(0, \infty)$  such that for every  $f$  in  $C_0((-a, a))$  we have  $f(\not{D}) \in C^*(\mathcal{Y} \subseteq M, E)$ .*

*Proof.* We repeat the proof [Roe16, Lem. 2.3] since this lemma is absolutely crucial for our purpose and in the reference, the statement is only proven under the stronger assumptions that  $\not{D}$  is associated to a Clifford bundle and the term  $R_0$  in (4.1) is positive on  $M \setminus \bar{Y}$ .

Let  $Y$  be a member of  $\mathcal{Y}$  such that  $\mathcal{D}$  is positive away from  $Y$ . For  $s$  in  $[0, \infty)$  we let  $U_s$  be the metric entourage of width  $s$  as in Example 2.9.

For an even function  $f$  in  $\mathcal{S}$  let  $\hat{f}$  denote the Fourier transform (in general a distribution) such that

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \hat{f}(t) \cos(tx) dt, \quad (4.2)$$

where the integral has to be interpreted appropriately. We assume  $\text{supp}(\hat{f}) \subseteq [-r, r]$  for some  $r$  in  $(0, \infty)$  and show that

$$\|f(\mathcal{D})\chi(M \setminus U_{b+r}[Y])\| \leq \sup_{|s| \geq a} |f(s)|, \quad (4.3)$$

where  $b$  in  $(0, \infty)$  is a bound on the diameter of the sets  $B_i$  used in the construction of the projection valued measure on  $L^2(M, E)$ . (This is the analogue of [Roe16, Lem. 2.5]).

The symmetric operator  $\mathcal{D}^2$  defines a hermitean sesquilinear form  $(\phi, \psi) \mapsto \langle \phi, \mathcal{D}^2 \psi \rangle$  on  $C_c^\infty(M \setminus \bar{Y}, E)$  which by assumption is bounded below by  $a^2$  for some  $a$  in  $(0, \infty)$ . Let  $A$  denote the square root of the Friedrichs extension of  $\mathcal{D}^2$  on  $L^2(M \setminus \bar{Y}, E)$ . By finite propagation speed of the solutions of the wave equation, for any  $h$  in  $L^2(M \setminus \bar{Y}, E)$  with  $\text{supp}(h) \in M \setminus U_r[Y]$  we have  $\cos(t\mathcal{D})h = \cos(tA)h$  for  $t \in [-r, r]$ . Replacing  $x$  by  $\mathcal{D}$  or  $A$  in (4.2) and since  $\text{supp}(\hat{f}) \subseteq [-r, r]$  we get  $f(\mathcal{D})h = f(A)h$ . Since  $\|f(A)\| \leq \sup_{|s| \geq a} |f(s)|$  we conclude (4.3).

One next derives the analogue of [Roe16, Lem. 2.6] by the same argument as in the reference: For arbitrary (not necessarily even)  $f$  in  $\mathcal{S}$  with  $\text{supp}(f) \in [-r, r]$  we have

$$\|f(\mathcal{D})\chi(M \setminus U_{2r+b}[Y])\| \leq 2 \sup_{|s| \geq a} |f(s)|. \quad (4.4)$$

To this end, we observe for odd  $f$  that

$$\|f(\mathcal{D})\chi(M \setminus U_{2r+b}[Y])\|^2 \leq \| |f|^2(\mathcal{D})\chi(M \setminus U_{2r+b}[Y]) \|^2$$

and that  $\text{supp}(\widehat{|f|^2}) \subseteq [-2r, 2r]$ . Then  $|f|^2$  is even so that we can apply the estimate already proven. A general function  $f$  can be written as a sum of an even and an odd function which accounts for the factor 2 in the right-hand side of (4.4).

Finally, we repeat the argument of [Roe16] that (4.4) implies Lemma 4.6. Assume that  $f$  is in  $C_0((-a, a))$ . As functions with compactly supported Fourier transform are dense in  $C_0(\mathbb{R})$ , given  $\epsilon$  in  $(0, \infty)$ , there exists  $g$  in  $\mathcal{S}$  with  $\hat{g}$  compactly supported and  $\|f - g\|_\infty \leq \epsilon$ , where we consider  $f$  as an element of  $\mathcal{S}$  through extension by zero. Then  $\|f(\mathcal{D}) - g(\mathcal{D})\| \leq \epsilon$ ,  $g$  has bounded propagation, and by (4.4),  $\|g(\mathcal{D})\chi(M \setminus U_{r+b}[Y])\| \leq \epsilon$  for  $r$  so large that  $\text{supp}(\hat{g}) \subseteq [-r, r]$ . We can conclude that  $f(\mathcal{D})$  is in distance  $2\epsilon$  from an element of  $C^*(\mathcal{Y} \subseteq M, E)$ . Since  $\epsilon$  is arbitrary we conclude that  $f(\mathcal{D}) \in C^*(\mathcal{Y} \subseteq M, E)$  as asserted.  $\square$

It follows from Lemma 4.6 that if  $\mathcal{D}$  is positive away from  $Y$ , then there exists  $a$  in  $(0, \infty)$  such that the homomorphism

$$\mathcal{S} \rightarrow C^*(M, E), \quad f \mapsto f(\mathcal{D})$$

restricts to a homomorphism

$$(\mathcal{D}_{\mathcal{Y}})_* : C_0((-a, a)) \rightarrow C^*(\mathcal{Y} \subseteq M, E).$$

As explained in Remark 2.49, it represents a class

$$[\mathcal{D}_{\mathcal{Y}}] \quad \text{in} \quad K_0^{\text{gr}}(C^*(\mathcal{Y} \subseteq M, E)) \quad (4.5)$$

such that  $j_*[\mathcal{D}_{\mathcal{Y}}] = [\mathcal{D}]$ .

In order to identify this class with a class of  $K\mathcal{X}_{-n}(M)$  we must incorporate the gradings and degrees in the definition of Roe categories and compare the graded  $K$ -theory of Roe algebra  $C^*(\mathcal{Y} \subseteq M, E)$  with the graded  $K$ -theory of a corresponding graded Roe category.

Generalizing the construction of the Roe category  $\mathbf{C}(X)$  in Section 2.5, for every bornological coarse space  $X$  we can consider the graded  $C^*$ -category  $\mathbf{C}[n](X)$  of graded locally finite  $X$ -controlled Hilbert spaces  $(H, \chi)$  determined on points of degree  $n$  and morphisms which are bounded operators that can be approximated in norm by controlled  $\mathbf{Cl}^n$ -equivariant operators. We get a functor

$$\mathbf{C}[n] : \mathbf{BC} \rightarrow (C^* \mathbf{Cat}^{\text{nu}})^{\text{gr}}.$$

We define the coarse  $K$ -homology functor of degree  $n$  as the composition

$$K\mathcal{X}[n] : \mathbf{BC} \xrightarrow{\mathbf{C}[n]} (C^* \mathbf{Cat}^{\text{nu}})^{\text{gr}} \xrightarrow{K^{\text{gr}}} \mathbf{Mod}(KU).$$

As observed in [BE17] (or using Example 2.51) we have a canonical equivalence of functors

$$K\mathcal{X}[n] \simeq \Sigma^n K\mathcal{X} : \mathbf{BC} \rightarrow \mathbf{Mod}(KU) \quad (4.6)$$

which first of all shows that  $K\mathcal{X}[n]$  is indeed a coarse homology theory, and that it is equivalent to  $K\mathcal{X}$  up to a shift.

In order to compare  $\mathbf{C}[n](M)$  and  $C^*(M, E)$  we form the intermediate graded  $C^*$ -category  $\tilde{\mathbf{C}}[n](M)$  of degree  $n$  whose objects are triples  $(H, \chi, U)$  with  $(H, \chi)$  in  $\mathbf{C}[n](M)$  and  $U : H \rightarrow L^2(M, E)$  a  $\mathbf{Cl}^n$ -equivariant controlled isometric embedding. The morphisms  $A : (H, \chi, U) \rightarrow (H', \chi', U')$  in  $\tilde{\mathbf{C}}[n](X)$  are the morphisms  $A : (H, \chi) \rightarrow (H', \chi')$  in  $\mathbf{C}[n](X)$ .

We now assume that  $M$  is not zero-dimensional. Then the graded  $M$ -controlled Hilbert space  $(L^2(M, E), \chi)$  is ample which means that any graded locally finite  $M$ -controlled

Hilbert space admits a controlled isometric embedding into it. Consequently, the forgetful functor

$$\mathcal{F} : \tilde{\mathbf{C}}[n](M) \rightarrow \mathbf{C}[n](M) , \quad (H, \chi, U) \mapsto (H, \chi) \quad (4.7)$$

is surjective on objects. Since it is fully faithful by definition, it is therefore an equivalence of graded  $C^*$ -categories.

We consider the Roe algebra  $C^*(M, E)$  as a graded  $C^*$ -category with a single object. We then have a functor

$$\mathcal{I} : \tilde{\mathbf{C}}[n](M) \rightarrow C^*(M, E) , \quad (A : (H, \chi, U) \rightarrow (H', \chi', U')) \mapsto U'AU^* . \quad (4.8)$$

One can show that  $C^*(M, E)$  is generated by the image of  $\mathcal{I}$  and it has been observed in [BE17] that

$$K^{\text{gr}}(\mathcal{I}) : K^{\text{gr}}(\tilde{\mathbf{C}}[n](M)) \rightarrow K^{\text{gr}}(C^*(M, E))$$

is an equivalence. The zig-zag of functors

$$\mathbf{C}[n](M) \xleftarrow{\mathcal{F}} \tilde{\mathbf{C}}[n](M) \xrightarrow{\mathcal{I}} C^*(M, E) \quad (4.9)$$

thus induces upon applying  $K^{\text{gr}}$  an equivalence

$$\kappa_M : \Sigma^n K\mathcal{X}(M) \stackrel{(4.6)}{\simeq} K\mathcal{X}[n](M) \xleftarrow{\mathcal{F}} K^{\text{gr}}(\tilde{\mathbf{C}}[n](M)) \xrightarrow{\mathcal{I}} K^{\text{gr}}(C^*(M, E)) .$$

The zig-zag (4.9) restricts to a zig-zag of ideals

$$\mathbf{C}[n](\mathcal{Y} \subseteq M) \xleftarrow{\mathcal{F}} \tilde{\mathbf{C}}[n](\mathcal{Y} \subseteq M) \xrightarrow{\mathcal{I}} C^*(\mathcal{Y} \subseteq M, E)$$

which in turn induces an equivalence

$$\kappa_{\mathcal{Y}} : \Sigma^n K\mathcal{X}(\mathcal{Y}) \simeq K^{\text{gr}}(C^*(\mathcal{Y} \subseteq M, E)) \quad (4.10)$$

such that we have commutative square

$$\begin{array}{ccc} \Sigma^n K\mathcal{X}(\mathcal{Y}) & \xrightarrow[\kappa_{\mathcal{Y}}]{\simeq} & K^{\text{gr}}(C^*(\mathcal{Y} \subseteq M, E)) , \\ \downarrow & & \downarrow \\ \Sigma^n K\mathcal{X}(M) & \xrightarrow[\kappa_M]{\simeq} & K^{\text{gr}}(C^*(M, E)) \end{array} \quad (4.11)$$

where the vertical maps are induced by the inclusions.

Recall the definition of the class  $[\mathcal{D}_{\mathcal{Y}}]$  in (4.5).

**Definition 4.7.** *If  $\mathcal{D}$  is positive away from  $\mathcal{Y}$ , then the coarse index class with support*

$$\text{ind}\mathcal{X}(\mathcal{D}, \text{on } \mathcal{Y}) \quad \text{in } K\mathcal{X}_{-n}(\mathcal{Y})$$

*is defined uniquely by the condition  $\kappa_{\mathcal{Y}}(\text{ind}\mathcal{X}(\mathcal{D}, \text{on } \mathcal{Y})) \simeq [\mathcal{D}_{\mathcal{Y}}]$ .*

We now formulate the basic properties of the coarse index class. For  $i = 0, 1$  let  $\mathcal{D}_i$  be a generalized Dirac operators of degree  $k_i$  on complete Riemannian manifolds  $M_i$ . If  $\mathcal{D}_1$  is positive away from  $\mathcal{Y}$ , then  $\mathcal{D}_0 \hat{\otimes} \text{id} + \text{id} \hat{\otimes} \mathcal{D}_1$  is positive away from  $M_0 \times \mathcal{Y}$ . Using the explicit description of the cup product from Remark 2.50, one can check that the image under the symmetric monoidal structure

$$K\mathcal{X}(M_0) \times K\mathcal{X}(\mathcal{Y}) \rightarrow K\mathcal{X}(M_0 \otimes \mathcal{Y})$$

of  $(\text{ind}\mathcal{X}(\mathcal{D}_0), \text{ind}\mathcal{X}(\mathcal{D}_1, \text{on } \mathcal{Y}))$  is the index  $\text{ind}\mathcal{X}(\mathcal{D}_0 \hat{\otimes} \text{id} + \text{id} \hat{\otimes} \mathcal{D}_1, \text{on } M_0 \times \mathcal{Y})$  of the product Dirac operator of degree  $k_0 + k_1$ .

We take  $M_0 = \mathbb{R}$  and  $\mathcal{D}_0 = \sigma\partial_t$  on the trivial bundle with fibre  $\mathbf{C}1^1$ . Then

$$\text{ind}\mathcal{X}(\sigma\partial_t) \in \pi_{-1}K\mathcal{X}(\mathbb{R}) \cong \pi_{-2}K\mathcal{X}(*) \cong KU_{-2}$$

is the inverse of the Bott element  $\beta$ . On the manifold  $\mathbb{R} \times M$ , there is a suspended Dirac operator  $\Sigma D$ , given by

$$\Sigma\mathcal{D} := \text{id}_{\text{pr}^*E} \hat{\otimes} \sigma\partial_t + \text{pr}^*\mathcal{D} \hat{\otimes} \text{id}_{\mathbf{C}1^1} \quad (4.12)$$

acting on sections of the bundle  $\text{pr}^*E \hat{\otimes} \mathbf{C}1^1$ . Here  $\sigma$  is the odd generator of the tensor factor  $\mathbf{C}1^1$  (acting by left multiplication) with  $\sigma^* = -\sigma$  and  $\sigma^2 = -1$ . A special case of the above statements on products is then the following.

Recall that the equivalence  $\delta^{MV} : K\mathcal{X}(\mathbb{R} \otimes \mathcal{Y}) \xrightarrow{\cong} \Sigma K\mathcal{X}(\mathcal{Y})$  (see (2.13)) induced by Mayer-Vietoris boundary for the decomposition of  $\mathbb{R} \otimes \mathcal{Y}$  into  $\mathbb{R}^- \times \mathcal{Y}$  and  $\mathbb{R}^+ \times \mathcal{Y}$ .

**Proposition 4.8** (Suspension theorem). *We have the equality*

$$\beta \cdot \delta^{MV} \text{ind}\mathcal{X}(\Sigma\mathcal{D}, \text{on } \mathbb{R} \times \mathcal{Y}) = \text{ind}\mathcal{X}(\mathcal{D}, \text{on } \mathcal{Y}) .$$

Here on the left, we multiplied using the  $KU$ -module structure on the values of  $K\mathcal{X}$ .

For  $i = 0, 1$  we consider generalized Dirac operators  $\mathcal{D}_i$  on complete Riemannian manifolds  $M_i$  which are positive away from big families  $\mathcal{Y}_i$ . Assume there exist subsets  $W_i$  in  $M_i$  such that there exists a Riemannian isometry  $e : M_0 \setminus W_0 \rightarrow M_1 \setminus W_1$  which is a morphism of bornological coarse spaces and covered by an isomorphism of all bundle data. Note that  $e$  is not necessarily an isomorphism of bornological coarse spaces, as  $e$  is not necessarily distance-preserving. We let  $\mathcal{W}$  and  $\mathcal{W}'$  be the big families generated by  $W$  and  $W'$ , respectively. By excision we get a morphism

$$e_* : K\mathcal{X}(\mathcal{Y}_0, \mathcal{Y}_0 \cap \mathcal{W}_0) \rightarrow K\mathcal{X}(\mathcal{Y}_1, \mathcal{Y}_1 \cap \mathcal{W}_1) . \quad (4.13)$$

We let

$$\pi_i : K\mathcal{X}(\mathcal{Y}_i) \rightarrow K\mathcal{X}(\mathcal{Y}_i, \mathcal{Y}_i \cap \mathcal{W}_i)$$

denote the projections. The following is [BE17, Thm. 10.5].

**Proposition 4.9** (The relative index theorem). *We have the equality*

$$e_*\pi_0(\text{ind}\mathcal{X}(\mathcal{D}_0, \text{on } \mathcal{Y}_0)) = \pi_1(\text{ind}\mathcal{X}(\mathcal{D}_1, \text{on } \mathcal{Y}_1)) .$$



## 4.2 Symbols with support

As before, let  $M$  be a complete Riemannian manifold. Classically, the symbol of a generalized Dirac operator on  $M$  is the analytic locally finite  $K$ -homology class  $\sigma^{\text{an}}(\not{D})$  in  $K^{\text{an}}(M)$ . In the present section, we propose to consider a refined symbol class  $\sigma(\not{D})$  in  $K^{\mathcal{X}}(M)$  such that  $p_M(\sigma(\not{D})) = \sigma^{\text{an}}(\not{D})$ , see (3.16) for  $p$ . If  $\not{D}$  is positive away from a big family  $\mathcal{Y}$ , then we introduce a further refined symbol class  $\sigma_{\mathcal{Y}}(\not{D})$  in  $K_{\mathcal{Y}}^{\mathcal{X}}(M)$  which in addition captures the reason for the fact that the  $a_{M,\mathcal{Y}}(\sigma(\not{D})) = \text{ind}(\not{D}, \text{on } \mathcal{Y})$  is supported on  $\mathcal{Y}$ .

Since by definition  $K_{\mathcal{Y}}^{\mathcal{X}}(M) \simeq \Sigma^{-1}K\mathcal{X}(\mathcal{O}_{\mathcal{Y}}^{\infty}(M))$  (see Definition 2.35) and  $K\mathcal{X}$  receives indices of Dirac type operators it is natural to represent  $\sigma_{\mathcal{Y}}(\not{D})$  as a coarse index of a generalized Dirac operator  $\tilde{\not{D}}$  naturally derived from  $\not{D}$  on the cone over  $M$ .

**Construction 4.10.** We assume that  $\not{D}$  has degree  $n$ . Then we consider the manifold  $\tilde{M} := \mathbb{R} \times M$  with the warped product metric

$$\tilde{g} = dt^2 + h(t)^2g$$

for some choice of a smooth function  $h : \mathbb{R} \rightarrow (0, \infty)$  with  $h(t) = 1$  for  $t \leq 0$  and  $h(t) \geq 1$  for  $t \geq 0$ . We let  $\tilde{\not{D}}$  be the degree  $n+1$  Dirac operator on the bundle  $\tilde{E} := \text{pr}_M^*E \hat{\otimes} \mathbf{Cl}^1 \rightarrow \tilde{M}$  explicitly given by the formula (see Remark 4.11 for the motivation)

$$\tilde{\not{D}} = \text{id}_{\text{pr}^*E} \hat{\otimes} \sigma \left( \partial_t + \frac{n \dot{h}(t)}{2 h(t)} \right) + \frac{1}{h(t)} \text{pr}^* \not{D} \hat{\otimes} \text{id}_{\mathbf{Cl}^1}, \quad (4.14)$$

where  $\sigma$  is the odd generator of the tensor factor  $\mathbf{Cl}^1$ .

**Remark 4.11.** Abstractly,  $\tilde{\not{D}}$  is the result obtained by adapting the suspension  $\Sigma \not{D}$  of  $\not{D}$  on  $\mathbb{R} \times M$  given in (4.12), to the metric  $\tilde{g}$ , using the adaptation procedure given in [Buna, Sec. 4]. The process of adaptation is uniquely fixed by the following requirements: (1) It is a local construction. (2) When applied to the spin Dirac operator on the product it produces the spin Dirac operator on the warped product. (3) Finally it is compatible with forming twisted Dirac operators and adding zero order terms. ■

We consider the big family

$$\tilde{M}_{\mathcal{Y}} := \mathbb{R} \times \mathcal{Y} \cup \{\mathbb{R}^+\} \times M. \quad (4.15)$$

on  $\tilde{M}$ , see Example 2.4 for notation.

**Lemma 4.12.** *If  $\not{D}$  is positive away from  $\mathcal{Y}$ , then  $\tilde{\not{D}}$  is positive away from  $\tilde{M}_{\mathcal{Y}}$ .*

*Proof.* Let  $Y$  be a member of  $\mathcal{Y}$  such that  $\mathcal{D}$  is positive on  $M \setminus Y$  and choose  $a$  in  $(0, \infty)$  as in Definition 4.2. We claim that  $\tilde{\mathcal{D}}$  is positive away from

$$(\mathbb{R} \times Y) \cup (\mathbb{R}^+ \times M) = (\mathbb{R} \times M) \setminus ((-\infty, 0) \times (M \setminus Y)) .$$

Indeed, let  $\tilde{\phi}$  be in  $C_c^\infty((-\infty, 0) \times (M \setminus Y), \tilde{E})$ . Then since  $h \equiv 1$  on the support of  $\tilde{\phi}$ , we have

$$\tilde{\mathcal{D}}\tilde{\phi} = (\text{id}_{\text{pr}^*E} \hat{\otimes} \sigma \partial_t + \mathcal{D} \hat{\otimes} \text{id}_{\text{C1}^1})\tilde{\phi} .$$

As the summands in the bracket commute (in the graded sense), we therefore get

$$\tilde{\mathcal{D}}^2\tilde{\phi} = \text{id}_{\text{pr}^*E} \hat{\otimes} (\sigma \partial_t)^2 \tilde{\phi} + \mathcal{D}^2 \hat{\otimes} \text{id}_{\text{C1}^1} \tilde{\phi},$$

so

$$\|\tilde{\mathcal{D}}\tilde{\phi}\|^2 = \langle \tilde{\phi}, \tilde{\mathcal{D}}^2\tilde{\phi} \rangle = -\langle \tilde{\phi}, \partial_t^2 \tilde{\phi} \rangle + \langle \tilde{\phi}, \mathcal{D}^2 \tilde{\phi} \rangle = \|\partial_t \tilde{\phi}\|^2 + \|\mathcal{D}\tilde{\phi}\|^2 \geq a^2 \|\tilde{\phi}\|^2 .$$

Here we used that  $\|\tilde{\mathcal{D}}\tilde{\phi}(t)\|_{\{t\} \times M}^2 \geq a^2 \|\tilde{\phi}(t)\|^2$  for each  $t$ .  $\square$

In view of Lemma 4.12 the Definition 4.7 provides the coarse index class  $\text{ind}\mathcal{X}(\tilde{\mathcal{D}}, \text{on } \tilde{M}_Y)$  in  $K\mathcal{X}_{-n-1}(\tilde{M}_Y)$ .

