Martin's Axiom and Weak Kurepa Hypothesis

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Abstract

I show that it is consistent relative to the consistency of a Mahlo cardinal that Martin's axiom holds at ω_2, but the weak Kurepa Hypothesis fails at ω_1. This answers a question posed by Honzik, Lambie-Hanson and Stejskalová. The consistency result is obtained by constructing a model where the weak Kurepa Hypothesis fails in any c.c.c. forcing extension.

Keywords: Martin's axiom, Kurepa tree, Suslin tree, Forcing with side conditions, Virtual model.

MSC: 03E35

1 Introduction

A *Kurepa* tree is an ω_1-tree with more than ω_1-many cofinal branches; by relaxing the notion of a Kurepa tree to let it have levels of size ω_1, we come to the notion of a *weak Kurepa tree*. The *Kurepa Hypothesis* (KH) states that there are Kurepa trees, and the *weak Kurepa Hypothesis* (wKH) states that there are weak Kurepa trees. So KH implies wKH. Notice that the Proper Forcing Axiom implies the failure of wKH, and the latter implies the failure of the Continuum Hypothesis (CH). It is natural and arguably important to investigate whether a compactness property can be made indestructible in a certain way. It was along with this line of research that the authors of [1] asked whether MA_{ω_2} is consistent with $\neg w$KH. The purpose of this paper is to answer that question in the affirmative.

In the early 90s, Jensen and Schlechta [2] proved that $\mathsf{MA}_{\omega_2} + \neg$KH is consistent from the existence of a Mahlo cardinal, modulo the consistency of ZFC. They proved more: after collapsing a Mahlo cardinal to ω_2 using the Levy collapse there are no Kurepa trees, and no further c.c.c. forcing can add such trees. Note that the large cardinal assumption is optimal, see [2]. Obviously, the Jensen–Schlechta model cannot witness $\neg w$KH, since CH holds in the generic extension. Therefore, any analogous forcing for $\neg w$KH should necessarily force $2^{\aleph_0} \geq \aleph_2$. We now know that it is possible to collapse an inaccessible cardinal with finite conditions using the techniques of generalised side conditions, which I will use to prove the following theorem.

Theorem 1.1. Assume that κ is a Mahlo cardinal. Then in a κ-c.c. generic extension, $\kappa = \omega_2$, wKH fails, and no c.c.c. forcing can force wKH.

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The following corollary is an immediate consequence of the above theorem using the well-established methods. So our concentration will be on the above theorem.

Corollary 1.2. Assume the consistency of a Mahlo cardinal. It is consistent that MA holds, $2^{\aleph_0}$ is arbitrary large, and $w\mathsf{KH}$ fails.

Some words about the proof

Our strategy for proving the main theorem will be similar to that used in [2] in the case of the Kurepa Hypothesis but with different media. More precisely, we collapse a Mahlo cardinal κ to ω_2 using the pure side condition forcing with countable virtual models. We then consider a c.c.c. forcing $\mathbb{Q}$ and assume, towards a contradiction, that it adds a weak Kurepa tree T which can be interpreted in almost all complete suborders of the collapse forcing we are interested in, say generic extension $V[\upsilon]$, for which $\upsilon < \kappa$ is inaccessible. Each $V[\upsilon]$ interprets only $\aleph_1$-many cofinal branches in the extension by $\mathbb{Q}$. Of course, $V[\upsilon]$ thinks that there are $\aleph_2$ many cofinal branches through T. However, some cofinal branches are not yet interpreted. A branch not interpreted in $V[\upsilon]$ is added, roughly speaking, by a c.c.c. forcing over an extension of $V[\upsilon]$, say $V^*[\upsilon]$. So there must be a Suslin tree in $V*[\upsilon]$ to which the branch is added. But $V^*[\upsilon]$ is an extension of $V[\upsilon]$ by a forcing with sufficiently many strongly generic conditions. Here is where we use our crucial lemma which says that a forcing with sufficiently many strongly generic conditions cannot add a Suslin subtree to a tree of height ω_1. Therefore the Suslin tree must be in the intermediate model $V[\upsilon]$. Since $V[\upsilon]$ is an υ-c.c. extension of the ground model, the Suslin tree, say S_υ has a name in V_υ. Now by standard arguments and the fact that there are stationarily many inaccessible cardinals like υ, there must be a name for two Suslin trees obtained at different stages, say $\upsilon < \nu$. But this is impossible, because then in $V[\nu]$, $S_\upsilon = S_\nu$ is Suslin because of S_ν while the marked cofinal branch to which S_υ was associated has now been interpreted in $V[\nu]$ and is a cofinal branch through S_υ.

Our key lemma is analogous to a lemma in [2]. However, the closedness of the Levy collapse plays a crucial role in this proof; here, our forcing contrasts with countably closed forcings in that it is strongly proper and thus has the ω_1 approximation property. Although our collapsing forcing is strongly opposed to the Levy collapse, it has relatively similar properties with respect to the so-called branch preservation lemmas. But with the additional advantage that it gets along much better with c.c.c. forcings than with countably closed ones, which is why Jensen–Schlechta's lemmas are a bit more sophisticated.

The structure of the paper

In Section 2 I will give the preliminary lemmata. In particular, I will prove a key lemma which is analogous to one used by Jensen and Schlechta in [2] and borrows ideas from the branch preservation lemma in Mitchell's [3]. Section 3 is devoted to the theory of virtual models and pure side condition forcing with virtual models due to Veličković. Finally, we will see the proof of the main theorem in the last section, i.e. Section 4. I will close this introduction by recalling the basic notions.

A remark

I will often omit the sub/superscripts if there is no confusion. In particular, we may or may not avoid the notation "ˇ" in a forcing name $\check{x}$; similarly for $\mathbb{P}$ in $\Vdash_{\mathbb{P}}$. Note that some of the facts could be proved in a more general way, but I decided to keep the paper as simple as possible. We will assume ZFC in our consistency results.

Trees

A tree T is a partially ordered set $T = (T, <_T)$ such that for every $t \in T$, $b_t := \{s \in T : s <_T t\}$ is well-ordered. The height of t in T is denoted by $\mathrm{ht}_T(t)$. The α-th level of a tree, denoted by T_α, consists exactly of the nodes of height α. The height of T is the smallest ordinal α such that T_α is empty. An ω_1-tree is a tree whose height is ω_1, but whose levels are all countable. A branch through T is a maximal chain in T. A branch is cofinal if its order type is ω_1. An ω_1-tree is called Suslin if it has no uncountable chains or antichains.

Definition 1.3. Assume that T is a tree of height ω_1, $\mathbb{P}$ is a forcing notion, and $\dot{b}$ is a $\mathbb{P}$-name for a cofinal branch through T. Let $\mathrm{S}(\dot{b}, \mathbb{P}) := \{t \in T : \exists p \in \mathbb{P}\ p \Vdash \text{“}t \in \dot{b}\text{”}\}$.

Forcing

Our notation and conventions for the theory of forcings are standard. In particular, for forcing conditions p, q, $p \leq q$ means that p is stronger than q; $p||q$ means that p and q are compatible; and $p \perp q$ means that p and q are incompatible. By a nontrivial forcing, we mean that it is nontrivial below any condition. For a set M and a filter G on a forcing $\mathbb{P}$, we let $M[G] := \{\dot{\tau}^G : \dot{\tau} \in X \text{ is a } \mathbb{P}\text{-name}\}$, where $\dot{\tau}^G$ is the interpretation of $\dot{\tau}$ by G. We denote the space of $\mathbb{P}$ names in V by $V^{\mathbb{P}}$. But we also abuse the language to let $V^{\mathbb{P}}$ denote unspecified generic extensions of V by $\mathbb{P}$.

