Classification of abelian finite-dimensional C^* -algebras by orthogonality

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ABSTRACT. The main goal of the article is to prove that if \mathcal{A}_1 and \mathcal{A}_2 are Birkhoff-James isomorphic C^* -algebras over the fields \mathbb{F}_1 and \mathbb{F}_2 , respectively and if \mathcal{A}_1 finite-dimensional, abelian of dimension greater than one, then $\mathbb{F}_1 = \mathbb{F}_2$ and \mathcal{A}_1 and \mathcal{A}_2 are (isometrically) *-isomorphic C^{*}-algebras. Furthermore, it is also proved that for a finite-dimensional C^* -algebra A, we have $\mathcal{L}^{\perp}_\mathcal{A}$ is the sum of minimal ideals which are not skew-fields and $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$ is the sum of minimal ideals which are skew-fields, where $\mathcal{L}_\mathcal{A}$ denotes the set of all left-symmetric elements in \mathcal{A} and for any subset $\mathcal{S} \subseteq \mathcal{A}$, the set \mathcal{S}^\perp represents the set of all elements of A which are Birkhoff-James orthogonal to S . A procedure to extract the minimal ideals which are (commutative) fields is also given.

1. Introduction

By Gelfand transform every unital abelian complex C^* -algebra $\mathcal A$ is $*$ -isomorphic to $C(X)$, the space of complex-valued continuous functions on some compact Hausdorff space. This translates the study of algebraic properties to the study of topological properties (and vice-versa): $C(X)$ is ∗-isomorphic to $C(Y)$ if and only if X and Y are homeomorphic topological spaces. Formally, Gelfand transform is a contravariant equivalence between the category of unital abelian C^* -algebras and the category of the space of compact Hausdorff spaces (see, e.g., [\[2,](#page-18-0) [5,](#page-19-0) [6\]](#page-19-1) for more information).

Recently Tanaka [\[23,](#page-19-2) Theorem 3.5, Corollary 3.6] showed that the same can be achieved by studying the geometrical properties rather than the topological ones: He characterized abelian complex C^* -algebras among all complex C^* -algebras by using only the underlying geometric structure. Moreover, he showed that two complex abelian C^* -algebras are $*$ -isomorphic if and only if their geometric structures are homeomorphic.

The geometric structure was initially defined in terms of *Birkhoff-James orthogonality* (see [\[22,](#page-19-3) Definition 3.4]). Figuratively speaking, suppose we obtain a cast of the closed unit ball of the C^* -algebra norm. We are allowed to examine it with a sufficiently long, infinitesimally thin needle by placing it tangentially in various directions onto the unit sphere of the norm, then translating it parallelly to the center of the ball and examining the points which the translated needle cuts out from the boundary. If x is the touching point of the needle and y is the cut-out point of the translated needle, then the tangentiality of the needle at point x in direction y is equivalent to $||x + \lambda y|| \ge ||x||$ for each scalar λ , that is, to Birkhoff-James orthogonality of x and y (for complex C^* -algebras, the probing needle has two real dimensions). The geometric structure was

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defined in terms of maximal faces of C^* -norm's unit ball and requires no knowledge of algebraic operations. Following the approach outlined above we, in [\[17,](#page-19-4) Theorem 2.1], completely classified the objects in the categories of real or complex finite-dimensional simple C^* -algebras by using only the relation of Birkhoff-James orthogonality, that is, by relying only on the norm structure alone. Another important aspect of our work was that we worked with real as well as complex C^* -algebras simultaneously and even gave a procedure to characterize the underlying field when the dimension of space is greater than one. The theory of real C^* -algebras is similar to complex C^* -algebras, see $[4, 6, 18, 21]$ $[4, 6, 18, 21]$ $[4, 6, 18, 21]$ $[4, 6, 18, 21]$ (and $[19]$ for a review of their applications) though being able to characterize the underlying field of a given real or complex C^* -algebra, from Birkhoff-James orthogonality alone, was still a bit surprising.

In this article we continue our work in the categories of finite-dimensional real or complex C^* algebras and characterize *(pseudo-)abelian* C^* -algebras together with the underlying fields when the dimension of C^* -algebra is greater than one. A few notations are in order. In the sequel, $\mathcal A$ will stand for a finite-dimensional C^{*}-algebra over the field $\mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \}$. We will denote the matrix block decomposition of a complex C^* -algebra $\mathcal A$ by

$$
(1.1) \qquad \mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_\ell}(\mathbb{C})
$$

for some positive integers $n_1 \leq \cdots \leq n_\ell$. Similarly, the matrix block decomposition of a real C^* -algebra A will be denoted by

(1.2)
$$
\mathcal{M}_{n_1}(\mathbb{K}_1) \oplus \cdots \oplus \mathcal{M}_{n_\ell}(\mathbb{K}_\ell)
$$

where $\mathbb{K}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ with real dimension d_1, \ldots, d_ℓ such that $d_i \leq \cdots \leq d_j$ whenever $n_i = \cdots = n_j$ (every A has such decomposition, see [\[6,](#page-19-1) Theorem 1.5, Theorem 8.4] for more information).

Recall that a non-zero two-sided ideal of A is minimal if it does not properly contain any other non-trivial two-sided ideal. We will refer to the sum of those minimal ideals of A which are skewfields as a pseudo-abelian summand of A and we will refer to the sum of abelian minimal ideals as an *abelian summand* of A . If C^* -algebra is decomposed as in (1.1) or (1.2) , then its minimal ideals coincide with the individual blocks, its pseudo-abelian summand coincides with the sum of all blocks of size one, and its abelian summand coincides with the sum of all blocks of size one over the real or complex field; see, e.g., Wedderburn-Artin theorem [\[7,](#page-19-8) V.4.6]. Therefore, if $\mathbb{F} = \mathbb{C}$ the pseudo-abelian summand coincides with the abelian summand of A . Also, if A is an abelian C^* -algebra, then its pseudo-abelian summand equals \mathcal{A} . However, in case $\mathbb{F} = \mathbb{R}$ the pseudo-abelian summand might contain a quaternionic block in which case it is not abelian. We will say that $\mathcal A$ is a pseudo-abelian C^* -algebra if it equals to its pseudo-abelian summand. In the case when $\mathbb{F} = \mathbb{C}$ this is the same as an abelian C^* -algebra.

Let us briefly discuss also the definition and basic properties of Birkhoff-James (BJ) orthogonality. For two vectors v, w in a normed space V over a field \mathbb{F} , v is said to be BJ orthogonal to w, denoted by $v \perp w$ if

$$
||v|| \le ||v + \lambda w|| \text{ for all } \lambda \in \mathbb{F}.
$$

One can easily see that this relation is homogeneous and that, equivalently, $v \perp w$ if and only if $f_v(w) = 0$ for some supporting functional f_v at v (that is, $||f_v|| = 1$ and $f_v(v) = ||v||$), see [\[13,](#page-19-9) Theorem 2.1 or [\[8,](#page-19-10) Equation 2.1]. Therefore, if we define the *outgoing neighborhood* of v by

(1.3)
$$
v^{\perp} = \{ w \in V; \ v \perp w \},
$$

then we have

(1.4)
$$
v^{\perp} = \bigcup \{ \ker f; \ f \text{ is supporting functional at } v \}.
$$

A bijective map $\phi: V \to V'$ is a *BJ isomorphism* between V and V' if

$$
v \perp w \iff \phi(v) \perp \phi(w) \quad \text{ for all } v, w \in V.
$$

Two normed spaces V and V' are *BJ isomorphic* if there exists a BJ isomorphism between them.

We can now state our main results. Recall that a finite-dimensional C^* -algebra is pseudo-abelian if it contains contains only blocks of size one in its matrix block decomposition.

Theorem 1.1. Let \mathcal{A}_1 and \mathcal{A}_2 be two BJ isomorphic C^{*}-algebras over the fields \mathbb{F}_1 and \mathbb{F}_2 . If \mathcal{A}_1 is finite-dimensional pseudo-abelian C^* -algebra with $\dim \mathcal{A}_1 \geq 2$, then the following are true:

- (1) $\mathbb{F}_1 = \mathbb{F}_2$
- (2) A_1 and A_2 are isomorphic as C^* -algebras, so in particular, A_2 is pseudo-abelian and $\dim \mathcal{A}_1 = \dim \mathcal{A}_2.$

It is immediate that if, in the Theorem [1.1,](#page-2-0) \mathcal{A}_1 is a finite-dimensional abelian C^* -algebra, then \mathcal{A}_2 is also abelian. We further remark that BJ orthogonality alone cannot determine the underlying field in one-dimensional C^{*}-algebras because the real C^{*}-algebra $\mathcal{M}_1(\mathbb{R}) = \mathbb{R}$ and the complex C^* -algebra $\mathcal{M}_1(\mathbb{C}) = \mathbb{C}$ are BJ isomorphic; see [\[17,](#page-19-4) Example 2.2].

The above theorem can be seen as a partial extension of a recent result [\[23,](#page-19-2) Corollary 3.6] which Tanaka proved for complex C^* -algebras: if two complex C^* -algebras \mathcal{A}_1 and \mathcal{A}_2 are BJ isomorphic and one of them is abelian, then they are isomorphic as C^* -algebras. Within Theorem [1.2](#page-3-0) below we will further generalize Theorem [1.1](#page-2-0) to include also the possibility when A_1, A_2 are BJ isomorphic but not pseudo-abelian; when combined, the two theorems imply that the pseudo-abelian summands of A_1 and of A_2 are isomorphic as C^* -algebras (provided A_1, A_2 are not one-dimensional). The characterization is based on the notion of left-symmetricity (see [\[16,](#page-19-11) [20\]](#page-19-12) and also [\[24\]](#page-19-13)). A vector v in a normed space V is *left-symmetric* if

$$
(v \perp w) \implies (w \perp v),
$$

and is right-symmetric if $(w \perp v) \implies (v \perp w)$. By using the outgoing neigborhood defined within [\(1.3\)](#page-2-1) and the *incoming neighborhood* $\perp v := \{w \in V; w \perp v\}$ it is easily seen that v is left-symmetric if and only if

$$
v^{\perp} \subseteq {}^{\perp}v,
$$

and is right-symmetric if and only if the reversed inclusion holds. For a subset S of V , we will use the notation $\mathcal{L}_{\mathcal{S}}$ for the set of all left-symmetric vectors relative to \mathcal{S} , i.e.

$$
\mathcal{L}_{\mathcal{S}} := \{ v \in \mathcal{S}; \ v^{\perp} \cap \mathcal{S} \subseteq {}^{\perp}v \cap \mathcal{S} \}.
$$

In particular, if $S = V$, then \mathcal{L}_V is the set of all left-symmetric vectors. We will also use the notations $\mathcal{L}_{\mathcal{S}}^{\perp} := \bigcap_{v \in \mathcal{L}_{\mathcal{S}}} v^{\perp}$ and $\mathcal{L}_{\mathcal{S}}^{\perp \perp} := \bigcap_{v \in \mathcal{L}_{\mathcal{S}}^{\perp}} v^{\perp}$.

Given a finite-dimensional C^* -algebra A, we will call the sum of all its minimal ideals which are not skew-fields (that is, the sum of all its blocks of sizes bigger than one) to be the nonpseudo-abelian summand of A. In particular, a finite-dimensioal C^* -algebra is a sum of its nonpseudo-abelian and its pseudo-abelian summands.

Theorem 1.2. Let A be a finite-dimensional C^{*}-algebra over field \mathbb{F} . Then, \mathcal{L}^{\perp}_A is the nonpseudoabelian summand and $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$ is the pseudo-abelian summand of \mathcal{A} .

The set $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$ is therefore a C^{*}-algebra and in fact classifies pseudo-abelian finite-dimensional C^* -algebras as follows:

Corollary 1.3. Let A be a finite-dimensional C^* -algebras over \mathbb{F} . Then A is a pseudo-abelian C^* -algebra if and only if $\mathcal{L}^{\perp \perp}_{\mathcal{A}} = \mathcal{A}$.