For the moment we write  $\tilde{M}_h$  and  $\tilde{\mathcal{D}}_h$  for the manifold and Dirac operator associated to the warping function  $h$ . The set of these functions is partially ordered and the constant function  $h \equiv 1$  is the minimal element. If  $h', h$  are two such functions such that

$$h' \leq h ,$$

then the identity map of  $\mathbb{R} \times M$  is a map of bornological coarse spaces  $q : \tilde{M}_h \rightarrow \tilde{M}_{h'}$ . We assume that  $\mathcal{D}$  is positive away from  $\mathcal{Y}$ . We add a subscript  $h$  or  $h'$  to the notation of the big family  $\tilde{M}_Y$  indicating from which space its coarse structure is induced.

**Proposition 4.13** ([Buna, Prop. 4.12]). *If  $\mathcal{D}$  is positive away from  $\mathcal{Y}$ , then we have the equality*

$$q_* \text{ind}\mathcal{X}(\tilde{\mathcal{D}}_h, \text{on } \tilde{M}_{h,Y}) = \text{ind}\mathcal{X}(\tilde{\mathcal{D}}_{h'}, \text{on } \tilde{M}_{h',Y}) \quad (4.16)$$

in  $K\mathcal{X}_{-n-1}(\tilde{M}_{h',Y})$ .

*Proof.* We emphasize that the all analysis of Dirac operators used in this proof is hidden in the suspension theorem Proposition 4.8 and the coarse relative index theorem, Proposition 4.9. Besides these results, the argument below only uses formal properties of coarse homology theories and some simple coarse geometric insights.

We consider the manifold  $\hat{M} := \mathbb{R} \times \mathbb{R} \times M$  and let  $(s, t)$  denote the coordinates on the factor  $\mathbb{R} \times \mathbb{R}$ . We choose a real-valued smooth increasing function  $\chi$  on  $\mathbb{R}$  such that  $\chi(s) = 0$  for  $s \leq 0$  and  $\chi(s) = 1$  for  $s \geq 1$ . We then consider the function on  $\hat{M}$  given by

$$\tilde{h}(s, t, m) := (1 - \chi(s))h'(t) + \chi(s)h(t) .$$

$\mathbb{R} \times (-\infty, 0] \times M$	$s = 1$	$[1, \infty) \times ([0, \infty) \times M)_h$
	$s = 0$	transition region
		$(-\infty, 0] \times ([0, \infty) \times M)_{h'}$

We view  $\hat{M}$  as a bornological coarse space with the warped product metric  $ds^2 + dt^2 + \tilde{h}g$ . We then get following morphisms of bornological coarse spaces

$$\mathbb{R} \otimes \tilde{M}_h \xrightarrow{\hat{q}} \hat{M} \xrightarrow{\hat{q}'} \mathbb{R} \otimes \tilde{M}_{h'} ,$$

all induced by the identity of the underlying sets.

The families

$$\hat{M}_{\mathcal{Y}} := \mathbb{R} \times \mathbb{R} \times \mathcal{Y}, \quad \hat{M}^{\{\leq a\}} := \{(-\infty, a]\} \times \mathbb{R} \times M, \quad \hat{M}^{\{\geq a\}} := \{[a, \infty)\} \times \mathbb{R} \times M$$

are all big families on  $\hat{M}$  and so are

$$\hat{M}_{\mathcal{Y}}^{\{\leq a\}} := \hat{M}^{\{\leq a\}} \cap \hat{M}_{\mathcal{Y}}, \quad \hat{M}_{\mathcal{Y}}^{\{\geq a\}} := \hat{M}^{\{\geq a\}} \cap \hat{M}_{\mathcal{Y}} .$$

We define  $\hat{\mathcal{D}}$  by a two-dimensional version of Construction 4.10. Notice that  $\hat{q}$  and  $\hat{q}'$  induce local isomorphisms of the Dirac operators on the subsets  $[1, \infty) \times \mathbb{R} \times M$  and  $(-\infty, 0] \times \mathbb{R} \times M$ , respectively.

We consider the the projections

$$\begin{aligned} \pi_{\leq 1} &: K\mathcal{X}(\mathbb{R} \otimes \tilde{M}_{h,\mathcal{Y}}) \longrightarrow K\mathcal{X}(\mathbb{R} \otimes \tilde{M}_{h,\mathcal{Y}}, \{(-\infty, 1]\} \otimes \tilde{M}_{h,\mathcal{Y}}) , \\ \pi'_{\geq 0} &: K\mathcal{X}(\mathbb{R} \otimes \tilde{M}_{h',\mathcal{Y}}) \longrightarrow K\mathcal{X}(\mathbb{R} \otimes \tilde{M}_{h',\mathcal{Y}}, \{[0, \infty)\} \otimes \tilde{M}_{h',\mathcal{Y}}) , \\ \tilde{\pi}_{\leq 1} &: K\mathcal{X}(\hat{M}_{\mathcal{Y}}) \longrightarrow K\mathcal{X}(\hat{M}_{\mathcal{Y}}, \hat{M}_{\mathcal{Y}}^{\{\leq 1\}}) , \\ \tilde{\pi}_{\geq 0} &: K\mathcal{X}(\hat{M}_{\mathcal{Y}}) \longrightarrow K\mathcal{X}(\hat{M}_{\mathcal{Y}}, \hat{M}_{\mathcal{Y}}^{\{\geq 0\}}) \end{aligned}$$

from the absolute to the relative coarse cohomology, and let

$$\begin{aligned} \hat{q}^{\text{Rel}} &: K\mathcal{X}(\mathbb{R} \otimes \tilde{M}_{h,\mathcal{Y}}, \{(-\infty, 1]\} \otimes \tilde{M}_{h,\mathcal{Y}}) \rightarrow K\mathcal{X}(\hat{M}_{\mathcal{Y}}, \hat{M}_{\mathcal{Y}}^{\{\leq 1\}}) \\ \hat{q}'^{\text{Rel}} &: K\mathcal{X}(\hat{M}_{\mathcal{Y}}, \hat{M}_{\mathcal{Y}}^{\{\geq 0\}}) \rightarrow K\mathcal{X}(\mathbb{R} \otimes \tilde{M}_{h',\mathcal{Y}}, \{[0, \infty)\} \otimes \tilde{M}_{h',\mathcal{Y}}) \end{aligned}$$

the maps in relative coarse  $K$ -homology. Then the relative index theorem, Proposition 4.9, provides the identities

$$\hat{q}_*^{\text{Rel}} \pi_{\leq 1,*} \text{ind} \mathcal{X}(\Sigma \tilde{\mathcal{D}}_h, \text{on } \mathbb{R} \times \tilde{M}_{h,\mathcal{Y}}) = \tilde{\pi}_{\leq 1,*} \text{ind} \mathcal{X}(\hat{\mathcal{D}}, \text{on } \hat{M}_{\mathcal{Y}}) , \quad (4.17)$$

$$\pi'_{\geq 0,*} \text{ind} \mathcal{X}(\Sigma \tilde{\mathcal{D}}_{h'}, \text{on } \mathbb{R} \times \tilde{M}_{h',\mathcal{Y}}) = \hat{q}'_*^{\text{Rel}} \tilde{\pi}_{\geq 0,*} \text{ind} \mathcal{X}(\hat{\mathcal{D}}, \text{on } \hat{M}_{\mathcal{Y}}) . \quad (4.18)$$

In the calculations below we use the following relation between Mayer-Vietoris boundaries and boundaries in pair sequences. Let  $X$  be a bornological coarse space with a big family  $\mathcal{Y}$  and let  $(W, W')$  be a coarsely excisive decomposition of  $X$ . Then we have a commutative diagram

$$\begin{array}{ccc}
K\mathcal{X}(\mathcal{Y}) & \xrightarrow{\delta^{MV}} & \Sigma K\mathcal{X}(\mathcal{Y} \cap W \cap W') \\
\pi \downarrow & \searrow \partial^{MV} & \downarrow \simeq \iota \\
K\mathcal{X}(\mathcal{Y}, \mathcal{Y} \cap \{W'\}) & & \\
j \uparrow \simeq & & \\
K\mathcal{X}(\mathcal{Y} \cap \{W\}, \mathcal{Y} \cap \{W\} \cap \{W'\}) & \xrightarrow{\partial} & \Sigma K\mathcal{X}(\mathcal{Y} \cap \{W\} \cap \{W'\}),
\end{array} \tag{4.19}$$

where  $\pi$  is the projection from the absolute to the relative coarse cohomology, and  $j$  and  $\iota$  are induced by the canonical inclusions. The morphism  $j$  is an equivalence by excision and  $\iota$  is an equivalence by coarse invariance and the assumption that  $(W, W')$  is coarsely excisive.

Pasting two instances of these diagrams, we get the commutative diagram

$$\begin{array}{ccc}
K\mathcal{X}(\mathbb{R} \otimes \tilde{M}_{h,\mathcal{Y}}) & \xrightarrow{-\delta_1^{MV}} & \Sigma K\mathcal{X}(\tilde{M}_{h,\mathcal{Y}}) \\
\pi_{\leq 1} \downarrow & \searrow -\partial_1^{MV} & \downarrow \simeq \iota_1 \\
K\mathcal{X}(\mathbb{R} \otimes \tilde{M}_{h,\mathcal{Y}}, \{(-\infty, 1]\} \otimes \tilde{M}_{h,\mathcal{Y}}) & & \\
j_1 \uparrow \simeq & & \\
K\mathcal{X}(\{[1, \infty)\} \otimes \tilde{M}_{h,\mathcal{Y}}, \{\{1\}\} \otimes \tilde{M}_{h,\mathcal{Y}}) & \xrightarrow{\partial} & \Sigma K\mathcal{X}(\{\{1\}\} \otimes \tilde{M}_{h,\mathcal{Y}}) \\
\hat{q}^{\text{rel}} \downarrow & & \hat{q}^0 \downarrow \\
K\mathcal{X}(\hat{M}_{\mathcal{Y}}^{\{\geq 1\}}, \hat{M}_{\mathcal{Y}}^{\{\geq 1\}} \cap \hat{M}_{\mathcal{Y}}^{\{\leq 1\}}) & \xrightarrow{\partial} & \Sigma K\mathcal{X}(\hat{M}_{\mathcal{Y}}^{\{\geq 1\}} \cap \hat{M}_{\mathcal{Y}}^{\{\leq 1\}}) \\
\tilde{j}_1 \downarrow \simeq & & \parallel \\
K\mathcal{X}(\hat{M}_{\mathcal{Y}}, \hat{M}_{\mathcal{Y}}^{\{\leq 1\}}) & & \\
\tilde{\pi}_{\leq 1} \uparrow & & \\
K\mathcal{X}(\hat{M}_{\mathcal{Y}}) & \xrightarrow{-\tilde{\delta}^{MV}} & \Sigma K\mathcal{X}(\hat{M}_{\mathcal{Y}}^{\{\leq 1\}} \cap \hat{M}_{\mathcal{Y}}^{\{\geq 1\}})
\end{array} \tag{4.20}$$

Here  $\delta_1^{MV}$  is the Mayer-Vietoris map for the coarsely excisive decomposition of  $\mathbb{R} \otimes \tilde{M}_{h,\mathcal{Y}}$  into  $(-\infty, 1] \times \tilde{M}_h$  and  $[1, \infty) \times \tilde{M}_h$ , while  $\tilde{\delta}^{MV}$  is the Mayer-Vietoris boundary for the pair of big families  $(\hat{M}_{\mathcal{Y}}^{\{\leq 1\}}, \hat{M}_{\mathcal{Y}}^{\{\geq 1\}})$  in  $\hat{M}$ . As above, the maps  $\hat{q}^?$  are all induced by  $\hat{q}$ . The additional minus signs come from the fact that the order of the partitions is swapped

in comparison to (4.19). Consequently we have

$$\begin{aligned}
-\hat{q}'_{*,1,*} \delta_{1,*}^{MV} \text{Ind} \mathcal{X}(\Sigma \tilde{\mathcal{D}}_h, \text{on } \mathbb{R} \times \tilde{M}_{h,\mathcal{Y}}) &\stackrel{(4.20)}{=} \partial_* \tilde{j}_{1,*}^{-1} \hat{q}'_{*,1,*} \text{Rel} \pi_{\leq 1,*} \text{Ind} \mathcal{X}(\Sigma \tilde{\mathcal{D}}_h, \text{on } \mathbb{R} \times \tilde{M}_{h,\mathcal{Y}}) \\
&\stackrel{(4.17)}{=} \partial_* \tilde{j}_{1,*}^{-1} \tilde{\pi}_{\leq 1,*} \text{Ind} \mathcal{X}(\hat{\mathcal{D}}, \text{on } \hat{M}_{\mathcal{Y}}) \\
&\stackrel{(4.20)}{=} -\tilde{\delta}_*^{MV} \text{Ind} \mathcal{X}(\hat{\mathcal{D}}, \text{on } \hat{M}_{\mathcal{Y}}) .
\end{aligned} \tag{4.21}$$

Similarly, we get the commutative diagram

$$\begin{array}{ccc}
K \mathcal{X}(\hat{M}_{\mathcal{Y}}) & \xrightarrow{\tilde{\delta}^{MV}} & \Sigma K \mathcal{X}(\hat{M}_{\mathcal{Y}}^{\{\leq 0\}} \cap \hat{M}_{\mathcal{Y}}^{\{\geq 0\}}) \\
\tilde{\pi}_{\geq 0} \downarrow & & \parallel \\
K \mathcal{X}(\hat{M}_{\mathcal{Y}}, \hat{M}_{\mathcal{Y}}^{\{\geq 0\}}) & & \\
\tilde{j}_0 \uparrow \simeq & & \\
K \mathcal{X}(\hat{M}_{\mathcal{Y}}^{\{\leq 0\}}, \hat{M}_{\mathcal{Y}}^{\{\leq 0\}} \cap \hat{M}_{\mathcal{Y}}^{\{\geq 0\}}) & \xrightarrow{\partial} & \Sigma K \mathcal{X}(\hat{M}_{\mathcal{Y}}^{\{\leq 0\}} \cap \hat{M}_{\mathcal{Y}}^{\{\geq 0\}}) \\
\hat{q}'_{*,\text{rel}} \downarrow & & \downarrow \hat{q}'_{*,0} \\
K \mathcal{X}(\{(-\infty, 0]\} \otimes \tilde{M}_{h',\mathcal{Y}}, \{\{0\}\} \otimes \tilde{M}_{h',\mathcal{Y}}) & \xrightarrow{\partial} & \Sigma K \mathcal{X}(\{\{0\}\} \otimes \tilde{M}_{h',\mathcal{Y}}) \\
j'_0 \downarrow \simeq & & \uparrow \iota'_0 \\
K \mathcal{X}(\mathbb{R} \otimes \tilde{M}_{h',\mathcal{Y}}, \{[0, \infty)\} \otimes \tilde{M}_{h',\mathcal{Y}}) & & \\
\pi'_{\geq 0} \uparrow & \nearrow \partial_0^{MV} & \\
K \mathcal{X}(\mathbb{R} \otimes \tilde{M}_{h',\mathcal{Y}}) & \xrightarrow{\delta_0^{MV}} & \Sigma K \mathcal{X}(\tilde{M}_{h',\mathcal{Y}}) ,
\end{array} \tag{4.22}$$

where  $\delta_0^{MV}$  is the Mayer-Vietoris boundary for the coarsely excisive partition of  $\mathbb{R} \otimes \tilde{M}_{h'}$  into  $(-\infty, 0] \times \tilde{M}_{h'}$  and  $[0, \infty) \times \tilde{M}_{h'}$ . Notice also that  $\hat{M}_{\mathcal{Y}}^{\{\leq 1\}} = \hat{M}_{\mathcal{Y}}^{\{\leq 0\}}$  and  $\hat{M}_{\mathcal{Y}}^{\{\geq 1\}} = \hat{M}_{\mathcal{Y}}^{\{\geq 0\}}$  so that the bottom arrow of (4.20) coincides with to top arrow of (4.22) up to sign.

We therefore get

$$\begin{aligned}
\hat{q}'_{*,0} \tilde{\delta}_*^{MV} \text{Ind} \mathcal{X}(\hat{\mathcal{D}}, \text{on } \hat{M}_{h,\mathcal{Y}}) &\stackrel{(4.22)}{=} \partial_* j'_{0,*}{}^{-1} \hat{q}'_{*,0} \text{Rel} \tilde{\pi}_{\geq 0,*} \text{Ind} \mathcal{X}(\hat{\mathcal{D}}, \text{on } \hat{M}_{h,\mathcal{Y}}) \\
&\stackrel{(4.18)}{=} \partial_* j'_{0,*}{}^{-1} \pi'_{\geq 0,*} \text{Ind} \mathcal{X}(\Sigma \tilde{\mathcal{D}}_{h'}, \text{on } \mathbb{R} \times \tilde{M}_{h',\mathcal{Y}}) \\
&\stackrel{(4.22)}{=} \iota'_{0,*} \delta_{0,*}^{MV} \text{Ind} \mathcal{X}(\Sigma \tilde{\mathcal{D}}_{h'}, \text{on } \mathbb{R} \times \tilde{M}_{h',\mathcal{Y}}) .
\end{aligned} \tag{4.23}$$

Notice that the diagram

$$\begin{array}{ccc}
\tilde{M}_h & \xrightarrow{q} & \tilde{M}_{h'} \\
(t,m) \mapsto (1,t,m) \downarrow & & \downarrow (t,m) \mapsto (0,t,m) \\
\mathbb{R} \otimes \tilde{M}_h & \xrightarrow{\hat{q}} \hat{M} \xrightarrow{\hat{q}'} & \mathbb{R} \otimes \tilde{M}_{h'}
\end{array}$$

does not strictly commute, but the two compositions are close to each other. We conclude that

$$\iota'_{0,*} q_* = \hat{q}'_{*0} \hat{q}_*^0 \iota_{1,*} : K\mathcal{X}(\tilde{M}_{h,\mathcal{Y}}) \rightarrow K\mathcal{X}(\{\{0\}\} \otimes \tilde{M}_{h',\mathcal{Y}}). \quad (4.24)$$

We now calculate

$$\begin{aligned} q_* \mathbf{Ind}(\tilde{\mathcal{D}}_h, \text{on } \tilde{M}_{h,\mathcal{Y}}) &\stackrel{\text{Proposition 4.8}}{=} \beta \cdot q_* \delta_{1,*}^{MV}(\mathbf{Ind}\mathcal{X}(\Sigma \tilde{\mathcal{D}}_h, \text{on } \mathbb{R} \times \tilde{M}_{h,\mathcal{Y}})) \\ &\stackrel{(4.24)}{=} \beta \cdot \iota'_{0,*} \hat{q}'_{*0} \hat{q}_*^0 \iota_{1,*} \delta_{1,*}^{MV} \mathbf{Ind}\mathcal{X}(\Sigma \tilde{\mathcal{D}}_h, \text{on } \mathbb{R} \times \tilde{M}_{h,\mathcal{Y}}) \\ &\stackrel{(4.21)}{=} \beta \cdot \iota'_{0,*} \hat{q}'_{*0} \tilde{\delta}_*^{MV} \mathbf{Ind}\mathcal{X}(\hat{\mathcal{D}}, \text{on } \hat{M}_{\mathcal{Y}}) \\ &\stackrel{(4.23)}{=} \beta \cdot \delta_{0,*}^{MV} \mathbf{Ind}\mathcal{X}(\Sigma \tilde{\mathcal{D}}_{h'}, \text{on } \mathbb{R} \times \tilde{M}_{h',\mathcal{Y}}) \\ &\stackrel{\text{Proposition 4.8}}{=} \mathbf{Ind}\mathcal{X}(\tilde{\mathcal{D}}_{h'}, \text{on } \tilde{M}_{h',\mathcal{Y}}), \end{aligned}$$

as desired.  $\square$

We now assume that  $\lim_{t \rightarrow \infty} h(t) = \infty$ . Then the identity of underlying sets is a map

$$\iota : \tilde{M} \rightarrow \mathcal{O}^\infty(M) \quad (4.25)$$

in **BC**. It sends  $\tilde{M}_{\mathcal{Y}}$  to  $\mathcal{O}_{\mathcal{Y}}^\infty(X)$ .

We assume that  $\mathcal{D}$  is positive away from  $\mathcal{Y}$ .

**Definition 4.14.** We define the symbol of  $\mathcal{D}$  with support in  $\mathcal{Y}$  as the class

$$\sigma_{\mathcal{Y}}(\mathcal{D}) := \iota_*(\beta \cdot \mathbf{ind}\mathcal{X}(\tilde{\mathcal{D}}, \text{on } \tilde{M}_{\mathcal{Y}})) \quad \text{in} \quad K_{\mathcal{Y},-n}^{\mathcal{X}}(M) \cong K\mathcal{X}_{-n+1}(\mathcal{O}_{\mathcal{Y}}^\infty(M)).$$

For  $\mathcal{Y} = \{M\}$  we write  $\sigma(\mathcal{D}) := \sigma_{\{M\}}(\mathcal{D})$  in  $K_{-n}^{\mathcal{X}}(M)$ . By Proposition 4.13 the class  $\iota_*(\mathbf{ind}\mathcal{X}(\tilde{\mathcal{D}}, \text{on } \tilde{M}_{\mathcal{Y}}))$  is independent of the choice of the warping function  $h$ . Consequently the symbol class  $\sigma_{\mathcal{Y}}(\mathcal{D})$  is well-defined.

Recall the Paschke morphism (3.18).

**Definition 4.15.** We define  $\sigma^{\text{an}}(\mathcal{D}) := p_M(\sigma(\mathcal{D}))$  in  $K_{-n}^{\text{an}}(M)$ .

