ON THE GEOMETRY OF STAR DOMAINS AND THE SPECTRA OF HODGE–LAPLACE OPERATORS UNDER DOMAIN APPROXIMATION

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ABSTRACT. We study the eigenvalues of Hodge–Laplace operators over bounded convex domains. Generalizing contributions by Guerini and Savo, we show that the Poincaré– Friedrichs constants of the Sobolev de Rham complexes increase with the degree of the forms. We establish that the spectra of Hodge–Laplace operators converge when the domain is approximated by a sequence of convex domains up to a bi-Lipschitz deformation of the boundary. As preparatory work with independent relevance, we study the gauge function and the reciprocal radial function of convex sets, providing new proofs for Lipschitz estimates by Vrećica and Toranzos for the reciprocal radial function and improving a Lipschitz estimate for the gauge function due to Beer.

1. INTRODUCTION

The spectrum of Hodge–Laplace operators in the L^2 de Rham complex has been extensively studied. Lower bounds for the eigenvalues of the Hodge–Laplace operators follow from upper bounds for the constants in Poincaré–Friedrichs inequalities of the exterior derivative operator. For example, the Poincaré–Friedrichs constants for the gradient and divergence operators correspond to the eigenvalues of the Neumann and Dirichlet Laplacians, respectively. We recall that the Poincaré–Friedrichs inequality holds for some domain Ω if for every $u \in H\Lambda^k(\Omega)$ there exists $w \in H\Lambda^k(\Omega)$ such that dw = du and

(1)
$$||w||_{L^2(\Omega)} \le C_{\mathrm{PF},k} ||du||_{L^2(\Omega)}$$

Estimates for the best possible Poincaré–Friedrichs inequality are of general interest because they characterize the stability of numerous partial differential equations in vector calculus and exterior calculus. The Poincaré–Friedrichs inequality for scalar functions, known as Poincaré inequality, has traditionally been a focus of study.

When the domain is convex, then the optimal Poincaré constants proportional to the diameter are well-known [31, 6]. More generally, the lowest non-zero eigenvalue of the Neumann–Laplacian, whose inverted square root is just the Poincaré constant for the gradient operator, already provides a lower bound for the spectra of Hodge–Laplace operators on k-forms with magnetic boundary conditions [5, 30] and the lowest eigenvalue of the Dirichlet–Laplacian [33]. In the Euclidean setting, the Hodge–Laplacian acts as a componentwise Laplacian but with boundary conditions that render an analysis via the exterior derivative more amenable. We also know explicit upper and lower bounds for the the lowest eigenvalues of Hodge–Laplace operators over convex domains [35, 27]. Guerini and Savo [22] have refined the comparison of different minimal eigenvalues of the Hodge–Laplace operators when the convex domain has smooth boundary: the smallest non-zero eigenvalues of the Hodge–Laplace operator are increasing in k, and if the domain is strictly convex, then they are even strictly increasing. However, their argument relies on the smoothness of the domain

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because they leverage curvature terms along the boundary. The modest primary goal of this work is to extend these observations: we establish that the Poincaré–Friedrichs constants are increasing in k even over non-smooth convex sets. In fact, we prove the convergence of the spectra of Hodge–Laplace operators under a bi-Lipschitz class of perturbations of the domain.

The main argument uses an approximation of the convex domain via strictly convex domains with smooth boundary. We therefore review pertinent elements of convex geometry and make some contributions of separate interest. In fact, we begin with the study of star domains. Roughly speaking, star domains are open bounded sets in Euclidean space whose entire boundary is visible from every point within an open subset. We study two scalar functions associated with a star domain Ω . The first is known as gauge function in the literature:

$$\boldsymbol{g}: \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \inf\{s > 0 \mid x \in s\Omega\}$$

The second seems to be new and equals the reciprocal of what is known as the radial function [7, 29]:

$$s: \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \|x\| \sup\left\{t > 0 \mid t \frac{x}{\|x\|} \in \Omega\right\}.$$

In our interpretation, the former describes how to transform the star domain Ω onto the open unit ball, whereas the latter describes how to transform in the opposite direction. We particularly emphasize their Lipschitz constants. These efforts reproduce results by Toranzos [36] and Vrećica [37] with new proofs. Moreover, we improve Beer's Lipschitz estimate for the gauge function [7] by a factor of two.

Eventually, this brings forth approximations of convex domains via a sequence of smoothly bounded strictly convex such that the gauge functions and reciprocal radial function converge as well. We construct bi-Lipschitz transformations from the original domain onto the approximating domains, ensuring that the Lipschitz constants remain uniformly bounded and that the transformations equals the identity away from an ever more narrowing band along the boundary. The convergence of the reciprocal radial function then implies the convergence of not only the Poincaré–Friedrichs constants of the Hodge–Laplace operators but also the sequences of eigenvalues. Here, the domain's convexity guarantees uniform bounds for the Poincaré–Friedrichs inequalities, Gaffney inequalities, and Sobolev–Poincaré inequalities.

We present what appears to be the first stability analysis of the Hodge–Laplace spectrum for this practically significant class of domain perturbations. We situate this research in a wider context. The smooth inner approximation of Lipschitz domains in the Hausdorff sense is entrenched in the analysis of boundary value problems [20, 8, 14]. However, we employ a different approximation for convex domains that enables an explicit convergence estimate of Poincaré–Friedrichs constants. This approach also contrasts with common approximations of Laplacians and Hodge–Laplacians over surfaces, where the transformations between the physical surface and the approximate surfaces have Lipschitz constants converging to one [15, 26, 11]. Furthermore, there is longstanding research on the spectrum of the Hodge–Laplace operator over manifolds when the Riemannian metric is subject to small continuous perturbations [4, 13]. By contrast, we examine the spectral convergence of the Hodge–Laplace operator between domains connected through bilipschitz transformations whose continuity constants are merely assumed to be uniformly bounded but which coincide with the identity on ever larger subdomains. Pulling back the (Euclidean) metric from the smooth domains onto the original convex domains, we can equivalently study the spectrum under essentially bounded perturbations of the Euclidean Riemannian metric over the domain localized towards the boundary.

This spectral convergence result is of evident theoretical interest in its own right. We deem it helpful in generalizing results over smooth convex domains to general convex domains. The stability of partial differential equations and the spectra of differential operators under domain perturbations is highly relevant to theoretical and computational engineering, where the calculus of differential forms has become an effective and popular tool [12, 3, 25]. Taking the present discussion as a starting point, questions for future research include how we can extend the spectral convergence estimates to general Lipschitz domains and how we can generalize the Guerini–Savo inequalities to L^p de Rham complexes.

This manuscript is organized as follows. We establish geometric results on convex domains in Section 2. We discuss the L^2 de Rham complex, the Hodge–Laplace operator and their spectra, and Poincaré–Friedrichs inequalities in Section 3. The main result is proven in Section 4. We finish with a translation of these results into vector calculus in Section 5, which we believe to be of particular interest for mathematical electromagnetism.

Notation. We use standard notation in this article. In particular, Lip(f) denotes the smallest Lipschitz constant, if finite, of any function $f: X \to Y$ over its domain X.

2. Domains and their convex kernel

This section discusses domains that are star-shaped with respect to a ball. This includes convex domains, which are the most important special case.

We use the following common notation: $B_{\rho}(z) \subseteq \mathbb{R}^n$ denotes the open ball with radius ρ centered at z, and $S_{\rho}(z)$ denotes the boundary of that ball, which is the sphere of radius ρ centered at z.

We say that a set $\mathcal{X} \subseteq \mathbb{R}^n$ is star-shaped with respect to $x_0 \in \mathcal{X}$ if for each $x \in \mathcal{X}$ and each $\lambda \in [0, 1]$ we have $(1 - \lambda)x + \lambda x_0 \in \mathcal{X}$. The set \mathcal{X} is called star-shaped with respect to some non-empty subset $\mathcal{B} \subseteq \mathcal{X}$ if \mathcal{X} is star-shaped with respect to every $x_0 \in \mathcal{B}$. A domain is convex if it is star-shaped with respect to each of its points. The convex kernel of a set \mathcal{X} is the largest subset \mathcal{B} with respect to which \mathcal{X} is star-shaped. One easily verifies that the convex kernel is convex and that every convex sets agrees with its convex kernel. A domain whose convex kernel contains an open set is also called a star domain. For convenience, we call a star domain centered if it contains an open set around the origin. Obviously, any star domain can be shifted onto a centered star domain. See Figure 1 for an illustration.

In what follows, we let $\Omega \subseteq \mathbb{R}^n$ be an open set.

2.1. Star domains as graph domains. Domains that are star-shaped with respect to a ball are Lipschitz domains, meaning that their boundary locally looks like the graph of a Lipschitz function, possibly after translation and rotation. We formally prove that observation.

Lemma 2.1. Suppose that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain that is star-shaped with respect to a ball B. Then Ω is a Lipschitz domain.

Proof. Without loss of generality, we assume that Ω is star-shaped with respect to a ball $B = B_{\rho}(0)$ and is itself contained within a ball of radius $R > \rho$ around the origin.



FIGURE 1. An example for domain that is star-shaped with respect to an interior ball, centered at x_0 and of radius ρ , and which is contained within some ball of radius R. Note that these two balls are not necessarily concentric.

Let $x \in \partial \Omega$. Without loss of generality, assume that x lies on the first coordinate axis. We let $H \subseteq \mathbb{R}^n$ be the coordinate hyperplane that is orthogonal to the first coordinate axis. We write $D_t(z) := H \cap B_t(z)$ for the disk of radius t around any point $z \in H$ relatively open within the hyperplane H.

For every $z \in D_{\rho}(0)$, there exists exactly one $x(z) \in \partial \Omega$ that lies on same side of H as x and whose orthogonal projection onto H equals z. By assumption, x(0) = x. The desired result follows if we show that x(z) is Lipschitz in z over $D_{\rho/2}(0)$.

Because Ω is star-shaped with respect to the ball $B_{\rho}(z)$ for any $z \in B_{\rho}(0)$, the convex closure of any $x \in \partial \Omega$ and $\overline{B_{\rho/2}(z)}$ is contained within $\overline{\Omega}$. Near x, this convex closure looks like a cone with angle $\gamma \in (0, \pi/2)$ that satisfies

$$\gamma \leq \gamma^* := \arctan\left(\frac{\rho}{2R}\right).$$

We define the open cylinder $Z = D_{\rho/2}(0) \times (0, 2R)$ and the cone

$$K(\gamma) := \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 < 0, \ x_2^2 + \dots + x_n^2 < \tan^2(\gamma) |x_1|^2 \right\}.$$

For all $z \in D_{\rho/2}(0)$ we have

$$(x(z) + K(\gamma)) \cap Z \subseteq \overline{\Omega}.$$

This means that x(z) is Lipschitz over $D_{\rho/2}$ with Lipschitz constant

$$L \le \tan\left(\frac{\pi}{2} - \gamma^*\right) = \cot(\gamma^*) = \frac{2R}{\rho}.$$

The proof is complete.

The radial projection onto the unit sphere from the boundary of a star domain onto the unit sphere defines a bi-Lipschitz transformation. When the unit sphere carries the arc metric and the domain boundary carries the Euclidean metric, then the following estimates hold. They are attained when the star domain is an open ball.

Lemma 2.2. Let Ω be a domain that is star-shaped with respect to the open ball of radius ρ around the origin. Suppose furthermore that $B_r(0) \subseteq \Omega \subseteq B_R(0)$. If $x, y \in \partial\Omega$ and $\alpha \in [0, \pi]$



FIGURE 2. Illustration of the geometric situation in the proof of Lemma 2.2. The angle γ is the angle of convex cone spanned by x and the interior ball $B_{\rho}(0)$.

is the angle between x and y, then

(2)
$$r\frac{2}{\pi}\boldsymbol{\alpha} \le \|\boldsymbol{x} - \boldsymbol{y}\| \le \frac{R^2}{\rho}\boldsymbol{\alpha}$$

Proof. Let $x, y \in \overline{\Omega}$ be distinct. If x and y lie on the same line through the origin, then

$$\boldsymbol{\alpha} = \frac{\pi}{2} \le \frac{\pi}{2} \frac{1}{r} \| \boldsymbol{x} - \boldsymbol{y} \|.$$

If x and y have an angle $\alpha \in (0, \pi)$, then the distance between the radial projections of x and y onto the sphere of radius r is at most ||x - y||. By the cosine law and the lower estimate for $\sqrt{2 - 2\cos(\alpha)}$ over $(0, \pi)$ we have

$$||x-y|| \ge r\sqrt{2-2\cos(\alpha)} \ge r\frac{2}{\pi}\alpha.$$

This proves the lower estimates.

Since Ω is star-shaped with respect to a ball of radius ρ around the origin, we know that the convex closure of x and $B_{\rho}(0)$ has an angle γ at x that satisfies

$$\frac{\rho}{R} \le \sin(\gamma) \le \frac{\rho}{r}.$$

Consider the triangle between x, y, and the origin. Within that triangle, we let $\beta > 0$ be the angle at x, and we again let $\alpha > 0$ be the angle at the origin. Let us assume for the time being that $\alpha \leq \pi/2 - \gamma$. It must hold that $\beta \geq \gamma$. By the law of sines,

$$\|x-y\| = \frac{\sin(\boldsymbol{\alpha})}{\sin(\beta)} \|y\| \le \frac{\sin(\boldsymbol{\alpha})}{\sin(\gamma)} \|y\| \le \frac{\sin(\boldsymbol{\alpha})}{\sin(\gamma)} R \le \frac{R^2}{\rho} \sin(\boldsymbol{\alpha}) \le \frac{R^2}{\rho} \boldsymbol{\alpha}$$

Let us now dispense with the assumption of an upper bound on $\boldsymbol{\alpha}$. Given any $x, y \in \partial\Omega$, we choose a sequence $x_0, \ldots, x_m \in \partial\Omega$ with $x_0 = x$ and $x_m = y$ within the two-dimensional subspace spanned by x and y such that the angles between successive members are all $\boldsymbol{\alpha}/m$. For m large enough, we estimate the distance between consecutive points and sum up the results. We now use the reverse triangle inequality. The desired result follows.

Star domains are spherical Lipschitz graph domains. We estimate the Lipschitz constant of the graph function when the unit sphere carries the arc length metric. This was shown by Toranzos [36, Theorem 1], and we hope the reader may deem our proof a simplification.



FIGURE 3. Illustration of the geometric situation in the proof of Lemma 2.3

Lemma 2.3. Let $\Omega \subseteq \mathbb{R}^n$ that is contained in the ball $B_R(0)$ and is star-shaped with respect to a ball $B_\rho(0)$. Then for all $x, y \in \partial \Omega$ with angle $\alpha \in [0, \pi]$ we have

(3)
$$\left| \|x\| - \|y\| \right| \le \frac{R}{\rho} \sqrt{R^2 - \rho^2} \cdot \boldsymbol{\alpha}.$$

Proof. We let $\boldsymbol{\alpha}^* \in (0, \pi/2)$ be defined by $\cos(\boldsymbol{\alpha}^*) = \rho/R$. Let $x, y \in \partial\Omega$ with $||x|| \ge ||y||$. Write $\boldsymbol{\alpha} \in [0, \pi]$ for the angle between x and y. We assume for the time being that $\boldsymbol{\alpha} \le \boldsymbol{\alpha}^*$.

We let $y' \in \Omega$ be the positive multiple of y which lies on the boundary of the convex closure of x and $B_{\rho}(0)$. Then $||x|| - ||y|| \le ||x|| - ||y'||$. It is evident that

$$||x|| = \rho \operatorname{sec}(\boldsymbol{\alpha}^*), \quad ||y|| \le \rho \operatorname{sec}(\boldsymbol{\alpha}^* - \boldsymbol{\alpha}).$$

The derivative of the secant $\sec'(t) = \frac{\sin(t)}{\cos^2(t)}$ is increasing over $(0, \pi/2)$. Hence,

$$||x|| - ||y|| \le \rho \sec'(\boldsymbol{\alpha}^*) \cdot \boldsymbol{\alpha}$$

We know from the geometric setting that

$$R\sin(\boldsymbol{\alpha}^*) = \sqrt{R^2 - \rho^2}, \quad \cos(\boldsymbol{\alpha}^*) = \rho/R.$$

This shows the desired claim for small angles α . Finally, recall that we have assumed that α is sufficiently small. In the general case, we partition the spherical arc from x to y into segments of equal length, and use the triangle inequality. The proof is complete.

2.2. **Positively homogeneous scalars.** We have developed transformations from the unit sphere onto the boundary of star domains. We want to extend these further to transformations from unit balls onto star domains. These transformations are *radial* in the sense explained below, and we describe their Lipschitz constants.

A function $f : \mathbb{R}^n \to \mathcal{X}$ taking values in a normed space \mathcal{X} is called *positively homogeneous* if for all $x \in \mathbb{R}^n$ and for all $t \in [0, \infty)$ we have

$$f(tx) = tf(x).$$

Several positively homogeneous functions have been studied in the context of convex sets and star domains.

We first review what is known as the gauge function [7] or Minkowski functional. Whenever $\Omega \subseteq \mathbb{R}^n$ is star-shaped domain with respect to the origin, we define the gauge function

$$g: \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \inf \{s > 0 \mid x \in s\Omega \}.$$

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This function is easily seen to be positively homogeneous. We may interpret it as a description how to transform the star domain into the unit ball: as we move along a direction, its value increases linearly from zero until it reaches one at the boundary of the domain.

