PROJECTIVE VERSIONS OF SPATIAL PARTITION QUANTUM GROUPS

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ABSTRACT. We generalize categories of spatial partitions in the sense of Cébron-Weber by introducing new base partitions. This allows us to construct additional examples of free orthogonal quantum groups but yields the same class of spatial partition quantum groups as before. Further, we use these new base partitions to show that the class of spatial partition quantum groups is closed under taking projective versions and in particular contains the projective version of all easy quantum groups. As an application, we determine the quantum groups corresponding to the categories of all spatial pair partitions and give explicit descriptions of the projective versions of easy quantum groups in terms of spatial partitions.

1. INTRODUCTION

Spatial partition quantum groups were first introduced by Cébron-Weber in [10] and are examples of compact quantum groups in the sense of Woronowicz [40, 42]. They generalize easy quantum groups by Banica-Speicher [6] and are obtained by replacing two-dimensional partitions with three-dimensional spatial partitions.

A spatial partition on m levels $p \in \mathcal{P}^{(m)}$ consists of $k \cdot m$ upper points and $\ell \cdot m$ lower points that are partitioned into disjoint subsets by lines. Further, we allow both upper and lower points to be colored uniformly along the levels. For example, we have

$$\mathord{\text{⊔}} \in \mathcal{P}^{(1)}, \qquad \mathord{\text{⋔}} \in \mathcal{P}^{(2)}, \qquad \mathord{\text{⋔}} \in \mathcal{P}^{(3)}.$$

Given spatial partitions on the same number of levels, we can construct new spatial partitions by forming their tensor product, involution and composition. A category of spatial partitions in the sense of Cébron-Weber is a set of spatial partitions that is closed under these operations and contains the base partitions $\mathord{\text{|}}^{(m)}$, $\mathord{\text{|}}^{(m)}$ and $\mathord{\text{⊓}}^{(m)}$, $\mathord{\text{⊔}}^{(m)}$. Here, $p^{(m)} \in \mathcal{P}^{(m)}$ denotes the spatial partition that is obtained by placing m copies of the partition p along each level.

By realizing spatial partitions as linear operators, categories of spatial partitions give rise to concrete C^*-tensor categories, which then correspond to spatial partition quantum groups via Woronowicz Tannaka-Krein duality [41]. In the case of partitions on one level, these quantum groups are exactly unitary easy quantum groups [6, 30] and include for example the free orthogonal and free unitary quantum groups [35], the quantum permutation group [36] or the hyperoctahedral quantum group [4].

Since easy quantum groups are based on partitions, they form a concrete class of quantum groups that can be studied and classified using combinatorics [17, 21, 16, 5, 37, 28]. In the case of orthogonal easy quantum groups, the classification has been completed in [29], whereas the classification in the unitary case is still ongoing. See for example [31, 15, 18] or the more recent work by Mang [23, 24, 25, 26].

Spatial partition quantum groups have so far been studied in [10, 14]. In [10], Cébron-Weber show that these quantum groups are closed under glued products, which implies that the class of spatial partition quantum groups is strictly larger than the class of easy quantum groups. Further, they provide a partial classification of categories of spatial pair partitions on two levels and discuss links to the quantum symmetries of finite-dimensional C^*-algebras.

In [14], the author shows that the category $P_2^{(2)}$ of all spatial pair partitions on two levels gives rise to the classical projective orthogonal group PO_n yielding a simpler example of a non-easy spatial partition quantum group. Additionally, an explicit description of the category of spatial partitions corresponding to the quantum symmetry group of $M_n(\mathbb{C}) \otimes \mathbb{C}^m$ is given.

Main results. In [10], Cébron-Weber ask about a generalization of the base partitions $\sqcap_\bullet^{(m)}$ and $\sqcap_\bullet^{(m)}$ that still allows the construction of compact matrix quantum groups from categories of spatial partitions. We answer this question by showing that the previous two base partitions can be replaced with any pairs of spatial partitions r and s satisfying the conjugate equations

$$\left[r^* \otimes \overset{\circ(m)}{\underset{\circ}{|}} \right] \cdot \left[\overset{\circ(m)}{\underset{\circ}{|}} \otimes s \right] = \overset{\circ(m)}{\underset{\circ}{|}}, \qquad \left[s^* \otimes \overset{\bullet(m)}{\underset{\bullet}{|}} \right] \cdot \left[\overset{\bullet(m)}{\underset{\bullet}{|}} \otimes r \right] = \overset{\bullet(m)}{\underset{\bullet}{|}}.$$

In the case of one level, the partitions $r = \sqcap$ and $s = \sqcap$ are the only solutions to these equations. However, in the case $m \geq 2$, we do not only obtain the previous base partitions $r = \sqcap_\bullet^{(m)}$ and $s = \sqcap_\circ^{(m)}$, but also twisted versions like

$$r = \, , \quad s = \qquad \text{and} \qquad r = \, , \quad s = \, .$$

We refer to Proposition 3.10 for a characterization of all possible solutions to these conjugate equations in the context of spatial partitions.

Using our new base partitions, we can now construct additional examples of free orthogonal quantum groups in the sense of Van Daele-Wang [11]. In the notation

of [1], these quantum groups are given by $O^+(F_\sigma)$ and with parameters

$$F_\sigma \colon (\mathbb{C}^n)^{\otimes m} \to (\mathbb{C}^n)^{\otimes m}, \quad F_\sigma(e_{i_1} \otimes \cdots \otimes e_{i_m}) = e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(m)}}$$

for all $n \in \mathbb{N}$ and $\sigma \in S_m$. See also Proposition 4.14 for more details and the precise statement in full generality.

Although we can construct new examples of spatial partition quantum groups, it turns out that using our new base partitions, we obtain the same class of quantum groups as defined by Cébron-Weber.

Theorem 1 (Corollary 4.17). *Let G be a spatial partition quantum group defined by any pair of spatial base partitions satisfying the conjugate equations. Then G is equivalent to a spatial partition quantum group in the sense of Cébron-Weber defined by the base partitions $\sqcap_\bullet^{(m)}$ and $\sqcap_\circ^{(m)}$.*

More generally, we show in Section 4.3 that permuting the points of any category of spatial partitions along the levels leaves the corresponding quantum group invariant, which makes it possible to reduce any base partitions to the case of $\sqcap_\bullet^{(m)}$ and $\sqcap_\circ^{(m)}$. Still, our new base partitions are useful on a combinatorial level and allow us to show that the class of spatial partition quantum groups is closed under taking projective versions.

Consider a compact matrix quantum group G with fundamental representation u and assume $\overline{u}$ is unitary. Then its projective version PG is the compact matrix quantum group defined by the representation $u \oplus \overline{u}$. If G is a classical group, then PG corresponds exactly to the quotient

$$PG = G/(G \cap \{\lambda I \mid \lambda \in \mathbb{C}\}).$$

Further, in the case quantum groups, projective versions have for example been studied in [2, 5, 7, 20]. In this context, our main result can be formulated as follows.

Theorem 2 (Corollary 5.9). *Let G be a spatial partition quantum group. Then PG is again a spatial partition quantum group. Moreover, its category of spatial partitions is given by $\mathrm{Flat}_{m,\circ\bullet}^{-1}(\mathcal{C})$, where $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ is the category of spatial partitions corresponding to G and $\mathrm{Flat}_{m,\circ\bullet}$ is the functor defined in Section 3.3.*

Note that the previous result does not only apply to projective versions PG but to any compact matrix quantum group defined by a $\oplus$-product of the fundamental representation u and a unitary conjugate representation $u^\bullet$. See Section 5.1 for further details.

As an application, we consider the categories $\mathcal{P}_2^{(m)}$ of all spatial pair partitions on m levels. In [6, 14], it is shown that $\mathcal{P}_2^{(1)}$ corresponds to the classical orthogonal group O_n and that $\mathcal{P}_2^{(2)}$ corresponds to its projective version PO_n. Using the previous theorem, we are now able to generalize these results in Section 5.2 to all $m \in \mathbb{N}$ and obtain

$$\mathcal{P}_2^{(m)} \quad \longleftrightarrow \quad \begin{cases} PO_n & \text{if } m \text{ is even,} \\ O_n & \text{if } m \text{ is odd.} \end{cases}$$

Finally, we consider projective versions of easy quantum groups in Section 5.3. Since easy quantum groups are a subclass of spatial partition quantum groups, our main result implies that their projective versions are again spatial partition quantum groups. Using a result of Gromada [20], we then derive sets of spatial partitions generating the categories of the projective versions of orthogonal easy quantum groups with degree of reflection two. This allows us to describe these quantum groups explicitly as universal C^*-algebras defined by finite sets of relations.

Overview. We begin in Section 2 with some preliminaries about compact matrix quantum groups, their representation categories and Woronowicz Tannaka-Krein duality. Then, we consider the combinatorics of spatial partitions in Section 3 and introduce our new base partitions. In particular, we characterize all possible base partitions and introduce two functors between categories of spatial partitions that are used to prove our main results.

In Section 4, we show how to construct compact matrix quantum groups from categories of spatial partitions containing our new base partitions and give a description of these quantum groups in terms of generators and relations. This allows us to construct the free orthogonal quantum groups $O^+(F_\sigma)$, before we show that our new base partitions yield the same class of spatial partition quantum groups as defined by Cébron-Weber.

Finally, we prove in Section 5 that the class of spatial partition quantum groups is closed under taking projective versions. Further, we determine the quantum groups corresponding to the categories $P_2^{(m)}$ of spatial pair partitions and give explicit descriptions of the projective versions of some orthogonal easy quantum groups in terms of spatial partitions.

2. Preliminaries

2.1. **Notation.** We begin by introducing some notations and conventions that will be used throughout the rest of the paper.

Let $\circ$ and $\bullet$ be two colors. Then we denote with

$$\{\circ, \bullet\}^* := \{1, \circ, \bullet, \circ\circ, \circ\bullet, \bullet\circ, \bullet\bullet, \circ\circ\circ, \dots\}$$

the free monoid generated by $\{\circ, \bullet\}$. It is given by the set of all finite words over $\{\circ, \bullet\}$ with concatenation as multiplication and the empty word 1 as the identity element. If $x \in \{\circ, \bullet\}^*$, then we denote with $|x|$ the length of the word x and with $x_i \in \{\circ, \bullet\}$ the i-th color. Further, let $\overline{\cdot} \colon \{\circ, \bullet\}^* \to \{\circ, \bullet\}^*$ be the anti-homomorphism defined by

$$\overline{\circ} = \bullet, \qquad \overline{\bullet} = \circ, \qquad \overline{x \cdot y} = \overline{y} \cdot \overline{x} \quad \forall x, y \in \{\circ, \bullet\}^*.$$

Next, consider a finite-dimensional Hilbert space V. Then we denote with $\overline{V}$ its conjugate Hilbert space where the conjugate of a vector $v \in V$ is denoted by $\overline{v} \in \overline{V}$. Using colors, we additionally define $V^\circ := V$, $V^\bullet := \overline{V}$ and $v^\circ := v$, $v^\bullet := \overline{v}$

for all $v \in V$. Further, we extend this notation to tensor products by defining $V^{\otimes x} := V^{x_1} \otimes \cdots \otimes V^{x_k}$ for all $x \in \{\circ, \bullet\}^*$ with $k := |x|$.

Let $n \in \mathbb{N}$. Then we define $[n] := \{1, \ldots, n\}$ and more generally $[\boldsymbol{n}] := [n_1] \times \cdots \times [n_m]$ for all $\boldsymbol{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$. Additionally, we introduce the Hilbert spaces $\mathbb{C}^{\boldsymbol{n}} := \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_m}$ with canonical bases given by $e_{\boldsymbol{i}} := e_{i_1} \otimes \cdots \otimes e_{i_m}$ for all $\boldsymbol{i} := (i_1, \ldots, i_m) \in [\boldsymbol{n}]$.

Next, consider two finite-dimensional Hilbert spaces V and W. Then $B(V, W)$ denotes the vector space of all linear operators $T \colon V \to W$ and we define $B(V) := B(V, V)$. Given an orthonormal basis of V indexed by J and an orthonormal basis of W indexed by I, we identify elements of $B(V, W)$ with matrices $(T_j^i)_{i \in I, j \in J}$. Moreover, if $T \colon V \to W$ is a linear operator, then its conjugate $\overline{T} \colon \overline{V} \to \overline{W}$ is given by $\overline{T}(\overline{v}) = \overline{T(v)}$ for all $v \in V$. Using colors, we additionally write $T^{\circ} := T, T^{\bullet} := \overline{T}$ and $T^{\otimes x} := T^{x_1} \otimes \cdots \otimes T^{x_k}$ for all $x \in \{\circ, \bullet\}^*$ with $k := |x|$.

Throughout the rest of the paper, we will use basic facts about C^*-algebras, their tensor products and universal C^*-algebras. For more information on these topics, we refer to [8]. In particular, we will denote with $\otimes$ the minimal tensor product of C^*-algebras.

Let A be a unital C^*-algebra. Then we identify elements $u \in B(V, W) \otimes A$ with A-valued matrices $(u_j^i)_{i \in I, j \in J}$ with respect to orthonormal bases of V and W index by J and I. Moreover, we embed $B(V, W)$ into $B(V, W) \otimes A$ via $T \mapsto T \otimes 1$ and define the anti-linear map $\overline{\cdot} \colon B(V, W) \otimes A \to B(\overline{V}, \overline{W}) \otimes A$ by

$$\overline{(T \otimes a)} := \overline{T} \otimes a^* \quad \forall T \in B(V, W), a \in A.$$

With respect to the conjugate bases of V and W, the matrix coefficients of $\overline{u}$ are then given by $\overline{u}_j^i = (u_j^i)^*$.

Finally, we introduce the Woronowicz tensor products $\circledcirc$ and $\circledcirc$ from [40]. These are the bilinear operators defined by

$$(T \otimes a) \circledcirc (S \otimes b) := TS \otimes a \otimes b \qquad \in B(V) \otimes A \otimes A,$$
$$(T \otimes a) \circledcirc (S \otimes b) := (T \otimes S) \otimes ab \quad \in B(V \otimes W) \otimes A$$

for all $T \otimes a \in B(V) \otimes A$ and $S \otimes b \in B(V) \otimes A$ or $S \otimes b \in B(W) \otimes A$ respectively. The matrix coefficients of both operators are given by

$$(u \circledcirc v)_j^i = \sum_k u_k^i \otimes v_j^k, \qquad (u \circledcirc v)_{j_1 j_2}^{i_1 i_2} = u_{j_1}^{i_1} \cdot v_{j_2}^{i_2}$$

with respect to orthonormal bases of V, W and the induced basis of $V \otimes W$.

2.2. Compact matrix quantum groups.
Compact quantum groups were first introduced by Woronowicz [40, 42] as a generalization of classical compact groups and provide the general framework for easy quantum groups and spatial partition quantum groups. In the following, we will restrict ourselves to compact matrix quantum groups, which are a class of compact quantum groups defined by a single unitary representation. We refer to [39, 3] for a detailed introduction to compact

matrix quantum groups and to [33, 27] for the general case of compact quantum groups.

Definition 2.1. Let V be a finite-dimensional Hilbert space, A be a unital C^*-algebra and $u \in B(V) \otimes A$. Then $G := (A, u)$ is called a *compact matrix quantum group* if

(1) A is generated as C^*-algebra by the matrix coefficients

$$\{(\varphi \otimes \mathrm{id}_A)(u) \mid \varphi \in B(V, \mathbb{C})\},$$

(2) u is unitary and $\overline{u}$ is invertible,

(3) there exists a $*$-homomorphism $\Delta \colon A \to A \otimes A$, such that

$$(\mathrm{id}_{B(V)} \otimes \Delta)(u) = u \,\textcircled{\scriptsize $\top$}\, u.$$

The element u is called the *fundamental representation* of G. Further, we denote the C^*-algebra A with $C(G)$ and the dense $*$-algebra generated by the matrix coefficients with $\mathcal{O}(G)$.

Note that we have chosen a basis-independent definition of compact matrix quantum groups. This allows us to keep track of the tensor product structure of the underlying Hilbert space V and does not force us to identify $\mathbb{C}^n \otimes \mathbb{C}^n$ with $\mathbb{C}^{n^2}$ in an arbitrary way.

Examples of compact matrix quantum groups are Wang's free orthogonal and free unitary quantum groups [35] or its deformations in the sense of Van Daele-Wang [11]. Using the notation of Banica [1], these quantum groups are defined as follows.

Definition 2.2. Let $\boldsymbol{n} \in \mathbb{N}^m$, $F \in B(\mathbb{C}^{\boldsymbol{n}})$ be invertible and denote with $\iota \colon \overline{\mathbb{C}^{\boldsymbol{n}}} \to \mathbb{C}^{\boldsymbol{n}}$ the linear isomorphism defined by $\iota(\overline{e_{\boldsymbol{i}}}) = e_{\boldsymbol{i}}$ for all $\boldsymbol{i} \in [\boldsymbol{n}]$. Define the universal unital C^*-algebras

$$A_o(F) := C^*(u_{\boldsymbol{ij}} \mid u \text{ is unitary and } u = (F\iota)\,\overline{u}\,(F\iota)^{-1}),$$

$$A_u(F) := C^*(u_{\boldsymbol{ij}} \mid u \text{ and } (F\iota)\,\overline{u}\,(F\iota)^{-1} \text{ are unitary})$$

generated by the coefficients of a matrix $u := (u_{\boldsymbol{j}}^{\boldsymbol{i}})_{\boldsymbol{i,j} \in [\boldsymbol{n}]}$. Then

$$O^+(F) := (A_o(F), u), \qquad U^+(F) := (A_u(F), u)$$

are the *free orthogonal* and the *free unitary* quantum groups with parameter F. Moreover, we define $O_{\boldsymbol{n}}^+ := O^+(\mathrm{id}_{\mathbb{C}^{\boldsymbol{n}}})$ and $U_{\boldsymbol{n}}^+ := U^+(\mathrm{id}_{\mathbb{C}^{\boldsymbol{n}}})$.