Theorem [1.1](#page-2-0) suggests that BJ orthogonality alone can determine whether a finite-dimensional C^* -algebra is abelian and, if not, to extract its abelian summand. We will provide a positive solution to this problem in the Section [5,](#page-17-0) once we develop the necessary machinery to formalize the procedure that isolates the quaternionic blocks in the pseudo-abelian summand.

Remark 1.4. (a) Finite-dimensional complex C^* -algebras are von-Neumann algebras (see, e.g., [\[2\]](#page-18-0) for more on von-Neumann algebras). Recall that an element p of von-Neumann algebra A is called a central projection if $p^2 = p = p^*$ (a projection) and p commutes with all other elements of A. A projection $p \in A$ is called abelian if pAp is commutative. In case of factor von-Neumann algebras $\mathcal{M}_n(\mathbb{C})$, central abelian projections exist only when $n = 1$. Thus, the abelian summand equals the complex linear span of abelian central projections. A non-zero projection p is minimal if the only non-zero projection $q \in A$ such that $q \leq p$ is $q = p$. Since p Ap is also a von Neumann algebra with p as an identity, it is easily seen that this is equivalent to the fact that pAp is a field. Thus, in case $\mathbb{F} = \mathbb{C}$, Theorem [1.2](#page-3-0) says that $\mathcal{L}^{\perp}_\mathcal{A}$ is the complex linear span of minimal non-central projections and $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$ is the complex linear span of abelian central projections (or abelian summand).

(b) In a related study [\[16,](#page-19-11) Theorem 3.2] the authors classified elements, left-symmetric relative to the positive cone of general complex C^* algebras. These are exactly scalar multiples of minimal projections.

A proof of Theorems [1.1](#page-2-0) and [1.2](#page-3-0) will be given in Section [4.](#page-16-0) In Section [2](#page-4-0) we characterize leftsymmetric elements and right-symmetric elements in Lemmas [2.2](#page-6-0) and [2.4.](#page-8-0) In Lemma [2.4](#page-8-0) we prove that right-symmetric elements are exactly scalar multiples of unitaries. This extends [\[24,](#page-19-13) Theorem 2.5] to general finite-dimensional C^* -algebras. As a consequence, every BJ isomorphism will map the set of scalar multiples of unitaries to itself. Section [3](#page-11-0) is devoted to developing the tools to prove Theorem [1.1.](#page-2-0) In [\(3.7\)](#page-12-0) and [\(3.9\)](#page-13-0), formulas to find the dimension of an pseudo-abelian C^* -algebra $\mathcal A$ are provided and in Corollary [3.3,](#page-12-1) a characterization of the underlying field of A is given, provided the dimension of A is greater than one. Lemma [3.5](#page-13-1) gives a procedure to find the number of blocks in the matrix block decomposition of an pseudo-abelian C^* -algebra. In section [5](#page-17-0) we extract the abelian summand and give a characterization of abelian C^* -algebras in terms of BJ orthogonality.

2. Symmetricity and smoothness

Any $A = \bigoplus_{k=1}^{\ell} A_k \in \bigoplus_{k=1}^{\ell} \mathcal{M}_{n_k}(\mathbb{K}_k)$ acts on a column vector $x = \bigoplus_{k=1}^{\ell} x_k \in \widehat{\mathbb{K}} = \bigoplus_{k=1}^{\ell} \mathbb{K}_k^{n_k}$ by $Ax = \bigoplus_{k=1}^{\ell} (A_k x_k)$. We let (a row vector) x^* be its conjugate transpose. If needed we will regard $\mathbb{K}_k^{n_k}$ as a right-vector space over the (skew) field \mathbb{K}_k ; the above action, when restricted to a summand $\mathbb{K}_k^{n_k}$, then induces a \mathbb{K}_k -linear operator. We regard $\widehat{\mathbb{K}}$ as a (right) F-vector space and equip it with a natural F-valued inner product by

$$
\Big\langle \bigoplus_{k=1}^{\ell} x_k, \bigoplus_{k=1}^{\ell} y_k \Big\rangle_{\mathbb{F}} = \sum_{k=1}^{\ell} \langle x_k, y_k \rangle_{\mathbb{F}} \quad \text{where} \quad \langle x_k, y_k \rangle_{\mathbb{F}} := \begin{cases} \text{Re } y_k^* x_k, & \text{if } \mathbb{F} = \mathbb{R} \\ y_k^* x_k, & \text{if } \mathbb{F} = \mathbb{C} \end{cases}
$$

Notice that $\langle x_k, y_k \rangle_{\mathbb{F}} = y_k^* x_k$ only in complex C^{*}-algebras in which case $\mathbb{K}_k = \mathbb{C}$ for each k and $\widehat{\mathbb{K}} = \mathbb{C}^{n_1 + \dots + n_\ell}$. Notice also that

(2.5)
$$
\langle x_k \lambda, y_k \lambda \rangle_{\mathbb{F}} = |\lambda|^2 \langle x_k, y_k \rangle_{\mathbb{F}}; \qquad \lambda \in \mathbb{K}_k
$$

which is clear if $\mathbb{F} = \mathbb{C}$ and is also clear if $\mathbb{F} = \mathbb{R}$ and x, y belong to \mathbb{R}^n or \mathbb{C}^n . However, if $x_k, y_k \in \mathbb{H}^n$ we have $\langle x_k \lambda, y_k \lambda \rangle_{\mathbb{F}} = \text{Re}\,\overline{\lambda}(y_k^* x_k)\lambda$. Here we decompose $y_k^* x_k \in \mathbb{K}_k = \mathbb{H}$ into its real and purely imaginary part and use that the conjugation $\lambda(y_k^*x_k)\lambda$ maps purely imaginary part again into purely imaginary part, so $\text{Re }\overline{\lambda}(y_k^* x_k) \lambda = \overline{\lambda} \text{Re}(y_k^* x_k) \lambda = |\lambda|^2 \text{Re}(y_k^* x_k)$.

This inner product defines a norm on $\widehat{\mathbb{K}}$ and the induced operator norm for $A = \bigoplus_{k=1}^{\ell} A_k \in$ $\bigoplus_{k=1}^{\ell} \mathcal{M}_{n_k}(\mathbb{K}_k)$ coincides with C^* -norm and satisfies

$$
\left\| \bigoplus_{k=1}^{\ell} A_k \right\| = \max \{ \|A_k\|; \ 1 \le k \le \ell \}.
$$

For its computation, we recall the singular value decomposition for $\mathcal{M}_n(\mathbb{K})$. It states that for $A \in \mathcal{M}_n(\mathbb{K})$, there exists K-orthonormal basis $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ (i.e., $x_i^* x_j = y_i^* y_j = \delta_{ij}$, Kronecker delta) such that $A = \sum_{n=1}^{\infty} A_n$ $i=1$ $\sigma_i y_i x_i^*$, where $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ are singular values of A arranged in the decreasing order, with $||A|| = \sigma_1$ (see [\[25,](#page-19-14) Theorem 7.2] for the singular value decomposition for $\mathcal{M}_n(\mathbb{H})$.

For $A = \bigoplus_{k=1}^{\ell} A_k \in \mathcal{A}$, we define

$$
M_0(A) = \{ x \in \mathbb{\hat{K}}; \ \|Ax\| = \|A\| \|x\| \}.
$$

We also define $M_0^*(A) = \bigoplus_{k=1}^{\ell} M_0^*(A_k)$ where

$$
M_0^*(A_k) = \begin{cases} M_0(A_k), & \text{if } \|A_k\| = \|A\| \\ 0_k, & \text{if } \|A_k\| < \|A\| \end{cases}.
$$

Notice that $M_0(A_k)$ is a K_k-subspace of $\mathbb{K}_k^{n_k}$ (see, e.g., [\[17,](#page-19-4) Lemma 3.1]), hence in particular an F-subspace, so $M_0^*(A)$ is also an F-subspace of K. We will prove in the next lemma that $M_0(A) = M_0^*(A)$; therefore $M_0(A)$ is also a F-subspace of K. We will use 0_{n_k} for the zero matrix in $\mathcal{M}_{n_k}(\mathbb{K}_k)$.

.

Lemma 2.1. Let $\mathcal{A} = \bigoplus_{k=1}^{\ell} \mathcal{M}_{n_k}(\mathbb{K}_k)$ be a C^* -algebra and let $A, B \in \mathcal{A}$. Then:

- (i) $B \in A^{\perp}$ if and only if there exists a normalized vector $x \in M_0(A)$ such that $\langle Ax, Bx \rangle_{\mathbb{F}} = 0$.
- (ii) If $A = \bigoplus_{k=1}^{\ell} A_k$ is its decomposition, then

$$
M_0^*(A_k) \subseteq M_0(A) = M_0^*(A) \subseteq \bigoplus_{k=1}^{\ell} M_0(A_k).
$$

Consequently, if $||A_i|| < \max{||A_k||}$; $1 \le k \le \ell$ } = $||A||$, then

$$
A^{\perp} = (A_1 \oplus \cdots \oplus A_{i-1} \oplus 0_{n_i} \oplus A_{i+1} \oplus \cdots \oplus A_{\ell})^{\perp}.
$$

Proof. For (i) we note that the BJ orthogonality of two vectors v and w in a normed space V depends only on the two-dimensional subspace generated by v and w. With this in mind, if $\mathbb{F} = \mathbb{R}$, consider the embedding of real C^{*}-algebras $M_n(\mathbb{C})$ and $M_n(\mathbb{H})$ into $M_{2n}(\mathbb{R})$ and $M_{4n}(\mathbb{R})$, respectively. This way, a C^* -algebra $\mathcal{A} = \bigoplus_{k=1}^{\ell} \mathcal{M}_{n_k}(\mathbb{K}_k)$ embeds into $\mathcal{M}_{\sum d_k n_k}(\mathbb{F})$ where $d_k = \dim_{\mathbb{R}} \mathbb{K}_k$ in case of $\mathbb{F} = \mathbb{R}$ and where $d_k = \dim_{\mathbb{C}} \mathbb{K}_k = 1$ in case of $\mathbb{F} = \mathbb{C}$ (since $\mathbb{K}_k = \mathbb{C}$ when $\mathbb{F} = \mathbb{C}$). Then, (i) follows by Stampfli-Magajna-Bhatia-Semrl classification (see $[3,$ Theorem 1 $]$ or, for some historical background, [\[17,](#page-19-4) Proposition 3.2] and [\[10,](#page-19-15) page 2716]) applied on $\mathcal{M}_{\sum d_k n_k}(\mathbb{F})$.

For (ii), we first show $M_0(A) = M_0^*(A)$. Let $x = \bigoplus_{k=1}^{\ell} x_k \in M_0(A)$. Then, $||Ax|| = ||A|| ||x||$ and so

$$
||A||2||x||2 = ||Ax||2 = \sum_{k=1}^{\ell} ||A_k x_k||2 \le \sum_{k=1}^{\ell} ||A_k||2 ||x_k||2 \le \max_{1 \le k \le \ell} ||A_k||2 (\sum_{k=1}^{\ell} ||x_k||2) = ||A||2 ||x||2.
$$

Now, it implies equalities overall so that

$$
||A_k x_k||^2 = ||A_k||^2 ||x_k||^2 = (\max_{1 \le k \le \ell} ||A_k||^2) ||x_k||^2 = ||A||^2 ||x_k||^2
$$
 for all $1 \le k \le \ell$.

