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ON SOBOLEV AND BESOV SPACES WITH HYBRID REGULARITY

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Abstract. The present article is concerned with the nonlinear approximation of functions in the Sobolev space H^q with respect to a tensor-product, or hyperbolic wavelet basis on the unit n-cube. Here, q is a real number, which is not necessarily positive. We derive Jackson and Bernstein inequalities to obtain that the approximation classes contain Besov spaces of hybrid regularity. Especially, we show that all functions that can be approximated by classical wavelets are also approximable by tensor-product wavelets at least at the same rate. In particular, this implies that for nonnegative regularity, the classical Besov spaces of regularity $B_\tau^{q+sn,\tau}$, with $\frac{1}{\tau} = s + \frac{1}{2}$, are included in the Besov spaces of hybrid regularity $\mathfrak{B}_\tau^{q,s,\tau}$, with isotropic regularity q and additional mixed regularity s.

1. Introduction

If we want to approximate a function, there are many different methods. The best known and understood method is linear approximation. In this setting, given a function $u \in V$, we take a sequence of nested, linear trial spaces $V_j \subseteq V$ and intend to quantify the best approximation error $\inf_{v_j \in V_j} \|u - v_j\|_V$ with respect to j. In order to guarantee a certain convergence order, specific constraints on the target function u have to be satisfied. For example, to approximate a function u in the (isotropic) Sobolev space $V = H^q(\square)$, with $\square := (0,1)^n$, by piecewise polynomial functions of order d defined on a quasi-uniform mesh with support length 2^{-j}, we require that $u \in H^s(\square)$ to expect the convergence order $2^{-(s-q)j}$ for $q \le s \le d$. When comparing the number of required trial functions with the accuracy, any function in $H^{q+ns}(\square)$ is asymptotically approximable by N terms at the rate N^{-s}.

In the case of sufficiently high regularity, this approach works perfectly fine. On the other hand, if only limited regularity of the function u is provided, the optimal convergence rate cannot be realised. Therefore, the framework of *nonlinear approximation* was developed to approximate a function adaptively, see e.g. [11] for an overview. In this setting, the trial spaces used for approximating u are no longer linear subspaces. In particular, for *best N-term approximation*, the trial space V_N is the space of linear combinations from a dictionary, consisting of at most N terms. It is easy to see that the best nonlinear approximant consisting of N terms is at least as good as every best linear approximant consisting of N terms, as we can always restrict V_N to be a linear space.

A basic tool for nonlinear approximation is a Riesz basis $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ for the space V, which can, if V is Sobolev space, be realised by a wavelet basis, cf. [6, 10, 11, 26] for example. With a Riesz basis at hand, approximating the function u in V by N terms from Ψ is equivalent to approximating the coefficient vector $\mathbf{u}$ in $\ell^2(\nabla)$. By using a wavelet basis one can show, cf. [11] and the references therein, that the requirements to achieve the rate N^{-s} are much weaker compared

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to linear approximation: In contrast to Sobolev regularity of order $q+sn$, only Besov regularity of order $q+sn$ and integrability $\tau := (\frac{1}{2}+s)^{-1}$ is required. Since then $\tau \leq 2$, we can conclude that these spaces contain the respective Sobolev space $H^{q+sn}(\square)$, but are in general much larger.

This classical result holds if the space $H^q(\square)$ is discretised by isotropic wavelets. The recent articles [3, 19, 25] show, however, that the classical Sobolev spaces $H^q(\square)$ can also be characterised by tensor-product or hyperbolic wavelets. This approach leads to a new perspective on the best N-term approximation, as it is well-known that tensor-product wavelets can approximate functions essentially better. This concept is known as the *Sparse Grid* [2, 13]. Here, the term *essentially* is understood as n-times as well up to logarithmic terms.

However, this is only true under certain additional requirements on the target function. For example, if $q = 0$, meaning that $H^q(\square) = L^2(\square)$, the target function needs to admit dominating mixed Sobolev smoothness. As a consequence, for the best N-term approximation, one needs to consider Besov spaces with dominating mixed regularity, which have, for instance, been studied in [18, 21, 27, 30].

In [22, 23], Besov spaces have been used for the approximation with tensor-product wavelets in $L^2(\square)$ and $H^1(\square)$. Therein, the respective approximation spaces have been defined as as the tensor product of quasi-Banach spaces, resulting in function spaces of Besov type with hybrid regularity. If $q \geq 0$, this procedure can easily be extended to $H^q(\square)$, but whenever q is strictly positive, the approximation spaces are of hybrid regularity. However, if one wishes to approximate a function in $H^q(\square)$ for negative q, which is necessary in the case of e.g. boundary integral equations, cf. [24, 28], these results do not carry over, as the resulting approximation spaces can no longer be written as the intersection of tensor-product spaces.

In contrast, the hybrid regularity Besov spaces on the whole space $\mathbb{R}^n$ have been introduced in terms of wavelet coefficients in [3]. Therein, upper and lower bounds on the Kolmogorov dictionary width and the best N-term approximation have been derived. However, those results require a difference in the isotropic part of the hybrid regularity. As we will see, this is also not the case when we consider the best N-term approximation in a Hilbert space $H^q(\square)$ with respect to a tensor-product wavelet basis.

In this article, we will characterise the approximation spaces $\mathcal{A}^s\big(H^q(\square)\big)$ consisting of functions $u \in H^q(\square)$, which can be approximated by N terms at the rate N^{-s}. As we will see, the resulting spaces contain the Besov spaces of hybrid regularity from [3]. Additionally, when requiring slightly more regularity in terms of logarithmic decay of the coefficients, with the help of [1] we can immediately conclude that these spaces can be nested between classical Besov spaces. However, to the authors' best knowledge, it was not known yet whether these hybrid regularity Besov spaces are embedded in the corresponding classical Besov spaces, which turns out to be true in the setting of best N-term approximation. Vice versa, also the opposite natural embedding of a classical Besov space of regularity $q+s$ into the hybrid regularity Besov space of isotropic regularity q and additional mixed regularity s will be proven.

The rest of this article is organised as follows: We introduce the multiscale hierarchy and state the requirements on the wavelets under consideration in Section 2. In Section 3, we define the function spaces used for the approximation with isotropic and tensor-product wavelets. This topic, together with a brief review of interpolation, is treated in Section 4. Afterwards, we compare the isotropic and hyperbolic approximation spaces in Section 5, and we state concluding remarks in Section 6.

Throughout this article, to avoid the repeated use of unspecified generic constants, we write $A \lesssim B$ if A is bounded by a uniform constant times B, where the constant does not depend on any parametres which A and B might depend on. Similarly, we write $A \gtrsim B$ if and only if $B \lesssim A$. Finally, if $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$.

2. Wavelet Bases

In this section, we define the wavelet bases under consideration and state their most important properties. Throughout the article, we assume that the scaling functions and the wavelets involved are compactly supported with support sizes depending on their levels. In particular, we assume that $\operatorname{diam}(\operatorname{supp} \phi_\lambda) \sim \operatorname{diam}(\operatorname{supp} \psi_\lambda) \sim 2^{-|\lambda|}$ holds for any one-dimensional scaling function ϕ_λ and wavelet ψ_λ on level $|\lambda|$. Moreover, we require that this property also holds for the dual scaling functions and the dual wavelets. The wavelets we are going to consider are, in general, biorthogonal and were first constructed in [4]. Later, this construction was also transferred to a finite interval in [7]. Alternatively, as shown in [5], it is also possible to use Daubechies wavelets [9, 10] on the interval. However, as for numerical applications there is a need for an efficient evaluation of the primal wavelets, we especially emphasise the constructions of [4, 7].