**Remark 4.16.** In this remark we explain how  $\mathbf{ind}\mathcal{X}(\mathcal{D})$  can be recovered from  $\sigma^{\text{an}}(\mathcal{D})$ . Let  $g$  be the Riemannian metric on  $M$ . Then we consider a Riemannian metric  $g'$  with  $g \leq g'$ , which is automatically complete, and let  $M'$  be the uniform bornological coarse space presented by the new metric. Then the identity map of underlying sets  $i : M' \rightarrow M$  is a contraction and therefore a morphism in **UBC**. Let  $\mathcal{D}'$  be the adaptation of  $\mathcal{D}$  to the new metric  $g'$  (see Remark 4.11). Using an analogous argument as in Proposition 4.13

based on  $\hat{M} = \mathbb{R} \times M$  with warped product metric  $dt^2 + \chi(t)g' + (1 - \chi(t))g$  one can show that  $i_* \text{ind} \mathcal{X}(\mathcal{D}') = \text{ind} \mathcal{X}(\mathcal{D})$ . We get a commutative diagram

$$\begin{array}{ccc}
& \xleftarrow{i} & \\
K^{\mathcal{X}}(M) & \xrightarrow{p_M} K^{\text{an}}(M) \xleftarrow{p_{M'}} K^{\mathcal{X}}(M') & , \quad \sigma(\mathcal{D}) \xrightarrow{p_M} \sigma^{\text{an}}(\mathcal{D}) \xleftarrow{p_{M'}} \sigma(\mathcal{D}') \\
\downarrow a_M & \swarrow a_{M'}^{\text{lf}} & \downarrow a_M \quad \downarrow a_{M'} \\
K^{\mathcal{X}}(M) & \xleftarrow{i_*} K^{\mathcal{X}}(M') & \text{ind} \mathcal{X}(\mathcal{D}) \xleftarrow{i_*} \text{ind} \mathcal{X}(\mathcal{D}')
\end{array}$$

We now choose the metric  $g'$  adapted to a locally finite triangulation of  $M$  so that  $p_{M'}$  is an equivalence and get the dotted arrow (see Example 2.34). We then have  $\text{ind} \mathcal{X}(\mathcal{D}) = a_{M'}^{\text{lf}}(\sigma^{\text{an}}(M))$ . ■

Let  $a_{M,\mathcal{Y}} : K_{\mathcal{Y}}^{\mathcal{X}}(M) \rightarrow K^{\mathcal{X}}(\mathcal{Y})$  be the index map (3.12).

**Lemma 4.17.** *We have the equality*

$$a_{M,\mathcal{Y}}(\sigma_{\mathcal{Y}}(\mathcal{D})) = \text{ind} \mathcal{X}(\mathcal{D}, \text{on } \mathcal{Y}) \quad \text{in } K^{\mathcal{X}}_{-n}(\mathcal{Y}) .$$

*Proof.* The first step in the construction of  $a_{M,\mathcal{Y}}$  is the application of the cone boundary  $\partial^{\text{cone}} : \mathcal{O}^{\infty}(M) \rightarrow \mathbb{R} \otimes M$  from (2.9). By Proposition 4.13, it sends the class  $\iota_*(\beta \cdot \text{ind} \mathcal{X}(\tilde{\mathcal{D}}, \text{on } \tilde{\mathcal{Y}}))$  to the class  $\beta \cdot \text{ind} \mathcal{X}(\Sigma \tilde{\mathcal{D}}, \text{on } \tilde{M}_{\mathcal{Y}})$  in  $K^{\mathcal{X}}_{-n+1}((\tilde{M}_{\mathcal{Y}})_{\mathbb{R} \otimes M})$ , where the subscript indicates that  $\tilde{M}_{\mathcal{Y}}$  has the bornological coarse structures induced from  $\mathbb{R} \otimes M$  and  $\iota$  is as in (4.25).

The second step is the application of the Mayer-Vietoris boundary for the decomposition  $(\{\mathbb{R}^-\} \times \mathcal{Y}, \{\mathbb{R}^+\} \times M)$ . Using that  $\Sigma \tilde{\mathcal{D}}$  is actually positive away from  $\mathbb{R} \times \mathcal{Y}$  and the suspension theorem, Proposition 4.8, the image of this class under the Mayer-Vietoris boundary is  $\text{ind} \mathcal{X}(\mathcal{D}, \text{on } \mathcal{Y})$  in  $K^{\mathcal{X}}_{-n}(\mathcal{Y})$ . □

**Remark 4.18.** Assume that  $\mathcal{D}$  is positive on  $M$ . Then we can take  $\mathcal{Y} = \{\emptyset\}$ . We get a class  $\sigma_{\{\emptyset\}}(\mathcal{D})$  in  $K^{\mathcal{X}}_{\{\emptyset\},-n}(M)$  which captures the reason for  $\text{ind} \mathcal{X}(\mathcal{D}) = 0$ , usually called the  $\rho$ -invariant, see Remark 3.27. ■

We consider a section  $\Psi$  in  $C^{\infty}(M, \text{End}_{\text{cl}^n}(E)^{\text{odd}})$ .

**Definition 4.19.** *We say that  $\Psi$  is a potential for  $\mathcal{D}$  if  $\Psi$  takes values in selfadjoint endomorphisms and the graded commutator  $[\mathcal{D}, \Psi]$  is an operator of order zero.*

Note that  $[\mathcal{D}, \Psi]$  is a zero-order operator if  $\Psi$  commutes in the graded sense with the symbol of  $\mathcal{D}$ .

If  $\Psi$  is a potential for  $\mathcal{D}$ , then we can form the new Dirac type operator  $\mathcal{D} + \Psi$  of degree  $n$ .

**Definition 4.20.** *We call  $\mathcal{D} + \Psi$  the Callias type operator associated to  $\mathcal{D}$  and  $\Psi$ .*

We will consider the condition that  $\mathcal{D} + \Psi$  is positive outside of a subset  $Y$  of  $M$ . But we want the stronger condition that this positivity is caused by  $\Psi$  alone. To this end we calculate

$$(\mathcal{D} + \Psi)^2 = \mathcal{D}^2 + [\mathcal{D}, \Psi] + \Psi^2 . \quad (4.26)$$

By  $\sigma(A)$  we denote the spectrum of a selfadjoint operator  $A$ . So  $\sigma(\Psi^2(m))$  below is the spectrum of the finite-dimensional operator  $\Psi^2(m)$  on the fibre  $E_m$  of the bundle  $E$  at  $m$  and  $\min \sigma(\Psi^2(m))$  is its minimal element.

**Definition 4.21.** *We say that  $\Psi$  is positive away from  $Y$  if*

$$\inf_{m \in M \setminus Y} \left( \min \sigma(\Psi^2(m)) - \|\nabla \Psi(m)\| \right) > 0 . \quad (4.27)$$

*If  $\mathcal{Y}$  is a big family then we say that  $\Psi$  is positive away from  $\mathcal{Y}$  if it is positive away from a member of  $\mathcal{Y}$ .*

**Lemma 4.22.** *If  $\Psi$  is positive away from  $Y$ , then  $\mathcal{D} + \Psi$  is positive away from  $Y$  in the sense of Definition 4.2.*

*Proof.* For  $\phi \in C_c^\infty(M \setminus \bar{Y}, E)$ , we have

$$\begin{aligned} \|(\mathcal{D} + \Psi)\phi\|^2 &= \|\mathcal{D}\phi\|^2 + \langle \phi, ([\mathcal{D}, \Psi] + \Psi^2)\phi \rangle \\ &\geq \inf_{m \in M \setminus Y} \left( \min \sigma(\Psi^2(m)) - \|[\mathcal{D}, \Psi]\| \right) \|\phi\|^2 . \end{aligned}$$

As  $\|[\mathcal{D}, \Psi]\| \leq \|\nabla \Psi\|$  pointwise, the result follows from the assumption of (4.27).  $\square$

As a consequence of Lemma 4.22, if  $\Psi$  is positive away from  $Y$ , then by Definition 4.7 we get a class

$$\text{ind} \mathcal{X}(\mathcal{D} + \Psi, \text{on } \mathcal{Y}) \quad \text{in} \quad K \mathcal{X}_{-n}(\mathcal{Y}) .$$

We abbreviate  $\tilde{\Psi} := \text{pr}^* \Psi \otimes \text{id}_{\mathbb{C}^1}$  in  $C^\infty(\mathbb{R} \times M, \tilde{E})$  and let  $\tilde{\mathcal{D}}$  as above be the adaptation of  $\Sigma \mathcal{D}$  to the warped product metric on  $\mathbb{R} \times M$  (see Construction 4.10). Then  $\tilde{\mathcal{D}} + \tilde{\Psi}$  is the adaptation of  $\Sigma(\mathcal{D} + \Psi)$ . Therefore by definition, with  $\iota$  as in (4.25)

$$\iota_*(\beta \cdot \text{ind} \mathcal{X}(\tilde{\mathcal{D}} + \tilde{\Psi})) = \sigma(\mathcal{D} + \Psi) \quad \text{in} \quad K_{-n}^{\mathcal{X}}(M) .$$



**Lemma 4.23.**  $\tilde{\Psi}$  is positive away from  $\mathbb{R} \times Y$ .

*Proof.* We note that

$$\|\nabla \tilde{\Psi}(t, m)\|^2 = h^{-2}(t) \|\nabla \Psi(m)\|^2 . \quad (4.28)$$

Since  $h^{-1}(t) \leq 1$  for all  $t$  in  $\mathbb{R}$  we have

$$\min(\sigma(\tilde{\Psi}(t, m)^2) - \|\nabla \tilde{\Psi}(t, m)\|) \geq \min(\sigma(\Psi(m)^2) - \|\nabla \Psi(m)\|)$$

for each  $m$  in  $M \setminus Y$  and each  $t$  in  $\mathbb{R}$ . Taking the infimum over  $\mathbb{R} \times (M \setminus Y)$ , the result follows from the positivity of  $\Psi$  away from  $Y$ .  $\square$

Assume that  $\Psi$  is positive away from  $Y$ .

**Definition 4.24.** We define the symbol with support of the Callias type operator  $\mathcal{D} + \Psi$  by

$$\sigma(\mathcal{D} + \Psi, \text{on } \mathcal{Y}) := \iota_*(\beta \cdot \text{ind } \mathcal{X}(\tilde{\mathcal{D}} + \tilde{\Psi}, \text{on } \mathcal{Y})) \quad \text{in } K_{-n}^{\mathcal{X}}(\mathcal{Y}) .$$

The symbol is well-defined by Lemma 4.23, Lemma 4.22 and Definition 4.7. Lemma 4.17 implies:

**Corollary 4.25.** We have the equality

$$a_{\mathcal{Y}}(\sigma(\mathcal{D} + \Psi, \text{on } \mathcal{Y})) = \text{ind } \mathcal{X}(\mathcal{D} + \Psi, \text{on } \mathcal{Y}) \quad \text{in } K \mathcal{X}_{-n}(\mathcal{Y}) .$$

**Remark 4.26.** Assume that  $\mathcal{D}$  is positive away from a big family  $\mathcal{Z}$ . Without further assumptions on  $\Psi$  the construction of the Callias type operator  $\mathcal{D} + \Psi$  destroys this positivity since we do not have control over the term  $[\mathcal{D}, \Psi]$  on  $(X \setminus Z) \cap Y$ . So we can not conclude that  $\mathcal{D} + \Psi$  is positive away from  $Y \cap Z$ .  $\blacksquare$

In order to improve on the point made in Remark 4.26 we adopt the following definition. Let  $\Psi$  be a potential and  $\mathcal{Z}$  be a big family on  $M$ . We consider the big family

$$\tilde{\mathcal{Y}}_{\mathcal{Z}} := \{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \cup \{\mathbb{R}^+\} \times \mathcal{Y} = \tilde{M}_{\mathcal{Z}} \cap (\mathbb{R} \times \mathcal{Y}) \quad (4.29)$$

in  $\tilde{M}$ .

**Definition 4.27.** We say  $\Psi$  is asymptotically constant away from  $\mathcal{Z}$  if

$$\lim_{Z \in \mathcal{Z}} \|\nabla \Psi|_{M \setminus Z}\|_{\infty} = 0 .$$

**Lemma 4.28.** *We assume that  $\tilde{D}$  is positive away from  $\mathcal{Z}$ , and that  $\tilde{\Psi}$  is positive away from  $\mathcal{Y}$  and asymptotically constant away from  $\mathcal{Z}$ . Then  $\tilde{D} + \tilde{\Psi}$  is positive away from  $\tilde{\mathcal{Y}}_{\mathcal{Z}}$ .*

*Proof.* By Lemma 4.22 and Lemma 4.23 we know that  $\tilde{D} + \tilde{\Psi}$  is positive away from  $\mathbb{R} \times \mathcal{Y}$ . We can therefore find a member  $Y$  of  $\mathcal{Y}$  and number  $b$  in  $(0, \infty)$  such that

$$\|(\tilde{D} + \tilde{\Psi})\tilde{\phi}\|^2 \geq b\|\tilde{\phi}\|^2$$

for any compactly supported section  $\tilde{\phi}$  of  $\mathbf{pr}^*E$  with support on  $\mathbb{R} \times (M \setminus \bar{Y})$ .

By (4.26), for any compactly supported smooth section  $\tilde{\phi}$  of  $\mathbf{pr}^*E$  we have

$$\|(\tilde{D} + \tilde{\Psi})\tilde{\phi}\|^2 \geq \|\tilde{D}\tilde{\phi}\|^2 - \|\tilde{\nabla}\Psi|_{\text{supp}(\tilde{\phi})}\|_{\infty}\|\tilde{\phi}\|^2 + \|\tilde{\Psi}\tilde{\phi}\|^2. \quad (4.30)$$

Assume that  $Z$  is a member of  $\mathcal{Z}$  and  $a$  is a number in  $(0, \infty)$  such that the quadratic form associated to  $\tilde{D}^2$  is bounded below by  $a^2$  on  $M \setminus Z$ . Then by (the proof of) Lemma 4.12, the quadratic form associated to  $\tilde{D}^2$  is bounded below by  $a^2$  on  $\mathbb{R}^- \times (M \setminus Z)$ . We now enlarge the member  $Z$  of  $\mathcal{Z}$  such that in addition  $\|\nabla\Psi|_{M \setminus Z}\|_{\infty} \leq a^2/2$ . Then by (4.28) and the fact that  $h$  is bounded below by 1 we have  $\|\tilde{\nabla}\tilde{\Psi}|_{\mathbb{R} \times (M \setminus Z)}\|_{\infty} \leq a^2/2$ . Consequently, if  $\tilde{\phi}$  is supported on  $(-\infty, 0) \times M \setminus \bar{Z}$ , then the right-hand side of (4.30) is larger than  $a^2/2 \cdot \|\tilde{\phi}\|^2$ .

Since any compactly supported section  $\tilde{\phi}$  of  $\mathbf{pr}^*E$  supported on  $\mathbb{R} \times (M \setminus \bar{Y}) \cup (-\infty, 0) \times (M \setminus \bar{Z})$  can be written as a sum of two sections, one compactly supported on  $\mathbb{R} \times (M \setminus \bar{Y})$  and one on  $(-\infty, 0) \times (M \setminus \bar{Z})$ , we see that for such sections

$$\|(\tilde{D} + \tilde{\Psi})\tilde{\phi}\|^2 \geq \min(b, \frac{a^2}{2})\|\tilde{\phi}\|^2.$$

This shows that  $\tilde{D} + \tilde{\Psi}$  is positive away from  $\tilde{\mathcal{Y}}_{\mathcal{Z}}$ , because  $\mathbb{R} \times (M \setminus \bar{Y}) \cup \mathbb{R}^- \times (M \setminus \bar{Z})$  contains the complement of the member  $\mathbb{R}^- \times (U_1[Y] \cap U_1[Z]) \cup \mathbb{R}^+ \times U_1[Y]$  of  $\tilde{\mathcal{Y}}_{\mathcal{Z}}$ , where  $U_1$  is the metric entourage of width 1  $\square$

**Definition 4.29.** *Under the assumptions of Lemma 4.28, we define*

$$\sigma_{\mathcal{Z}}(\tilde{D} + \tilde{\Psi}, \text{on } \mathcal{Y}) := \iota_*(\beta \cdot \text{ind}\mathcal{X}(\tilde{D} + \tilde{\Psi}, \text{on } \tilde{\mathcal{Y}}_{\mathcal{Z}}))$$

in  $K_{\mathcal{Z}, -n}^{\mathcal{X}}(\mathcal{Y})$ .

**Lemma 4.30.** *Under the assumptions of Lemma 4.28 we have*

$$a_{\mathcal{Y}, \mathcal{Z}}(\sigma_{\mathcal{Z}}(\tilde{D} + \tilde{\Psi}, \text{on } \mathcal{Y})) = \text{ind}\mathcal{X}(\tilde{D} + \tilde{\Psi}, \text{on } \mathcal{Y} \cap \mathcal{Z}) \quad (4.31)$$

in  $K_{\mathcal{X}, -n}(\mathcal{Y} \cap \mathcal{Z})$ .

*Proof.* We have a diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{i} & \mathbb{R} \times M \\ & \searrow \iota & \nearrow \partial^{\text{cone}} \\ & \mathcal{O}^\infty(M) & \end{array} .$$

in **BC** all induced by the identity maps of the underlying sets. In view of the definition of  $a_{\mathcal{Y}, \mathcal{Z}}$  in terms of the cone boundary and the suspension equivalence we get

$$\kappa_* a_{\mathcal{Y}, \mathcal{Z}}(\sigma_{\mathcal{Z}}(\not{D} + \Psi, \text{on } \mathcal{Y})) = \partial^{MV} i_*(\beta \cdot \text{ind} \mathcal{X}(\tilde{\not{D}} + \tilde{\Psi}, \text{on } \tilde{\mathcal{Y}}_{\mathcal{Z}})) ,$$

where  $\partial^{MV}$  is the Mayer-Vietoris boundary for the decomposition of  $\tilde{\mathcal{Y}}_{\mathcal{Z}}$  into  $\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z})$  and  $\{\mathbb{R}^+\} \times \mathcal{Y}$  and  $\kappa : \mathcal{Y} \cap \mathcal{Z} = \{0\} \times (\mathcal{Y} \cap \mathcal{Z}) \rightarrow \{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \cap \{\mathbb{R}^+\} \times \mathcal{Y}$  consist of coarse equivalences and induces an equivalence in coarse  $K$ -homology. By Proposition 4.13 we have

$$i_* \text{ind} \mathcal{X}(\tilde{\not{D}} + \tilde{\Psi}, \text{on } \tilde{\mathcal{Y}}_{\mathcal{Z}}) = \text{ind} \mathcal{X}(\Sigma(\not{D} + \Psi), \text{on } \tilde{\mathcal{Y}}_{\mathcal{Z}}) .$$

The right-hand side actually comes from an index supported on the smaller family  $\mathbb{R} \times (\mathcal{Y} \cap \mathcal{Z})$ . The equality (4.31) now follows from the suspension theorem Proposition 4.8, as the equivalence  $\Sigma^{-1} K \mathcal{X}(\mathbb{R} \otimes (\mathcal{Y} \cap \mathcal{Z})) \simeq K \mathcal{X}(\mathcal{Y} \cap \mathcal{Z})$  is induced by the Mayer-Vietoris boundary.  $\square$

### 4.3 Callias type operators and pairings

As in the previous section we consider a generalized Dirac operator  $\not{D}$  of degree  $n$  on a bundle  $E \rightarrow M$  which is positive away from a big family  $\mathcal{Z}$  (see Definition 4.2) and a potential  $\Psi$  in  $C^\infty(M, \text{End}_{\mathbf{Cl}^n}(E)^{\text{odd}})$  which is very positive away from a second big family  $\mathcal{Y}$  (see Definition 4.31 below) and asymptotically constant away from  $\mathcal{Z}$  (see Definition 4.27). In this section, we express the coarse index  $\text{ind} \mathcal{X}(\not{D} + \Psi, \text{on } \mathcal{Y} \cap \mathcal{Z})$  in  $K \mathcal{X}(\mathcal{Y} \cap \mathcal{Z})$  and the symbol  $\sigma_{\mathcal{Z}}(\not{D} + \Psi, \text{on } \mathcal{Y})$  in  $K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y})$  of the Callias type operator  $\not{D} + \Psi$  in terms of the coarse corona pairing  $\cap^{\mathcal{X}}$  and the coarse symbol pairing  $\cap^{\mathcal{X}\sigma}$  introduced in Definition 3.35 and Definition 3.40, respectively.

We assume that  $n = k + l$  for integers  $k, l$  and  $E \cong E_0 \hat{\otimes} V$ . Here  $V$  is a finite-dimensional graded Hilbert space with a right action of  $\mathbf{Cl}^l$  such that the generators of  $\mathbf{Cl}^l$  act by anti-selfadjoint operators. Furthermore  $E_0 \rightarrow M$  is a graded hermitean bundle of  $\mathbf{Cl}^k$ -modules carrying a Dirac operator  $\not{D}_0$  of degree  $k$ . We assume that  $\not{D} = \not{D}_0 \hat{\otimes} \text{id}_V$  and that  $\Psi = \text{id}_{E_0} \hat{\otimes} \Psi_0$  for some map  $\Psi_0 : M \rightarrow \text{End}_{\mathbf{Cl}^l}(V)^{\text{odd}}$  with selfadjoint values. In particular, the graded Roe algebra of  $\mathbf{Cl}^{k+l}$ -invariant operators associated to the bundle  $E \rightarrow M$  is given by

$$C^*(M, E) \cong \text{End}(V) \hat{\otimes} C^*(M, E_0) ,$$

where  $\text{End}(V)$  denotes the graded algebra of  $\mathbf{Cl}^l$ -equivariant operators on  $V$  and  $C^*(M, E_0)$  is the graded Roe algebra of  $\mathbf{Cl}^k$ -equivariant operators on  $E_0 \rightarrow M$ . We consider a big family  $\mathcal{Y}$  on  $M$ . Recall Definition 4.21 of the notion of local positivity of  $\Psi_0$ .

**Definition 4.31.** We say that  $\Psi_0$  is very positive away from  $\mathcal{Y}$  if it is positive away from  $\mathcal{Y}$  and  $e^{-\Psi^2}$  belongs to  $\text{End}(V) \hat{\otimes} C_u(\mathcal{Y})$ .