A complementary definition is what we call expansion function or reciprocal radial function. When $\Omega \subseteq \mathbb{R}^n$ is a bounded star domain, then the expansion function is defined on unit sphere as

$$\boldsymbol{s}: \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \|x\| \sup\left\{t > 0 \mid t \frac{x}{\|x\|} \in \Omega\right\}.$$

Apparently, its restriction to the unit sphere equals

 $\boldsymbol{s}: S^1(0) \to \mathbb{R}, \quad \hat{u} \mapsto \sup \left\{ t > 0 \mid t\hat{u} \in \Omega \right\}.$

Notice that $\mathbf{s}(\hat{u})$ with $\hat{u} \in S_1(0)$ is the norm of the intersection point of $\partial\Omega$ and the halfray from the origin in direction \hat{u} . The expansion function is the positively homogeneous extension of that function to the entire Euclidean space:

$$s: \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \|x\|\psi\left(\frac{x}{\|x\|}\right).$$

Its relation to the gauge function is the identity

$$\boldsymbol{s}(x) = \frac{\|x\|^2}{\boldsymbol{g}(x)}.$$

If the domain is convex, then this is reflected in the gauge function.

Lemma 2.4. If Ω is a convex domain whose interior contains the origin, then its gauge function is convex.

Proof. Suppose that Ω is a convex domain but that \boldsymbol{g} is not convex. Then there exist $t \in [0, 1]$ and $x, y \in \overline{\Omega}$ such that

$$t\boldsymbol{g}(x) + (1-t)\boldsymbol{g}(y) < \boldsymbol{g}\left(tx + (1-t)y\right).$$

Write w := tx + (1 - t)y. After rescaling and using positive homogeneity, we may assume $w \in \partial \Omega$ and so g(w) = 1 without loss of generality. Now, without loss of generality, g(x) < 1.

For all $\epsilon > 0$ there exists $s \in [0, 1]$ and $w' \in (1, \infty)w$ such that $w' = s(1+\epsilon)x + (1-s)y$ and g(w') > g(w). But then $w' \notin \overline{\Omega}$. So $\overline{\Omega}$ is not convex. Since the closure of convex domains is convex, neither is Ω . The desired proof follows by contradiction.

We study the Lipschitz constants of these domains. The following auxiliary lemma bounds the local variation of the magnitude of the boundary points.

Lemma 2.5. Assume that Ω be a bounded star domain that is star-shaped with respect to the ball $B_{\rho}(0)$ and is contained in the ball $B_R(0)$. Let $x \in \partial \Omega$ and let $\gamma \in (0, \pi/2)$ with $\sin(\gamma) = \rho/||x||$. Let $y \in \partial \Omega$ with $||x|| \ge ||y||$ and suppose that the angle α between x and y is at most $\pi/2 - \gamma$. Then

$$||y|| \ge \frac{\sin(\gamma)}{\sin(\gamma + \alpha)} ||x||,$$

where $\gamma \in (0, \pi/2)$ with $\sin(\gamma) = ||x||/\rho$.

Proof. Let now $y' \in \overline{\Omega}$ the intersection of the ray from the origin through y with the convex closure of x and $\overline{B_{\rho}(0)}$. Since $\alpha \leq \pi/2 - \gamma$, we know that $\|y\| \geq \|y'\|$. Consider the triangle whose vertices are x, y', and the origin. The angle at x is γ and its opposing side has length $\|y'\|$. The angle at y' is $\pi - \gamma - \alpha$. Using the sine theorem,

$$\|y'\| = \frac{\sin(\gamma)}{\sin(\pi - \gamma - \boldsymbol{\alpha})} \|x\| = \frac{\sin(\gamma)}{\sin(\gamma + \boldsymbol{\alpha})} \|x\|.$$

This shows the auxiliary lemma.

Theorem 2.6. Let Ω be a bounded star domain that is star-shaped with respect to the ball $B_{\rho}(0)$ and is contained in the ball $B_R(0)$. The expansion function s satisfies

(4)
$$\forall x, y \in \mathbb{R}^n : |\boldsymbol{s}(x) - \boldsymbol{s}(y)| \le \frac{R^2}{\rho} ||x - y||$$

The gauge function g satisfies

(5)
$$\forall x, y \in \mathbb{R}^n : |\boldsymbol{g}(x) - \boldsymbol{g}(y)| \le \frac{1}{\rho} ||x - y||.$$

Proof. Let $x, y \in \mathbb{R}^n$. If one of them is zero, then the claim is obvious, so let us assume that both x and y are non-zero. Let $\boldsymbol{\alpha} \in [0, \pi]$ be their angle. If $\boldsymbol{\alpha} = 0$ or $\boldsymbol{\alpha} = \pi$, then

$$|\mathbf{s}(x) - \mathbf{s}(y)| \le R ||x - y||, \quad |\mathbf{g}(x) - \mathbf{g}(y)| \le \frac{1}{\rho} ||x - y||$$

Consider now the case $0 < \alpha < \pi$. Without loss of generality, $||x|| \ge ||y||$. There exists $0 < \gamma < \pi/2$ such that $\sin(\gamma) = \rho/||x||$. We first assume more specifically that $0 < \alpha < \pi - \gamma$ so that Lemma 2.5 is applicable. There exists t > 0 such that ||y|| = t||x|| and we know

$$\frac{\sin(\gamma)}{\sin(\gamma + \alpha)} \le t \le 1.$$

For any $c, d \in \mathbb{R}$ there exist $a, b \in \mathbb{R}$ such that

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c, \quad \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} = d.$$

When c = ||x|| and d = ||y||, then $(a, b)^t$ is the gradient of a linear function over the subspace spanned by x and y, and that linear function agrees with s over the semi-ray $[0, \infty)x \cup [0, \infty)y$. The coefficients satisfy

$$a = \|x\| = \frac{\rho}{\sin(\gamma)}, \quad b = \frac{1}{\sin(\boldsymbol{\alpha})} \left(\|y\| - \|x\|\cos(\boldsymbol{\alpha})\right) = \|y\|\csc(\boldsymbol{\alpha}) - \|x\|\cot(\boldsymbol{\alpha}).$$

On the one hand, we have $\csc(\alpha) - \cot(\alpha) \le 1$ when $0 < \alpha < \pi/2$. On the other hand,

$$0 > \frac{\sin(\gamma)}{\sin(\gamma + \alpha)} \csc(\alpha) - \cot(\alpha) = -\cot(\gamma + \alpha) > -\cot(\gamma).$$

The norm of the gradient is the Lipschitz constant. We find

$$||x|| \sqrt{1 + \cot(\gamma)^2} = ||x|| \frac{1}{\sin(\gamma)} \le \frac{R^2}{\rho}$$

That shows (4) when the angle is small.

When $c = ||x||^{-1}$ and $d = ||y||^{-1}$, then $(a, b)^t$ is the again gradient of a linear function over the subspace spanned by x and y, and that linear function agrees with **g** over the semi-ray $[0, \infty)x \cup [0, \infty)y$. Specifically,

$$a = \frac{1}{\|x\|} = \frac{\sin(\gamma)}{\rho}, \quad b = \frac{1}{\sin(\alpha)\|x\|} \left(\frac{1}{t} - \cos(\alpha)\right).$$

We simplify

$$b = \frac{1}{\sin(\alpha) \|x\|} \left(\frac{\sin(\gamma + \alpha)}{\sin(\gamma)} - \cos(\alpha) \right)$$
$$= \frac{1}{\sin(\alpha)\rho} \left(\sin(\gamma + \alpha) - \cos(\alpha) \sin(\gamma) \right) = \frac{\sin(\alpha) \cos(\gamma)}{\sin(\alpha)\rho} = \frac{\cos(\gamma)}{\rho}$$

The norm of the gradient is the Lipschitz constant and equals $1/\rho$. That shows (5) when the angle is small.

In particular, the set $\mathbb{R}^n \setminus \{0\}$ has an open cover over each of which members the Lipschitz continuity is satisfied with the desired upper bound. Whenever $0 < \alpha < \pi$, then the straight line segment from x to y does not pass through the origin and is covered by a finite number of open sets over each of which the Lipschitz condition is satisfied with the desired upper bound. The proof is complete.

Remark 2.7. Beer showed that the gauge function of star domains has Lipschitz constant at most $2/\rho$, where $\rho > 0$ is the radius of a ball in the convex kernel of the domain. Theorem 2.6 improves that estimate by a factor of two.

The expansion function is not defined on unbounded star domains: a reasonable generalization would require it to assume infinite values. However, the gauge function is defined even over unbounded star domains. For completeness, we show that our Lipschitz estimate extends to the gauge functions of unbounded star domains as well.

Proposition 2.8. Let Ω be a star domain that is star-shaped with respect to the ball $B_{\rho}(0)$. Then the gauge function \boldsymbol{g} satisfies

(6)
$$\forall x, y \in \mathbb{R}^n : |\boldsymbol{g}(x) - \boldsymbol{g}(y)| \le \frac{1}{\rho} ||x - y||.$$

Proof. We define the sequence of domains $\Omega_m := \Omega \cap B_{m\rho}(0)$ and write g_m for the corresponding sequences of gauge functions. These domains are star-shaped with respect to Ω_1 , and so they have the common Lipschitz constant ρ^{-1} as per Theorem 2.6. It remains to show the sequence of gauge functions converges pointwise to g.

Let $\hat{u} \in S_1(0)$ be a unit vector. If $\boldsymbol{g}(\hat{u}) > 0$, then $(0, \infty)\hat{u}$ crosses the boundary at $t\hat{u}$ for some unique t > 0, and we conclude $\boldsymbol{g}(\hat{u}) = \boldsymbol{g}_m(\hat{u})$ for any $m\rho > t$. If $\boldsymbol{g}(\hat{u}) = 0$, then $(0, \infty)\hat{u}$ cannot touch the boundary and lies within Ω , and we conclude that $\boldsymbol{g}_m(\hat{u}) = (m\rho)^{-1}$ converges to zero. By positive homogeneity, (6) holds.

We also characterize the gauge function of unbounded sets: the gauge function is zero except at the origin if and only if the star domain is unbounded.

Lemma 2.9. The gauge function of a star domain attains the value zero if and only if the star domain is unbounded.

Proof. Suppose that Ω is a star domain and \boldsymbol{g} is its gauge function. If \boldsymbol{g} is strictly positive over the unit circle, then the domain is bounded. Otherwise, there exists a sequence \hat{u}_m of unit vectors such that $\boldsymbol{g}(\hat{u}_m)$ converges to zero. As the unit sphere is compact, these unit vectors have an accumulation point \hat{u}_{\star} . By Lipschitz continuity of \boldsymbol{g} , it follows that $\boldsymbol{g}(\hat{u}_{\star}) = 0$.

Remark 2.10. We mention in passing that among the functions associated with the star domains is also the so-called radial function [29]. The expansion function and the radial function agree on the unit sphere, but the expansion function is positively homogeneous of degree one whereas the radial function is positively homogeneous of degree negative one.

2.3. Bi-Lipschitz parametrizations. As seen earlier, we can parameterize the boundary of star domains over the unit sphere. We extend this to parameterizing the entire star domain over the open unit ball.

We produce two different positively homogeneous transformations of Euclidean space. Suppose that Ω is a centered star domain. On the one hand, we define the vector gauge transformation

$$\boldsymbol{G}: \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \boldsymbol{g}(x) \, \frac{x}{\|x\|}.$$

On the other hand, we define the *expansion transformation*

$$\boldsymbol{S}: \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \psi\left(\frac{x}{\|x\|}\right) x.$$

We study the mapping properties and Lipschitz constants of these positively homogeneous functions. The first and most important observation is that they are mutual inverses of each other.

Lemma 2.11. Let Ω be a centered star domain. Then G and S are invertible with

$$G^{-1} = S$$

Proof. We first show that $G(S(\hat{u})) = \hat{u}$ for any unit vector $\hat{u} \in S_1(0)$. We know that $x = S(\hat{u})$ is the positive multiple of \hat{u} that lies on $\partial\Omega$. Since g equals one along $\partial\Omega$, it follows that $G(x) = \hat{u}$.

Next, $G(S(t\hat{u})) = tG(S(\hat{u})) = t\hat{u}$ holds for all t > 0 because both mappings are positively homogeneous. This shows G(S(x)) = x for all $x \in \mathbb{R}^n$. Since both mappings are bijective, it now also follows that S(G(x)) = x for all $x \in \mathbb{R}^n$. The result is proven.

We estimate the Lipschitz constants of these transformations. The restriction to the unit sphere recovers a result first proven by Vrećica [37, Theorem 1].

Theorem 2.12. Let Ω be a domain contained in the ball $B_R(0)$, containing the ball $B_r(0)$, and star-shaped with respect to the ball $B_\rho(0)$. Then the expansion transformation $\mathbf{S} : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism that restricts to

(7)
$$\boldsymbol{S}: B_1(0) \to \Omega.$$

For all $x, y \in \mathbb{R}^n$ it holds that

(8)
$$r||x-y|| \le ||\mathbf{S}(x) - \mathbf{S}(y)|| \le \frac{R^2}{\rho} ||x-y||.$$

Proof. Let $x, y \in \mathbb{R}^n$. There exist unit vectors $\hat{u}, \hat{v} \in S_1(0)$ and $s, t \in [0, \infty)$ with $x = s\hat{u}$ and $y = t\hat{v}$. We first address a few special cases. If $\hat{u} = -\hat{v}$ or if $\hat{u} = \hat{v}$, then we easily check that

$$\rho \|s\hat{u} - t\hat{v}\| = \rho |s - t| \le |\mathbf{S}(s\hat{u}) - \mathbf{S}(t\hat{v})| \le R|s - t| = R \|s\hat{u} - t\hat{v}\|.$$

If $\boldsymbol{s}(\hat{u}) = \boldsymbol{s}(\hat{v})$, then

$$\rho \|x - y\| \le |\boldsymbol{S}(x) - \boldsymbol{S}(y)| \le R \|x - y\|$$

So let us assume now that \hat{u} and \hat{v} have an angle $0 < \alpha < \pi$ and that $\mathbf{s}(\hat{u}) \neq \mathbf{s}(\hat{v})$. Without loss of generality, $\mathbf{s}(\hat{u}) > \mathbf{s}(\hat{v})$. We use the lower bound in Lemma 2.5: writing γ for the angle of the cone from x to the interior ball $B_{\rho}(0)$, and provided that $\alpha < \pi/2 - \gamma$, we have

$$\frac{\sin(\gamma)}{\sin(\gamma+\alpha)}\boldsymbol{s}(\hat{u}) \le \boldsymbol{s}(\hat{v}).$$

Due to positive homogeneity of S, we may assume without loss of generality, after rescaling, that s = 1. We study the extrema of the quotient

$$\Delta(t) := \frac{\|\mathbf{S}(\hat{u}) - t\mathbf{S}(\hat{v})\|^2}{\|\hat{u} - t\hat{v}\|^2}$$

Notice that

$$\|\hat{u} - t\hat{v}\|^2 = |1 - t\cos(\alpha)|^2 + |t\sin(\alpha)|^2$$
$$\|\boldsymbol{S}(\hat{u}) - t\boldsymbol{S}(\hat{v})\|^2 = |\boldsymbol{s}(\hat{u}) - t\boldsymbol{s}(\hat{v})\cos(\alpha)|^2 + |t\boldsymbol{s}(\hat{v})\sin(\alpha)|^2$$

We verify the limit behaviors

(9)
$$\lim_{t \to 0} \Delta(t) = \boldsymbol{s}(\hat{u})^2, \quad \lim_{t \to \infty} \Delta(t) = \boldsymbol{s}(\hat{v})^2.$$

Let us write $\kappa = \mathbf{s}(\hat{v})/\mathbf{s}(\hat{u})$ and notice $\kappa < 1$. One finds that

$$\Delta(t) = \frac{\mathbf{s}(\hat{u})^2 - 2t\mathbf{s}(\hat{u})\mathbf{s}(\hat{v}) + t^2\mathbf{s}(\hat{v})^2}{1 - 2t\cos(\alpha) + t^2} = \mathbf{s}(\hat{u})^2 \frac{1 - 2t\kappa + t^2\kappa^2}{1 - 2t\cos(\alpha) + t^2}.$$

The denominator is always positive. We identify critical points of $\Delta(t)$. The derivative is a positive multiple of

$$(-2\kappa\cos(\alpha) + 2t\kappa^2) (1 - 2t\cos(\alpha) + t^2) - (1 - 2t\kappa\cos(\alpha) + t^2\kappa^2) (-2\cos(\alpha) + 2t)$$

= 2 (t(\kappa^2 - 1) - (\kappa - 1)\cos(\alpha))
= 2(\kappa - 1) (t(\kappa + 1) - \cos(\alpha)).