2.1. **Univariate Wavelet Bases.**

We consider a sequence of nested, finite-dimensional, and asymptotically dense function spaces

$$V_{j_0} \subseteq V_{j_0+1} \subseteq \ldots \subseteq V_{j-1} \subseteq V_j \subseteq V_{j+1} \ldots \subseteq V,$$

which are used to discretise a vector space V consisting of functions (or distributions) on the unit interval $[0, 1]$. Typically, $V = H^q([0, 1])$ is a Sobolev space with regularity q. We assume that the function spaces V_j can be generated by shifts and dyadic dilations of a scaling function ϕ, with possible modifications at the endpoints of the interval. Moreover, for a suitable index set Δ_j, we assume that

$$\Phi_j := \{\phi_\lambda : \lambda \in \Delta_j\}$$

is a basis set of V_j with $\|\phi_\lambda\|_{L^2([0,1])} \sim 1$ and $\operatorname{diam}(\operatorname{supp} \phi_\lambda) \sim 2^{-j}$. Here, the index $\lambda = (j, k)$ contains information of the level j and the location k, particularly meaning that $\phi_\lambda(x) = 2^{\frac{j}{2}} \phi(2^j x - k)$. In the easiest case, one can think of ϕ as the constant function 1, and of ϕ_λ as a properly scaled, dyadic indicator function, i.e., $\phi_\lambda = 2^{\frac{j}{2}} \mathbb{1}_{[2^{-j}k,\, 2^{-j}(k+1)]}$.

We say that the spaces V_j have the approximation order d if they contain locally all polynomials up to the order d. Moreover, V_j are said to have the regularity $\gamma := \sup\{s \in \mathbb{R} : V_j \subseteq H^s([0, 1])\}$.

2.1.1. *Multiscale Bases on* $[0, 1]$.

As the spaces V_j are nested, we may write

$$V_j = V_{j-1} \oplus W_j \tag{1}$$

with the complement or difference space W_j. One can show that, if the scaling function ϕ generates a shift-invariant space, cf. [4, 7], there exist mother wavelet functions ψ such that

$$\Psi_j := \{\psi_\lambda : \lambda \in \nabla_j\}$$

is a basis set of W_j. Also herein, ∇_j is a suitable index set and ψ_λ is a properly scaled and translated copy of a mother wavelet (again with possible modifications at the endpoints of the interval). For convenience, if $\lambda \in \Delta_j$ or $\lambda \in \nabla_j$, let us denote its level by $|\lambda| := j$. Furthermore, due to (1), both ϕ and all mother wavelets ψ can be obtained by translated copies of the refined function ϕ.

From (1), we recursively obtain the multiscale decomposition

$$V_j = V_{j_0} \oplus W_{j_0+1} \oplus \ldots \oplus W_j,$$

provided that $j_0 < j$. If we also define $W_{j_0} := V_{j_0}$, $\nabla_{j_0} := \Delta_{j_0}$, and $\psi_\lambda := \phi_\lambda$ for $\lambda \in j_0$, as the function spaces V_j are asymptotically dense, we conclude that the set

$$\Psi := \{\psi_\lambda : \lambda \in \nabla\}, \qquad \nabla := \bigcup_{j=j_0}^{\infty} \nabla_j, \tag{2}$$

spans a dense subset of V.

If we follow the construction of [4, 7], and use the appropriate scaling $\|\psi_\lambda\|_{L^2([0,1])} \sim 1$, the set Ψ forms a Riesz basis of $L^2([0,1])$, meaning that

$$\left\| \sum_{\lambda\in\nabla} c_\lambda \psi_\lambda \right\|_{L^2([0,1])}^2 \sim \sum_{\lambda\in\nabla} |c_\lambda|^2.$$

Hence, there exists a biorthogonal multiresolution analysis

$$\tilde{V}_{j_0} \subseteq \tilde{V}_{j_0+1} \subseteq \ldots \subseteq \tilde{V}_{j-1} \subseteq \tilde{V}_j \subseteq \tilde{V}_{j+1} \ldots \subseteq V'$$

which is also a Riesz basis for $L^2([0,1])$ and asymptotically dense in V'.

The spaces $\tilde{V}_j := \{\tilde{\phi}_\lambda : \lambda \in \Delta_j\}$ admit the regularity $\tilde{\gamma} > 0$ and the approximation order $\tilde{d}$. This fact provides us the number of vanishing moments of the wavelet ψ, which means that $\langle p, \psi\rangle = 0$ for any polynomial p up to the order $\tilde{d}$. Finally, there exists also a unique biorthogonal wavelet basis $\tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in \nabla\}$, satisfying

$$\langle \tilde{\psi}_{\lambda'}, \psi_\lambda \rangle = \delta_{\lambda,\lambda'}. \tag{3}$$

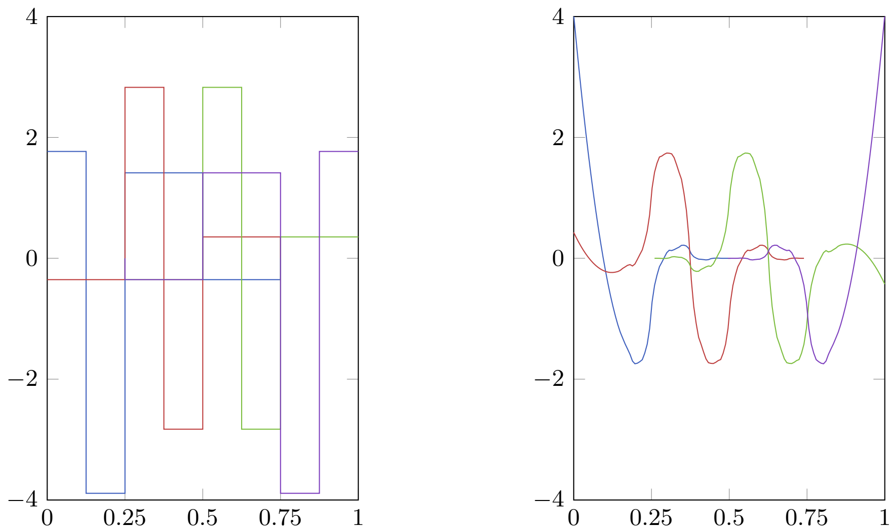

FIGURE 1. Primal wavelets (left) and dual wavelets (right) according to the construction in [7]. In this setting, we have $d = 1$, $\gamma = \frac{1}{2}$, and $\tilde{d} = 3$.

For the setting of piecewise constant wavelets with three vanishing moments $(d, \tilde{d}) = (1, 3)$ and piecewise linear wavelets with two vanishing moments $(d, \tilde{d}) = (2, 2)$, an illustration is provided in Figure 1 and Figure 2, respectively. Notice that in the latter figure, the primal wavelets depicted discretise functions with with zero boundary conditions at $x = 0$ and nonzero boundary conditions at $x = 1$.

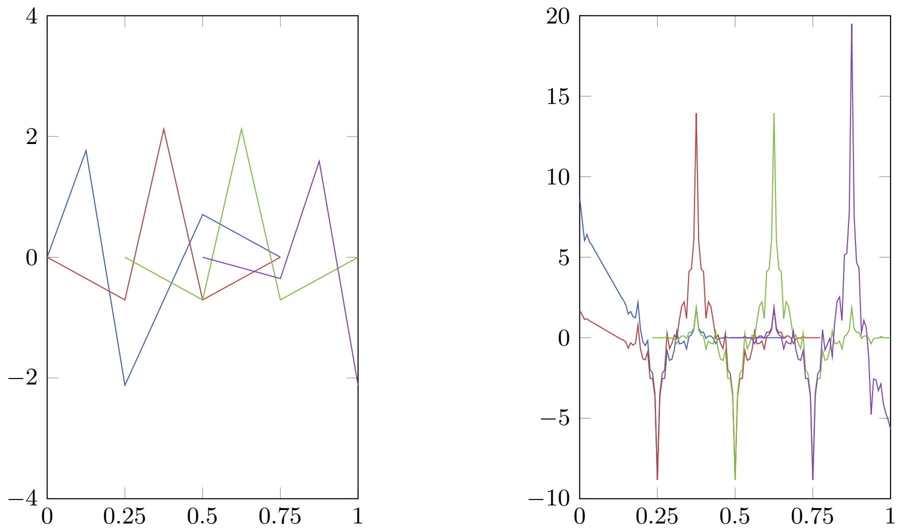

FIGURE 2. Primal wavelets (left) and dual wavelets (right) with complementary boundary conditions according to the construction in [8]. In this setting, we have $d = \tilde{d} = 2$, and $\gamma = \frac{3}{2}$. The primal wavelets satisfy zero boundary conditions at $x = 0$ and the dual ansatz functions at $x = 1$.