Additional examples of compact matrix quantum groups are the quantum permutation group S_n^+ [36], the hyperoctahedral quantum groups H_n^+ [4] or more generally easy quantum groups [6] and spatial partition quantum groups [10]. We introduce spatial partition quantum groups in Section 4, which includes easy quantum groups as a special case. For a more detailed introduction to easy quantum groups, we refer to [38, 39].

As in the case of classical groups, one can generalize the notion of subgroup and isomorphic groups to the setting of compact matrix quantum groups.

Definition 2.3. Let G and H be compact matrix quantum groups with fundamental representations u on V and v on W. Then

(1) H is a *subgroup* of G and we write $H \subseteq G$ if there exists a unitary $Q \colon W \to V$ and a unital $*$-homomorphism $\varphi \colon C(G) \to C(H)$ with
$$\varphi(u) = QvQ^{-1},$$

(2) G and H are *isomorphic* and we write $G = H$ if $H \subseteq G$ and the previous $*$-homomorphism φ is a $*$-isomorphism,

(3) G and H are *isomorphic as compact quantum groups* if there exists a unital $*$-isomorphism $\varphi \colon C(G) \to C(H)$ with
$$(\varphi \otimes \varphi) \circ \Delta_G = \Delta_H \circ \varphi.$$

Note that an isomorphism between compact quantum groups is more general and does not respect the fundamental representation of a compact matrix quantum group. In particular, it allows us to compare quantum groups with fundamental representations of different dimensions.

Finally, we introduce representations of compact matrix quantum groups, which will be discussed in more detail in the following.

Definition 2.4. Let G be a compact matrix quantum group. A *representation* of G on a finite-dimensional Hilbert space V is an invertible element $v \in B(V) \otimes C(G)$ that satisfies
$$(\mathrm{id}_{B(V)} \otimes \Delta)(v) = v \,\underline{\scriptstyle\textcircled{\tiny T}}\, v.$$

Consider a compact matrix quantum group G on a Hilbert space V. Then its fundamental representation $u \in B(V) \otimes C(G)$ and the trivial representation $1 \in B(\mathbb{C}) \otimes C(G)$ are always representations. Further, if v and w are representations, then both $\overline{v}$ and $v \,\textcircled{\tiny T}\, w$ are again representations.

2.3. **Representation categories and rigidity.** Next, we recall that representations of a compact matrix quantum group have the structure of a tensor category, which allows us to formulate Woronowicz Tannaka-Krein duality in the next section. We refer to [41, 33, 27, 19] for more on the representation theory of compact matrix quantum groups and to [13] for tensor categories in general.

We begin by introducing intertwiners between representations.

Definition 2.5. Let G be a compact matrix quantum group with representations v on V and w on W. Then the space of *intertwiners* between v and w is given by
$$\mathrm{Hom}(v, w) := \{T \in B(V, W) \mid Tv = wT\}.$$

Consider a compact matrix quantum group G. Then we can construct a category $\mathrm{Rep}(G)$ with objects given by finite-dimensional unitary representations of G and morphisms $\mathrm{Hom}(v, w)$ given by intertwiners between v and w. Additionally, we can use the $\textcircled{\tiny T}$-operator and the adjoint T^* of intertwiners to define a monoidal and a $*$-structure on this category. The following proposition summarizes its most important properties.

Proposition 2.6. *The finite-dimensional unitary representations of a compact matrix quantum group are a concrete monoidal $*$-category in following the sense:*

(1) $\mathrm{id}_V \in \mathrm{Hom}(v, v)$ *for every representation v on V.*

(2) $\mathrm{Hom}(v, w)$ *is a linear subspace of $B(V, W)$ for all representations v on V and w on W.*

(3) If $S \in \mathrm{Hom}(v, w)$ and $T \in \mathrm{Hom}(w, x)$, then $ST \in \mathrm{Hom}(v, x)$.

(4) If $T \in \mathrm{Hom}(v, w)$, then $T^* \in \mathrm{Hom}(w, v)$.

(5) If $S \in \mathrm{Hom}(v, w)$ and $T \in \mathrm{Hom}(x, y)$, then $S \otimes T \in \mathrm{Hom}(v \oplus x, w \oplus y)$.

In addition to the previous properties, the representation category of a compact matrix quantum group is rigid in the sense that every unitary representation has a conjugate representation.

Definition 2.7. Let G be a compact matrix quantum group with unitary representations u° on V and $u^\bullet$ on W. Then $u^\bullet$ is *conjugate* to u° if there exist intertwiners $R \in \mathrm{Hom}(1, u^\circ \oplus u^\bullet)$ and $S \in \mathrm{Hom}(1, u^\bullet \oplus u^\circ)$ satisfying the conjugate equations

$$(R^* \otimes \mathrm{id}_V) \cdot (\mathrm{id}_V \otimes S) = \mathrm{id}_V, \qquad (S^* \otimes \mathrm{id}_W) \cdot (\mathrm{id}_W \otimes R) = \mathrm{id}_W.$$

Note that the representation $u^\bullet$ is also called a dual of u° and the intertwiners R and S are called duality morphisms. We refer to [13] and [27] for the definition of rigidity in a more general context.

In [41], it is shown that the representation category of a compact matrix quantum groups is generated by tensor powers of its fundamental representation u and its conjugate representation. Thus, it will be enough to consider only the fundamental representation $u^\circ := u$ with conjugate $u^\bullet$ in the following. Further, we introduce the notation $u^x := u^{x_1} \oplus \ldots \oplus u^{x_k}$ for all $x \in \{\circ, \bullet\}^*$ with $k := |x| > 0$. In the case $x = 1$, we define $u^1 := 1$ as the trivial representation.

Note that the conjugate of a representation is not unique but is only determined up to unitary equivalence. Furthermore, the representation $\overline{u^\circ}$ is in general not conjugate to u° since it is not necessarily unitary. However, we show in the following that any conjugate representation $u^\bullet$ is equivalent to $\overline{u^\circ}$ and this equivalence is defined by the solutions R and S of the conjugate equations.

Proposition 2.8. *Consider a compact matrix quantum group with fundamental representation u° on V and let $F \in B(\overline{V})$ be invertible. If $u^\bullet := F \overline{u^\circ} F^{-1}$ is unitary, then u° and $u^\bullet$ are conjugate with duality morphisms R and S given by*

$$R^{ij} = F^j_i, \qquad S^{ij} = (\overline{F}^{-1})^j_i$$

with respect to any basis of V and its conjugate basis.

Proof. A proof can be found in [19]. One checks directly that

$$u^\circ (u^\circ)^* = 1 \iff R \in \mathrm{Hom}(1, u^\circ \oplus u^\bullet),$$
$$u^\bullet (u^\bullet)^* = 1 \iff S \in \mathrm{Hom}(1, u^\bullet \oplus u^\circ).$$

Thus, R and S are intertwiners. Further, they satisfy the conjugate equations since

$$(R^* \otimes \mathrm{id}_V) \cdot (\mathrm{id}_V \otimes S) = \overline{F}^{-1}\overline{F}, \qquad (S^* \otimes \mathrm{id}_{\overline{V}}) \cdot (\mathrm{id}_{\overline{V}} \otimes R) = FF^{-1}.$$

$\square$

Proposition 2.9. *Let G be a compact matrix quantum group with unitary representation u° on V. Assume $u^\bullet$ is a unitary representation on $\overline{V}$ that is conjugate to u° with duality morphisms R and S. Then $u^\bullet = F\overline{u^\circ}F^{-1}$, where $F \in B(\overline{V})$ is given by $F^i_j = R^{ji}$ with respect to any basis of V and its conjugate basis.*

Proof. The representation $\overline{u^\circ}$ is equivalent to a unitary representation that is conjugate to u°, see [27, Example 2.2.3]. Further, [27, Proposition 2.2.4] shows that conjugates are unique up to equivalence. Hence, there exists an invertible $\widetilde{F} \in B(\overline{V})$ such that $u^\bullet = \widetilde{F}\overline{u^\circ}\widetilde{F}^{-1}$. The previous proposition then implies that

$$\widetilde{R} \in \mathrm{Hom}(1, u^\circ \oplus u^\bullet), \qquad \widetilde{S} \in \mathrm{Hom}(1, u^\bullet \oplus u^\circ),$$

where $\widetilde{R}$ and $\widetilde{S}$ are the solutions to the conjugate equations defined by $\widetilde{F}$. As in [18, Theorem 3.4.6], we compute

$$(\widetilde{S}^* \otimes \mathrm{id}_{\overline{V}}) \cdot (\mathrm{id}_{\overline{V}} \otimes R) = F\widetilde{F}^{-1} \in \mathrm{Hom}(u^\bullet, u^\bullet),$$

which yields

$$u^\bullet = F\widetilde{F}^{-1}u^\bullet\widetilde{F}F^{-1} = F\overline{u^\circ}F^{-1}.$$

$\square$

2.4. **Woronowicz Tannaka-Krein duality.** In the case of classical groups, Tannaka-Krein duality [12] allows the reconstruction of a compact group from an abstract category of representations. A similar result for compact matrix quantum groups was first proven by Woronowicz in [41].

There exist multiple proofs of Woronowicz Tannaka-Krein duality at various levels of abstraction, see for example [41, 27, 22, 19]. In the following, we choose the approach of Gromada [19], which is most suitable for the setting of categories defined by partitions. Thus, we begin by introducing the notion of an abstract two-colored representation category that captures the properties of the representation category of a compact matrix quantum group discussed in the previous section.

Definition 2.10. Let V be a finite-dimensional Hilbert space. A *two-colored representation category* $\mathcal{C}$ is a collection of linear subspaces $\mathcal{C}(x, y) \subseteq B(V^{\otimes x}, V^{\otimes y})$ for all $x, y \in \{\circ, \bullet\}^*$ satisfying the following properties:
  (1) $\mathrm{id}_{V^{\otimes x}} \in \mathcal{C}(x, x)$ for all $x \in \{\circ, \bullet\}^*$.
  (2) If $S \in \mathcal{C}(x, y)$ and $T \in \mathcal{C}(y, z)$, then $ST \in \mathcal{C}(x, z)$.
  (3) If $T \in \mathcal{C}(x, y)$, then $T^* \in \mathcal{C}(y, x)$.
  (4) If $S \in \mathcal{C}(w, x)$ and $T \in \mathcal{C}(y, z)$, then $S \otimes T \in \mathcal{C}(wy, xz)$.

(5) There exist $R \in \mathcal{C}(1, \circ\bullet)$ and $S \in \mathcal{C}(1, \bullet\circ)$ satisfying the conjugate equations

$$(R^* \otimes \mathrm{id}_V) \cdot (\mathrm{id}_V \otimes S) = \mathrm{id}_V, \qquad (S^* \otimes \mathrm{id}_{\overline{V}}) \cdot (\mathrm{id}_{\overline{V}} \otimes R) = \mathrm{id}_{\overline{V}}.$$

Woronowicz Tannaka-Krein duality now states that given any two-colored representation category $\mathcal{C}$, there exists a unique compact matrix quantum group whose representation category $\mathrm{Rep}(G)$ is given by $\mathcal{C}$.

Theorem 2.11 (Woronowicz Tannaka-Krein duality)**.** *Let $\mathcal{C}$ be a two-colored representation category on a Hilbert space V. Then there exists a unique compact matrix quantum group G with fundamental representation u° on V and unitary representation $u^\bullet$ on $\overline{V}$, such that*

$$\mathrm{Hom}(u^x, u^y) = \mathcal{C}(x, y) \quad \forall x, y \in \{\circ, \bullet\}^*.$$

First, let us comment on the uniqueness of the unitary representation $u^\bullet$ in the previous theorem. Since $\mathcal{C}$ contains a pair of morphisms R and S satisfying the conjugate equations, the representations u° and $u^\bullet$ are conjugate. Proposition 2.9 then implies that $u^\bullet = F\overline{u^\circ}F^{-1}$ with $F^i_j = R^{ji}$. Thus, the representation $u^\bullet$ is uniquely determined by the category $\mathcal{C}$.

Next, consider the uniqueness of the quantum group G. The proof of Woronowicz Tannaka-Krein duality in [19] shows that all relations of the dense $*$-algebra $\mathcal{O}(G)$ are spanned by the intertwiner relations

$$Tu^x = u^y T \qquad \forall x, y \in \{\circ, \bullet\}^*, \; T \in \mathcal{C}(x, y).$$

Hence, $\mathcal{O}(G)$ is uniquely determined by the category $\mathcal{C}$ up to $*$-isomorphism. In contrast, the C^*-algebra $C(G)$ is not unique but it is always possible to choose $C(G)$ as the maximal C^*-completion of $\mathcal{O}(G)$. In this case, G is uniquely defined by the following universal property.

Proposition 2.12. *Let G be the compact matrix quantum group in Theorem 2.11 and H be a compact matrix quantum group with fundamental representation w° on W and unitary representation $w^\bullet$ on $\overline{W}$. If there exists a unitary $Q \colon V \to W$, such that*

$$\mathrm{Hom}(u^x, u^y) \subseteq (Q^{-1})^{\otimes y} \cdot \mathrm{Hom}(w^x, w^y) \cdot Q^{\otimes x} \quad \forall x, y \in \{\circ, \bullet\}^*,$$

then H is a subgroup G via $u^\circ \mapsto Q^{-1}w^\circ Q$.

Motivated by Woronowicz Tannaka-Krein duality and the previous proposition, we introduce the following notion of equivalence between compact matrix quantum groups.

Definition 2.13. Let G be a compact matrix quantum group with fundamental representation u° and conjugate representation $u^\bullet$. A compact matrix quantum group H with fundamental representation w° on W and unitary representation $w^\bullet$ on $\overline{W}$ is called *equivalent* to G if there exists a unitary $Q \colon V \to W$ such that

$$\mathrm{Hom}(u^x, u^y) = (Q^{-1})^{\otimes y} \cdot \mathrm{Hom}(w^x, w^y) \cdot Q^{\otimes x} \quad \forall x, y \in \{\circ, \bullet\}^*.$$

One can check that the previous definition is indeed an equivalence relation between compact matrix quantum groups. Further, Woronowicz Tannaka-Krein duality and the previous discussion show that two compact matrix quantum groups G and H are equivalent if and only if $\mathcal{O}(G)$ and $\mathcal{O}(H)$ are $*$-isomorphic. Moreover, this isomorphism is given by $u = Q^{-1}wQ$, where u and w are the fundamental representations of G and H respectively.

3. COMBINATORICS OF SPATIAL PARTITIONS

In the following, we introduce spatial partitions, their categories and our new base partitions as purely combinatorial objects. These will then be used again in Section 4 to define spatial partition quantum groups. Further, we construct two functors between categories of spatial partitions that will be used to prove our main results in Section 4 and Section 5.

3.1. **Spatial partitions.** We begin with spatial partitions, which are the main combinatorial objects to define categories of spatial partitions. In contrast to [10], we will not focus on spatial partitions with only white points, but we will consider colored partitions as described in [10, Remark 2.7].

Definition 3.1. Let $m \in \mathbb{N}$. A *spatial partition* on m *levels* is a tuple $(x, y, \{B_i\})$, where $x, y \in \{\circ, \bullet\}^*$ and $\{B_i\}$ is a decomposition of the points

$$\{1, \ldots, |x| + |y|\} \times \{1, \ldots, m\}$$

into disjoint subsets called *blocks*. We call x the *upper colors* and y the *lower colors* of the partition.

Given a spatial partition, we can visualize it as a colored string-diagram as follows. First, we draw an upper layer consisting of the points (i, j) with $1 \le i \le |x|$ and a lower layer consisting of the points (i, j) with $|x| < i \le |y|$, where in both layers i increases from the left to the right and j from the front to the back. Then we connect all points in the same blocks by lines and assign each upper point (i, j) the color x_i and each lower point $(|x| + i, j)$ the color y_i.

Example 3.2. Consider the spatial partition on two levels with upper colors $x = \circ\bullet$, lower colors $y = \circ\circ\bullet$ and blocks

$$\{(1, 1), (3, 2)\}, \{(1, 2), (3, 1)\}, \{(2, 1), (4, 1)\}, \{(2, 2), (4, 2), (5, 2)\}, \{(5, 1)\}.$$

Visualizing it as string-diagram yields
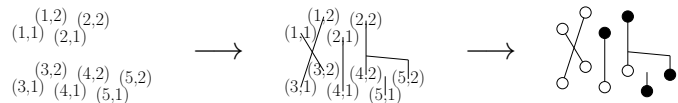
Note that spatial partitions on one level with only white points are exactly the partitions appearing for example in the definition of the Temperly-Liebe algebra [32], the Brauer algebra [9] or more generally orthogonal easy quantum groups [4].

In the following, we denote with $\mathcal{P}^{(m)}$ the set of all spatial partitions on m levels and with $\mathcal{P}^{(m)}(x, y)$ the set of all spatial partitions on m levels with upper colors x and lower colors y. Further, we introduce the following spatial partitions that will be used throughout the rest of the paper.

Definition 3.3.

(1) Let $x \in \{\circ, \bullet\}^*$. Then we denote with $\mathrm{id}_x \in \mathcal{P}^{(1)}(x, x)$ the *identity partition* on x. It is the spatial partition on one level with upper and lower colors x, where each upper point is directly connected to the corresponding lower point, e.g.

$$\mathrm{id}_\circ = \Big|, \qquad \mathrm{id}_\bullet = \Big|, \qquad \mathrm{id}_{\circ\bullet\circ} = \Big| \Big| \Big|.$$

(2) Let $p \in \mathcal{P}^{(1)}(x, y)$ be a spatial partition. Then we denote with $p^{(k)} \in \mathcal{P}^{(k)}(x, y)$ the *amplification* of p. It is obtained by placing k copies of p behind each other, e.g.

$$\mathrm{id}_\circ^{(2)} = \Big|\Big|, \qquad \ulcorner\!\urcorner^{(3)} = \quad.$$

(3) Let $x, y \in \{\circ, \bullet\}$ and $\sigma \in S_m$ be a permutation on $\{1, \ldots, m\}$. Then we denote with $\sigma_{xy} \in \mathcal{P}^{(m)}(1, xy)$ the spatial partition with lower points xy and blocks given by $\{(1, i), (2, \sigma(i))\}$ for all $1 \leq i \leq m$. For example, we have

$$(1)(2)_{\circ\bullet} = \quad, \qquad (12)_{\circ\bullet} = \quad, \qquad (132)_{\circ\circ} = \quad,$$

where the corresponding permutations are written in cycle notation. Similarly, we denote with $\sigma_y^x \in \mathcal{P}^{(m)}(x, y)$ the rotated version of σ_{xy} with upper color x and lower color y, e.g.

$$(1)(2)_\bullet^\circ = \quad, \qquad (12)_\bullet^\circ = \quad, \qquad (132)_\circ^\circ = \quad.$$

Given spatial partitions on a fixed number of levels m, we can use the following operations to construct new spatial partitions.
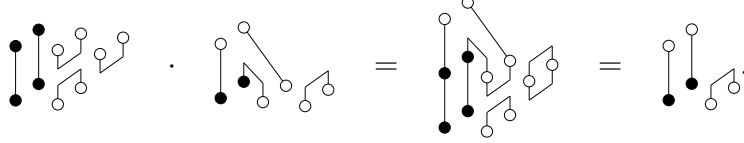
Definition 3.4.

