Corrigendum to the equivalent statement of the Laplacian Spread Conjecture *

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Abstract: For a graph G, let $\alpha(G)$ denote its second smallest Laplacian eigenvalue. The Laplacian Spread Conjecture states that $\alpha(G)+\alpha(\overline{G}) \geq 1$, where $\overline{G}$ is the complement of G. In this paper, we have corrected two conclusions: First, the necessary and sufficient condition for $\alpha(G)+\alpha(\overline{G}) \geq 1$ is $\| \bigtriangledown_x - \bigtriangledown_y \|^2 \geq 1$ rather than $\| \bigtriangledown_x - \bigtriangledown_y \|^2 \geq 2$ which has been proved in [1] as demonstrated in our study. Second, we show that the Laplacian spread of balanced digraph Γ satisfies $LS(\Gamma) \leq n - \frac{1}{2}$ but not $LS(\Gamma) \leq n-1$ in [5], since inequality $\| \bigtriangledown_x - \bigtriangledown_y \|^2 \geq 2$ does not hold.

Keywords: Laplacian Spread Conjecture; balanced digraphs; eigenvalues.

MSC: 05C50.

1. Introduction

All concepts used in this paper can be found in [6] and in the articles cited below, unless defined otherwise. Let G be an undirected and unweighted simple graph with vertex set $V(G) = \{1, 2, \dots, n\}$ and edge set $E(G) = \{(i,j) | i, j \in V(G)\}$. The *adjacency matrix* of G denoted by $A(G) = [a_{ij}]$, is a square matrix whose entries are indexed by $n \times n$ and $a_{ij} = 1$ if $\{i,j\} \in E(G)$ and 0 otherwise, where n is the *order* of G, i.e. the number of the vertices. Obviously, $A(G)^T = A(G)$. We denote the *Laplacian matrix* of G by $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G and $D(G)$ is a diagonal matrix of vertex degrees, i.e. whose *ith* diagonal entry is $d(i)$, where $d(i)$ denotes the *degree* of vertex i.

If $x, y \in \mathbb{R}^n$ are vectors, we denote their *inner product* by $< x, y >= \sum_{i \in V(G)} x_i y_i$, and denote the *norm* of x by $\| x \| = \sqrt{< x, x >}$. We say two vectors $x, y \in \mathbb{R}^n$ are *orthonormal* if $\| x \| = \| y \| = 1$ and $< x, y >= 0$.

We denote the Laplacian eigenvalues of G by

$$0 = \lambda_n(G) \leq \lambda_{n-1}(G) \leq \cdots \leq \lambda_1(G).$$

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If we denote the *complement* of G by $\overline{G}$, then it is well-known that the eigenvalues of $L(\overline{G})$ are

$$0 = \lambda_n(\overline{G}) \leq n - \lambda_1(G) \leq \cdots \leq n - \lambda_{n-1}(G).$$

In [9], Fiedler defined the *algebraic connectivity* of the graph G by $\alpha(G) = \lambda_{n-1}(G)$. A related and useful quantity is $\beta(G) = \lambda_1(G) = n - \alpha(\overline{G})$, where $\overline{G}$ denotes the complement of G. The *Laplacian spread* of G is defined as $\beta(G) - \alpha(G)$. Clearly, $\beta(G) - \alpha(G) \leq n$. In [10] and [11], it was conjectured that n can be replaced with $n - 1$.

Laplacian Spread Conjecture (LSC): For any graph G of order $n \geq 2$, the following holds:

$$\beta(G) - \alpha(G) \leq n - 1, \tag{1}$$

or equivalently $\alpha(G) + \alpha(\overline{G}) \geq 1$, with equality if and only if G or $\overline{G}$ is isomorphic to the join of an isolated vertex and a disconnected graph of order $n - 1$.

We refer [1] [2] [3] [4] for more information about Laplacian Spread Conjecture. In this paper, we corrected an equivalent statement to the LSC has been proved in [1]. Also, we show that the Laplacian spread has a bound of $n - \frac{1}{2}$ for all balanced digraphs of order n.