2.1.2. *Multiscale Transforms.* By considering the basis sets Φ_j, Ψ_j as row vectors, we may, due to the relation (1), write

$$\Phi_{j-1} = \Phi_j \mathbf{M}_{j,0}, \quad \Psi_j = \Phi_j \mathbf{M}_{j,1}, \tag{4}$$

$$\tilde{\Phi}_{j-1} = \tilde{\Phi}_j \tilde{\mathbf{M}}_{j,0}, \quad \tilde{\Psi}_j = \tilde{\Phi}_j \tilde{\mathbf{M}}_{j,1}. \tag{5}$$

Herein, the matrices $\mathbf{M}_{j,0}, \tilde{\mathbf{M}}_{j,0} \in \mathbb{R}^{|\Delta_j| \times |\Delta_{j-1}|}$ and $\mathbf{M}_{j,1}, \tilde{\mathbf{M}}_{j,1} \in \mathbb{R}^{|\Delta_j| \times |\nabla_j|}$ are called the *refinement masks*. By the local supports and the chosen scaling, the matrices $\left[\mathbf{M}_{j,0}, \mathbf{M}_{j,1}\right]$ and $\left[\tilde{\mathbf{M}}_{j,0}, \tilde{\mathbf{M}}_{j,1}\right]$ are uniformly stable.

If $u \in V_j$, we can write

$$u = \sum_{\lambda \in \Delta_j} c_\lambda \phi_\lambda = \sum_{\lambda \in \Delta_{j-1}} c_\lambda \phi_\lambda + \sum_{\lambda \in \nabla_j} d_\lambda \psi_\lambda.$$

In view of (4), this is equivalent to

$$u = \Phi_j \mathbf{c}_j = \left[\Phi_{j-1}, \Psi_j\right] \begin{bmatrix} \mathbf{c}_{j-1} \\ \mathbf{d}_j \end{bmatrix} = \Phi_j \left[\mathbf{M}_{j,0}, \mathbf{M}_{j,1}\right] \begin{bmatrix} \mathbf{c}_{j-1} \\ \mathbf{d}_j \end{bmatrix},$$

where $\mathbf{c}_j := [c_\lambda]_{\lambda \in \Delta_j}$, $\mathbf{d}_j := [d_\lambda]_{\lambda \in \nabla_j}$. On the other hand, remarking that $\mathbf{c}_j = \langle \tilde{\Phi}_j, u \rangle$ and $\mathbf{d}_j = \langle \tilde{\Psi}_j, u \rangle$, we obtain that

$$\begin{bmatrix} \mathbf{c}_{j-1} \\ \mathbf{d}_j \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{M}}_{j,0}^\intercal \\ \tilde{\mathbf{M}}_{j,1}^\intercal \end{bmatrix} \mathbf{c}_j.$$

Hence, we conclude that $[\tilde{\mathbf{M}}_{j,0}, \tilde{\mathbf{M}}_{j,1}]^\intercal [\mathbf{M}_{j,0}, \mathbf{M}_{j,1}] = \mathbf{I}$, and therefore

$$\tilde{\mathbf{M}}_{j,0}^\intercal \mathbf{M}_{j,0} = \mathbf{I}, \qquad \tilde{\mathbf{M}}_{j,1}^\intercal \mathbf{M}_{j,1} = \mathbf{I},$$

$$\tilde{\mathbf{M}}_{j,0}^\intercal \mathbf{M}_{j,1} = \mathbf{0}, \qquad \tilde{\mathbf{M}}_{j,1}^\intercal \mathbf{M}_{j,0} = \mathbf{0}.$$

By applying the multiscale transform recursively, there follows

$$\big[\Phi_{j_0}, \Psi_{j_0+1}, \dots, \Psi_j\big] = \Phi_j \mathbf{T}_j := \Phi_j \prod_{\ell=j_0+1}^{j} \begin{bmatrix} [\mathbf{M}_{\ell,0}, \mathbf{M}_{\ell,1}] & \\ & \mathbf{I}_{|\Delta_j|-|\Delta_\ell|} \end{bmatrix}, \qquad (6)$$

and accordingly, on the dual side,

$$\begin{bmatrix} \mathbf{c}_{j_0} \\ \mathbf{d}_{j_0+1} \\ \vdots \\ \mathbf{d}_j \end{bmatrix} = \tilde{\mathbf{T}}_j^\intercal \mathbf{c}_j := \prod_{\ell=j_0+1}^{j} \begin{bmatrix} \begin{bmatrix} \tilde{\mathbf{M}}_{\ell,0}^\intercal \\ \tilde{\mathbf{M}}_{\ell,1}^\intercal \end{bmatrix} & \\ & \mathbf{I}_{|\Delta_j|-|\Delta_\ell|} \end{bmatrix} \mathbf{c}_j. \qquad (7)$$

2.2. **Multivariate Wavelet Bases.** In this subsection, we will briefly review the construction of multivariate wavelet bases on the unit cube $\square = [0,1]^n$. There are two well-established ways to create wavelet bases on $\square$. On the one hand, one can construct *isotropic* wavelet bases, i.e., bases consisting of functions whose supports are shape-regular cuboids. On the other hand, one can use the tensor product of wavelets on different levels, resulting in so-called *anisotropic* or *hyperbolic* wavelet bases. In both settings, we also start with a multiresolution analysis where on each level j, the n-fold tensor product of the spaces V_j defined in Section 2.1 is involved.

2.2.1. *Isotropic Wavelet Bases.* Let us first address the bivariate case. It is trivial to deduce that for any $j \geq j_0$, the basis set $\Phi_j \otimes \Phi_j$ discretises the space $V_j \otimes V_j$. As for $j \geq j_0 + 1$ we also have

$$V_j \otimes V_j = \big(V_{j-1} \otimes V_{j-1}\big) \oplus \big(V_{j-1} \otimes W_j\big) \oplus \big(W_j \otimes V_{j-1}\big) \oplus \big(W_j \otimes W_j\big),$$

we can define

$$\Theta_j := \Theta_j^{(0,1)} \cup \Theta_j^{(1,0)} \cup \Theta_j^{(1,1)} := \big(\Phi_{j-1} \otimes \Psi_j\big) \cup \big(\Psi_j \otimes \Phi_{j-1}\big) \cup \big(\Psi_j \otimes \Psi_j\big)$$

as a basis of the complement space $(V_j \otimes V_j) \ominus (V_{j-1} \otimes V_{j-1})$.

This procedure can be generalised to the n-variate case as well by defining

$$\Theta_j^{\mathbf{e}} := \Theta_j^{e_1} \otimes \cdots \otimes \Theta_j^{e_n}, \qquad \Theta_j^{e} = \begin{cases} \Phi_{j-1}, & e = 0, \\ \Psi_j, & e = 1, \end{cases}$$

by which the basis set

$$\Theta_j := \bigcup_{\mathbf{e} \in \{0,1\}^n \setminus \{\mathbf{0}\}} \Theta_j^{\mathbf{e}}$$

spans the complement space. If we define $\Theta_{j_0} := \Phi_{j_0} \otimes \ldots \otimes \Phi_{j_0}$, then there holds

$$\operatorname{span}\left\{ \bigcup_{j=j_0}^{m} \Theta_j \right\} = V_m \otimes \cdots \otimes V_m,$$

and these sets form a Riesz basis of $L^2(\square)$ for $m \to \infty$.

For convenience, let us also define the index set $\nabla_j^{\mathbf{e}}$ as the set of indices of wavelets $\theta_\mu \in \Theta_j^{\mathbf{e}}$, as well as

$$\nabla_j := \bigcup_{\mathbf{e} \in \{0,1\}^n \setminus \{\mathbf{0}\}} \nabla_j^{\mathbf{e}}, \qquad \nabla := \bigcup_{j \geq j_0} \nabla_j,$$

with the canonical adaptation for ∇_{j_0}. Finally, similar to the univariate case, we define the level of an index as $|\mu| := j$ for any $\mu \in \nabla_j$.