Therefore, if $||A_i|| < \max_{1 \le k \le \ell} ||A_k||$ we have $x_i = 0$, while if $||A_i|| = \max_{1 \le k \le \ell} ||A_k||$, we have $x_i \in M_0(A_i)$. It implies $M_0(A) \subseteq M_0^*(A)$. Conversely, if $x = \bigoplus_{k=1}^{\ell} x_k \in M_0^*(A)$, let Λ be the collection of all indices *i* such that $||A_i|| = ||A||$. By definition of $M_0^*(A)$ we have $x_k = 0$ if $k \notin \Lambda$, so that

$$
||Ax||^2 = \sum_{k=1}^{\ell} ||A_k x_k||^2 = \sum_{i \in \Lambda} ||A_i x_i||^2 = ||A||^2 \sum_{i \in \Lambda} ||x_i||^2 = ||A||^2 ||x||^2.
$$

This proves $x \in M_0(A)$, hence $M_0^*(A) \subseteq M_0(A)$. The other containment in (ii) follows directly from the definitions, while the last statement of the lemma follows from (i) and (ii). \Box

We now begin to investigate algebraic properties of elements of A with BJ orthogonality. Let us record a trivial observation which classifies the 0 element in terms of BJ orthogonality. We will tacitly used it in many subsequent lemmas when claiming that BJ orthogonality alone determines a relevant property:

$$
A=0\Longleftrightarrow A\perp A.
$$

Lemma 2.2. Let $\mathcal{A} = \mathcal{M}_{n_1}(\mathbb{K}_1) \oplus \cdots \oplus \mathcal{M}_{n_\ell}(\mathbb{K}_\ell)$ with $n_1 = \cdots = n_p = 1$ and $n_{p+1}, \ldots, n_\ell \ge 2$ for some $p \ge 0$. If $p = 0$, then A has no non-zero left-symmetric elements. If $p \ge 1$, then the following are equivalent for a non-zero element $A \in \mathcal{A}$:

- (i) A is left-symmetric element of A ,
- (ii) The block decomposition of A has only one non-zero entry and it belongs to $\mathcal{M}_1(\mathbb{K}_i)$ for some $i \in [1, p]$.

Proof. Let A be a normalized left-symmetric and let i be such that $||A_i|| = ||A|| = 1$. Without loss of generality, $A_i = \sum_i = \text{diag}(\sigma_1^i, \dots, \sigma_{n_i}^i)$ is already in its singular value decomposition form where $1 = \|A_i\| = \sigma_1^i \geq \cdots \geq \sigma_{n_i}^i \geq 0$. Consider now a matrix $B_i = e_2^i (e_2^i)^* \in \mathcal{M}_{n_i}(\mathbb{K}_i)$ and let $B = (\bigoplus_{k=1}^{i-1} 0_{n_k}) \oplus B_i \oplus (\bigoplus_{k=i+1}^{\ell} 0_{n_k}).$ Notice that A attains its norm on $x = (\bigoplus_{k=1}^{i-1} 0_k) \oplus e_1^i \oplus e_k^i$ $(\bigoplus_{k=i+1}^{\ell} 0_k)$ and

$$
\langle Ax, Bx \rangle_{\mathbb{F}} = \langle A_i e_1^i, B_i e_1^i \rangle_{\mathbb{F}} = 0,
$$

so $A \perp B$ by Lemma [2.1.](#page-5-0) Being left-symmetric, this implies $B \perp A$. Note also that B attains its norm only on vectors $y \in (\bigoplus_{k=1}^{i-1} 0_k) \oplus e_2^i \mathbb{K}_i \oplus (\bigoplus_{k=i+1}^{\ell} 0_k)$, and maps them into themselves, so $B \perp A$ forces

$$
0 = \langle By, Ay \rangle_{\mathbb{F}} = \langle B_i e_2^i, A_i e_2^i \rangle_{\mathbb{F}} = \sigma_2^i.
$$

Similarly, $\sigma_j^i = 0$ for all $2 \leq j \leq n_i$. Thus, $A_i = \text{diag}(1, 0, \ldots, 0) = e_1^i (e_1^i)^*$. If now $n_i > 1$, consider $B_i = (ae_1^i + be_2^i)(ae_1^i + be_2^i)^* - \frac{1}{3}$ $\frac{1}{3}(-be_1^i + ae_2^i)(-be_1^i + ae_2^i)^*$ where $(a,b) = (-1/2, \sqrt{3}/2)$ and $B = (\bigoplus_{k=1}^{i-1} 0_{n_k}) \oplus B_i \oplus (\bigoplus_{k=i+1}^{\ell} 0_{n_k})$. Notice that A attains its norm on $x = (\bigoplus_{k=1}^{i-1} 0_k) \oplus$ $e_1^i \oplus (\bigoplus_{k=i+1}^{\ell} 0_k)$ and that

$$
\langle Ax, Bx \rangle_{\mathbb{F}} = \langle A_i e_1^i, B_i e_1^i \rangle_{\mathbb{F}} = 0,
$$

so $A \perp B$. Notice also that B attains its norm only on a multiple of $y = (\bigoplus_{k=1}^{i-1} 0_k) \oplus (ae_1^i +$ be_2^i \oplus $(\bigoplus_{k=i+1}^{\ell} 0_k)$ and that $\langle By, Ay \rangle_{\mathbb{F}} = \langle B_i y_i, A_i y_i \rangle_{\mathbb{F}} = a^2 \neq 0$, so $B \not\perp A$. Hence, A is not left-symmetric, a contradiction.

The only possibilities left for A are of form $\alpha_1 \oplus \cdots \oplus \alpha_p \oplus A_{p+1} \oplus \cdots \oplus A_{\ell}$, with $||A_k|| < ||A||$ for $k > p$ (we identified the 1-by-1 summands with scalars $\alpha_k \in \mathbb{K}_k$). We first claim that each of A_k is zero for $k > p$. Namely, we clearly have

$$
0^p\oplus \mathcal{M}_{n_{p+1}}(\mathbb{K}_{p+1})\oplus \cdots \oplus \mathcal{M}_{n_\ell}(\mathbb{K}_\ell) \subseteq (\alpha_1\oplus \cdots \oplus \alpha_p\oplus A_{p+1}\oplus \cdots \oplus A_\ell)^{\perp},
$$

(here, 0^p denotes p repeated zeros).

Since A is left-symmetric, it implies that $X = (\bigoplus_{k=1}^{p+i-1} 0_{n_k}) \oplus A_{p+i} \oplus (\bigoplus_{k=p+i+1}^{\ell} 0_{n_k})$ satisfies $X \perp A$. Clearly if $X \neq 0$ it achieves its norm only inside its unique non-zero block. Then, however, $X \perp A$ is equivalent to $A_{p+i} \perp A_{p+i}$, so that $A_{p+i} = 0$ since the only matrix in $\mathcal{M}_{n_{p+i}}(\mathbb{K}_{p+i})$ which is orthogonal to all matrices is a zero matrix.

Hence, the only possibilities left for A are of form $\alpha_1 \oplus \cdots \oplus \alpha_p \oplus 0_{n_{p+1}} \oplus \cdots \oplus 0_{n_\ell}$. We claim that only one of α_i can be non-zero. Without loss of generality $|\alpha_1| = ||A||$. Assume $\alpha_2 \neq 0$ and notice that

$$
0\oplus \overline{\alpha_2} \oplus 0^{p-2}\oplus 0_{n_{p+1}}\oplus \cdots \oplus 0_{n_\ell}\in (\alpha_1\oplus \alpha_2\oplus \cdots \oplus \alpha_p\oplus 0_{n_{p+1}}\oplus \cdots \oplus 0_{n_\ell})^\perp
$$

,

but they are not mutually BJ orthogonal. Indeed, $\alpha_2 = 0$, and the only possibility for left-symmetric A is that all, except possibly one, of its blocks are zero and the non-zero block is of size 1-by-1.

It is easily observed that these are indeed left-symmetric elements, namely: given $A = \alpha \oplus \alpha$ $\{0\}^{p-1} \oplus (\bigoplus_{k=p}^{\ell} 0_{n_k}),$ choose any $B = \beta \oplus (\bigoplus_{k=2}^{\ell} B_k) \in A^{\perp}.$ Note that, modulo a multiplication by scalars, $x = 1 \oplus 0 \oplus \cdots \oplus 0$ is the only norm-attaining vector for A. Then, $0 = \langle Ax, Bx \rangle_{\mathbb{F}} =$ $\langle \alpha \cdot 1, \beta \cdot 1 \rangle_{\mathbb{F}} = \text{Re}(\bar{\beta}\alpha)$ (or it equals $\bar{\beta}\alpha$ if $\mathbb{F} = \mathbb{C}$). Now, if $|\beta| = ||B||$, then x is also a normattaining vector for B and we have $B \perp A$. If $|\beta| \neq ||B||$, then it is clear that $B \perp A$. It implies A is left-symmetric.

We now characterize right-symmetric elements. The following lemma which generalizes [\[17,](#page-19-4) Lemma 3.3] will be useful.

Lemma 2.3. Let $\mathcal{A} = \bigoplus_{k=1}^{\ell} \mathcal{M}_{n_k}(\mathbb{K}_k)$ be a C^* -algebra over \mathbb{F} . For elements $A = \bigoplus_{k=1}^{\ell} A_k$ and $B=\bigoplus_{k=1}^\ell B_k$ we have $A^\perp\subseteq B^\perp$ if and only if for all $1\leq i\leq \ell$, we have $M_0^*(A_i)\subseteq M_0^*(B_i)$ and there exist non-zero $\alpha_i \in \mathbb{F}$ having the same modulus such that $A_ix_i = \alpha_i(B_ix_i)$ for all $x_i \in M_0^*(A_i)$.

Proof. Let $A^{\perp} \subseteq B^{\perp}$. If $||A_i|| < ||A||$, then $M_0^*(A_i) = \{0_i\} \subseteq M_0^*(B_i)$. Assume $||A_i|| = ||A||$, let $x_i \in M_0(A_i)$ be a normalized vector and consider the hyperplane $H = \{X = \bigoplus_{k=1}^{\ell} X_k \in$ $\mathcal{A}; \ \ \langle X_i x_i, A_i x_i \rangle_{\mathbb{F}} = 0$ contained in $A^{\perp} \subseteq B^{\perp}$. By [\[13,](#page-19-9) Theorem 2.1] (whose proof works over real as well as complex normed spaces), we have $|F(B)| = ||F|| ||B||$, where $F(X) = \langle A_i x_i, X_i x_i \rangle_F$ is an F-linear functional on A. Hence

$$
|\langle A_i x_i, B_i x_i \rangle_{\mathbb{F}}| = |F(B)| = ||F|| ||B|| = ||A_i x_i|| \cdot ||x_i|| \cdot ||B||,
$$

(where last equality follows as an application of Cauchy-Schwarz inequality and the fact that $||X_i|| \leq ||X||$. Now, we have

$$
||A_ix_i|| \cdot ||x_i|| \cdot ||B|| = |\langle A_ix_i, B_ix_i\rangle_{\mathbb{F}}| \le ||A_ix_i|| ||B_ix_i|| \le ||A_ix_i|| \cdot ||x_i|| \cdot ||B_i|| \le ||A_ix_i|| \cdot ||x_i|| \cdot ||B||.
$$

So, we get equality throughout, so

 $||B|| = ||B_i||$

and $||B_ix_i|| = ||B_i|| ||x_i||$. Also, by the condition of equality in Cauchy-Schwarz inequality, we get $A_ix_i = \lambda_i(B_ix_i)$ for some $\lambda_i = \lambda_i(x_i) \in \mathbb{F}$. Now, we prove $\lambda_i(x_i)$ is independent of $x_i \in M_0(A_i)$. First note that $|\lambda_i(x_i)| = ||A_i||/||B_i|| = ||A||/||B||$. Let $x_i, y_i \in M_0(A_i)$ be normalized vectors such that $A_i x_i = \mu_i(B_i x_i)$ and $A_i y_i = \nu_i(B_i y_i)$. We first observe that there exists a normalized vector $w_i \in M_0(A_i)$ such that

$$
\frac{\mu_i + \nu_i}{2} = \frac{\langle B_i x_i, A_i x_i \rangle_{\mathbb{F}} + \langle B_i y_i, A_i y_i \rangle_{\mathbb{F}}}{2} = \langle B_i w_i, A_i w_i \rangle_{\mathbb{F}};
$$

in the last step we have used the convexity of $\{\langle B_ix_i, A_ix_i\rangle_{\mathbb{F}}; \ \|x_i\|=1, x_i\in M_0(A_i)\},\$ the numerical range of the compression of $A_i^*B_i$ to the subspace $M_0(A_i)$ (recall that $A_i^*B_i$ is embedded into a suitable $\mathcal{M}_n(\mathbb{R}), \mathcal{M}_{2n}(\mathbb{R})$ or $\mathcal{M}_{4n}(\mathbb{R})$). Again, $w_i \in M_0(A_i)$ implies that $A_iw_i = \tau_i(B_iw_i)$ for some τ_i with $|\tau_i| = ||A||/||B||$. Thus, $|\mu_i + \nu_i| = 2||A||/||B||$. But $|\mu_i| = |\nu_i| = ||A||/||B||$ and hence $\mu_i = \nu_i$. So, $\lambda_i(x_i)$ is independent of x_i .