The critical point is $t_0 = \cos(\alpha)/(\kappa + 1)$. We substitute and simplify:

$$\begin{aligned} \Delta(t_0) &= \mathbf{s}(\hat{u})^2 \frac{1 - 2t_0 \kappa \cos(\alpha) + t_0^2 \kappa^2}{1 - 2t_0 \cos(\alpha) + t_0^2} \\ &= \mathbf{s}(\hat{u})^2 \frac{(\kappa + 1)^2 - 2\kappa(\kappa + 1)\cos^2(\alpha) + \kappa^2\cos^2(\alpha)}{(\kappa + 1)^2 - 2(\kappa + 1)\cos^2(\alpha) + \cos^2(\alpha)} \\ &= \mathbf{s}(\hat{u})^2 \frac{\cos^2(\alpha) + (\kappa + 1)^2\sin^2(\alpha)}{\kappa^2\cos^2(\alpha) + (\kappa + 1)^2\sin^2(\alpha)} \\ &= \mathbf{s}(\hat{u})^2 \frac{1 + (\kappa + 1)^2\tan^2(\alpha)}{\kappa^2 + (\kappa + 1)^2\tan^2(\alpha)}. \end{aligned}$$

The last expression is decreasing in κ . Moreover, the derivative of

$$\Delta'(t) = 2(\kappa - 1)\mathbf{s}(\hat{u})^2 \frac{t(\kappa + 1) - \cos(\alpha)}{(1 - 2t\cos(\alpha) + t^2)^2}$$

is a positive multiple of

$$(\kappa-1)\Big((\kappa+1)-4\left(t(\kappa+1)-\cos(\alpha)\right)\left(t-\cos(\alpha)\right)\Big).$$

Therefore, $\Delta''(t_0)$ is a positive multiple of $(\kappa - 1)(\kappa + 1) = \kappa^2 - 1 > 0$. That means that $\Delta''(t_0)$ is a local maximum. The value $\Delta(t_0)$ will be largest of κ is smallest. Using the lower bound for $s(\hat{v})$, we estimate

$$\frac{1 + (\kappa + 1)^2 \tan^2(\alpha)}{\kappa^2 + (\kappa + 1)^2 \tan^2(\alpha)} \\
\leq \frac{\cos^2(\alpha) \sin^2(\gamma + \alpha) + \sin^2(\alpha) \left(\sin(\gamma) \sin(\gamma + \alpha) + \sin^2(\gamma + \alpha)\right)}{\cos^2(\alpha) \sin^2(\gamma) + \sin^2(\alpha) \left(\sin(\gamma) \sin(\gamma + \alpha) + \sin^2(\gamma + \alpha)\right)} \\
\leq \frac{\cos^2(\alpha) \sin^2(\gamma + \alpha)}{\cos^2(\alpha) \sin^2(\gamma)} \leq \frac{1}{\sin^2(\gamma)}.$$

The desired upper bound follows from $||x|| \sin(\gamma) = \rho$. Since the global maximum is attained at t_0 , the limit values (9) constitute the lower bounds of $\Delta(t)$.

Lastly, consider the general case where the angle between \hat{u} and \hat{v} is $0 < \alpha < \pi$. Then $s\hat{u}$ and $t\hat{v}$ are connected by a straight line segment that does not pass the origin. It is covered by a finite number of angular sectors over which \boldsymbol{S} satisfies (8). This completes the proof. \Box

The unit ball can be viewed as reference star domains. We study transformations between general star domains. Such transformations are easily obtained via composing transformations between star domains and the unit ball.

Corollary 2.13. Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be bounded centered star domains that are contained in the balls $B_{R_1}(0)$ and $B_{R_2}(0)$, respectively, and that are star-shaped with respect to the balls $B_{\rho_1}(0)$ and $B_{\rho_2}(0)$, respectively. Then there exists a positively homogeneous bi-Lipschitz mapping

(10)
$$\Xi: \mathbb{R}^n \to \mathbb{R}^n$$

which maps Ω_1 onto Ω_2 and such that for all $x, y \in \mathbb{R}^n$ it holds that

(11)
$$\|\Xi(x) - \Xi(y)\| \le \frac{R_2^2}{\rho_2 \rho_1} \|x - y\|, \quad \|\Xi^{-1}(x) - \Xi^{-1}(y)\| \le \frac{R_1^2}{\rho_2 \rho_1} \|x - y\|$$

We bring forth a different class of transformations, tailored to our applications: these are the identity over some interior set of the domains and satisfy Lipschitz estimates close to the boundary.

Theorem 2.14. Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be bounded centered star domains such that Ω_1 is starshaped with respect to the ball $B_{\rho}(0)$ and contained in $B_R(0)$. Assume that there exists $\mu > 0$ such that $\mathbf{s}_1, \mathbf{s}_2 : \mathbb{R}^n \to \mathbb{R}$ are their respective expansion functions and that

(12)
$$\forall \hat{x} \in S_1(0) : |\mathbf{s}_1(\hat{x}) - \mathbf{s}_2(\hat{x})| < \mu.$$

If $2\mu < R$, then there exists a mapping

(13)
$$\Xi: \Omega_1 \to \Omega_2$$

that equals the identity over an interior set,

(14)
$$x \in \Omega_1 \setminus B_{2\mu\rho/R}(\partial\Omega_1) : \Xi(x) = x$$

and that satisfies the Lipschitz condition

(15)
$$\operatorname{Lip}(\Xi) \le \frac{R}{2\rho} + \frac{(R+\mu)^2}{(\rho-\mu)\rho}, \quad \operatorname{Lip}(\Xi^{-1}) \le \frac{R^2}{(\rho-2\mu)\rho}.$$

Proof. Let $\hat{x} \in S_1(0)$ be a unit vector. Write $T = 1 - 2\mu/R$. If t < T, then $s_1(t\hat{x}) < s_1(\hat{x}) - 2\mu$. In particular, $s_1(t\hat{x}) < s_2(\hat{x}) - \mu$.

If $x \in B_T(0)$ and $\hat{z} \in S_1(0)$, then

$$\|\mathbf{S}_1(x) - \mathbf{S}_1(\hat{z})\| \ge \rho \|x - \hat{z}\| \ge \rho(1 - T) = 2\mu \frac{\rho}{R}.$$

We consider the following transformation $\Theta: B_1(0) \to \Omega_2$ on the unit ball:

$$\Theta(x) = \begin{cases} \mathbf{S}_1(x) & \text{if } \|x\| \le T, \\ \frac{1 - \|x\|}{1 - T} \mathbf{S}_1(x) + \frac{\|x\| - T}{1 - T} \mathbf{S}_2(x) & \text{if } \|x\| \ge T. \end{cases}$$

Consider any $\hat{z} \in S_1(0)$. The mapping Θ equals S_1 over $[0, T]\hat{z}$ and thus maps $[0, T]\hat{z}$ onto the straight line from the origin to $S_1(T\hat{z})$. The line segment $[T, 1)\hat{z}$ is mapped linearly onto the straight line segment from $S_1(T\hat{z})$ to $S_2(\hat{z})$. By construction, $||S_1(T\hat{z})|| \leq ||S_2(\hat{z})||$. We conclude that Θ maps $[0, 1)\hat{z}$ onto the straight line segment from the origin to $S_2(\hat{z})$ in a bijective manner. Hence, Θ goes from $B_1(0)$ onto Ω_2 while mapping each vector onto a multiple of it.

Consequently, $\Psi : \Omega_1 \to \Omega_2$ with $\Psi := \Theta \circ S_1^{-1}$ is bijective. It equals the identity over $S_1(B_T(0))$. The desired Lipschitz continuity will follow if we prove that Θ is Lipschitz continuous.

Let $x, y \in B_1(0)$. We see that

$$\Theta(x) - \Theta(y) = \frac{1 - \|x\|}{1 - T} \mathbf{S}_1(x) + \frac{\|x\| - T}{1 - T} \mathbf{S}_2(x) - \frac{1 - \|y\|}{1 - T} \mathbf{S}_1(x) - \frac{\|y\| - T}{1 - T} \mathbf{S}_2(x) + \frac{1 - \|y\|}{1 - T} \mathbf{S}_1(x) + \frac{\|y\| - T}{1 - T} \mathbf{S}_2(x) - \frac{1 - \|y\|}{1 - T} \mathbf{S}_1(y) - \frac{\|y\| - T}{1 - T} \mathbf{S}_2(y) = \frac{\|y\| - \|x\|}{1 - T} (\mathbf{S}_1(x) - \mathbf{S}_2(x)) + \frac{1 - \|y\|}{1 - T} (\mathbf{S}_1(x) - \mathbf{S}_1(y)) + \frac{\|y\| - T}{1 - T} (\mathbf{S}_2(x) - \mathbf{S}_2(y)).$$

We easily obtain the upper bound:

$$\|\Theta(x) - \Theta(y)\| \le \|x - y\| \frac{\|S_1(x) - S_2(x)\|}{1 - T} + \max\left(\operatorname{Lip}(S_1), \operatorname{Lip}(S_2)\right) \|x - y\|.$$

This upper estimate is finished by observing

$$\frac{\|\boldsymbol{S}_1(x) - \boldsymbol{S}_2(x)\|}{1 - T} \le \frac{\mu}{2\mu/R} \le \frac{R}{2}.$$

We also verify a lower bound via

$$\|\Theta(x) - \Theta(y)\| \ge \left\|\frac{1 - \|y\|}{1 - T} \left(\mathbf{S}_1(x) - \mathbf{S}_1(y)\right) + \frac{\|y\| - T}{1 - T} \left(\mathbf{S}_2(x) - \mathbf{S}_2(y)\right)\right\| - \frac{\|y\| - \|x\|}{1 - T} \|\mathbf{S}_1(x) - \mathbf{S}_2(x)\|.$$

If $1 - ||y|| \ge ||y|| - T$, then

$$\begin{aligned} \left\| \frac{1 - \|y\|}{1 - T} \left(\mathbf{S}_1(x) - \mathbf{S}_1(y) \right) + \frac{\|y\| - T}{1 - T} \left(\mathbf{S}_2(x) - \mathbf{S}_2(y) \right) \right\| \\ \ge \|\mathbf{S}_1(x) - \mathbf{S}_1(y)\| - \frac{\|y\| - T}{1 - T} \|\mathbf{S}_1(x) - \mathbf{S}_2(x)\| - \frac{\|y\| - T}{1 - T} \|\mathbf{S}_1(y) - \mathbf{S}_2(y)\| \end{aligned}$$

and an analogous inequality holds if $1 - ||y|| \le ||y|| - T$. By assumption, $|S_1 - S_2| < \mu$ over $B_1(0)$. The Lipschitz continuity of Θ^{-1} over $\Theta(B_1(0) \setminus B_T(0))$ follows for $\mu > 0$ small enough.

The global Lipschitz continuity of Θ and its inverse is now evident because $B_1(0)$ is convex. The proof is complete.

2.4. Oriented distance functions of star domains. We review another scalar function associated with general domains. The distance function of a domain gives a description from the outside but does not provide much about its inside. As a remedy, we study the *oriented distance function* [19]. The oriented distance function of an open set $\Omega \subseteq \mathbb{R}^n$ is defined as

(16)
$$\boldsymbol{d}_{\Omega}(x) = \begin{cases} \inf_{\substack{y \in \partial \Omega}} \|x - y\| & \text{if } x \notin \Omega, \\ -\inf_{\substack{y \in \partial \Omega}} \|x - y\| & \text{if } x \in \overline{\Omega}. \end{cases}$$

Lemma 2.15. The oriented distance function is Lipschitz with optimal Lipschitz constant 1. If Ω is convex, then the oriented distance function is convex.

Proof. Over Ω and over $\mathbb{R}^n \setminus \overline{\Omega}$, the oriented distance function is the minimum or maximum, respectively, of a family of Lipschitz functions with Lipschitz constant 1, and hence it has got Lipschitz constant 1 over either of these two sets. Consider now $x_0 \in \Omega$ and $x_1 \notin \Omega$. There exists minimal $t_0 \in [0, 1]$ and maximal $t_1 \in [0, 1]$ with

$$z_0 := (1 - t_0)x_0 + t_0 x_1 \in \partial \Omega, \quad z_1 := (1 - t_1)x_0 + t_1 x_1 \in \partial \Omega.$$

We then verify

$$\begin{aligned} |\boldsymbol{d}(x_0) - \boldsymbol{d}(x_1)| &\leq |\boldsymbol{d}(x_0) - \boldsymbol{d}(z_0)| + |\boldsymbol{d}(z_0) - \boldsymbol{d}(z_1)| + |\boldsymbol{d}(z_1) - \boldsymbol{d}(x_1)| \\ &\leq ||x_0 - z_0|| + ||z_1 - x_1|| \leq ||x_0 - x_1||. \end{aligned}$$

One easily sees that this the best Lipschitz constant.

Let us now assume that Ω is convex. Given any $x, y \in \Omega$, we have

$$B_{|\boldsymbol{d}(x)|}(x) \subseteq \Omega, \quad B_{|\boldsymbol{d}(y)|}(y) \subseteq \Omega,$$

and hence for all $t \in [0, 1]$ we conclude

$$B_{t|\boldsymbol{d}(x)|+(1-t)|\boldsymbol{d}(y)|}(tx+(1-t)y) \subseteq \Omega,$$

which shows that d is convex over Ω . If instead $x, y \notin \Omega$, then for any $\epsilon > 0$ we can choose $x_{\epsilon}, y_{\epsilon} \in \overline{\Omega}$ such that

$$||x - x_{\epsilon}|| \le \mathbf{d}(x) + \epsilon, \quad ||y - y_{\epsilon}|| \le \mathbf{d}(y) + \epsilon$$

For any $t \in [0, 1]$ we write $z_{t,\epsilon} = tx_{\epsilon} + (1 - t)y_{\epsilon} \in \Omega$. We observe

$$d(tx + (1-t)y) \le ||tx + (1-t)y - tx_{\epsilon} + (1-t)y_{\epsilon}|| \le t||x - x_{\epsilon}|| + (1-t)||y - y_{\epsilon}|| \le td(x) + (1-t)d(y) + 2\epsilon.$$

The convexity of d over $\mathbb{R}^n \setminus \Omega$ follows because $\epsilon > 0$ was arbitrary.

Lastly, suppose that $x \notin \Omega$ and $y \in \Omega$ and let $z \in \partial \Omega$ be on the straight line segment from x to y with

$$z = (1 - t_0)x + t_0 y$$

for some $t_0 \in [0, 1]$. Then

$$t_0 = \frac{\|x - z\|}{\|x - y\|}, \quad 1 - t_0 = \frac{\|z - y\|}{\|x - y\|}.$$

There exists an affine hyperplane through z such that Ω is on only one side of H. We let $h_x, h_y \ge 0$ be the heights of x and y at this supporting hyperplane. There exists $c \ge 0$ such that $h_x = c ||x - z||$ and $h_y = c ||z - y||$. It follows that

$$(1-t_0)\mathbf{d}(x) + t_0\mathbf{d}(y) \ge (1-t_0)h_x - t_0h_y$$

$$\ge (1-t_0)c||x-z|| - t_0c||z-y||$$

$$\ge c\frac{||z-y|| ||x-z||}{||x-y||} - c\frac{||x-z|| ||z-y||}{||x-y||} \ge 0 = \mathbf{d}(z).$$

Let $t \in [0, 1]$. Then we use the convexity of **d** over Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ to see

$$d((1-t)x + tz) \leq (1-t)d(x) + td(z) \leq (1-t)d(x) + t(1-t_0)d(x) + tt_0d(y), d((1-t)z + ty) \leq (1-t)d(z) + td(y) \leq (1-t)(1-t_0)d(x) + (1-t)t_0d(y) + td(y).$$

The proof is complete.

2.5. Smooth approximation of star domains. We approximate a given star domain Ω by star domain with smooth boundary. If the domain Ω is convex, then our approximate domain will be even strictly convex. The following proposition, whose proof is standard, leads to the smooth approximation of bounded convex domains. We also point out Błocki [10] for a related result.

Proposition 2.16. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain that contains the ball $B_{\rho}(0)$ and contained the ball $B_R(0)$. Then there exists a family $\Omega_{\epsilon} \subseteq \mathbb{R}^n$ of bounded strictly convex domains with smooth boundaries, indexed over $\epsilon > 0$, such that the expansion functions satisfy

(17)
$$\forall \hat{x} \in S_1(0) : \| \boldsymbol{s}_{\Omega}(\hat{x}) - \boldsymbol{s}_{\Omega_{\epsilon}}(\hat{x}) \| < \epsilon$$

when $\epsilon > 0$ small enough. Here, \mathbf{s}_{Ω} is the expansion function of Ω and the $\mathbf{s}_{\Omega_{\epsilon}}$ are the expansion functions of the Ω_{ϵ} .

Proof. We let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a positive smooth function with unit integral and supported within the unit ball, and for every $\epsilon > 0$ we set $\eta_{\epsilon}(z) := \epsilon^{-n} \eta(z/\epsilon)$.

We write $d : \mathbb{R}^n \to \mathbb{R}$ be the oriented distance function. Let $d_{\epsilon} : \mathbb{R}^n \to \mathbb{R}$ be their mollification with a smoothing term of radius ϵ . Then each d_{ϵ} is smooth and also convex: for all $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$ we find

$$\begin{aligned} \boldsymbol{d}_{\epsilon} \left((1-t)x + ty \right) &\leq \int_{B_{\epsilon}(0)} \eta_{\epsilon}(z) \boldsymbol{d}_{\epsilon} \left((1-t)x + ty + z \right) \\ &\leq (1-t) \int_{B_{\epsilon}(0)} \eta_{\epsilon}(z) \boldsymbol{d} \left(x + z \right) + t \int_{B_{\epsilon}(0)} \eta_{\epsilon}(z) \boldsymbol{d} \left(y + z \right) \\ &\leq (1-t) \boldsymbol{d}_{\epsilon} \left(x \right) + t \boldsymbol{d}_{\epsilon} \left(y \right). \end{aligned}$$

The sum of a convex function and a strictly convex function is strictly convex. Thus,

$$\boldsymbol{d}_{\epsilon}'(x) := \boldsymbol{d}_{\epsilon}(x) + \epsilon^2 \|x\|^2$$

is smooth and strictly convex. Consequently, the sublevel sets of d'_{ϵ} are strictly convex. Since d'_{ϵ} is smooth, its sublevel sets have smooth boundary.