If  $\Psi_0$  is very positive away from  $\mathcal{Y}$ , then it generates a homomorphism

$$\Psi_{0,*} : \mathcal{S} \rightarrow \text{End}(V) \hat{\otimes} C_u(\mathcal{Y})$$

which represents a class  $[\Psi_0]$  in  $K_0^{\text{gr}}(\text{End}(V) \hat{\otimes} C_u(\mathcal{Y})) \cong K^l(\mathcal{Y})$ , see Remark 2.49.

Recall the coarse symbol pairing Definition 3.40. Let  $\mathcal{Z}$  be a second big family on  $M$ .

**Theorem 4.32.** *Assume:*

1.  $\mathcal{D}_0$  is positive away from  $\mathcal{Z}$ .
2.  $\Psi_0$  is asymptotically constant away from  $\mathcal{Z}$ .
3.  $\Psi_0$  is very positive away from  $\mathcal{Y}$ .
4.  $\|\nabla\Psi_0\|$  is uniformly bounded on  $M$ .

Then we have the equality

$$\sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y}) = [\Psi_0] \cap^{\mathcal{X}\sigma} \sigma_{\mathcal{Z}}(\mathcal{D}_0) \tag{4.32}$$

in  $K_{\mathcal{Z},-n}^{\mathcal{X}}(\mathcal{Y})$ .

**Remark 4.33.** Note that the right-hand side of (4.32) is defined without assuming 2. But this assumption is necessary in order to get the left-hand side well-defined. Indeed, the conditions imply that Definition 4.29 applies and provides the class

$$\sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y}) \quad \text{in} \quad K_{\mathcal{Z},-n}^{\mathcal{X}}(\mathcal{Y}) .$$

Both sides of (4.32) do not require the technical condition 4 which is only needed in the proof of this equality. ■

*Proof.* We apply Construction 4.10 to obtain the operators  $\tilde{\mathcal{D}}$  and  $\tilde{\mathcal{D}} + \tilde{\Psi}$  on  $\tilde{M}$ . Since  $\mathcal{D}$  is positive away from  $\mathcal{Z}$ , Lemma 4.12 implies that  $\tilde{\mathcal{D}}$  is positive away from the big family  $\tilde{M}_{\mathcal{Z}} \subset \tilde{M}$  as defined in (4.15). Hence by Lemma 4.6, there exists  $a$  in  $(0, \infty)$  such that

$$\tilde{\mathcal{D}}_{0,*} : C_0((-a, a)) \rightarrow C^*(\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0)$$

is well-defined. Similarly, as by Lemma 4.22,  $D + \Psi$  is positive away from  $\mathcal{Y}$ , the operator  $\tilde{\mathcal{D}} + \tilde{\Psi}$  on  $\tilde{M}$  is positive away from the big family  $\tilde{\mathcal{Y}}_{\mathcal{Z}}$  defined in (4.29), and we get a well-defined  $*$ -homomorphism

$$(\tilde{\mathcal{D}} + \tilde{\Psi})_* : C_0((-a, a)) \rightarrow C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}) .$$

We also have

$$\Psi_{0,*} : C_0((-a, a)) \rightarrow \text{End}(V) \hat{\otimes} C_u(\mathcal{Y}) .$$

Recall from Section 2.4 that the coproduct  $\Delta : \mathcal{S} \rightarrow \mathcal{S} \hat{\otimes} \mathcal{S}$  is the  $*$ -homomorphism determined uniquely by

$$f \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} X) ,$$

where  $X$  is the unbounded multiplier of  $\mathcal{S}$  given by  $(Xf)(x) = xf(x)$ . The multiplier  $X$  restricts to a bounded multiplier on each of the subalgebras  $C_0((-a, a))$ . In particular we get a restriction

$$\Delta_a : C_0((-a, a)) \rightarrow C_0((-a, a)) \hat{\otimes} C_0((-a, a))$$

of the coproduct, making the diagram

$$\begin{array}{ccc} C((-a, a)) & \xrightarrow{\epsilon} & \mathcal{S} \\ \Delta_a \downarrow & & \downarrow \Delta \\ C((-a, a)) \hat{\otimes} C((-a, a)) & \xrightarrow{\epsilon \otimes \epsilon} & \mathcal{S} \hat{\otimes} \mathcal{S} \end{array} \quad (4.33)$$

commute, where  $\epsilon : C_0((-a, a)) \rightarrow \mathcal{S}$  is the extension-by-zero map.

In view of the above and Remarks 2.49, 2.50, the composition

$$C_0((-a, a)) \xrightarrow{\Delta_a} C_0((-a, a)) \hat{\otimes} C_0((-a, a)) \xrightarrow{\Psi_{0,*} \hat{\otimes} \tilde{\mathcal{D}}_{0,*}} \text{End}(V) \hat{\otimes} C_u(\mathcal{Y}) \hat{\otimes} C^*(\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0)$$

represents the product

$$[\Psi_0] \cup [\tilde{\mathcal{D}}_0] \quad \text{in} \quad K_0^{\text{gf}}(\text{End}(V) \hat{\otimes} C_u(\mathcal{Y}) \hat{\otimes} C^*(\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0)) .$$

In analogy to Lemma 3.39 (with the same arguments and using the analogue of Corollary 3.11 for  $A$  in  $C^*(\tilde{M}, \tilde{E})$ ), there is a well-defined pairing

$$\hat{\mu} : \text{End}(V) \hat{\otimes} C_u(\mathcal{Y}) \hat{\otimes} C^*(\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0) \rightarrow \frac{C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E})}{C^*(\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M}, \tilde{E})} \quad (4.34)$$

defined as the matrix extension of  $\hat{\mu}(f \hat{\otimes} A) := [\chi(\text{pr}^* f \hat{\otimes} \text{id})(\text{id} \hat{\otimes} A)]$ .

**Lemma 4.34.** *The diagram*

$$\begin{array}{ccc} C((-a, a)) & \xrightarrow{(\tilde{\mathcal{D}} + \tilde{\Psi})^*} & C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}) \\ \Delta_a \downarrow & & \downarrow p_0 \\ C_0((-a, a)) \hat{\otimes} C((-a, a)) & & \\ \Psi_{0,*} \hat{\otimes} \tilde{\mathcal{D}}_{0,*} \downarrow & & \\ \text{End}(V) \hat{\otimes} C_u(\mathcal{Y}) \hat{\otimes} C^*(\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0) & \xrightarrow{\hat{\mu}} & \frac{C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E})}{C^*(\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M}, \tilde{E})} \end{array} \quad (4.35)$$

commutes, where the right vertical map  $p_0$  is the quotient projection.

*Proof.* We have a sequence of horizontal ideal inclusions and projections

$$\begin{array}{ccccc} C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}) & \longrightarrow & C^*(\tilde{M}, \tilde{E}) & \longrightarrow & \mathcal{M}(C^*(\tilde{M}, \tilde{E})) \\ \downarrow p_0 & & \downarrow & & \downarrow p \\ \frac{C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E})}{C^*(\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M}, \tilde{E})} & \xrightarrow{!} & \frac{C^*(\tilde{M}, \tilde{E})}{C^*(\{\mathbb{R}^-\} \times M \subseteq \tilde{M}, \tilde{E})} & \longrightarrow & \frac{\mathcal{M}(C^*(\tilde{M}, \tilde{E}))}{C^*(\{\mathbb{R}^-\} \times M \subseteq \tilde{M}, \tilde{E})} \end{array}$$

where  $\mathcal{M}(\cdot)$  denotes taking the multiplier algebra. In order to see that the marked arrow is injective we observe that

$$C^*(\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M}, \tilde{E}) = C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}) \cap C^*(\{\mathbb{R}^-\} \times M \subseteq \tilde{M}, \tilde{E}) .$$

The diagram (4.35) therefore maps injectively to the diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{(\tilde{\mathcal{D}} + \tilde{\Psi})_*} & \mathcal{M}(C^*(\tilde{M}, \tilde{E})) \\ \Delta \downarrow & & \downarrow p \\ \mathcal{S} \hat{\otimes} \mathcal{S} & & \\ \Psi_{0,*} \hat{\otimes} \tilde{\mathcal{D}}_{0,*} \downarrow & & \\ \text{End}(V) \hat{\otimes} C_u(\mathcal{Y}) \hat{\otimes} C^*(\tilde{M}, \tilde{E}_0) & \xrightarrow{\hat{\mu}} & \frac{\mathcal{M}(C^*(\tilde{M}, \tilde{E}))}{C^*(\{\mathbb{R}^-\} \times M \subseteq \tilde{M}, \tilde{E})} , \end{array} \quad (4.36)$$

where  $\hat{\mu}$  is the extension of (4.34) defined by the same formula. We must check the commutativity of (4.36). By continuity and a density argument it suffices to check this commutativity on the elements  $e^{-tx^2}$  and  $xe^{-tx^2}$  of  $\mathcal{S}$  for all  $t$  in  $(0, \infty)$ . The coproducts of these elements are given by

$$\Delta(e^{-tx^2}) = e^{-tx^2} \hat{\otimes} e^{-tx^2} , \quad \Delta(xe^{-tx^2}) = xe^{-tx^2} \hat{\otimes} e^{-tx^2} + e^{-tx^2} \hat{\otimes} xe^{-tx^2} . \quad (4.37)$$

The first relation in (4.37) implies the first equality in

$$\hat{\mu}((\Psi_{0,*} \hat{\otimes} \tilde{\mathcal{D}}_{0,*})(\Delta(e^{-tx^2}))) = \hat{\mu}(e^{-t\Psi_0^2} \hat{\otimes} e^{-t\tilde{\mathcal{D}}_0^2}) = p(e^{-t\tilde{\Psi}^2})p(e^{-t\tilde{\mathcal{D}}^2}) ,$$

and the second is a reformulation of the definition  $\hat{\mu}$  in terms of  $p$ .

By the usual finite propagation speed argument (use Lemma 4.6 with  $\mathcal{Y} = \{\tilde{M}\}$ ) we have  $e^{-t\tilde{\mathcal{D}}^2} \in C^*(\tilde{M}, \tilde{E})$ . In contrast, the operator  $e^{-t\tilde{\Psi}^2}$  is not contained in  $C^*(\tilde{M}, \tilde{E})$ , but it is fortunately a multiplier for this algebra. For this reason we passed to multiplier algebras above.

The maps  $t \mapsto e^{-tx^2}$  and  $t \mapsto xe^{-tx^2}$  both are smooth functions  $(0, \infty) \rightarrow \mathcal{S}$ . Consequently, postcomposing these functions with any  $*$ -homomorphism  $\mathcal{S} \rightarrow A$  yields smooth functions  $(0, \infty) \rightarrow A$ . In particular,

$$t \mapsto \tilde{\Psi}_*(e^{-tx^2}) = e^{-t\tilde{\Psi}^2} \quad \text{and} \quad t \mapsto \tilde{\mathcal{D}}_*(e^{-tx^2}) = e^{-t\tilde{\mathcal{D}}^2}$$

are smooth functions with values in the right upper corner of (4.36). We will denote their derivatives suggestively by  $-\tilde{\Psi}^2 e^{-t\tilde{\Psi}^2}$  and  $-\tilde{\mathcal{D}}^2 e^{-t\tilde{\mathcal{D}}^2}$ , respectively.

Since  $\Psi_0$  is very positive away from  $\mathcal{Y}$  by Assumption 3, we have  $e^{-t\Psi_0^2} \in \text{End}(V) \hat{\otimes} C_u(\mathcal{Y})$ . Using only boundedness and uniform continuity, the argument given in the proof of Lemma 3.39 shows that

$$e^{-t\tilde{\Psi}_0^2} = \text{pr}^* e^{-t\Psi_0^2} \in \text{End}(V) \hat{\otimes} \ell_{\{\mathbb{R}^-\}}^\infty \times_M(\tilde{M}) .$$

By Corollary 3.11, for all  $s, t$  in  $(0, \infty)$  the commutator of  $e^{-s\tilde{\Psi}^2}$  and  $e^{-t\tilde{\mathcal{D}}^2}$  is contained in  $C^*(\{\mathbb{R}^-\} \times M \subseteq \tilde{M}, \tilde{E})$ . Consequently,  $p(e^{-s\tilde{\Psi}^2})$  and  $p(e^{-s\tilde{\mathcal{D}}^2})$  commute, and so do their derivatives. For example,  $p(e^{-s\tilde{\mathcal{D}}^2})$  commutes with  $p(\tilde{\Psi}^2 e^{-t\tilde{\Psi}^2})$ .

We must show the equality of semigroups

$$p(e^{-t\tilde{\Psi}^2})p(e^{-t\tilde{\mathcal{D}}^2}) = p(e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) . \quad (4.38)$$

To this end we consider the smooth function

$$(0, \infty) \times (0, \infty) \ni (s, t) \mapsto p(e^{-s\tilde{\Psi}^2})p(e^{-s\tilde{\mathcal{D}}^2})p(e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) \in \frac{\mathcal{M}(C^*(\tilde{M}, \tilde{E}))}{C^*(\{\mathbb{R}^-\} \times M \subseteq \tilde{M}, \tilde{E})} .$$

We calculate

$$\begin{aligned} & -\frac{\partial}{\partial s} p(e^{-s\tilde{\Psi}^2})p(e^{-s\tilde{\mathcal{D}}^2})p(e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) \\ &= p(\tilde{\Psi}^2 e^{-s\tilde{\Psi}^2})p(e^{-s\tilde{\mathcal{D}}^2})p(e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) + p(e^{-s\tilde{\Psi}^2})p(\tilde{\mathcal{D}}^2 e^{-s\tilde{\mathcal{D}}^2})p(e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) \\ &= p(e^{-s\tilde{\mathcal{D}}^2})p(\tilde{\Psi}^2 e^{-s\tilde{\Psi}^2})p(e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) + p(e^{-s\tilde{\Psi}^2})p(\tilde{\mathcal{D}}^2 e^{-s\tilde{\mathcal{D}}^2})p(e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) \\ &= p(e^{-s\tilde{\mathcal{D}}^2})p(\tilde{\Psi}^2 e^{-s\tilde{\Psi}^2} \cdot e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) + p(e^{-s\tilde{\Psi}^2})p(\tilde{\mathcal{D}}^2 e^{-s\tilde{\mathcal{D}}^2} \cdot e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) \\ &\stackrel{!}{=} p(e^{-s\tilde{\mathcal{D}}^2})p(e^{-s\tilde{\Psi}^2} \cdot \tilde{\Psi}^2 e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) + p(e^{-s\tilde{\Psi}^2})p(e^{-s\tilde{\mathcal{D}}^2} \cdot \tilde{\mathcal{D}}^2 e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) \\ &= p(e^{-s\tilde{\mathcal{D}}^2})p(e^{-s\tilde{\Psi}^2})p(\tilde{\Psi}^2 e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) + p(e^{-s\tilde{\Psi}^2})p(e^{-s\tilde{\mathcal{D}}^2})p(\tilde{\mathcal{D}}^2 e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) \\ &= p(e^{-s\tilde{\Psi}^2})p(e^{-s\tilde{\mathcal{D}}^2})p((\tilde{\Psi}^2 + \tilde{\mathcal{D}}^2)e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) \\ &= -\frac{\partial}{\partial t} p(e^{-s\tilde{\Psi}^2})p(e^{-s\tilde{\mathcal{D}}^2})p(e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}) \end{aligned} \quad (4.39)$$

In the marked equality, we used the identities

$$\begin{aligned} \tilde{\Psi}^2 e^{-s\tilde{\Psi}^2} \cdot e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2} &= e^{-s\tilde{\Psi}^2} \cdot \tilde{\Psi}^2 e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2} , \\ \tilde{\mathcal{D}}^2 e^{-s\tilde{\mathcal{D}}^2} \cdot e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2} &= e^{-s\tilde{\mathcal{D}}^2} \cdot \tilde{\mathcal{D}}^2 e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2} \end{aligned} \quad (4.40)$$

which require justification. Recall here that we use  $\tilde{\Psi}^2 e^{-s\tilde{\Psi}^2}$  as a symbol for the operator  $\tilde{\Psi}_*(x^2 e^{-sx^2}) = -\partial_s e^{-s\tilde{\Psi}^2}$ , and it is not a priori clear that the right hand sides of (4.40) are even defined. Let  $\text{dom}(\tilde{\Psi}^2)$  denote the domain of the unbounded selfadjoint operator

determined by  $\tilde{\Psi}^2$  equipped with the graph norm. Then  $e^{-s\tilde{\Psi}^2} : \text{dom}(\tilde{\Psi}^2) \rightarrow H$  is a norm-differentiable family of operators with (negative) derivative

$$\text{dom}(\tilde{\Psi}^2) \xrightarrow{\tilde{\Psi}^2} H \xrightarrow{e^{-s\tilde{\Psi}^2}} H .$$

This composition happens to have a continuous extension to a bounded operator on  $H$  which we denoted by  $\tilde{\Psi}^2 e^{-s\tilde{\Psi}^2}$  above. We apply Lemma 4.38 below with  $H = L^2(\tilde{M}, \tilde{E})$ ,  $A = t^{1/2}\tilde{\mathcal{D}}$ ,  $B = t^{1/2}\tilde{\Psi}$ ,  $D = C_c^\infty(\tilde{M}, \tilde{E})$ . Note that  $\{\tilde{\mathcal{D}}, \tilde{\Psi}\}$  is bounded by Assumption 4. Since  $e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2} : H \rightarrow \text{dom}((\tilde{\mathcal{D}} + \tilde{\Psi})^2)$  is continuous, Lemma 4.38.2 implies that  $e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}$  corestricts to a bounded map from  $H$  to  $\text{dom}(\tilde{\Psi}^2)$ . Hence we can write  $\tilde{\Psi}^2 e^{-s\tilde{\Psi}^2} \cdot e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}$  as the composition of bounded operators

$$H \xrightarrow{e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}} \text{dom}(\tilde{\Psi}^2) \xrightarrow{\tilde{\Psi}^2} H \xrightarrow{e^{-s\tilde{\Psi}^2}} H .$$

This finishes the justification for the first equality in (4.40). For the second equality, we argue similarly using the claim that  $e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2}$  restricts to a bounded map from  $H$  to  $\text{dom}(\tilde{\mathcal{D}}^2)$ .

By (4.39), for every  $t$  in  $(0, \infty)$  the function

$$(0, 1) \ni u \mapsto p(e^{-tu\tilde{\Psi}^2})p(e^{-tu\tilde{\mathcal{D}}^2})p(e^{-(1-u)t(\tilde{\mathcal{D}}+\tilde{\Psi})^2})$$

is constant. This together with the following two norm limits

$$\lim_{u \downarrow 0} e^{-tu\tilde{\mathcal{D}}^2} e^{-tu\tilde{\Psi}^2} e^{-(1-u)t(\tilde{\mathcal{D}}+\tilde{\Psi})^2} = e^{-t(\tilde{\mathcal{D}}+\tilde{\Psi})^2} \quad (4.41)$$

and

$$\lim_{u \uparrow 1} e^{-tu\tilde{\mathcal{D}}^2} e^{-tu\tilde{\Psi}^2} e^{-(1-u)t(\tilde{\mathcal{D}}+\tilde{\Psi})^2} = e^{-t\tilde{\Psi}^2} e^{-t\tilde{\mathcal{D}}^2} \quad (4.42)$$

implies (4.38).

We now justify the limits (4.41) and (4.42). For the first, we use the decomposition

$$e^{-tu\tilde{\mathcal{D}}^2} e^{-tu\tilde{\Psi}^2} e^{-(1-u)t(\tilde{\mathcal{D}}+\tilde{\Psi})^2} = e^{-tu\tilde{\mathcal{D}}^2} e^{-tu\tilde{\Psi}^2} e^{-t\frac{1}{2}(\tilde{\mathcal{D}}+\tilde{\Psi})^2} \cdot e^{-t(\frac{1}{2}-u)(\tilde{\mathcal{D}}+\tilde{\Psi})^2} .$$

The second factor on the right hand side converges to  $e^{-t\frac{1}{2}(\tilde{\mathcal{D}}+\tilde{\Psi})^2}$  and we must show that

$$\lim_{u \downarrow 0} e^{-tu\tilde{\mathcal{D}}^2} e^{-tu\tilde{\Psi}^2} e^{-t\frac{1}{2}(\tilde{\mathcal{D}}+\tilde{\Psi})^2} = e^{-t\frac{1}{2}(\tilde{\mathcal{D}}+\tilde{\Psi})^2} .$$

We use the decomposition

$$H \xrightarrow{e^{-t\frac{1}{2}(\tilde{\mathcal{D}}+\tilde{\Psi})^2}} \text{dom}(\tilde{\mathcal{D}}^2) \cap \text{dom}(\tilde{\Psi}^2) \xrightarrow{e^{-tu\tilde{\Psi}^2}} \text{dom}(\tilde{\mathcal{D}}) \xrightarrow{e^{-tu\tilde{\mathcal{D}}^2}} H . \quad (4.43)$$

The first map is a bounded operator by Lemma 4.38.2 (applied to  $A = t^{1/2}\tilde{\mathcal{D}}$ ,  $B = t^{1/2}\tilde{\Psi}$ ). The last map is a continuous family of bounded operators which tends to the inclusion  $\text{dom}(\tilde{\mathcal{D}}) \rightarrow H$  as  $u \downarrow 0$ . For the middle map in (4.43) we apply Lemma 4.39 with  $A = \tilde{\mathcal{D}}$



and  $B = \tilde{\Psi}$ ,  $H = L^2(\tilde{M}, \tilde{E})$  and  $D = C_c^\infty(\tilde{M}, \tilde{E})$ . The assumptions on the functions of  $B$  are easily verified. Lemma 4.39.3 states that the middle map is a continuous family of bounded operators which tends to the inclusion if  $u \downarrow 0$ . This finishes the verification of (4.41).

For (4.42) we consider the adjoint and use the decomposition

$$H \xrightarrow{e^{-tu\tilde{D}^2}} \text{dom}(\tilde{D}) \xrightarrow{e^{-tu\tilde{\Psi}^2}} \text{dom}(\tilde{D}) \cap \text{dom}(\tilde{\Psi}) \stackrel{\text{Lemma 4.38.2}}{=} \text{dom}(\tilde{D} + \tilde{\Psi}) \xrightarrow{e^{-(1-u)t(\tilde{D}+\tilde{\Psi})^2}} H . \quad (4.44)$$

As  $u \uparrow 1$  the first map converges in norm to  $e^{-t\tilde{D}^2} : H \rightarrow \text{dom}(\tilde{D})$ . The second map is continuous by Lemma 4.39.2 and converges to  $e^{-t\tilde{\Psi}^2} : \text{dom}(\tilde{D}) \rightarrow \text{dom}(\tilde{D}) \cap \text{dom}(\tilde{\Psi})$  as  $u \uparrow 1$ . Finally the last map converges in norm to the inclusion  $\text{dom}(\tilde{D} + \tilde{\Psi}) \rightarrow H$ . This gives (4.42).