Let X be a set. A condition $p \in \mathbb{P}$ is called *strongly* $(X, \mathbb{P})$*-generic*, if for every $q \leq p$ there exists $q \restriction X \in X \cap \mathbb{P}$ such that for every $r \in X \cap \mathbb{P}$ with $r \leq q \restriction X$, r is compatible with q. The forcing $\mathbb{P}$ is *strongly proper* for a family $\mathscr{S}$ if for every $X \in \mathscr{S}$ and every $p \in X$, there exists a strongly $(X, \mathbb{P})$-generic condition $q \leq p$. It is called *strongly proper* if it is strongly proper for a club in $\mathscr{P}([\mathbb{P}]^\omega)$.

We assume that the reader is familiar with the notion of a complete suborder and the quotient of a forcing notion by a filter on a complete suborder of it.

Notation 1.4. For a cardinal θ, we let H_θ denote the collection of sets whose transitive closure are of size less than θ.

2 Preliminary lemmata

Let us begin with the following standard lemma.

Lemma 2.1. Suppose $\mathbb{P}$ is a forcing and $A \subseteq \mathbb{P}$ is a maximal antichain. Let $F : A \to V^{\mathbb{P}}$ be a function. Then there is $\dot{\tau} \in V^{\mathbb{P}}$ such that for every $p \in A$, $p \Vdash_{\mathbb{P}} \text{“}\check{F}(p) = \dot{\tau}\text{”}$.

$\boxed{2.1}$

Lemma 2.1 immediately implies the *Existential Completeness Lemma* for the forcing relation, which is given below in a concise form

$$p \Vdash_{\mathbb{P}} \text{“}\exists x\, \phi(x)\text{”} \Longrightarrow \exists \dot{\tau} \in V^{\mathbb{P}} \big(p \Vdash_{\mathbb{P}} \text{“}\phi(\dot{\tau})\text{”}\big).$$

Lemma 2.2 (Jensen–Schlechta [2])**.** Assume that $\mathbb{P}$ is a c.c.c. forcing and that T is a tree of height ω_1. Suppose that $\mathbb{P}$ forces $\dot{b}$ to be a new cofinal branch through $\check{T}$. Then $\mathrm{S}(\dot{b},\mathbb{P}) \subseteq T$ is a Suslin tree in V that acquires a cofinal branch in $V^{\mathbb{P}}$, namely $\dot{b}$.[1]

Proof. Since $\dot{b}$ is forced to be a branch, it is clear that $\mathrm{S}(\dot{b},\mathbb{P})$ is downward closed, i.e., if $t \in T$ and $s \in \mathrm{S}(\dot{b},\mathbb{P})$ are such that $t \leq_T s$, then $t \in \mathrm{S}(\dot{b},\mathbb{P})$. On the other hand, since $\mathbb{P}$ preserves ω_1 and $\dot{b}$ is forced to have length ω_1, $\mathrm{S}(\dot{b},\mathbb{P})$ has height ω_1. For every $s \in \mathrm{S}(\dot{b},\mathbb{P})$, fix a condition $p_s \in \mathbb{P}$ that forces $s \in \dot{b}$. Now if $s_0, s_1 \in \mathrm{S}(\dot{b},\mathbb{P})$ are incomparable in T, then p_{s_0} is incompatible with p_{s_1} in $\mathbb{P}$. Therefore, any uncountable antichain in $\mathrm{S}(\dot{b},\mathbb{P})$ would give an uncountable antichain in $\mathbb{P}$, which is a contradiction. Therefore, $\mathrm{S}(\dot{b},\mathbb{P})$ does not have any uncountable antichain. It is clear that $\mathbb{P}$ forces $\dot{b} \subseteq \check{\mathrm{S}}(\dot{b},\mathbb{P})$. It remains to show that $\mathrm{S}(\dot{b},\mathbb{P})$ does not have any cofinal branch in V. Assume towards a contradiction that $b_* \in V$ is a cofinal branch through T. If there is $s_* \in b_*$ such that

$$b_* = \{s \in \mathrm{S}(\dot{b},\mathbb{P}) : s \leq s_* \text{ or } s_* \leq s\},$$

then letting $p \in \mathbb{P}$ be a condition forcing $s_* \in \dot{b}$, we have $p \Vdash$ “$\check{b}_* \subseteq \dot{b}$”. Since a branch is maximal, we have $p \Vdash$ “$\check{b}_* = \dot{b}$”. This contradicts the fact that $\dot{b}$ is forced to be a new branch. Therefore, there is an unbounded set $X \subseteq \omega_1$ such that for every $\alpha \in X$, there are nodes $t_*^{\alpha} <_{\mathrm{S}(\dot{b},\mathbb{P})} t^{\alpha}$ such that $t_*^{\alpha} \in b_*$, $\mathrm{ht}(t_*^{\alpha}) = \alpha$ and $t^{\alpha} \notin b_*$. Now $\{t^{\alpha} : \alpha \in X\}$ is an uncountable antichain in $\mathrm{S}(\dot{b},\mathbb{P})$. A contradiction! $\boxed{2.2}$

The following is standard. We give a proof for completeness.

Lemma 2.3. Assume that $\mathbb{P} * \dot{\mathbb{Q}}$ is a nontrivial two-step iteration forcing. Let $X \in V$. Let $\dot{f} : X \to 2$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a function. Let $x \in X$. Suppose that $(p,\dot{q})$ does not decide $\dot{f}(x)$. Then, there are $p_0^*, p_1^* \leq p$ and a $\mathbb{P}$-name $\dot{q}^*$ such that

- $\mathbb{1}_{\mathbb{P}} \Vdash$ “$\dot{q}^* \leq_{\dot{\mathbb{Q}}} \dot{q}$”, and
- $(p_i^*, \dot{q}^*) \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}}$ “$\dot{f}(x) = i$”, for every $i = 0,1$.

Proof. Since $(p,\dot{q})$ does not decide $\dot{f}(x)$, there are conditions $(p_0,\dot{q}_0),(p_1,\dot{q}_1) \leq (p,\dot{q})$ in $\mathbb{P} * \dot{\mathbb{Q}}$ such that for every $i = 0,1$,

$$(p_i,\dot{q}_i) \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \text{“}\dot{f}(x) = i\text{”}.$$

We can extend p_i to p_i^* such that $p_0^* \perp p_1^*$ in $\mathbb{P}$ as $\mathbb{P}$ is nontrivial, and hence non-atomic. Now let A be a maximal antichain below p containing p_0^*, p_1^*. By Lemma 2.1, there is a $\mathbb{P}$-name $\dot{\tau}$ such that

[1] See Definition 1.3 for the definition of $\mathrm{S}(\dot{b},\mathbb{P})$.

- $p_i^* \Vdash_{\mathbb{P}}$ "$\dot{\tau} = \dot{q}_i$", for every $i = 0, 1$, and
- $p' \Vdash_{\mathbb{P}}$ "$\dot{\tau} = \dot{q}$", for every $p' \in B \setminus \{p_0^*, p_1^*\}$.

Therefore,

$$p \Vdash_{\mathbb{P}} \text{“}\dot{\tau} \leq_{\dot{\mathbb{Q}}} \dot{q}\text{”}.$$

Now let B be a maximal antichain containing p. By Lemma 2.1 again, there is a $\mathbb{P}$-name $\dot{q}^*$ such that

- $p \Vdash_{\mathbb{P}}$ "$\dot{q}^* = \dot{\tau}$", and
- $p' \Vdash_{\mathbb{P}}$ "$\dot{q}^* = \dot{q}$", for every $p' \in A \setminus \{p\}$.