2.2.2. *Tensor-Product Wavelet Bases.* Another approach is to use the tensor product of one-dimensional wavelets on all the different levels. This approach is straightforward: for a given multiindex $\mathbf{j} \geq \mathbf{j}_0+\mathbf{1}$, we define the index set $\boldsymbol{\nabla}_{\mathbf{j}} := \nabla_{j_1}\times\cdots\times\nabla_{j_n}$, and for $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n) \in \boldsymbol{\nabla}_{\mathbf{j}}$, we define the wavelet function

$$\psi_{\boldsymbol{\lambda}}(\boldsymbol{x}) := \big(\psi_{\lambda_1}\otimes\cdots\otimes\psi_{\lambda_n}\big)(\boldsymbol{x}) = \psi_{\lambda_1}(x_1)\ldots\psi_{\lambda_n}(x_n).$$

All such wavelets on a level $\mathbf{j}$ span the corresponding complement space, i.e.,

$$\Psi_{\mathbf{j}} := \big\{\psi_{\boldsymbol{\lambda}} : \boldsymbol{\lambda}\in\boldsymbol{\nabla}_{\mathbf{j}}\big\}$$

is a basis set of $W_{j_1}\otimes\cdots\otimes W_{j_n}$. If we also re-define $\psi_\lambda := \phi_\lambda$ for $\lambda\in\nabla_{j_0}$ as it was done in the univariate case, and extend the above definition to any multiindex $\mathbf{j}\geq\mathbf{j}_0$, then we deduce out of (2) that the span of

$$\Psi := \{\psi_{\boldsymbol{\lambda}} : \boldsymbol{\lambda}\in\boldsymbol{\nabla}\} = \bigcup_{\mathbf{j}\geq\mathbf{j}_0}\Psi_{\mathbf{j}}, \quad \boldsymbol{\nabla} := \bigcup_{\mathbf{j}\geq\mathbf{j}_0}\boldsymbol{\nabla}_{\mathbf{j}},$$

is dense in $V\otimes\cdots\otimes V$ and forms a Riesz basis.

2.3. **Auxillary Results.**

In this section, we state and prove three lemmata. When comparing the different function spaces in Section 5.2, they will be crucial.

Lemma 2.1. *Let $0<p\leq 1$ and $\mathbf{A}\in\mathbb{R}^{m\times n}$. Then, there holds*

$$\|\mathbf{A}\|_p \leq \big\|\mathbf{A}^{\odot p}\big\|_1^{\frac{1}{p}} = \left[\max_{1\leq j\leq n}\sum_{i=1}^m \big|a_{i,j}\big|^p\right]^{\frac{1}{p}}, \tag{8}$$

where $\mathbf{A}^{\odot p}$ is the component-wise p-th power of $\mathbf{A}$.

Proof. Let $\mathbf{u}\in\mathbb{R}^n$. In view of the subadditivity

$$|x+y|^p \leq |x|^p+|y|^p, \quad 0<p\leq 1, \tag{9}$$

we conclude

$$\begin{aligned}\|\mathbf{A}\mathbf{u}\|_p^p &= \sum_{i=1}^m \big|[\mathbf{A}\mathbf{u}]_i\big|^p = \sum_{i=1}^m \left|\sum_{j=1}^n a_{i,j}u_j\right|^p \\ &\leq \sum_{i=1}^m\sum_{j=1}^n \big|a_{i,j}\big|^p\big|u_j\big|^p \\ &\leq \|\mathbf{u}\|_p^p \max_{1\leq j\leq n}\sum_{i=1}^m\big|a_{i,j}\big|^p.\end{aligned}$$

This yields (8). □

Remark 2.2. *The subadditivity* (9) *implies that the function $\|\cdot\|_p^p$ is subadditive. Hence, the quasi-triangle inequality*

$$\|\mathbf{u}+\mathbf{v}\|_p \leq 2^{\frac{1}{p}}\big(\|\mathbf{u}\|_p+\|\mathbf{v}\|_p\big)$$

follows, cf. [12].

Lemma 2.3. *Let $m\geq j_0$ and consider the wavelet transforms $\mathbf{T}_m$ and $\tilde{\mathbf{T}}_m$ given by* (6) *and* (7)*, respectively. Then, for $0<p\leq 2$, there holds*

$$\|\mathbf{T}_m^\intercal\|_p, \|\tilde{\mathbf{T}}_m^\intercal\|_p \lesssim 1, \qquad \|\mathbf{T}_m\|_p, \|\tilde{\mathbf{T}}_m\|_p \lesssim 2^{m(\frac{1}{p}-\frac{1}{2})}.$$

Proof. We will show the theorem on the primal side only, as for the dual side, the same arguments can be used. First, we remark that, for $p = 2$, both statements are true since both Φ and Ψ are uniformly stable bases of $L^2([0,1])$.

Let us consider the first inequality. We will start by showing this for $p = 1$, from which the statement follows for all $1 \leq p \leq 2$ by interpolation. In view of (6), for each $\lambda \in \bigcup_{j=j_0}^{m} \nabla_j$, there holds

$$\psi_\lambda = \sum_{\mu \in \Delta_m} t_{\mu,\lambda} \phi_\mu. \tag{10}$$

Therefore, to estimate

$$\|\mathbf{T}_m^\intercal\|_1 = \|\mathbf{T}_m\|_\infty = \max_{\mu \in \Delta_m} \sum_{j=j_0}^{m} \sum_{\lambda \in \nabla_j} |t_{\mu,\lambda}|,$$

we fix $\mu \in \Delta_m$. Since ϕ_μ is compactly supported, on each level $j_0 \leq j \leq m$, there are at most $\mathcal{O}(1)$ wavelets ψ_λ with $\operatorname{supp} \psi_\lambda \cap \operatorname{supp} \phi_\mu \neq \emptyset$, by which

$$\|\mathbf{T}_m\|_\infty \lesssim \max_{\mu \in \Delta_m} \sum_{j=j_0}^{m} \max_{\lambda \in \nabla_j} |t_{\mu,\lambda}|.$$

Since $\|\phi_\mu\|_{L^\infty} \sim 2^{\frac{m}{2}} \sim 2^{\frac{m-j}{2}} \|\psi_\lambda\|_{L^\infty}$, we conclude that for any $\lambda \in \nabla_j$, there must hold

$$|t_{\mu,\lambda}| \lesssim 2^{\frac{j-m}{2}}, \tag{11}$$

by which

$$\|\mathbf{T}_m\|_\infty \lesssim \sum_{j=j_0}^{m} 2^{\frac{j-m}{2}} \lesssim 1.$$

Let us finally treat the case $0 < p < 1$. By Lemma 2.1 and the arguments above, we have

$$\|\mathbf{T}_m^\intercal\|_p^p \leq \max_{\mu \in \Delta_m} \sum_{j=j_0}^{m-1} \sum_{\lambda \in \nabla_j} \left|t_{\mu,\lambda}\right|^p \lesssim \sum_{j=j_0}^{m} 2^{p\frac{j-m}{2}} \lesssim 1,$$

by which $\|\mathbf{T}_m^\intercal\|_p$ is bounded independently from m as well. This proves the first inequality.

To show the second inequality, we will first show that $\|\mathbf{T}_m\|_1 \lesssim 2^{\frac{m}{2}}$. If this holds true, then for $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2}$, i.e., $1 - \theta = \frac{2}{p} - 1$, we can conclude by interpolation, that

$$\|\mathbf{T}_m\|_p \leq \|\mathbf{T}_m\|_1^{1-\theta} \|\mathbf{T}_m\|_2^{\theta} \lesssim 2^{\frac{m}{2}(\frac{2}{p}-1)} \cdot 1^\theta = 2^{m(\frac{1}{p}-\frac{1}{2})},$$

provided that $1 \leq p \leq 2$.