Also, $|\alpha_i| = ||A||/||B||$ for all i, when $M_0^*(A_i) \neq \{0_i\}$ and in case $M_0^*(A_i) = \{0_i\}$, we can choose any α_i , so in particular $\alpha_i = ||A||/||B||$. The converse follows from Lemma [2.1](#page-5-0) (i).

As an application of Lemma [2.1,](#page-5-0) we also get a classification of right-symmetric elements. It turns out that they are nothing but scalar multiples of unitary elements. For simple complex finite-dimensional C^* -algebra, this follows from [\[24,](#page-19-13) Theorem 2.5] (but see also [\[17,](#page-19-4) Lemma 3.7]).

Lemma 2.4. The following are equivalent in a finite-dimensional C^* -algebra A over \mathbb{F} :

- (i) A is a scalar multiple of some unitary U.
- (ii) There does not exist a non-zero $B \in \mathcal{A}$ such that $A^{\perp} \subset B^{\perp}$.
- (iii) A is right-symmetric in A .

Proof. Without loss of generality, let $\mathcal{A} = \mathcal{M}_{n_1}(\mathbb{K}_1) \oplus \cdots \oplus \mathcal{M}_{n_\ell}(\mathbb{K}_\ell)$.

(i) \iff (ii). Notice that the unitaries in $\mathcal{M}_{n_1}(\mathbb{K}_1) \oplus \cdots \oplus \mathcal{M}_{n_\ell}(\mathbb{K}_\ell)$ are of form $U_1 \oplus \cdots \oplus U_\ell$ where U_i is a unitary matrix in $\mathcal{M}_{n_i}(\mathbb{K}_i)$. Therefore, $A = A_1 \oplus \cdots \oplus A_\ell$ is a multiple of unitary if and only if $M_0^*(A_i) = \mathbb{K}_i^{n_i}$ for each i. Also, by Lemma [2.3,](#page-7-0) there exists non-zero B such that $A^{\perp} \subsetneq B^{\perp}$ if and only if $\dim_{\mathbb{K}_i}(M_0^*(A_i)) < \dim_{\mathbb{K}_i}(M_0^*(B_i))$ for some *i*. The two claims combined give the wanted equivalence.

(i) \iff (iii). Assume $A = \bigoplus_{k=1}^{\ell} A_k$ is right-symmetric. We first show that each $A_k \in$ $\mathcal{M}_{n_k}(\mathbb{K}_k)$ is of the same norm. Suppose on the contrary. Without loss of generality $\|A_1\|$ < $\max\{\|A_2\|,\ldots,\|A_\ell\|\} = \|A_i\| = 1$ where $i \neq 1$ and consider $X = X_1 \oplus (A_2/2) \oplus \cdots \oplus (A_\ell/2)$ with $||X_1|| = 1$ and X_1 is BJ orthogonal to A_1 . Then, we have $X \perp A$. Due to right-symmetry of A this implies $A \perp X$ so there exists a normalized vector $x \in M_0(A) \subseteq \text{Span}\{M_0(A_2), \ldots, M_0(A_\ell)\}$ $\{(0, x_2, \ldots, x_\ell); \ \|A(0, x_2, \ldots, x_\ell)\| = \|A\| \cdot \|(0, x_2, \ldots, x_\ell)\| \text{ and } x_k \in \mathbb{K}_k^{n_k}\}\$ such that

$$
0 = \langle Ax, Xx \rangle_{\mathbb{F}} = \sum_{k=2}^{\ell} \frac{1}{2} \langle A_k x_k, A_k x_k \rangle_{\mathbb{F}} = \frac{1}{2} \sum_{k} ||A_k x_k||^2 = \frac{1}{2} ||Ax||^2 \neq 0,
$$

which is a contradiction. This implies $||A_1|| = \max\{||A_2||, \ldots, ||A_\ell||\} = ||A||$. Similarly, we get $||A_i|| = ||A||$ for all $1 \le i \le \ell$. Now, we prove that each A_i is a scalar multiple of a unitary matrix.

Let $1 \leq i \leq \ell$ be fixed. Recall that $(X_1 \oplus \cdots \oplus X_\ell) \mapsto (U_1 X_1 V_1^*, \ldots, U_\ell X_\ell V_\ell^*)$ is an isometry for unitaries U_k and V_k in $\mathcal{M}_{n_k}(\mathbb{K}_k)$, and hence induces a BJ isomorphism, so we can assume with no loss of generality that

$$
A_i = \Sigma_i = \text{diag}\,(\sigma_1^i, \dots, \sigma_{n_i}^i)
$$

with $\sigma_1^i = \cdots = \sigma_j^i > \sigma_{j+1}^i \geq \cdots \geq \sigma_{n_i}^i \geq 0$ for some $j \in \{1, \ldots, n_i\}$. We claim that $j = n_i$ i.e. A_i is a scalar matrix $||A_i|| = ||A||$, and this will imply that A is a scalar multiple of unitary.

Assume, if possible, $j \leq n_i - 1$ and denote $\sigma := \frac{\sigma_{j+1}^i}{\sigma_s^i}$ $\frac{j+1}{\sigma_j^i} \in [0,1)$, and consider $B = B_1 \oplus \cdots \oplus B_\ell$ where $B_k = A_k$ if $k \neq i$ and

$$
B_i = e_1^i (e_1^i)^* + \dots + e_{j-1}^i (e_{j-1}^i)^* + x_j^i (y_j^i)^* + x_{j+1}^i (y_{j+1}^i)^*,
$$

where $\{e_j^i; 1 \leq j \leq n_i\}$ denotes the standard basis of $\mathbb{K}_i^{n_i}$ and

$$
x_j^i = \frac{e_j^i - e_{j+1}^i}{\sqrt{2}}, \quad y_j^i = \frac{\sigma e_j^i + e_{j+1}^i}{\sqrt{1 + \sigma^2}}, \quad x_{j+1}^i = \frac{e_j^i + e_{j+1}^i}{\sqrt{2}}, \quad y_{j+1}^i = \frac{e_j^i - \sigma e_{j+1}^i}{\sqrt{1 + \sigma^2}}.
$$

Notice that B is already in its singular value decomposition and achieves its norm on y_j^i , which is mapped into $By_j^i = x_j^i$ while $Ay_j^i = \frac{\sigma_j^i}{\sqrt{1}}$ $\frac{\sigma_{j+1}}{\sqrt{1+\sigma^2}}(e^i_j+e^i_{j+1})$ is clearly orthogonal to x^i_j in $\langle \cdot, \cdot \rangle_{\mathbb{F}}$. Thus, $B \perp A$ by Lemma [2.1.](#page-5-0) Because of right-symmetricity of A we then have $A \perp B$, so there exists a normalized $w = w_1 \oplus \cdots \oplus w_\ell \in M_0(A) = M_0^*(A) = \bigoplus_{1}^{\ell} M_0(A_k)$ with $\|Aw\| = \|A\|$ and $0 = \sum_{k} \langle A_k w_k, B_k w_k \rangle_{\mathbb{F}}$. Due to $B_k = A_k$ whenever $k \neq i$ we see that

$$
0 = \sum_{k \neq i} ||A_k w_k||^2 + \langle A_i w_i, B_i w_i \rangle_{\mathbb{F}} = ||A||^2 \sum_{k \neq i} ||w_k||^2 + \langle A_i w_i, B_i w_i \rangle_{\mathbb{F}}
$$

= $||A||^2 \sum_{k \neq i} ||w_k||^2 + ||A_i||\langle w_i, B_i w_i \rangle_{\mathbb{F}}$
= $||A||^2 \sum_{k \neq i} ||w_k||^2 + ||A||\langle w_i, B_i w_i \rangle_{\mathbb{F}};$

in the one but last equality we used that the restriction of A_i to the K_i-subspace $\mathbb{K}_i^j \oplus 0_{n_i-j} =$ $M_0(A_i) = M_0^*(A_i) \ni w_i$ is a σ_1^i -multiple of identity. Notice also that the compression of B_i to this subspace equals $\begin{pmatrix} I_{j-1} & 0 \\ 0 & -I_{j-1} \end{pmatrix}$ 0 $\frac{1-\sigma}{\sqrt{2}\sqrt{1+\sigma^2}}$ $\overline{ }$ and is positive-definite. This gives $w_k = 0$ for every k, a contradiction.

Conversely, if A is a scalar multiple of a unitary then it achieves its norm on every non-zero vector. Consider an arbitrary $B \perp A$; then B achieves its norm on some vector y with $\langle By, Ay \rangle_{\mathbb{F}} = 0$. Since A also achieves its norm on the same vector we see that $A \perp B$ also holds, so that A is a rightsymmetric element.

We note that Lemma [2.4](#page-8-0) implies that the set of scalar multiples of unitary elements is invariant under any BJ isomorphism between two finite-dimensional C^* -algebras. We end this section by proving that same holds for smooth elements of finite-dimensional C^* -algebras also. We call $A = \bigoplus_{k=1}^{\ell} A_k \in \mathcal{A}$ to be smooth if there exists exactly one index i such that $||A_i|| = ||A||$ and $\dim_{\mathbb{K}_i}(M_0(A_i)) = 1$. For example, $A = (\bigoplus_{k=1}^{i-1} 0_{n_k}) \oplus E_{st}^i \oplus (\bigoplus_{k=i+1}^{\ell} 0_{n_k})$ are smooth elements for all matrix units $E_{st}^i \in \mathcal{M}_{n_i}(\mathbb{K}_i)$, which can be easily seen by writing them as $E_{st}^i = e_s^i (e_t^i)^*$ and using the fact that $||E_{st}^ix|| = |(e_t^i)^*x| = |x_t^i| \le ||x||$ for $x = \sum_{j=1}^{n_i} x_j^i e_j^i \in \mathbb{K}_i^{n_i}$, with inequality being strict except if $x = x_t^i e_t^i$.

There is a well known notion of smoothness in general Banach space V that states that a vector $v \in V$ is smooth if and only if there exists a unique normalized functional f on V such that $f(v) = ||v||$ (such f is called a supporting functional for v). We prove below that the two definitions are equivalent. However before we do that let us note that our definition of smoothness on finite-dimensional C^* -algebras is a special case of Holub's condition, see [\[11\]](#page-19-16). The equivalence of Holub's condition and smoothness has been studied by many authors, for a brief survey see [\[9\]](#page-19-17). In particular, it is known that the two definitions are equivalent for finite-dimensional simple or abelian C^* -algebras, see [\[11,](#page-19-16) Theorem 2.1], [\[1,](#page-18-3) Theorem 3.1], [\[12,](#page-19-18) Theorem 3.3], and [\[14,](#page-19-19) Corollary 3.3, [\[15,](#page-19-20) Corollary 2.2]. We show below that the same holds for any finite-dimensional C^* -algebra.