Therefore,

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}\dot{q}^* \leq_{\dot{\mathbb{Q}}} \dot{q}\text{”}.$$

Furthermore, $p_i^* \Vdash_{\mathbb{P}}$ "$\dot{q}^* = \dot{\tau} = \dot{q}_i$", for every $i = 0, 1$. Since $(p_i^*, \dot{q}^*) \leq_{\mathbb{P}*\dot{\mathbb{Q}}} (p_i, \dot{q}_i)$, for every $i = 0, 1$, we have

$$(p_i^*, \dot{q}^*) \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \text{“}\dot{f}(x) = i\text{”}$$

2.3

We want to show that a strongly proper forcing cannot add a new Suslin subtree to a tree of height ω_1. This is analogous to a lemma in Jensen–Schlechta's paper that σ-closed forcing cannot add a new Suslin tree to a tree of height ω_1. The Jensen–Schlechta's lemma uses a beautiful argument as in the famous branch preserving lemma due to Silver. We use an argument similar to Mitchell's beautiful argument in [3].

For the rest of this subsection, let T be a rooted tree of height ω_1, $\mathbb{P}$ be a nontrivial forcing that is strongly proper for a stationary set $\mathscr{S} \subseteq \mathscr{P}([\mathbb{P}]^\omega)$. Suppose that $\dot{S}$ is forced by $\mathbb{P}$ to be a Suslin subtree of T. Let θ be an uncountable regular cardinal with $\mathbb{P}, \dot{S} \in H_\theta$. Let also $\dot{\mathbb{S}}$ denote the canonical forcing structure of $\dot{S}$. Let $\mathbb{Q}$ denote the poset whose underlying set is $\dot{\mathbb{S}}$ and $\dot{\sigma} \leq_{\mathbb{Q}} \dot{\tau}$ if and only if $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\sigma} \leq_{\dot{\mathbb{S}}} \dot{\tau}$". Let

$$\mathscr{S}^* := \{M \prec H_\theta : |M| = \aleph_0 \wedge \mathbb{P}, \dot{S} \in M\}.$$

Lemma 2.4. $\mathbb{Q}$ is proper for $\mathscr{S}^*$, i.e., for every $M \in \mathscr{S}^*$ and every $\dot{\sigma}_0 \in \mathbb{Q} \cap M$, there is an $(M, \mathbb{Q})$-generic condition $\dot{\sigma}_1 \leq_{\mathbb{Q}} \dot{\sigma}_0$.

Proof. Assume that $M \in \mathscr{S}^*$ and $\dot{\sigma}_0 \in M \cap \mathbb{Q}$ are given. Let $\delta := M \cap \omega_1$. Using Lemma 2.1, we may assume without loss of generality that for some $p_0 \in M \cap \mathbb{P}$ and some $s_0 \in M \cap T$,

- $p_0 \Vdash_{\mathbb{P}}$ "$\dot{\sigma}_0 = \check{s}_0$" and
- $p \perp p_0 \Rightarrow p \Vdash_{\mathbb{P}}$ "$\mathrm{ht}(\dot{\sigma}_0)$" $= 0$.

Using Lemma 2.1 again, we find and fix $\dot{\sigma}_1 \in \dot{S}$ such that for some strongly $(M, \mathbb{P})$-generic condition $p_1 \leq p_0$ and some $s_1 \in T_\delta$ with $s_0 <_T s_1$, we have

- $p_1 \Vdash$ "$\dot{\sigma}_1 = \check{s}_1$".
- $p \perp p_1 \Rightarrow p \Vdash_{\mathbb{P}}$ "$\dot{\sigma}_1 = \dot{\sigma}_0$".

Notice that $\dot\sigma_1 \leq_{\mathbb{Q}} \dot\sigma_0$. We will show that $\dot\sigma_1$ is $(M,\mathbb{Q})$-generic. Fix $\dot\sigma_1' \leq_{\mathbb{Q}} \dot\sigma_1$. Let $D \subseteq \mathbb{Q}$ in M be open and dense. We will show that there is some condition in $D\cap M$ compatible with $\dot\sigma_1'$.

Claim 2.5. $\mathbb{1}_{\mathbb{P}} \Vdash$ "$D[\dot G]$ is dense in $\dot{\mathbb{S}}$".

Proof. Let $G \subseteq \mathbb{P}$ be an arbitrary V-generic filter. Assume that $s \in \mathbb{S} := \dot{\mathbb{S}}^G$. Let $\dot\sigma$ be a $\mathbb{P}$-name such that $\dot\sigma^G = s$. We can find a $\mathbb{P}$-name $\dot\sigma' \in \dot{\mathbb{S}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot\sigma' \leq_{\dot{\mathbb{S}}} \dot\sigma$". Therefore, there is $\dot\sigma'' \in D$ with $\dot\sigma'' \leq_{\mathbb{Q}} \dot\sigma'$. In other words, $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot\sigma'' \leq_{\dot{\mathbb{S}}} \dot\sigma$"'. So $s'' := \dot\sigma''^G \in D[G]$ and $s'' \leq_{\mathbb{S}} s$. $\boxed{2.5}$

Since p_1 is strongly $(M,\mathbb{P})$-generic, there is $p_1 \restriction M \in M\cap\mathbb{P}$ be such that any extension of $p_1 \restriction M$ in M is compatible with p_1. In particular, $p_1, p_1 \restriction M$ are compatible. Fix such a condition $p_1 \restriction M$. Let G be a V-generic filter $G \subseteq \mathbb{P}$ with $p_1, p_1 \restriction M \in G$. Working in $V[G]$, $D[G]$ is dense in $\mathbb{S}$, by the above claim. On the other hand, $D[G] \in M[G]$ and every condition in $\mathbb{S}$ is $(M[G],\mathbb{S})$-generic, since $\mathbb{S}$ has the countable chain condition. So there is $u \in D[G]\cap M[G]$ with $u <_T s_1$. Notice that since G contains an $(M,\mathbb{P})$-generic condition, namely p_1, we have $M[G]\cap T = M\cap T$, and hence $u \in M$. So $s_0 \leq_T u <_T s_1$. Let $p_2' \leq p_1, p_1 \restriction M$ and $\dot\sigma_2 \in D$ be such that $p_2' \Vdash$ "$\check u = \dot\sigma_2'$". By elementarity, there is $p_2 \in M$ with $p_2 \leq p_1 \restriction M$ and there is $\dot\sigma_2 \in D\cap M$ such that $p_2 \Vdash$ "$\dot\sigma_2 = \check u$". Since D is open and $\dot\sigma_0 \in M$, without loss of generality and by applying by applying Lemma 2.1, we may assume that $p \Vdash$ "$\dot\sigma = \dot\sigma_0$", for every $p \perp p_2$. So $\dot\sigma_2 \leq_{\mathbb{Q}} \dot\sigma_0$.

Claim 2.6. $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot\sigma_1' || \dot\sigma_2$".

Proof. Let G be a V-generic filter over $\mathbb{P}$. Let $S := \dot S^G$. Assume towards a contradiction that $\dot\sigma_1'^G \perp \dot\sigma_2^G$. Therefore, G cannot contain both p_2 and p_1 simultaneously, since, as we have seen above, we would have

$$u = \dot\sigma_2^G \leq_S s_1 = \dot\sigma_1'^G \leq_S \dot\sigma_1'^G.$$

If G has none of them, then $\dot\sigma_0^G = \dot\sigma_1^G = \dot\sigma_2^G \leq_S \dot\sigma_1'^G$. If G has p_2 but not p_1, we then have

$$\dot\sigma_0^G = \dot\sigma_1^G \leq_S u = \dot\sigma_2^G \leq_S \dot\sigma_1'^G.$$

If G has p_1 but not p_2, then $\dot\sigma_0^G = \dot\sigma_2^G \leq_S s_1 = \dot\sigma_1^G$. In either case, $\dot\sigma_1'^G$ is compatible with $\dot\sigma_2^G$. Therefore, $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot\sigma_1' || \dot\sigma_2$ in $\dot{\mathbb{S}}$". $\boxed{2.6}$

Returning to our main proof, by the Existential Completeness Lemma, there is $\dot\sigma \in \dot{\mathbb{S}}$ such that

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{"}\dot\sigma \leq_{\dot{\mathbb{S}}} \dot\sigma_1', \dot\sigma_2\text{"}.$$

In other words, $\dot\sigma \leq_{\mathbb{Q}} \dot\sigma_1', \dot\sigma_2$. Therefore, $\dot\sigma_2 \in D\cap M$ is compatible with $\dot\sigma_1'$ in $\mathbb{Q}$. $\boxed{2.4}$

Lemma 2.7. $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot S \in V$".