Let $j_0 \leq j \leq m$ and $\lambda \in \nabla_j$. As the wavelet ψ_λ satisfies $\operatorname{diam}(\operatorname{supp} \psi_\lambda) \lesssim 2^{-j}$, and $\operatorname{diam}(\operatorname{supp} \phi_\mu) \lesssim 2^{-m}$, there are at most $\mathcal{O}(2^{m-j})$ nontrivial coefficients $t_{\mu,\lambda}$ in (10). Hence, by (11), there holds

$$\sum_{\mu \in \Delta_m} |t_{\mu,\lambda}| \lesssim 2^{m-j} \cdot 2^{\frac{j-m}{2}} = 2^{\frac{m-j}{2}} \leq 2^{\frac{m}{2}},$$

by which also $\|\mathbf{T}_m\|_1 \lesssim 2^{\frac{m}{2}}$.

Finally, for $0 < p < 1$, we again make use Lemma 2.1, which implies that

$$\|\mathbf{T}_m\|_p^p \leq \max_{j_0 \leq j \leq m} \max_{\lambda \in \nabla_j} \sum_{\mu \in \Delta_m} |t_{\mu,\lambda}|^p \lesssim \max_{j_0 \leq j \leq m} 2^{m-j} 2^{p\frac{j-m}{2}} = 2^{m(1-\frac{p}{2})}.$$

After taking the p-th root, we arrive at $\|\mathbf{T}_m\|_p \lesssim 2^{m(\frac{1}{p}-\frac{1}{2})}$, which is what we wanted to show. □

Remark 2.4. *In the proof of Lemma 2.3, the crucial step is exploiting the compact support of the wavelets and their duals. As already stated, such wavelets exist and were constructed on the real line and adapted to the interval, cf.* [5, 9, 10] *for orthonormal wavelets and* [4, 7] *for the biorthogonal spline wavelets, which are numerically more feasible.*

On the other hand, the criterion of the compact support may be relaxed: Indeed, what we reqire is that the row- and column sums (if $p < 1$ for the p-th power) of the matrix can be controlled sufficiently well.

In Section 5.2, we will need to estimate of Kronecker products of matrices. Such estimates are derived in [17, 20, 22], for example. However, in the special case we are interested in, the proof is rather elementary, so we provide it for the reader's convenience.

Lemma 2.5. *Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{k\times \ell}$. For $p \in \{1, 2, \infty\}$, there holds*

$$\|\mathbf{A} \otimes \mathbf{B}\|_p = \|\mathbf{A}\|_p \|\mathbf{B}\|_p.$$

Proof. For the $\|\cdot\|_2$-norm, the claim follows from the fact that the eigenvalues of $\mathbf{A} \otimes \mathbf{B}$ are given by the products of any two eigenvalues of $\mathbf{A}$ and $\mathbf{B}$.

For the $\|\cdot\|_\infty$-norm, there holds

$$\begin{aligned}
\|\mathbf{A} \otimes \mathbf{B}\|_\infty &= \max_{1\le r_1\le n} \max_{1\le r_2\le \ell} \sum_{t_1=1}^{m} \sum_{t_2=1}^{k} \left|a_{r_1,t_1} b_{r_2,t_2}\right| \\
&= \left[\max_{1\le r_1\le n} \sum_{t_1=1}^{m} \left|a_{r_1,t_1}\right|\right] \left[\max_{1\le r_2\le \ell} \sum_{t_2=1}^{k} \left|b_{r_2,t_2}\right|\right] \\
&= \|\mathbf{A}\|_\infty \|\mathbf{B}\|_\infty.
\end{aligned}$$

For the $\|\cdot\|_1$-norm, the claim follows with exactly the same arguments. □

3. Function Spaces

The goal of this section is to characterise the function spaces used in the setting of best N-term approximation. We will see that these spaces can be characterised by isotropic or tensor-product wavelets, or in the case of Sobolev spaces, by both of them. For the sake of convenience, for $\boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\mathbf{j}}$, we shall denote in the following

$$|\boldsymbol{\lambda}|_1 := |\mathbf{j}|_1, \qquad |\boldsymbol{\lambda}|_\infty := |\mathbf{j}|_\infty.$$

3.1. Sobolev Spaces.

In Section 2, we have assumed that all θ_μ and $\psi_{\boldsymbol{\lambda}}$ are scaled such that they are normalised in $L^2(\square)$. However, if we define

$$\begin{aligned}
\psi_{\boldsymbol{\lambda}}^{(p)} &:= 2^{|\boldsymbol{\lambda}|_1\left(\frac{1}{p}-\frac{1}{2}\right)}\psi_{\boldsymbol{\lambda}}, & \tilde{\psi}_{\boldsymbol{\lambda}}^{(p)} &:= 2^{|\boldsymbol{\lambda}|_1\left(\frac{1}{2}-\frac{1}{p}\right)}\tilde{\psi}_{\boldsymbol{\lambda}}, \\
\theta_\mu^{(p)} &:= 2^{n|\mu|\left(\frac{1}{p}-\frac{1}{2}\right)}\theta_\mu, & \tilde{\theta}_\mu^{(p)} &:= 2^{n|\mu|\left(\frac{1}{2}-\frac{1}{p}\right)}\tilde{\theta}_\mu,
\end{aligned}$$

we get a primal basis which is normalised in $L^p(\square)$, and a dual basis which is normalised in $L^{p'}(\square)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. With the above notation at hand, every function $u \in L^p(\square)$ admits unique expansions

$$\begin{aligned}
u &= \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} u_{\boldsymbol{\lambda}}^{(p)} \psi_{\boldsymbol{\lambda}}^{(p)}, & u_{\boldsymbol{\lambda}}^{(p)} &= \langle \tilde{\psi}_{\boldsymbol{\lambda}}^{(p)}, u\rangle, \\
u &= \sum_{\mu\in\bigcirc} u_\mu^{(p)} \theta_\mu^{(p)}, & u_\mu^{(p)} &= \langle \tilde{\theta}_\mu^{(p)}, u\rangle.
\end{aligned}$$

To keep the notation simple, we always assume that a coefficient $u_{\boldsymbol{\lambda}}$ or u_μ without suffix p corresponds to an $L^2(\square)$-normalised expansion of u.

We intend next to characterise certain function spaces. It is well established, see e.g. [6, 26], that univariate wavelet coefficients of a function characterise the norm of this function with respect to a range of function spaces. However, in a multivariate setting, we can also use a tensor-product wavelet basis to characterise isotropic function spaces, i.e., there holds

$$\|u\|_{H^s(\square)}^2 \sim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2s|\boldsymbol{\lambda}|_\infty} \big|\langle\tilde{\psi}_{\boldsymbol{\lambda}}, u\rangle\big|^2 \sim \sum_{\mu\in\bigcirc} 2^{2s|\mu|}\big|\langle\tilde{\theta}_\mu, u\rangle\big|^2, \quad -\tilde{\gamma} < s < \gamma, \tag{12}$$

$$\|u\|_{H^t(\square)}^2 \sim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2t|\boldsymbol{\lambda}|_\infty} \big|\langle u, \psi_{\boldsymbol{\lambda}}\rangle\big|^2 \sim \sum_{\mu\in\bigcirc} 2^{2s|\mu|}\big\langle u, \theta_\mu\rangle\big|^2, \quad -\gamma < t < \tilde{\gamma}, \tag{13}$$

cf. [1, 6, 16, 19, 25, 26]. In particular, a function u is contained in an isotropic Sovolev space if the coefficients of its (tensor-product) wavelet expansion decay sufficiently fast. Note that we have slightly abused the notation here, as the primal and dual wavelets may characterise function spaces with complementary boundary conditions.

On the other hand, in [16], there was also shown that the coeffcients with respect to a tensor-product basis characterise the dominating mixed regularity of a function u. Shortly after, in [14], Sobolev spaces of *hybrid regularity* were introduced. Roughly speaking, these spaces $\mathfrak{H}^{q,s}$, which are sometimes also called *Griebel-Knapek spaces*, contain all functions u which, with respect to $H^q(\square)$, admit mixed derivatives up to the order s. In terms of tensor products, we define these spaces in the following way.