The function $||x||^2$ is at most $4R^2$ over 2Ω . If $x \in \Omega$ and $2\epsilon < |\mathbf{d}(x)|$, then $\mathbf{d}_{\epsilon}(x) < -\epsilon$. If $x \notin \Omega$ and $2\epsilon < |\mathbf{d}(x)|$, then $\mathbf{d}_{\epsilon}(x) > \epsilon$. Consequently, $|\mathbf{d}_{\epsilon}| > \epsilon$ outside $B_{2\epsilon}(\partial\Omega)$. Provided that $4\epsilon R^2 < 1$, we must have \mathbf{d}'_{ϵ} non-zero outside $B_{2\epsilon}(\partial\Omega)$. Therefore, $||\mathbf{s}_{\Omega} - \mathbf{s}_{\Omega_{\epsilon}}|| < 2\epsilon$. After re-indexing, the proof is complete.

Theorem 2.17. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain that is contained in $B_R(0)$ and is star-shaped with respect to $B_\rho(0)$. There exists a family $\Omega_\epsilon \subseteq \mathbb{R}^n$ of bounded strictly convex domains with smooth boundary together with a family of mappings

(18)
$$\Phi_{\epsilon}: \Omega \to \Omega_{\epsilon}$$

that satisfy

(19)
$$\forall x \in \Omega \setminus B_{2\rho/R\epsilon}(\partial\Omega) : \Phi_{\epsilon}(x) = x$$

and are bi-Lipschitz with

(20)
$$\operatorname{Lip}\left(\Phi_{\epsilon}\right) \leq \frac{R}{2\rho} + \frac{(R+\epsilon)^{2}}{(\rho-\epsilon)\rho}, \quad \operatorname{Lip}\left(\Phi_{\epsilon}^{-1}\right) \leq \frac{R^{2}}{(\rho-2\epsilon)\rho}.$$

Proof. Proposition 2.16 produces a family of strictly convex bounded domains Ω_{ϵ} with smooth boundary. The expansion function \mathbf{s}_{Ω} of Ω and the expansion functions $\mathbf{s}_{\Omega_{\epsilon}}$ of Ω_{ϵ} satisfy $|\mathbf{s}_{\Omega} - \mathbf{s}_{\Omega_{\epsilon}}| < \epsilon$ over the unit sphere. Theorem 2.14 now shows the desired claim. \Box

3. Analysis of the de Rham complex

We review notions of de Rham complexes of differential forms with coefficients in Lebesgue spaces. Apart from basic functional analysis, the reader is assumed to be familiar with the theory of linear operators in Hilbert spaces ([34, 40]) and exterior calculus ([28]). Much of the material outlined in this section appears in various forms within the literature on Hilbert complexes [1, 2, 26].

We let $\Lambda^k(\mathbb{R}^n)$ be the exterior product of the vector space \mathbb{R}^n . That space carries a Hilbert space structure that is induced from \mathbb{R}^n , and is thus a normed space.

Throughout this section, we assume that $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain. We write $L^p(\Omega)$ for the Lebesgue space with exponent $1 \leq p \leq \infty$. Equipped with the canonical norm, this becomes a Banach space.

A measurable differential form over Ω is any measurable function $u : \Omega \to \Lambda^k(\mathbb{R}^n)$. For each such measurable function, the pointwise norm $|u| : \Omega \to \mathbb{R}$ is measurable again. We let $L^p \Lambda^k(\Omega)$ be the space of those measurable mappings $u : \Omega \to \Lambda^k(\mathbb{R}^n)$ for which $|u| \in L^p(\Omega)$. We equip $L^p \Lambda^k(\Omega)$ with the canonical norm:

$$||u||_{L^p(\Omega)} := \left(\int_{\Omega} |u(x)|^p dx\right)^{\frac{1}{p}}, \quad u \in L^p \Lambda^k(\Omega),$$

with the obvious modification when $p = \infty$. Hence, $L^p \Lambda^k(\Omega)$ contains exactly those differential k-forms over Ω whose coefficients are p-integrable. When p = 2, then this norm is induced by an inner product, making $L^2 \Lambda^k(\Omega)$ is a Hilbert space. Occasionally, we will use the abbreviations $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ for the L^2 scalar product and norm, respectively, when there is no danger of ambiguity.

We also write $C^{\infty}\Lambda^k(\Omega)$ for the smooth differential forms, which are the smooth mappings from Ω into $\Lambda^k(\mathbb{R}^n)$, and we write $C_c^{\infty}\Lambda^k(\Omega)$ for those smooth differential forms whose support is compact in Ω .

We define

$$H\Lambda^k(\Omega) := \left\{ u \in L^2\Lambda^k(\Omega) \mid d^k u \in L^2\Lambda^{k+1}(\Omega) \right\}$$

where the exterior derivative is taken in the sense of distributions. The linear operator

(21)
$$d^k : H\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \to L^2\Lambda^{k+1}(\Omega)$$

is densely-defined and closed. Moreover, it satisfies the differential property

$$d^{k+1}d^k = 0$$

The operator (21) can be defined equivalently as the closure of the linear mapping d^k : $C^{\infty}\Lambda^k(\Omega) \to C^{\infty}\Lambda^{k+1}(\Omega)$. Since the operator is closed, it defines the Hilbert space $H\Lambda^k(\Omega)$ together with its graph norm

$$||u||_{H\Lambda^k(\Omega)} := \left(\int_{\Omega} |u(x)|^2 + |d^k u(x)|^2 dx\right)^{\frac{1}{2}}$$

Writing $\tilde{u} \in L^2 \Lambda^k(\mathbb{R}^n)$ for the trivial extension of any $u \in L^2 \Lambda^k(\Omega)$, we also define

$$H_0\Lambda^k(\Omega) := \left\{ u \in H\Lambda^k(\Omega) \mid \tilde{u} \in H\Lambda^k(\mathbb{R}^n) \right\}$$

which is a closed subspace of $H\Lambda^k(\Omega)$. We have a closed densely-defined linear operator

(22)
$$d^k: H_0\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \to L^2\Lambda^{k+1}(\Omega),$$

which can be defined equivalently as the closure of the mapping $d^k : C_c^{\infty} \Lambda^k(\Omega) \to C_c^{\infty} \Lambda^{k+1}(\Omega)$. The focus of interest is the L^2 de Rham complex over the open set Ω , which is the differential complex

$$(23a) \quad 0 \to H\Lambda^0(\Omega) \xrightarrow{d^0} H\Lambda^1(\Omega) \xrightarrow{d^1} \dots \xrightarrow{d^{n-2}} H\Lambda^{n-1}(\Omega) \xrightarrow{d^{n-1}} H\Lambda^n(\Omega) \to 0.$$

We also define the differential complex with boundary conditions

(23b)
$$0 \to H_0 \Lambda^0(\Omega) \xrightarrow{d^0} H_0 \Lambda^1(\Omega) \xrightarrow{d^1} \dots \xrightarrow{d^{n-2}} H_0 \Lambda^{n-1}(\Omega) \xrightarrow{d^{n-1}} H_0 \Lambda^n(\Omega) \to 0.$$

All these constructions have dual counterparts. Towards that purpose, we recall the Hodge star operator

$$\star: L^2 \Lambda^k(\Omega) \to L^2 \Lambda^{n-k}(\Omega),$$

which is an isometry. This allows us to define

$$H^*\Lambda^k(\Omega) := \star H\Lambda^{n-k}(\Omega), \quad H^*_0\Lambda^k(\Omega) := \star H_0\Lambda^{n-k}(\Omega).$$

The exterior coderivative is the densely-defined closed operator

(24)
$$\delta^k : H^* \Lambda^k(\Omega) \subseteq L^2 \Lambda^k(\Omega) \to L^2 \Lambda^{k-1}(\Omega), \quad u \mapsto \star^{-1} d^{n-k} \star u.$$

We also use the restriction

(25)
$$\delta^k : H_0^* \Lambda^k(\Omega) \subseteq L^2 \Lambda^k(\Omega) \to L^2 \Lambda^{k-1}(\Omega).$$

The exterior coderivative satisfies the differential property

$$\delta^{k-1}\delta^k = 0$$

We have the dual de Rham complex

(26a)
$$0 \leftarrow H_0^* \Lambda^0(\Omega) \xleftarrow{\delta^1} H_0^* \Lambda^1(\Omega) \xleftarrow{\delta^2} \dots \xleftarrow{\delta^{n-1}} H_0^* \Lambda^{n-1}(\Omega) \xleftarrow{\delta^n} H_0^* \Lambda^n(\Omega) \leftarrow 0$$

and the corresponding complex with boundary conditions

(26b)
$$0 \leftarrow H^*\Lambda^0(\Omega) \xleftarrow{\delta^1} H^*\Lambda^1(\Omega) \xleftarrow{\delta^2} \dots \xleftarrow{\delta^{n-1}} H^*\Lambda^{n-1}(\Omega) \xleftarrow{\delta^n} H^*\Lambda^n(\Omega) \leftarrow 0.$$

The most critical property here is the duality in the sense of unbounded operators. With the appropriate choice of indices, the operator d^k in (21) is the adjoint of the operator δ^{k+1} in (25), and the operator d^k in (22) is the adjoint of the operator δ^{k+1} in (24). We have the following integration by parts formulas:

$$\forall u \in H\Lambda^k(\Omega) : \forall v \in H_0^*\Lambda^{n-k}(\Omega) : \int_\Omega \langle d^k u, v \rangle = \int_\Omega \langle u, \delta^{k+1}v \rangle, \\ \forall u \in H_0\Lambda^k(\Omega) : \forall v \in H^*\Lambda^{n-k}(\Omega) : \int_\Omega \langle d^k u, v \rangle = \int_\Omega \langle u, \delta^{k+1}v \rangle.$$

The L^2 de Rham complex gives rise to the Hodge–Laplace operators. We focus on the L^2 de Rham complex (23a) and its adjoint differential complex (26b). Let

$$\operatorname{dom}(\Delta_k) := \left\{ \begin{array}{ll} u \in H\Lambda^k(\Omega) \cap H_0^{\star}\Lambda^k(\Omega) & | & \begin{array}{l} \delta^k u \in H\Lambda^{k-1}(\Omega), \\ d^k u \in H_0^{\star}\Lambda^{k+1}(\Omega) \end{array} \right\}.$$

The k-th Hodge-Laplace operator is the unbounded operator

(27)
$$\Delta_k : \operatorname{dom}(\Delta_k) \subseteq L^2 \Lambda^k(\Omega) \to L^2 \Lambda^k(\Omega), \quad u \mapsto \delta^{k+1} d^k u + d^{k-1} \delta^k u.$$

This operator is densely-defined, closed, and self-adjoint.

We could also base a Hodge–Laplace-type operator on the differential complex (26a) and its adjoint differential complex (23b), but this would be the same operator up to an isometry and shall henceforth not be our specific concern. 3.1. Closed range condition and potential operators. The theory of Hilbert de Rham complexes becomes more amenable once an additional condition is satisfied. We say that the *closed range condition* is satisfied if the differential operators $d^k : H\Lambda^k(\Omega) \to L^2\Lambda^{k+1}(\Omega)$ in the de Rham complex have closed range.

We first notice that the closed range condition holds for all the differential operators in the complex (23a) if and only if it holds for any of the other differential complexes (26b), (23b), or (26a), as follows via known relationships within the theory of Hilbert spaces.

For practical purposes, we recall that the closed range condition holds if Ω is a bounded Lipschitz domain [30], which includes bounded convex domains.

Densely-defined closed operators with closed range allow for generalized inverses known as pseudoinverses [9]. By the bounded inverse theorem we recognize that the closed range condition is equivalent to the existence of bounded linear operators

(28)
$$d_k^{\dagger} : L^2 \Lambda^{k+1}(\Omega) \to H \Lambda^k(\Omega) \subseteq L^2 \Lambda^k(\Omega)$$

that satisfy

$$d^{k}u = d^{k}d_{k}^{\dagger}d^{k}u, \quad u \in H\Lambda^{k}(\Omega),$$

ker $d_{k}^{\dagger} = \left(d^{k}H\Lambda^{k}(\Omega)\right)^{\perp},$
ran $(d_{k}^{\dagger})^{\perp} = \ker\left(d^{k}:H\Lambda^{k}(\Omega)\subseteq L^{2}\Lambda^{k}(\Omega)\to L^{2}\Lambda^{k+1}(\Omega)\right).$

These potential operators act as generalized inverses of the exterior derivatives: they vanish on the orthogonal complement of the range of the exterior derivative and produce the minimum-norm preimage of any exterior derivative. By duality, the closed range condition is also equivalent to the existence of bounded linear operators

(29)
$$\delta_k^{\dagger} : L^2 \Lambda^{k-1}(\Omega) \to H_0^* \Lambda^k(\Omega) \subseteq L^2 \Lambda^k(\Omega)$$

that satisfy

$$\delta^{k} u = \delta^{k} \delta^{\dagger}_{k} \delta^{k} u, \quad u \in H_{0}^{\star} \Lambda^{k}(\Omega),$$

$$\ker \delta^{\dagger}_{k} = \left(\delta^{k} H_{0}^{\star} \Lambda^{k}(\Omega)\right)^{\perp},$$

$$\operatorname{ran}(\delta^{\dagger}_{k})^{\perp} = \ker \left(\delta_{k} : H_{0}^{\star} \Lambda^{k}(\Omega) \subseteq L^{2} \Lambda^{k}(\Omega) \to L^{2} \Lambda^{k-1}(\Omega)\right)$$

These act as potential operators for the exterior coderivative in the analogous manner.

The operators d_k^{\dagger} and $\bar{\delta}_{k+1}^{\dagger}$ are mutually adjoint, as follows from d^k and δ^{k+1} being mutually adjoint, and they have the same operator norm:

(30)
$$C_{\mathrm{PF},\Omega,k} = \sup_{f \in L^2 \Lambda^{k+1}(\Omega) \setminus \{0\}} \frac{\|d_k^{\dagger}f\|_{L^2 \Lambda^k(\Omega)}}{\|f\|_{L^2(\Omega)}} = \sup_{g \in L^2 \Lambda^k(\Omega) \setminus \{0\}} \frac{\|\delta_{k+1}^{\dagger}g\|_{L^2(\Omega)}}{\|g\|_{L^2(\Omega)}}$$

These operator norms agree with what is known as the Poincaré–Friedrichs constant. The *Poincaré–Friedrichs constant* $C_{\text{PF},\Omega,k}$ appears in the Poincaré–Friedrichs inequality: for all $u \in H\Lambda^k(\Omega)$ there exists $w \in H\Lambda^k(\Omega)$ such that $d^kw = d^ku$ and

$$||w||_{L^2(\Omega)} \le C_{\mathrm{PF},\Omega,k} ||d^k u||_{L^2(\Omega)}.$$

Exactly the same constant appears in the dual version of the inequality: for all $u \in H_0^* \Lambda^{k+1}(\Omega)$ there exists $w \in H_0^* \Lambda^{k+1}(\Omega)$ such that $\delta^{k+1} w = \delta^{k+1} u$ and

$$||w|| \le C_{\mathrm{PF},\Omega,k} ||\delta^{k+1}u||$$

These operators are potential operators corresponding to the differential operators in the de Rham complex. Importantly, they also provide a potential operator for the Hodge–Laplace operator. We introduce the bounded linear operator

(31)
$$\Delta_k^{\dagger} : L^2 \Lambda^k(\Omega_k) \to \operatorname{dom}(\Delta_k) \subseteq L^2 \Lambda^k(\Omega), \quad u \mapsto \delta_{k+1}^{\dagger} d_k^{\dagger} u + d_{k-1}^{\dagger} \delta_k^{\dagger} u.$$

By construction, this operator is self-adjoint. It acts as a generalized inverse of the Hodge– Laplace operator and satisfies

$$\Delta_k u = \Delta_k \Delta_k^{\dagger} \Delta_k u, \quad u \in \operatorname{dom}(\Delta_k),$$
$$\ker \Delta_k^{\dagger} = (\Delta_k \operatorname{dom}(\Delta_k))^{\perp},$$
$$\operatorname{ran}(\Delta_k^{\dagger})^{\perp} = \ker \left(\Delta_k : \operatorname{dom}(\Delta_k) \subseteq L^2 \Lambda^k(\Omega) \to L^2 \Lambda^k(\Omega)\right)$$

Note that by construction

$$d^k \Delta_k^\dagger = \delta_{k+1}^\dagger, \quad \delta^k \Delta_k^\dagger = d_{k-1}^\dagger$$

This operator thus satisfies the operator norm identities

$$\sup_{f \in L^2 \Lambda^k(\Omega) \setminus \{0\}} \frac{\|\Delta_k^{\dagger} f\|_{L^2(\Omega)}}{\|f\|_{L^2 \Lambda^k(\Omega)}} = \max(C_{\mathrm{PF},\Omega,k}, C_{\mathrm{PF},\Omega,k-1})^2,$$
$$\sup_{f \in L^2 \Lambda^k(\Omega) \setminus \{0\}} \frac{\|\delta^k \Delta_k^{\dagger} f\|_{L^2(\Omega)}}{\|f\|_{L^2 \Lambda^k(\Omega)}} = C_{\mathrm{PF},\Omega,k-1},$$
$$\sup_{f \in L^2 \Lambda^k(\Omega) \setminus \{0\}} \frac{\|d^k \Delta_k^{\dagger} f\|_{L^2(\Omega)}}{\|f\|_{L^2 \Lambda^k(\Omega)}} = C_{\mathrm{PF},\Omega,k}.$$

The main purpose of this manuscript is to estimate the Poincaré–Friedrichs constants from above.