Proof. Assume towards a contradiction that $\mathbb{P}$ forces that $\dot S$ is not in V. Notice that $\mathbb{P} * \dot{\mathbb{S}}$ adds a new cofinal branch through T. Let $\dot b : \check T \to 2$ be a $\mathbb{P} * \dot{\mathbb{S}}$-name which is forced to be the characteristic function of a cofinal branch through $\dot S$. Fix $M \in \mathscr{S}^*$ with $\dot b \in M$. By Lemma 2.4, there is an $(M,\mathbb{Q})$-generic condition $\dot\sigma' \in \mathbb{Q}$. Let $\delta := M \cap \omega_1$. Let $p' \in \mathbb{P}$ be a strongly $(M,\mathbb{P})$-generic condition. We can extend $(p',\dot\sigma')$ to a condition $(p,\dot\sigma)$ such that $\dot\sigma \leq_{\mathbb{Q}} \dot\sigma'$ and for some $s \in T_\delta$, $(p,\dot\sigma) \Vdash_{\mathbb{P}*\dot{\mathbb{S}}}$ "$\dot b(s) = 1$". Since p' is strongly $(M,\mathbb{P})$-generic, there is $p \restriction M \in M \cap \mathbb{P}$ such that every $r \leq p \restriction M$ in $M \cap \mathbb{P}$ is compatible with p. Let

$$D := \{\dot\tau \in \dot{\mathbb{S}} : \exists p_0^*, p_1^* \leq p \restriction M,\ \exists t \in T \text{ such that } (p_i^*, \dot\tau) \Vdash \text{“}\dot b(t) = i\text{”}\}.$$

Claim 2.8. D is dense in $\mathbb{Q}$.

Proof. Suppose that $\dot\sigma_0 \in \dot{\mathbb{S}}$. Since $\dot b$ is forced to be new, there is $t \in T$ such that $(p \restriction M, \dot\sigma_0)$ does not decide $\dot b(t)$. By Lemma 2.3, there is $\dot\sigma_1 \in \dot{\mathbb{S}}$ with $\dot\sigma_1 \leq_{\mathbb{Q}} \dot\sigma_0$ so that for some $p_0^*, p_1^* \leq p \restriction M$, we have $(p_i^*, \dot\sigma_1) \Vdash$ "$\dot b(t) = i$", for $i = 0,1$. So $\dot\sigma_1 \in D$. 2.8

Notice that $D \in M$. By Lemma 2.4, there is $\dot\tau \in D \cap M$ such that $\dot\tau$ is compatible with $\dot\sigma$ in $\leq_{\mathbb{Q}}$ as witnessed by, say $\dot\pi$. Let p_0^*, p_1^*, t in M witness that $\dot\tau \in D$. Fix $p_i \leq p_i^*, p$, for $i = 0,1$. So for $i = 0,1$, we have $(p_i, \dot\pi) \leq (p,\dot\sigma), (p_i^*, \dot\tau)$. Therefore, $(p,\dot\sigma)$ does not decide $\dot b(t)$. But this is a contradiction, since

$$\left[(p,\dot\sigma) \Vdash_{\mathbb{P}*\dot{\mathbb{S}}} \text{“}\dot b(t) = i\text{”}\right] \iff t <_T s.$$

2.7

Putting it all together, we have proven the following.

Proposition 2.9. Suppose that T is a tree of height ω_1. Let G be a V-generic filter on a forcing $\mathbb{P}$ which is strongly proper for a set stationary in $\mathscr{P}([\mathbb{P}]^\omega)$. Let $S \in V[G]$ be a Suslin subtree of T. Then $S \in V$.

2.9

3 Forcing with virtual models

The notion of a virtual elementary submodel was invented by Veličković in 2014 to iterate semi-proper forcings with finite conditions. More precisely, he used it in his iteration theorem [7]. Although the virtual models are easier to use in the iteration of proper forcings, this case has not been covered anywhere except in a more complicated and technical context in [4, 5]. Therefore, we define it here and prove the relevant properties either in whole or in part, but with sufficient reference and guidance. Needless to say, the credit goes entirely to Veličković without further mention.

Virtual models

We are interested in transitive set models of ZFC. We call them *suitable*. For a suitable model $\mathfrak{A}$ and an ordinal $\alpha \in \mathfrak{A}$, let $\mathfrak{A}_\alpha = V_\alpha \cap \mathfrak{A}$. We let

$$E_{\mathfrak{A}} := \{\alpha \in \mathrm{Ord} \cap \mathfrak{A} : \mathfrak{A}_\alpha \prec \mathfrak{A}\}.$$

Note that $E_{\mathfrak{A}}$ is a closed subset of $\mathrm{Ord}^{\mathfrak{A}}$, however it could be empty. It is easily seen that $E_{\mathfrak{A}} \cap \alpha$ is uniformly definable in $\mathfrak{A}$ with parameter α, for any $\alpha \in E_{\mathfrak{A}}$. In particular, if $M \prec \mathfrak{A}$, and $\alpha \in M$, then $E_{\mathfrak{A}} \cap \alpha \in M$. The first two lemmas are technical and used in many situations regarding virtual models. Both can be proved using the Tarski–Vaught criterion.

Lemma 3.1. Suppose $\mathfrak{A}$ is a suitable structure and that $M \prec \mathfrak{A}$. If $\alpha \in E_{\mathfrak{A}}$ and $(M \cap \mathrm{Ord}^{\mathfrak{A}}) \setminus \alpha \neq \emptyset$, then $\min(M \cap \mathrm{Ord}^{\mathfrak{A}} \setminus \alpha) \in E_{\mathfrak{A}}$.

Proof. See [5, Lemma 3.1]. 3.1

Let $A, B \neq \emptyset$. We let $\mathrm{Hull}(A,B) := \{f(b) : f \in A \text{ is a function and } b \in [B]^{<\omega} \cap \mathrm{dom}(f)\}$.

Lemma 3.2. Suppose that $\mathfrak{A}$ is a suitable structure, M is an elementary submodel of $\mathfrak{A}$ and X is a subset of $\mathfrak{A}$.

1. Let $\delta := \sup(M \cap \mathrm{Ord})$, and assume that $X \cap \mathfrak{A}_\delta \neq \emptyset$. Then $X \cap \mathfrak{A}_\delta \subseteq \mathrm{Hull}(M,X)$.
2. $M \preceq \mathrm{Hull}(M,X) \preceq \mathfrak{A}$.
3. $\mathrm{Hull}(M,X)$ is minimal with respect to the above properties.

Proof. See [5, Lemma 3.3] 3.2

Definition 3.3. A submodel M of $\mathfrak{A}$ is an α-model in $\mathfrak{A}$ if

1. $\mathrm{Hull}(M, \mathfrak{A}_\alpha)$ is transitive and
2. $M, \mathfrak{A}_\alpha \prec \mathrm{Hull}(M, \mathfrak{A}_\alpha)$.

Definition 3.4. M is a virtual model in $\mathfrak{A}$ if it is an α-model for some $\alpha \in E_{\mathfrak{A}}$.