(1) Let $p \in \mathcal{P}^{(m)}(x, y)$ and $q \in \mathcal{P}^{(m)}(w, z)$. Then the *tensor product* $p \otimes q \in \mathcal{P}^{(m)}(xw, yz)$ is obtained by placing p and q next to each other, e.g.

$$\quad \otimes \quad = \quad.$$

(2) Let $p \in \mathcal{P}^{(m)}(x, y)$. Then the *involution* $p^* \in \mathcal{P}^{(m)}(y, x)$ is obtained by swapping the upper and lower points, e.g.

$$\left(\quad\right)^* = \quad.$$

(3) Let $p \in \mathcal{P}^{(m)}(y, z)$ and $q \in \mathcal{P}^{(m)}(x, y)$. Then the *composition* $pq \in \mathcal{P}^{(m)}(x, z)$ is obtained by first placing q on top of p and identifying the lower points of q with the upper points of p. Then these points are removed and the resulting blocks are simplified, e.g.


Finally, we introduce the following properties of spatial partitions that will be used later.

Definition 3.5. Let $p \in \mathcal{P}^{(m)}(x, y)$ be a spatial partition. Then p is called
  (1) *invertible* if there exists a spatial partition $p^{-1} \in P^{(m)}(y, x)$ such that $p^{-1}p = \mathrm{id}_x$ and $pp^{-1} = \mathrm{id}_y$.
  (2) $\boldsymbol{n}$-*graded* for some $\boldsymbol{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$ if $n_k = n_\ell$ for all points (i, k) and (j, ℓ) that are in the same block in p.
  (3) *pair partition* if every block of p has size two.

Note that the inverse of a spatial partition $p \in P^{(m)}(x, y)$ is always given by $p^{-1} = p^*$ and in this case p and p^* are both pair partitions whose blocks form a bijection between the corresponding upper and lower points, e.g.

$$p = \text{\raisebox{-1em}{\includegraphics[height=3em]{p}}}, \qquad p^{-1} = \text{\raisebox{-1em}{\includegraphics[height=3em]{pinv}}}.$$
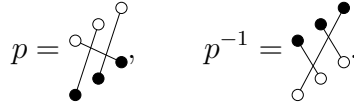
In particular, if p is invertible, then $|x| = |y|$. In the following, we present a proof of this statement for the case $|x| = |y| = 1$, which will be relevant later.

Lemma 3.6. *Let $p \in \mathcal{P}^{(m)}(x, y)$ be invertible with $|x| = |y| = 1$. Then $p = \sigma_y^x$ and $p^{-1} = (\sigma^{-1})_x^y$ for a permutation $\sigma \in S_m$.*

Proof. As in [17], we denote with $t(p)$ the number of through-blocks of p, i.e. the number of blocks of p that contain both an upper and a lower point. Then $t(p) \leq m$ and equality holds if and only if $p = \sigma_y^x$ for a permutation $\sigma \in S_m$. Further, $t(qp) \leq \min(t(q), t(p))$ for any $q \in \mathcal{P}^{(m)}(y, x)$, see [17, Remark 2.6]. Thus, if p is invertible, then

$$m = t(\mathrm{id}_x^{(m)}) = t(p^{-1} \cdot p) \leq \min(t(p^{-1}), t(p)) \leq t(p) \leq m.$$

This shows $t(p) = m$ and similarly one obtains $t(p^{-1}) = m$. Thus, both p and p^{-1} are of the form σ_y^x and $(\sigma^{-1})_x^y$ for a permutation $\sigma \in S_m$. $\qquad\square$

3.2. **Categories of spatial partitions.** Next, we introduce categories of spatial partitions as purely combinatorial objects. These will be used again in Section 4, where we interpret categories of spatial partitions as representation categories of compact matrix quantum groups.

Definition 3.7. A *category of spatial partitions* on m levels is a subset $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ that

(1) is closed under compositions, tensor products and involutions,

(2) contains $\mathrm{id}_\circ$ and $\mathrm{id}_\bullet$.

If $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ is a category of spatial partitions, then we denote with $\mathcal{C}(x,y)$ the set of all spatial partitions in $\mathcal{C}$ with upper colors x and lower colors y.

Note that in contrast to [10], we do not include the base partitions $\sqcap_\bullet^{(m)}$ and $\sqcap_\circ^{(m)}$ in the definition. Thus, our definition is more general and includes all categories of spatial partitions in the sense of Cébron-Weber as special case. However, we will come back to the role of these base partitions later. Examples of categories of spatial partitions include

(1) the set $\mathcal{P}^{(m)}$ of all spatial partitions on m levels,

(2) the set $\mathcal{P}_2^{(m)}$ of all spatial pair partitions on m levels,

(3) the set $\mathcal{NC} \subseteq \mathcal{P}^{(1)}$ of all non-crossing partitions, i.e. partitions that can be drawn without crossing lines in two dimensions.

Additional examples of categories of spatial partitions can for example be found in [29, 31, 10]. As in the previous examples, categories of spatial partitions are often defined by specifying a property of spatial partitions. However, it is also possible to define categories of spatial partition by a generating set.

Definition 3.8. Let $\mathcal{C}_0$ be a set of spatial partitions. Then we denote with $\mathcal{C} := \langle \mathcal{C}_0 \rangle$ the category *generated* by $\mathcal{C}_0$. It is the smallest category of spatial partitions containing $\mathcal{C}_0$ and consists of all finite combinations of elements in $\mathcal{C}_0$ and the base partitions $\mathrm{id}_\circ$ and $\mathrm{id}_\bullet$.

Consider again the categories of spatial partitions from the previous example. Using generators, these categories can be written as

$$\mathcal{P}^{(1)} = \big\langle \sqcap, \times, \sqcup \big\rangle, \qquad \mathcal{P}_2^{(1)} = \big\langle \sqcap, \times \big\rangle, \qquad \mathcal{NC} = \big\langle \sqcap, \sqcup \big\rangle,$$

see [29] for further details. Moreover, generators for general $\mathcal{P}^{(m)}$ and $\mathcal{P}_2^{(m)}$ can be found in [10].

In the original definition of spatial partition quantum groups in [10], Cébron-Weber include the additional base partitions $\sqcap_\bullet^{(m)}$ and $\sqcap_\circ^{(m)}$. These are used in the construction of spatial partitions quantum groups and guarantee that the representation $\overline{u}$ is unitary. However, as discussed in Section 2.3 and Section 2.4, it is only required that there exists a conjugate representation $u^\bullet$ that is not necessarily given by $\overline{u}$. This is equivalent to the existence of solutions to the conjugate equations in the sense of Definition 2.7. Thus, translating the conjugate equations into the setting of spatial partitions yields the following definition.

Definition 3.9. Let $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ be a category of spatial partitions. Then $\mathcal{C}$ is called *rigid* if it contains a pair of duality partitions $r \in \mathcal{C}(1, \circ\bullet)$ and $s \in \mathcal{C}(1, \bullet\circ)$ satisfying

the conjugate equations

$$(r^* \otimes \mathrm{id}_\circ) \cdot (\mathrm{id}_\circ \otimes s) = \mathrm{id}_\circ, \qquad (s^* \otimes \mathrm{id}_\bullet) \cdot (\mathrm{id}_\bullet \otimes r) = \mathrm{id}_\bullet .$$

In the case of spatial partitions on one level, the conjugate equations can be visualized diagrammatically as follows:
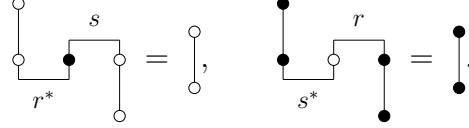

In particular, this shows that $r = \sqcap$ and $s = \sqcap$ are the only solutions in the case $m = 1$. Further, one checks that the duality partitions $r = \sqcap^{(m)}$ and $s = \sqcap^{(m)}$ of Cébron-Weber satisfy the conjugate equations, which implies that categories of spatial partitions in the sense of [10] are always rigid. However, for $m \geq 2$, there exist additional solutions given by spatial partitions like

The following proposition characterizes all possible solutions of the conjugate equations in the context of spatial partitions.

Proposition 3.10. *Duality partitions in the sense of Definition 3.9 are exactly of the form $r = \sigma_{\circ\bullet}$ and $s = \sigma_{\bullet\circ}^{-1}$ for a permutation $\sigma \in S_m$.*

Proof. Fix a number of levels m and assume $r \in \mathcal{P}^{(m)}(1, \circ\bullet)$ and $s \in \mathcal{P}^{(m)}(1, \bullet\circ)$ are a pair of duality partitions. Define the new partitions $r' \in \mathcal{P}^{(m)}(\circ, \bullet)$ and $s' \in P^{(m)}(\bullet, \circ)$ by moving the lower points $(1, i)$ for $1 \leq i \leq m$ to the upper layer. Then the conjugate equations (visualized only for one level) yield
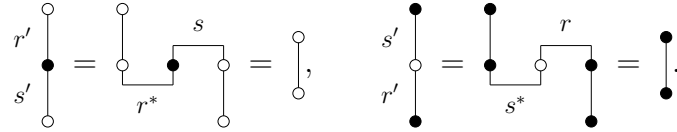
Thus, r' and s' are a pair of inverse partitions and Lemma 3.6 implies that they are the form $r' = \sigma_\bullet^\circ$ and $s' = (\sigma^{-1})_\circ^\bullet$ for a permutation $\sigma \in S_m$. Hence, we obtain $r = \sigma_{\circ\bullet}$ and $s = \sigma_{\bullet\circ}^{-1}$ by moving the upper points $(1, i)$ for $1 \leq i \leq m$ back to the lower layer.

Conversely, let $\sigma \in S_m$ and define the partition $r = \sigma_{\circ\bullet}$ and $s = \sigma_{\bullet\circ}^{-1}$. By tracing the blocks in the first conjugate equation, one checks that each point $(1, i)$ in r is connected to $(2, \sigma(i))$, which is again connected to $(2, \sigma^{-1}(\sigma(i))) = (2, i)$ in s. Thus, the first conjugate equation

$$(r^* \otimes \mathrm{id}_\circ) \cdot (\mathrm{id}_\circ \otimes s) = \mathrm{id}_\circ$$

is satisfied. Similarly, one checks that the second conjugate equation is satisfied, which shows that r and s are duality partitions. $\qquad \square$

So far, we have only considered categories of spatial partition $\mathcal{C}$ as purely combinatorial objects. However, we can also give them the structure of a category in the usual sense by defining the set of objects $\mathrm{Ob}(\mathcal{C}) := \{\circ, \bullet\}^*$ and the set of morphisms

$$\mathrm{Hom}(x, y) := \mathcal{C}(x, y) = \mathcal{C} \cap \mathcal{P}^{(m)}(x, y) \qquad \forall x, y \in \{\circ, \bullet\}^*.$$

Moreover, using the language of categories, the tensor product and involution give categories of spatial partitions the structure of strict monoidal †-categories. See [13] for precise definitions of these terms.

Using this framework, we can now show that the duality of the objects $\circ$ and $\bullet$ in the sense of Definition 3.9 implies the duality of x and $\overline{x}$ for all objects $x \in \{\circ, \bullet\}^*$. Thus, our notation of rigidity agrees with the general notation of rigidity as for example defined in [13] and [27].

Proposition 3.11. *Let $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ be a rigid category of spatial partitions and $x \in \{\circ, \bullet\}^*$. Then $\mathcal{C}$ contains a pair of duality partitions $r \in \mathcal{C}(1, x\overline{x})$ and $s \in \mathcal{C}(1, \overline{x}x)$ satisfying the conjugate equations for x and $\overline{x}$, i.e.*

$$(r^* \otimes \mathrm{id}_x) \cdot (\mathrm{id}_x \otimes s) = \mathrm{id}_x, \qquad (s^* \otimes \mathrm{id}_{\overline{x}}) \cdot (\mathrm{id}_{\overline{x}} \otimes r) = \mathrm{id}_{\overline{x}}.$$

Proof. The statement follows directly by inductively applying [13, Proposition 1.10.7] in the context of spatial partitions. Alternatively, the duality partitions r and s can be constructed explicitly by nesting the duality partitions for $\circ$ and $\bullet$. $\square$

3.3. **Spatial partition functors.** Next, we introduce functors between categories of spatial partitions. These allow us to transform spatial partitions while preserving their basic operations and will be used in the proofs our main results in Section 4 and Section 5.

Definition 3.12. Let $\mathcal{C} \subseteq \mathcal{P}^{(n)}$ and $\mathcal{D} \subseteq \mathcal{P}^{(m)}$ be categories of spatial partitions. A *spatial partition functor* $F \colon \mathcal{C} \to \mathcal{D}$ is given by a function $F \colon \{\circ, \bullet\}^* \to \{\circ, \bullet\}^*$ and functions $F \colon \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ for all $x, y \in \{\circ, \bullet\}^*$ satisfying the following properties:

(1) $F(1) = 1$ and $F(xy) = F(x)F(y)$ for all $x, y \in \{\circ, \bullet\}^*$
(2) F preserves the category operators, i.e.

$$F(p \cdot q) = F(p) \cdot F(q), \quad F(p^*) = F(p)^*, \quad F(p \otimes q) = F(p) \otimes F(q)$$

for all (composable) $p, q \in \mathcal{C}$,
(3) $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ for all $x \in \{\circ, \bullet\}^*$.

Note that a functor does not necessarily preserve the number of levels of spatial partitions. Using the language of categories, we also introduce the notion of fully faithful functors.

Definition 3.13. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a spatial partition functor. Then F is called *fully faithful* if the functions $F \colon \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ are bijections for all $x, y \in \{\circ, \bullet\}^*$.

A simple example of a spatial partition functor is obtained by decoloring spatial partitions. Formally, we define $F \colon \mathcal{P}^{(m)} \to \mathcal{P}^{(m)}$ by

$$F(x) := \circ^{|x|} \qquad\qquad \forall x \in \{\circ, \bullet\}^*,$$
$$F(p) := (\circ^{|x|}, \circ^{|y|}, \{B_i\}) \qquad \forall p := (x, y, \{B_i\}) \in \mathcal{P}^{(m)}.$$

Since F preserves the block structure of spatial partitions, one can check directly that it satisfies all properties in Definition 3.12 and that F is fully faithful. However, F is not surjective in the sense that its image does not contain $\overset{\bullet(m)}{\underset{\bullet}{|}}$. In particular, this shows that the image of a spatial partition functor is not necessarily again a category of spatial partitions.

Next, we present two additional examples of spatial partition functors that will be used again in the context of spatial partition quantum groups in Section 4 and Section 5.

3.3.1. *Permuting levels.* Let $\sigma, \tau \in S_m$ be permutations. Then we obtain a functor $\mathrm{Perm}_{\sigma,\tau} \colon \mathcal{P}^{(m)} \to \mathcal{P}^{(m)}$ by permuting the levels of white points using σ and the levels of black points using τ. Consider for example $\sigma = (12)$ and $\tau = (1)(2)$. Then
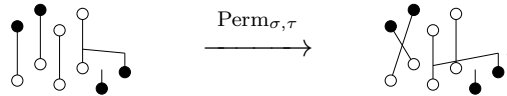
Using the permutation partitions introduced in Definition 3.3, we can define $\mathrm{Perm}_{\sigma,\tau}$ as follows.

Definition 3.14. Let $\sigma, \tau \in S_m$ and define

$$q_{\sigma,\tau}^{\circ} := \sigma_{\circ}^{\circ}, \qquad q_{\sigma,\tau}^{\bullet} := \tau_{\bullet}^{\bullet}, \qquad q_{\sigma,\tau}^{xy} := q_{\sigma,\tau}^x \otimes q_{\sigma,\tau}^y \quad \forall x, y \in \{\circ, \bullet\}^*, |xy| > 1.$$

Then the *permutation functor* $\mathrm{Perm}_{\sigma,\tau} \colon \mathcal{P}^{(m)} \to \mathcal{P}^{(m)}$ is given by

$$\mathrm{Perm}_{\sigma,\tau}(x) := x \qquad\qquad \forall x \in \{\circ, \bullet\}^*,$$
$$\mathrm{Perm}_{\sigma,\tau}(p) := q_{\sigma,\tau}^y \cdot p \cdot (q_{\sigma,\tau}^x)^{-1} \qquad \forall p \in \mathcal{P}^{(m)}(x, y).$$

Example 3.15. Consider again the spatial partition from the initial example with $\sigma := (12)$ and $\tau = (1)(2)$. Then
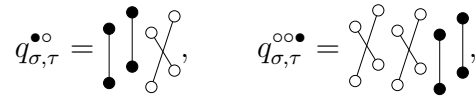
which yields
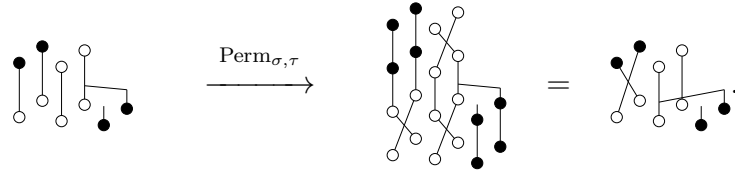
Next, we verify that the permutation functor $\mathrm{Perm}_{\sigma,\tau}$ is indeed a spatial partition functor in the sense of Definition 3.12 that additionally is fully faithful.