For the odd part we have

$$\hat{\mu}((\tilde{\Psi}_{0,*} \hat{\otimes} \tilde{D}_{0,*})(\Delta(xe^{-tx^2}))) = p(\tilde{\Psi}e^{-t\tilde{\Psi}^2})p(e^{-t\tilde{D}^2}) + p(e^{-t\tilde{\Psi}^2})p(\tilde{D}e^{-t\tilde{D}^2})$$

and therefore must show that

$$p(\tilde{\Psi}e^{-t\tilde{\Psi}^2})p(e^{-t\tilde{D}^2}) + p(\tilde{D}e^{-t\tilde{D}^2})p(e^{-t\tilde{\Psi}^2}) = p((\tilde{D} + \tilde{\Psi})e^{-t(\tilde{D}+\tilde{\Psi})^2}) . \quad (4.45)$$

We show

$$p(\tilde{\Psi}e^{-t\tilde{\Psi}^2})p(e^{-t\tilde{D}^2}) = p(\tilde{\Psi}e^{-t(\tilde{D}+\tilde{\Psi})^2}) , \quad (4.46)$$

where the right-hand side is defined by Lemma 4.38.1. For  $\epsilon$  in  $(0, \infty)$ , we observe using the even case that

$$\begin{aligned} p(e^{-\epsilon\tilde{\Psi}^2}\tilde{\Psi}e^{-t\tilde{\Psi}^2})p(e^{-t\tilde{D}^2}) &= p(\tilde{\Psi}e^{-\epsilon\tilde{\Psi}^2})p(e^{-t\tilde{\Psi}^2})p(e^{-t\tilde{D}^2}) \\ &= p(\tilde{\Psi}e^{-\epsilon\tilde{\Psi}^2})p(e^{-t(\tilde{D}+\tilde{\Psi})^2}) \\ &= p(e^{-\epsilon\tilde{\Psi}^2}\tilde{\Psi}e^{-t(\tilde{D}+\tilde{\Psi})^2}) \end{aligned}$$

Since  $\tilde{\Psi}e^{-t(\tilde{D}+\tilde{\Psi})^2}$  and  $\tilde{\Psi}e^{-t\tilde{\Psi}^2}$  corestrict to bounded maps from  $H$  to  $\text{dom}(\tilde{\Psi})$  (we use Lemma 4.38.1 for the first case) we can take the limit  $\epsilon \downarrow 0$  and get (4.46). The identity

$$p(\tilde{D}e^{-t\tilde{D}^2})p(e^{-t\tilde{\Psi}^2}) = p(\tilde{D}e^{-t(\tilde{D}+\tilde{\Psi})^2}) \quad (4.47)$$

can be shown by an analogous argument.  $\square$

**Corollary 4.35.** *We have the equality*

$$\hat{\mu}_*([\Psi_0] \cup [\tilde{D}_0]) = p_{0,*}([\tilde{D} + \tilde{\Psi}]) \quad \text{in} \quad K_0^{\text{gr}} \left( \frac{C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E})}{C^*(\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M}, \tilde{E})} \right) .$$

Recall that on the one hand  $\sigma_{\mathcal{Z}}(\tilde{D} + \Psi, \text{on } \mathcal{Y})$  is obtained in Section 4.2 from the class represented by  $(\tilde{D} + \tilde{\Psi})_*$  on  $\tilde{\mathcal{Y}}_{\mathcal{Z}}$  in  $K^{\text{gr}}(C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, E))$  via an identification of the  $K$ -theory of this Roe algebra with the  $K$ -theory of the  $C^*$ -category going into the definition of the coarse  $K$ -homology functor  $K(\mathbf{C}(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M})) \simeq K\mathcal{X}(\tilde{\mathcal{Y}}_{\mathcal{Z}})$ . On the other hand the coarse symbol pairing  $-\cap^{\mathcal{X}\sigma} -$  is defined in Definition 3.40 in terms of the latter  $C^*$ -categories directly. The remaining work in this proof of Theorem 4.32 consists of transporting the construction of  $-\cap^{\mathcal{X}\sigma} -$  through these identifications until it can directly be compared with the class represented by  $\hat{\mu}(\Psi_{0,*} \hat{\otimes} \tilde{D}_{0,*})\Delta_a$ .

The formula (3.26) for  $\mu$  also defines pairings

$$\hat{\mu}_0 : C_u(\mathcal{Y}) \hat{\otimes} C^*(\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0) \rightarrow \frac{C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0)}{C^*(\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M}, \tilde{E}_0)},$$

$$\tilde{\mu}' : C_u(\mathcal{Y}) \otimes \tilde{\mathbf{C}}[n](\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}) \rightarrow \frac{\tilde{\mathbf{C}}[n](\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M})}{\tilde{\mathbf{C}}[n](\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M})}$$

and

$$\mu' : C_u(\mathcal{Y}) \otimes \mathbf{C}[n](\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}) \rightarrow \frac{\mathbf{C}[n](\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M})}{\mathbf{C}[n](\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M})},$$

where the notation  $\mathbf{C}[n](-)$  and  $\tilde{\mathbf{C}}[n](-)$  was introduced in Section 4.1.

Recall the functor  $\mathcal{I}$  from (4.8).

**Lemma 4.36.** *The following square commutes:*

$$\begin{array}{ccc} C_u(\mathcal{Y}) \otimes \tilde{\mathbf{C}}[n](\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}) & \xrightarrow{\tilde{\mu}'} & \frac{\tilde{\mathbf{C}}[n](\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M})}{\tilde{\mathbf{C}}[n](\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M})} \\ \downarrow 1 \otimes \mathcal{I} & & \downarrow \mathcal{I} \\ C_u(\mathcal{Y}) \hat{\otimes} C^*(\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0) & \xrightarrow{\hat{\mu}_0} & \frac{C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0)}{C^*(\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M}, \tilde{E}_0)}. \end{array}$$

*Proof.* Let  $A : (H, \chi, U) \rightarrow (H', \chi', U')$  be a morphism in  $\tilde{\mathbf{C}}[n](\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M})$  and  $f$  be in  $C_u(\mathcal{Y})$ . Let  $\tilde{\chi}$  denote the projection-valued measure on  $L^2(\tilde{M}, \tilde{E}_0)$ . Then we have

$$\begin{aligned} (1 \otimes \mathcal{I}) \circ \tilde{\mu}'(f \otimes A) &= \mathcal{I}([\chi'(\text{pr}^* f)A]) = [U' \chi'(\text{pr}^* f)AU^*] \\ \hat{\mu}_0 \circ (1 \otimes \mathcal{I})(f \otimes A) &= \hat{\mu}_0(f \otimes U'AU^*) = [\tilde{\chi}(\text{pr}^* f)U'AU^*] \end{aligned}$$

The difference of the two representatives is given by

$$U' \chi'(\text{pr}^* f)AU^* - \tilde{\chi}(\text{pr}^* f)U'AU^* = (U' \chi'(\text{pr}^* f) - \tilde{\chi}(\text{pr}^* f)U')AU^* .$$

Since  $f$  is uniformly continuous we have  $\mathbf{pr}^* f \in \ell_{\{\mathbb{R}^-\} \times M}^\infty(\tilde{M})$  (see Remark 3.19). Since  $U'$  is controlled for the coarse structure on  $M$ , Lemma 3.9 implies that the above difference is in  $C^*(\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M}, \tilde{E})$ .  $\square$

The morphism  $\mathcal{F}$  from (4.7) and the map  $\iota$  from (4.25) induce the respective vertical arrows in the following diagram

$$\begin{array}{ccc}
C_u(\mathcal{Y}) \hat{\otimes} \tilde{\mathbf{C}}[n](\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}) & \xrightarrow{\tilde{\mu}'} & \frac{\tilde{\mathbf{C}}[n](\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M})}{\tilde{\mathbf{C}}[n](\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M})} \\
\downarrow \text{id} \hat{\otimes} \mathcal{F} & & \downarrow \mathcal{F} \\
C_u(\mathcal{Y}) \hat{\otimes} \mathbf{C}[n](\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}) & \xrightarrow{\mu'} & \frac{\mathbf{C}[n](\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M})}{\mathbf{C}[n](\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M})} \\
\downarrow \text{id} \hat{\otimes} \iota & & \downarrow \iota \\
C_u(\mathcal{Y}) \hat{\otimes} \mathbf{C}[n](\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M)) & \xrightarrow{\mu} & \frac{\mathbf{C}[n](\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M))}{\mathbf{C}[n](\mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z}) \subseteq \mathcal{O}^\infty(M))}
\end{array}$$

which obviously commutes. We tensor with  $\text{End}(V)$ , apply  $\Sigma^{-l} K^{\text{gr}}$  and immediately use the equivalence  $\Sigma^{-l} K^{\text{gr}}(\text{End}(V) \hat{\otimes} -) \simeq K^{\text{gr}}(-)$  in order to save space. We get a commutative diagram

$$\begin{array}{ccccc}
& & K^{\text{gr}}(C_u(\mathcal{Y})) \times K^{\text{gr}}(\mathbf{C}[n](\mathcal{O}_{\mathcal{Z}}^\infty(M))) & & \\
& & \uparrow \text{id} \times (\iota \circ \mathcal{F}) & \searrow \hat{\boxtimes} & \\
& & K^{\text{gr}}(C_u(\mathcal{Y}) \hat{\otimes} \mathbf{C}[n](\mathcal{O}_{\mathcal{Z}}^\infty(M))) & & \\
K^{\text{gr}}(C_u(\mathcal{Y})) \times K^{\text{gr}}(\tilde{\mathbf{C}}[n](\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M})) & & \uparrow \text{id} \hat{\otimes} \iota \circ \mathcal{F} & \xrightarrow{\mu} & K^{\text{gr}}\left(\frac{\mathbf{C}[n](\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M))}{\mathbf{C}[n](\mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z}) \subseteq \mathcal{O}^\infty(M))}\right) \\
\downarrow \text{id} \times \mathcal{I} \simeq & & \downarrow \text{id} \hat{\otimes} \mathcal{I} \simeq & \searrow \tilde{\mu}' & \uparrow \iota \circ \mathcal{F} \\
K^{\text{gr}}(C_u(\mathcal{Y})) \times K^{\text{gr}}(C^*(\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0)) & & K^{\text{gr}}(C_u(\mathcal{Y}) \hat{\otimes} \tilde{\mathbf{C}}[n](\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M})) & \xrightarrow{\tilde{\mu}'} & K^{\text{gr}}\left(\frac{\tilde{\mathbf{C}}[n](\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M})}{\tilde{\mathbf{C}}[n](\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M})}\right) \\
& & \downarrow \text{id} \hat{\otimes} \mathcal{I} \simeq & \searrow \hat{\mu} & \downarrow \mathcal{I} \\
& & \Sigma^{-l} K^{\text{gr}}(\text{End}(V) \hat{\otimes} C_u(\mathcal{Y}) \hat{\otimes} C^*(\tilde{M}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E}_0)) & \xrightarrow{\hat{\mu}} & \Sigma^{-l} K^{\text{gr}}\left(\frac{C^*(\tilde{\mathcal{Y}}_{\mathcal{Z}} \subseteq \tilde{M}, \tilde{E})}{C^*(\{\mathbb{R}^-\} \times (\mathcal{Y} \cap \mathcal{Z}) \subseteq \tilde{M}, \tilde{E})}\right)
\end{array}$$

We consider the class  $([\Psi_0], [\tilde{D}_0])$  in the lower left corner. Its image under the up-right composition is by definition  $\mu_*([\Psi_0] \hat{\boxtimes} \beta^{-1} \sigma_{\mathcal{Z}}(\tilde{D}_0))$  while its image under the right-up

composition is  $\iota_* \mathcal{F}_* \hat{\mu}_*([\Psi_0] \cup [\tilde{D}_0])$ . This gives the first equality in the following display, while Corollary 4.35 implies the second:

$$\beta^{-1} \mu([\Psi_0] \hat{\boxtimes} \sigma_{\mathcal{Z}}(\tilde{D}_0)) = \iota_* \mathcal{F}_* \mathcal{I}_*^{-1} \hat{\mu}_*([\Psi_0] \cup [\tilde{D}_0]) = \iota_* \mathcal{F}_* \mathcal{I}_*^{-1}([\tilde{D} + \tilde{\Psi}]) .$$

In view of the Definition 3.40 of the coarse symbol pairing and Definition 4.7 of the coarse index, Definition 4.14 of the symbol, and (4.9) we can conclude that

$$[\Psi_0] \cap^{\mathcal{X}\sigma} \sigma_{\mathcal{Z}}(\tilde{D}_0) = \beta \iota_* \text{ind } \mathcal{X}(\tilde{D} + \tilde{\Psi}, \text{ on } \tilde{\mathcal{Y}}_{\mathcal{Z}}) = \sigma_{\mathcal{Z}}(\tilde{D} + \Psi, \text{ on } \mathcal{Y})$$

as asserted in (4.32). □

## 4.4 Technical Results

In this section, we provide some technical results that were used in the proof of the crucial Lemma 4.34.

Let  $H$  be a Hilbert space and  $A, B$  be two unbounded selfadjoint operators on  $H$ .

**Assumption 4.37.** *We assume that there is a linear subspace  $D$  of  $\text{dom}(A) \cap \text{dom}(B)$  preserved by both  $A$  and  $B$  such that  $A|_D, B|_D$  and  $(A|_D + B|_D)^k$  are essentially selfadjoint for  $k = 1, 2$ . We let  $A + B$  denote the unique selfadjoint extension of  $A|_D + B|_D$ .*

If  $C$  is a selfadjoint operator on  $H$  with  $D \subseteq \text{dom}(C)$  such that  $C|_D$  is essentially selfadjoint, then  $\text{dom}(C)$  is the closure of  $D$  with respect to the graph norm  $\|x\|_C := \|Cx\| + \|x\|$ . For  $k, l$  in  $\mathbb{N}$  we equip  $\text{dom}(A^k) \cap \text{dom}(B^l)$  with the norm

$$\|x\|_{A^k, B^l} := \|x\|_{A^k} + \|x\|_{B^l} .$$

**Lemma 4.38.** *Let  $A$  and  $B$  satisfy Assumption 4.37 and assume in addition that*

$$\{A, B\} := AB + BA : D \rightarrow H$$

*extends to a bounded operator on  $H$ .*

1. *We have an isomorphism  $\text{dom}(A + B) = \text{dom}(A) \cap \text{dom}(B)$ .*
2. *We have an isomorphism  $\text{dom}((A + B)^2) = \text{dom}(A^2) \cap \text{dom}(B^2)$ .*
3. *The operator  $A : D \rightarrow D$  extends to a bounded operator*

$$A : \text{dom}(A^2) \cap \text{dom}(B^2) \rightarrow \text{dom}(A) \cap \text{dom}(B) .$$

*Proof.* For two norms  $\| - \|, \| - \|'$  on the same domain we write  $\|x\| \lesssim \|x\|'$  for the statement that there exists a constant  $C$  in  $(0, \infty)$  such that  $\|x\| \leq C\|x\|'$  for all  $x$  in the domain.

By the triangle inequality we have

$$\| - \|_{A+B} \lesssim \| - \|_{A,B} \quad (4.48)$$

on  $D$ . Using that the restrictions of  $A, B$  and  $A + B$  are essentially selfadjoint on  $D$  we can further conclude that

$$\mathbf{dom}(A) \cap \mathbf{dom}(B) \subseteq \mathbf{dom}(A + B) .$$

This is a non-trivial consequence of our assumptions and does not follow directly from (4.48) which would only imply that the closure of  $D$  in the norm  $\| - \|_{A,B}$  is contained in  $\mathbf{dom}(A + B)$ . Namely, if  $\phi$  is in  $\mathbf{dom}(A) \cap \mathbf{dom}(B)$ , then by the symmetry of  $A$  and  $B$  the linear functionals  $D \ni \psi \mapsto \langle A\psi, \phi \rangle$  and  $D \ni \psi \mapsto \langle B\psi, \phi \rangle$  are bounded. Then also  $D \ni \psi \mapsto \langle (A + B)\psi, \phi \rangle$  is bounded which implies  $\phi \in \mathbf{dom}(A + B)^*$  and hence  $\phi \in \mathbf{dom}(A + B)$  since  $(A + B)|_D$  is essentially selfadjoint.

For the reverse inclusion

$$\mathbf{dom}(A + B) \subseteq \mathbf{dom}(A) \cap \mathbf{dom}(B) ,$$

it suffices to show

$$\| - \|_{A,B} \lesssim \| - \|_{A+B} , \quad (4.49)$$

on  $D$ , as  $\mathbf{dom}(A + B)$  is the closure of  $D$  with respect to  $\| - \|_{A+B}$ . We calculate for  $x$  in  $D$

$$\begin{aligned} \|(A + B)x\|^2 &= \langle (A + B)x, (A + B)x \rangle \\ &= \langle x, (A + B)^2 x \rangle \\ &= \langle x, (A^2 + B^2 + \{A, B\})x \rangle \\ &\geq \|Ax\|^2 + \|Bx\|^2 - \|\{A, B\}\| \|x\|^2 \end{aligned}$$

which implies (4.49). This finishes the argument for Assertion 1.

The argument for Assertion 2 is a more complicated version of the argument above. It is clear from  $(A + B)^2 = A^2 + B^2 + \{A, B\}$  and the boundedness of  $\{A, B\}$  that

$$\|x\|_{(A+B)^2} \lesssim \|x\|_{A^2, B^2} \quad (4.50)$$

on  $D$ . Using that the restrictions of  $A^2, B^2$  and  $(A + B)^2$  to  $D$  are essentially selfadjoint, we can further conclude

$$\mathbf{dom}(A^2) \cap \mathbf{dom}(B^2) \subseteq \mathbf{dom}((A + B)^2) .$$

To get the converse inclusion, it suffices to show the reverse estimate

$$\|x\|_{A^2, B^2} \lesssim \|x\|_{(A+B)^2} \quad (4.51)$$

on  $D$ . For  $x$  in  $D$  we have

$$\begin{aligned}
\|(A+B)^2x\|^2 &= \langle x, (A+B)^4x \rangle \\
&= \langle x, (A^2 + B^2 + \{A, B\})^2x \rangle \\
&= \langle x, (A^4 + B^4 + A^2B^2 + B^2A^2 + R)x \rangle \\
&= \|A^2x\|^2 + \|B^2x\|^2 + \langle x, (A^2B^2 + B^2A^2)x \rangle + \langle x, Rx \rangle,
\end{aligned}$$

where

$$R := A^2\{A, B\} + \{A, B\}A^2 + B^2\{A, B\} + \{A, B\}B^2 + \{A, B\}^2.$$

Because of

$$(AB)^*AB + (BA)^*BA = A^2B^2 + B^2A^2 + \{A, B\}^2 - A\{A, B\}B - B\{A, B\}A,$$

we get

$$\langle x, (A^2B^2 + B^2A^2)x \rangle \geq -\langle x, (\{A, B\}^2 - A\{A, B\}B - B\{A, B\}A)x \rangle.$$

Therefore

$$\|(A+B)^2x\|^2 \geq \|A^2x\|^2 + \|B^2x\|^2 - \langle x, (\{A, B\}^2 - A\{A, B\}B - B\{A, B\}A)x \rangle + \langle x, Rx \rangle.$$

Using the Cauchy-Schwarz inequality, the boundedness of  $\{A, B\}$  and estimates of the form  $|uv| \leq \epsilon u^2 + \frac{4}{\epsilon}v^2$  for every  $\epsilon$  in  $(0, \infty)$  several times we obtain the desired estimate (4.51). This finishes the verification of Assertion 2.

For Assertion 3 we use the non-trivial consequence of the proof of Assertion 2 that  $\text{dom}(A^2) \cap \text{dom}(B^2)$  is the closure of  $D$  with respect to the norm  $\| - \|_{A^2, B^2}$ . It is clear that  $A$  restricts to a bounded operator  $A : \text{dom}(A^2) \cap \text{dom}(B^2) \rightarrow \text{dom}(A)$ . It suffices to show that

$$\|BAx\| \lesssim \|x\|_{A^2, B^2} \tag{4.52}$$

on  $D$ . We calculate

$$\begin{aligned}
\langle BAx, BAx \rangle &= -\langle BAx, ABx \rangle + \langle BAx, \{A, B\}x \rangle \\
&= -\langle ABAx, Bx \rangle + \langle BAx, \{A, B\}x \rangle \\
&= \langle BA^2x, Bx \rangle - \langle \{A, B\}Ax, Bx \rangle + \langle BAx, \{A, B\}x \rangle \\
&= \langle A^2x, B^2x \rangle - \langle \{A, B\}Ax, Bx \rangle + \langle BAx, \{A, B\}x \rangle.
\end{aligned}$$

We get the estimate

$$\|BAx\|^2 \leq \|A^2x\| \|B^2x\| + \|\{A, B\}\| \|Ax\| \|Bx\| + \|BAx\| \|\{A, B\}\| \|x\|$$

which implies (4.52). □

Note that the assumptions for Lemma 4.38 are invariant under exchanging  $A$  and  $B$ . In the following lemma we break this symmetry.