2. An equivalent statement of LSC

More recently, M. Einollahzadeh and M.M. Karkhaneei have proved that LSC holds for all graphs G in [8]. In paper [1], the authors shown that the inequation (1) is equivalent to the following statement: For any two orthonormal vectors $x, y \in \mathbb{R}^n$ with zero mean and $n \geq 2$,

$$\| \triangledown_x - \triangledown_y \|^2 \geq 2,$$

where $\nabla_x \in \mathbb{R}^{\binom{n}{2}}$ be the vector whose ij-th entry is $|x_i - x_j|$, for all $i < j$.

Theorem 2.1. ([1]) *The following statements are equivalent.*

(i) For any graph G of order $n \geq 2$,

$$\alpha(G) + \alpha(\overline{G}) \geq 1.$$

(ii) For any two orthonormal vectors $x, y \in \mathbb{R}^n$ with zero mean and $n \geq 2$,

$$\| \triangledown_x - \triangledown_y \|^2 \geq 2.$$

In fact, the equivalent statement of $\alpha(G)+\alpha(\overline{G})\geq 1$ is $\parallel \bigtriangledown_x - \bigtriangledown_y \parallel^2 \geq 1$ rather than $\parallel \bigtriangledown_x - \bigtriangledown_y \parallel^2 \geq 2$ in Theorem 2.1. Fortunately, this "clerical error" does not materially affect the conclusion of the article.

For any $x \in \mathbb{R}^n$, let $\nabla_x \in \mathbb{R}^{\binom{n}{2}}$ be the vector whose ij-th entry is $|x_i - x_j|$, for all $i < j$. Let I, J and $\mathbf{e}$ denote unit matrix, the all-one matrix and all-one vector, respectively. For the sake of convenience and enhanced readability, we are now ready to rephrase and demonstrate Theorem 2.1 as follows.

Theorem 2.2. *The following statements are equivalent.*

(i) For any graph G of order $n \geq 2$,

$$\alpha(G)+\alpha(\overline{G}) \geq 1.$$

(ii) For any two orthonormal vectors $x, y \in \mathbb{R}^n$ with zero mean and $n \geq 2$,

$$\parallel \bigtriangledown_x - \bigtriangledown_y \parallel^2 \geq 1. \tag{2}$$

where $\nabla_x \in \mathbb{R}^{\binom{n}{2}}$ be the vector whose ij-th entry is $|x_i - x_j|$, for all $i < j$.

Proof. $(i) \Rightarrow (ii)$ Let $x, y \in \mathbb{R}^n$ be two orthonormal vectors with zero mean. Then, define G to be a graph with vertex set $V = \{1, 2, ..., n\}$, where

$$(i, j) \in E \Leftrightarrow (x_i - x_j)(y_i - y_j) < 0.$$

Let N be an oriented incidence matrix of G, and $\overline{N}$ be an oriented incidence matrix of $\overline{G}$. Now,

$$\begin{aligned}
\parallel \bigtriangledown_x - \bigtriangledown_y \parallel^2 &= \sum_{1\leq i<j\leq n} (\mid x_i - x_j \mid - \mid y_i - y_j \mid)^2 \\
&= \frac{1}{2}\sum_{i,j=1}^{n} (\mid x_i - x_j \mid - \mid y_i - y_j \mid)^2 \\
&= \frac{1}{2}\sum_{(i,j)\in E} [(x_i - x_j) + (y_i - y_j)]^2 + \frac{1}{2}\sum_{(i,j)\notin E} [(x_i - x_j) - (y_i - y_j)]^2 \\
&= \frac{1}{2}\sum_{(i,j)\in E} [(x_i + y_i) - (x_j + y_j)]^2 + \frac{1}{2}\sum_{(i,j)\notin E} [(x_i - y_i) - (x_j - y_j)]^2 \\
&= \frac{1}{2}(x+y)^T N N^T (x+y) + \frac{1}{2}(x-y)^T \overline{N}\,\overline{N}^T (x-y) \\
&= \frac{1}{2}(x+y)^T L(G)(x+y) + \frac{1}{2}(x-y)^T L(\overline{G})(x-y) \\
&\geq \frac{1}{2}\alpha(G)\|x+y\|^2 + \frac{1}{2}\alpha(\overline{G}) \parallel x - y \parallel^2
\end{aligned}$$

$$
\begin{aligned}
&= \alpha(G) + \alpha(\overline{G}) \\
&\geq 1.
\end{aligned}
$$