Definition 3.1. We define $\mathfrak{H}^{q,s}(\square)$ for $q \geq 0$ and $s \in \mathbb{R}$ as the space

$$\mathfrak{H}^{q,s}(\square) := \bigcap_{i=1}^{n} \bigotimes_{j=1}^{n} H^{s+\delta_{i,j}q}([0,1]). \tag{14}$$

For $q < 0$, we define the space $\mathfrak{H}^{q,s}(\square)$ via the duality $\mathfrak{H}^{q,s}(\square) = \big(\mathfrak{H}^{-q,-s}(\square)\big)'$.

Remark 3.2. *In the two-dimensional setup,* (14) *simply means that*

$$\mathfrak{H}^{q,s}(\square) = \big(H^{q+s}([0,1]) \otimes H^s([0,1])\big) \cap \big(H^s([0,1]) \otimes H^{q+s}([0,1])\big).$$

By using the norm equivalences (12) and (13), we see that we can also characterise the spaces $\mathfrak{H}^{q,s}(\square)$ in terms of wavelet coefficients.

Theorem 3.3. *There holds*

$$\|u\|_{\mathfrak{H}^{q,s}(\square)}^2 \sim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2q|\boldsymbol{\lambda}|_\infty+2s|\boldsymbol{\lambda}|_1} \big|\langle\tilde{\psi}_{\boldsymbol{\lambda}}, u\rangle\big|^2, \quad -\tilde{\gamma} < s, q+s < \gamma, \tag{15}$$

$$\|u\|_{\mathfrak{H}^{q,s}(\square)}^2 \sim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2q|\boldsymbol{\lambda}|_\infty+2s|\boldsymbol{\lambda}|_1} \big|\langle u, \psi_{\boldsymbol{\lambda}}\rangle\big|^2, \quad -\gamma < s, q+s < \tilde{\gamma}. \tag{16}$$

Proof. Let $q \geq 0$ and $s \in \mathbb{R}$. For the sake of simplicity, we fix the dimension to $n = 2$, but we emphasise that the same arguments also work in higher dimensions.

We will basically follow the arguments that were used in [19] for classical Sobolev spaces. By standard tensor-product arguments, for $u = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} u_{\boldsymbol{\lambda}}\psi_{\boldsymbol{\lambda}}$, there holds with (12), and $u_{\boldsymbol{\lambda}} := \langle\tilde{\psi}_{\boldsymbol{\lambda}}, u\rangle$,

$$\|u\|_{H^{q+s}([0,1])\otimes H^s([0,1])}^2 \sim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2(q+s)|\lambda_{\mathrm{x}}|+2s|\lambda_{\mathrm{y}}|}|u_{\boldsymbol{\lambda}}|^2,$$

$$\|u\|_{H^s([0,1])\otimes H^{q+s}([0,1])}^2 \sim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2s|\lambda_{\mathrm{x}}|+2(q+s)|\lambda_{\mathrm{y}}|}|u_{\boldsymbol{\lambda}}|^2,$$

provided that $-\tilde{\gamma} < s, q+s < \gamma$. Therfore, as $q+s \geq s$, there holds

$$2^{2(q+s)|\lambda_{\rm x}|+2s|\lambda_{\rm y}|} + 2^{2s|\lambda_{\rm x}|+2(q+s)|\lambda_{\rm y}|} \sim 2^{2(q+s)|\boldsymbol{\lambda}|_\infty + 2s\min\{|\lambda_{\rm x}|,|\lambda_{\rm y}|\}} = 2^{2q|\boldsymbol{\lambda}|_\infty + 2s|\boldsymbol{\lambda}|_1}.$$

Hence, (15) follows for $q \geq 0$. With exactly the same arguments, we can also show (16) for $q \geq 0$.

For $q < 0$, we use a duality argument to show both asymptotic inequalities. By duality, there holds

$$\|u\|_{\mathfrak{H}^{q,s}(\square)} = \sup_{\|v\|_{\mathfrak{H}^{-q,-s}(\square)}=1} \langle u, v\rangle.$$

Hence, writing $v = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \tilde{v}_{\boldsymbol{\lambda}}\tilde{\psi}_{\boldsymbol{\lambda}}$, we obtain by the biorthogonality that

$$\begin{aligned}
\|u\|_{\mathfrak{H}^{q,s}(\square)} &= \sup_{\|v\|_{\mathfrak{H}^{-q,-s}(\square)}=1} \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} u_{\boldsymbol{\lambda}}\tilde{v}_{\boldsymbol{\lambda}} \\
&\leq \left[\sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2q|\boldsymbol{\lambda}|_\infty+2s|\boldsymbol{\lambda}|_1}|u_{\boldsymbol{\lambda}}|^2\right]^{\frac{1}{2}} \sup_{\|v\|_{\mathfrak{H}^{-q,-s}(\square)}=1} \left[\sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{-2q|\boldsymbol{\lambda}|_\infty-2s|\boldsymbol{\lambda}|_1}|\tilde{v}_{\boldsymbol{\lambda}}|^2\right]^{\frac{1}{2}} \\
&\sim \left[\sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2q|\boldsymbol{\lambda}|_\infty+2s|\boldsymbol{\lambda}|_1}|u_{\boldsymbol{\lambda}}|^2\right]^{\frac{1}{2}},
\end{aligned}$$

where we have used (16) for $-q \geq 0$.

For the lower bound, we remark that there holds

$$\begin{aligned}
&\left[\sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2q|\boldsymbol{\lambda}|_\infty+2s|\boldsymbol{\lambda}|_1}|u_{\boldsymbol{\lambda}}|^2\right]^{\frac{1}{2}} \\
&\quad= \sup_{\|[2^{-q|\boldsymbol{\lambda}|_\infty-s|\boldsymbol{\lambda}|_1}\tilde{v}_{\boldsymbol{\lambda}}]_{\boldsymbol{\lambda}}\|_{\ell^2(\boldsymbol{\nabla})}=1} \langle \mathbf{u}, \tilde{\mathbf{v}}\rangle_{\ell^2(\boldsymbol{\nabla})} \\
&\quad= \sup_{\|[2^{-q|\boldsymbol{\lambda}|_\infty-s|\boldsymbol{\lambda}|_1}\tilde{v}_{\boldsymbol{\lambda}}]_{\boldsymbol{\lambda}}\|_{\ell^2(\boldsymbol{\nabla})}=1} \left\langle u, \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \tilde{v}_{\boldsymbol{\lambda}}\tilde{\psi}_{\boldsymbol{\lambda}}\right\rangle \\
&\quad\leq \|u\|_{\mathfrak{H}^{q,s}(\square)} \sup_{\|[2^{-q|\boldsymbol{\lambda}|_\infty-s|\boldsymbol{\lambda}|_1}\tilde{v}_{\boldsymbol{\lambda}}]_{\boldsymbol{\lambda}}\|_{\ell^2(\boldsymbol{\nabla})}=1} \left\|\sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \tilde{v}_{\boldsymbol{\lambda}}\tilde{\psi}_{\boldsymbol{\lambda}}\right\|_{\mathfrak{H}^{-q,-s}(\square)} \\
&\quad\sim \|u\|_{\mathfrak{H}^{q,s}(\square)}.
\end{aligned}$$

Note that we have again used (16) for $-q \geq 0$. This shows (15).

The same arguments also allow us to show (16) for $q < 0$, completing the proof of this theorem. □

Remark 3.4. *For a smaller range of parameters, the statement of Theorem 3.3 has already been established in* [15]*. Moreover, we note that the one-sided upper bound in* (15) *can be extended for $-\tilde{d} < s, q+s < \gamma$, whereas the one-sided lower bound can be extended to $-\tilde{\gamma} < s, q+s < d$. Similarly, we can also extend the upper and lower bounds in* (16) *up to $-d$ and $\tilde{d}$, respectively.*

We note that there holds $\theta_\mu, \psi_{\boldsymbol{\lambda}} \in \mathfrak{H}^{0,\gamma}(\square) \subseteq \mathfrak{H}^{\gamma,0}(\square) = H^\gamma(\square)$ by the tensor product structure. With the above norm equivalences, it is easy to see that $\mathfrak{H}^{0,\gamma}(\square)$ is continuously embedded into $\mathfrak{H}^{q,s}(\square)$ if either $q \geq 0$ and $s < \gamma - q$, or if $q < 0$ and $s < \gamma$, see also [3]. Furthermore, we can derive the following proposition.