**Lemma 4.39.** *We keep Assumption 4.37 and assume in addition that*

$$[0, \infty) \ni \epsilon \mapsto e^{-\epsilon B^2} x$$

*is a differentiable  $\text{dom}(A)$ -valued function for all  $x$  in  $D$ .*

1. *The family of commutators  $([A, e^{-\epsilon B^2}] : D \rightarrow H)_{\epsilon \in (0, \infty)}$  extends to a norm-continuous family*

$$[0, \infty) \ni \epsilon \mapsto [A, e^{-\epsilon B^2}] \in B(H)$$

*vanishing at  $\epsilon = 0$ .*

2. *The family of bounded operators  $(e^{-\epsilon B^2})_{\epsilon \in (0, \infty)}$  on  $H$  restricts to a norm continuous family of bounded operators*

$$(0, \infty) \ni \epsilon \mapsto e^{-\epsilon B^2} : \text{dom}(A) \rightarrow \text{dom}(A) \cap \text{dom}(B) .$$

3. *The family of bounded operators  $(e^{-\epsilon B^2})_{\epsilon \in [0, \infty)}$  on  $H$  restricts to a continuous family of bounded operators*

$$[0, \infty) \ni \epsilon \mapsto e^{-\epsilon B^2} : \text{dom}(A^2) \cap \text{dom}(B^2) \rightarrow \text{dom}(A) .$$

*Proof.* On  $D$  we have the relation  $[A, B^2] = (\{A, B\}B - B\{A, B\})$  and therefore

$$e^{-r\epsilon B^2} [A, B^2] e^{-(1-r)\epsilon B^2} = e^{-r\epsilon B^2} (\{A, B\}B - B\{A, B\}) e^{-(1-r)\epsilon B^2} .$$

Decomposing the right-hand side as

$$e^{-r\epsilon B^2} \cdot \{A, B\} \cdot B e^{-(1-r)\epsilon B^2} - e^{-r\epsilon B^2} B \cdot \{A, B\} \cdot e^{-(1-r)\epsilon B^2}$$

we see that it extends to a norm-continuous family

$$(0, \infty) \times (0, 1) \ni (\epsilon, r) \mapsto \epsilon e^{-r\epsilon B^2} [A, B^2] e^{-(1-r)\epsilon B^2} \in B(H) .$$

We claim the integral of this family over  $r$  in  $[0, 1]$  exists and defines a norm-continuous family

$$[0, \infty) \ni \epsilon \mapsto \int_0^1 \epsilon e^{-r\epsilon B^2} [A, B^2] e^{-(1-r)\epsilon B^2} dr \in B(H)$$

vanishing at  $\epsilon = 0$ . In order to estimate the norm of the integrand we split the interval of integration into the halves  $[0, 1/2]$  and  $[1/2, 1]$ . We further fix  $c$  in  $(0, \infty)$ . We see that there exists a constant  $C$  in  $(0, \infty)$  such that for all  $r$  in  $[0, 1/2]$  and  $\epsilon$  in  $(0, c)$

$$\epsilon \|e^{-r\epsilon B^2} \{A, B\} B e^{-(1-r)\epsilon B^2}\| \leq \epsilon^{1/2} \|e^{-r\epsilon B^2}\| \| \{A, B\} \| \epsilon^{1/2} \|B e^{-(1-r)\epsilon B^2}\| \leq C \epsilon^{1/2}$$

and

$$\epsilon \|e^{-r\epsilon B^2} B \{A, B\} e^{-(1-r)\epsilon B^2}\| \leq \epsilon^{1/2} r^{-1/2} \| \epsilon^{1/2} r^{1/2} B e^{-r\epsilon B^2} \| \| \{A, B\} \| \| e^{-(1-r)\epsilon B^2} \| \leq C \epsilon^{1/2} r^{-1/2} .$$

These estimates imply the claim for the integral over  $[0, 1/2]$ . The other half of the integral is discussed similarly.

For all  $x, y$  in  $D$  and  $\epsilon$  in  $(0, \infty)$  we have by our assumption that

$$-\partial_r \langle y, [A, e^{-r\epsilon B^2}]x \rangle = \langle y, \epsilon e^{-rB^2} [A, B^2] e^{-(1-r)\epsilon B^2} x \rangle .$$

Integrating and using that  $D$  is dense in  $H$  we get the equality

$$[A, e^{-\epsilon B^2}] = -\epsilon \int_0^1 e^{-r\epsilon B^2} [A, B^2] e^{-(1-r)\epsilon B^2} dr$$

on  $D$ . By continuous extension we get a norm-continuous family

$$[0, \infty) \ni \epsilon \rightarrow [A, e^{-\epsilon B^2}] \in B(H)$$

vanishing at  $\epsilon = 0$ . This finishes the proof of Assertion 1.

We now show Assertion 2. We know that

$$(0, \infty) \ni \epsilon \mapsto e^{-\epsilon B^2} : H \rightarrow \text{dom}(B)$$

is a norm-continuous family of bounded operators. We must show that

$$(0, \infty) \ni \epsilon \mapsto e^{-\epsilon B^2} : \text{dom}(A) \rightarrow \text{dom}(A)$$

is a norm-continuous family of bounded operators. To this end we must see that

$$(0, \infty) \ni \epsilon \mapsto Ae^{-\epsilon B^2} : \text{dom}(A) \rightarrow H$$

is a norm-continuous family of bounded operators. This follows from

$$Ae^{-\epsilon B^2} = [A, e^{-\epsilon B^2}] + e^{-\epsilon B^2} A , \tag{4.53}$$

and Assertion 1.

We now show Assertion 3. We know that

$$[0, \infty) \ni \epsilon \mapsto e^{-\epsilon B^2} : \text{dom}(B^2) \rightarrow H$$

is a continuous family of bounded operators. We must show that

$$[0, \infty) \ni \epsilon \mapsto Ae^{-\epsilon B^2} : \text{dom}(A^2) \cap \text{dom}(B^2) \rightarrow H$$

is a continuous family of bounded operators. This follows from (4.53) and the decomposition

$$\text{dom}(A^2) \cap \text{dom}(B^2) \xrightarrow{A} \text{dom}(B) \xrightarrow{e^{-\epsilon B^2}} H ,$$

where the first map is continuous by Lemma 4.38.3 □



## 4.5 The Dirac-goes-to-Dirac principle

We consider a complete Riemannian manifold  $M$  with a uniformly continuous and controlled smooth function  $f : M \rightarrow \mathbb{R}$  such that 0 is a regular value. Then  $N := f^{-1}(\{0\})$  is an embedded smooth manifold which decomposes  $M$  into closed subspaces  $M_{\pm} := f^{-1}(\mathbb{R}^{\pm})$ . The pair of subsets  $(M_+, M_-)$  is a coarsely and uniformly excisive decomposition of  $M$ . If  $\mathcal{Y}, \mathcal{Z}$  are a big families on  $M$ , then by Lemma 2.39 this decomposition gives rise to a Mayer-Vietoris fibre sequence

$$K_{\mathcal{Z}}^{\mathcal{X}}(M_- \cap \mathcal{Y}) \oplus K_{\mathcal{Z}}^{\mathcal{X}}(M_+ \cap \mathcal{Y}) \rightarrow K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}) \xrightarrow{\delta^{MV}} \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(N \cap \mathcal{Y}) .$$

We assume that  $\mathcal{D}$  is a Dirac operator of degree  $n$  which is positive away from a big family  $\mathcal{Z}$  (see Definition 4.2), and that  $\Psi$  is a potential which is very positive away from  $\mathcal{Y}$  (see Definition 4.31) and asymptotically constant away from  $\mathcal{Z}$  (see Definition 4.27). We then consider the Callias-type operator  $\mathcal{D} + \Psi$ . We shall assume that  $N$  has a tubular neighbourhood on which all data has a product structure. This means that near  $N$  the operator has the form  $\Sigma(\mathcal{D} + \Psi)_{|N}$  for some Callias type operator  $(\mathcal{D} + \Psi)_{|N}$  of degree  $n - 1$  on  $N$ , see (4.12) for notation and Remark 4.40 below for more details.

**Remark 4.40.** One can reconstruct  $(\mathcal{D} + \Psi)_{|N}$  as follows. Assume that  $\mathcal{D} + \Psi$  acts on the bundle  $E \rightarrow M$  of graded right  $\mathbf{Cl}^n$ -modules. Let  $\sigma$  in  $\text{End}_{\mathbf{Cl}^n}(E_{|N})$  be the Clifford multiplication by the normal vector pointing in direction  $M_+$ . We let  $e_n$  be the  $n$ th generator of  $\mathbf{Cl}^n$  and  $z$  be the grading of  $E$ . Then  $iz\sigma e_n$  is an even selfadjoint involution. We let  $E_0$  be the 1-eigensubbundle of  $E_{|N}$  for  $iz\sigma e_n$ . The grading  $z$  induces by restriction a grading  $z_{|N}$  of  $E_0$ . The right action of the subalgebra  $\mathbf{Cl}^{n-1}$  generated by remaining generators  $e_1, \dots, e_{n-1}$  of  $\mathbf{Cl}^n$  preserves the subbundle and therefore induces a right action on  $E_0$ . The left Clifford multiplication by tangent vectors along  $N$  also preserves  $E_0$  and induces a Clifford bundle structure on  $E_0 \rightarrow N$ . Since  $\sigma$  and hence  $iz\sigma e_n$  are parallel by the product structure assumption, the connection of  $E$  induces a connection  $\nabla_0$  on  $E_0$ . We let  $\mathcal{D}_{|N}$  denote the Dirac operator defined by this Dirac bundle structure on  $E_0$ . The restriction of  $\Psi$  to  $N$  commutes with  $iz\sigma e_n$  (it anticommutes with  $z$  and  $\sigma$  and commutes with  $e_n$ ) and therefore restricts to an endomorphism  $\Psi_{|N}$  on  $E_0$ . In order to see that  $\mathcal{D} + \Psi$  is isomorphic to  $\Sigma(\mathcal{D}_{|N} + \Psi_{|N})$  near  $N$  we define an isomorphism

$$E_0 \hat{\otimes} \mathbf{Cl}^1 \rightarrow E_{|N} , \quad v \otimes (a + be_n) \mapsto av + bve_n , \quad (4.54)$$

where we consider  $\mathbf{Cl}^1$  as generated by  $e_n$  and  $a, b$  are in  $\mathbb{C}$ . This map is compatible with the gradings, the right  $\mathbf{Cl}^n$ -actions, the left Clifford multiplication tangent vectors of  $M$  and the restriction of the connection along  $N$ . We define the Clifford multiplication by  $\sigma$  on  $E_0 \hat{\otimes} \mathbf{Cl}^1$  by  $\sigma(v \hat{\otimes} c) = \pm v \hat{\otimes} e_n c$ , where  $v$  is homogenous of parity  $\pm$ . Since we assume a product structure of all data near  $N$  it is then clear that  $\mathcal{D} + \Psi$  is isomorphic to  $\Sigma(\mathcal{D} + \Psi)_{|N}$  in a neighbourhood of  $N$ .  $\blacksquare$

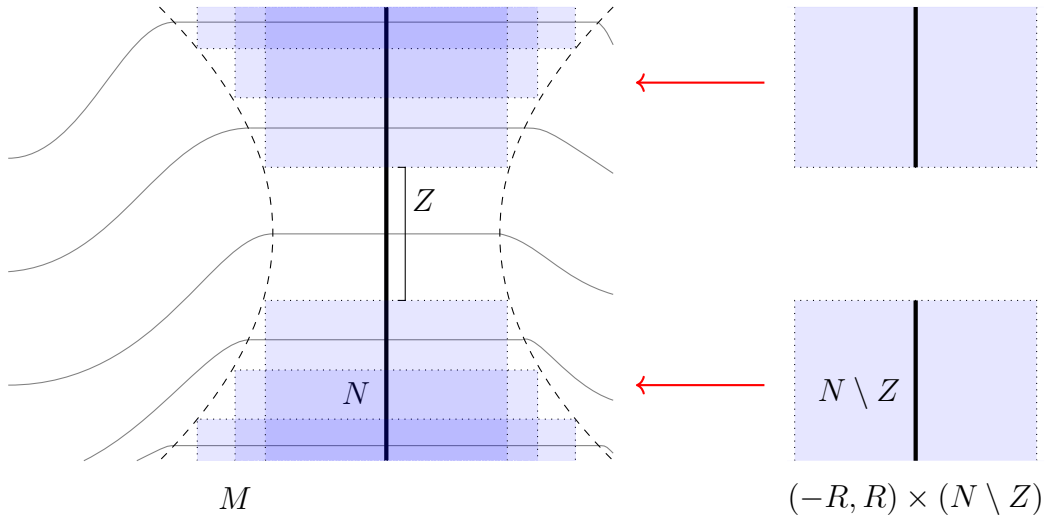
The assumptions on  $\Psi$  imply that  $\Psi_{|N}$  is very positive away from  $N \cap \mathcal{Y}$  and is asymptotically constant away from  $N \cap \mathcal{Z}$ . Below in Theorem 4.44 we will need the condition that  $\mathcal{D}_{|N}$  is positive away from  $N \cap \mathcal{Z}$ .

**Remark 4.41.** In general we can not deduce the local positivity of  $\mathcal{D}|_N$  from the local positivity of  $\mathcal{D}$ . For example, consider the spin Dirac operator on the Riemannian product  $M := \mathbb{R} \times S^1$ , where  $S^1$  has the bounding spin structure. We further consider the decomposition along  $N := \mathbb{R} \times S^0$ . Then  $\mathcal{D}$  is positive everywhere, i.e, away from  $\emptyset$ , but  $\mathcal{D}|_N$  is not.

On the other hand, if the positivity of  $\mathcal{D}$  away from  $\mathcal{Z}$  is caused by the zero order term in the Weizenboeck formula (4.1), then because of the assumption of a product structure,  $\mathcal{D}|_N$  is positive away from  $N \cap \mathcal{Z}$ . See Lemma 4.43 below for a further example. ■

Let  $U$  be a tubular neighbourhood of  $N$  on which all data has a product structure. Assume that  $\mathcal{D}$  is positive away from the big family  $\mathcal{Z}$ .

**Definition 4.42.** We say that the width of  $U$  goes to  $\infty$  away from  $\mathcal{Z}$  if for every  $R$  in  $(0, \infty)$  there exists a member  $Z$  of  $\mathcal{Z}$  and an isometric embedding  $(-R, R) \times (N \setminus Z) \rightarrow U \setminus Z$  compatible with the product structure of the bundle data.



**Lemma 4.43.** If the width of  $U$  goes to  $\infty$  away from  $\mathcal{Z}$  and  $\mathcal{D}$  is positive away from  $\mathcal{Z}$ , then  $\mathcal{D}|_N$  is positive away from  $N \cap \mathcal{Z}$ .

*Proof.* Since  $\mathcal{D}$  is positive away from  $\mathcal{Z}$ , according to Definition 4.2 there exists  $a$  in  $(0, \infty)$  and a member  $Z$  of  $\mathcal{Z}$  such that  $\|\mathcal{D}\phi\|^2 \geq a^2\|\phi\|^2$  for each  $\phi$  in  $C_c^\infty(M \setminus \bar{Z}, E)$ .

Using the identification (4.54), we may consider sections  $\phi$  in  $C^\infty(M, E)$  supported on the image of  $(-R, R) \times (N \setminus \bar{Z})$  in product form  $\phi = f \hat{\otimes} \psi$  with  $f$  in  $C_c^\infty((-R, R)) \otimes \mathbb{C}1$  and

$\psi$  in  $C_c^\infty(N \setminus \bar{Z}, E_0)$ . For such a section  $\phi$ , we have

$$\begin{aligned} \|\not{D}\phi\|_{L^2(M,E)}^2 &= \langle \not{D}^2\phi, \phi \rangle_{L^2(M,E)} = \langle -f'' \otimes \psi + f \otimes \not{D}_{|N}^2\psi, f \otimes \psi \rangle_{L^2(M,E)} \\ &= \|f'\|_{L^2((-R,R),\mathbf{Cl}^1)}^2 \|\psi\|_{L^2(N,E_0)}^2 + \|f\|_{L^2((-R,R),\mathbf{Cl}^1)}^2 \|\not{D}_{|N}\psi\|_{L^2(N,E_0)}^2. \end{aligned} \quad (4.55)$$

For every  $R$  in  $(0, \infty)$ , we can find  $f$  in  $C_c^\infty((-R, R)) \otimes \mathbf{Cl}^1$  with  $\|f\|_{L^2}^2 = 1$  and  $\|f'\|_{L^2}^2 \leq \frac{2}{R^2}$ . Choose  $R$  such that  $\frac{2}{R^2} \leq \frac{a^2}{2}$  and enlarge the member  $Z$  so that there exists an isometric embedding  $(-R, R) \times (N \setminus Z) \rightarrow U \setminus Z$  and choose a function  $f$  as above corresponding to this  $R$ . Then by (4.55), for all  $\psi$  in  $C_c^\infty(N \setminus \bar{Z}, E_0)$ , we have

$$\begin{aligned} \|\not{D}_{|N}\psi\|_{L^2(N,E_0)}^2 &= \|\not{D}\phi\|_{L^2(M,E)}^2 - \|f'\|_{L^2((-R,R),\mathbf{Cl}^1)}^2 \|\psi\|_{L^2(N,E_0)}^2 \\ &\geq a^2 \|\phi\|_{L^2(M,E)}^2 - \frac{2}{R^2} \|\psi\|_{L^2(N,E_0)}^2 \\ &\geq a^2 \|f\|_{L^2((-R,R),\mathbf{Cl}^1)}^2 \|\psi\|_{L^2(N,E_0)}^2 - \frac{a^2}{2} \|\psi\|_{L^2(N,E_0)}^2 \\ &= \frac{a^2}{2} \|\psi\|_{L^2(N,E_0)}^2 \end{aligned}$$

Hence  $\not{D}_{|N}$  is positive away from  $N \cap \mathcal{Z}$ . □

The following theorem is called the Dirac-goes-to-Dirac principle and has been observed in various variations, a first instance goes back to [BDT89].

**Theorem 4.44.** *We assume that  $\not{D}$  is positive away from  $\mathcal{Z}$ , that  $\not{D}_{|N}$  is positive away from  $N \cap \mathcal{Z}$ , and that  $\Psi$  is very positive away from  $\mathcal{Y}$ . If  $N$  has a tubular neighbourhood with product structure of uniform width, then*

$$\delta^{MV}(\sigma_{\mathcal{Z}}(\not{D} + \Psi, \text{on } \mathcal{Y})) = \beta^{-1} \sigma_{N \cap \mathcal{Z}}((\not{D} + \Psi)_{|N}, \text{on } N \cap \mathcal{Y})$$

in  $K_{\mathcal{Z}, -n-1}^{\mathcal{X}}(N \cap \mathcal{Y})$ ,

*Proof.* We apply the relative index theorem (Proposition 4.9) to the following data: We consider

$$M_0 := \tilde{M}, \quad \mathcal{Y}_0 := \tilde{\mathcal{Y}}_{\mathcal{Z}}, \quad \not{D}_0 := \tilde{\not{D}} + \tilde{\Psi}, \quad (4.56)$$

see (4.29). Furthermore, we let

$$M_1 := (\mathbb{R} \otimes N)^\sim, \quad \mathcal{Y}_1 := (\mathbb{R} \times (N \cap \mathcal{Y}))_{\mathbb{R} \times (N \cap \mathcal{Z})}^\sim, \quad \not{D}_1 := (\Sigma(\not{D} + \Psi)_{|N})^\sim,$$

where  $(\dots)^\sim$  refers to applying the Construction 4.10. Let  $U \cong I \times N$  be the tubular neighborhood of  $N$  with uniform width. By assumption, we have an isomorphism of all data over  $\mathbb{R}^+ \times U \cong \mathbb{R}^+ \times I \times N$ . Let  $W_0 \subset M_0$  and  $W_1 \subset M_1$  be the complements of these sets and let  $\mathcal{W}_i := \{W_i\}$  be the big family generated by  $W_i$ . One checks that the identification  $e : M_0 \setminus W_0 \rightarrow M_1 \setminus W_1$  is a morphism of bornological coarse spaces.

Then by Proposition 4.9, we have

$$e_*\pi_0(\text{ind}\mathcal{X}(\not{D}_0, \text{on } \mathcal{Y}_0)) = \pi_1(\text{ind}\mathcal{X}(\not{D}_1, \text{on } \mathcal{Y}_1)) , \quad (4.57)$$

in  $K\mathcal{X}(\mathcal{Y}_1, \mathcal{Y}_1 \cap \mathcal{W}_1)$ , where  $\pi_i : K\mathcal{X}(\mathcal{Y}_i) \rightarrow K\mathcal{X}(\mathcal{Y}_i, \mathcal{Y}_i \cap \mathcal{W}_i)$  are the projections.

We consider the Mayer-Vietoris boundary for the coarsely excisive compositions of  $M_i$  given by  $(M_{0,-}, M_{0,+})$  with

$$M_{0,\pm} = \mathbb{R} \times M_{\pm} ,$$

and  $(M_{1,-}, M_{1,+})$  with

$$M_{1,\pm} = \mathbb{R} \times \mathbb{R}^{\pm} \times N .$$

By naturality of the Mayer-Vietoris boundaries we get the commutative diagram

$$\begin{array}{ccc} K\mathcal{X}(\mathcal{Y}_0, \mathcal{Y}_0 \cap \mathcal{W}_0) & \xrightarrow{\delta_0^{MV}} & \Sigma K\mathcal{X}(\tilde{N} \cap \mathcal{Y}_0, \tilde{N} \cap \mathcal{Y}_0 \cap \mathcal{W}_0) \\ \downarrow e_* & & \downarrow \simeq \\ K\mathcal{X}(\mathcal{Y}_1, \mathcal{Y}_1 \cap \mathcal{W}_1) & \xrightarrow{\delta_1^{MV}} & \Sigma K\mathcal{X}(\tilde{N} \cap \mathcal{Y}_1, \tilde{N} \cap \mathcal{Y}_1 \cap \mathcal{W}_1) , \end{array}$$

so by (4.57), we get

$$\delta_0^{MV}(\pi_0(\text{ind}\mathcal{X}(\not{D}_0, \text{on } \mathcal{Y}_0))) = \delta_1^{MV}(\pi_1(\text{ind}\mathcal{X}(\not{D}_1, \text{on } \mathcal{Y}_1))) \quad (4.58)$$

in  $K\mathcal{X}_{-n-1}(\tilde{N} \cap \mathcal{Y}_0, \tilde{N} \cap \mathcal{Y}_0 \cap \mathcal{W}_0) \cong K\mathcal{X}_{-n-1}(\tilde{N} \cap \mathcal{Y}_1, \tilde{N} \cap \mathcal{Y}_1 \cap \mathcal{W}_1)$ .

**Lemma 4.45.** *We have*

$$\delta_1^{MV} \text{ind}\mathcal{X}(\not{D}_1, \text{on } \mathcal{Y}_1) = \beta^{-1} \text{ind}\mathcal{X}(((\not{D} + \Psi)|_N)^\sim, \text{on } \tilde{N} \cap \mathcal{Y}_1) . \quad (4.59)$$

in  $K\mathcal{X}_{-n-1}(\tilde{N} \cap \mathcal{Y}_1)$ .