$(ii) \Rightarrow (i)$ Let a, b be real unit eigenvectors of G and $\overline{G}$, respectively, corresponding to $\alpha(G)$ and $\alpha(\overline{G})$. Since $L(G)\mathbf{e} = 0$ and $L(\overline{G})\mathbf{e} = 0$, it follows that $a, b \in \mathbf{e}^{\perp}$. Furthermore, since $L(\overline{G}) = (nI - J) - L(G)$, it follows that b is an eigenvector of $L(G)$ corresponding to $\beta(G)$, which implies that $a \perp b$. Hence,

$$
x = \frac{a+b}{\sqrt{2}} \quad \textit{and} \quad y = \frac{a-b}{\sqrt{2}}
$$

are orthonormal vectors with zero mean. Now,

$$
\begin{aligned}
\alpha(G) + \alpha(\overline{G}) &= a^T L(G) a + b^T L(\overline{G}) b \\
&= \sum_{(i,j)\in E} (a_i - a_j)^2 + \sum_{(i,j)\notin E} (b_i - b_j)^2 \\
&\geq \sum_{i,j=1}^{n} \min\{(a_i - a_j)^2, (b_i - b_j)^2\} \\
&= \frac{1}{2} \sum_{i,j=1}^{n} [(a_i - a_j)^2 + (b_i - b_j)^2 - \mid (a_i - a_j)^2 - (b_i - b_j)^2 \mid] \\
&= \frac{1}{4} \sum_{i,j=1}^{n} [\mid (a_i - a_j) + (b_i - b_j) \mid - \mid (a_i - a_j) - (b_i - b_j) \mid]^2 \\
&= \frac{1}{2} \sum_{i,j=1}^{n} (\mid \frac{a_i - a_j}{\sqrt{2}} + \frac{b_i - b_j}{\sqrt{2}} \mid - \mid \frac{a_i - a_j}{\sqrt{2}} - \frac{b_i - b_j}{\sqrt{2}} \mid)^2 \\
&= \frac{1}{2} \sum_{i,j=1}^{n} (\mid \frac{a_i + b_i}{\sqrt{2}} - \frac{a_j + b_j}{\sqrt{2}} \mid - \mid \frac{a_i - b_i}{\sqrt{2}} - \frac{a_j - b_j}{\sqrt{2}} \mid)^2 \\
&= \frac{1}{2} \sum_{i,j=1}^{n} (\mid x_i - x_j \mid - \mid y_i - y_j \mid)^2 \\
&= \sum_{1 \leq i < j \leq n} (\mid x_i - x_j \mid - \mid y_i - y_j \mid)^2 \\
&= \parallel \bigtriangledown_x - \bigtriangledown_y \parallel^2 \\
&\geq 1.
\end{aligned}
$$