Proposition 3.5. *The spaces $\mathfrak{H}^{q,s}(\square)$ are Hilbert spaces when equipped with the inner product*

$$\langle u, v\rangle_{\mathfrak{H}^{q,s}(\square)} := \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} 2^{2q|\boldsymbol{\lambda}|_\infty+2s|\boldsymbol{\lambda}|_1}\langle\tilde{\psi}_{\boldsymbol{\lambda}}, u\rangle\langle\tilde{\psi}_{\boldsymbol{\lambda}}, v\rangle.$$

Moreover, we have the Gelfand triples $\mathfrak{H}^{q,s}(\square) \hookrightarrow H^q(\square) \hookrightarrow \mathfrak{H}^{q,-s}(\square)$ for $s \geq 0$.

3.2. **Besov Spaces.** When dealing with best N-term approximation, one has to consider Besov spaces, which are defined by three indices. Primarily, we have the regularity α and the integrability p, and secondarily, a fine index τ.

Definition 3.6. For $\alpha > 0$ and $0 < p, \tau < \infty$, we define the quantity

$$|\mathbf{u}|_{\mathbf{b}^{\alpha,p}_\tau} := \left[\sum_{m\geq j_0} 2^{\tau m\alpha}\left[\sum_{\mu\in\bigcirc_m}|u_\mu|^p\right]^{\frac{\tau}{p}}\right]^{\frac{1}{\tau}}, \tag{17}$$

and $\mathbf{b}^{\alpha,p}_\tau$ as the space containing all vectors for which the above quantity (17) is finite. For $\max\{p,\tau\} = \infty$, we apply the usual modifications. Then, the *Besov space* $B^{\alpha,p}_\tau(\square)$ is defined as the space containing all functions $u = \Theta\mathbf{u}$, for which the norm

$$\|u\|_{B^{\alpha,p}_\tau(\square)} := \|u\|_{L^p(\square)} + |\mathbf{u}^{(p)}|_{\mathbf{b}^{\alpha,p}_\tau}$$

is finite.

Remark 3.7. *Classically, the Besov spaces $B^{\alpha,p}_\tau(\square)$ are defined by the decay behaviour of a function's moduli of continuity. However, if the wavelets involved satisfy*

(1) *$\theta_\mu \in B^{\beta,p}_\tau(\square)$ for some $\beta > \alpha$ and*
(2) *θ_μ has $r > \max\{\alpha, n(1/p-1)\}$ vanishing moments,*

then the classical Besov seminorm and the discrete Besov seminorm (17) *are equivalent, cf.* [11] *and the references therein. Besides, the Besov spaces can also be characterised by a Littlewood-Paley decomposition, cf.* [29].

As we will see, the classical Besov spaces and also the Besov spaces of dominating mixed smoothness will not be the right spaces for the approximation with respect to a tensor-product basis in an isotropic energy space. For this, we need to consider Besov spaces of hybrid regularity as introduced in [3], which are characterised by the decay of the wavelet coefficients.

Definition 3.8. For given $q \geq 0$, $s > 0$, and $0 < p, \tau < \infty$, we define

$$|\mathbf{u}|_{\mathfrak{b}^{q,s,p}_\tau} := \left[\sum_{\mathbf{j}\geq\mathbf{j}_0} 2^{\tau(q|\mathbf{j}|_\infty+s|\mathbf{j}|_1)}\left[\sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}_\mathbf{j}}|u_{\boldsymbol{\lambda}}|^p\right]^{\frac{\tau}{p}}\right]^{\frac{1}{\tau}}, \tag{18}$$

with the usual modifications in the case $\max\{p,\tau\} = \infty$, and the space $\mathfrak{b}^{q,s,p}_\tau$ as the space containing all vectors for which the quantity (18) is finite. The *Besov space of hybrid regularity* $\mathfrak{B}^{q,s,p}_\tau(\square)$ is defined as the space containing all functions $u = \Psi\mathbf{u}$, for which the norm

$$\|u\|_{\mathfrak{B}^{q,s,p}_\tau(\square)} := \|u\|_{L^p(\square)} + |\mathbf{u}^{(p)}|_{\mathfrak{b}^{q,s,p}_\tau} \tag{19}$$

is finite.

Remark 3.9. *The above definitions are given for nonnegative α, q, and s only, since for $\alpha < 0$, $B^{\alpha,p}_\tau(\square)$ does not need to be included in $L^p(\square)$, and likewise in the hybrid case. Nevertheless, an extension of the seminorms* (17) *and* (18) *to negative α, q, and s, respectively, is straightforward.*

Remark 3.10. *At the first glance, the expression* (18) *looks different from that in* [3]*. This can, however, be explained by the fact that the wavelet bases in* [3] *are normalised in* $L^\infty(\square)$*. Moreover, if* $p = \tau = 2$*, then we have the identities*

$$\mathfrak{B}_2^{q,s,2}(\square) = \mathfrak{H}^{q,s}(\square), \quad B_2^{\alpha,2}(\square) = H^\alpha(\square).$$

Indeed, this can be immediately concluded from (17) and (12), or (18) and Theorem 3.3, respectively. In particular, we have extended the norm equivalences from [14, 15] to negative q, and we have shown that in the case $p = \tau = 2$, the hybrid regularity Besov spaces from [3] agree with the hybrid regularity Sobolev spaces.

4. Approximation and Interpolation

The aim of this section is to briefly summarise the interpolation between vector spaces and the N-term approximation, which is a kind of nonlinear approximation. Our goal is to characterise the approximation spaces for the anisotropic tensor-product wavelet basis. All the results provided here can also be found in [11].

4.1. **Interpolation.** First, we want to specify the abstract interpolation between normed vector spaces by means of the K-functional. We consider two normed vector spaces, $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$, and we assume that W is continuously embedded in V, that is, $\|\cdot\|_V \lesssim \|\cdot\|_W$. Moreover, let $|\cdot|_W$ be a seminorm on W. Then, for $t > 0$, we define the K-functional by

$$K(u, t) := \inf_{w \in W} \|u - w\|_V + t|w|_W.$$

For $\theta \in (0, 1)$, and $\tau \in (0, \infty]$, we define

$$|u|_{(V,W)_{\theta,\tau}} := \left(\int_0^\infty \frac{1}{t} \left[t^{-\theta} K(u, t) \right]^\tau \mathrm{d}t \right)^{\frac{1}{\tau}}, \qquad \tau < \infty,$$

and

$$|u|_{(V,W)_{\theta,\infty}} := \sup_{t>0} t^{-\theta} K(u, t).$$

Finally, we define the interpolation spaces between V and W as

$$(V, W)_{\theta,\tau} := \left\{ u \in V : |u|_{(V,W)_{\theta,\tau}} < \infty \right\}.$$

One can show that there holds

$$|u|_{(V,W)_{\theta,\tau}} \lesssim \|u\|_V^{1-\theta} |u|_W^\theta, \qquad u \in W \subseteq V.$$

4.2. **Approximation.** Another question is the following: Given a Riesz basis $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ of a space V, for which subspace $W \subseteq V$ can we approximate all functions $u \in W$ with at most N terms at the rate N^{-s}? This is a topic of *nonlinear approximation* and, in particular, best N-term approximation, as we consider trial spaces

$$V_N := \left\{ \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda : c_\lambda \in \mathbb{R},\ |\Lambda| \leq N \right\}.$$

Such trial spaces are nonlinear, as for $u, v \in V_N$ there holds in general only $u + v \in V_{2N}$, cf. [11].