Lemma 2.5. Let A be a finite-dimensional C^{*}-algebra over \mathbb{F} . Then, $A \in \mathcal{A}$ is a smooth element if and only if there exists a unique normalized functional f on A such that $f(A) = ||A||$. Furthermore, for a smooth element $A \in \mathcal{A}$, there exists a unique i such that if $x = \bigoplus_{k=1}^{\ell} x_k \in M_0(A)$, then $x_k = 0$ for all $k \neq i$ and we have

$$
A^{\perp} = (0_{n_1} \oplus \cdots \oplus 0_{n_{i-1}} \oplus (A_i x_i) x_i^* \oplus 0_{n_{i+1}} \oplus \cdots \oplus 0_{n_\ell})^{\perp}.
$$

Proof. First, let A be a smooth element of A. By definition there exists exactly one i such that $||A_i|| = ||A||$ which, by Lemma [2.1,](#page-5-0) gives the statement about the vectors in $M_0(A)$. Moreover, we also have $\dim_{\mathbb{K}_i}(M_0(A_i)) = 1$, so there exists a vector $x_i \in \mathbb{K}_i^{n_i}$ such that

(2.6)
$$
M_0(A) = 0 \oplus \cdots \oplus 0 \oplus x_i \mathbb{K}_i \oplus 0 \oplus \cdots \oplus 0
$$

Then, (i) of Lemma [2.1](#page-5-0) and identity [\(2.5\)](#page-4-1), by which $\langle A(x_i\gamma), B(x_i\gamma)\rangle_{\mathbb{F}} = |\gamma|^2 \langle Ax_i, Bx_i\rangle_{\mathbb{F}}$ for each normalized vector $x_i \gamma \in x_i \mathbb{K}_i = M_0(A_i)$ imply

$$
A^{\perp} = (0_{n_1} \oplus \cdots \oplus 0_{n_{i-1}} \oplus (A_i x_i) x_i^* \oplus 0_{n_{i+1}} \oplus \cdots \oplus 0_{n_\ell})^{\perp}.
$$

This proves the last statement.

To prove the first one, let A be a smooth element which attains its norm on i -th component and define an F-linear functional f on A as $f: (\bigoplus_{k=1}^{\ell} X_k) \mapsto \frac{1}{\|A_i\|} \langle A_i x_i, X_i x_i \rangle_{\mathbb{F}}$. Then, by Cauchy-Schwarz, f is a normalized functional and $f(A) = ||A_i|| = ||A||$; also, $A^{\perp} =$ Ker f. By [\(1.4\)](#page-2-2), the kernel of every supporting functional of A is contained in $A^{\perp} =$ Ker f so f is a unique supporting functional of A.

Conversely, let $A \in \mathcal{A}$ be such that there exists a unique normalized supporting functional for A. Assume there exist $j \neq i$ such that $||A_i|| = ||A_i|| = ||A||$. Then there would be normalized vectors $x \in \mathbb{K}_i^{n_i}$ and $y \in \mathbb{K}_j^{n_j}$ j^{α_j} such that $||A_ix|| = ||A||$ and $||A_jy|| = ||A||$ so $f_x: (\bigoplus_{k=1}^{\ell} X_k) \to \frac{1}{||A_i||} \langle A_ix, X_i \rangle_{\mathbb{F}}$ and $f_y: (\bigoplus_{k=1}^{\ell} X_k) \to \frac{1}{\|A_j\|} \langle A_j y, X_j y \rangle_{\mathbb{F}}$ would be two distinct supporting (normalized) functionals for A, which contradicts our assumption that A has a unique supporting (normalized) functional for A. So, there exists a unique i such that $||A_i|| = ||A||$. Similar arguments prove that there does not exist two \mathbb{K}_i -linearly independent vectors $x, y \in M_0(A_i) \subseteq \mathbb{K}_i^{n_i}$. Thus, $\dim_{\mathbb{K}_i}(M_0(A)) = 1$. \Box

The next lemma will show that smooth elements are preserved under BJ isomorphism.

Lemma 2.6. Let A be a finite-dimensional C^{*}-algebra over \mathbb{F} . Then $A \in \mathcal{A}$ is a smooth element if and only if there does not exist $B \in \mathcal{A}$ such that $B^{\perp} \subsetneq A^{\perp}$.

Proof. We assume the usual matrix decompositions (1.1) – (1.2) of A. Let $A \in \mathcal{A}$ be smooth, achieving its norm only on *i*-th component, and let $B = \bigoplus_{k=1}^{\ell} B_k$ satisfy $B^{\perp} \subsetneq A^{\perp}$. Notice that $M_0^*(B_k)$ is a \mathbb{K}_k -subspace of $\mathbb{K}_k^{n_k}$ for each k. By Lemma [2.3,](#page-7-0) $B^{\perp} \subsetneq A^{\perp}$ implies $\dim_{\mathbb{K}_k}(M_0^*(B_k)) \leq$ $\dim_{\mathbb{K}_k}(M_0^*(A_k))$. Thus, for all $k \neq i$ we have $\dim_{\mathbb{K}_k}(M_0^*(B_k)) = 0$, i.e., $B_k = 0$, while for $k = i$

we have $\dim_{\mathbb{K}_i}(M_0^*(B_i)) \leq 1$ with $M_0^*(B_i) \subseteq M_0^*(A_i)$. So, either $B_i = 0$ or $M_0^*(B_i) = M_0^*(A_i)$ and $B_i|_{M_0^*(A_i)} = \alpha A_i|_{M_0^*(A_i)}$ for some non-zero $\alpha \in \mathbb{F}$. The former case gives $B^{\perp} = 0^{\perp} = A$, while the later case gives $B^{\perp} = A^{\perp}$ by (i) of Lemma [2.1.](#page-5-0) So $B^{\perp} \subsetneq A^{\perp}$ is not possible.

Conversely, assume there does not exist $B \in \mathcal{A}$ such that $B^{\perp} \subsetneq A^{\perp}$ and suppose, if possible, that A is not smooth. Then either there exists two distinct i, j such that $||A_i|| = ||A_i|| = ||A||$ or there exists j with $||A_j|| = ||A||$ but $\dim_{\mathbb{K}_j}(M_0(A_j)) > 1$. In the first case, by Lemma [2.1\(](#page-5-0)i),

$$
\left(\left(\bigoplus_{k=1}^i 0_{n_k} \right) \oplus A_i \oplus \left(\bigoplus_{k=i+1}^\ell 0_{n_k} \right) \right)^\perp \neq \left(\left(\bigoplus_{k=1}^j 0_{n_k} \right) \oplus A_j \oplus \left(\bigoplus_{k=j+1}^\ell 0_{n_k} \right) \right)^\perp
$$

are both properly contained in A^\perp because only the first contains $\big(\bigoplus_{k=1}^j 0_{n_k}\big) \oplus A_j \oplus \big(\bigoplus_{k=j+1}^\ell 0_{n_k}\big)$. In the second case, let $x, y \in M_0(A_i)$ be K_j-linearly independent. By applying Gram-Schmidt we can assume that their \mathbb{K}_j -valued inner product, $\langle x, y \rangle := y^*x = 0 \in \mathbb{K}_j$. Then, by Lemma [2.1\(](#page-5-0)i),

$$
\left(\left(\bigoplus_{k=1}^i 0_{n_k} \right) \oplus (Ax)x^* \oplus \left(\bigoplus_{k=i+1}^\ell 0_{n_k} \right) \right)^\perp \neq \left(\left(\bigoplus_{k=1}^i 0_{n_k} \right) \oplus (Ay)y^* \oplus \left(\bigoplus_{k=i+1}^\ell 0_{n_k} \right) \right)^\perp
$$

are both properly contained in A^{\perp} because only the first one contains $(\bigoplus_{k=1}^{i} 0_{n_k}) \oplus (Ay)y^* \oplus$ $(\bigoplus_{k=i+1}^{\ell} 0_{n_k})$. Either case is contradictory. So A is a smooth element.

3. BJ ORTHOGONALITY IN PSEUDO-ABELIAN C^* -ALGEBRA

In the next lemma we show that BJ orthogonality characterizes the underlying field in case of finite-dimensional abelian C^* -algebra $\mathcal{A} = \bigoplus_{k=1}^{\ell} \mathcal{M}_1(\mathbb{F}) = \mathbb{F}^{\ell}$ over the field \mathbb{F} . As usual, we prefer to write its elements as sequences, though we might still use ⊕ notation. We will use the notation

 $\mathcal{R}_{\mathcal{A}} = \{A; \; A \text{ is right-symmetric element in } \mathcal{A}\}.$

Lemma 3.1. Let $\mathcal{A} = \mathbb{F}^{\ell}$ be a finite-dimensional abelian C^* -algebra over the field \mathbb{F} with $\ell \geq 2$. Then, the set

$$
\{A^{\perp}; A \in \mathcal{R}_{\mathcal{A}} \setminus \{0\}\}\
$$

has finitely many elements in case of $\mathbb{F} = \mathbb{R}$ and infinitely many in case of $\mathbb{F} = \mathbb{C}$. They are indexed by the tuples $(1, \pm 1, \ldots, \pm 1)$ in case of \mathbb{R}^{ℓ} and are indexed by $(1, e^{i\theta_2}, \ldots, e^{i\theta_{\ell}})$ for $\theta_k \in [0, 2\pi)$ in case of \mathbb{C}^{ℓ} .

Proof. Any non-zero right symmetric element is a multiple of unitary by Lemma [2.4.](#page-8-0) Also, A^{\perp} = $(\lambda A)^{\perp}$ for $\lambda \in \mathbb{F} \setminus \{0\}$ because BJ orthogonality is homogeneous. So, we have

$$
\{A^{\perp}; A \in \mathcal{R}_{\mathcal{A}}\} = \{(1, \alpha_2, \dots, \alpha_\ell)^\perp; |\alpha_i| = 1\} \cup \{\mathbb{F}^\ell = 0^\perp\}.
$$

As for the fact that $(1, \alpha_2, \dots, \alpha_\ell)$ with $|\alpha_i| = 1$ have different outgoing neighborhoods, notice that if $A = (1, \alpha_2, \dots, \alpha_\ell)$ and $B = (1, \beta_2, \dots, \beta_\ell)$ satisfy $A^\perp = B^\perp$, then by Lemma [2.3,](#page-7-0) there exists $\lambda \in \mathbb{F}$ such that

$$
(1, \alpha_2, \ldots, \alpha_\ell) = \lambda(1, \beta_2, \ldots, \beta_\ell),
$$

which implies $\lambda = 1$ and $A = B$.

We remark that Lemma [3.1](#page-11-1) does not hold if $\ell = 1$ because $\mathcal{M}_1(\mathbb{R})$ and $\mathcal{M}_1(\mathbb{C})$ are BJ isomorphic (see [\[17,](#page-19-4) Example 2.2]). In the next lemma we give a complete characterization of \mathbb{F}^{ℓ} among the finite-dimensional pseudo-abelian C^* -algebras. We also provide a formula for the dimension of a complex abelian C^* -algebra which uses nothing but BJ orthogonality relation. It is simpler than the one valid in general normed spaces, see [\[8,](#page-19-10) Theorem 1.1 and Remark 1.2]. For simplicity we denote 1-by-1 blocks $\mathcal{M}_1(\mathbb{K})$ simply as K.

Lemma 3.2. Let $\mathcal{A} = \mathbb{K}_1 \oplus \cdots \oplus \mathbb{K}_\ell$ be a finite-dimensional pseudo-abelian C^* -algebra over the field \mathbb{F} . If $\mathcal{A} = \mathbb{F}^{\ell}$, then

(3.7)
$$
\dim \mathcal{A} = |\{A^{\perp}; A \in \mathcal{L}_{\mathcal{A}} \setminus \{0\}\}|.
$$

However, if $\mathbb{F} = \mathbb{R}$ and one of $\mathbb{K}_i \in \{\mathbb{C}, \mathbb{H}\},$ then $\{A^{\perp}; A \in \mathcal{L}_A \setminus \{0\}\}$ is an infinite set.

Proof. By Lemma [2.2,](#page-6-0) the set of all non-zero left-symmetric elements in $\mathcal A$ is given by (recall that 0^n denotes *n* repeated zeros)

$$
\{(0^{k-1}, \alpha_k, 0^{\ell-k}); \ 1 \leq k \leq \ell \text{ and } \alpha_k \in \mathbb{K}_k \setminus \{0\}\}.
$$

Now, all non-zero F-multiples of an element share the same outgoing neighbourhood. We further note that, by Lemma [2.3\(](#page-7-0)i), $i \neq j$ and $\alpha_i \in \mathbb{K}_i \setminus \{0\}, \alpha_j \in \mathbb{K}_j \setminus \{0\}$ imply

$$
A^{\perp} = (0, \ldots, 0, \alpha_i, 0, \ldots, 0)^{\perp} \neq (0, \ldots, 0, \alpha_j, 0, \ldots, 0)^{\perp} = B^{\perp},
$$

(because $M_0(A) \neq M_0(B)$). Therefore, if $\mathcal{A} = \mathbb{R}^{\ell}$ or \mathbb{C}^{ℓ} , then [\(3.7\)](#page-12-0) holds.