*Proof.* The identity of underlying sets induces a map of bornological coarse spaces  $(\mathbb{R} \otimes N)^\sim \rightarrow \mathbb{R} \otimes \tilde{N}$ . By Proposition 4.13 the induced map in coarse  $K$ -homology sends  $\text{ind}\mathcal{X}(\not{D}_1, \text{on } \mathcal{Y}_1)$  to  $\text{ind}\mathcal{X}(\Sigma((\not{D} + \Psi)|_N)^\sim, \text{on } \mathcal{Y}_1)$ . By naturality of the Mayer-Vietoris boundaries we get the first equality in

$$\begin{aligned} \delta_1^{MV}(\text{ind}\mathcal{X}(\not{D}_1, \text{on } \mathcal{Y}_1)) &= \delta_1^{MV}(\text{ind}\mathcal{X}(\Sigma((\not{D} + \Psi)|_N)^\sim, \text{on } \mathcal{Y}_1)) \\ &= \beta^{-1} \text{ind}\mathcal{X}(((\not{D} + \Psi)|_N)^\sim, \text{on } \tilde{N} \cap \mathcal{Y}_1) , \end{aligned}$$

while the second one follows from the suspension theorem (Proposition 4.8).  $\square$

Combining (4.58) with (4.59) and using the definition of  $\not{D}_0$  and  $\mathcal{Y}_0$  in (4.56), we get

$$\pi_0(\delta_0^{MV}(\text{ind}\mathcal{X}(\tilde{\not{D}} + \tilde{\Psi}, \text{on } \tilde{\mathcal{Y}}_Z))) = \beta^{-1} \pi_1(\text{ind}\mathcal{X}(((\not{D} + \Psi)|_N)^\sim, \text{on } \tilde{N} \cap \mathcal{Y}_1))$$

Here we implicitly employed the fact that  $\pi_i$  intertwines the relative with the absolute Mayer-Vietoris boundary. Applying the map of coarse spaces  $\iota : \tilde{M} \rightarrow \mathcal{O}^\infty(M)$  to this equation and using naturality of the Mayer-Vietoris boundary, Definition 4.29 yields the following result.

**Corollary 4.46.** *We have*

$$\pi_0(\delta^{MV}(\sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y}))) = \pi_0(\beta^{-1}\sigma_{N \cap \mathcal{Z}}((\mathcal{D} + \Psi)|_N, \text{on } N \cap \mathcal{Y})) . \quad (4.60)$$

in  $K\mathcal{X}_{-n-1}(\mathcal{O}_{\mathcal{Z}}^\infty(N \cap \mathcal{Y}), \mathcal{O}_{\mathcal{Z}}^\infty(N \cap \mathcal{Y}) \cap \mathcal{W}_0)$ .

We now observe that every member of  $\mathcal{O}_{\mathcal{Z}}^\infty(N \cap \mathcal{Y}) \cap \mathcal{W}_0$  is contained in a member of  $\mathcal{O}^-(N \cap \mathcal{Y} \cap \mathcal{Z})$ . Since these subsets are flasque the map

$$K\mathcal{X}(\mathcal{O}_{\mathcal{Z}}^\infty(N \cap \mathcal{Y}) \cap \mathcal{W}_0) \rightarrow K\mathcal{X}(\mathcal{O}_{\mathcal{Z}}^\infty(N \cap \mathcal{Y}))$$

vanishes and the projection  $\pi_0$  is a split injection. We can therefore omit  $\pi_0$  in (4.60) and get the desired equality.  $\square$

**Remark 4.47.** Even if we do not know that  $\mathcal{D}|_N$  is positive away from  $N \cap \mathcal{Z}$  we have a class  $\delta^{MV}\sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y})$  in  $K_{\mathcal{Z}}^{\mathcal{X}}(N \cap \mathcal{Y})$  whose image under  $K_{\mathcal{Z}}^{\mathcal{X}}(N \cap \mathcal{Y}) \rightarrow K^{\mathcal{X}}(N \cap \mathcal{Y})$  is  $\beta^{-1}\sigma((\mathcal{D} + \Psi)|_N, \text{on } N \cap \mathcal{Y})$ .  $\blacksquare$

## 4.6 Decompositions and Mayer-Vietoris

We consider a bornological coarse space  $M$  (we use the symbol  $M$  since it will later be the bornological coarse space associated to a complete Riemannian manifold). Let  $\tilde{f} : M \rightarrow \mathbb{R}$  be a controlled function. Then we define the big families

$$\mathcal{M}_{\pm} := \tilde{f}^{-1}(\{\mathbb{R}^{\pm}\})$$

on  $M$  and set  $\mathcal{Y} := \mathcal{M}_- \cap \mathcal{M}_+$ . We consider the function

$$f := \frac{\tilde{f}}{\sqrt{1 + \tilde{f}^2}} \quad \text{in } \ell_{\mathcal{Y}}^\infty(M) \quad (4.61)$$

and observe that

$$\left(\frac{f+1}{2}\right)^2 - \frac{f+1}{2} \in \ell^\infty(\mathcal{Y}) ,$$

see Definition 3.1 for notation. The class of  $\frac{f+1}{2}$  in  $C(\partial^{\mathcal{Y}}X) = \frac{\ell_{\mathcal{Y}}^\infty(M)}{\ell^\infty(\mathcal{Y})}$  is a projection and represents a  $K$ -theory class

$$p := \left[ \frac{f+1}{2} \right] \quad (4.62)$$

in  $K_0(C(\partial^{\mathcal{Y}}X))$ , see Remark 2.48.

We have the pairing (3.21)

$$p \cap^{\mathcal{X}} - : K\mathcal{X}(M) \rightarrow \Sigma K\mathcal{X}(\mathcal{Y}) .$$

On the other hand, the decomposition of  $M$  into  $(\mathcal{M}_-, \mathcal{M}_+)$  induces in view of (2.11) a Mayer-Vietoris fibre sequence

$$K\mathcal{X}(\mathcal{M}_-) \oplus K\mathcal{X}(\mathcal{M}_+) \rightarrow K\mathcal{X}(M) \xrightarrow{\partial^{MV}} \Sigma K\mathcal{X}(\mathcal{Y}) .$$

**Lemma 4.48.** *We have an equivalence  $p \cap^{\mathcal{X}} - \simeq \partial^{MV} : K\mathcal{X}(M) \rightarrow \Sigma K\mathcal{X}(\mathcal{Y})$ .*

*Proof.* The origin of the Mayer-Vietoris boundary map for coarse  $K$ -homology  $K\mathcal{X}$  is the excisiveness of the  $\mathbf{Mod}(KU)$ -valued functor  $K\mathcal{X}$  on  $\mathbf{BC}$ . It sends the square

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{M}_+ \\ \downarrow & & \downarrow \\ \mathcal{M}_- & \longrightarrow & M \end{array} \quad (4.63)$$

of big families or objects in  $\mathbf{BC}$  to the left square in

$$\begin{array}{ccccc} K\mathcal{X}(\mathcal{Y}) & \longrightarrow & K\mathcal{X}(\mathcal{M}_+) & \dashrightarrow & K\mathcal{X}(\mathcal{M}_+, \mathcal{Y}) \\ \downarrow & & \downarrow & & \downarrow \simeq \\ K\mathcal{X}(\mathcal{M}_-) & \longrightarrow & K\mathcal{X}(M) & \dashrightarrow & K\mathcal{X}(M, \mathcal{M}_-) \end{array} \quad (4.64)$$

in  $\mathbf{Mod}(KU)$  which has the property of being cocartesian. It extends to a map of fibre sequences as indicated so that the right vertical map is an equivalence. The Mayer-Vietoris boundary map is by definition

$$\partial^{MV} : K\mathcal{X}(M) \rightarrow K\mathcal{X}(M, \mathcal{M}_-) \xleftarrow{\simeq} K\mathcal{X}(\mathcal{M}_+, \mathcal{Y}) \xrightarrow{\partial} \Sigma K\mathcal{X}(\mathcal{Y}) , \quad (4.65)$$

where  $\partial$  is the boundary map associated to the upper fibre sequence. Since we want to compare this map with a construction of  $p \cap^{\mathcal{X}} -$  defined on the level of  $C^*$ -categories we must also express the  $\partial^{MV}$  in terms of  $C^*$ -categorical constructions.

The square (4.63) yields an excisive square of  $C^*$ -categories [BE23, Lemma 6.10]

$$\begin{array}{ccc} \mathbf{C}(\mathcal{Y} \subseteq M) & \longrightarrow & \mathbf{C}(\mathcal{M}_+ \subseteq M) \\ \downarrow & & \downarrow \\ \mathbf{C}(\mathcal{M}_- \subseteq M) & \longrightarrow & \mathbf{C}(M) \end{array} \quad (4.66)$$

Applying the  $K$ -theory functor for  $C^*$ -categories we get a square

$$\begin{array}{ccc} K(\mathbf{C}(\mathcal{Y} \subseteq M)) & \longrightarrow & K(\mathbf{C}(\mathcal{M}_+ \subseteq M)) \\ \downarrow & & \downarrow \\ K(\mathbf{C}(\mathcal{M}_- \subseteq M)) & \longrightarrow & K(\mathbf{C}(M)) \end{array}$$

which corresponds to the left push-out square in (4.64). Unfolding the definition of  $\nu$  in (3.19) and (3.22) the map  $p \cap^{\mathcal{X}} -$  is given by the following composition

$$\begin{aligned} K\mathcal{X}(M) &\simeq K(\mathbf{C}(M)) \xrightarrow{p \otimes \text{id}} K\left(\frac{\ell_{\mathcal{Y}}^{\infty}(M)}{\ell^{\infty}(\mathcal{Y})} \otimes \mathbf{C}(M)\right) \\ &\xrightarrow{\nu} K\left(\frac{\mathbf{C}(M)}{\mathbf{C}(\mathcal{Y} \subseteq M)}\right) \xrightarrow{\partial} \Sigma K(\mathbf{C}(\mathcal{Y} \subseteq M)) \simeq \Sigma K\mathcal{X}(\mathcal{Y}) \end{aligned}$$

We now argue with the commutative diagram

$$\begin{array}{ccc} K(\mathbf{C}(M)) & \longrightarrow & K\left(\frac{\mathbf{C}(M)}{\mathbf{C}(\mathcal{M}_- \subseteq M)}\right) \\ \downarrow \nu \circ (p \otimes \text{id}) & & \uparrow \simeq \\ K\left(\frac{\mathbf{C}(M)}{\mathbf{C}(\mathcal{Y} \subseteq M)}\right) & \longleftarrow & K\left(\frac{\mathbf{C}(\mathcal{M}_+ \subseteq M)}{\mathbf{C}(\mathcal{Y} \subseteq M)}\right) \\ & \searrow \partial \quad \swarrow \partial & \\ & \Sigma K(\mathbf{C}(\mathcal{Y} \subseteq M)) & \end{array}$$

where the horizontal arrows are induced by the obvious projections and inclusions. The lower triangle commutes by the naturality of the boundary operator for  $C^*$ -algebra  $K$ -theory for maps of exact sequences of  $C^*$ -categories. The upper right vertical morphism is the left vertical equivalence in (4.64). The composition along the left part reflects the definition of  $p \cap^{\mathcal{X}} -$ , and the other composition is the definition (4.65) of  $\partial^{MV}$ .  $\square$

We now assume that  $M$  is a uniform bornological coarse space and that  $\tilde{f}$  is in addition uniformly continuous. We define the closed subspaces  $M_{\pm} := f^{-1}(\mathbb{R}^{\pm})$  and set  $N := M_+ \cap M_-$ .

Then we have  $p \in K_0(C(\partial_u^{\mathcal{Y}} M))$  and  $\partial p \in K^1(\mathcal{Y})$ , where  $\partial$  is the boundary map in  $K$ -theory associated to the exact sequence of  $C^*$ -algebras (3.8) and we use the grading convention  $\pi_{-*}K(C_u(\mathcal{Y})) \simeq K^*(\mathcal{Y})$ .

Let  $\mathcal{Z}$  be a second big family. We have the coarse symbol pairing

$$\partial p \cap^{\mathcal{X}\sigma} - : K_{\mathcal{Z}}^{\mathcal{X}}(M) \rightarrow \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y})$$

introduced in Definition 3.40. By Lemma 2.39.2 the decomposition  $(M_+, M_-)$  (which is coarsely and uniformly excisive) of  $M$  induces a Mayer-Vietoris fibre sequence

$$K_{\mathcal{Z}}^{\mathcal{X}}(M_+) \oplus K_{\mathcal{Z}}^{\mathcal{X}}(M_-) \rightarrow K_{\mathcal{Z}}^{\mathcal{X}}(M) \xrightarrow{\delta^{MV}} \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(N).$$

We let

$$i^{\mathcal{X}} : K_{\mathcal{Z}}^{\mathcal{X}}(N) \rightarrow K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}) \quad \text{and} \quad i : K\mathcal{X}(N \cap \mathcal{Z}) \rightarrow K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z}) \quad (4.67)$$

denote the maps induced by the obvious inclusions. The square

$$\begin{array}{ccc} K_{\mathcal{Z}}^{\mathcal{X}}(N) & \xrightarrow{i^{\mathcal{X}}} & K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}) \\ \downarrow a_{N, N \cap \mathcal{Z}} & & \downarrow a_{\mathcal{Y}, \mathcal{Y} \cap \mathcal{Z}} \\ K\mathcal{X}(N \cap \mathcal{Z}) & \xrightarrow{i} & K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z}) \end{array} \quad (4.68)$$

commutes by the naturality of the index map Definition 2.40.

**Lemma 4.49.** *We have an equivalence*

$$\partial p \cap^{\mathcal{X}\sigma} \simeq i^{\mathcal{X}} \circ \delta^{MV} : K_{\mathcal{Z}}^{\mathcal{X}}(M) \rightarrow \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}) .$$

*Proof.* We consider the following morphism of exact sequences of  $C^*$ -categories

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ C_u(\mathcal{Y}) \otimes \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(M)) & \xrightarrow{\mu} & \frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y}) \subseteq \mathcal{O}^{\infty}(M))}{\mathbf{C}(\mathcal{O}^{\infty}(\mathcal{Y} \cap \mathcal{Z}) \subseteq \mathcal{O}^{\infty}(M))} & \longleftarrow & \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y}) \subseteq \mathcal{O}^{\infty}(M)) \\ \downarrow & & \downarrow & & \downarrow \\ C_{u,\mathcal{Y}}(X) \otimes \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(M)) & \longrightarrow & \frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(M) \subseteq \mathcal{O}^{\infty}(M))}{\mathbf{C}(\mathcal{O}^{\infty}(\mathcal{Y} \cap \mathcal{Z}) \subseteq \mathcal{O}^{\infty}(M))} & \longleftarrow & \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(M) \subseteq \mathcal{O}^{\infty}(M)) \\ \downarrow & & \downarrow & & \downarrow \\ C_u(\partial_u^{\mathcal{Y}} M) \otimes \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(M)) & \xrightarrow{!} & \frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(M) \subseteq \mathcal{O}^{\infty}(M))}{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y}) \subseteq \mathcal{O}^{\infty}(M))} & \longequal{\quad} & \frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(M) \subseteq \mathcal{O}^{\infty}(M))}{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(\mathcal{Y}) \subseteq \mathcal{O}^{\infty}(M))} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

where the left vertical sequence is obtained by tensoring (3.8) with  $\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^{\infty}(M))$ . To explain the arrow marked “!”, notice that we have a map

$$\partial^{\mathcal{O}^{\infty}(\mathcal{Y})} \mathcal{O}^{\infty}(M) \xrightarrow{\text{pr}} \partial^{\mathcal{Y}} M \xrightarrow{c} \partial_u^{\mathcal{Y}} M ,$$

where the first map is the extension of the controlled map  $\text{pr} : \mathcal{O}^{\infty}(M) \rightarrow M$  to a continuous map between the coronas (see Remark 3.8) and the second map is the comparison map (3.9). The arrow marked “!” is then the composition  $\nu \circ ((c \circ \text{pr})^* \otimes \text{id})$ , where  $\nu$  is the appropriate version of (3.19).

Applying  $K$ -theory and using the naturality of the boundary map in  $K$ -theory for morphisms of exact sequences of  $C^*$ -categories we get the two lower right commutative squares



in

$$\begin{array}{c}
K(\partial^{\mathcal{O}^\infty(\mathcal{Y})}\mathcal{O}^\infty(M)) \times K_{\mathcal{Z}}^{\mathcal{X}}(M) \\
\uparrow K((\text{copr})^* \times \text{id}) \\
K(\partial_u^{\mathcal{Y}}M) \times K_{\mathcal{Z}}^{\mathcal{X}}(M) \\
\downarrow \partial \times \text{id} \\
\Sigma K(\mathcal{Y}) \times K_{\mathcal{Z}}^{\mathcal{X}}(M) \\
\downarrow \\
\Sigma K(C_u(\mathcal{Y}) \otimes \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(M))) \\
\downarrow \\
\Sigma K\left(\frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M))}{\mathbf{C}(\mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z}) \subseteq \mathcal{O}^\infty(M))}\right) \\
\downarrow \\
\Sigma K(\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M)))
\end{array}
\begin{array}{c}
\downarrow \\
K(C(\partial^{\mathcal{O}^\infty(\mathcal{Y})}\mathcal{O}^\infty(M)) \otimes \mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(M))) \\
\downarrow K(\nu) \\
K\left(\frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(M))}{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M))}\right) \\
\downarrow \\
K\left(\frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(M) \subseteq \mathcal{O}^\infty(M))}{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M))}\right) \\
\downarrow \partial \\
K\left(\frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(M) \subseteq \mathcal{O}^\infty(M))}{\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M)}\right) \\
\downarrow \partial \\
\Sigma K\left(\frac{\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M))}{\mathbf{C}(\mathcal{O}^-(\mathcal{Y} \cap \mathcal{Z}) \subseteq \mathcal{O}^\infty(M))}\right) \\
\downarrow \partial \\
\Sigma K(\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M)))
\end{array}$$

The left horizontal maps are given by the symmetric monoidal structure of the  $K$ -theory functor for  $C^*$ -categories and the left squares commute by bi-exactness of this structure. Finally, the top right square commutes by functoriality of  $K$ , by definition of the arrow marked “!”.

If we fix  $p$  in  $K_0(\partial_u^{\mathcal{Y}}M)$ , then the up-right-down composition is

$$(c \circ \text{pr})^* p \cap^{\mathcal{X}} - : K_{\mathcal{Z}}^{\mathcal{X}}(M) \rightarrow \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}) \stackrel{\text{def}}{=} \Sigma K(\mathbf{C}(\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) \subseteq \mathcal{O}^\infty(M))).$$

By Lemma 4.48 we get

$$(c \circ \text{pr})^* p \cap^{\mathcal{X}} - \simeq \tilde{\delta}^{MV} : K_{\mathcal{Z}}^{\mathcal{X}}(M) \rightarrow \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}),$$

where  $\tilde{\delta}^{MV}$  is the Mayer-Vietoris boundary for the decomposition of  $\mathcal{O}_{\mathcal{Z}}^\infty(M)$  into  $\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{M}_+)$  and  $\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{M}_-)$ . The down-right composition is

$$\partial p \cap^{\mathcal{X}\sigma} - : K_{\mathcal{Z}}^{\mathcal{X}}(M) \rightarrow \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{O}_{\mathcal{Z}}^\infty(M)) \rightarrow \Sigma K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}).$$

We have a map of squares

$$\begin{array}{ccc}
\mathcal{O}_{\mathcal{Z}}^\infty(N) & \longrightarrow & \mathcal{O}_{\mathcal{Z}}^\infty(M_+) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathcal{Z}}^\infty(M_-) & \longrightarrow & \mathcal{O}_{\mathcal{Z}}^\infty(M)
\end{array}
\rightarrow
\begin{array}{ccc}
\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{Y}) & \longrightarrow & \mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{M}_+) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathcal{Z}}^\infty(\mathcal{M}_-) & \longrightarrow & \mathcal{O}_{\mathcal{Z}}^\infty(M)
\end{array}$$

which implies that  $i^{\mathcal{X}} \circ \delta^{MV} \simeq \tilde{\delta}^{MV}$ . Combining these equivalences we get the assertion of the lemma.  $\square$

We now assume that  $M$  is represented by a complete Riemannian manifold, that  $\tilde{f} : M \rightarrow \mathbb{R}$  is smooth, and that 0 is a regular value. Then  $N$  is an embedded codimension-one submanifold of  $M$  with trivial normal bundle which separates  $M$  into the two components  $M_-$  and  $M_+$ . We equip  $N$  with the induced Riemannian metric. Then the inclusion  $i : N \rightarrow M$  is a morphism of bornological coarse spaces. We assume that  $N$  has a product tubular neighbourhood of uniform width.

If  $\mathcal{D}_0$  is a generalized Dirac operator on  $M$  which is positive away from a big family  $\mathcal{Z}$  with symbol  $\sigma_{\mathcal{Z}}(\mathcal{D}_0)$  in  $K_{\mathcal{Z}}^{\mathcal{X}}(M)$ , then by Theorem 4.44 we know that

$$\delta^{MV}(\sigma_{\mathcal{Z}}(\mathcal{D}_0)) = \beta^{-1} \sigma_{N \cap \mathcal{Z}}(\mathcal{D}_{0|N})$$

in  $K_{\mathcal{Z}}^{\mathcal{X}}(N)$  provided that  $\mathcal{D}_0$  has a product structure on the product tubular neighbourhood of uniform width and  $\mathcal{D}_{0|N}$  is also positive away from  $N \cap \mathcal{Z}$ . We can conclude a coarse version of Roe's partitioned manifold index theorem [Roe88].

**Corollary 4.50.** *Under the assumption made above, we have an equality*

$$i(\text{ind}\mathcal{X}(\mathcal{D}_{0|N}, \text{on } N \cap \mathcal{Z})) \simeq \beta \cdot \partial^{MV} \text{ind}\mathcal{X}(\mathcal{D}_0, \text{on } \mathcal{Z})$$

in  $K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z})$ .