Proposition 3.16. $\mathrm{Perm}_{\sigma,\tau}$ *is a fully faithful spatial partition functor.*

Proof. Since $\mathrm{Perm}_{\sigma,\tau}$ preserves colors, it follows immediately that it respects the concatenation of colors. Further, Lemma 3.6 implies $q_{\sigma,\tau}^* = q_{\sigma,\tau}^{-1}$, which yields

$$\mathrm{Perm}_{\sigma,\tau}(p)^* = ((q_{\sigma,\tau}^x)^{-1})^* \cdot p^* \cdot (q_{\sigma,\tau}^y)^* = \mathrm{Perm}_{\sigma,\tau}(p^*) \qquad \forall p \in \mathcal{P}^{(m)}(x,y).$$

Similarly, one verifies directly that

$$\mathrm{Perm}_{\sigma,\tau}(p_1 \cdot p_2) = \mathrm{Perm}_{\sigma,\tau}(p_1) \cdot \mathrm{Perm}_{\sigma,\tau}(p_2),$$
$$\mathrm{Perm}_{\sigma,\tau}(p_1 \otimes p_2) = \mathrm{Perm}_{\sigma,\tau}(p_1) \otimes \mathrm{Perm}_{\sigma,\tau}(p_2)$$

for all (composable) $p_1, p_2 \in \mathcal{P}^{(m)}$. Hence, $\mathrm{Perm}_{\sigma,\tau}$ respects the category operations. Additionally, we have

$$\mathrm{Perm}_{\sigma,\tau}(\mathrm{id}_x) = q_{\sigma,\tau}^x \cdot \mathrm{id}_x \cdot (q_{\sigma,\tau}^x)^{-1} = \mathrm{id}_x \quad \forall x \in \{\circ, \bullet\}^*,$$

which shows that $\mathrm{Perm}_{\sigma,\tau}$ is spatial partition functor. Finally, one checks that

$$\mathrm{Perm}_{\sigma,\tau}(\mathrm{Perm}_{\sigma^{-1},\tau^{-1}}(p)) = \mathrm{Perm}_{\sigma^{-1},\tau^{-1}}(\mathrm{Perm}_{\sigma,\tau}(p)) = p \qquad \forall p \in \mathcal{P}^{(m)},$$

which implies that $\mathrm{Perm}_{\sigma,\tau}$ is bijective on the sets $\mathcal{P}^{(m)}(x,y)$. Thus, it is fully faithful. $\qquad\square$

Finally, we introduce $\boldsymbol{n}$-graded permutations and show that $\mathrm{Perm}_{\sigma,\tau}$ maps categories of spatial partitions to categories of spatial partitions while preserving rigidity.

Definition 3.17. Let $\boldsymbol{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$. A permutation $\sigma \in S_m$ is called $\boldsymbol{n}$-*graded* if $n_i = n_{\sigma(i)}$ for all $1 \le i \le m$.

Proposition 3.18. *Let* $\sigma, \tau \in S_m$ *be* $\boldsymbol{n}$-*graded permutations and* $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ *be a* $\boldsymbol{n}$-*graded rigid category of spatial partitions. Then* $\mathrm{Perm}_{\sigma,\tau}(\mathcal{C})$ *is again a* $\boldsymbol{n}$-*graded rigid category of spatial partitions.*

Proof. Define $\mathcal{D} := \mathrm{Perm}_{\sigma,\tau}(\mathcal{C})$. Since $\mathrm{Perm}_{\sigma,\tau}$ respects the category operations, it follows directly that $\mathcal{D}$ is closed under compositions, tensor products and involutions. Further, $\mathrm{Perm}_{\sigma,\tau}(\mathrm{id}_x) = \mathrm{id}_x \in \mathcal{D}$ for all $x \in \{\circ, \bullet\}^*$, which shows that $\mathcal{D}$ is a category of spatial partitions. Since $\mathcal{C}$ rigid, there exists a pair of duality partitions $r \in \mathcal{C}(1, \circ\bullet)$ and $s \in \mathcal{C}(1, \bullet\circ)$ satisfying the conjugate equations. Then $\mathrm{Perm}_{\sigma,\tau}(r) \in \mathcal{D}(1, \circ\bullet)$ and $\mathrm{Perm}_{\sigma,\tau}(s) \in \mathcal{D}(1, \bullet\circ)$ also satisfy the conjugate equations, which implies that $\mathcal{D}$ is again rigid. Finally, observe that permuting the levels of a $\boldsymbol{n}$-graded spatial partition by a $\boldsymbol{n}$-graded permutation yields again a $\boldsymbol{n}$-graded spatial partition. Thus, $\mathcal{D}$ is also a $\boldsymbol{n}$-graded category of spatial partitions. $\qquad\square$

3.3.2. *Flattening partitions.* Next, we introduce the functor $\mathrm{Flat}_{m,z}$ that flattens spatial partitions along the levels and is used in Section 5 in the context of projective versions of spatial partition quantum groups. Its definition is motivated by [10, Remark 2.4 & 2.8], where Cébron-Weber consider the bijection

$$\{1, \ldots k + \ell\} \times \{1, \ldots m\} \cong \{1, \ldots, m \cdot (k + \ell)\}$$

that identifies a point (i, j) with the point $m \cdot (i - 1) + j$. This bijection induces a bijection between the sets of spatial partitions $\mathcal{P}^{(m)}(\circ^k, \circ^l)$ and $\mathcal{P}^{(1)}(\circ^{mk}, \circ^{ml})$ by rearranging the points accordingly, e.g.

$$
\text{[diagram]} \in \mathcal{P}^{(3)}(\circ^1, \circ^2) \quad \cong \quad \text{[diagram]} \in \mathcal{P}^{(1)}(\circ^3, \circ^6).
$$

It respects the composition, tensor product and involution of spatial partitions, but it fails to preserve the base partitions as defined in [10] when applied to categories of spatial partitions.

Next, we generalize the previous bijection to the case of m levels and colored spatial partitions. Note that we reverse the order of black points in the following definition since in the context of quantum groups the representation $\overline{u \oplus w}$ is equivalent to $\overline{w} \oplus \overline{u}$. See Section 5 for further information.

Definition 3.19. Let $m, d \geq 1$ and $x, y \in \{\circ, \bullet\}^*$ with $k := |x|$ and $\ell := |y|$. Define the bijections

$$
\varphi_{m,d}^{x,y} \colon \{1, \ldots, k + \ell\} \times \{1, \ldots, m \cdot d\} \to \{1, \ldots, (k + \ell) \cdot d\} \times \{1, \ldots, m\}
$$

given by

$$
(i, j + k \cdot m) \mapsto \begin{cases} (i \cdot d - d + k + 1, j) & \text{if } (xy)_i = \circ, \\ (i \cdot d - k, j) & \text{if } (xy)_i = \bullet, \end{cases}
$$

for all $1 \leq i \leq k + \ell$, $1 \leq j \leq m$ and $0 \leq k < d$.

In the special case $x = \circ^k$, $y = \circ^\ell$ and $m = 1$, we obtain exactly the bijection of Cébron-Weber from the beginning of the section. A more general case on four levels is visualized in the following example.

Example 3.20. Consider four levels of points with upper colors $x = \circ$ and lower colors $y = \circ\bullet$. Then applying $\varphi := \varphi_{2,2}^{x,y}$ to these points yields

$$
\text{[diagram]} \quad \xrightarrow{\varphi} \quad \text{[diagram]} \quad \cong \quad \text{[diagram]}
$$

where the relabeled points are again rearranged according to our usual convention.

By extending this bijection to the blocks of spatial partitions, we obtain for example

$$
\text{[diagram]} \quad \xrightarrow{\varphi} \quad \text{[diagram]} .
$$

In particular, we observe that black points are flattened in reverse order. Using these bijections $\varphi_{m,d}^{x,y}$ from Definition 3.19, we can now define the functor $\mathrm{Flat}_{m,z}$.

Definition 3.21. Let $m \in \mathbb{N}$ and $z \in \{\circ, \bullet\}^*$ with $d := |z| > 1$. Define the spatial partition functor $\mathrm{Flat}_{m,z} \colon \mathcal{P}^{(m \cdot d)} \to \mathcal{P}^{(m)}$ by
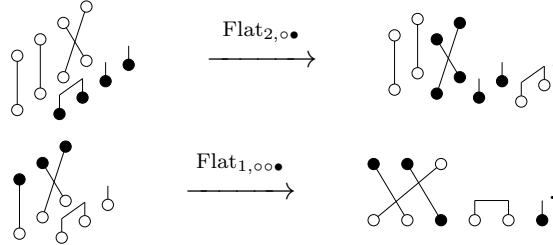
$$\mathrm{Flat}_{m,z}(\circ) := z, \quad \mathrm{Flat}_{m,z}(\bullet) := \overline{z} = \overline{z_d} \cdots \overline{z_1}$$

and

$$\mathrm{Flat}_{m,z}(p) := \left(\mathrm{Flat}_{m,z}(x), \mathrm{Flat}_{m,z}(y), \{\varphi_{m,d}^{x,y}(B_i)\} \right)$$

for all spatial partitions $p := (x, y, \{B_i\}) \in \mathcal{P}^{(m \cdot d)}$.

If $z = \circ \cdots \circ$, then $\mathrm{Flat}_{m,z}$ preserves colors and only applies the permutations $\varphi_{m,|z|}^{x,y}$ as in the previous example. However, in the general case, we obtain for example

It remains to verify that $\mathrm{Flat}_{m,z}$ is indeed a spatial partition functor.

Proposition 3.22. $\mathrm{Flat}_{m,z}$ *is a fully faithful spatial partition functor.*

Proof. By definition, $\mathrm{Flat}_{m,z}$ respects the concatenation of colors. Further, $\mathrm{Flat}_{m,z}$ respects compositions and involutions since it permutes the points of spatial only depending on the upper colors x and lower colors y. Moreover, $\mathrm{Flat}_{m,z}$ moves points at back levels in consecutive groups of size m to the front, which also implies that it respects tensor products. Finally,

$$\mathrm{Flat}_{m,z}(\mathrm{id}_\circ^{(m \cdot |z|)}) = \mathrm{id}_z^{(m)}, \qquad \mathrm{Flat}_{m,z}(\mathrm{id}_\bullet^{(m \cdot |z|)}) = \mathrm{id}_{\overline{z}}^{(m)}$$

yields that $\mathrm{Flat}_{m,z}$ maps identity partitions to identity partitions. Thus, $\mathrm{Flat}_{m,z}$ is a spatial partition functor. Additionally, it is fully faithful since the maps $\varphi_{m,d}^{x,y}$ are bijections. $\qquad\square$

The following proposition now shows that the pre-image under $\mathrm{Flat}_{m,z}$ preserves categories of spatial partitions, rigidity and gradings. This will be used again in Section 5, where it allows us to relate the category of a spatial partition quantum group to the category of its projective versions. Note that the proof relies on our generalization of base partitions and the statement would not hold in the setting of spatial partition quantum groups defined by Cébron-Weber.

Proposition 3.23. *Let* $\boldsymbol{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$ *and* $z \in \{\circ, \bullet\}^*$ *with* $d := |z| \geq 1$. *If* $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ *is a* $\boldsymbol{n}$*-graded rigid category of spatial partitions, then*

$$\mathrm{Flat}_{m,z}^{-1}(\mathcal{C}) := \{p \in \mathcal{P}^{(m \cdot d)} \mid \mathrm{Flat}_{m,z}(p) \in \mathcal{C}\}$$

is a $(\boldsymbol{n} \cdots \boldsymbol{n})$*-graded rigid category of spatial partitions, where*

$$(\boldsymbol{n} \cdots \boldsymbol{n}) := (n_1, \ldots, n_m, \ldots, n_1, \ldots, n_m) \in \mathbb{N}^{m \cdot d}.$$

Proof. Define $\mathcal{D} := \mathrm{Flat}_{m,z}^{-1}(\mathcal{C})$. Since $\mathrm{Flat}_{m,z}$ respects all operations on spatial partitions, it follows that $\mathcal{D}$ is closed under compositions, involutions and tensor products. Similarly, $\mathrm{Flat}_{m,z}$ maps identity partitions to identity partitions, which implies $\mathrm{id}_x \in \mathcal{D}$ for all $x \in \{\circ, \bullet\}^*$. Thus, $\mathcal{D}$ is a category of spatial partitions. Now, assume $\mathcal{C}$ is rigid. The Proposition 3.11 shows that there exist spatial partitions $r \in \mathcal{C}(1, z\bar{z})$ and $s \in \mathcal{C}(1, \bar{z}z)$ satisfying the conjugate equations for z and $\bar{z}$. Since $\mathrm{Flat}_{m,z}$ is fully faithful, there exist $\widetilde{r} \in \mathcal{D}(1, \circ\bullet)$ and $\widetilde{s} \in \mathcal{D}(1, \bullet\circ)$ with $\mathrm{Flat}_{m,z}(\widetilde{r}) = r$ and $\mathrm{Flat}_{m,z}(\widetilde{s}) = s$. These satisfy the conjugate equations for $\circ$ and $\bullet$ such that $\mathcal{D}$ is also rigid. Finally, assume $\mathcal{C}$ is $\boldsymbol{n}$-graded. Since $\mathrm{Flat}_{m,z}$ only moves levels in consecutive groups of size m to the front, it follows immediately that $\mathcal{D}$ is $(\boldsymbol{n} \cdots \boldsymbol{n})$-graded. $\qquad\square$

4. From spatial partitions to quantum groups

In the following, we use spatial partitions from the previous section to define spatial partition quantum groups. We then describe these quantum groups in terms of universal C^*-algebras and show that the free orthogonal quantum groups $O^+(F_\sigma)$ can be constructed from our new base partitions. Finally, we show that all resulting quantum groups are invariant under permuting levels, which implies that our new base partitions yield the same class of quantum groups as in [10].

Our definition of spatial partition quantum groups follows the one in [10], which generalizes the definition of easy quantum groups in [6, 30]. However, we discuss additional technicalities that arise when combining colors with our new base partitions.

4.1. **Spatial partition quantum groups.** In order to define spatial partition quantum groups, we need to associate a linear operator $T_p^{(\boldsymbol{n})}$ to each spatial partition $p \in \mathcal{P}^{(m)}$. However, we first have to introduce some additional notation.

Definition 4.1. Let $p \in \mathcal{P}^{(m)}(x, y)$ be a spatial partition and $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_{|x|}, \boldsymbol{j}_1, \ldots, \boldsymbol{j}_{|y|} \in \mathbb{N}^m$ be multi-indices of the form $\boldsymbol{i}_k = (i_{k,1}, \ldots, i_{k,m})$ for $1 \leq k \leq |x|$ and $\boldsymbol{j}_k = (j_{k,1}, \ldots, j_{k,m})$ for $1 \leq k \leq |y|$. Label each upper point (k, ℓ) of p with the index $i_{k,\ell}$ and each lower point $(|x| + k, \ell)$ with the index $j_{k,\ell}$. Then define

$$(\delta_p)_{\boldsymbol{j}_1, \ldots, \boldsymbol{j}_{|y|}}^{\boldsymbol{i}_1, \ldots, \boldsymbol{i}_{|x|}} := \begin{cases} 1 & \text{if all points in each block have equal labels,} \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.2. Consider the spatial partition $p := \raisebox{-0.5ex}{\includegraphics[height=2ex]{p}} \in \mathcal{P}^{(2)}(\circ, \circ\bullet)$ and the indices $\boldsymbol{i}_1 := (i_{1,1}, i_{1,2}), \boldsymbol{j}_1 := (j_{1,1}, j_{1,2}), \boldsymbol{j}_2 := (j_{2,1}, j_{2,2}) \in \mathbb{N}^2$. Then

$$(\delta_p)_{\boldsymbol{j}_1, \boldsymbol{j}_2}^{\boldsymbol{i}_1} = \raisebox{-2ex}{\includegraphics[height=5ex]{diag}} = \delta_{i_{1,1}j_{1,2}}\delta_{i_{1,2}j_{1,1}}\delta_{j_{2,1}j_{2,2}},$$

where $\delta_{k\ell}$ denotes the usual Kronecker delta.

Now, we can assign linear maps to spatial partitions as follows.

Definition 4.3. Let $p \in \mathcal{P}^{(m)}(x, y)$ be a $\boldsymbol{n}$-graded spatial partition. Then define the linear map

$$T_p^{(\boldsymbol{n})} \colon (\mathbb{C}^{\boldsymbol{n}})^{\otimes x} \to (\mathbb{C}^{\boldsymbol{n}})^{\otimes y}, \quad \left(T_p^{(\boldsymbol{n})}\right)^{\boldsymbol{i}_1, \ldots, \boldsymbol{i}_\ell}_{\boldsymbol{j}_1, \ldots, \boldsymbol{j}_k} := \left(\delta_p\right)^{\boldsymbol{j}_1, \ldots, \boldsymbol{j}_k}_{\boldsymbol{i}_1, \ldots, \boldsymbol{i}_\ell},$$

where $k := |x|$, $\ell := |y|$ and the coordinates are with respect to the canonical bases

$$\left(e^{x_1}_{\boldsymbol{j}_1} \otimes \cdots \otimes e^{x_k}_{\boldsymbol{j}_k}\right)_{\boldsymbol{j}_1, \ldots, \boldsymbol{j}_k \in [\boldsymbol{n}]}, \quad \left(e^{y_1}_{\boldsymbol{i}_1} \otimes \cdots \otimes e^{y_\ell}_{\boldsymbol{i}_\ell}\right)_{\boldsymbol{i}_1, \ldots, \boldsymbol{i}_\ell \in [\boldsymbol{n}]}$$

of $(\mathbb{C}^{\boldsymbol{n}})^{\otimes x}$ and $(\mathbb{C}^{\boldsymbol{n}})^{\otimes y}$ respectively.