The proof is complete. □

3. **Balanced digraphs**

Let Γ be a directed graph of order n with vertex set $V(\Gamma) = \{1, 2, \ldots, n\}$ and edge set $E(\Gamma) = \{(\overrightarrow{i, j}) | i, j \in V(\Gamma)\}$. The *adjacency matrix* of Γ denoted by $A(\Gamma) = [a_{ij}]$, is a square matrix whose entries are indexed by $n \times n$ and $a_{ij} = 1$ if $(\overrightarrow{i, j}) \in E(\Gamma)$ and 0 otherwise. Note that $A(\Gamma)^T = A(\Gamma)$ are generally not true since $(\overrightarrow{i, j}) \in E(\Gamma)$ not necessarily imply $(\overrightarrow{j, i}) \in E(\Gamma)$. Then, we define the *Laplacian matrix* of Γ by $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $A(\Gamma)$ is the adjacency matrix of Γ and $D(\Gamma)$ is a diagonal matrix whose *ith* diagonal entry is $d^+(i)$, where $d^+(i)$ denotes the *out-degree* of vertex $i \in V(\Gamma)$, which is equal to the number of edges of the form $(\overrightarrow{i, j}) \in E(\Gamma)$. Similarly, we denote the *in-degree* of vertex $i \in V(\Gamma)$ by $d^-(i)$, which is equal to the number of edges of the form $(\overrightarrow{j, i}) \in V(\Gamma)$. It is worth noting that every undirected graph can be regarded as a directed graph with bidirectional edges.

In this section, we denote $\Gamma = (V, E)$ be a *balanced digraph*, that is, $d^+(i) = d^-(i)$ for all $i \in V$. Briefly, let L denote the Laplacian matrix of Γ. Then, $L\mathbf{e} = 0$ and $L^T\mathbf{e} = 0$, which implies that $\mathbf{e}$ is an eigenvector of the Hermitian (symmetric) part of the Laplacian: $H(L) = \frac{1}{2}(L + L^T)$ corresponding to the zero eigenvalue.

The *algebraic connectivity* of Γ is given by

$$\alpha(\Gamma) = \min_{x \perp e, \|x\|=1} x^T L x,$$

and another related useful quantity is

$$\beta(\Gamma) = \max_{x \perp e, \|x\|=1} x^T L x.$$

We define the *Laplacian spread* of a digraph Γ by

$$LS(\Gamma) = \beta(\Gamma) - \alpha(\Gamma).$$

Since Γ is balanced, $H(L)$ is an M-matrix, which implies that its eigenvalues are nonnegative. Hence, $\beta(\Gamma)$ is equal to the maximum eigenvalue of $H(L)$ and $\alpha(\Gamma)$ is equal to the second smallest eigenvalue of $H(L)$. Moreover, since Γ is balanced if and only if $\overline{\Gamma}$ is balanced, it follows that $\alpha(\overline{\Gamma})$ is equal to the second smallest eigenvalue of $H(\overline{L})$, where $\overline{L}$ denotes the Laplacian matrix of $\overline{\Gamma}$ and $\overline{\Gamma}$ be the *complement* of Γ.

In this section, we will point out that the conclusion in [5] is wrong: the Laplacian spread of balanced digraphs satisfy $LS(\Gamma) \leq n - 1$. The reason is that authors in [5] only proved conclusion that $\| \bigtriangledown_x - \bigtriangledown_y \|^2 \geq 2$ if and only if $\alpha(\Gamma) + \alpha(\overline{\Gamma}) \geq 1$ for a balanced digraph Γ. However, the given example demonstrates that inequality $\| \bigtriangledown_x - \bigtriangledown_y \|^2 \geq 2$ does not hold.

Example 3.1. *Let* $a = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0, \cdots, 0)^T, b = (\frac{1}{\sqrt{n^2-n}}, \cdots, \frac{1}{\sqrt{n^2-n}}, \frac{1-n}{\sqrt{n^2-n}})^T \in \mathbb{R}^n$, *then* a, b *are two orthonormal vectors with zero mean, and*

$$x = \frac{a+b}{\sqrt{2}}, y = \frac{a-b}{\sqrt{2}}$$

are also two orthonormal vectors with zero mean. Hence

$\bigtriangledown_a = (\sqrt{2}, \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{2}}, \quad \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \quad 0, \cdots, 0, 0)^T$, *and*

$\bigtriangledown_b = (0, \cdots, 0, \frac{n}{\sqrt{n^2-n}}, \quad 0, \cdots, 0, \frac{n}{\sqrt{n^2-n}}, \quad 0, \cdots, 0, \frac{n}{\sqrt{n^2-n}})^T$.