Finally, let $\mathcal{A} = \mathbb{K}_1 \oplus \cdots \oplus \mathbb{K}_{\ell}$ be C^* -algebra over \mathbb{R} with $\mathbb{K}_1 \in \{\mathbb{C}, \mathbb{H}\}$. Then

$$
A_{\lambda} = (1 + \lambda \mathbf{i}, 0, \dots, 0); \qquad (\mathbf{i}^2 = -1 \in \mathbb{R})
$$

are left-symmetric elements for all $\lambda \in \mathbb{R} \setminus \{0\}$ by Lemma [2.2.](#page-6-0) However, $A^{\perp}_{\lambda} \neq A^{\perp}_{\mu}$ for $\lambda \neq \mu$ because $(1 - \frac{1}{\lambda}i, 0, \dots, 0) \in (1 + \mu i, 0, \dots, 0)^{\perp}$ if and only if $\mu = \lambda$.

As a direct consequence of Lemma [3.1](#page-11-1) with its proof, and Lemma [3.2,](#page-12-2) we get the following corollary which characterizes the underlying fields in an pseudo-abelian C^* -algebra with the help of BJ orthogonality.

Corollary 3.3. Let A be pseudo-abelian C^* -algebra over the field \mathbb{F} with $\dim_{\mathbb{F}} A \geq 2$. Then $\mathbb{F} = \mathbb{C}$ if and only if

 $|\{A^{\perp}; A \in \mathcal{L}_{\mathcal{A}}\}| < \infty$ and $|\{A^{\perp}; A \in \mathcal{R}_{\mathcal{A}}\}| = \infty$.

Moreover, $\mathbb{F} = \mathbb{R}$ if and only if either both sets are infinite or else they are both finite.

Since the above corollary characterizes the underlying field and complex pseudo-abelian finitedimensional C^{*}-algebras are completely determined by their dimension, which is given by formula [\(3.7\)](#page-12-0), we already got a complete BJ characterization of complex finite-dimensional pseudo-abelian C^* -algebras. It remains to focus on real finite-dimensional pseudo-abelian C^* -algebras, where we still need to compute the number of blocks over reals, over complexes, and over quaternions. This will be done by carefully counting the cardinality associated with finite collections of smooth elements $A_1, \ldots, A_s \in \mathcal{A}$. By convention, if $s = 0$ we let $\bigcap_{k=1}^s A_k^{\perp} := \mathcal{A}$.

Lemma 3.4. Let $A \in \{M_1(\mathbb{C}), M_1(\mathbb{H})\}$ be a real C^* -algebra. Then, there exist finitely many smooth elements A_1, \ldots, A_s in A such that

$$
|\{B^\perp; \, B\in \mathcal{L}_{\bigcap_{k=1}^s A_k^\perp\setminus\{0\}}\}|<\infty.
$$

If s is the minimal such number, then $s = 1$ in case of $\mathcal{A} = \mathcal{M}_1(\mathbb{C})$ and $s = 3$ in the case of $\mathcal{A} = \mathcal{M}_1(\mathbb{H})$. In both cases, $\bigcap_{k=1}^s A_k^{\perp}$ is a one-dimensional real vector space.

Proof. As usual, $\mathbb K$ will denote either the field $\mathbb C$ or the (skew) field $\mathbb H$. Recall from Lemma [2.1](#page-5-0) that, for $A = (a), B = (b) \in \mathcal{M}_1(\mathbb{K})$ (1-by-1 matrices), we have $A \perp B$ if and only if $\text{Re}\langle A \cdot 1, B \cdot 1 \rangle =$ $\text{Re}(\overline{b}a) = 0$ (here, $B \cdot 1$ is matrix B applied on a vector $1 \in \mathbb{K}$). Therefore, A^{\perp} coincides with the kernel of the R-linear functional $f_A \colon \mathcal{M}_1(\mathbb{K}) \to \mathbb{R}$, given by $f_A \colon X = (x) \mapsto \text{Re}(\overline{x}a) = \text{Re}(\overline{a}x)$. Notice also that the map $W \mapsto f_W$ from $\mathcal{M}_1(\mathbb{K})$ to $\text{Hom}_{\mathbb{R}}(\mathcal{M}_1(\mathbb{K}), \mathbb{R})$ is R-linear with zero kernel, because for $W = (w)$, we have $f_W(W) = \text{Re}(\bar{w}w) = |w|^2 = 0$ if and only if $W = 0$. It implies that $A^{\perp} = \ker f_A$ and $B^{\perp} = \ker f_B$ are different whenever A, B are R-linearly independent. Therefore, if $V \subseteq M_1(\mathbb{K})$ is a real subspace of dimension at least two, and $A = (a), B = (b) \in V$ are R-linearly independent, then $(A + \lambda_1 B)$ and $(A + \lambda_2 B)$ are R-linearly independent for $\lambda_1 \neq \lambda_2$ and as such $(A + \lambda_1 B)^{\perp} \neq (A + \lambda_2 B)^{\perp}$. Thus, the cardinality of $\{A^{\perp}; A \in \mathcal{V}\}\$ is infinite. This shows that $s \geq 1$ in case $\mathbb{K} = \mathbb{C}$ and $s \geq 3$ in case of $\mathbb{K} = \mathbb{H}$. The inequalities are achieved, for example by using $\{i\}$ in case of $\mathbb C$ and $\{i, j, k\}$ in case of $\mathbb K = \mathbb H$.

Lemma 3.5. Let $\mathcal{A} = \mathbb{K}_1 \oplus \cdots \oplus \mathbb{K}_\ell$ be a real finite-dimensional pseudo-abelian C^* -algebra. Then, there exists finitely many smooth elements A_1, \ldots, A_s in A such that

(3.8)
$$
|\{B^{\perp};\ B\in\mathcal{L}_{\bigcap_{k=1}^{s}A_{k}^{\perp}}\setminus\{0\}\}|<\infty.
$$

If s is minimal such number, then $\bigcap_{k=1}^s A_k^{\perp} = (\mathbb{R}\alpha_1,\ldots,\mathbb{R}\alpha_\ell)$ for some unimodular numbers $\alpha_k \in \mathbb{K}_k$ and the cardinality of the set in [\(3.8\)](#page-13-2) is equal to ℓ , the number of matrix blocks in A. Furthermore,

$$
\dim \mathcal{A} = s + \ell.
$$

Proof. In case of a real C^{*}-algebra $\mathcal{A} = \mathbb{R}^{\ell}$ we have, by Lemma [3.2,](#page-12-2) $s = 0$; clearly also dim $\mathcal{A} = \ell$, which, again by Lemma [3.2,](#page-12-2) equals the cardinality of (3.8) , and the statement follows by inserting $\alpha_k = 1$. We now consider the remaining cases of a real C^{*}-algebra when one of the blocks is $\mathbb C$ or $\mathbb H$ with $\mathbb F = \mathbb R$. Without loss of generality, let $\mathcal A = \mathbb R^r \oplus \mathbb C^c \oplus \mathbb H^h$ for some $r, c, h \geq 0$. Now, if we take A_k 's to be all the elements in the finite set $\{(0^j, \mu, 0^{\ell-j-1})\;;\;r+1\leq j\leq \ell,\ \mu\in\{\mathbf{i},\mathbf{j},\mathbf{k}\}\cap\mathbb{K}_j\},\$ then A_k are smooth elements. It is straightforward that $(\alpha_1, \ldots, \alpha_\ell) \in \bigcap_{k=1}^s A_k^{\perp}$ is possible only if all $\alpha_k \in \mathbb{R}$, so

$$
\bigcap_{k=1}^s A_k^{\perp} = \mathbb{R}^r \oplus (\mathbb{R} + 0i)^c \oplus (\mathbb{R} + 0i + 0j + 0k)^h.
$$

By Lemma [2.2,](#page-6-0) all left-symmetric elements in $\bigcap_{k=1}^s A_k^{\perp} \setminus \{0\}$ are of the form $(0^{j-1}, \alpha, 0^{\ell-j})$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Note that $(0^{j-1}, \alpha, 0^{\ell-j})^{\perp}$ equals $(0^{j-1}, 1, 0^{\ell-j})^{\perp}$ So,

$$
\{B^{\perp};\ B\in \mathcal{L}_{\bigcap_{k=1}^s A_k^{\perp}}\setminus\{0\}\}=\{B^{\perp};\ B=(0^{j-1},1,0^{\ell-j})\text{ for }1\leq j\leq \ell\},\
$$

which is a finite set, and (3.8) holds for some finite s.

Let s be minimal such and let $A_1, \ldots, A_s \in \mathcal{A}$ be the corresponding smooth elements for which [\(3.8\)](#page-13-2) holds. Being smooth, Lemma [2.5](#page-10-0) implies that

$$
A_i^{\perp} = (0^{j-1}, \mu_{A_i}, 0^{\ell - j})^{\perp}
$$

for some j (depending on i) and a unimodular $\mu_{A_i} \in \mathbb{K}_j$. Hence, A_i^{\perp} is an R-vector subspace of $\mathcal{A} = \mathbb{K}_1 \oplus \cdots \oplus \mathbb{K}_\ell$ having only j-th block different from \mathbb{K}_j . It implies that $\bigcap_{k=1}^s A_k^{\perp}$ is also an R-vector subspace of A . Moreover, by Lemma [3.4,](#page-13-3) its j-th block is at most one-dimensional real vector space, else $\bigcap_{k=1}^{\ell} A_k$ contains infinitely many left-symmetric elements with pairwise distinct outgoing neighborhoods (relative to j-th block $\mathcal{M}_1(\mathbb{K}_j) = \mathbb{K}_j$, hence also relative to \mathcal{A}), which would contradict the choice of s. Thus, $\bigcap_{k=1}^s A_k^{\perp} = \bigoplus_{k=1}^{\ell} \mathbb{R} \alpha_k$ for some numbers $\alpha_k \in \mathbb{K}_k$ which we can assume to be either unimodular or 0.

To finish, define a map $sgn_0: \mathbb{K} \to \mathbb{K}$ by $sgn_0(\alpha) = 1$ if $\alpha = 0$, else $sgn_0(\alpha) = \frac{\overline{\alpha}}{|\alpha|}$ and observe that $x \to (\text{sgn}_0(\alpha_1) \oplus \cdots \oplus \text{sgn}_0(\alpha_\ell))x$ is an isometry of A, so induces a BJ isomorphism. With its help we can achieve that $\bigcap_{k=1}^s A_k^{\perp} = \bigoplus_{k=1}^{\ell} \mathbb{R} \alpha_k$ with each $\alpha_k \in \{0,1\}$. This, in turn, is BJ isomorphic to \mathbb{R}^m , where $m \leq \ell$ is the number of non-zero α_i . But since \mathbb{R}^{ℓ} also contains finitely many leftsymmetric elements with pairwise distinct outgoing neighborhoods then, by the minimality of s, we have $m = \ell$ and so $\alpha_k = 1$ for each k. Thus, $\bigcap_{k=1}^s A_k^{\perp} = \bigoplus_{k=1}^{\ell} \mathbb{R} \alpha_k$ for some unimodular numbers $\alpha_k \in \mathbb{K}_k$ and it contains exactly ℓ non-zero left-symmetric elements with pairwise distinct outgoing neighborhoods. This shows that the set in (3.8) has cardinality ℓ . By Lemma [3.4,](#page-13-3) we furthermore have

$$
(3.10) \t\t s = c + 3h.
$$

Now, the dimension of A is clearly equal to

$$
\dim_{\mathbb{R}} \mathcal{A} = r + 2c + 4h.
$$

while the number of blocks satisfies $\ell = r + c + h$. This implies $s = \dim \mathcal{A} - \ell$.

We say that a subset $S \subseteq A$ has property \mathcal{FL} if $|\{A^{\perp}; A \in \mathcal{L}_{S}\}| < \infty$.