*Proof.* Since the index map Definition 2.40 is a natural transformation between  $\mathbf{Mod}(KU)$ -valued excisive functors it commutes with Mayer-Vietoris maps. This is used in the marked equality below. We calculate

$$\begin{aligned} i(\text{ind}\mathcal{X}(\mathcal{D}_{0|N}, \text{on } N \cap \mathcal{Z})) &\stackrel{\text{Lemma 4.17}}{=} i(a_{N, \mathcal{Z}}(\sigma_{N \cap \mathcal{Z}}(\mathcal{D}_{0|N}))) \\ &\stackrel{\text{Theorem 4.44}}{=} \beta \cdot i(a_{N, \mathcal{Z}}(\delta^{MV} \sigma_{\mathcal{Z}}(\mathcal{D}_0))) \\ &\stackrel{!}{=} \beta \cdot \partial^{MV} a_{M, \mathcal{Z}}(\sigma_{\mathcal{Z}}(\mathcal{D}_0)) \\ &\stackrel{\text{Lemma 4.17}}{=} \beta \cdot \partial^{MV} \text{ind}\mathcal{X}(\mathcal{D}_0, \text{on } \mathcal{Z}) . \end{aligned}$$

$\square$

**Remark 4.51.** Corollary 4.50 leads to the usual application to obstructions against positive scalar curvature in the coarse equivalence class of  $M$ . We assume that  $i : N \rightarrow M$  is a coarse embedding with a uniform tubular neighbourhood along  $N$ . Then  $i : K\mathcal{X}(N) \rightarrow K\mathcal{X}(\mathcal{Y})$  is an equivalence. Assume that  $M$  is spin and let  $\mathcal{D}_0$  be the spin Dirac operator. Then  $\mathcal{D}_{0|N}$  is again the spin Dirac operator on  $N$ . If  $\text{ind}\mathcal{X}(\mathcal{D}_{0|N}) \neq 0$ , then by Corollary 4.50 and since  $i$  is an equivalence also  $\text{ind}\mathcal{X}(\mathcal{D}_0) \neq 0$ .

We call two complete Riemannian metrics  $g, g'$  on  $M$  coarsely equivalent if  $(M, g)$  and  $(M, g')$  represent the same bornological coarse space. If  $g'$  is coarsely equivalent to  $g$  and  $\mathcal{D}'_0$  is the associated Dirac operator, then  $\text{ind}\mathcal{X}(\mathcal{D}_0) = \text{ind}\mathcal{X}(\mathcal{D}'_0)$ . We use here that two coarsely equivalent metrics induce the same index classes for the spin Dirac operators. This can again be shown similarly as Proposition 4.13 using the suspension theorem and the relative index theorem.

If  $\text{ind}\mathcal{X}(\mathcal{D}_{0|N}) \neq 0$ , we can conclude that  $M$  can not have a metric  $g'$  with uniformly positive scalar curvature in the coarse equivalence class of  $M$ . It is important to note here that it is not required that  $g'$  has a uniform tube at  $N$ , or even a product structure at  $N$  at all. Otherwise the conclusion would be trivial since then the metric on  $N$  induced by  $g'$  has uniformly positive scalar curvature and therefore  $\text{ind}\mathcal{X}(\mathcal{D}_{0|N}) = 0$ .

Using the supports we obtain a new type of obstruction. We say that a class in  $K\mathcal{X}(N)$  comes from the big family  $\mathcal{Y}$  in  $N$  if it belongs to the image of  $K\mathcal{X}(\mathcal{Y}) \rightarrow K\mathcal{X}(N)$ .

Let  $\mathcal{Z}$  be a big family in  $M$ .

**Corollary 4.52.** *If  $\text{ind}\mathcal{X}(\mathcal{D}_{0|N})$  does not come from  $N \cap \mathcal{Z}$ , then there is no complete metric  $g'$  in the coarse equivalence class of  $g$  of uniform positive scalar curvature on  $M \setminus Z$  for any member  $Z$  of  $\mathcal{Z}$ . ■*

Assume that  $\mathcal{D}_0$  is a generalized Dirac operator of degree  $k$  on a bundle  $E_0 \rightarrow M$  which is positive away from a big family  $\mathcal{Z}$  and that  $\mathcal{D}_{0|N}$  is positive away from  $N \cap \mathcal{Z}$ . We further assume that  $\tilde{f}$  is asymptotically constant away from  $\mathcal{Z}$ . We define  $\Psi_0 : M \rightarrow \text{End}_{\mathbf{Cl}^1}(\mathbf{Cl}^1)$  by  $\Psi_0(m) := \tilde{f}(m)i\sigma$ , where  $\sigma$  in  $\mathbf{Cl}^1$  is the odd, anti-selfadjoint generator with  $\sigma^2 = -1$ . Then  $\Psi_0$  is very positive away from  $\mathcal{Y}$  and asymptotically constant away from  $\mathcal{Z}$ . We define the bundle  $E := E_0 \hat{\otimes} \mathbf{Cl}^1$  of  $\mathbf{Cl}^{k+1}$ -modules,  $\mathcal{D} := \mathcal{D}_0 \hat{\otimes} \text{id}_{\mathbf{Cl}^1}$  and  $\Psi := \text{id}_{E_0} \hat{\otimes} \Psi_0$ .

The function  $\Psi_0$  defines a class  $[\Psi_0]$  in  $K^1(\mathcal{Y})$ . By Theorem 4.32 we have

$$\sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{Y}) = [\Psi_0] \cap^{\mathcal{X}\sigma} \sigma_{\mathcal{Z}}(\mathcal{D}_0) .$$

Combining (2.27), (2.34) and (4.61) we have the well-known formula for the  $K$ -theoretic boundary map

$$[\Psi_0] = \partial p \quad \text{in} \quad K^1(\mathcal{Y}) . \quad (4.69)$$

**Proposition 4.53.** *In  $K\mathcal{X}(\mathcal{Y} \cap \mathcal{Z})$  we have*

$$\text{ind}\mathcal{X}(\mathcal{D} + \Psi, \text{on } \mathcal{Y} \cap \mathcal{Z}) = \beta^{-1}i(\text{ind}\mathcal{X}(\mathcal{D}_{0|N}, \text{on } N \cap \mathcal{Z}))$$

*Proof.* We have the following chain of equalities:

$$\begin{aligned}
\text{ind } \mathcal{X}(\not{D} + \Psi, \text{ on } \mathcal{Y} \cap \mathcal{Z}) &\stackrel{(4.31)}{=} a_{\mathcal{Y}, \mathcal{Z}}(\sigma_{\mathcal{Z}}(\not{D} + \Psi, \text{ on } \mathcal{Y})) \\
&\stackrel{\text{Theorem 4.32, (4.69)}}{=} a_{\mathcal{Y}, \mathcal{Z}}(\partial p \cap^{\mathcal{X}} \sigma_{\mathcal{Z}}(\not{D}_0)) \\
&\stackrel{\text{Proposition 3.43}}{=} p \cap^{\mathcal{X}} a_{M, \mathcal{Z}}(\sigma_{\mathcal{Z}}(\not{D}_0)) \\
&\stackrel{\text{Lemma 4.17}}{=} p \cap^{\mathcal{X}} \text{ind } \mathcal{X}(\not{D}_0, \text{ on } \mathcal{Z}) \\
&\stackrel{\text{Lemma 4.48}}{=} \partial^{MV} \text{ind } \mathcal{X}(\not{D}_0, \text{ on } \mathcal{Z}) \\
&\stackrel{\text{Corollary 4.50}}{=} \beta^{-1} i(\text{ind } \mathcal{X}(\not{D}_{0|N}, \text{ on } N \cap \mathcal{Z})) .
\end{aligned}$$

□

**Remark 4.54.** If  $N$  is compact, then  $K(\mathcal{Y}) \simeq K\mathcal{X}(\ast) \simeq KU$  and  $K\mathcal{X}(N) \simeq KU$  and under these indentifications  $\text{ind } \mathcal{X}(\not{D} + \Psi, \text{ on } \mathcal{Y})$  and  $\text{ind } \mathcal{X}(\not{D}_{0|N})$  are the usual Fredholm indices. If  $k+1$  is even, then the equality of indices  $\text{ind}(\not{D} + \Psi) = \text{ind}(\not{D}_{0|N})$  in  $KU_{-k-1}$  is a special case of Roe's partitioned manifold index theorem [Roe88], and also of the index theorem for Callias type operators shown in [Bun95]. ■

## 4.7 The multi-partitioned index theorem

Let  $M$  be a uniform bornological coarse space with a big family  $\mathcal{Z}$ . Assume that we have a finite family  $\tilde{f} := (\tilde{f}_i)_{i=1, \dots, n}$  of uniformly continuous and controlled functions  $\tilde{f}_i : M \rightarrow \mathbb{R}$ . Then we can iterate the constructions from Section 4.6. To  $\tilde{f}_i$  we associate the big families  $\mathcal{M}_{i, \pm}$ , as in (4.62),  $\mathcal{Y}_i$ , and the class and  $p_i$  in  $K^0(\partial_u^{\mathcal{Y}_i} X)$ . For a subset  $I$  of the poset  $\langle n \rangle := \{1, \dots, n\}$ , we further define

$$\mathcal{Y}_I := \bigcap_{i \in I} \mathcal{Y}_i .$$

If  $i \notin I$ , then the decomposition  $(\mathcal{Y}_I \cap \mathcal{M}_{i, -}, \mathcal{Y}_I \cap \mathcal{M}_{i, +})$  of  $\mathcal{Y}_I$  induces a Mayer-Vietoris boundary

$$\partial_i^{MV} : \Sigma^{|I|} K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}_I) \rightarrow \Sigma^{|I|+1} K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}_{I \cup \{i\}}) .$$

We avoid to use the embeddings (4.67) by considering excision with respect to the decomposition into two big families instead to the closed subspaces themselves, see e.g. the right triangle in (4.19). We have  $\partial p_i \in K^1(\mathcal{Y}_i)$  and can form the product

$$\partial p_{i_k} \cup \dots \cup \partial p_{i_1} \quad \text{in} \quad K^{|I|}(\mathcal{Y}_I) , \quad \text{for} \quad I = \{i_1 < \dots < i_k\} .$$

By an iterated application of Lemma 3.46 and the version of Lemma 4.49 for  $M$  replaced by a big family we get:

**Corollary 4.55.** For  $I = \{i_1 < \dots < i_k\} \subseteq \langle n \rangle$ , we have an equivalence of functors

$$(\partial p_{i_k} \cup \dots \cup \partial p_{i_1}) \cap^{\mathcal{X}\sigma} \simeq \partial_{i_k}^{MV} \circ \dots \circ \partial_{i_1}^{MV} : K_{\mathcal{Z}}^{\mathcal{X}}(X) \rightarrow \Sigma^{|I|} K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{Y}_I) .$$

Let  $\mathcal{W}$  be a big family on  $M$  with the property  $\tilde{f}(\mathcal{W}) \subseteq \mathcal{B}_{\mathbb{R}^n}$  so that  $\tilde{f}$  becomes a morphism  $\tilde{f} : (M, \mathcal{W}) \rightarrow (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$  in  $\mathbf{BC}^{(2)}$ . We then get an induced map

$$\partial \tilde{f} : \partial^{\mathcal{W}} M \rightarrow \partial_h \mathbb{R}^n \rightarrow S^{n-1} .$$

If we perform the constructions above with the coordinate functions  $(x_i)_{i=1, \dots, n}$  of  $\mathbb{R}^n$  instead of  $(f_i)_{i=1, \dots, n}$ , then we get big families  $\tilde{\mathcal{Y}}_i$  on  $\mathbb{R}^n$  and classes  $\partial \tilde{p}_i \in K^1(\tilde{\mathcal{Y}}_i)$  such that  $(\partial \tilde{p}_n \cup \dots \cup \partial \tilde{p}_1) \in K^n(\tilde{\mathcal{Y}}_{\langle n \rangle}) \cong K_c^n(\mathbb{R}^n) \cong \mathbb{Z}$  is a generator. In view of the well-understood long exact sequence in  $K$ -theory

$$K_c^*(\mathbb{R}^n) \rightarrow K^*(\mathbb{R}^n) \rightarrow K^*(S^{n-1}) \xrightarrow{\partial} K_c^{*+1}(\mathbb{R}^n)$$

there exists a unique class  $\tilde{u}$  in  $K^{n-1}(S^{n-1})$  such that  $\tilde{u}|_* = 0$  (i.e.,  $\tilde{u}$  is in the reduced summand) and  $\partial \tilde{u} = (\partial \tilde{p}_n \cup \dots \cup \partial \tilde{p}_1)$ . We set  $u := (\partial \tilde{f})^* \tilde{u}$  in  $K^n(\mathcal{Y}_{\langle n \rangle})$  and conclude that  $\partial u = (\partial p_n \cup \dots \cup \partial p_1)$ .

**Corollary 4.56.** The following square commutes:

$$\begin{array}{ccc} K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{W}) & \xrightarrow{\partial_n^{MV} \circ \dots \circ \partial_1^{MV}} & \Sigma^n K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{W} \cap \mathcal{Y}_{\langle n \rangle}) \\ \downarrow a_{\mathcal{W}, \mathcal{Z}} & & \downarrow a_{\mathcal{W} \cap \mathcal{Y}_{\langle n \rangle}, \mathcal{Z}} \\ K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{W} \cap \mathcal{Z}) & \xrightarrow{u \cap^{\mathcal{X}}} & \Sigma^n K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{W} \cap \mathcal{Y}_{\langle n \rangle} \cap \mathcal{Z}) \end{array} .$$

We assume that  $M$  comes from a complete Riemannian manifold. We consider a Callias type generalized Dirac operator  $\mathcal{D} + \Psi$  such that  $\Psi$  is very positive away from the big family  $\mathcal{W}$  and asymptotically constant away from the big family  $\mathcal{Z}$ , and  $\mathcal{D}$  is positive away from  $\mathcal{Z}$ . We assume that  $\tilde{f}$  is smooth and has 0 in  $\mathbb{R}^n$  as a regular value, and we set  $N := \tilde{f}^{-1}(\{0\})$ .

**Corollary 4.57.** If  $N$  has a tubular neighbourhood of uniform width with product structures, then in  $K_{\mathcal{Z}}^{\mathcal{X}}(\mathcal{W} \cap \mathcal{Y}_{\langle n \rangle} \cap \mathcal{Z})$  we have the equality

$$u \cap^{\mathcal{X}} \text{ind} \mathcal{X}(\mathcal{D} + \Psi, \text{on } \mathcal{W} \cap \mathcal{Z}) \simeq \beta^{-n} i(\text{ind} \mathcal{X}((\mathcal{D} + \Psi)|_N, \text{on } N \cap \mathcal{W} \cap \mathcal{Z})) .$$

*Proof.* We have

$$\begin{aligned}
u \cap^{\mathcal{X}} \text{ind} \mathcal{X}(\mathcal{D} + \Psi, \text{on } \mathcal{W} \cap \mathcal{Z}) &\stackrel{(4.31)}{=} u \cap^{\mathcal{X}} a_{\mathcal{W}, \mathcal{Z}}(\sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{W})) \\
&\stackrel{\text{Corollary 4.56}}{=} a_{\mathcal{W} \cap \mathcal{Y}_{\langle n \rangle}, \mathcal{Z}}(\partial_n^{MV} \circ \dots \circ \partial_1^{MV} \sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{W})) \\
&= a_{\mathcal{W} \cap \mathcal{Y}_{\langle n \rangle}, \mathcal{Z}}(i^{\mathcal{X}} \delta_n^{MV} \circ \dots \circ \delta_1^{MV} \sigma_{\mathcal{Z}}(\mathcal{D} + \Psi, \text{on } \mathcal{W})) \\
&\stackrel{\text{Theorem 4.44}}{=} \beta^{-n} a_{\mathcal{W} \cap \mathcal{Y}_{1, \dots, n}, \mathcal{Z}}(i^{\mathcal{X}} \sigma_{N \cap \mathcal{Z}}((\mathcal{D} + \Psi)|_N, \text{on } N \cap \mathcal{W} \cap \mathcal{Z})) \\
&\stackrel{(4.68)}{=} \beta^{-n} i a_{N \cap \mathcal{W}, \mathcal{Z}}(\sigma_{N \cap \mathcal{Z}}((\mathcal{D} + \Psi)|_N, \text{on } N \cap \mathcal{W} \cap \mathcal{Z})) \\
&\stackrel{(4.31)}{=} \beta^{-n} i(\text{ind} \mathcal{X}((\mathcal{D} + \Psi)|_N, \text{on } N \cap \mathcal{W} \cap \mathcal{Z})) .
\end{aligned}$$

□

**Example 4.58.** We assume that  $\mathcal{W} \cap \mathcal{Y}_{\langle n \rangle} \cap \mathcal{Z} \subseteq \mathcal{B}$ . Then for every member  $W$  in  $\mathcal{W} \cap \mathcal{Y}_{\langle n \rangle} \cap \mathcal{Z}$  the restriction  $\tilde{f}_W : W \rightarrow \mathbb{R}^n$  is proper. Consequently we get a map

$$t_{\tilde{f}} : K \mathcal{X}(\mathcal{W} \cap \mathcal{Y}_{\langle n \rangle} \cap \mathcal{Z}) \xrightarrow{\tilde{f}_*} K \mathcal{X}(\mathbb{R}^n) \xrightarrow{\tilde{u} \cap^{\mathcal{X}} -} \Sigma^n K \mathcal{X}(\mathcal{B}_{\mathbb{R}^n}) \simeq \Sigma^n KU \quad (4.70)$$

If  $\mathcal{D} + \Psi$  is a Callias type operator of degree  $k$  as above, then following [SZ18, Def. 1.3] we define the multipartitioned index by

$$t_{\tilde{f}}(u \cap^{\mathcal{X}} \text{ind} \mathcal{X}(\mathcal{D} + \Psi, \text{on } \mathcal{W} \cap \mathcal{Z})) \in \pi_{-n-k} KU .$$

We furthermore have  $N \cap \mathcal{W} \cap \mathcal{Z} \subseteq N \cap \mathcal{B}$  and therefore

$$\tilde{f}_{|N, *} \text{ind} \mathcal{X}((\mathcal{D} + \Psi)|_N, \text{on } N \cap \mathcal{W} \cap \mathcal{Z}) \in K \mathcal{X}_{n-k}(\mathcal{B}_{\mathbb{R}^n}) \simeq \pi_{n-k} KU .$$

The following consequence of Corollary 4.57 is a generalization of the multi-partition index theorem [Zei17, Thm. 2.7], [SZ18, Thm. 1.4].

**Corollary 4.59.** *Under the assumptions of Corollary 4.57 we have*

$$t_{\tilde{f}}(u \cap^{\mathcal{X}} \text{ind} \mathcal{X}(\mathcal{D} + \Psi, \text{on } \mathcal{W} \cap \mathcal{Z})) = \beta^{-n} \tilde{f}_{|N, *} \text{ind} \mathcal{X}((\mathcal{D} + \Psi)|_N, \text{on } N \cap \mathcal{W} \cap \mathcal{Z}) .$$

## 4.8 Slant products

In this section we interpret the main construction from [EWZ22]. Let  $X, Y$  be two uniform bornological coarse spaces. Let  $\mathcal{B}_Y$  be the bornology of  $Y$  and consider the big family  $X \times \mathcal{B}_Y$  on  $X \otimes Y$ . Pull-back of functions along the projection  $\text{pr} : X \times Y \rightarrow Y$  induces a map of instances of the exact sequence of  $C^*$ -algebras (3.8)

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_u(\mathcal{B}_Y) & \longrightarrow & C_{u, \mathcal{B}_Y}(Y) & \longrightarrow & C(\partial_u^{\mathcal{B}_Y} Y) \longrightarrow 0 . \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_u(X \otimes \mathcal{B}_Y) & \longrightarrow & C_{u, X \otimes \mathcal{B}_Y}(X \otimes Y) & \longrightarrow & C(\partial_u^{X \otimes \mathcal{B}_Y}(X \otimes Y)) \longrightarrow 0
\end{array}$$

This gives a commutative square

$$\begin{array}{ccc} K(\partial_u^{\mathcal{B}_Y} Y) & \xrightarrow{\partial} & \Sigma K(\mathcal{B}_Y) \\ \downarrow \partial_h \text{pr}^* & & \downarrow \text{pr}^* \\ K(\partial_u^{X \otimes \mathcal{B}_Y} (X \otimes Y)) & \xrightarrow{\partial} & \Sigma K(X \otimes \mathcal{B}_Y) \end{array} .$$

If  $\theta$  is a class in  $K_k(\partial_h^{\mathcal{B}_Y} Y)$ , then by Proposition 3.43 we get a commutative diagram

$$\begin{array}{ccccc} K^{\mathcal{X}}(X \otimes Y) & \xrightarrow{\text{pr}^* \partial \theta \cap \mathcal{X}^{\sigma_-}} & \Sigma^{k+1} K^{\mathcal{X}}(X \otimes \mathcal{B}_Y) & \xrightarrow{\kappa^\sigma} & \Sigma^{k+1} K^{\mathcal{X}}(X) \\ \downarrow a_{X \otimes Y} & & \downarrow a_{X \otimes \mathcal{B}_Y} & & \downarrow a_X \\ K^{\mathcal{X}}(X \otimes Y) & \xrightarrow{(\partial_h \text{pr})^* \theta \cap \mathcal{X}^-} & \Sigma^{k+1} K^{\mathcal{X}}(X \otimes \mathcal{B}_Y) & \xrightarrow{\kappa} & \Sigma^{k+1} K^{\mathcal{X}}(X) \end{array} .$$

The maps  $\kappa$  and  $\kappa^\sigma$  are well-defined since the restriction of  $\text{pr}$  to any member of  $X \otimes \mathcal{B}_Y$  is proper and hence a morphism in **UBC** and **BC**. The horizontal compositions are the slant products first defined in [EWZ22] and denoted there by  $/\theta$ . The fact that the outer square commutes yields a map of fibre sequences

$$\begin{array}{ccccc} \text{Fib}(a_{X \otimes Y}) & \longrightarrow & K^{\mathcal{X}}(X \otimes Y) & \xrightarrow{a_{X \otimes Y}} & K^{\mathcal{X}}(X \otimes Y) \\ \downarrow \text{dotted} & & \downarrow & & \downarrow \\ \Sigma^{k+1} \text{Fib}(a_X) & \longrightarrow & \Sigma^{k+1} K^{\mathcal{X}}(X) & \xrightarrow{a_X} & \Sigma^{k+1} K^{\mathcal{X}}(X) \end{array} ,$$

where the dotted arrow is determined by the universal property of the fibre. The commutativity expresses the compatibility [EWZ22, Thm. 4.10] of the slant product with the Higson-Roe sequence, see Remark 3.27. Most of the formal properties of the slant product stated in [EWZ22] are consequences of the formal properties of the pairings stated in Section 3.4 and Section 3.5.

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