We now concentrate on a particular case. Namely, $V_\kappa = (V_\kappa, \in)$, where κ is inaccessible and we only consider countable virtual models. Thus let us fix an inaccessible cardinal κ. We denote E_{V_κ} by E_κ.

Definition 3.5. For each $\alpha \in E_\kappa$, we let $\mathscr{C}_\alpha$ denote the set of countable α-models in V_κ. We also let $\mathscr{C}_{<\alpha} := \bigcup\{\mathscr{C}_\beta : \beta \in E \cap \alpha\}$, for $\alpha \in E_\kappa \cup \{\kappa\}$. The collections $\mathscr{C}_{\leq\alpha}, \mathscr{C}_{\geq\alpha}$, etc are defined in the obvious way. We shall write $\mathscr{C}$ for $\mathscr{C}_{<\kappa}$.

Notation 3.6. If $M \in \mathscr{C}$, then let $\eta(M)$ be the (unique) ordinal α such that $M \in \mathscr{C}_\alpha$.

The uniqueness can be proved using a simple counting argument. Observe that if M is a countable virtual model and $\alpha \in E_\kappa$, then $|\mathrm{Hull}(M, V_\alpha)| = |V_\alpha|$. So $M \in V_{\mathrm{next}(\alpha)}$, where $\mathrm{next}(\alpha)$ is the least element of $E_\kappa \setminus (\alpha+1)$.

Definition 3.7. A virtual model M is called *standard* if $M \prec V_{\eta(M)}$.

Notation 3.8. We let $\mathscr{C}_{\mathrm{st}}$ denote the class of all countable standard virtual models in V_κ.

Definition 3.9. Suppose $M,N \in \mathscr{C}$ and $\alpha \in E$. An isomorphism $\sigma : M \to N$ is called an α-*isomorphism* if there is an isomorphism $\bar{\sigma}$: Hull$(M,V_\alpha) \to$ Hull(N,V_α) extending σ. We say that M and N are α-*isomorphic* and write $M \cong_\alpha N$ if there is an α-isomorphism between them.

Definition 3.10. Suppose $\alpha, \beta \in E$ and M is a β-model. Let $\overline{\mathrm{Hull}(M,V_\alpha)}$ be the transitive collapse of Hull(M,V_α), and let π be the collapse map. We define $M \restriction \alpha$ to be $\pi[M]$, i.e. the image of M under the collapse map of Hull(M,V_α).

We call $M \restriction \alpha$, the projection of M to α.

The following is easy and we use it without referring to it.

Fact 3.11. Suppose that $\alpha \leq \beta$ are in E. Assume that $M,N \in \mathscr{C}$.

1. If $M \cong_\beta N$, then $M \cong_\alpha N$.
2. $(M \restriction \beta) \restriction \alpha = M \restriction \alpha$.

3.11

Notice that if $\alpha > \eta(M)$, then $M \restriction \alpha = M$.

For a virtual model M, by $\mathscr{C}^M$, we mean $M \cap \mathscr{C}^{\mathrm{Hull}(M,V_{\eta(M)})}$, where $\mathscr{C}^{\mathrm{Hull}(M,V_{\eta(M)})}$ is the set of virtual models relativized to Hull$(M,V_{\eta(M)})$. Note that although Hull$(M,V_{\eta(M)})$ is a suitable model, $E_{\mathrm{Hull}(M,V_{\eta(M)})}$ could be bounded in $\mathsf{Ord} \cap$ Hull$(M,V_{\eta(M)})$.

Definition 3.12. Suppose $M,N \in \mathscr{C}$ and $\alpha \in E$. We write $M \in_\alpha N$ if there is $M' \in \mathscr{C}^N$ such that $M' \cong_\alpha M$. If this happens, we say that M is α-*in* N.

the important thing here is to be careful that although the witness M' in the above definition might not be in $\mathscr{C}$. and hence not a virtual model in the sense of V_κ, M is a virtual model in the sense of Hull$(M,V_{\eta(M)})$. Therefore, the expression $M \cong_\alpha M'$ makes sense. Moreover, there is no confusion since Hull$(M,V_{\eta(M)})$ is transitive.

Fact 3.13. Suppose $M,N \in \mathscr{C}$ with $M \in N$. Let $\alpha \in E$, and suppose $N' \in \mathscr{C}^A$, for some $A \in \mathscr{A}_\alpha$, and $\sigma : N \to N'$ is an α-isomorphism. Then M and $\sigma(M)$ are α-isomorphic.

Proof. See [5, Proposition 2.14]. 3.13

The following is straightforward.

Fact 3.14. Let $\alpha, \beta \in E_\kappa$ with $\alpha \leq \beta$. Suppose $M,N \in \mathscr{C}_{\geq \beta}$ and $M \in_\beta N$. Then $M \restriction \alpha \in_\alpha N \restriction \alpha$.

3.14

Note that $\mathscr{C}$ is closed under projections. The collections $\mathscr{C}_{<\alpha}$, $\mathscr{C}_{\leq \alpha}$, and $\mathscr{C}_{\geq \alpha}$ are defined in the obvious way.

Fact 3.15. $\mathscr{C}_{\mathrm{st}}$ contains a club in $\mathscr{P}_{\omega_1}(V_\lambda)$.

Proof. Let $M \prec (V_\kappa, \in, E_\kappa)$ be countable. Since E is a club, $\eta = \sup(M \cap \lambda) \in E$. By Lemma 3.2, $\mathrm{Hull}(M, V_\eta) = V_\eta$. Thus $M \in \mathscr{C}_{\mathrm{st}}$. 3.15

Fact 3.16. Let $\alpha \in E_\kappa$. Suppose $M, N, P \in \mathscr{C}$ and $M \in_\alpha N \in_\alpha P$. Then $M \in_\alpha P$.

Proof. See [5, Proposition 2.25]. Hint: use Fact 3.13. 3.16

Definition 3.17. Assume that $M \in \mathscr{C}$. Let $\alpha \in E_\kappa \cap (\eta(M)+1)$. M is said to be *active* at α if $M \cap E_\kappa \cap \alpha$ is unbounded in $E_\kappa \cap \alpha$.

Notation 3.18. Suppose $M \in \mathscr{C}$. Let $\mathfrak{a}(M) = \{\alpha \in E : M \text{ is active at } \alpha\}$.

Let $\alpha \in E$ and let $\mathscr{M}$ be a set of virtual models. We let

$$\mathscr{M} \restriction \alpha = \{M \restriction \alpha : M \in \mathscr{M}\} \quad \text{and} \quad \mathscr{M}^\alpha = \{M \restriction \alpha :\ M \in \mathscr{M} \text{ and } \alpha \in \mathfrak{a}(M)\}.$$

Definition 3.19. Let $\alpha \in E$ and let $\mathscr{M} \subseteq \mathscr{C}$. We say $\mathscr{M}$ is an α*-chain* (or $\in_\alpha$-chain) if for all $M, N \in \mathscr{M}$, either $M \in_\alpha N$ or $M \cong_\alpha N$ or $N \in_\alpha M$.

It is easily seen that if $\mathscr{M}$ is an α-chain, then $\mathscr{M} \restriction \alpha$ is an α-chain as well.

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Definition 3.20. Let $\alpha \in E$. A set p is a condition in $\mathbb{P}_\alpha$ if

1. $p \subseteq \mathscr{C}_{\leq \alpha}$ is finite, and
2. for every $\delta \in E_\kappa \cap (\alpha+1)$, p^δ is a δ-chain.

A condition p is stronger than q if for every $\delta \in E_\kappa \cap (\alpha+1)$, $q^\delta \subseteq p^\delta$.