A spatial partition quantum group is a compact matrix quantum group where the intertwiner spaces between tensor powers of its fundamental representation u° and its conjugate representation $u^\bullet$ are spanned by linear maps associated with a category of spatial partitions.

In contrast to [10], the conjugate representation $u^\bullet$ will no longer be given by $\overline{u^\circ}$ but depend on the corresponding category of spatial partitions. Further, we include a possible change of basis in the definition such that the notion of spatial partition quantum group is compatible with isomorphisms of compact matrix quantum groups.

Definition 4.4. Let G be a compact matrix quantum group with fundamental representation u° on V and unitary representation $u^\bullet$ on $\overline{V}$. Then G is a *spatial partition quantum group* if there exists a $\boldsymbol{n}$-graded rigid category of spatial partitions $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ and a unitary $Q \colon \mathbb{C}^{\boldsymbol{n}} \to V$, such that

$$\operatorname{Hom}(u^x, u^y) = \operatorname{span}\left\{Q^{\otimes y} \cdot T_p^{(\boldsymbol{n})} \cdot (Q^{-1})^{\otimes x} \mid p \in \mathcal{C}(x, y)\right\} \quad \forall x, y \in \{\circ, \bullet\}^*.$$

Since the category $\mathcal{C}$ contains a pair of spatial partitions solving the conjugate equations, it follows that $u^\bullet$ is conjugate to u°. Before we discuss how this conjugate representation $u^\bullet$ depends on $\mathcal{C}$, we first show that for every rigid category of spatial partitions, there exists a corresponding spatial partition quantum group. This requires the following proposition, which states that the mapping $p \mapsto T_p^{(\boldsymbol{n})}$ is almost functorial.

Proposition 4.5. *Let $\boldsymbol{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$ and $p, q \in \mathcal{P}^{(m)}$ be $\boldsymbol{n}$-graded spatial partitions. Then*

(1) $T_p^{(\boldsymbol{n})} \otimes T_q^{(\boldsymbol{n})} = T_{p \otimes q}^{(\boldsymbol{n})}$

(2) $\left(T_p^{(\boldsymbol{n})}\right)^* = T_{p^*}^{(\boldsymbol{n})}$

(3) $T_p^{(\boldsymbol{n})} \cdot T_q^{(\boldsymbol{n})} = N^\alpha \cdot T_{pq}^{(\boldsymbol{n})}$, *where $N := n_1 \cdots n_m$ and $\alpha \in \mathbb{N}$ denotes the number of removed loops when composing p and q (if the composition is defined).*

Proof. See [10, Proposition 3.7]. □

Using the previous proposition, we can now show the existence of spatial partition quantum groups using Woronowicz Tannaka-Krein duality.

Theorem 4.6. *Let $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ be a $\boldsymbol{n}$-graded rigid category of spatial partitions. Then there exists a unique compact matrix quantum group $G_{\boldsymbol{n}}(\mathcal{C})$ with fundamental representation u° on $V := \mathbb{C}^{\boldsymbol{n}}$ and a unitary representation $u^\bullet$ on $\overline{V}$, such that*

$$\operatorname{Hom}(u^x, u^y) = \operatorname{span}\left\{T_p^{(\boldsymbol{n})} \mid p \in \mathcal{C}(x, y)\right\} \quad \forall x, y \in \{\circ, \bullet\}^*.$$

Proof. Define the linear subspaces

$$\widehat{\mathcal{C}}(x, y) := \operatorname{span}\left\{T_p^{(\boldsymbol{n})} \mid p \in \mathcal{C}(x, y)\right\} \subseteq B(V^{\otimes x}, V^{\otimes y})$$

for all $x, y \in \{\circ, \bullet\}^*$. Then Proposition 4.5 implies that $\widehat{\mathcal{C}}$ is closed under composition, involution and tensor products as in Definition 2.10. Further, one checks that $T_{\operatorname{id}_x}^{(\boldsymbol{n})} = \operatorname{id}_{V^{\otimes x}} \in \widehat{\mathcal{C}}(x, x)$ for all $x \in \{\circ, \bullet\}^*$. Since $\mathcal{C}$ is rigid, there exists a pair of spatial partitions $r \in \mathcal{C}(1, \circ\bullet)$ and $s \in \mathcal{C}(1, \bullet\circ)$ satisfying the conjugate equations. By Proposition 4.5, the operators $R := T_r^{(\boldsymbol{n})}$ and $S := T_s^{(\boldsymbol{n})}$ satisfy again the conjugate equations for u° and $u^\bullet$ because no closed loops are removed when composing r and s. Thus, the subspaces $\widehat{\mathcal{C}}(x, y)$ form a two-colored representation category in the sense of Definition 2.10 and the statement follows by applying Theorem 2.11.   □

Remark 4.7. As discussed in Section 2.4, the quantum group $G_{\boldsymbol{n}}(\mathcal{C})$ from the previous theorem is uniquely determined by the following universal property. Let H be a compact matrix quantum group with fundamental representation w° on V and unitary representation $w^\bullet$ on $\overline{V}$. If there exists a unitary $Q \colon \mathbb{C}^{\boldsymbol{n}} \to V$, such that

$$\operatorname{Hom}(u^x, u^y) \subseteq (Q^{-1})^{\otimes y} \cdot \operatorname{Hom}(w^x, w^y) \cdot Q^{\otimes x} \quad \forall x, y \in \{\circ, \bullet\}^*,$$

then H is a subgroup of G via the map $u^\circ \mapsto Q^{-1} w^\circ Q$.

Remark 4.8. It follows directly from the definition that a compact matrix quantum group is a spatial partition quantum group if and only if it is equivalent to $G_{\boldsymbol{n}}(\mathcal{C})$ for a $\boldsymbol{n}$-graded rigid category of spatial partitions $\mathcal{C}$.

Next, we come back to the conjugate representations $u^\bullet$ of spatial partition quantum groups and show that $u^\bullet$ can be expressed in terms of u° and unitaries $F_\sigma^{(\boldsymbol{n})}$.

Definition 4.9. Let $\sigma \in S_m$ be a $\boldsymbol{n}$-graded permutation. Then define the linear map $F_\sigma^{(\boldsymbol{n})} \colon \mathbb{C}^{\boldsymbol{n}} \to \mathbb{C}^{\boldsymbol{n}}$ by

$$F_\sigma^{(\boldsymbol{n})}(e_{i_1} \otimes \cdots \otimes e_{i_m}) = e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(m)}} \quad \forall (i_1, \ldots, i_m) \in [\boldsymbol{n}].$$

Note that the mapping $\sigma \mapsto F_\sigma^{(\boldsymbol{n})}$ defines a unitary representation of $\boldsymbol{n}$-graded permutations on $\mathbb{C}^{\boldsymbol{n}}$, i.e. we have $F_\sigma^{(\boldsymbol{n})} \cdot F_\tau^{(\boldsymbol{n})} = F_{\sigma\tau}^{(\boldsymbol{n})}$ and $\left(F_\sigma^{(\boldsymbol{n})}\right)^* = \left(F_\sigma^{(\boldsymbol{n})}\right)^{-1} = F_{\sigma^{-1}}^{(\boldsymbol{n})}$ for all $\boldsymbol{n}$-graded $\sigma, \tau \in S_m$. Additionally, we write $\overline{F}_\sigma^{(\boldsymbol{n})}$ for the corresponding conjugate operator.

Using the linear maps $F_\sigma^{(\boldsymbol{n})}$, we can now describe the conjugate representations $u^\bullet$ of spatial partition quantum groups of the form $G_{\boldsymbol{n}}(\mathcal{C})$.

Proposition 4.10. *Let $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ be a $\boldsymbol{n}$-graded rigid category of spatial partitions containing duality partitions $\sigma_{\circ\bullet}$ and $\sigma_{\bullet\circ}^{-1}$. Then the conjugate representation $u^{\bullet}$ of $G_{\boldsymbol{n}}(\mathcal{C})$ is given by $u^{\bullet} = \overline{F}_{\sigma}^{(\boldsymbol{n})} \, \overline{u^{\circ}} \left(\overline{F}_{\sigma}^{(\boldsymbol{n})} \right)^{-1}$.*

Proof. Denote with u° the fundamental representation of $G_{\boldsymbol{n}}(\mathcal{C})$ and define $r := \sigma_{\circ\bullet}$ and $s := \sigma_{\bullet\circ}^{-1}$. Since r and s are a pair of duality partitions, Proposition 4.5 implies that $R := T_r^{(\boldsymbol{n})}$ and $S := T_s^{(\boldsymbol{n})}$ satisfy the conjugate equations for u° and $u^{\bullet}$. Hence, Proposition 2.9 yields $u^{\bullet} = F\overline{u^{\circ}}F^{-1}$ with F defined by $F_{\boldsymbol{j}}^{\boldsymbol{i}} = R^{\boldsymbol{j},\boldsymbol{i}}$ for all $\boldsymbol{i}, \boldsymbol{j} \in [\boldsymbol{n}]$. Next, we compute

$$F_{\boldsymbol{j}}^{\boldsymbol{i}} = \left(T_r^{(\boldsymbol{n})}\right)^{\boldsymbol{j},\boldsymbol{i}} = (\delta_r)_{\boldsymbol{j},\boldsymbol{i}} = \delta_{j_1 i_{\sigma(1)}} \cdots \delta_{j_m i_{\sigma(m)}} = \left(F_{\sigma}^{(\boldsymbol{n})}\right)_{\boldsymbol{j}}^{\boldsymbol{i}}$$

for all $\boldsymbol{i} := (i_1, \ldots, i_m), \boldsymbol{j} := (i_1, \ldots, j_m) \in [\boldsymbol{n}]$. Thus, $F = F_{\sigma}^{(\boldsymbol{n})}$. $\square$

The previous proposition shows that the conjugate representation $u^{\bullet}$ of $G_{\boldsymbol{n}}(\mathcal{C})$ is uniquely determined by the category $\mathcal{C}$. Since any spatial partition quantum group is equivalent to a $G_{\boldsymbol{n}}(\mathcal{C})$, it follows that the conjugate representation $u^{\bullet}$ of a general spatial partition quantum group is uniquely determined by the corresponding category $\mathcal{C}$ and the change of basis Q.

Since the linear maps $\overline{F}_{\sigma}^{(\boldsymbol{n})}$ are unitary, the previous proposition additionally implies that the representation $\overline{u^{\circ}} = \left(\overline{F}_{\sigma}^{(\boldsymbol{n})}\right)^{-1} u^{\bullet} \overline{F}_{\sigma}^{(\boldsymbol{n})}$ is again unitary. This shows that $\overline{u^{\circ}}$ is a conjugate representation of u° and that every spatial partition quantum group is a subgroup of $U_{\boldsymbol{n}}^{+}$. We will come back to this fact in Section 4.3, where we show the intertwiners spaces of a spatial partition quantum groups are still spanned by spatial partition when we replace $u^{\bullet}$ by $\overline{u^{\circ}}$.

4.2. **Presentations of spatial partitions quantum groups.** Next, we describe the C^*-algebras $C(G_{\boldsymbol{n}}(\mathcal{C}))$ from the previous section as universal C^*-algebras. If the category of spatial partitions $\mathcal{C}$ is generated by a finite set of spatial partitions, then we obtain a universal C^*-algebras defined by a finite set of relations. In particular, this allows us to show that the quantum groups $O^+(F_{\sigma}^{(\boldsymbol{n})})$ are spatial partition quantum groups.

We begin by extending the notation $G_{\boldsymbol{n}}(\mathcal{C})$ to arbitrary sets of spatial partitions that generate a rigid category of spatial partitions.

Definition 4.11. Let $\mathcal{C}_0 \subseteq \mathcal{P}^{(m)}$ be a set of $\boldsymbol{n}$-graded spatial partitions with duality partitions $\sigma_{\circ\bullet}, \sigma_{\bullet\circ}^{-1} \in \langle \mathcal{C}_0 \rangle$. Denote with A the universal unital C^*-algebra generated by the entries of a matrix $u := (u_{\boldsymbol{j}}^{\boldsymbol{i}})_{\boldsymbol{i},\boldsymbol{j}\in[\boldsymbol{n}]}$ and relations

  (1) $u^{\circ} := u$ and $u^{\bullet} := \overline{F}_{\sigma}^{(\boldsymbol{n})}\overline{u}\left(\overline{F}_{\sigma}^{(\boldsymbol{n})}\right)^{-1}$ are unitary,

  (2) $T_p^{(\boldsymbol{n})} u^x = u^y T_p^{(\boldsymbol{n})}$ for all $x, y \in \{\circ, \bullet\}^*$ and $p \in \mathcal{C}_0 \cap \mathcal{P}^{(m)}(x, y)$.

Then $G_{\boldsymbol{n}}(\mathcal{C}_0) := (A, u)$ is the *spatial partitions quantum groups defined by $\mathcal{C}_0$*.

First, note that the compact matrix quantum group $G_{\boldsymbol{n}}(\mathcal{C}_0)$ from the previous definition is well-defined. Indeed, A is generated by the elements $u_{\boldsymbol{j}}^{\boldsymbol{i}}$ and by the first relations, u is unitary and $\overline{u}$ is invertible. Further, one checks that both relations are compatible with the comultiplication, see also Definition 2.2 and [41].

Next, we show that the C^*-algebra $C(G_{\boldsymbol{n}}(\mathcal{C}_0))$ and thus the quantum group $G_{\boldsymbol{n}}(\mathcal{C}_0)$ does not depend on the particular choice of duality partitions $\sigma_{\circ\bullet}, \sigma_{\bullet\circ}^{-1} \in \langle\mathcal{C}_0\rangle$. However, we first need the following lemma.

Lemma 4.12. *Let $\mathcal{C}_0 \subseteq \mathcal{P}^{(m)}$ be a set of $\boldsymbol{n}$-graded spatial partitions generating a rigid category of spatial partitions $\mathcal{C} := \langle\mathcal{C}_0\rangle$ and consider the quantum group $G_{\boldsymbol{n}}(\mathcal{C}_0)$. Then*

$$\operatorname{span}\left\{T_p^{(\boldsymbol{n})} \mid p \in \mathcal{C}(x,y)\right\} \subseteq \operatorname{Hom}(u^x, u^y) \quad \forall x, y \in \{\circ, \bullet\}^*.$$

Proof. Proposition 2.6 and Proposition 4.5 imply that $T_{p\otimes q}^{(\boldsymbol{n})}$, $T_{p^*}^{(\boldsymbol{n})}$ and $T_{pq}^{(\boldsymbol{n})}$ are also intertwiners of $G_{\boldsymbol{n}}(\mathcal{C}_0)$ for all (composable) $p, q \in \mathcal{C}_0$. Thus, we obtain inductively that $T_p^{(\boldsymbol{n})}$ is an intertwiner for all $p \in \mathcal{C}$. The statement follows by applying the linearity of intertwiner spaces from Proposition 2.6. $\square$

Now, consider the quantum group $G_{\boldsymbol{n}}(\mathcal{C}_0)$ and assume $\mathcal{C} := \langle\mathcal{C}_0\rangle$ contains a different pair of duality partitions $r := \tau_{\circ\bullet}$ and $s := \tau_{\bullet\circ}^{-1}$ for $\tau \in S_m$. Then the linear maps $R := T_r^{(\boldsymbol{n})}$ and $S := T_s^{(\boldsymbol{n})}$ are intertwiners by the previous proposition and the proof of Proposition 4.10 shows that $u^\bullet = \overline{F}_\tau^{(\boldsymbol{n})} \overline{u} \left(\overline{F}_\tau^{(\boldsymbol{n})}\right)^{-1}$. Therefore, $u^\bullet$ and the C^*-algebra $C(G_{\boldsymbol{n}}(\mathcal{C}_0))$ do not depend on the particular choice of duality partitions in Definition 4.11.

If $\mathcal{C}_0 = \mathcal{C}$ is already a category of spatial partitions, then the notation $G_{\boldsymbol{n}}(\mathcal{C}_0)$ of the previous definition and $G_{\boldsymbol{n}}(\mathcal{C})$ of Theorem 4.6 overlap. However, the following proposition shows that both quantum groups coincide in this case and that more generally $G_{\boldsymbol{n}}(\langle\mathcal{C}_0\rangle)$ and $G_{\boldsymbol{n}}(\mathcal{C}_0)$ agree.

Proposition 4.13. *Let $\mathcal{C}_0 \subseteq \mathcal{P}^{(m)}$ be a set of $\boldsymbol{n}$-graded spatial partitions generating a rigid category of spatial partitions $\mathcal{C} := \langle\mathcal{C}_0\rangle$. Then the compact matrix quantum groups $G_{\boldsymbol{n}}(\mathcal{C})$ and $G_{\boldsymbol{n}}(\mathcal{C}_0)$ from Theorem 4.6 and Definition 4.11 are isomorphic.*

Proof. Denote with u and w the fundamental representations of $G_{\boldsymbol{n}}(\mathcal{C}_0)$ and $G_{\boldsymbol{n}}(\mathcal{C})$ respectively. Then Lemma 4.12 and the universal property of $G_{\boldsymbol{n}}(\mathcal{C})$ yield the inclusion $G_{\boldsymbol{n}}(\mathcal{C}_0) \subseteq G_{\boldsymbol{n}}(\mathcal{C})$ via $w \mapsto u$. Conversely, $\mathcal{C}_0 \subseteq \mathcal{C}$ and Proposition 4.10 show that w satisfies all defining relations of u. Thus, the universal property of $C(G_{\boldsymbol{n}}(\mathcal{C}_0))$ yields the inverse inclusion $G_{\boldsymbol{n}}(\mathcal{C}) \subseteq G_{\boldsymbol{n}}(\mathcal{C}_0)$ via $u \mapsto w$. Therefore, both quantum groups are isomorphic. $\square$

Using the description of spatial partition quantum groups as universal C^*-algebras, we can now construct the free orthogonal quantum groups $O^+(F_\sigma^{(\boldsymbol{n})})$ from spatial partitions.