From the proof of Theorem 1, it is easy to see that

$$\| \bigtriangledown_x - \bigtriangledown_y \|^2 = \sum_{1 \le i < j \le n} \min\{(a_i - a_j)^2, (b_i - b_j)^2\} = \frac{1}{2} + \frac{1}{2} = 1,$$

and conflict with $\| \bigtriangledown_x - \bigtriangledown_y \|^2 \ge 2$ *for any two orthonormal vectors* $x, y \in \mathbb{R}^n$ *with zero mean.*

As a means of enhancement, let's prove the Laplacian spread conjecture abuot the version of balanced digraphs: For any balanced digraph Γ of order $n \ge 2$, the Laplacian spread satisfies:

$$LS(\Gamma) = \beta(\Gamma) - \alpha(\Gamma) \le n - \frac{1}{2},$$

or equivalently $\alpha(\Gamma) + \alpha(\overline{\Gamma}) \ge \frac{1}{2}$, where $\overline{\Gamma}$ denotes the complement of Γ.

Lemma 3.2. *Let* Γ *be a balanced digraph of order* $n \ge 2$ *and let* $x \in \mathbb{R}^n$. *Then,*

$$x^T L x = \frac{1}{2} \sum_{(i,j) \in E} (x_i - x_j)^2.$$

Proof. Let N be an oriented incidence matrix of graph $\widehat{\Gamma}$, where $\widehat{\Gamma}$ denotes undirected simple graph corresponding to Γ. Then $2H(L) = NN^T$ since Γ is balanced, hence

$$\begin{aligned} x^T L x &= x^T H(L) x \\ &= \frac{1}{2} x^T N N^T x \\ &= \frac{1}{2} \sum_{(i,j) \in E} (x_i - x_j)^2. \end{aligned}$$

□

As in section 2, let's denote unit matrix, the all-one matrix and all-one vector by I, J and $\mathbf{e}$, respectively, and let $\nabla_x \in \mathbb{R}^{\binom{n}{2}}$ be an $\binom{n}{2}$-dimensional vector for any $x \in \mathbb{R}^n$. We are now ready to prove the main result of this section.

Theorem 3.3. *The following statements are equivalent:*

(i) For any balanced digraph Γ *of order* $n \geq 2$*,*

$$\alpha(\Gamma) + \alpha(\overline{\Gamma}) \geq \frac{1}{2}.$$

(ii) For any two orthonormal vectors $x, y \in \mathbb{R}^n$ *with zero mean and* $n \geq 2$*,*

$$\| \nabla_x - \nabla_y \|^2 \geq 1.$$

Proof. $(i) \Rightarrow (ii)$ Let $x, y \in \mathbb{R}^n$ be two orthonormal vectors with zero mean and Γ be a digraph with vertex set $V = \{1, 2, ..., n\}$, where

$$(\overrightarrow{i, j}) \in E \Leftrightarrow (x_i - x_j)(y_i - y_j) < 0.$$

Note that Γ is a bidirectional digraph since $(\overrightarrow{i, j}) \in E$ if and only if $(\overrightarrow{j, i}) \in E$, and hence Γ is a balanced digraph. Now,

$$\begin{aligned}
\| \nabla_x - \nabla_y \|^2 &= \sum_{1 \leq i < j \leq n} (| x_i - x_j | - | y_i - y_j |)^2 \\
&= \frac{1}{2} \sum_{i,j=1}^{n} (| x_i - x_j | - | y_i - y_j |)^2 \\
&= \frac{1}{2} \sum_{(i,j) \in E} ((x_i - x_j) + (y_i - y_j))^2 + \frac{1}{2} \sum_{(i,j) \notin E} ((x_i - x_j) - (y_i - y_j))^2 \\
&= \frac{1}{2} \sum_{(i,j) \in E} ((x_i + y_i) - (x_j + y_j))^2 + \frac{1}{2} \sum_{(i,j) \notin E} ((x_i - y_i) - (x_j - y_j))^2 \\
&= (x + y)^T L (x + y) + (x - y)^T \overline{L} (x - y) \\
&\geq \alpha(\Gamma) \|x + y\|^2 + \alpha(\overline{\Gamma}) \| x - y \|^2 \\
&= 2(\alpha(\Gamma) + \alpha(\overline{\Gamma})) \\
&\geq 1.
\end{aligned}$$