Notice that if $\alpha \leq \beta$ are in $E_\kappa \cup \{\kappa\}$, then $\mathbb{P}_\alpha$ is a suborder of $\mathbb{P}_\beta$. Therefore, we let

$$\mathbb{P}_\kappa := \bigcup\{\mathbb{P}_\alpha : \alpha \in E_\kappa\},$$

and we use the notation $p \leq q$ if and only if p is stronger than q in some $\mathbb{P}_\alpha$. It is clear that $\leq$ is transitive and non-atomic, but it is not separative.

Lemma 3.21. Let $\alpha, \beta \in E_\kappa \cup \{\kappa\}$. Assume that $\alpha \leq \beta$. Define the function $\rho_{\beta,\alpha} : \mathbb{P}_\beta \to \mathbb{P}_\alpha$ be defined by $\rho_{\beta,\alpha}(p) = p \restriction \alpha$. Then

1. $\alpha = \beta \Rightarrow \rho_{\beta,\alpha} = id$.
2. $\rho_{\beta,\alpha}$ is order preserving. and $\rho_{\beta,\alpha}(\mathbb{1}_{\mathbb{P}_\beta}) = \mathbb{1}_{\mathbb{P}_\alpha}$
3. for every $\gamma \in E_\kappa \cup \{\kappa\}$ with $\beta \leq \gamma$, $\rho_{\gamma,\beta} \circ \rho_{\beta,\alpha} = \rho_{\gamma,\alpha}$.
4. for every $p \in \mathbb{P}_\beta$ and every $q \in \mathbb{P}_\alpha$ with $q \leq \rho_{\beta,\alpha}(p)$, there is $r \in \mathbb{P}_\beta$ such that r is a greatest lower bound of p and q, and $\rho_{\beta,\alpha}(r) = q$.

Proof. 1 and 2 are straightforward. 3 follows from Fact 3.11. To prove 4, let r be defined by $r = p \cup q$. To show that $r \in \mathbb{P}_\beta$, we need to show that for every $\delta \in E_\kappa \cap (\beta+1)$, r^δ is a δ-chain. Fix such a δ. Notice that if $\delta > \alpha$, then $q^\delta = \emptyset$; and if $\delta \leq \alpha$, then have $q^\delta \supseteq p^\delta$. Therefore,

$$r^\delta = \begin{cases} p^\delta & \text{if } \delta > \alpha \\ q^\delta & \text{if } \delta \leq \alpha \end{cases}$$

is a δ-chain. It is clear that $r \leq p, q$, $\rho_{\beta,\alpha}(r) = q$, and that r is a greatest lower bound p and q. $\boxed{3.21}$

Lemma 3.22. Assume that $\alpha \in E$ is of uncountable cofinality. Then $\mathbb{P}_\alpha$ is forcing equivalent to $\mathbb{P}_\alpha \cap V_\alpha$.

Proof. Define $\rho : \mathbb{P}_\alpha \to \mathbb{P}_\alpha \cap V_\alpha$ by $\rho(p) = \rho_{\alpha,\gamma(p)}(p)$, where

$$\gamma(p) := \sup(\bigcup\{\mathfrak{a}(M) \cap \alpha : M \in p\}).$$

Since α is of countable cofinality, the set of active points of a model is a closed set, and p is a finite set of countable models, we have $\gamma(p) \in E_\kappa \cap \alpha$. So ρ is well-defined. It is clear that $\rho \restriction \mathbb{P}_\alpha \cap V_\alpha = id$. Lemma 3.21 implies that ρ is a forcing projection.

To conclude the proof, we show that the quotient forcing of $\mathbb{P}_\alpha$ by any generic filter on $\mathbb{P}_\alpha \cap V_\alpha$ is trivial. For which it is enough to show that for every $p, q \in \mathbb{P}_\alpha$ such that $\rho(p), \rho(q)$ are compatible, then p and q are compatible. Let $r \leq \rho(p), \rho(q)$. Define r by

$$s := r \cup \{M \in p \cup q : M \notin V_\alpha\}.$$

Note that there is no countable model active at α, hence $s^\alpha = \emptyset$. For every $\delta \in E_\kappa \cap \alpha$, on the other hand, we have $s^\delta = r^\delta$. So s is a condition in $\mathbb{P}_\alpha$. It is clear that $s \leq p, q$. $\boxed{3.22}$

The following is immediate.

Corollary 3.23. $\mathbb{P}_\kappa$ is strongly proper for $\{V_\alpha : \alpha \in E \wedge \mathrm{cof}(\alpha) > \omega\}$. Moreover, $\mathbb{1}_{\mathbb{P}_\kappa}$ is strongly $(X, \mathbb{P}_\kappa)$-generic for every set X such that $\mathbb{P}_\kappa \cap X = \mathbb{P}_\kappa \cap V_\alpha$, for some $\alpha \in E$ of uncountable cofinality. Therefore, $\mathbb{P}_\kappa$ has the κ-chain condition.

$\boxed{3.23}$

In fact, $\mathbb{P}_\kappa$ has the κ-Knaster property; we leave the proof to the reader.

Definition 3.24. Let $p \in \mathbb{P}_\kappa$. Assume that $N \in p$ and $M \in p \cap \mathscr{C}_{\text{st}}$. We let

$$\varepsilon(N, M) := \sup\{\alpha \in \mathfrak{a}(N) : N \restriction \alpha \in_\alpha M\}.$$

We define $N \restriction M$ only if $\varepsilon(N, M) \neq 0$. Since both N and M are active at $\varepsilon(N, M)$, we will also have

$$N \restriction \varepsilon(N, M) \in_{\varepsilon(N,M)} M \restriction \varepsilon(N, M).$$

Now by Fact 3.13, we have $N \restriction \varepsilon(N,M) \in_{\varepsilon(N,M)} M$. Let

$$\varepsilon^*(N,M) := \min(E_\kappa \cap M \setminus \varepsilon(N,M)).$$

Now let $N \restriction M := N^*$ where $N^* \in M$ is the unique $\varepsilon^*(N,M)$-model which is $\varepsilon(N,M)$-isomorphic to N. Finally, let $p \restriction M := \{N \restriction M : N \in p\}$.

Notation 3.25. For every $p \in \mathbb{P}_\kappa$, we let $\ell(p)$ be the least $\alpha \in E$ such that $p \in \mathbb{P}_{\ell(p)}$.

Notice that $\eta(M) \leq \ell(p)$, for every p with $M \in p$.

Lemma 3.26. $p \restriction M \in M \cap \mathbb{P}_{<\eta(M)}$ and $p \leq p \restriction M$.

Proof. It is clear that $p \restriction M \in M$. To see why $p \restriction M$ is a condition, it is enough- by elementarity of M- that if $N_1, N_2 \in p \restriction M$ are active at $\delta \in M \cap E_\kappa$ and $N_1 \cap \omega_1 < N_2 \cap \omega_1$, then $N_1 \restriction \delta \in_\delta N_2 \restriction \delta$. There are $N_1', N_2' \in p$ such that $N_1 = N_1' \restriction M$ and $N_2 = N_2' \restriction M$. Let us set $\varepsilon_i := \varepsilon(N_i', M)$, for every $i = 1, 2$. By the definition of ε_i, we have $\delta \in \mathfrak{a}(M) \cap \mathfrak{a}(N_i')$, for every $i = 1, 2$. Since no countable model is active at a point of uncountable cofinality, we have $\delta \in M \cap \min(\varepsilon_1^*, \varepsilon_2^*)$, which in turn implies that $\delta < \varepsilon_1, \varepsilon_2$. So $\delta \in \mathfrak{a}(N_1') \cap \mathfrak{a}(N_2')$. Now since p is a condition and $N_i \cap \omega_1 = N_i' \cap \omega_1$, for every $i = 1, 2$, we must have $N_1' \restriction \delta \in_\delta N_2' \restriction \delta$. But then

$$N_1 \restriction \delta = N_1' \restriction \varepsilon_1 \restriction \delta = N_1' \restriction \delta \in_\delta N_2' \restriction \delta = N_2' \restriction \varepsilon_2 \restriction \delta = N_2 \restriction \delta.$$

To see that $p \leq p \restriction M$, observe that if $N \in (p \restriction M)^\delta$. Then M is active at δ and $\delta \in M$. We can find $N' \in p$ such that N' is $\varepsilon(N', M)$-isomorphic to N. As we have seen above, $\delta \in \mathfrak{a}(N') \cap \varepsilon(N', M)$ and $N \restriction \delta = N' \restriction \delta \in p^\delta$. 3.26

Lemma 3.27. Suppose that $q \leq p \restriction M$ is a condition in $M \cap \mathbb{P}_\kappa$. Then $p \cup q \in \mathbb{P}_{\ell(p)}$ and $p \cup q \leq p, q$. In fact, $p \cup q$ is the greatest lower bound of p and q.