Proposition 4.14. *Let $\sigma \in S_m$ be a $\boldsymbol{n}$-graded permutation and $\mathcal{C} := \left\langle \sigma_{\circ\bullet}, \sigma_{\bullet\circ}^{-1}, \overset{\bullet(m)}{\underset{\circ}{|}} \right\rangle$. Then $G_{\boldsymbol{n}}(\mathcal{C})$ and $O^+(F_\sigma^{(\boldsymbol{n})})$ are isomorphic.*

Proof. Denote with $u =: u^\circ$ the fundamental representation of $G_{\boldsymbol{n}}(\mathcal{C})$ and with $w =:$ w° the fundamental representation of $O^+(F_\sigma^{(\boldsymbol{n})})$. Further, denote with $\iota\colon \overline{\mathbb{C}^{\boldsymbol{n}}} \to \mathbb{C}^{\boldsymbol{n}}$ the linear isomorphism given by $\iota(\overline{e_i}) = e_i$ for all $i \in [\boldsymbol{n}]$. Then u is a unitary and Proposition 4.10 yields that its conjugate representation $u^\bullet$ is given by $u^\bullet = \overline{F}_\sigma^{(\boldsymbol{n})}\, \overline{u}\, \big(\overline{F}_\sigma^{(\boldsymbol{n})}\big)^{-1}$. Since $\iota = T_{\mathfrak{f}^{(m)}}^{(\boldsymbol{n})} \in \mathrm{Hom}(u^\bullet, u^\circ)$, we have $\iota u^\bullet = u^\circ \iota$, which implies

$$u = u^\circ = \iota u^\bullet \iota^{-1} = \big(\iota \overline{F}_\sigma^{(\boldsymbol{n})}\big)\, \overline{u}\, \big(\iota \overline{F}_\sigma^{(\boldsymbol{n})}\big)^{-1} = \big(F_\sigma^{(\boldsymbol{n})}\iota\big)\, \overline{u}\, \big(F_\sigma^{(\boldsymbol{n})}\iota\big)^{-1}.$$

Thus, the universal property of $C(O^+(F_\sigma^{(\boldsymbol{n})}))$ yields the inclusion $G_{\boldsymbol{n}}(\mathcal{C}) \subseteq O^+(F_\sigma^{(\boldsymbol{n})})$ via $w \mapsto u$.

Conversely, define $w^\bullet := \iota^{-1}w\iota$, which is unitary since both w and ι are unitary. Further, we have $\iota w^\bullet = \iota\iota^{-1}w\iota = w^\circ\iota$, which shows $T_{\mathfrak{f}^{(m)}}^{(\boldsymbol{n})} = \iota \in \mathrm{Hom}(w^\bullet, w^\circ)$. By the definition of $O^+(F_\sigma^{(\boldsymbol{n})})$ and the argument before, we also have

$$w^\bullet = \iota^{-1}w\iota = \big(\iota^{-1}F_\sigma^{(\boldsymbol{n})}\iota\big)\, \overline{w}\, \big(\iota^{-1}F_\sigma^{(\boldsymbol{n})}\iota\big)^{-1} = \overline{F}_\sigma^{(\boldsymbol{n})}\, \overline{w^\circ}\, \big(\overline{F}_\sigma^{(\boldsymbol{n})}\big)^{-1}.$$

Hence, Proposition 2.8 shows that w° and $w^\bullet$ are conjugate via intertwiners $R \in \mathrm{Hom}(1, w^{\circ\bullet})$ and $S \in \mathrm{Hom}(1, w^{\bullet\circ})$ given by

$$R^{i,j} = (\overline{F}_\sigma^{(\boldsymbol{n})})_i^j, \quad S^{i,j} = (F_{\sigma^{-1}}^{(\boldsymbol{n})})_i^j \qquad \forall i, j \in [\boldsymbol{n}].$$

As in proof of Proposition 4.10, this is equivalent to $R = T_{\sigma_{\circ\bullet}}^{(\boldsymbol{n})}$ and $S = T_{\sigma_{\bullet\circ}^{-1}}^{(\boldsymbol{n})}$. Therefore, w satisfies all the defining relations of $C(G_{\boldsymbol{n}}(\mathcal{C}_0))$ with $\mathcal{C}_0 := \{\sigma_{\circ\bullet}, \sigma_{\bullet\circ}^{-1}, \mathfrak{f}^{(m)}\}$. Since $G_{\boldsymbol{n}}(\mathcal{C}_0) = G_{\boldsymbol{n}}(\mathcal{C})$ by Proposition 4.13, the universal property of $C(G_{\boldsymbol{n}}(\mathcal{C}_0))$ yields the inverse inclusion $O^+(F_\sigma^{(\boldsymbol{n})}) \subseteq G_{\boldsymbol{n}}(\mathcal{C})$ via $u \mapsto w$. $\qquad\square$

Note that $O^+(F)$ and $O^+(QFQ^T)$ are isomorphic for any unitary Q, see for example [33]. Since $F_\tau^{(\boldsymbol{n})}F_\sigma^{(\boldsymbol{n})}\big(F_\tau^{(\boldsymbol{n})}\big)^T = F_{\tau\sigma\tau^{-1}}^{(\boldsymbol{n})}$, this implies that $O^+(F_\sigma^{(\boldsymbol{n})})$ and $O^+(F_{\tau\sigma\tau^{-1}}^{(\boldsymbol{n})})$ are isomorphic for all $\sigma, \tau \in S_m$. Thus, $O^+(F_\sigma^{(\boldsymbol{n})})$ depends only on the conjugacy class of σ.

4.3. **Permuting levels.** In Section 3.3.1, we introduced the functor $\mathrm{Perm}_{\sigma,\tau}$ that permutes the levels of a spatial partition depending on the color of the points. In the following, we show that $\mathrm{Perm}_{\sigma,\tau}$ leave the quantum groups $G_{\boldsymbol{n}}(\mathcal{C})$ invariant. This implies that our new duality partitions yield the same class of spatial partition quantum groups as defined by Cébron-Weber in [10].

The main work to prove this result will be done in the following technical lemma about the linear maps $T_{\mathrm{Perm}_{\sigma,\tau}(p)}^{(\boldsymbol{n})}$.

Lemma 4.15. *Let $\sigma, \tau \in S_m$ be $\boldsymbol{n}$-graded permutations. Consider a compact matrix quantum group with fundamental representation u° on $\mathbb{C}^{\boldsymbol{n}}$ and unitary representation $u^\bullet$ on $\overline{\mathbb{C}^{\boldsymbol{n}}}$. Define the unitary representations $\hat{u}^\circ := u^\circ$ and $\hat{u}^\bullet := \big(\overline{F}_{\tau\sigma^{-1}}^{(\boldsymbol{n})}\big)^{-1} u^\bullet\, \overline{F}_{\tau\sigma^{-1}}^{(\boldsymbol{n})}$.*

Then

$$T^{(\boldsymbol{n})}_{\mathrm{Perm}_{\sigma,\tau}(p)} \in \mathrm{Hom}(u^x, u^y) \iff \left(F^{(\boldsymbol{n})}_\sigma\right)^{\otimes y} T^{(\boldsymbol{n})}_p \left(\left(F^{(\boldsymbol{n})}_\sigma\right)^{-1}\right)^{\otimes x} \in \mathrm{Hom}(\hat{u}^x, \hat{u}^y)$$

for all $p \in \mathcal{P}^{(m)}(x,y)$.

Proof. Let $p \in \mathcal{P}^{(m)}(x,y)$ and recall from Definition 3.14 that

$$\mathrm{Perm}_{\tau,\rho}(p) := q^y_{\sigma,\tau} \cdot p \cdot (q^x_{\sigma,\tau})^{-1}.$$

Define the linear operators $Q^z_{\sigma,\tau} := T^{(\boldsymbol{n})}_{q^z_{\sigma,\tau}}$ for all $z \in \{\circ, \bullet\}^*$. Then

$$(Q^\circ_{\sigma,\tau})^{\boldsymbol{i}}_{\boldsymbol{j}} = (\delta_{q^\circ_{\sigma,\tau}})^{\boldsymbol{j}}_{\boldsymbol{i}} = \delta_{j_1 i_{\sigma(1)}} \dots \delta_{j_m i_{\sigma(m)}} = (F^{(\boldsymbol{n})}_\sigma)^{\boldsymbol{i}}_{\boldsymbol{j}}$$

for all $\boldsymbol{i} := (i_1, \dots, i_m), \boldsymbol{j} := (j_1, \dots, j_m) \in [\boldsymbol{n}]$. Hence, $Q^\circ_{\sigma,\tau} = F^{(\boldsymbol{n})}_\sigma$ and similarly one shows $Q^\bullet_{\sigma,\tau} = \overline{F}^{(\boldsymbol{n})}_\tau$. Since $F^{(\boldsymbol{n})}_\sigma$ and $\overline{F}^{(\boldsymbol{n})}_\tau$ come from a unitary representation, it follows inductively that $Q^z_{\sigma,\tau}$ defines a unitary representation of $\boldsymbol{n}$-graded permutations in $S_m \times S_m$. Thus, we can write

$$T^{(\boldsymbol{n})}_{\mathrm{Perm}_{\tau,\rho}(p)} = Q^y_{\sigma,\tau} \cdot T^{(\boldsymbol{n})}_p \cdot (Q^x_{\sigma,\tau})^{-1} = (Q^y_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^y_{\sigma,\sigma}) \cdot T^{(\boldsymbol{n})}_p \cdot (Q^x_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^x_{\sigma,\sigma})^{-1}.$$

Hence, $T^{(\boldsymbol{n})}_{\mathrm{Perm}_{\tau,\rho}(p)} \in \mathrm{Hom}(u^x, u^y)$ if and only if

$$(Q^y_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^y_{\sigma,\sigma}) \cdot T^{(\boldsymbol{n})}_p \cdot (Q^x_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^x_{\sigma,\sigma})^{-1} \cdot u^x$$
$$= u^y \cdot (Q^y_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^y_{\sigma,\sigma}) \cdot T^{(\boldsymbol{n})}_p (Q^x_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^x_{\sigma,\sigma})^{-1},$$

which is again equivalent to

$$Q^y_{\sigma,\sigma} \cdot T^{(\boldsymbol{n})}_p \cdot (Q^x_{\sigma,\sigma})^{-1} \cdot (Q^x_{\mathrm{id},\tau\sigma^{-1}})^{-1} \cdot u^x \cdot Q^x_{\mathrm{id},\tau\sigma^{-1}}$$
$$= (Q^y_{\mathrm{id},\tau\sigma^{-1}})^{-1} \cdot u^y \cdot Q^y_{\mathrm{id},\tau\sigma^{-1}} \cdot Q^y_{\sigma,\sigma} \cdot T^{(\boldsymbol{n})}_p \cdot (Q^x_{\sigma,\sigma})^{-1}.$$

Since

$$\left(F^{(\boldsymbol{n})}_\sigma\right)^{\otimes z} = Q^z_{\sigma,\sigma}, \quad \hat{u}^z = (Q^z_{\mathrm{id},\tau\sigma^{-1}})^{-1} u^z Q^z_{\mathrm{id},\tau\sigma^{-1}} \qquad \forall z \in \{\circ, \bullet\}^*,$$

we conclude that

$$T^{(\boldsymbol{n})}_{\mathrm{Perm}_{\sigma,\tau}(p)} \in \mathrm{Hom}(u^x, u^y) \iff \left(F^{(\boldsymbol{n})}_\sigma\right)^{\otimes y} T^{(\boldsymbol{n})}_p \left(\left(F^{(\boldsymbol{n})}_\sigma\right)^{-1}\right)^{\otimes x} \in \mathrm{Hom}(\hat{u}^x, \hat{u}^y).$$

$\square$

Using the previous lemma, we can now show that the spatial partition functor $\mathrm{Perm}_{\sigma,\tau}$ leaves the quantum groups $G_{\boldsymbol{n}}(\mathcal{C})$ invariant.

Theorem 4.16. *Let* $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ *be a* $\boldsymbol{n}$*-graded rigid category of spatial partitions and* $\sigma, \tau \in S_m$ *be* $\boldsymbol{n}$*-graded permutations. Then* $G_{\boldsymbol{n}}(\mathcal{C})$ *and* $G_{\boldsymbol{n}}(\mathrm{Perm}_{\sigma,\tau}(\mathcal{C}))$ *are isomorphic.*

Proof. Define $\mathcal{D} := \mathrm{Perm}_{\sigma,\tau}(\mathcal{C})$ and denote with u°, $u^\bullet$ and w°, $w^\bullet$ the fundamental representation and conjugate representation of $G_{\boldsymbol{n}}(\mathcal{C})$ and $G_{\boldsymbol{n}}(\mathcal{D})$ respectively. Since the functor $\mathrm{Perm}_{\sigma,\tau}$ is fully faithful and does not change colors, we have

$$\mathrm{Hom}(w^x, w^y) = \mathrm{span}\left\{ T_p^{(\boldsymbol{n})} \mid p \in \mathcal{D}(x,y) \right\} = \mathrm{span}\left\{ T_{\mathrm{Perm}_{\sigma,\tau}(p)}^{(\boldsymbol{n})} \mid p \in \mathcal{C}(x,y) \right\}$$

for all $x, y \in \{\circ, \bullet\}^*$. Define $\hat{w}^\circ := w^\circ$ and $\hat{w}^\bullet := \left(\overline{F}_{\tau\sigma^{-1}}^{(\boldsymbol{n})}\right)^{-1} w^\bullet \overline{F}_{\tau\sigma^{-1}}^{(\boldsymbol{n})}$. Then Lemma 4.15 yields

$$\mathrm{span}\left\{ \left(F_\sigma^{(\boldsymbol{n})}\right)^{\otimes y} T_p^{(\boldsymbol{n})} \left(\left(F_\sigma^{(\boldsymbol{n})}\right)^{-1}\right)^{\otimes x} \mid p \in \mathcal{C}(x,y) \right\} \subseteq \mathrm{Hom}(\hat{w}^x, \hat{w}^y),$$

which is equivalent to

$$\mathrm{Hom}(u^x, u^y) \subseteq \left(\left(F_\sigma^{(\boldsymbol{n})}\right)^{-1}\right)^{\otimes y} \cdot \mathrm{Hom}(\hat{w}^x, \hat{w}^y) \cdot \left(F_\sigma^{(\boldsymbol{n})}\right)^{\otimes x}.$$

Thus, the universal property of $G_{\boldsymbol{n}}(\mathcal{C})$ and $\hat{w}^\circ = w^\circ$ yield the inclusion

$$G_{\boldsymbol{n}}(\mathcal{D}) \subseteq G_{\boldsymbol{n}}(\mathcal{C}), \quad u^\circ \mapsto \left(F_\sigma^{(\boldsymbol{n})}\right)^{-1} w^\circ F_\sigma^{(\boldsymbol{n})}.$$

Since $\mathcal{C} = \mathrm{Perm}_{\sigma^{-1},\tau^{-1}}(\mathcal{D})$, the same argument also yields the inverse inclusion

$$G_{\boldsymbol{n}}(\mathcal{C}) \subseteq G_{\boldsymbol{n}}(\mathcal{D}), \quad w^\circ \mapsto \left(F_{\sigma^{-1}}^{(\boldsymbol{n})}\right)^{-1} u^\circ F_{\sigma^{-1}}^{(\boldsymbol{n})}.$$

Therefore, $G_{\boldsymbol{n}}(\mathcal{C})$ and $G_{\boldsymbol{n}}(\mathcal{D})$ are isomorphic. $\qquad\square$

The previous theorem might be generally useful for determining the quantum groups associated with concrete categories of spatial partitions. However, as an immediate consequence, we obtain that the class of spatial partition quantum groups defined with our new duality partitions agrees with the class of spatial partition quantum groups defined by Cébron-Weber in [10].

Corollary 4.17. *Let G be a spatial partition quantum group. Then G is equivalent to $G_{\boldsymbol{n}}(\mathcal{C})$ for a $\boldsymbol{n}$-graded rigid category of spatial partitions $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ containing the duality partitions $\sqcap_\bullet^{(m)}$ and $\sqcap_\bullet^{(m)}$.*

Proof. As discussed in Remark 4.8, any spatial partition quantum group G is equivalent to $G_{\boldsymbol{n}}(\mathcal{D})$ for a $\boldsymbol{n}$-graded rigid category of spatial partitions $\mathcal{D} \subseteq \mathcal{P}^{(m)}$ containing some duality partitions $\sigma_{\circ\bullet}$ and $\sigma_{\bullet\circ}^{-1}$. By the previous theorem, this quantum group is again equivalent to $G_{\boldsymbol{n}}(\mathcal{C})$, where $\mathcal{C} := \mathrm{Perm}_{\mathrm{id},\sigma^{-1}}(\mathcal{D})$ contains the duality partitions

$$\mathrm{Perm}_{\mathrm{id},\sigma^{-1}}(\sigma_{\circ\bullet}) = \sqcap_\bullet^{(m)}, \qquad \mathrm{Perm}_{\mathrm{id},\sigma^{-1}}(\sigma_{\bullet\circ}^{-1}) = \sqcap_\bullet^{(m)}.$$

$\qquad\square$

In Section 4, we have shown that for every spatial partition quantum group, the representation $\overline{u}$ is unitary and conjugate to u. However, it remained open if we can choose $u^\bullet = \overline{u}$ and obtain again a spatial partition quantum group. The previous corollary now answers this positively. Since $\sqcap_\bullet^{(m)} = (\mathrm{id}_{S_m})_{\circ\bullet}$, the previous corollary

and Proposition 4.10 show that any spatial partition quantum group is equivalent to a spatial partition quantum group with

$$u^\bullet = \overline{F}^{(\boldsymbol{n})}_{\mathrm{id}_{S_m}} \cdot \overline{u} \cdot \left(\overline{F}^{(\boldsymbol{n})}_{\mathrm{id}_{S_m}}\right)^{-1} = \overline{u}.$$

Thus, we can always choose $u^\bullet = \overline{u}$ without loss of generality in the setting of spatial partition quantum groups.