3.2. Gaffney inequalities and consequences. The Hilbert spaces $H\Lambda^k(\Omega)$ and $H_0^*\Lambda^k(\Omega)$ are defined via the graph norms of the exterior derivative or the exterior coderivative: their members have just enough regularity to ensure that they and their (co)differentials have coefficients in $L^2(\Omega)$. While they are widely known as Sobolev differential forms, they a priori do not relate to the classical Sobolev spaces.

However, $H^1(\Omega) := H^1 \Lambda^k(\Omega)$ is the classical Sobolev space of first order. We write $H^1 \Lambda^k(\Omega)$ for the subspace of $L^2 \Lambda^k(\Omega)$ whose coefficients are in the Sobolev space $H^1(\Omega)$. Clearly,

(32)
$$H^{1}\Lambda^{k}(\Omega) \subseteq H\Lambda^{k}(\Omega) \cap H_{0}^{\star}\Lambda^{k}(\Omega).$$

It is an intricate question whether that inclusion is onto, and the answer is generally negative. However, under stronger assumptions on the domain, such as when the domain is convex, a positive answer is formalized in Gaffney's inequality.

Theorem 3.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain. Let $u \in H\Lambda^k(\Omega) \cap H_0^*\Lambda^k(\Omega)$. Then $u \in H^1\Lambda^k(\Omega)$, and

(33)
$$\|\nabla u\|_{L^{2}(\Omega)} \leq \|d^{k}u\|_{L^{2}(\Omega)} + \|\delta^{k}u\|_{L^{2}(\Omega)}.$$

Proof. See, e.g., [18], [30].

Let us now assume that Ω is indeed bounded and convex. The validity of Gaffney's inequality has numerous implications. By the Rellich embedding theorem, the Gaffney inequality implies the compact embedding of $H\Lambda^k(\Omega) \cap H_0^*\Lambda^k(\Omega)$ into $L^2(\Omega)$. It follows that the potential operators

$$\Delta_k^{\dagger}: L^2 \Lambda^k(\Omega) \to L^2 \Lambda^k(\Omega)$$

are compact. By the spectral theorem for compact self-adjoint operators, the separable space $L^2\Lambda^k(\Omega)$ has a countable orthonormal basis consisting of eigenvectors of the bounded operator Δ^{\dagger} , and the operator has a countable spectrum of eigenvalues with finite multiplicity. Correspondingly, Δ_k is a self-adjoint operator with a countable spectrum of eigenvalues with finite multiplicity. The non-zero¹ eigenvalues form an ascending sequence

$$0 < \lambda_{k,1} \leq \lambda_{k,2} \leq \ldots$$

Its non-zero eigenvalues are the reciprocals of Δ_k^{\dagger} 's non-zero eigenvalues. The smallest non-zero eigenvalue relates to the Poincaré–Friedrichs constants:

(34)
$$\lambda_{k,1} = \max\left(C_{\mathrm{PF},\Omega,k}, C_{\mathrm{PF},\Omega,k-1}\right)^{-2}.$$

Compact operators between Hilbert spaces have a singular decomposition. It is now evident how the singular value decomposition of the derivatives looks like: there exists a countable orthonormal system $u_{k,j} \in H\Lambda^k(\Omega)$ that spans the orthogonal complement of the kernel of (21), another countable orthonormal system $u_{k+1,j}^* \in H\Lambda^{k+1}(\Omega)$ that spans the range of (21), and a non-increasing null sequence $\theta_{k,j} \in \mathbb{R}$ such that (28) and (29) satisfy

$$d_k^{\dagger} = \sum_{j=1}^{\infty} \theta_{k,j} u_{k,j} \langle u_{k+1,j}^{\star}, \cdot \rangle, \quad \delta_{k+1}^{\dagger} = \sum_{j=1}^{\infty} \theta_{k,j} u_{k+1,j}^{\star} \langle u_{k,j}, \cdot \rangle.$$

The unbounded operators (21) and (25) have corresponding singular value decompositions

$$d^{k} = \sum_{j=1}^{\infty} \theta_{k,j}^{-1} u_{k+1,j}^{\star} \langle u_{k,j}, \cdot \rangle, \quad \delta^{k+1} = \sum_{j=1}^{\infty} \theta_{k,j}^{-1} u_{k,j} \langle u_{k+1,j}^{\star}, \cdot \rangle.$$

We write the Hodge–Laplace operator and its potential operator as

$$\Delta_k = \sum_{j=1}^{\infty} \theta_{k,j}^{-2} u_{k,j} \langle u_{k,j}, \cdot \rangle + \sum_{j=1}^{\infty} \theta_{k-1,j}^{-2} u_{k,j}^{\star} \langle u_{k,j}^{\star}, \cdot \rangle,$$

$$\Delta_k^{\dagger} = \sum_{j=1}^{\infty} \theta_{k,j}^2 u_{k,j} \langle u_{k,j}, \cdot \rangle + \sum_{j=1}^{\infty} \theta_{k-1,j}^2 u_{k,j}^{\star} \langle u_{k,j}^{\star}, \cdot \rangle.$$

We thus have partitioned the non-zero eigenvalues of the Hodge–Laplace operator into two orthogonal components: one belonging to $\delta^{k+1}d^k$, the other belonging to $d^{k-1}\delta^k$.

Remark 3.2. Gaffney's inequality implies the compact embedding of $H\Lambda^k(\Omega) \cap H_0^*\Lambda^k(\Omega)$ into $L^2\Lambda^k(\Omega)$ via the Rellich embedding theorem. Compact embeddings in three-dimensional vector calculus have been a staple in the literature [38, 32, 41].

¹If the domain is convex, then all eigenvalues are non-zero except one single eigenvalue of the scalar Neumann Laplacian, but we will not need this fact.

3.3. Sobolev–Poincaré inequalities. Even though the focus of our discussion are differential forms with coefficients in the Lebesgue space $L^2(\Omega)$, we will need a Sobolev–Poincaré inequality for differential forms based on different Lebesgue spaces. Importantly, we need explicitly knowledge how the constant in that inequality depends on the domain. We utilize the following explicit result of Chua and Wheeden, slightly adapted for differential forms instead of scalar functions.

Theorem 3.3 ([16]). Let $n \ge 1$ and $0 \le k \le n$. Let $q \in [1, \infty)$ such that

$$\frac{1}{2} - \frac{1}{n} \le \frac{1}{q}$$

Then there exists a constant $C_{n,k,q}$ depending only on n, k, and q such that for all bounded convex domains $\Omega \subseteq \mathbb{R}^n$ and $u \in H^1\Lambda^k(\Omega)$ we have

$$\|u - \operatorname{Avg} u\|_{L^{q}(\Omega)} \leq C_{n,k,q} \operatorname{vol}(\Omega)^{\frac{1}{q} - \frac{1}{2}} \operatorname{diam}(\Omega) \|\nabla u\|_{L^{2}(\Omega)},$$

where $\operatorname{Avg} u := \int_{\Omega} u \, dx \in \Lambda^k(\mathbb{R}^n)$ denotes the average of u over the domain Ω .

We dwell a bit further with this embedding and its consequences. Consider a Lebesgue exponent $q \in [1, \infty)$ with $1/2 - 1/n \leq 1/q$. We assume that Ω is a bounded convex domain. We make the technical observation that for any $u \in L^2 \Lambda^k(\Omega)$ it holds that

$$\|\operatorname{Avg} u\|_{L^{q}(\Omega)} = \operatorname{vol}(\Omega)^{\frac{1}{q}} \cdot (\operatorname{Avg} u) = \operatorname{vol}(\Omega)^{\frac{1}{q} - \frac{1}{2}} \|\operatorname{Avg} u\|_{L^{2}(\Omega)}.$$

This in combination with the Poincaré–Sobolev inequality and Gaffney's inequality yields

$$\begin{aligned} \|u\|_{L^{q}(\Omega)} &\leq \|u - \operatorname{Avg} u\|_{L^{q}(\Omega)} + \|\operatorname{Avg} u\|_{L^{q}(\Omega)} \\ &\leq C_{n,k,q} \operatorname{vol}(\Omega)^{\frac{1}{q} - \frac{1}{2}} \operatorname{diam}(\Omega) \|\nabla u\|_{L^{2}(\Omega)} + \operatorname{vol}(\Omega)^{\frac{1}{q} - \frac{1}{2}} \|\operatorname{Avg} u\|_{L^{2}(\Omega)} \\ &\leq C_{n,k,q} \operatorname{vol}(\Omega)^{\frac{1}{q} - \frac{1}{2}} \operatorname{diam}(\Omega) \left(\|d^{k}u\|_{L^{2}(\Omega)} + \|\delta^{k}u\|_{L^{2}(\Omega)} \right) \\ &\quad + \operatorname{vol}(\Omega)^{\frac{1}{q} - \frac{1}{2}} \|u\|_{L^{2}(\Omega)}. \end{aligned}$$

Let us now study the special case where $u \in L^2\Lambda^k(\Omega)$ is any eigenvector of the Hodge– Laplace operator with some eigenvalue $\lambda > 0$. This is also an eigenvector of the potential operator Δ_k^{\dagger} with eigenvalue λ^{-1} . We verify that

$$d^{k}u = d^{k}\lambda\lambda^{-1}u = d^{k}\lambda\Delta_{k}^{\dagger}u = \lambda\delta_{k+1}^{\dagger}u,$$

$$\delta^{k}u = \delta^{k}\lambda\lambda^{-1}u = \delta^{k}\lambda\Delta_{k}^{\dagger}u = \lambda d_{k-1}^{\dagger}u.$$

From the structure of the eigendecomposition of the Hodge–Laplace operator, we know the following: if u is as singular vector of δ_{k+1}^{\dagger} to the singular value $\theta_{k,j}$, then

$$\lambda \delta_{k+1}^{\dagger} u = \theta_{k,j}^{-1} u$$

and if u is as singular vector of d_{k-1}^{\dagger} to the singular value $\theta_{k-1,j}$, then

$$\lambda d_{k-1}^{\dagger} u = \theta_{k-1,j}^{-1} u$$

Hence, the eigenvectors of the Hodge–Laplace operator over a bounded convex domain enjoy a modicum of additional regularity, having Lebesgue norms of a higher exponent that is uniformly bounded in terms of the L^2 norm. This bounds the strengths of its singularities. The estimate depends on lower and upper bounds of the domain volume and the eigenvalue. We can control how that constant depends on the geometry.

4. Approximation of Poincaré-Friedrichs constants and Spectra

We study the spectra of Hodge–Laplace operators over convex domains. We transfer results known for convex domains with smooth boundary to general convex domains, with possibly merely Lipschitz boundary, via a geometric approximation argument.

The main result of this manuscript addresses the convergence of the Poincaré–Friedrichs constants under smooth approximation of the domains. The approximation is up to a uniformly bi-Lipschitz deformation close to the boundary. The convexity enters the proof via the uniform control of the Poincaré–Sobolev and Gaffney inequalities.

Theorem 4.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain. Assume that $C_0, C_1 > 0$ are constants and that $\Omega_{\epsilon} \subseteq \mathbb{R}^n$ be a family of convex domains together with a family of mappings

$$(35) \qquad \qquad \Phi_{\epsilon}: \Omega \to \Omega_{\epsilon}$$

that are Lipschitz with

(36)
$$\operatorname{Lip}(\Phi_{\epsilon}) \le C_0, \quad \operatorname{Lip}(\Phi_{\epsilon}^{-1}) \le C_0$$

and satisfy

(37)
$$\forall x \in \Omega \setminus B_{C_1\epsilon}(\partial \Omega) : \Phi_{\epsilon}(x) = x.$$

If $C_{\mathrm{PF},k,\epsilon}$ is the k-th Poincaré–Friedrichs constants of the Ω_{ϵ} and $C_{\mathrm{PF},k}$ is the k-th Poincaré– Friedrichs constants of the Ω , then

$$\lim_{\epsilon \to 0} C_{\mathrm{PF},k,\epsilon} = C_{\mathrm{PF},k}.$$

Proof. For notational convenience, let us write $\Omega_0 := \Omega$ and let $\Phi_0 : \Omega_0 \to \Omega_0$ be the identity. With that notation, (36) and (37) hold for $\epsilon = 0$ too. We assume without loss of generality that $0 \in \Omega$ such that there exists $\rho > 0$ with $B_{\rho}(0) \subseteq \Omega_{\epsilon}$ for all $\epsilon > 0$ small enough. That gives a lower bound for the volume of the domains Ω_{ϵ} . At the same time, if $\Omega \subseteq B_R(0)$ for some R > 0, then vol $(\Omega_{\epsilon}) \subseteq$ vol $(B_{R+\epsilon}(0))$ is an upper bound for the volume. Hence, for $\epsilon > 0$ small enough, the diameters and volumes of the domains Ω_{ϵ} are close to the diameter and volume, respectively, of the domain Ω .

We write $\mathbf{J}_{\epsilon} : L^p \Lambda^k(\Omega_{\epsilon}) \to L^p \Lambda^k(\Omega_{\epsilon})$ for the pullback operation on differential forms from Ω_{ϵ} onto Ω , which is a bounded operation for all Lebesgue exponents $1 \leq p < \infty$. Notice that \mathbf{J}_{ϵ} is invertible and that the operator norm is uniformly bounded via the Lipschitz constants of Φ_{ϵ} and its inverse.

Let us denote the Poincaré–Friedrichs constant of each domain Ω_{ϵ} by $C_{\text{PF},k,\epsilon}$. Throughout this proof, the bounded linear operator (28) over the domain Ω_{ϵ} is written

$$\boldsymbol{Q}_{k,\epsilon}: L^2 \Lambda^{k+1}(\Omega_{\epsilon}) \to L^2 \Lambda^k(\Omega_{\epsilon}).$$

The operator norm of $\mathbf{Q}_{k,\epsilon}$ equals $C_{\mathrm{PF},k,\epsilon}$. For any $f_{\epsilon} \in L^2 \Lambda^{k+1}(\Omega_{\epsilon})$ with $d^{k+1}f_{\epsilon} = 0$ we have $\mathbf{Q}_{k,\epsilon}f_{\epsilon} \in H\Lambda^k(\Omega_{\epsilon}) \cap H_0^*\Lambda^k(\Omega_{\epsilon})$ with $d^k \mathbf{Q}_{k,\epsilon}f_{\epsilon} = f_{\epsilon}$ and $\delta^k \mathbf{Q}_{k,\epsilon}f_{\epsilon} = 0$. We define the operators

$$\boldsymbol{P}_{k,\epsilon}: L^2 \Lambda^{k+1}(\Omega) \to L^2 \Lambda^k(\Omega), \quad u \mapsto \boldsymbol{J}_{\epsilon} \boldsymbol{Q}_{k,\epsilon} \boldsymbol{J}_{\epsilon}^{-1} u$$

It is clear that the Poincaré–Friedrichs constants of the potential operators $Q_{k,\epsilon}$ and their transformations $P_{k,\epsilon}$ are uniformly bounded for $\epsilon > 0$ small enough. It remains to study their limit behavior as ϵ goes to zero.

We use the singular value decomposition of these compact operators. For each $\epsilon > 0$, there exists an orthonormal system $u_{k,\epsilon,j} \in H\Lambda^k(\Omega_{\epsilon})$ that spans the orthogonal complement of the kernel of (21) over Ω_{ϵ} , another orthonormal system $u_{k+1,\epsilon,j}^{\star} \in H\Lambda^{k+1}(\Omega_{\epsilon})$ that spans the range of (21) over Ω_{ϵ} , and a non-increasing null sequence $\theta_{k,\epsilon,j} \in \mathbb{R}$ such that

$$\boldsymbol{Q}_{k,\epsilon} = \sum_{j=1}^{\infty} \theta_{k,\epsilon,j} u_{k,\epsilon,j} \langle u_{k+1,\epsilon,j}^{\star}, \cdot \rangle.$$

This implies the singular value decompositions

$$\boldsymbol{P}_{k,\epsilon} = \sum_{j=1}^{\infty} \theta_{k,\epsilon,j} \boldsymbol{J}_{\epsilon} \boldsymbol{u}_{k,\epsilon,j} \langle \boldsymbol{J}_{\epsilon} \boldsymbol{u}_{k+1,\epsilon,j}^{\star}, \cdot \rangle$$

It remains to show that $\theta_{k,\epsilon,j} = C_{\mathrm{PF},k,\epsilon}$ converges to $\theta_{k,0,j} = C_{\mathrm{PF},k,0}$.