Proof. Let $r = p \cup q$. Let $\delta \in E$. We are done if $q^\delta = \emptyset$. So let us assume that q^δ is nonempty, which in turn implies that M is active at δ. It is easily seen that

$$r^\delta = q^\delta \cup \{M \restriction \delta\} \cup \{N : N \in p^\delta : M \restriction \delta \in_\delta N\}.$$

It is obvious that r^δ is a δ-chain. 3.27

The following is straightforward.

Corollary 3.28. $\mathbb{P}_\kappa$ is strongly proper for $\mathscr{C}_{\text{st}}$. In fact, for every standard model M and every condition p with $M \in p$, p is strongly $(M, \mathbb{P}_\kappa)$-generic.

3.28

Assume that $G_\kappa \subseteq \mathbb{P}_\kappa$ is a filter, let $\mathscr{M} := \bigcup G_\kappa$. Then for every $\delta \in E_\kappa$, $\mathscr{M}^\delta$ is a δ-chain. Let $\delta \in E_\kappa$. Using Fact 3.16, it is easy to see that

$$\{M \cap V_\delta : M \in \mathscr{M}^{\mathrm{next}(\gamma)}\}$$

is an $\in$-chain. If G_κ is V-generic, then, since ω_1 is preserved, the length of the above chain must be ω_1. So the size of V_δ is ω_1 in $V[G_\kappa]$. Therefore, $\kappa = \omega_2$ in V. By standard arguments, $2^{\aleph_1} = \aleph_2$ in $V[G_\kappa]$. On the other hand, by the κ-chain condition if the continuum hypothesis held in $V[G_\kappa]$, then all the reals must have been added by some initial stage, say $\mathbb{P}_\alpha$. But then as we have seen above G_κ is $V[G_\alpha]$-generic for $\mathbb{P}_\kappa/G_\alpha$, where $G_\alpha := G_\kappa \cap \mathbb{P}_\alpha$. The poset $\mathbb{P}_\kappa/G_\alpha$ is strongly proper for a stationary set by Proposition 3.29, and this is a contradiction. Since a poset which is strongly proper forcing for a stationary set necessarily adds reals.

Proposition 3.29. Let $\alpha \in E$. Suppose $G_\alpha \subseteq \mathbb{P}_\alpha$ is a V-generic filter. The forcing $\mathbb{P}_\kappa/G_\alpha$ is strongly proper for a stationary set.

Proof. Let

$$\mathscr{S} := \{M \cap \mathbb{P}_\kappa : M \in \mathscr{C}_{\geq\alpha} \text{ and } M \restriction \alpha \in \bigcup G_\alpha\}.$$

It is easy to see that $\mathscr{S}$ is stationary in $\mathscr{P}([\mathbb{P}_\kappa]^\omega)$. Let $M \in \mathscr{C}_{\geq\alpha}$ and $M \restriction \alpha \in \bigcup G_\alpha$. Suppose that $p \in \mathbb{P}_\kappa/G_\alpha \cap M$. Then it is not hard to see that $p^M \leq p$ is strongly $(M, \mathbb{P}_\kappa/G_\alpha)$-generic. 3.29

4 The main theorem

We prove our main theorem in this section. We shall use the forcing $\mathbb{P}_\kappa$ constructed in the previous section. We will use its main properties without referring them:

- $\mathbb{P}_\kappa$ is κ-c.c.
- $\mathbb{P}_\kappa$ is strongly proper.
- $\mathbb{P}_\kappa$ is strongly proper for $\{V_\alpha : \alpha \in E_\kappa \text{ such that } \mathrm{cof}(\alpha) > \omega\}$.
- $\mathbb{P}_\kappa$ forces $\kappa = \dot{\aleph}_2 = \dot{\mathfrak{c}} = 2^{\aleph_1}$.
- $\mathbb{P}_\kappa/G_\alpha$ is strongly proper for some stationary set in $[\mathbb{P}_\kappa/G_\alpha]^\omega$, for every $\alpha \in E_\kappa$ and every V-generic filter $G_\alpha \subseteq \mathbb{P}_\alpha$.

Notice that one can also use the Neeman forcing with finite sequences of countable elementary submodels of V_κ as small nodes and $\{V_\alpha : \alpha \in E_\kappa \wedge \mathrm{cof}(\alpha) > \omega\}$ as transitive nodes, see [6].

Theorem 4.1. Assume that κ is Mahlo. Then in a strongly proper and κ-c.c. generic extension, $\kappa = \omega_2$, wKH fails, and no c.c.c. forcing can force wKH.

Proof. Let $\mathbb{P}_\kappa$ be Veličković forcing with finite chains of countable virtual models, i.e. the forcing from Section 3. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}_\kappa$-name for a c.c.c. forcing. It will be convenient to denote $\dot{\mathbb{Q}}$ by $\dot{\mathbb{Q}}_\kappa$. We write the proof through several steps to make it more readable.

(*Step 0: the general setting*) Assume towards a contradiction that $\dot{\mathbb{Q}}_\kappa$ forces *w*KH over $V^{\mathbb{P}_\kappa}$. So suppose that $\dot{\mathbb{Q}}_\kappa$ forces, over $V^{\mathbb{P}_\kappa}$, $\dot{T}$ to be a weak Kurepa tree whose underlying set is ω_1, and suppose that this is witnessed by a sequence $\langle \dot{b}_\alpha : \alpha < \kappa \rangle$ of pair-wise distinct cofinal branches through $\dot{T}$. Since $\mathbb{P}_\kappa$ forces $2^{\aleph_1} = \aleph_2$, we may assume that the above sequence enumerates all the cofinal branches of $\dot{T}$. Moreover, by standard arguments, we may treat $\dot{T}$ and each $\dot{b}_\alpha$ as $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$-names as well. Since $\mathbb{P}_\kappa$ has the κ-chain condition, $\dot{\mathbb{Q}}_\kappa$ is forced to be c.c.c. and that $\dot{T}$ is forced to be a tree on ω_1^V (as $\mathbb{P}_\kappa$ preserves ω_1), we may assume without loss of generality that $\{\dot{b}_\alpha : \alpha < \kappa\} \cup \{\dot{T}\} \subseteq V_\kappa$. On the other hand, since $\mathbb{P}_\kappa$ forces that $\mathfrak{c} = \aleph_2$ and that $\dot{\mathbb{Q}}_\kappa$ is c.c.c., we may also assume, without loss of generality, that $\dot{\mathbb{Q}}_\kappa \subseteq V_\kappa$.