5. Projective spatial partition quantum groups

In the following, we use the functor $\mathrm{Flat}_{m,z}$ from Section 3.3.2 to show that the class of spatial partition quantum groups is closed under taking projective versions or more generally taking tensor powers of u° and $u^\bullet$. We then use this result to compute the spatial partition quantum groups corresponding to the categories $P_2^{(m)}$ of spatial pair partition on m levels. Further, we show that all projective versions of easy quantum groups are again spatial partition quantum groups. A result of Gromada [20] then allows us to describe these projective versions explicitly in terms of generators and relations, if the underlying quantum group has a degree of reflection two.

5.1. **Tensor powers of spatial partition quantum groups.** If w is a unitary representation of a compact matrix quantum group, then its matrix coefficients can be used to define a new compact matrix quantum group. Now, assume u is the fundamental representation of a quantum group G and $\overline{u}$ is also unitary. Then $w := u \textcircled{\tiny{T}} \overline{u}$ yields the projective version of G denoted by PG. If G is a classical matrix group, then PG corresponds exactly to the classical projective version

$$PG := G/(G \cap \{\lambda I \mid \lambda \in \mathbb{C}\}).$$

For more information on the projective versions of compact matrix quantum groups, see for example [7, 20, 3].

More generally, we can use any powers of the fundamental representation $u =: u^\circ$ and a unitary representation $u^\bullet$ to construct new quantum groups.

Definition 5.1. Let $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$ and G be a compact matrix quantum group with fundamental representation u° and unitary representation $u^\bullet$. Then define $G^z := (A, w)$, where $A \subseteq C(G)$ is the C^*-algebra generated by the matrix coefficients of $w := u^z$.

Note that if $u^\bullet = \overline{u^\circ}$, then the projective version PG is exactly given by $G^{\circ \bullet}$. Further, we have shown in the previous sections that $\overline{u^\circ}$ is unitary for all spatial partition quantum groups and that we can choose $u^\bullet = \overline{u^\circ}$ without loss of generality. Thus, the projective version of every spatial partition quantum group G is well-defined and given by $G^{\circ \bullet}$.

In the following, we show that if G is a spatial partition quantum group with corresponding category $\mathcal{C} \subseteq \mathcal{P}^{(m)}$, then G^z is again a spatial partition quantum group for all $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$. Moreover, the corresponding category of G^z

is given by $\mathrm{Flat}_{m,z}^{-1}(\mathcal{C})$, where $\mathrm{Flat}_{m,z}$ is the functor from Section 3.3.2. Here, the fact that $\mathrm{Flat}_{m,z}^{-1}(\mathcal{C})$ is again a rigid category of spatial partition relies on our new duality partition, see Proposition 3.23.

Since the functor $\mathrm{Flat}_{m,z}^{-1}$ does not preserve colors, our first step is to show that the representations w^x and $u^{\mathrm{Flat}_{m,z}(x)}$ are equivalent for all $x \in \{\circ, \bullet\}^*$, where $w^\circ := u^z$ and $w^\bullet := \overline{u^z}$.

Lemma 5.2. *Let $z \in \{\circ, \bullet\}^*$ with $d := |z| \geq 1$ and consider a compact matrix quantum group with fundamental representation u° on V and unitary representation $u^\bullet$ on $\overline{V}$. Define $w^\circ := u^z$, $w^\bullet := \overline{u^z}$ and the unitaries*

$$Q_\circ \colon V^{\otimes z} \to V^{\otimes z}, \quad v_1 \otimes \cdots \otimes v_d \mapsto v_1 \otimes \cdots \otimes v_d \quad \forall v_1, \ldots, v_d \in V,$$

$$Q_\bullet \colon \overline{V^{\otimes z}} \to V^{\otimes \overline{z}}, \quad \overline{v_1 \otimes \cdots \otimes v_d} \mapsto \overline{v_d} \otimes \cdots \otimes \overline{v_1} \quad \forall v_1, \ldots, v_d \in V.$$

Then

$$Q_x := \bigotimes_{i=1}^{|x|} Q_{x_i} \in \mathrm{Hom}\left(w^x, u^{\mathrm{Flat}_{m,z}(x)}\right)$$

for all $x \in \{\circ, \bullet\}^*$ and $m \in \mathbb{N}$.

Proof. First, consider the case $x = \circ$. Then $w^\circ = u^z = u^{\mathrm{Flat}_{m,z}(\circ)}$ and $Q_\circ = \mathrm{id}_V^{\otimes z} \in \mathrm{Hom}(u^z, u^z)$. Next, consider the case $x = \bullet$ and choose an orthonormal basis $(v_i)_{i \in I}$ of V inducing canonical bases for $\overline{V^{\otimes z}}$ and $V^{\otimes \overline{z}}$ indexed by $i_1, \ldots, i_d \in I$. With respects to these bases, the matrix entries of $w^\bullet = \overline{u^z}$ are given by

$$(\overline{u^z})_{j_1,\ldots,j_d}^{i_1,\ldots,i_d} = \left((u^z)_{j_1,\ldots,j_d}^{i_1,\ldots,i_d}\right)^* = \left((u^{z_1})_{j_1}^{i_1} \cdots (u^{z_d})_{j_d}^{i_d}\right)^* = (u^{\overline{z_d}})_{j_d}^{i_d} \cdots (u^{\overline{z_1}})_{j_1}^{i_1}$$

for all $i_1, \ldots, i_d, j_1, \ldots, j_d \in I$. On the other hand, we have $u^{\mathrm{Flat}_{m,z}(\bullet)} = u^{\overline{z}}$ and

$$(u^{\overline{z}})_{j_1,\ldots,j_d}^{i_1,\ldots,i_d} = (u^{\overline{z_d}})_{j_1}^{i_1} \cdots (u^{\overline{z_d}})_{j_d}^{i_d}.$$

Thus, performing a change of basis using $Q_\bullet$ yields $\overline{u^z} = Q_\bullet^{-1} \cdot u^{\overline{z}} \cdot Q_\bullet$, which is equivalent to $Q_\bullet \in \mathrm{Hom}(w^\bullet, u^{\mathrm{Flat}_{m,z}(\bullet)})$.

Finally, consider an arbitrary $x \in \{\circ, \bullet\}^*$. The previous computations show that $Q_{x_i} \in \mathrm{Hom}(w^{x_i}, u^{\mathrm{Flat}_{m,z}(x_i)})$ for all $1 \leq i \leq |x|$. Since intertwiner spaces are closed under tensor products and $\mathrm{Flat}_{m,z}$ is a functor, this implies

$$Q_x := \bigotimes_{i=1}^{|x|} Q_{x_i} \in \mathrm{Hom}\left(\oplus_{i=1}^{|x|} w^{x_i}, \oplus_{i=1}^{|x|} u^{\mathrm{Flat}_{m,z}(x_i)}\right) = \mathrm{Hom}\left(w^x, u^{\mathrm{Flat}_{m,z}(x)}\right).$$

$\square$

Remark 5.3. As before, let $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$ and G be a compact matrix quantum group with fundamental representation u° and unitary representation $u^\bullet$. Since both $u^{\overline{z}}$ and $Q_\bullet$ are unitary, the previous lemma also shows that

$$w^\bullet := \overline{u^z} = Q_\bullet^{-1} \cdot u^{\overline{z}} \cdot Q_\bullet.$$

is unitary. Therefore, Proposition 2.8 implies that $w^\bullet = \overline{u^z}$ is conjugate to the fundamental representation $w^\circ = u^z$ of G^z.

Remark 5.4. Let u and v be unitary representations of a compact matrix quantum group G and assume there exists a unitary $Q \in \mathrm{Hom}(u, v)$. Then $u = Q^{-1}vQ$ and we can write

$$\mathrm{Hom}(u, w) = \mathrm{Hom}(v, w) \cdot Q, \qquad \mathrm{Hom}(w, u) = Q^{-1} \cdot \mathrm{Hom}(w, v)$$

for any unitary representation w of G.

Next, we show that for any spatial partition p the linear maps $T^{(\boldsymbol{n})}_{\mathrm{Flat}_{m,z}(p)}$ and $T^{(\boldsymbol{n}\cdots\boldsymbol{n})}_p$ agree up to a change of basis. Here, $\boldsymbol{n}\cdots\boldsymbol{n}$ denotes the d-fold repetition of $\boldsymbol{n}$ as in Proposition 3.23.

Lemma 5.5. *Let* $\boldsymbol{n} \in \mathbb{N}^m$ *and* $z \in \{\circ, \bullet\}^*$ *with* $d := |z| \geq 1$. *Then*

$$Q_y^{-1} \cdot T^{(\boldsymbol{n})}_{\mathrm{Flat}_{m,z}(p)} \cdot Q_x = (S^{-1})^{\otimes y} \cdot T^{(\boldsymbol{n}\cdots\boldsymbol{n})}_p \cdot S^{\otimes x}$$

for all $p \in \mathcal{P}^{(m \cdot d)}(x, y)$, *where* Q_x, Q_y *are defined in Lemma 5.2 for* $V = \mathbb{C}^n$ *and*

$$S\colon (\mathbb{C}^{\boldsymbol{n}})^{\otimes z} \to \mathbb{C}^{\boldsymbol{n}\cdots\boldsymbol{n}}, \quad e^{z_1}_{\boldsymbol{i}_1} \otimes \cdots \otimes e^{z_d}_{\boldsymbol{j}_d} \mapsto e_{\boldsymbol{i}_1} \otimes \cdots \otimes e_{\boldsymbol{i}_d} \quad \forall \boldsymbol{i}_1, \ldots, \boldsymbol{i}_d \in [\boldsymbol{n}].$$

Proof. Let $p \in \mathcal{P}^{(m \cdot d)}(x, y)$. Define $k := |x|$, $\ell := |y|$ and let $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_\ell, \boldsymbol{j}_1, \ldots, \boldsymbol{j}_k \in [\boldsymbol{n}\cdots\boldsymbol{n}]$ be of the form

$$\boldsymbol{i}_\alpha := (i_{\alpha,1,1}, \ldots, i_{\alpha,1,m}, \ldots, i_{\alpha,d,1}, \ldots, i_{\alpha,d,m}),$$

$$\boldsymbol{j}_\alpha := (j_{\alpha,1,1}, \ldots, j_{\alpha,1,m}, \ldots, j_{\alpha,d,1}, \ldots, j_{\alpha,d,m}).$$

For $1 \leq \beta \leq d$, define the slices

$$\boldsymbol{i}^{(\beta)}_\alpha := \begin{cases} (i_{\alpha,\beta,1}, \ldots, i_{\alpha,\beta,m}) & \text{if } x_\alpha = \circ, \\ (i_{\alpha,\beta,m}, \ldots, i_{\alpha,\beta,1}) & \text{if } x_\alpha = \bullet, \end{cases} \quad \boldsymbol{j}^{(\beta)}_\alpha := \begin{cases} (j_{\alpha,\beta,1}, \ldots, j_{\alpha,\beta,m}) & \text{if } y_\alpha = \circ, \\ (j_{\alpha,\beta,m}, \ldots, j_{\alpha,\beta,1}) & \text{if } y_\alpha = \bullet. \end{cases}$$

Then by the definition of S, we have

$$\left((S^{-1})^{\otimes y} \cdot T^{(\boldsymbol{n}\cdots\boldsymbol{n})}_p \cdot S^{\otimes x}\right)^{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell}_{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k} = \left(T^{(\boldsymbol{n}\cdots\boldsymbol{n})}_p\right)^{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell}_{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k} = (\delta_p)^{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k}_{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell}.$$

On the other hand, the definition of Q_x and Q_y yields that

$$\left(Q_y^{-1} \cdot T^{(\boldsymbol{n})}_{\mathrm{Flat}_{m,z}(p)} \cdot Q_x\right)^{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell}_{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k} = \left(T^{(\boldsymbol{n})}_{\mathrm{Flat}_{m,z}(p)}\right)^{\boldsymbol{i}^{(1)}_1,\ldots,\boldsymbol{i}^{(d)}_1,\ldots,\boldsymbol{i}^{(1)}_\ell,\ldots,\boldsymbol{i}^{(d)}_\ell}_{\boldsymbol{j}^{(1)}_1,\ldots,\boldsymbol{j}^{(d)}_1,\ldots,\boldsymbol{j}^{(1)}_k,\ldots,\boldsymbol{j}^{(d)}_k}$$

which is given by

$$\left(T^{(\boldsymbol{n})}_{\mathrm{Flat}_{m,z}(p)}\right)^{\boldsymbol{i}^{(1)}_1,\ldots,\boldsymbol{i}^{(d)}_1,\ldots,\boldsymbol{i}^{(1)}_\ell,\ldots,\boldsymbol{i}^{(d)}_\ell}_{\boldsymbol{j}^{(1)}_1,\ldots,\boldsymbol{j}^{(d)}_1,\ldots,\boldsymbol{j}^{(1)}_k,\ldots,\boldsymbol{j}^{(d)}_k} = (\delta_{\mathrm{Flat}_{m,z}(p)})^{\boldsymbol{j}^{(1)}_1,\ldots,\boldsymbol{j}^{(d)}_1,\ldots,\boldsymbol{j}^{(1)}_k,\ldots,\boldsymbol{j}^{(d)}_k}_{\boldsymbol{i}^{(1)}_1,\ldots,\boldsymbol{i}^{(d)}_1,\ldots,\boldsymbol{i}^{(1)}_\ell,\ldots,\boldsymbol{i}^{(d)}_\ell}.$$

The statement of the lemma now follows, since

$$(\delta_p)^{\boldsymbol{j}_1,\ldots,\boldsymbol{j}_k}_{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_\ell} = (\delta_{\mathrm{Flat}_{m,z}(p)})^{\boldsymbol{j}^{(1)}_1,\ldots,\boldsymbol{j}^{(d)}_1,\ldots,\boldsymbol{j}^{(1)}_k,\ldots,\boldsymbol{j}^{(d)}_k}_{\boldsymbol{i}^{(1)}_1,\ldots,\boldsymbol{i}^{(d)}_1,\ldots,\boldsymbol{i}^{(1)}_\ell,\ldots,\boldsymbol{i}^{(d)}_\ell}$$

by the definition of $\mathrm{Flat}_{m,z}$. $\qquad\square$

Using the previous two lemmas, we can now prove our main theorem and show that quantum groups of the form $G_n(\mathcal{C})^z$ are again spatial partition quantum groups.

Theorem 5.6. *Let $\mathcal{C} \subseteq \mathcal{P}^{(m)}$ be a $\boldsymbol{n}$-graded rigid category of spatial partitions and $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$. Then $G_{\boldsymbol{n}}(\mathcal{C})^z$ is equivalent to $G_{\boldsymbol{n}\cdots\boldsymbol{n}}(\mathcal{D})$ with $\mathcal{D} := \mathrm{Flat}_{m,z}^{-1}(\mathcal{C})$.*

Proof. Denote with u°, $u^\bullet$ and $\hat{w}^\circ$, $\hat{w}^\bullet$ the fundamental representation and conjugate representation of $G_{\boldsymbol{n}}(\mathcal{C})$ and $G_{\boldsymbol{n}\cdots\boldsymbol{n}}(\mathcal{D})$ respectively. Further, denote with $w^\circ := (u^\circ)^z$ the fundamental representation of G^z and with $w^\bullet := \overline{(u^\circ)^z}$ its conjugate representation, see Remark 5.3. Then Lemma 5.2 in combination with Remark 5.4 and fact that $\mathrm{Flat}_{m,z}$ is fully faithful imply that

$$\begin{aligned}
\mathrm{Hom}(w^x, w^y) &= Q_y^{-1} \cdot \mathrm{Hom}(u^{\mathrm{Flat}_{m,z}(x)}, u^{\mathrm{Flat}_{m,z}(y)}) \cdot Q_x \\
&= \mathrm{span}\left\{ Q_y^{-1} \cdot T_p^{(\boldsymbol{n})} \cdot Q_x \mid p \in \mathcal{C}(\mathrm{Flat}_{m,z}(x), \mathrm{Flat}_{m,z}(y)) \right\} \\
&= \mathrm{span}\left\{ Q_y^{-1} \cdot T_{\mathrm{Flat}_{m,z}(p)}^{(\boldsymbol{n})} \cdot Q_x \mid p \in \mathcal{D} \right\}
\end{aligned}$$

for all $x, y \in \{\circ, \bullet\}^*$. By Lemma 5.5, we have

$$Q_y^{-1} \cdot T_{\mathrm{Flat}_{m,z}(p)}^{(\boldsymbol{n})} \cdot Q_x = (S^{-1})^{\otimes y} \cdot T_p^{(\boldsymbol{n}\cdots\boldsymbol{n})} \cdot S^{\otimes x}$$

for a unitary $S \colon (\mathbb{C}^{\boldsymbol{n}})^{\otimes z} \to \mathbb{C}^{\boldsymbol{n}\cdots\boldsymbol{n}}$, which yields

$$\begin{aligned}
\mathrm{Hom}(w^x, w^y) &= \mathrm{span}\left\{ (S^{-1})^{\otimes y} \cdot T_p^{(\boldsymbol{n}\cdots\boldsymbol{n})} \cdot S^{\otimes x} \mid p \in \mathcal{D}(x, y) \right\} \\
&= (S^{-1})^{\otimes y} \cdot \mathrm{Hom}(\hat{w}^x, \hat{w}^y) \cdot S^{\otimes x}.
\end{aligned}$$

Therefore, $G_{\boldsymbol{n}}(\mathcal{C})^z$ and $G_{\boldsymbol{n}\cdots\boldsymbol{n}}(\mathcal{D})$ are equivalent. $\square$

Since every spatial partition quantum group is equivalent to a quantum group of the form $G_{\boldsymbol{n}}(\mathcal{C})$, we can use the following proposition to transfer the previous theorem to all spatial partition quantum groups.