We first study the difference between the potentials when the source term has higher regularity. Let $q \in [1, \infty)$ with $\frac{1}{2} - \frac{1}{n} \leq \frac{1}{q}$. Recall that for all bounded open subsets $U \subseteq \mathbb{R}^n$ we have $L^q \Lambda^k(U) \subseteq L^2 \Lambda^k(U)$ with

$$\forall u \in L^q \Lambda^k(U) : \|u\|_{L^2(\mathbb{R}^n)} \le \operatorname{vol}(U)^{\frac{1}{2} - \frac{1}{q}} \|u\|_{L^q(\mathbb{R}^n)}.$$

Let $f \in L^q \Lambda^{k+1}(\Omega)$ for some q > 2. We define $f_{\epsilon} \in L^q \Lambda^{k+1}(\Omega_{\epsilon})$ via the pullback $f_{\epsilon} := \mathbf{J}_{\epsilon}^{-1} f$. As the pullback commutes with the exterior derivative, $f_{\epsilon} \in H \Lambda^{k+1}(\Omega_{\epsilon})$ with $d^{k+1} f_{\epsilon} = 0$. We define

$$w_{\epsilon} := \boldsymbol{Q}_{k,\epsilon} f_{\epsilon}, \quad u_{\epsilon} := \boldsymbol{J}_{\epsilon} w_{\epsilon} = \boldsymbol{P}_{k,\epsilon} f.$$

This implies $w_{\epsilon} \in H_0^* \Lambda^k(\Omega_{\epsilon})$ with $\delta^k w_{\epsilon} = 0$. We estimate its Lebesgue norms with higher exponent as in Section 3.3, which yields

$$\|w_{\epsilon}\|_{L^{q}(\Omega_{\epsilon})} \leq C_{n,k,q} \operatorname{vol}(\Omega_{\epsilon})^{\frac{1}{q}-\frac{1}{2}} \operatorname{diam}(\Omega_{\epsilon}) \|\nabla w_{\epsilon}\|_{L^{2}(\Omega_{\epsilon})}$$

where $\operatorname{Avg} w_{\epsilon} := \int_{\Omega_{\epsilon}} w_{\epsilon} \, dx \in \Lambda^{k}(\mathbb{R}^{n})$ denotes the average of u over the domain Ω_{ϵ} . Now,

$$\|\nabla w_{\epsilon}\|_{L^{2}(\Omega_{\epsilon})} \leq \|d^{k}w_{\epsilon}\|_{L^{2}(\Omega_{\epsilon})}$$

Additionally,

$$\begin{aligned} \|\operatorname{Avg} w_{\epsilon}\|_{L^{q}(\Omega_{\epsilon})} &\leq \operatorname{vol}(\Omega)^{\frac{1}{q}-\frac{1}{2}} \|\operatorname{Avg} w_{\epsilon}\|_{L^{2}(\Omega_{\epsilon})} \\ &\leq \operatorname{vol}(\Omega)^{\frac{1}{q}-\frac{1}{2}} C_{\operatorname{PF},k,\epsilon} \|f_{\epsilon}\|_{L^{2}(\Omega_{\epsilon})} + \operatorname{vol}(\Omega_{\epsilon})^{\frac{1}{q}-\frac{1}{2}} C_{\operatorname{PF},\Omega_{\epsilon},k} \|f_{\epsilon}\|_{L^{2}(\Omega_{\epsilon})} \\ &\leq C_{n,k,q} \operatorname{diam}(\Omega_{\epsilon}) \|f_{\epsilon}\|_{L^{q}(\Omega_{\epsilon})} + C_{\operatorname{PF},\Omega_{\epsilon},k} \|f_{\epsilon}\|_{L^{q}(\Omega_{\epsilon})}. \end{aligned}$$

Note that each w_{ϵ} is characterized equivalently as

$$\int_{\Omega_{\epsilon}} w_{\epsilon} \, dx = 0 \quad \text{if } k = 0,$$
$$\int_{\Omega_{\epsilon}} \langle w_{\epsilon}, d^{k-1}\xi_{\epsilon} \rangle \, dx = 0, \quad \xi_{\epsilon} \in H\Lambda^{k-1}(\Omega_{\epsilon}) \text{ if } k \ge 1,$$
$$\int_{\Omega_{\epsilon}} \langle d^{k}w_{\epsilon}, d^{k}v_{\epsilon} \rangle \, dx = \int_{\Omega_{\epsilon}} \langle f_{\epsilon}, d^{k}v_{\epsilon} \rangle \, dx, \quad v_{\epsilon} \in H\Lambda^{k}(\Omega_{\epsilon}).$$

Applying the pullback along $\Phi_{\epsilon} : \Omega \to \Omega_{\epsilon}$, we write $u_{\epsilon} := \Phi_{\epsilon} w_{\epsilon}$ and rewrite these equations as:

$$\int_{\Omega} u_{\epsilon} \det \left(\mathbf{J} \Phi_{\epsilon} \right) \, dx = 0 \text{ if } k = 0,$$

$$\int_{\Omega} \langle \Phi_{\epsilon}^{-*} u_{\epsilon}, \Phi_{\epsilon}^{-*} d^{k-1} \xi \rangle \det \left(\mathbf{J} \Phi_{\epsilon} \right) \, dx = 0, \quad \xi \in H \Lambda^{k-2}(\Omega) \text{ if } k \ge 1,$$

$$\int_{\Omega} \langle \Phi_{\epsilon}^{-*} d^{k} u_{\epsilon}, \Phi_{\epsilon}^{-*} d^{k} v \rangle \det \left(\mathbf{J} \Phi_{\epsilon} \right) \, dx = \int_{\Omega} \langle f, \Phi_{\epsilon}^{-*} d^{k} v \rangle \det \left(\mathbf{J} \Phi_{\epsilon} \right) \, dx, \quad v \in H \Lambda^{k-1}(\Omega).$$

In what follows, we abbreviate

$$\langle u_1, u_2 \rangle_{\epsilon} = \langle \Phi_{\epsilon}^{-*} u_1, \Phi_{\epsilon}^{-*} u_2 \rangle \det (\mathbf{J} \Phi_{\epsilon})$$

whenever u_1 and u_2 are differential forms of the same degree. We now estimate the difference $u - u_{\epsilon}$. If k = 0, then

$$\int_{\Omega} u_{\epsilon} dx = \int_{\Omega} u_{\epsilon} - u_{\epsilon} \det \left(\mathbf{J} \Phi_{\epsilon} \right) dx$$

=
$$\int_{B_{\epsilon}(\partial\Omega) \cap \Omega} u_{\epsilon} - u_{\epsilon} \det \left(\mathbf{J} \Phi_{\epsilon} \right) dx$$

=
$$\int_{B_{\epsilon}(\partial\Omega) \cap \Omega} u - u_{\epsilon} dx \leq \operatorname{vol} \left(\Omega \cap B_{C_{1}\epsilon}(\partial\Omega) \right)^{\frac{1}{2}} \left(1 + C_{0}^{n} \right) \|u_{\epsilon}\|_{L^{2}(\Omega)}.$$

If instead $k \ge 1$, then for all $\xi \in H\Lambda^{k-1}(\Omega)$ we verify

$$\int_{\Omega} \langle u_{\epsilon}, d^{k-1}\xi \rangle \, dx = \int_{\Omega} \langle u_{\epsilon}, d^{k-1}\xi \rangle_{\epsilon} - \langle u_{\epsilon}, d^{k-1}\xi \rangle \, dx$$
$$= \int_{B_{\epsilon}(\Omega)\cap\Omega} \langle u_{\epsilon}, d^{k-1}\xi \rangle_{\epsilon} - \langle u_{\epsilon}, d^{k-1}\xi \rangle \, dx$$
$$= \operatorname{vol}\left(\Omega \cap B_{C_{1}\epsilon}(\partial\Omega)\right)^{\frac{1}{2}-\frac{1}{q}} \left(1 + C_{0}^{n+2k}\right) \|u_{\epsilon}\|_{L^{q}(\Omega)} \|d^{k-1}\xi\|_{L^{2}(\Omega)}$$

Lastly, for all $v \in H\Lambda^{k-1}(\Omega)$ we have

$$\int_{\Omega} \langle d^{k} u_{0} - d^{k} u_{\epsilon}, d^{k} v \rangle_{0} dx$$
$$= \int_{\Omega} \langle d^{k} u_{\epsilon}, d^{k} v \rangle_{\epsilon} - \langle d^{k} u_{\epsilon}, d^{k} v \rangle_{0} dx + \int_{\Omega} \langle f, d^{k} v - \det \left(\mathbf{J} \Phi_{\epsilon} \right) \Phi_{\epsilon}^{-*} d^{k} v \rangle dx.$$

We estimate

$$\begin{aligned} \left| \int_{\Omega} \langle d^{k} u_{\epsilon}, d^{k} v \rangle_{\epsilon} - \langle d^{k} u_{\epsilon}, d^{k} v \rangle_{0} dx \right| \\ &\leq \operatorname{vol} \left(\Omega \cap B_{C_{1}\epsilon}(\partial \Omega) \right)^{\frac{1}{2} - \frac{1}{q}} \left(1 + C_{0}^{n+2(k+1)} \right) \| u_{\epsilon} \|_{L^{q}(\Omega)} \| d^{k} v \|_{L^{2}(\Omega)} \\ &\left| \int_{\Omega} \langle f, d^{k} v - \det \left(\mathbf{J} \Phi_{\epsilon} \right) \Phi_{\epsilon}^{-*} d^{k} v \rangle dx \right| \\ &\leq \operatorname{vol} \left(\Omega \cap B_{C_{1}\epsilon}(\partial \Omega) \right)^{\frac{1}{2} - \frac{1}{q}} \left(1 + C_{0}^{n+k} \right) \| f \|_{L^{q}(\Omega)} \| d^{k} v \|_{L^{2}(\Omega)}. \end{aligned}$$

We know from the Lipschitz estimate of the expansion function ${\boldsymbol S}$ that

$$\boldsymbol{S}\left(B_{1-C_1\epsilon/\rho}(0)\right)\cap B_{C_1\epsilon}(\partial\Omega)=\emptyset.$$

One thus estimates

$$\operatorname{vol}(B_{C_1\epsilon}(\partial\Omega)) \leq \operatorname{vol}(\Omega) - (1 - C_1\epsilon/\rho)\operatorname{vol}(\Omega) = C_1\epsilon/\rho\operatorname{vol}(\Omega).$$

That means

$$\lim_{\epsilon \to 0} \sup_{\substack{f \in L^q \Lambda^k(\Omega) \\ f \neq 0}} \frac{\|\boldsymbol{P}_{k,\epsilon}f - \boldsymbol{P}_{k,0}f\|_{L^2(\Omega)}}{\|f\|_{L^q(\Omega)}} = 0.$$

We thus conclude that the potentials $P_{k,\epsilon}$ converge to $P_{k,0}$ as bounded operators from $L^q \Lambda^k(\Omega)$ into $L^2 \Lambda^{k+1}(\Omega)$ when ϵ goes to zero.

It remains to show that the operator norms of $\mathbf{P}_{k,\epsilon}$ converge to the operator norm of $\mathbf{P}_{k,0}$ as operators from $L^2\Lambda^k(\Omega)$ into $L^2\Lambda^{k-1}(\Omega)$. For every $\epsilon > 0$ there exists $g_{\epsilon} \in L^2\Lambda^k(\Omega)$ with unit L^2 norm such that

$$\sup_{\substack{g \in L^2 \Lambda^k(\Omega) \\ q \neq 0}} \frac{\|\boldsymbol{P}_{k,\epsilon}g\|_{L^2 \Lambda^{k+1}(\Omega)}}{\|g\|_{L^2 \Lambda^k(\Omega)}} = \|\boldsymbol{P}_{k,\epsilon}g_{\epsilon}\|_{L^2 \Lambda^{k+1}(\Omega)} = \theta_{k,\epsilon,1}.$$

We want to show that the $\theta_{k,\epsilon,1}$ converge to $\theta_{k,0,1}$ as ϵ goes to zero. We prove this by contradiction. To this end, we consider any strictly descending null sequence $\epsilon_m > 0$.

First, we bound the limes inferior of the sequence from below. Since $P_{k,0}$ is compact, there exists $g_0 \in L^2 \Lambda^{k+1}(\Omega)$ with $\|g_0\|_{L^2(\Omega)} = 1$ such that

$$\|P_{k,0}g_0\|_{L^2(\Omega)} = \theta_{k,0,1}$$

As discussed in Section 3.3, we know $g_0 \in L^q \Lambda^{k+1}(\Omega)$. Hence,

$$\lim_{m \to \infty} \|\boldsymbol{P}_{k,\epsilon_m} g_0 - \boldsymbol{P}_{k,0} g_0\|_{L^2(\Omega)} = 0$$

This implies that for any $\mu > 0$ we have for m large enough we can ensure

$$\begin{aligned} \theta_{k,0,1} &= \| \boldsymbol{P}_{k,0} g_0 \|_{L^2(\Omega)} \leq \| \boldsymbol{P}_{k,\epsilon_m} g_0 \|_{L^2(\Omega)} + \| \boldsymbol{P}_{k,\epsilon_m} g_0 - \boldsymbol{P}_{k,0} g_0 \|_{L^2(\Omega)} \\ &< \| \boldsymbol{P}_{k,\epsilon_m} g_0 \|_{L^2(\Omega)} + \mu \\ &= \theta_{k,\epsilon_m,1} + \mu. \end{aligned}$$

This shows that

$$\liminf_{m \to \infty} \theta_{k,\epsilon_m,1} \ge \theta_{k,0,1}.$$

Second, we bound the limes superior of the sequence from above. As a consequence of the Banach–Alaoglu theorem, the unit ball in $L^2\Lambda^{k+1}(\Omega)$ is weakly compact. We hence assume without loss of generality, by switching to a subsequence, that g_{ϵ_m} converges weakly to some $g \in L^2\Lambda^{k+1}(\Omega)$. Because $\mathbf{P}_{k,0}$ is compact, it maps weakly convergent sequences to strongly convergent sequences, and hence $\mathbf{P}_{k,0}g_{\epsilon_m}$ converges to $\mathbf{P}_{k,0}g$ in $L^2\Lambda^k(\Omega)$. We now observe

$$\begin{aligned} \theta_{k,\epsilon_m,1} &= \|\boldsymbol{P}_{k,\epsilon_m} g_{\epsilon_m}\|_{L^2(\Omega)} \leq \|\boldsymbol{P}_{k,0} g_{\epsilon_m}\|_{L^2(\Omega)} + \|\boldsymbol{P}_{k,\epsilon_m} g_{\epsilon_m} - \boldsymbol{P}_{k,0} g_{\epsilon_m}\|_{L^2(\Omega)} \\ &\leq \theta_{k,0,1} + \|\boldsymbol{P}_{k,\epsilon_m} g_{\epsilon_m} - \boldsymbol{P}_{k,0} g_{\epsilon_m}\|_{L^2(\Omega)}. \end{aligned}$$

The operators $\boldsymbol{P}_{k,\epsilon_m}$ converge to $\boldsymbol{P}_{k,0}$ as operators from $L^q \Lambda^{k+1}(\Omega)$ into $L^2 \Lambda^k(\Omega)$.

Moreover, each $J_{\epsilon_m}^{-1}g_{\epsilon_m}$ is an eigenvector of the k-th Hodge–Laplace operator over Ω_{ϵ_m} to the smallest eigenvalue of that operator. An upper bound for that eigenvalue follows from any lower bound for $\theta_{k,\epsilon_m,1}$. By the previous discussion, we already know that, say, $\theta_{k,\epsilon_m,1} \geq \theta_{k,0,1}/2$ for ϵ_m small enough.

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This shows that g_{ϵ_m} is bounded in $L^q \Lambda^k(\Omega)$, as that norm depends on $\theta_{k,\epsilon_m,1}^{-1}$. We conclude that $\theta_{k,\epsilon_m,1}$ is a bounded sequence such that

$$\limsup_{n \to \infty} \theta_{k, \epsilon_m, 1} \le \theta_{k, 0, 1}.$$

The combination of the estimates for the limes inferior and the limes superior proves that $\theta_{k,\epsilon_m,1}$ converges to $\theta_{k,0,1}$. The argument is complete.

The convergence of the lowest eigenvalues under approximation of the domain will be generalized to the higher eigenvalues as well. Let us first examine some consequences of this special case. Over strictly convex domains with smooth boundary, the Poincaré–Friedrichs inequalities agree with the smallest eigenvalues of the Hodge–Laplace operators. Here, they form a sequence strictly decreasing in the degree of the forms.

Lemma 4.2. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded strictly convex domain with smooth boundary. Then the Poincaré–Friedrichs constants $C_{\text{PF},k}$ of the k-th Hodge–Laplace operators satisfy

$$C_{\mathrm{PF},k} < C_{\mathrm{PF},1} < \dots < C_{\mathrm{PF},n}.$$

Proof. We use [22, Theorem 2.6]: The smallest eigenvalues of the Hodge–Laplace eigenvalues satisfy

$$\lambda_0 = \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n$$

The eigenvalues are related to the Poincaré–Friedrichs inequalities via

$$\lambda_k = \min\left(C_{\mathrm{PF},k}^{-2}, C_{\mathrm{PF},k-1}^{-2}\right)$$

We already know that $C_{\text{PF},n-1}^{-2} = \lambda_n$. If $C_{\text{PF},k-1}^{-2} = \lambda_k$ for some $2 \leq k \leq n$, then

$$C_{\text{PF},k-1}^{-2} = \lambda_k > \lambda_{k-1} = \min\left(C_{\text{PF},k-1}^{-2}, C_{\text{PF},k-2}^{-2}\right).$$

Repeated application of this observation gives the desired result.

We extend that observation to general convex domains.