Lemma 3.6. Let A be a finite-dimensional pseudo-abelian C^* -algebra over $\mathbb R$ with $\dim \mathcal A \geq 2$. Choose the minimal integer s and the corresponding smooth elements A_1, \ldots, A_s as in Lemma [3.5,](#page-13-1) and define the set

$$
\Xi := \{ A \in \mathcal{L}_{\mathcal{A}}; \; \exists \, m \, \text{ such that } A^{\perp} \cap \bigcap_{k \neq m} A_k^{\perp} \text{ has property } \mathcal{FL} \}.
$$

Then,

$$
\Omega = \bigcap_{A \in \Xi} A^{\perp}
$$

consists exactly of elements in A that are zero in nonreal blocks of A. Hence, the cardinality of

$$
\{A^{\perp}; A \in \Omega \cap \mathcal{L}_{\mathcal{A}} \setminus \{0\}\}\
$$

coincides with the number of real 1-by-1 blocks in A.

Proof. Let $\mathcal{A} = \mathbb{R}^r \oplus \mathbb{C}^c \oplus \mathbb{H}^h$. Using Lemma [3.5,](#page-13-1)

$$
\bigcap_{k=1}^s A_k^{\perp} = \bigoplus_{k=1}^{\ell} \mathbb{R} \alpha_k
$$

for some unimodular numbers $\alpha_k \in \mathbb{K}_k$. Now, without loss of generality (by multiplying with a suitable unitary element, i.e., applying a suitable isometry), $\alpha_k = 1$ for each $1 \leq k \leq \ell$. Since A_k are smooth there exist X_k such that $A_k^{\perp} = X_k^{\perp}$, where collection of all X_k takes the form $(0^{q-1}, i, 0^{\ell-q})$ in complex blocks, i.e., for $r+1 \le q \le r+c$, and the form $(0^{q-1}, \mu_{j,q}, 0^{\ell-q})$ with $\text{span}\{\mu_{1,q},\mu_{2,q},\mu_{3,q}\} = \text{span}\{\mathbf{i},\mathbf{j},\mathbf{k}\}\ (1 \leq j \leq 3)$ in quaternionic blocks, i.e., for $r+c+1 \leq q \leq \ell$ (because we assumed $\alpha_k = 1$). Without loss of generality, we assume $A_k = X_k$.

Consider q such that $\mathbb{K}_q \in \{\mathbb{C}, \mathbb{H}\}.$ We examine only the slightly more challenging case of $\mathbb{K}_q = \mathbb{H}$ in the sequel. Then there exists m such that $A_m = (0^{q-1}, \mu_{1,q}, 0^{\ell-q})$. Let us replace A_m with

$$
A = (0^{q-1}, 1, 0^{\ell-q}).
$$

Now, for each unimodular $\mu \in \mathbb{K}_q$ we have $\mu^{\perp} = \ker f_{\mu}$ (here, μ^{\perp} denotes the relative outgoing neighborhood inside the q-th block $\mathcal{M}_1(\mathbb{K}_q) = \mathbb{K}_q$ and $f_\mu(x) = \text{Re}(\overline{\mu}x)$ is R-linear functional on \mathbb{K}_q) is a three-dimensional subspace of \mathbb{K}_q . Then, if μ_1, μ_2, μ_3 are R-linearly independent, then $f_{\mu_1}, f_{\mu_2}, f_{\mu_3}$ are also R-linearly independent. Now, since $\mu_{1,q}, \mu_{2,q}, \mu_{3,q}$ are purely imaginary and R-linearly independent, it implies that $1, \mu_{2,q}, \mu_{3,q}$ are R-linearly independent, so $1^\perp \cap \mu_{2,q}^\perp \cap \mu_{3,q}^\perp$ is a one-dimensional subspace in $\mathcal{M}_1(\mathbb{K}_q)$ (again, we are using here the relative outgoing neighborhoods, within q-th block $\mathcal{M}_1(\mathbb{K}_q)$ only). So, $A^\perp \cap \bigcap_{k \neq m} A_k^\perp = \bigoplus_{k \neq q} \mathbb{R} \alpha_k \oplus \mathbb{R} \beta$ for some unimodular $\beta \in \mathbb{K}_q$ (because by replacing A_m^{\perp} with A^{\perp} affects the q-th block only), which contains only finitely many left-symmetric elements with pairwise distinct outgoing neighborhoods. It implies

$$
(0^{q-1}, 1, 0^{\ell-j}), (0^{q-1}, \mu_{q,j}, 0^{\ell-j}) \in \Xi \text{ for } j \in \{1, 2, 3\}.
$$

Since span $(1, \mu_{1,q}, \mu_{2,q}, \mu_{3,q}) = \mathbb{K}_q$, then $\Omega = \bigcap_{A \in \Xi} A^{\perp}$ can only contain elements with zero entries in q-th block. Since \mathbb{K}_q was arbitrary non-real block, then Ω can only contain elements with zero entry in all non-real blocks of A , that is,

$$
\Omega \subseteq \mathbb{R}^r \oplus 0^c \oplus 0^h.
$$

Now we consider those i such that $\mathbb{K}_i = \mathbb{R}$. By the minimality of s and the fact that we could assume $A_k = X_k$, each A_k has zero entries in real blocks. Therefore, if we replace some A_m (which has a non-zero entry only in the q -th block) with a left-symmetric A , as outlined by the procedure, then we claim that A also will have a non-zero entry in the q -th block: In case this block is complex, then by the minimality of s, A_m is the only element among $\{A_1, \ldots, A_s\}$ with non-zero entry in q-th block. Now, if its substitute, $A \in \Xi$ would have a zero entry in q-th block, then $(0^{q-1} \oplus M_1(\mathbb{C}) \oplus 0^{\ell-q}) \subseteq A^{\perp} \cap \bigcap_{k \neq m} A_k^{\perp}$. This, by Lemma [3.4,](#page-13-3) contradicts the fact that $A^{\perp}\cap\bigcap_{k\neq m}A_k^{\perp}$ has property \mathcal{FL} .

The arguments when the q -th block is quaternionic are similar; the only difference is that by the minimality of s (and Lemma [3.4\)](#page-13-3) we now have three matrices among $\{A_1, \ldots, A_s\}$ with non-zero entries in this block, and they must be R-linearly independent. Then the substitute, $A \in \Xi$ must again have non-zero entries only in q-th block.

Therefore. $\mathbb{R}^r \oplus 0^c \oplus 0^h \subseteq A^{\perp} \cap \bigcap_{k \neq m} A_k^{\perp}$ for every $A \in \Xi$ and so $\mathbb{R}^r \oplus 0^c \oplus 0^h \subseteq \Omega \subseteq \mathbb{R}^r \oplus 0^c \oplus 0^h$, as claimed. The claim about the number of non-zero left symmetric elements with property \mathcal{FL} inside Ω is now clear.

4. Proofs of Main results

Proof of Theorem [1.2.](#page-3-0) Without loss of generality, $A = \bigoplus_{k=1}^{\ell} M_{n_k}(\mathbb{K}_k)$ with $n_1 = \cdots = n_p = 1$ and $n_{p+1}, \ldots, n_\ell \neq 1$ for some $p \geq 0$. If $p = 0$, then, by Lemma [2.2,](#page-6-0) $\mathcal{L}_{\mathcal{A}} = \{0\}$, so $\mathcal{L}_{\mathcal{A}}^{\perp} = \mathcal{A}$. This matches with the sum of minimal ideals which are not skew-fields. Also, $\mathcal{L}^{\perp\perp}_{\mathcal{A}} = \{0\}$, which agrees with the statement when there are no skew-field minimal ideals.

If $p \neq 0$, then, by Lemma [2.2,](#page-6-0)

$$
\mathcal{L}_{\mathcal{A}} = \bigcup_{1 \leq i \leq p} \big(\bigoplus_{k=1}^{i-1} 0_{n_k} \big) \oplus \mathcal{M}_1(\mathbb{K}_i) \oplus \big(\bigoplus_{k=i+1}^{\ell} 0_{n_k} \big).
$$

Note that $M_1(\mathbb{K}_i)^{\perp} = \{0_{n_i}\}\)$ because $A_i \not\perp A_i$ for any non-zero $A_i \in \mathcal{M}_1(\mathbb{K}_i)$. Thus, if $A =$ $\alpha_1\oplus\cdots\oplus\alpha_p\oplus A_{p+1}\oplus\cdots\oplus A_\ell\in\mathcal{L}_\mathcal{A}^\bot$, then the left-symmetric element $\big(\bigoplus_{k=1}^{j-1}0_{n_k}\big)\oplus\alpha_j\oplus\big(\bigoplus_{k=j+1}^\ell0_{n_k}\big)$ is orthogonal to A; giving that $\alpha_j = 0$ for all $1 \leq j \leq p$. Hence,

$$
\mathcal{L}_{\mathcal{A}}^{\perp} := \bigcap_{A \in \mathcal{L}_{\mathcal{A}}} A^{\perp} = \big(\bigoplus_{k=1}^{p} 0_{n_k}\big) \oplus \bigoplus_{k=p+1}^{\ell} \mathcal{M}_{n_k}(\mathbb{K}_k),
$$

which coincides with the sum of minimal ideals that are not skew-fields. Moreover,

$$
\mathcal{L}_{\mathcal{A}}^{\perp\perp} := \bigcap_{A \in \mathcal{L}_{\mathcal{A}}^{\perp}} A^{\perp} = \bigoplus_{k=1}^{p} \mathcal{M}_1(\mathbb{K}_k) \oplus \big(\bigoplus_{k=p+1}^{\ell} 0_{n_k}\big),
$$

which is the sum of skew-field minimal ideals. \Box

Proof of Theorem [1.1.](#page-2-0) Assume there is a BJ isomorphism between A and A' and A is a finitedimensional pseudo-abelian C^* -algebra. If ϕ is a BJ isomorphism between A and \mathcal{A}' , we get $\mathcal{L}_{\mathcal{A}'} = \phi(\mathcal{L}_{\mathcal{A}}), \ \mathcal{R}_{\mathcal{A}'} = \phi(\mathcal{R}_{\mathcal{A}})$ and $\phi(0) = 0$ (since $x = 0$ is the only element with $x \perp x$). Using Corollary [1.3,](#page-3-1) we get \mathcal{A}' is pseudo-abelian. Using BJ isomorphism of \mathcal{A} and \mathcal{A}' , we have that the cardinalities of $\{A^{\perp}; A \in \mathcal{B} \text{ is left-symmetric}\}\$ and $\{A^{\perp}; A \in \mathcal{B} \text{ is right-symmetric}\}\$ are same for $\mathcal{B} \in \{A, A'\}$. Then, using Corollary [3.3,](#page-12-1) we get $\mathbb{F} = \mathbb{F}'$. For $\mathbb{F} = \mathbb{C}$, then result follows because the dimensions of A and A' are same using [\(3.7\)](#page-12-0). Now, we consider the case $\mathbb{F} = \mathbb{F}' = \mathbb{R}$.

Let $\mathcal{A} = \mathbb{R}^r \oplus \mathbb{C}^c \oplus \mathbb{H}^h$ and $\mathcal{A}' = \mathbb{R}^{r'} \oplus \mathbb{C}^{c'} \oplus \mathbb{H}^{h'}$. Then, by Lemmas [2.6](#page-10-1) and [3.6](#page-14-0) we have

$$
r=r'.
$$

By Lemma [3.5,](#page-13-1) we also have that the number of blocks in $\mathcal A$ and $\mathcal A'$ are same, i.e.,

$$
r + c + h = r' + c' + h'.
$$

The minimal number s such that there exists smooth elements A_1, \ldots, A_s for which [\(3.8\)](#page-13-2) holds is also preserved under a BJ isomorphism. Hence, by (3.9) , the dimensions of A and \mathcal{A}' are same, so

$$
r + 2c + 4h = r' + 2c' + 4h'.
$$

It implies, $r = r', c = c'$ and $h = h'$. Consequently, A and A' are C^* -isomorphic.