Proposition 5.7. *Let G and H be compact matrix quantum groups with fundamental representations u°, $\hat{u}^\circ$ and conjugate representations $u^\bullet$, $\hat{u}^\bullet$ respectively. If G and H are equivalent, then G^z and H^z are equivalent for all $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$.*

Proof. Since G and H are equivalent, there exists a unitary S such that

$$\mathrm{Hom}(u^x, u^y) = S^{\otimes y} \cdot \mathrm{Hom}(\hat{u}^x, \hat{u}^y) \cdot (S^{-1})^{\otimes x} \qquad \forall x, y \in \{\circ, \bullet\}^*.$$

Denote with $w^\circ := (u^\circ)^z$ and $\hat{w}^\circ := (\hat{u}^\circ)^z$ the fundamental representations of G^z and H^z. Then Remark 5.3 shows that $w^\bullet := \overline{(u^\circ)^z}$ and $\hat{w}^\bullet := \overline{(\hat{u}^\circ)^z}$ are the corresponding conjugate representations. Let $x, y \in \{\circ, \bullet\}^*$ and define $\widetilde{x} := \mathrm{Flat}_{1,z}(x)$ and $\widetilde{y} :=$

$\mathrm{Flat}_{1,z}(y)$. Then Lemma 5.2 and Remark 5.4 show that

$$\begin{aligned}
\mathrm{Hom}(w^x, w^y) &= Q_y^{-1} \cdot \mathrm{Hom}(u^{\widetilde{x}}, u^{\widetilde{y}}) \cdot Q_x \\
&= Q_y^{-1} \cdot S^{\otimes \widetilde{y}} \cdot \mathrm{Hom}(\hat{u}^{\widetilde{x}}, \hat{u}^{\widetilde{y}}) \cdot (S^{-1})^{\otimes \widetilde{x}} \cdot Q_x \\
&= Q_y^{-1} \cdot S^{\otimes \widetilde{y}} \cdot Q_{\widetilde{y}} \cdot \mathrm{Hom}(\hat{w}^x, \hat{w}^y) \cdot Q_{\widetilde{x}}^{-1} \cdot (S^{-1})^{\otimes \widetilde{x}} \cdot Q_x.
\end{aligned}$$

Thus, G^z and H^z are also equivalent. $\qquad\square$

Corollary 5.8. *Let G be a spatial partition quantum group and $z \in \{\circ, \bullet\}^*$ with $|z| \geq 1$. Then G^z is a spatial partition quantum group.*

Proof. Since G is a spatial partition quantum group, it is equivalent to $G_{\boldsymbol{n}}(\mathcal{C})$ for a $\boldsymbol{n}$-graded rigid category of spatial partitions $\mathcal{C} \subseteq \mathcal{P}^{(m)}$. Then the previous proposition shows that G^z is equivalent to $G_{\boldsymbol{n}}(\mathcal{C})^z$, which is a spatial partition quantum group by Theorem 5.6. $\qquad\square$

As a special case, it follows that the class of spatial partition quantum groups is closed under taking projective versions.

Corollary 5.9. *Let G be a spatial partition quantum group. Then PG is a spatial partition quantum group.*

Proof. By Corollary 4.17 and the discussion at the end of Section 4.3, we can assume that the conjugate representation of G is given by $u^\bullet = \overline{u^\circ}$. In this case, $PG = G^{\circ\bullet}$, which a spatial partition quantum group by Corollary 5.8. $\qquad\square$

5.2. **Categories of all spatial pair partitions.** As a first application of the previous theorem, we consider the quantum groups associated with the categories $\mathcal{P}_2^{(m)}$ of all spatial pair partitions on m levels, i.e. spatial partition on m levels with every block of size two.

It is shown in [6] that the category $\mathcal{P}_2^{(1)}$ of pair partitions on one level corresponds to the classical orthogonal group O_n. Further, the author shows in [14] that the category $\mathcal{P}_2^{(2)}$ of spatial pair partition on two levels corresponds to the classical projective orthogonal group PO_n. In the following, we determine quantum groups in the remaining cases $m \geq 3$ and show that $G_{(n,\ldots,n)}(P_2^{(m)})$ is equivalent to $O_n^{\circ\cdots\circ}$.

Note that in contrast to our definition, the categories in [6, 14] are defined in terms of colorless partitions. However, in our setting, $\mathcal{P}_2^{(m)}$ contains the spatial partition $\mathring{|}^{(m)}$ that corresponds to the C^*-algebraic relations $u_j^i = (u_j^i)^*$ making the generators self-adjoint. Thus, our quantum groups $G_{(n,\ldots,n)}(\mathcal{P}_2^{(m)})$ agree with the corresponding orthogonal quantum groups in [6, 14]. See also [30] for more on the relation between orthogonal and unitary easy quantum groups.

Proposition 5.10. *Let $m \geq 1$ and $n \in \mathbb{N}$. Then $G_{(n,\ldots,n)}(\mathcal{P}_2^{(m)})$ is equivalent to O_n^z with $z := \circ^m$.*

Proof. Since the functor $\mathrm{Flat}_{1,z}$ does not change the size of blocks, pair partitions are mapped to pair partitions and we have

$$\mathrm{Flat}_{1,z}^{-1}(\mathcal{P}_2^{(1)}) = \mathcal{P}_2^{(m)}.$$

Since $G_n(\mathcal{P}_2^{(1)}) = O_n$, Theorem 5.6 implies that $G_{(n,\dots,n)}(\mathcal{P}_2^{(m)})$ is equivalent to O_n^z. $\square$

By relaxing our notion of isomorphism of compact matrix quantum groups, we can give a more explicit description of the resulting quantum groups $O_n^{\circ\cdots\circ}$.

Proposition 5.11. *Let $n \in \mathbb{N}$ and $z := \circ^m$ with $m \geq 1$. Then*

$$O_n^z = \begin{cases} O_n & \text{if } m \text{ is odd,} \\ PO_n & \text{if } m \text{ is even,} \end{cases}$$

as compact quantum groups.

Proof. Denote with u the fundamental representation of O_n. First, we assume that m is odd, i.e. $m = 2k + 1$. Then, consider the inclusion $C(O_n^z) \hookrightarrow C(O_n)$, which is injective and respects the comultiplication. Since u is orthogonal, we compute

$$\sum_{j_1,\dots,j_k \in [n]} u_{j_0}^{i_0} u_{j_1}^1 u_{j_1}^1 \dots u_{j_k}^1 u_{j_k}^1 = u_{j_0}^{i_0} \underbrace{\left(\sum_{j_1=1}^{n} u_{j_1}^1 u_{j_1}^1\right)}_{=1} \dots \underbrace{\left(\sum_{j_k=1}^{n} u_{j_k}^1 u_{j_k}^1\right)}_{=1} = u_{j_0}^{i_0} \in C(O_n^z)$$

for all $i_0, j_0 \in [n]$. Thus, the inclusion is also surjective and defines an isomorphism of compact quantum groups.

Next, we assume that m is even, i.e. $m = 2k$. Then, we have an inclusion $C(O_n^z) \hookrightarrow C(PO_n)$ and compute

$$\sum_{j_2,\dots,j_k \in [n]} u_{j_0}^{i_0} u_{j_1}^{i_1} u_{j_2}^1 u_{j_2}^1 \dots u_{j_k}^1 u_{j_k}^1 = u_{j_0}^{i_0} u_{j_1}^{i_1} \underbrace{\left(\sum_{j_2=1}^{n} u_{j_2}^1 u_{j_2}^1\right)}_{=1} \dots \underbrace{\left(\sum_{j_k=1}^{n} u_{j_k}^1 u_{j_k}^1\right)}_{=1} = u_{j_0}^{i_0} u_{j_1}^{i_1}$$

for all $i_0, j_0, i_1, j_1 \in [n]$. Thus, $u_{j_0}^{i_0} u_{j_1}^{i_1} \in C(O_n^z)$, which shows that the inclusion is surjective and defines an isomorphism of compact quantum groups. $\square$

5.3. **Projective easy quantum groups.** Consider an easy quantum group G. Then its projective version PG might not be an easy quantum group in general. However, since easy quantum groups are a subclass of spatial partition quantum groups, it follows from Theorem 5.6 that projective versions of easy quantum groups are again spatial partition quantum groups. Thus, it is always possible to describe projective versions of easy quantum groups in terms of spatial partitions.

If G corresponds to a category of partitions $\mathcal{C}$, then the proof of Theorem 5.6 shows the category of its projective version PG is given by $\mathrm{Flat}_{1,\circ\bullet}^{-1}(\mathcal{C})$. This is particularly useful if the category $\mathcal{C}$ can be characterized via an abstract property as in the case of spatial pair partitions in the previous section. On the other hand,

if the category $\mathcal{C}$ is only given by a set of generators, then it might be difficult in general to obtain a generating set for the category of its projective version.

However, in the special case when G is an orthogonal quantum group with degree of reflection two, Gromada [20] provides a result that allows us to convert generators of $\mathcal{C}$ to generators of its projective version.

Proposition 5.12 ([20]). *Consider an orthogonal compact matrix quantum group* $G \subseteq O^+(F)$ *with fundamental representation on* $\mathbb{C}^n$ *and degree of reflection two. Consider an admissible generating set* S *of* $\mathrm{Rep}(G)$ *containing the duality morphisms. Then* $\mathrm{Rep}(PG)$ *is generated by* S *and* $\mathrm{id}_{\mathbb{C}^n} \otimes S \otimes \mathrm{id}_{\mathbb{C}^n}$.

Here, an orthogonal compact matrix group with fundamental representation u has *degree of reflection two*, if

$$\mathrm{Hom}(u^{\oplus k}, u^{\oplus \ell}) = \{0\} \quad \forall k, \ell \in \mathbb{N}, \ k + \ell \text{ even.}$$

Further, a linear map $T \in S$ is called *admissible*, if T is of the form $T \colon (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes \ell}$ with k and ℓ even. Note that we can always use the duality morphisms to bring any intertwiner of a degree of reflection two quantum group into admissible form. See [20] or [27] for further details.

Next, we reformulate Proposition 5.12 in the setting of spatial partition quantum groups, before we apply it to concrete examples of easy quantum groups.

Proposition 5.13. *Let* $n \in \mathbb{N}$ *and* $\mathcal{C} \subseteq \mathcal{P}^{(1)}$ *be a category of spatial partitions generated by* $\big|_{\circ}^{\bullet}$ *and a set* $\mathcal{C}_0 \subseteq \mathcal{P}^{(1)}$ *such that*

    (1) $\mathcal{C}_0$ *contains the partition* $\sqcap$,

    (2) every $p \in \mathcal{C}_0$ *has only white points,*

    (3) if $p \in \mathcal{C}_0$ *has upper colors* x *and lower colors* y*, then* $|x|$ *and* $|y|$ *are even.*

Then $PG_n(\mathcal{C})$ *is equivalent to* $G_{(n,n)}(\mathcal{D})$*, where* $\mathcal{D} \subseteq \mathcal{P}^{(2)}$ *is generated by* $\big|_{\circ}^{\bullet(2)}$*,* $\mathrm{Flat}_{1,\circ\bullet}^{-1}(\mathcal{C}_0)$ *and* $\mathrm{Flat}_{1,\circ\bullet}^{-1}(\mathrm{id}_\circ \otimes \mathcal{C}_0 \otimes \mathrm{id}_\circ)$.

Proof. Since $\mathcal{C}$ contains the partitions $\big|_{\circ}^{\bullet}$ and $\sqcap$, it follows that $\mathcal{C}$ also contains the partitions $\sqcup$ and $\sqcap$. Thus, Proposition 4.14 implies that $G_n(\mathcal{C}) \subseteq O_n^+$ is an orthogonal quantum group and we have $\overline{u} = u$ after identify $\mathbb{C}^n$ with $\overline{\mathbb{C}^n}$. Moreover, it follows from Definition 4.11 that the intertwiners $S := \{T_p^{(n)} \mid p \in \mathcal{C}_0\}$ generate the category $\mathrm{Rep}(G)$ in the sense of [20]. Therefore, we can apply Proposition 5.12 and obtain that the representation category $\mathrm{Rep}(PG)$ is generated S and $\mathrm{id}_{\mathbb{C}^n} \otimes S \otimes \mathrm{id}_{\mathbb{C}^n}$ in the sense of [20]. The proof Theorem 5.6 and in particular Lemma 5.5 now show that we can use the functor $\mathrm{Flat}_{1,\circ\bullet}$ translate these new generators back to spatial partitions. Further, we have to add again the spatial partition $\big|_{\circ}^{\bullet(2)}$ to obtain an orthogonal quantum group. Thus, $PG_n(\mathcal{C})$ is equivalent to $G_{(n,n)}(\mathcal{D})$ where $\mathcal{D}$ is generated by $\big|_{\circ}^{\bullet(2)}$, $\mathrm{Flat}_{1,\circ\bullet}^{-1}(\mathcal{C}_0)$ and $\mathrm{Flat}_{1,\circ\bullet}^{-1}(\mathrm{id}_\circ \otimes \mathcal{C}_0 \otimes \mathrm{id}_\circ)$. $\qquad\square$

Now, we can apply the previous proposition to easy quantum group with degree of reflection two. Figure 1 presents the result for easy quantum groups described

easy quantum group	generators $\mathcal{C}_0$	projective generators $\mathcal{D}_0$
O_n		
O_n^*		
O_n^+		
H_n		
H_n^*		
H_n^+		
S_n'		
$S_n'^+$		
B_n'		
$B_n'^+$		
$B_n^{\#*}$		
$B_n^{\#+}$		

FIGURE 1. Generators for categories of easy quantum groups and their projective version. The spatial partitions $\overset{\bullet}{\underset{\circ}{|}}$ and $\overset{\bullet(2)}{|}$ are omitted.

in [29], see also the appendix of [34]. It contains the generators for the corresponding categories of spatial partitions as well as their projective versions. Note that the spatial partitions $\overset{\bullet}{\underset{\circ}{|}}$ and $\overset{\bullet(2)}{|}$ are included implicitly.

Figure 1 does not include the quantum groups S_n, B_n, S_n^+ and B_n^+ since these do not have a degree of reflection two. However, their projective versions depend only on the even part of their respective categories of spatial partitions, which are exactly the categories of S_n', B_n', $S_n'^+$ and $B_n'^+$.

We can now use Figure 1 and Definition 4.11 to describe the C^*-algebras $A := C(G_{(n,n)}(\mathcal{D}))$ of the previous projective quantum groups as universal unital C^*-algebras generated by a finite set of relations. Since the fundamental representation of a projective easy quantum group is defined on $\mathbb{C}^n \otimes \overline{\mathbb{C}^n}$, it follows that A is generated by the coefficients of the matrix $u := \left(u_{j_1j_2}^{i_1i_2}\right)$ with $i_1, i_2, j_1, j_2 \in [n]$. Further, one checks that the category $\mathcal{D}$ always contains $\overset{\curvearrowright}{\curvearrowright} = (12)_{\circ\circ}$ such that the duality partitions of $\mathcal{D}$ are given by $(12)_{\circ\bullet}$, $(12)_{\bullet\circ}$ Thus, we have $u^\bullet := \overline{F}_{(12)}\overline{u}\,\overline{F}_{(12)}^{-1}$, which can be written using matrix coefficients as $(u^\bullet)_{j_1j_2}^{i_1i_2} = \left(u_{j_2j_1}^{i_2i_1}\right)^*$. The C^*-algebra A then satisfies the relations

$$uu^* = u^*u = 1, \qquad u^\bullet(u^\bullet)^* = (u^\bullet)^*u^\bullet = 1$$

making both u and $u^\bullet$ unitary. Further, the intertwiner $T^{(n,n)}_{\uparrow^{(2)}}$ yields the relation $T^{(n,n)}_{\uparrow^{(2)}} u^\bullet = u T^{(n,n)}_{\uparrow^{(2)}}$ or equivalently $\left(u^{i_2 i_1}_{j_2 j_1}\right)^* = u^{i_1 i_2}_{j_1 j_2}$. Finally, the C^*-algebra A satisfies the relations $T^{(n,n)}_p u^{\oplus k} = u^{\oplus \ell} T^{(n,n)}_p$ for all $p \in \mathcal{D}_0$. For the partitions given in Figure 1, these relations can be explicitly written as follows, where again all indices are quantified over $[n]$.

$$1 = \sum_{\ell_1, \ell_2} u^{i_1 i_2}_{\ell_1 \ell_2}$$

$$u^{i_1 i_4}_{j_1 j_2} = \sum_{\ell_1, \ell_2} u^{i_1 i_2}_{j_1 \ell_1} u^{i_3 i_4}_{\ell_2 j_2}$$

$$\delta_{i_1 i_2} = \sum_\ell u^{i_1 i_2}_{\ell \ell}$$

$$\delta_{i_2 i_3} u^{i_1 i_4}_{j_1 j_2} = \sum_\ell u^{i_1 i_2}_{j_1 \ell} u^{i_3 i_4}_{\ell j_2}$$

$$u^{i_2 i_1}_{j_1 j_2} = u^{i_1 i_2}_{j_2 j_1}$$

$$u^{i_1 i_3}_{j_1 j_2} u^{i_2 i_4}_{j_3 j_4} = u^{i_3 i_4}_{j_1 j_3} u^{i_1 i_2}_{j_2 j_4}$$

$$\delta_{i_1 i_2} u^{i_1 i_1}_{j_1 j_2} = \delta_{j_1 j_2} u^{i_1 i_2}_{j_1 j_1}$$

$$\delta_{i_2 i_3} u^{i_1 i_2}_{j_1 j_2} u^{i_2 i_4}_{j_3 j_4} = \delta_{j_2 j_3} u^{i_1 i_2}_{j_1 j_2} u^{i_3 i_4}_{j_2 j_4}$$

$$\sum_k u^{i_2 k}_{j_1 j_2} = \sum_\ell u^{i_1 i_2}_{\ell j_1}$$

$$\sum_k u^{i_1 i_3}_{j_1 j_2} u^{k i_4}_{j_3 j_4} = \sum_\ell u^{i_1 i_2}_{j_1 \ell} u^{i_3 i_4}_{j_2 j_4}$$

$$\delta_{i_1 i_4} u^{i_3 i_2}_{j_1 j_2} = \sum_\ell u^{i_1 i_2}_{\ell j_2} u^{i_3 i_4}_{j_1 \ell}$$

$$\delta_{i_2 i_5} u^{i_1 i_4}_{j_1 j_2} u^{i_3 i_6}_{j_3 j_4} = \sum_\ell u^{i_1 i_2}_{j_1 \ell} u^{i_3 i_4}_{j_3 j_2} u^{i_5 i_6}_{\ell j_4}$$

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