(*Step 1: analysing the complete suborders and quotients*) Since κ is Mahlo, we can fix a stationary subset $U \subseteq \kappa$ consisting of inaccessible cardinals such that for every $\upsilon \in U$,

$$\mathbb{V}_\upsilon := (V_\upsilon, \in, \dot{\mathbb{Q}} \cap V_\upsilon, \langle \dot{b}_\alpha : \alpha < \upsilon \rangle, \dot{T}) \prec (V_\kappa, \in, \dot{\mathbb{Q}}, \langle \dot{b}_\alpha : \alpha < \kappa \rangle, \dot{T}).$$

Therefore, U is in particular a subset of E_κ. We fix a $\upsilon \in U$ until the further notice. We first show that

1. $(\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa) \cap V_\upsilon$ is a complete suborder of $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$.
2. $(\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa) \cap V_\upsilon = (\mathbb{P}_\kappa \cap V_\upsilon) * (\dot{\mathbb{Q}}_\kappa \cap V_\upsilon)$.

By the elementarity of $\mathbb{V}_\upsilon$, any incompatible pair of conditions in $(\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa) \cap V_\upsilon$ remain incompatible in $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$. Suppose that $A \subseteq (\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa) \cap V_\upsilon$ is a maximal antichain. Then A must belong to V_υ, as $\mathbb{P}_\kappa$ is κ-Knaster. So by elementarity of $\mathbb{V}_\upsilon$, A is maximal in $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$. So we proved the first item. The second item is clear. Notice that by Lemma 3.22, $\mathbb{P}_\kappa \cap V_\upsilon$ is forcing equivalent to $\mathbb{P}_\upsilon$. Thus it makes sense to denote $(\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa) \cap V_\upsilon$ by $\mathbb{P}_\upsilon * \dot{\mathbb{Q}}_\upsilon$. By the first item above, $\mathbb{P}_\upsilon$ forces that $\dot{\mathbb{Q}}_\upsilon$ has the countable chain condition.

For every V-generic filter $G_\upsilon * \dot{H}_\upsilon \subseteq \mathbb{P}_\upsilon * \dot{\mathbb{Q}}_\upsilon$, we abuse language and define the following quotient forcing over $V[G_\upsilon * \dot{H}_\upsilon]$,

$$\mathbb{R}_\upsilon := (\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa)/(G_\upsilon * \dot{H}_\upsilon).$$

(*Step 2: the interpreted and uninterpreted branches*) Suppose that $G_\kappa * \dot{H}_\kappa$ is a V-generic filter on $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$. By the previous step,

$$G_\upsilon * \dot{H}_\upsilon := (G_\kappa * \dot{H}_\kappa) \cap V_\upsilon$$

is a V-generic filter on $\mathbb{P}_\upsilon * \dot{H}_\upsilon$. Therefore, $G_\kappa * \dot{H}_\kappa$ is also a $V[G_\upsilon * \dot{H}_\upsilon]$-generic filter on $\mathbb{R}_\upsilon$. Consequently, it follows that for every $\nu \in U \cap \upsilon$,

$$\dot{b}_\nu^{G_\nu * \dot{H}_\nu} = \dot{b}_\nu^{G_\kappa * \dot{H}_\kappa} \subseteq \dot{T}^{G_\nu * \dot{H}_\nu} = \dot{T}^{G_\kappa * \dot{H}_\kappa}.$$

Notice that G_κ is also $V[G_\upsilon * \dot{H}_\upsilon]$-generic. So we can form the model $V[G_\upsilon * \dot{H}_\upsilon][G_\kappa]$, which is an inner model of $V[G_\kappa * \dot{H}_\kappa]$.

Claim 4.2. $b_\upsilon \notin V[G_\upsilon * \dot{H}_\upsilon][G_\kappa]$

Proof. If b_υ belongs to $V[G_\upsilon * \dot{H}_\upsilon][G_\kappa]$, it has to belong to $V[G_\upsilon * \dot{H}_\upsilon]$ for $V[G_\upsilon * \dot{H}_\upsilon]$ is a c.c.c. extension of $V[G_\upsilon]$, and hence $\mathbb{P}_\kappa/G_\upsilon$ is strongly proper for a stationary set in $V[G_\upsilon * \dot{H}_\upsilon]$. So $\mathbb{P}_\kappa/G_\upsilon$ cannot add a cofinal branch through a tree of height ω_1. So

$$b_\upsilon \in H_{\omega_2}^{V[G_\upsilon * \dot{H}_\upsilon]}.$$

Since $\upsilon \in U$, $\langle \dot{b}_\nu^{G_\upsilon} : \nu < \upsilon \rangle$ enumerates all the cofinal branches through T in $V[G_\upsilon * \dot{H}_\upsilon]$. So $b_\upsilon = b_\nu^{G_\upsilon * \dot{H}_\upsilon}$, for some $\nu < \upsilon$. This contradicts our assumption that the cofinal branches are forced to be distinct. 4.2

(*Step 3: applying branch preserving lemmas*) We have

$$V[G_\kappa] \subseteq V[G_\upsilon * \dot{H}_\upsilon][G_\kappa] \subseteq V[G_\kappa * \dot{H}_\kappa].$$

Therefore, $V[G_\kappa * \dot{H}_\kappa]$ is a c.c.c. extension of $V[G_\upsilon * \dot{H}_\upsilon][G_\kappa]$. Now since $b_\upsilon \notin V[G_\upsilon * \dot{H}_\upsilon][G_\kappa]$, we can apply Lemma 2.2. So $\mathrm{S}(\dot{b}_\upsilon, \mathbb{R}_\upsilon) \subseteq T$ is a Suslin tree in $V[G_\upsilon * \dot{H}_\upsilon][G_\kappa]$. By Proposition 2.9, $\mathrm{S}(\dot{b}_\upsilon, \mathbb{R}_\upsilon) \in V[G_\upsilon * \dot{H}_\upsilon]$. Consequently, there is a $\mathbb{P}_\upsilon * \dot{\mathbb{Q}}_\upsilon$-name $\dot{S}_\upsilon$ for a Suslin subtree of T such that

$$\dot{S}_\upsilon^{G_\upsilon * \dot{H}_\upsilon} = \mathrm{S}(\dot{b}_\upsilon, \mathbb{R}_\upsilon).$$

However, $\mathbb{P}_\upsilon * \dot{\mathbb{Q}}_\upsilon$ has the υ-chain condition in V, since υ is an inaccessible limit point of E_κ. Therefore, without loss of generality, we may assume that $\dot{S}_\upsilon \in V_\upsilon$.

(*Step 4: drawing a contradiction*) Since $\upsilon \in U$ was arbitrary, with Fodor's lemma and a simple counting argument, there is a κ-sized set $U^* \subseteq U$ such that for every $\nu, \upsilon \in U^*$,

$$\dot{S}_\nu = \dot{S}_\upsilon.$$

Now fix $\nu < \upsilon$ in U^*. We have

$$\mathrm{S}(\dot{b}_\nu, \mathbb{R}_\nu) = \dot{S}_\nu^{G_\upsilon * \dot{H}_\upsilon} = \dot{S}_\upsilon^{G_\upsilon * \dot{H}_\upsilon} = \mathrm{S}(\dot{b}_\upsilon, \mathbb{R}_\upsilon).$$

Let us denote the above identical tree by S. We now want to draw a contradiction. On the one hand, we do know that S is Suslin in $V[G_\upsilon * \dot{H}_\upsilon]$, since so is

$$\mathrm{S}(\dot{b}_\upsilon, \mathbb{R}_\upsilon) = \mathrm{S}.$$

On the other hand, S has a cofinal branch in $V[G_\upsilon * \dot{H}_\upsilon]$, as

$$b_\nu = \dot{b}_\nu^{G_\upsilon * \dot{H}_\upsilon} \in V[G_\upsilon * \dot{H}_\upsilon]$$

is a cofinal branch through $\mathrm{S}(\dot{b}_\nu, \mathbb{R}_\nu) = \mathrm{S}$.

4.1

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