5. Extraction of abelian summand

Recall from Theorem [1.2](#page-3-0) that, given a finite-dimensional C^* -algebra \mathcal{A} ,

$$
\mathcal{L}_{\mathcal{A}}^{\perp\perp} = \mathbb{C}^c; \quad \text{if } \mathbb{F} = \mathbb{C},
$$

or

$$
\mathcal{L}_{\mathcal{A}}^{\perp\perp} = \mathbb{R}^r \oplus \mathbb{C}^c \oplus \mathbb{H}^h; \quad \text{if } \mathbb{F} = \mathbb{R},
$$

where r, c, h are the numbers of 1-by-1 real, complex and quaternionic blocks in the matrix block decomposition, respectively. Notice that in case of complex C^* -algebra, its pseudo-abelian and abelian summands coincide and equal to \mathbb{C}^c . However, for a real C^* - algebra its abelian summand equals $\mathbb{R}^r \oplus \mathbb{C}^c$ and differs from the pseudo-abelian summand when $h > 0$. Now, we give a procedure to classify the abelian summand in case of real C^* -algebra. Recall that \mathbb{F} , the underlying field of \mathcal{A} , can be determined using BJ orthogonality alone when $\dim{\mathcal{L}}_{\mathcal{A}}^{\perp\perp} \geq 2$, see Corollary [3.3](#page-12-1) and note that the dimension of the pseudo-abelian C^* -algebra $L^{\perp\perp}_{\mathcal{A}}$ can be computed using [\(3.9\)](#page-13-0) (this was proven for real C^* -algebra but it holds even for complex C^* -algebra since then $s = 0$ and (3.9) reduces to [\(3.7\)](#page-12-0)). When dim $\mathcal{L}_{\mathcal{A}}^{\perp\perp} = 1$, then the pseudo-abelian and abelian summand of $\mathcal A$ coincides and equal to $\mathbb C$ (if $\mathbb F = \mathbb C$) or $\mathbb R$ (if $\mathbb F = \mathbb R$), respectively.

Clearly, to extract the abelian summand we only need to work within the pseudo-abelian summand $\mathcal{L}^{\perp\perp}_{\mathcal{A}}$ and we only need to consider real C^* -algebras, that is, $\mathbb{F} = \mathbb{R}$. We will require the smooth points in $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$. Since $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$ is a C^* -algebra, its smooth points can be described by BJ orthogonality alone, see Lemma [2.6.](#page-10-1) We will also require a property similar to the property \mathcal{FL} , which was defined just prior Lemma [3.6.](#page-14-0) We say that a subset $S \subseteq \mathcal{L}_{\mathcal{A}}^{\perp\perp}$ has a relative property ${\mathcal {FL}}$ with respect to pseudo-abelian summand, ${\mathcal {PFL}}$ for short, if

$$
|\{X^{\perp} \cap \mathcal{L}_{\mathcal{A}}^{\perp\perp}; \ \ X \in \mathcal{L}_{\mathcal{S}}\}| < \infty.
$$

Now, we consider the following procedure to extract quaternionic 1-by-1 blocks in $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$: Start with an arbitrary finite set $\Omega \subseteq \mathcal{L}_{\mathcal{A}}^{\perp\perp}$, with property \mathcal{PFL} , that consists of smooth points relative to $\mathcal{L}^{\perp\perp}_{\mathcal{A}}$ and is of minimal possible cardinality (it exists by Lemma [3.5\)](#page-13-1). By Lemma [2.1](#page-5-0) we replace every element $S \in \Omega$ by $\hat{S} \in \mathcal{L}_{\mathcal{A}}^{\perp\perp}$, which is left symmetric relative to $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$ and satisfies $S^{\perp} \cap \mathcal{L}_{\mathcal{A}}^{\perp\perp}$ $\hat{S}^{\perp}\cap\mathcal{L}_{\mathcal{A}}^{\perp\perp}$. This way we achieve that each element of Ω is left-symmetric relative to $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$ and, as such, belongs to a single block of $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$ (see Lemma [2.2\(](#page-6-0)ii)) It follows from the proof of equation [\(3.10\)](#page-14-1) in Lemma [3.5](#page-13-1) that, due to its minimal cardinality, $|\Omega| = c + 3h$ and no element in Ω belongs to a real block of $\mathcal{L}^{\perp\perp}_{\mathcal{A}}$, while each complex block of $\mathcal{L}^{\perp\perp}_{\mathcal{A}}$ has one and each quaternionic block of $\mathcal{L}^{\perp\perp}_{\mathcal{A}}$ has three representatives in Ω .

Thus, if $|\Omega| \leq 2$, then $\mathcal{L}_{\mathcal{A}}^{\perp \perp}$ is the abelian part (and A is abelian if and only if $\mathcal{A} = \mathcal{L}_{\mathcal{A}}^{\perp \perp}$). If $|\Omega| \geq 3$, let Ω' be the collection of all 3-subsets (i.e., subsets of cardinality 3) of Ω . For each $\{X_1, X_2, X_3\} \in \Omega'$ we select (if they exist) all non-zero $X \in \mathcal{L}_{\mathcal{A}}^{\perp\perp}$, left-symmetric relative to $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$, such that the three sets

$$
(5.11) \qquad \bigcap_{S\in\Omega\backslash\{X_1\}} S^\perp\cap X^\perp\cap \mathcal{L}_\mathcal{A}^{\perp\perp}, \qquad \bigcap_{S\in\Omega\backslash\{X_2\}} S^\perp\cap X^\perp\cap \mathcal{L}_\mathcal{A}^{\perp\perp}, \qquad \bigcap_{S\in\Omega\backslash\{X_3\}} S^\perp\cap X^\perp\cap \mathcal{L}_\mathcal{A}^{\perp\perp}
$$

have property \mathcal{PFL} . By Lemma [3.4](#page-13-3) every quaternionic 1-by-1 block contains such a representative triple in Ω (e.g., if $\{X_1, X_2, X_3\} = (0 \oplus \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \oplus 0) \subseteq 0 \oplus \mathbb{H} \oplus 0$; we can take $X = (0 \oplus 1 \oplus 0) \in \mathbb{H}$ $0 \oplus \mathbb{H} \oplus 0$). Conversely if, for a triple $\{X_1, X_2, X_3\} \subseteq \Omega$, we can find such a left-symmetric X, then, by Lemma [2.2\(](#page-6-0)ii), X belongs to a single block of $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$. It is then easy to see that if $\{X_1, X_2, X_3\}$ do not belong to the same block (necessarily quternionic), then, by the minimal cardinality of Ω , at least one of the three sets in (5.11) will not have the property PFL , a contradiction.

One also sees that each X must belong to the same quaternionic block containing X_1, X_2, X_3 and at least one, say X_0 , is not in their R-linear span. Then, $X_1^{\perp} \cap X_2^{\perp} \cap X_3^{\perp} \cap X_0^{\perp}$ vanishes on this quaternionic block. Therefore, the common outgoing neighborhood of all those triples, together with all the adjourned vertices X, and intersected by $\mathcal{L}^{\perp\perp}_{\mathcal{A}}$, is the abelian summand of \mathcal{A} .

To summarize the complete extraction of abelian summand: Start with $\mathcal{L}^{\perp\perp}_{\mathcal{A}}$. If $\dim \mathcal{L}^{\perp\perp}_{\mathcal{A}} = 1$ (c.f. [\(3.9\)](#page-13-0)) or if $\dim \mathcal{L}_{\mathcal{A}}^{\perp\perp} \geq 2$ and $\mathbb{F} = \mathbb{C}$ (see Corollary [3.3\)](#page-12-1), then $\mathcal{L}_{\mathcal{A}}^{\perp\perp}$ is the abelian summand. Otherwise, apply the above procedure to get it.

Corollary 5.1. Let A be a finite-dimensional C^* -algebra over \mathbb{F} . Then, the following are equivalent:

- (i) A is abelian.
- (ii) $A = \mathcal{L}^{\perp \perp}_{\mathcal{A}}$ and it contains no quaterninic blocks.
- (iii) $A = \mathcal{L}^{\perp \perp}_A$ and if $A_1, \ldots, A_s \in \mathcal{A}$ is any (hence every) minimal tuple with the property $\mathcal{F}L$, then it contains no triple for which a non-zero left symmetric X would exist so that (5.11) would have property FL.

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REFERENCES

- [1] T.J. Abatzoglou, Norm derivatives on spaces of operators, Mathematische Annalen, 239 (1979), 129–135.
- [2] B. Blackadar, Operator Algebras-Theory of C^{*}-Algebras and von Neumann Algebras, Springer, 2006.
- [3] R. Bhatia, P. Semrl, *Orthogonality of matrices and some distance problems*, Linear Algebra Appl. 287 (1999), 77–86.
- [4] C. Constantinescu, C*-algebras. Vol. 1-5, volume 58-62 of North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam, 2001.
- [5] J.B. Conway, A Course in Functional Analysis, Second edition, Grad. Texts in Math., 96, Springer, New York, 1990.
- [6] K.R. Goodearl, Notes on real and complex C^{*}-algebras, Shiva Math. Ser., 5, Shiva Publishing Ltd., Nantwich, 1982.
- [7] L.C. Grove, Algebra, Pure Appl. Math., 110, Academic Press, Inc., New York, 1983.
- [8] A. Guterman, B. Kuzma, S. Singla, S. Zhiliana, Birkhoff–James classification of norm's properties, Adv. Oper. Theory 9 (2024), Article no. 43, a special issue dedicated to to Professor Chi-Kwong Li on the occasion of his 65th birthday.
- [9] P. Grover, S. Singla, Birkhoff-James orthogonality and applications: A survey, Operator Theory, Functional Analysis and Applications, eds. M. A. Bastos, L. Castro, A. Y. Karlovich, Oper. Theory Adv. Appl., Birkhäuser Cham, 282 (2021), 293–315.
- [10] P. Grover, S. Singla, Subdifferential set of the joint numerical radius of a tuple of matrices, Linear Multilinear Algebra 71 (2023), no. 17, 2709–2718.
- [11] J. Hennefeld, Smooth, compact operators, Proceedings of the American Mathematical Society, 77 (1979), 87–90.
- [12] J.R. Holub, On the metric geometry of ideals of operators on Hilbert space, Mathematische Annalen, 201 (1973), 157–163.
- [13] R.C. James, Orthogonality and linear functionals in normed linear spaces. Trans. Amer. Math. Soc. 61 (1947), 265–292.
- [14] D.J. Kečkić, Gateaux derivative of $B(H)$ norm, Proceedings of the American Mathematical Society, 133 (2005), 2061–2067.
- [15] D.J. Kečkić, Orthogonality and smooth points in $C(K)$ and $C_b(\Omega)$, Eurasian Mathematical Journal, 3 (2012), 44–52.
- [16] N. Komuro, K.-S. Saito, R. Tanaka, On symmetry of Birkhoff orthogonality in the positive cones of C^{*}-algebras with applications, J. Math. Anal. Appl. 474 (2019), no. 2, 1488–1497
- [17] B. Kuzma and S. Singla, Non-linear classification of finite-dimensional simple C*-algebras, accepted for publication in Filomat.
- [18] B. Li, Real operator algebras, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.
- [19] J. Rosenberg, Structure and applications of real C^{*}-algebras, Operator algebras and their applications, 235– 258. Contemp. Math., 671 American Mathematical Society, Providence, RI, 2016.
- [20] D. Sain, P. Ghosh, K. Paul, On symmetry of Birkhoff–James orthogonality of linear operators on finitedimensional real Banach spaces, Oper. Matrices 11 (2017) 1087–1095.
- [21] H. Schröder, K-theory for real C^{*}-algebras and applications, volume 290 of Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow, 1993.
- [22] R. Tanaka, Nonlinear equivalence of Banach spaces based on Birkhoff-James orthogonality, II, J. Math. Anal. Appl. 514 (2022), no. 1, Paper No. 126307, 19 pp.
- [23] R. Tanaka, A Banach space theoretical characterization of abelian C^{*}-algebras, Proc. Amer. Math. Soc. Ser. B 10 (2023), 208–218.
- [24] A. Turnšek, On operators preserving James' orthogonality, Linear Algebra Appl. 407 (2005), 189–195.
- [25] F. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251 (1997), 21–57.

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