Theorem 4.3. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain. Then the Poincaré–Friedrichs constants $C_{\text{PF},k}$ of the k-th Hodge–Laplace operators satisfy

$$C_{\mathrm{PF},k} \le C_{\mathrm{PF},1} \le \dots \le C_{\mathrm{PF},n}$$

Proof. According to Theorem 2.17, there exists a sequence $\Omega_{\epsilon} \subseteq \mathbb{R}^n$ of strictly convex domains with smooth boundary and a family of bi-Lipschitz homeomorphisms $\Phi_{\epsilon} : \Omega \to \Omega_{\epsilon}$ that satisfy the conditions of Theorem 4.1. Writing $C_{\mathrm{PF},k,\epsilon}$ for the k-th Poincaré–Friedrichs constant of Ω_{ϵ} ,

$$\lim_{\epsilon \to 0} C_{\mathrm{PF},k,\epsilon} = C_{\mathrm{PF},k}$$

holds. The desired now follows in combination with Lemma 4.2

Remark 4.4. The results by Guerini and Savo [22] generalize numerous inequalities of Hodge–Laplace eigenvalues when the domain is smooth. It is well-known that the smallest Neumann eigenvalue over a Lipschitz domain is at least the smallest Dirichlet eigenvalue. That the smallest non-zero Neumann eigenvalue over a convex set is, more generally, a lower bound for the spectra of the Hodge–Laplace operators was previously shown by Mitrea [30, Corollary 5.10], who also uses an approximation argument and refers to Bao and Zhou [5] for the case for three-dimensional convex domains. However, that the Poincaré–Friedrichs

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constants of smooth domains constitute a non-increasing sequence is used within the proofs in [22]. We recover this fact from the results stated in that reference when the smallest eigenvalues of the Hodge–Laplace operators are a strictly increasing sequence. These results are only provided for smoothly bounded domains: the convexity of the domain enters the argument via curvature terms along the (smooth) domain boundary. An alternative route towards the result for convex Lipschitz domains might use an intrinsic notion of principal curvatures in a non-smooth setting. Going beyond the Hilbert space setting, it is an open question how these inequalities can be extended to the Poincaré–Friedrichs constants of L^p de Rham complexes.

For completeness of the exposition, we collect a few explicit upper and lower bound for the Poincaré–Friedrichs inequalities of bounded convex domains in Euclidean space. In the original sources, these are stated in terms of eigenvalues of the Hodge–Laplace operators, whereas we can now restate them in terms of Poincaré–Friedrichs constants.

Proposition 4.5. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain. Suppose that Ω contains the ball $B_{\rho}(0)$ and is contained in the ball $B_R(0)$. Then

(38)
$$\operatorname{vol}(\Omega)^{\frac{1}{n}}C_{\mathrm{PF},0}(B_1(0)) \le C_{\mathrm{PF},0} \le \operatorname{diam}(\Omega)/\pi$$

(39)
$$\rho C_{\text{PF},n-1}(B_1(0)) \le C_{\text{PF},n-1} \le R C_{\text{PF},n-1}(B_1(0))$$

If 0 < k < n - 1, then

(40)
$$C_0 \le \frac{\sqrt{n}e^{3/2}}{\sqrt{k(n-k)}} \cdot \operatorname{diam}(\Omega)$$

If 0 < k < n-1, $\gamma > 0$ and $\mathsf{M} \in \mathbb{R}^{n \times n}$ is a matrix with singular values $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n$ such that $\mathsf{M}B_1(0) \subseteq \Omega \subseteq \gamma \mathsf{M}B_1(0)$ and $\mathsf{vol}(\mathsf{M}B_1(0))$ has the maximal possible volume, then

(41)
$$((k+1)(n+2)\gamma^n)^{-\frac{1}{2}}\sigma_{k+1} \le C_k \le \gamma \sqrt{\binom{n}{k}}\sigma_{k+1}$$

Proof. The upper bound $C_{\text{PF},0} \leq \text{diam}(\Omega)^2/\pi$ for the Poincaré constant is well-known. Weinberger showed that among all *n*-dimensional domains of a given volume, the ball with that volume maximizes the smallest non-zero Neumann eigenvalue [39], which implies the lower bound for $C_{\text{PF},0}$.

The operator $\delta_n : H_0^* \Lambda^n(B_\rho(0)) \to L^2 \Lambda^{n-1}(B_\rho(0))$ is a restriction of $\delta_n : H_0^* \Lambda^n(\Omega) \to L^2 \Lambda^{n-1}(\Omega)$, which in turn is a restriction of $\delta_n : H_0^* \Lambda^n(B_R(0)) \to L^2 \Lambda^{n-1}(B_R(0))$. That implies the lower and upper bound for $C_{\mathrm{PF},n-1}$.

As per Theorem 4.3, the smallest eigenvalue of the Hodge–Laplace operator Δ_k equals $C_{\text{PF},k-1,0}$. Now the first upper bounds for $C_{\text{PF},k}$ are a consequence of Théorème 1.3 in [21]. Finally, the last set of lower and upper bounds is a consequence of Theorem 1.1 and subsequent comments in [35].

Remark 4.6. The last set of inequalities, based on Savo's work, relate the Poincaré–Friedrichs constants to the principal radii of a pair of homothetic ellipsoids, one contained in the convex domain and the other containing the convex domain, under the condition that the inner ellipsoid has the maximal possible volume. The last ellipsoid is unique, known as John ellipsoid, and allows the choice of $\gamma \leq n$.

We continue the perturbation theory of Hodge–Laplace spectra that was initiated with Theorem 4.1. While our main interest is in the Poincaré–Friedrichs constants of the Hilbert– de Rham complex, we address the convergence of the Hodge–Laplace spectra over convex domains under small domain perturbations. We prove a slightly stronger result: the optimal potential operators over the approximate domains converge, in a certain sense, to the optimal potential operators of the original domain. Since these operators are compact, this implies the convergence of their singular values. The convergence of the eigenvalues of the respective Hodge–Laplace operators is a consequence.

Theorem 4.7. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain. Assume that $C_0, C_1 > 0$ are constants and that $\Omega_{\epsilon} \subseteq \mathbb{R}^n$ be a family of convex domains together with a family of mappings

(42)
$$\Phi_{\epsilon}: \Omega \to \Omega_{\epsilon}$$

that are Lipschitz with

(43)
$$\operatorname{Lip}(\Phi_{\epsilon}) \le C_0, \quad \operatorname{Lip}(\Phi_{\epsilon}^{-1}) \le C_0$$

and satisfy

(44)
$$\forall x \in \Omega \setminus B_{C_1\epsilon}(\partial \Omega) : \Phi_{\epsilon}(x) = x.$$

Then the optimal potential operators over the domain Ω ,

(45)
$$d_k^{\dagger}: L^2 \Lambda^{k+1}(\Omega) \to L^2 \Lambda^k(\Omega),$$

and the optimal potential operators over the domains Ω_{ϵ} ,

(46)
$$d_{k,\epsilon}^{\dagger}: L^2 \Lambda^{k+1}(\Omega_{\epsilon}) \to L^2 \Lambda^k(\Omega_{\epsilon}),$$

satisfy the convergence in operator norm

(47)
$$\lim_{\epsilon \to 0} \Phi_{\epsilon}^* d_{k,\epsilon}^{\dagger} \Phi_{\epsilon}^{-*} = d_k^{\dagger}$$

Proof. This proof is to be read as a continuation of the proof of Theorem 4.1.

We use the singular value decomposition of the compact potential operators. In what follows, we simplify the notation and write these as

$$\boldsymbol{P}_{k,\epsilon} = \sum_{j=1}^{\infty} \theta_{k,\epsilon,j} h_{k,\epsilon,j} \langle g_{k,\epsilon,j}, \cdot \rangle,$$

where we abbreviate $h_{k,\epsilon,j} := \mathbf{J}_{\epsilon} u_{k,\epsilon,j}$ and $g_{k,\epsilon,j} := \mathbf{J}_{\epsilon} u_{k+1,\epsilon,j}^{\star}$.

Let $\{\epsilon_m\}_m$ be a null sequence. We have seen previously that we can assume without loss of generality, after switching a subsequence, that the unit vectors $g_{k,\epsilon_m,1}$ converges weakly to some $g \in L^2 \Lambda^k(\Omega)$ of norm at most one, and that $\theta_{k,\epsilon_m,1}$ converges to $\theta_{k,0,1}$. Obviously,

$$\begin{split} \langle \boldsymbol{P}_{k,0}g_{k,\epsilon_m,1}, \boldsymbol{P}_{k,0}g_{k,\epsilon_m,1} \rangle &= \langle (\boldsymbol{P}_{k,0} - \boldsymbol{P}_{k,\epsilon_m})g_{k,\epsilon_m,1}, \boldsymbol{P}_{k,0}g_{k,\epsilon_m,1} \rangle \\ &+ \langle \boldsymbol{P}_{k,\epsilon_m}g_{k,\epsilon_m,1}, (\boldsymbol{P}_{k,0} - \boldsymbol{P}_{k,\epsilon_m})g_{k,\epsilon_m,1} \rangle \\ &+ \langle \boldsymbol{P}_{k,\epsilon_m}g_{k,\epsilon_m,1}, \boldsymbol{P}_{k,\epsilon_m}g_{k,\epsilon_m,1} \rangle. \end{split}$$

Once more using the compactness of $P_{k,0}$ and the uniform bounds on the L^q norms of the $g_{k,\epsilon_m,1}$, one finds the limit behavior

$$\langle \boldsymbol{P}_{k,0}g, \boldsymbol{P}_{k,0}g \rangle = \lim_{m \to \infty} \langle \boldsymbol{P}_{k,0}g_{k,\epsilon_m,1}, \boldsymbol{P}_{k,0}g_{k,\epsilon_m,1} \rangle$$

$$= \lim_{m \to \infty} \langle \boldsymbol{P}_{k,\epsilon_m}g_{k,\epsilon_m,1}, \boldsymbol{P}_{k,\epsilon}g_{k,\epsilon_m,1} \rangle$$

$$= \lim_{m \to \infty} \theta_{k,\epsilon_m,1}^2 \|g_{k,\epsilon_m,1}\| = \lim_{m \to \infty} \theta_{k,\epsilon_m,1}^2 = \theta_{k,0,1}^2$$

Since g has a norm of at most one, we conclude that it must be a unit vector. If a sequence of unit vectors in a Hilbert space converges weakly to some unit vector, then that weak limit must be a strong limit. Hence, g is the limit of $g_{k,\epsilon_m,1}$ as m goes to infinity. From

$$h_{k,\epsilon_m,1} - h_{k,0,1} = \mathbf{P}_{k,\epsilon_m} g_{k,\epsilon_m,1} - \mathbf{P}_{k,0} g_{k,0,1}$$

= $\mathbf{P}_{k,\epsilon_m} g_{k,\epsilon_m,1} - \mathbf{P}_{k,\epsilon_m} g_{k,0,1} + \mathbf{P}_{k,\epsilon_m} g_{k,0,1} - \mathbf{P}_{k,0} g_{k,0,1},$

we now easily conclude that also $g_{k,0,1}$ is the limit of $h_{k,\epsilon_m,1}$ as m goes to infinity.

We know use an induction argument. Let us assume that there exists $b \ge 2$ such that for every null sequence $\{\epsilon_m\}_m$ we can switch to a subsequence such that for all $1 \le j \le b-1$ we have the limits,

$$\lim_{m \to \infty} \theta_{k,\epsilon_m,j} = \theta_{k,0,j}, \qquad \lim_{m \to \infty} g_{k,\epsilon_m,j} = g_{k,0,j}, \qquad \lim_{m \to \infty} h_{k,\epsilon_m,j} = h_{k,0,j}.$$

Here, the last two limit is in the strong sense.

We study the sequence of singular values $\theta_{k,\epsilon_m,b}$ and the corresponding vectors $g_{k,\epsilon_m,b}$. First, for any $1 \leq j \leq b-1$ we verify the asymptotic orthogonality

(48a)
$$\lim_{m \to \infty} \langle g_{k,0,b}, g_{k,\epsilon_m,j} \rangle = \langle g_{k,0,b}, g_{k,0,j} \rangle = 0,$$

(48b)
$$\lim_{m \to \infty} \langle g_{k,\epsilon_m,b}, g_{k,0,j} \rangle = \lim_{m \to \infty} \langle g_{k,\epsilon_m,b}, g_{k,0,j} - g_{k,\epsilon_m,j} \rangle = 0.$$

First, we once again bound the limes inferior of the sequence $\theta_{k,\epsilon_m,b}$ from below. By assumption,

$$\|P_{k,0}g_{k,0,b}\|_{L^2(\Omega)} = \theta_{k,0,b}.$$

We know that $g_{k,0,b} \in L^q \Lambda^{k+1}(\Omega)$. Hence,

$$\lim_{m \to \infty} \|\boldsymbol{P}_{k,\epsilon_m} g_{k,0,b} - \boldsymbol{P}_{k,0} g_{k,0,b}\|_{L^2(\Omega)} = 0$$

In conjunction with the asymptotic orthogonality (48), we conclude that for any $\mu > 0$ we have for *m* large enough:

$$\begin{aligned} \|\boldsymbol{P}_{k,0}g_{k,0,b}\|_{L^{2}(\Omega)} &\leq \|\boldsymbol{P}_{k,\epsilon_{m}}g_{k,0,b}\|_{L^{2}(\Omega)} + \|\boldsymbol{P}_{k,\epsilon_{m}}g_{k,0,b} - \boldsymbol{P}_{k,0}g_{k,0,b}\|_{L^{2}(\Omega)} \\ &< \theta_{k,\epsilon_{m},b} + \mu. \end{aligned}$$

This means that

$$\theta_{k,\epsilon_m,b} \ge \theta_{k,0,b} - \mu_{b}$$

Because $\mu > 0$ was arbitrary, it follows that

$$\liminf_{m \to \infty} \theta_{k,\epsilon_m,1} \ge \theta_{k,0,1}$$

We now bound the limes superior of the sequence $\theta_{k,\epsilon_m,b}$ from above. As a consequence of the Banach–Alaoglu theorem, the unit ball in $L^2 \Lambda^{k+1}(\Omega)$ is weakly compact. We hence assume without loss of generality, by switching to a subsequence, that $g_{k,\epsilon_m,b}$ converges weakly to some $g \in L^2 \Lambda^{k+1}(\Omega)$. Since $\mathbf{P}_{k,0}$ is compact, $\mathbf{P}_{k,0}g_{\epsilon_m}$ converges to $\mathbf{P}_{k,0}g$ in $L^2 \Lambda^k(\Omega)$. Therefore,

$$\theta_{k,\epsilon_m,b} = \|\boldsymbol{P}_{k,\epsilon_m}g_{k,\epsilon_m,b}\|_{L^2(\Omega)}$$

$$\leq \|\boldsymbol{P}_{k,0}g_{k,\epsilon_m,b}\|_{L^2(\Omega)} + \|\boldsymbol{P}_{k,\epsilon_m}g_{k,\epsilon_m,b} - \boldsymbol{P}_{k,0}g_{k,\epsilon_m,b}\|_{L^2(\Omega)}.$$

Due to the asymptotic orthogonality (48), we find

(49)
$$\lim_{m \to \infty} \| \boldsymbol{P}_{k,0} \boldsymbol{g}_{k,\epsilon_m,b} \|_{L^2(\Omega)} \le \theta_{k,\epsilon_m,b}.$$

By assumption, each $J_{\epsilon_m}^{-1}g_{k,\epsilon_m,b}$ is an eigenvector of the k-th Hodge–Laplace operator over Ω_{ϵ_m} to the eigenvalue $\theta_{k,\epsilon_m,b}^{-2}$, which is bounded by $4\theta_{k,0,b}^{-2}$ for ϵ_m small enough as shown above. We thus find an upper bound on the L^q norm of $g_{k,\epsilon_m,b}$, uniform for ϵ_m small enough, and use that the operators $\mathbf{P}_{k,\epsilon_m}$ converge to $\mathbf{P}_{k,0}$ as operators from $L^q \Lambda^{k+1}(\Omega)$ into $L^2 \Lambda^k(\Omega)$. Hence, the sequence $\theta_{k,\epsilon_m,b}$ is bounded and satisfies

$$\limsup_{n \to \infty} \theta_{k, \epsilon_m, b} \le \theta_{k, 0, b}.$$

Since the limes inferior and superior agree, $\theta_{k,\epsilon_m,b}$ converges to $\theta_{k,0,b}$.

We now study the convergence of the sequence $g_{k,\epsilon_m,b}$. Similar to above,

$$\langle \boldsymbol{P}_{k,0}g_{k,\epsilon_m,b}, \boldsymbol{P}_{k,0}g_{k,\epsilon_m,b} \rangle = \langle (\boldsymbol{P}_{k,0} - \boldsymbol{P}_{k,\epsilon_m})g_{k,\epsilon_m,b}, \boldsymbol{P}_{k,0}g_{k,\epsilon_m,b} \rangle + \langle \boldsymbol{P}_{k,\epsilon_m}g_{k,\epsilon_m,b}, (\boldsymbol{P}_{k,0} - \boldsymbol{P}_{k,\epsilon_m})g_{k,\epsilon_m,b} \rangle + \langle \boldsymbol{P}_{k,\epsilon_m}g_{k,\epsilon_m,b}, \boldsymbol{P}_{k,\epsilon_m}g_{k,\epsilon_m,b} \rangle.$$

We have uniform bounds on the L^q norms of the $g_{k,\epsilon_m,b}$ for ϵ_m small enough. Together with the compactness of $P_{k,0}$, it follows that

$$\langle \boldsymbol{P}_{k,0}g, \boldsymbol{P}_{k,0}g \rangle = \lim_{m \to \infty} \langle \boldsymbol{P}_{k,0}g_{k,\epsilon_m,b}, \boldsymbol{P}_{k,0}g_{k,\epsilon_m,b} \rangle$$

$$= \lim_{m \to \infty} \langle \boldsymbol{P}_{k,\epsilon_m}g_{k,\epsilon_m,b}, \boldsymbol{P}_{k,\epsilon}g_{k,\epsilon_m,b} \rangle$$

$$= \lim_{m \to \infty} \theta_{k,\epsilon_m,b}^2 ||g_{k,\epsilon_m,b}|| = \lim_{m \to \infty} \theta_{k,\epsilon_m,b}^2 = \theta_{k,0,b}^2$$

We know that g has a norm of at most one, being the weak limit of unit vectors. Moreover, g is orthogonal to $g_{k,0,j}$ for $1 \le j \le b-1$, as seen from

$$\langle g, g_{k,0,j} \rangle = \lim_{m \to \infty} \langle g_{k,\epsilon_m,b}, g_{k,0,j} \rangle = \lim_{m \to \infty} \langle g_{k,\epsilon_m,b}, g_{k,\epsilon_m,j} \rangle + \langle g_{k,\epsilon_m,b}, g_{k,\epsilon_m,j} - g_{k,0,j} \rangle = 0.$$

Since g realizes the operator norm of $P_{k,0}$ on the orthogonal complement of $g_{k,\epsilon_m,j}$, we see that it must be a unit vector. Arguing as above, since g is a unit vector that is the weak limit of a sequence of unit vectors, g is the (strong) limit of that sequence. Finally, from

$$h_{k,\epsilon_m,b} - h_{k,0,b} = \mathbf{P}_{k,\epsilon_m} g_{k,\epsilon_m,b} - \mathbf{P}_{k,0} g_{k,0,b}$$
$$= \mathbf{P}_{k,\epsilon_m} g_{k,\epsilon_m,b} - \mathbf{P}_{k,\epsilon_m} g_{k,0,b} + \mathbf{P}_{k,\epsilon_m} g_{k,0,b} - \mathbf{P}_{k,0} g_{k,0,b},$$

we now easily conclude that also $h_{k,\epsilon_m,b}$ is the limit of $g_{k,0,b}$ as m goes to infinity.