$(ii) \Rightarrow (i)$ Let a, b be real unit eigenvectors of $H(L)$ and $H(\overline{L})$, respectively, corresponding to $\alpha(\Gamma)$ and $\alpha(\overline{\Gamma})$. Since $H(L)\mathbf{e} = 0$ and $H(\overline{L})\mathbf{e} = 0$, it follows that $a, b \in \mathbf{e}^{\perp}$. Furthermore, since $H(\overline{L}) = (nI - J) - H(L)$, it follows that b is an eigenvector of $H(L)$ corresponding to $\beta(\Gamma)$, which implies that $a \perp b$. Hence,

$$x = \frac{a + b}{\sqrt{2}} \quad \textit{and} \quad y = \frac{a - b}{\sqrt{2}}$$

are also orthonormal vectors with zero mean. Now,

$$
\begin{aligned}
\alpha(\Gamma)+\alpha(\overline{\Gamma}) &= a^T H(L)a + b^T H(\overline{L})b \\
&= a^T La + b^T \overline{L}b \\
&= \frac{1}{2}\sum_{(i,j)\in E}(a_i-a_j)^2 + \frac{1}{2}\sum_{(i,j)\notin E}(b_i-b_j)^2 \\
&\geq \frac{1}{2}\sum_{i,j=1}^{n}\min\{(a_i-a_j)^2,(b_i-b_j)^2\} \\
&= \frac{1}{4}\sum_{i,j=1}^{n}[(a_i-a_j)^2+(b_i-b_j)^2- \mid (a_i-a_j)^2-(b_i-b_j)^2 \mid] \\
&= \frac{1}{8}\sum_{i,j=1}^{n}[\mid (a_i-a_j)+(b_i-b_j) \mid - \mid (a_i-a_j)-(b_i-b_j) \mid]^2 \\
&= \frac{1}{4}\sum_{i,j=1}^{n}(\mid \frac{a_i-a_j}{\sqrt{2}}+\frac{b_i-b_j}{\sqrt{2}} \mid - \mid \frac{a_i-a_j}{\sqrt{2}}-\frac{b_i-b_j}{\sqrt{2}} \mid)^2 \\
&= \frac{1}{4}\sum_{i,j=1}^{n}(\mid \frac{a_i+b_i}{\sqrt{2}}-\frac{a_j+b_j}{\sqrt{2}} \mid - \mid \frac{a_i-b_i}{\sqrt{2}}-\frac{a_j-b_j}{\sqrt{2}} \mid)^2 \\
&= \frac{1}{2}\sum_{1\leq i<j\leq n}(\mid x_i-x_j \mid - \mid y_i-y_j \mid)^2 \\
&= \frac{1}{2}\parallel \bigtriangledown_x - \bigtriangledown_y \parallel^2 \\
&\geq \frac{1}{2}\,.
\end{aligned}
$$

The proof is complete. □

Since the inequation (1) was proven in [8], it follows that the statement (2) is always true. Therefore, we have the following corollary.

Corollary 3.4. *If Γ is a balanced digraph of order $n \geq 2$, then the Laplacian spread satisfies*

$$LS(\Gamma) \leq n - \frac{1}{2}.$$

Proof. Since $LS(\Gamma) = \beta(\Gamma) - \alpha(\Gamma) = n - \alpha(\overline{\Gamma}) - \alpha(\Gamma) = n - (\alpha(\overline{\Gamma}) + \alpha(\Gamma))$, thus $LS(\Gamma) \leq n - \frac{1}{2}$ by Theorem 3.3. □

4. **Declaration of competing interest**

There is no competing interest.

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