This allows us to use an induction argument that proves the following: For every $b \ge 1$ and any null sequence $\{\epsilon_m\}_m$, switching to a subsequence ensures the limits

(50)
$$\lim_{m \to \infty} \theta_{k,\epsilon_m,j}^2 = \theta_{k,0,j}^2, \qquad \lim_{m \to \infty} g_{k,\epsilon_m,j} = g_{k,0,j}, \qquad \lim_{m \to \infty} h_{k,\epsilon_m,j} = h_{k,0,j},$$
for any $1 \le j \le b$.

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With that observation in place, we finish the proof by showing the convergence of $\mathbf{P}_{k,\epsilon}$ to $\mathbf{P}_{k,0}$ in the operator norm. It suffices to show that every null sequence $\{\epsilon_m\}_m$ contains a subsequence such that $\mathbf{P}_{k,0}$ is the limit of $\mathbf{P}_{k,\epsilon_m}$ in the operator norm. To this end, given any null sequence $\{\epsilon_m\}_m$ and any $\mu > 0$, we choose $b \ge 1$ such that $\theta_{k,0,b} < \mu$. We extract a subsequence such that (50) holds. For some choice of M, every $m \ge M$ satisfies

$$\sum_{\substack{j=1\\j=1}^{b}}^{b} |\theta_{k,\epsilon_{m},j} - \theta_{k,0,j}| + ||g_{k,\epsilon_{m},j} - g_{k,0,j}|| + ||h_{k,\epsilon_{m},j} - h_{k,0,j}|| < \mu,$$

$$\sup_{\substack{f \in L^{2}\Lambda^{k+1}(\Omega)\\||f||=1}} \left| \sum_{j=1}^{b} \theta_{k,\epsilon_{m},j} h_{k,\epsilon_{m},j} \langle g_{k+1,\epsilon_{m},j}, f \rangle - \theta_{k,0,j} h_{k,0,j} \langle g_{k+1,0,j}, f \rangle \right| < \mu.$$

In particular, $\theta_{k,\epsilon_m,b} < 2\mu$ holds then. Therefore, summing up,

$$\|\boldsymbol{P}_{k,\epsilon} - \boldsymbol{P}_{k,0}\| < 4\mu.$$

We conclude that any null sequence $\{\epsilon_m\}_m$ can be restricted to a subsequence such that P_{k,ϵ_m} converges to $P_{k,0}$ in operator norm. This shows the desired result.

Corollary 4.8. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain. Assume that $C_0, C_1 > 0$ are constants and that $\Omega_{\epsilon} \subseteq \mathbb{R}^n$ be a family of convex domains together with a family of mappings

(51)
$$\Phi_{\epsilon}: \Omega \to \Omega_{\epsilon}$$

that are Lipschitz with

(52)
$$\operatorname{Lip}(\Phi_{\epsilon}) \leq C_0, \quad \operatorname{Lip}(\Phi_{\epsilon}^{-1}) \leq C_0$$

and satisfy

(53)
$$\forall x \in \Omega \setminus B_{C_1\epsilon}(\partial \Omega) : \Phi_{\epsilon}(x) = x.$$

Then the non-zero eigenvalues of the Hodge–Laplace operators over Ω_{ϵ} ,

$$(54) 0 < \lambda_{1,\epsilon} < \lambda_{2,\epsilon} < \dots$$

converge to the non-zero eigenvalues of the Hodge–Laplace operators over Ω ,

$$(55) 0 < \lambda_1 < \lambda_2 < \dots$$

5. Vector calculus

In order to make these results available to a wider audience, we also provide them in the language of the vector calculus, used prominently in the discussion of mathematical electromagnetism [32, 17, 38, 41, 23, 24]. Our main focus is on curl and divergence.

Let $\Omega \subseteq \mathbb{R}^3$ be an open set. Recall that $L^2(\Omega)$ is the space of scalar-valued squareintegrable functions defined on Ω and write $L^2(\Omega) := L^2(\Omega)^3$ for vector-valued functions with each component in $L^2(\Omega)$. We write $H^1(\Omega)$ for the space of scalar-valued $L^2(\Omega)$ functions with weak gradients in $L^2(\Omega)$. Similarly, $H(\operatorname{curl}, \Omega)$ denotes the space of vector-valued $L^2(\Omega)$ functions with weak curls in $L^2(\Omega)$, and $H(\operatorname{div}, \Omega)$ denotes the space of vector-valued $L^{2}(\Omega)$ functions with weak divergences in $L^{2}(\Omega)$. Formally,

$$H^{1}(\Omega) = \{ u \in L^{2}(\Omega) \mid \text{grad} \ u \in L^{2}(\Omega) \},\$$
$$H(\operatorname{curl}, \Omega) = \{ u \in L^{2}(\Omega) \mid \operatorname{curl} u \in L^{2}(\Omega) \},\$$
$$H(\operatorname{div}, \Omega) = \{ u \in L^{2}(\Omega) \mid \operatorname{div} u \in L^{2}(\Omega) \}.$$

Writing $\tilde{u} \in L^2(\mathbb{R}^n)$ for the trivial extension of any $u \in L^2(\Omega)$, and $\tilde{u} \in L^2(\mathbb{R}^n)$ for the trivial extension of any $u \in L^2(\Omega)$, the spaces with boundary conditions are defined via

$$H_0^1(\Omega) = \{ u \in L^2(\Omega) \mid \text{grad} \ u \in L^2(\Omega) \},$$
$$H_0(\text{curl}, \Omega) = \{ u \in L^2(\Omega) \mid \text{curl} \ u \in L^2(\Omega) \},$$
$$H_0(\text{div}, \Omega) = \{ u \in L^2(\Omega) \mid \text{div} \ u \in L^2(\Omega) \}.$$

These spaces form the de Rham complex of 3D vector calculus

$$H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl},\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div},\Omega) \xrightarrow{\text{div}} L^2(\Omega)$$

and its dual differential complex

$$L^{2}(\Omega) \xleftarrow{\operatorname{div}} \boldsymbol{H}_{0}(\operatorname{div}, \Omega) \xleftarrow{\operatorname{curl}} \boldsymbol{H}_{0}(\operatorname{curl}, \Omega) \xleftarrow{\operatorname{grad}} H^{1}_{0}(\Omega).$$

The fundamental stability properties of this differential complex are characterized by the Poincaré–Friedrichs inequalities. These characterize the existence of potentials for the respective differential operators.

We say that a Poincaré inequality holds when the following is true: there exists a constant $C_{\text{grad},\Omega}$ such that for every $u \in H^1(\Omega)$ there exists $w \in H^1(\Omega)$ satisfying grad w = grad u and

(56a)
$$\|w\|_{L^2(\Omega)} \le C_{\operatorname{grad},\Omega} \|\operatorname{grad} u\|_{L^2(\Omega)}$$

The analogues for the curl operator and the divergence operator are also known as Weber inequalities. We say that a Poincaré–Friedrichs–Weber inequality holds for the curl operator when the following is true: there exists a constant $C_{\operatorname{curl},\Omega}$ such that for every $\boldsymbol{u} \in \boldsymbol{H}(\operatorname{curl},\Omega)$ there exists $\boldsymbol{w} \in \boldsymbol{H}(\operatorname{curl},\Omega)$ satisfying $\operatorname{curl} \boldsymbol{w} = \operatorname{curl} \boldsymbol{u}$ and

(56b)
$$\|\boldsymbol{w}\|_{L^2(\Omega)} \le C_{\operatorname{curl},\Omega} \|\operatorname{curl} \boldsymbol{u}\|_{L^2(\Omega)}$$

Likewise, we say that a Poincaré–Friedrichs–Weber inequality holds for the divergence operator when the following is true: there exists a constant $C_{\text{div},\Omega}$ such that for every $\boldsymbol{u} \in \boldsymbol{H}(\text{div},\Omega)$ there exists $\boldsymbol{w} \in \boldsymbol{H}(\text{div},\Omega)$ satisfying div $\boldsymbol{w} = \text{div} \boldsymbol{u}$ and

(56c)
$$\|\boldsymbol{w}\|_{L^2(\Omega)} \leq C_{\operatorname{div},\Omega} \|\operatorname{div} \boldsymbol{u}\|_{L^2(\Omega)}.$$

We remark that if any of the inequalities (56) hold, then the set of possible constants includes a minimum, being an intersection of closed intervals.

The main outcome of this manuscript, in the language of vector calculus, reads as follows.

Theorem 5.1. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded convex domain. Then (56) holds and the smallest possible constants satisfy

$$\frac{\operatorname{diam}(\Omega)}{\pi} \ge C_{\operatorname{grad},\Omega} \ge C_{\operatorname{curl},\Omega} \ge C_{\operatorname{div},\Omega},$$

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References

- D. N. ARNOLD, R. S. FALK, AND R. WINTHER, Finite element exterior calculus, homological techniques, and applications, Acta Numer., 15 (2006), pp. 1–155.
- [2] —, Finite element exterior calculus: from Hodge theory to numerical stability, Bull. Amer. Math. Soc. (N.S.), 47 (2010), pp. 281–354.
- [3] D. N. ARNOLD AND S. W. WALKER, The Hellan-Herrmann-Johnson method with curved elements, SIAM J. Numer. Anal., 58 (2020), pp. 2829–2855.
- G. A. BAKER AND J. DODZIUK, Stability of spectra of Hodge-de Rham Laplacians, Mathematische Zeitschrift, 224 (1997), pp. 327–345.
- [5] G. BAO AND Z. ZHOU, An inverse problem for scattering by a doubly periodic structure, Trans. Am. Math. Soc., 350 (1998), pp. 4089–4103.
- [6] M. BEBENDORF, A note on the Poincaré inequality for convex domains, Z. Anal. Anwendungen, 22 (2003), pp. 751–756.
- [7] G. A. BEER, Starshaped sets and the Hausdorff metric, Pac. J. Math., 61 (1975), pp. 21–27.
- [8] C. BERNARDI, M. COSTABEL, M. DAUGE, AND V. GIRAULT, Continuity properties of the inf-sup constant for the divergence, SIAM Journal on Mathematical Analysis, 48 (2016), pp. 1250–1271.
- [9] F. J. BEUTLER, The operator theory of the pseudo-inverse. I: Bounded operators, II: Unbounded operators with arbitrary range, J. Math. Anal. Appl., 10 (1965), pp. 451–470, 471–493.
- [10] Z. BLOCKI, Smooth exhaustion functions in convex domains, Proceedings of the American Mathematical Society, 125 (1997), pp. 477–484.
- [11] A. BONITO, A. DEMLOW, AND J. OWEN, A priori error estimates for finite element approximations to eigenvalues and eigenfunctions of the Laplace-Beltrami operator, SIAM Journal on Numerical Analysis, 56 (2018), pp. 2963–2988.
- [12] S. C. BRENNER AND L. R. SCOTT, The mathematical theory of finite element methods, vol. 15 of Texts in Applied Mathematics, Springer, New York, third ed., 2008.
- [13] N. CHARALAMBOUS AND Z. LU, The spectrum of continuously perturbed operators and the Laplacian on forms, Differential Geometry and its Applications, 65 (2019), pp. 227–240.
- [14] T. CHAUMONT-FRELET, S. NICAISE, AND J. TOMEZYK, Uniform a priori estimates for elliptic problems with impedance boundary conditions, Commun. Pure Appl. Anal., 19 (2020), pp. 2445–2471.
- [15] S. H. CHRISTIANSEN, Résolution des équations intégrales pour la diffraction d'ondes acoustiques et électromagnétiques - Stabilisation d'algorithmes itératifs et aspects de l'analyse numérique, theses, Ecole Polytechnique X, Jan. 2002.
- [16] S.-K. CHUA AND R. L. WHEEDEN, Weighted Poincaré inequalities on convex domains, Math. Res. Lett., 17 (2010), pp. 993–1011.
- [17] M. COSTABEL, A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains, Math. Methods Appl. Sci., 12 (1990), pp. 365–368.
- [18] G. CSATO, B. DACOROGNA, AND S. SIL, On the best constant in Gaffney inequality, J. Funct. Anal., 274 (2018), pp. 461–503.
- [19] M. C. DELFOUR AND J.-P. ZOLÉSIO, Shape analysis via oriented distance functions, J. Funct. Anal., 123 (1994), pp. 129–201.
- [20] P. GRISVARD, *Elliptic problems in nonsmooth domains*, vol. 69 of Class. Appl. Math., Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), reprint of the 1985 hardback ed. ed., 2011.
- [21] P. GUERINI, Spectre du Laplacien de Hodge-de Rham: estimées sur les variétés convexes, Bulletin of the London Mathematical Society, 36 (2004), pp. 88–94.
- [22] P. GUERINI AND A. SAVO, Eigenvalue and gap estimates for the Laplacian acting on p-forms, Trans. Amer. Math. Soc., 356 (2004), pp. 319–344.
- [23] R. HIPTMAIR, Finite elements in computational electromagnetism, Acta Numer., 11 (2002), pp. 237–339.
- [24] R. HIPTMAIR, Maxwell's equations: continuous and discrete, in Computational electromagnetism. Lectures from the CIME school, Cetraro, Italy, June 9–14, 2014, Cham: Springer; Florence: Fondazione CIME, 2015, pp. 1–58.
- [25] M. HOLST AND M. LICHT, Geometric transformation of finite element methods: theory and applications, Appl. Numer. Math., 192 (2023), pp. 389–413.

- [26] M. HOLST AND A. STERN, Geometric variational crimes: Hilbert complexes, finite element exterior calculus, and problems on hypersurfaces, Found. Comput. Math., 12 (2012), pp. 263–293.
- [27] W. G. KOUASSY AND M. KOUROUMA, Estimates for the first non-zero eigenvalue of the Hodge Laplacian acting on differential forms defined on a Riemannian manifold, Afrika Matematika, 31 (2019), p. 333–366.
- [28] J. M. LEE, Introduction to Smooth Manifolds, vol. 218 of Graduate Texts in Mathematics, Springer, New York, second ed., 2013.
- [29] Y. LIN AND Y. WU, Lipschitz star bodies, Acta Math. Sci., Ser. B, Engl. Ed., 43 (2023), pp. 597–607.
- [30] M. MITREA, Dirichlet integrals and Gaffney-Friedrichs inequalities in convex domains, Forum Math., 13 (2001), pp. 531–567.
- [31] L. E. PAYNE AND H. F. WEINBERGER, An optimal Poincaré inequality for convex domains, Arch. Rational Mech. Anal., 5 (1960), pp. 286–292.
- [32] R. PICARD, An elementary proof for a compact imbedding result in generalized electromagnetic theory, Math. Z., 187 (1984), pp. 151–164.
- [33] G. PÓLYA, Remarks on the foregoing paper, Journal of Mathematics and Physics, 31 (1952), pp. 55–57.
- [34] M. REED AND B. SIMON, Methods of Modern Mathematical Physics, vol. 1, Academic press New York, 1972.
- [35] A. SAVO, Hodge-Laplace eigenvalues of convex bodies, Trans. Am. Math. Soc., 363 (2011), pp. 1789– 1804.
- [36] F. A. TORANZOS, Radial functions of convex and star-shaped bodies, Am. Math. Mon., 74 (1967), pp. 278–280.
- [37] S. VREĆICA, A note on starshaped sets, Publ. Inst. Math., Nouv. Sér., 29 (1981), pp. 283–288.
- [38] C. WEBER, A local compactness theorem for Maxwell's equations, Math. Methods Appl. Sci., 2 (1980), pp. 12–25.
- [39] H. F. WEINBERGER, An isoperimetric inequality for the n-dimensional free membrane problem, Journal of Rational Mechanics and Analysis, 5 (1956), pp. 633–636.
- [40] D. WERNER, Funktionalanalysis, Springer-Lehrb., Berlin: Springer Spektrum, 8th revised edition ed., 2018.
- [41] K. J. WITSCH, A remark on a compactness result in electromagnetic theory, Math. Methods Appl. Sci., 16 (1993), pp. 123–129.

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