We next define the N-term error

$$E_N(u) := \inf_{v_N \in V_N} \|u - v_N\|_V,$$

the seminorms

$$|u|_{\mathcal{A}^s_\tau(V)} := \left(\sum_{N>0} \frac{1}{N} \left[N^s E_N(u)\right]^\tau\right)^{\frac{1}{\tau}}, \quad 0 < \tau < \infty,$$
$$|u|_{\mathcal{A}^s_\infty(V)} := \sup_{N \geq 0} N^s E_N(u),$$

and the approximation spaces

$$\mathcal{A}^s_\tau(V) := \left\{u \in V : |u|_{\mathcal{A}^s_\tau(V)} < \infty\right\}.$$

One can show that the approximation spaces $\mathcal{A}^s_\tau(V)$ can, under some circumstances, be fully characterised by interpolation. We just quote the result. For a proof, see [11] and the references therein.

Theorem 4.1. *Consider a vector space V and a subspace $W \subseteq V$, and assume that there is a number $r > 0$ such that there holds a Jackson inequality*

$$E_N(u) \lesssim N^{-r}|u|_W$$

and a Bernstein inequality

$$|u_N|_W \lesssim N^r \|u\|_V.$$

Then, for each $s \in (0, r)$ and each $\tau \in (0, \infty]$, there holds

$$\mathcal{A}^s_\tau(V) = (V, W)_{\frac{s}{r},\tau}$$

with equivalent norms.

For best N-term approximation with respect to the isotropic wavelet basis $\Theta = \{\theta_\mu : \mu \in \bigcirc\}$ on the unit cube, it is well known that one can derive a Jackson and Bernstein inequality between the spaces $H^q(\square)$ and $B^{q+rn,\tau}_\tau(\square)$ where $\frac{1}{\tau} = r + \frac{1}{2}$, cf. [11]. Therefore, the following theorem holds.

Theorem 4.2. *Consider the space $H^q(\square)$ and the properly scaled Riesz basis $\Theta = \{\theta_\mu : \mu \in \bigcirc\}$. Then, for $\frac{1}{\tau} = r + \frac{1}{2}$ and $0 < \kappa \leq \infty$, there holds*

$$A^s_\kappa\big(H^q(\square)\big) = \big(H^q(\square),\ B^{q+rn,\tau}_\tau(\square)\big)_{\frac{s}{r},\kappa}, \quad 0 < s < r,$$

provided that $\Theta \subseteq B^{q+rn+\varepsilon,\tau}_\tau(\square)$ for some $\varepsilon > 0$ and that the wavelets admit $\tilde{d} > \max\{q + rn, n(r - \frac{1}{2})\}$ vanishing moments.

4.3. **Approximation with Tensor-Product Wavelets.** With the results of the previous two subsections, we can now classify the approximation spaces for any function space $H^q(\square)$ with respect to an anisotropic tensor-product wavelet basis.

Theorem 4.3. *For $r > 0$, $q \in (-\tilde{\gamma}, \gamma)$, and τ such that*

$$\frac{1}{\tau} = r + \frac{1}{2},$$

there holds the Jackson inequality

$$\inf_{v_N \in V_N} \|u - v_N\|_{H^q(\square)} \lesssim N^{-r}|u|_{\mathfrak{B}^{q,r,\tau}_\tau(\square)}. \tag{20}$$

Proof. We use an argument similar to [11, 22]. First, we remark that

$$\Psi^q := \left\{2^{-q|\boldsymbol{\lambda}|_\infty}\psi_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\right\} \tag{21}$$

is a Riesz basis for $H^q(\square)$ due to the norm equivalence (12). Thus, the best N-term approximation to any function u asymptotically corresponds to the N terms with the largest coefficients in absolute value. Hence, if $\mathbf{u}^\star = (\mathbf{u}^\star(k))_k$ is a descending (in absolute value) reordering of the sequence $(2^{q|\boldsymbol{\lambda}|_\infty}u_{\boldsymbol{\lambda}})_{\boldsymbol{\lambda}}$, then there holds

$$\inf_{v_N \in V_N} \|u - v_N\|^2_{H^q(\square)} \sim \sum_{k=N+1}^{\infty} |\mathbf{u}^\star(k)|^2.$$

Therefore, it is enough to require that the quantity

$$\|\mathbf{u}^\star\|_{\ell^\tau_\infty} := \sup_{k\in\mathbb{N}} k^{\frac{1}{\tau}} |\mathbf{u}^\star(k)|$$

is uniformly bounded. Since the embedding $\ell^\tau \subseteq \ell^\tau_\infty$ is continuous, we have that

$$\inf_{v_N\in V_N} \|u - v_N\|_{H^q(\square)} \lesssim N^{-r} \|\mathbf{u}^\star\|_{\ell^\tau(\mathbb{N})}. \tag{22}$$

By rescaling the coefficients, we obtain that

$$\begin{aligned}
\|\mathbf{u}^\star\|^\tau_{\ell^\tau(\mathbb{N})} &= \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \left|2^{q|\boldsymbol{\lambda}|_\infty} u_{\boldsymbol{\lambda}}\right|^\tau = \sum_{\mathbf{j}\geq\mathbf{j}_0} 2^{\tau q|\mathbf{j}|_\infty} \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}_{\mathbf{j}}} \left|2^{|\mathbf{j}|_1\left(\frac{1}{\tau}-\frac{1}{2}\right)} u^{(\tau)}_{\boldsymbol{\lambda}}\right|^\tau \\
&= \sum_{\mathbf{j}\in\mathbb{N}_0^2} 2^{\tau(q|\mathbf{j}|_\infty + r|\mathbf{j}|_1)} \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}_{\mathbf{j}}} \left|u^{(\tau)}_{\boldsymbol{\lambda}}\right|^\tau \\
&= |u|^\tau_{\mathfrak{B}^{q,r,\tau}_\tau(\square)}.
\end{aligned}$$

Together with (22), this implies (20). □

Theorem 4.4. *Assume that the assumptions of Theorem 4.3 hold. Then, we also have the Bernstein inequality*

$$|u_N|_{\mathfrak{B}^{q,r,\tau}_\tau(\square)} \lesssim N^r \|u_N\|_{H^q(\square)}, \qquad u_N \in V_N. \tag{23}$$

Proof. We will again make use of the norm equivalence (12) for the wavelet basis. Since $u_N \in V_N$, we may write u_N as a linear combination

$$u_N = \sum_{\boldsymbol{\lambda}\in\Lambda} u_{\boldsymbol{\lambda}} \psi_{\boldsymbol{\lambda}}$$

of at most $|\Lambda| \leq N$ terms. Again, by rescaling, we have that

$$\begin{aligned}
|u_N|^\tau_{\mathfrak{B}^{q,r,\tau}_\tau(\square)} &= \sum_{\boldsymbol{\lambda}\in\Lambda} 2^{\tau(q|\boldsymbol{\lambda}|_\infty + r|\boldsymbol{\lambda}|_1)} \left|u^{(\tau)}_{\boldsymbol{\lambda}}\right|^\tau = \sum_{\boldsymbol{\lambda}\in\Lambda} 2^{\tau(q|\boldsymbol{\lambda}|_\infty + r|\boldsymbol{\lambda}|_1)} \left|2^{|\boldsymbol{\lambda}|_1\left(\frac{1}{2}-\frac{1}{p}\right)} u_{\boldsymbol{\lambda}}\right|^\tau \\
&= \sum_{\boldsymbol{\lambda}\in\Lambda} 2^{\tau q|\boldsymbol{\lambda}|_\infty} \left|u_{\boldsymbol{\lambda}}\right|^\tau \lesssim \left(\sum_{\boldsymbol{\lambda}\in\Lambda} 1\right)^{1-\frac{\tau}{2}} \left(\sum_{\boldsymbol{\lambda}\in\Lambda} 2^{2q|\mathbf{j}|_\infty} \left|u_{\boldsymbol{\lambda}}\right|^2\right)^{\frac{\tau}{2}} \\
&\sim N^{r\tau} \cdot \|u\|^\tau_{H^q(\square)}.
\end{aligned}$$

By taking the τ-th root, we get (23). □

With the Jackson and the Bernstein inequality at hand, Theorem 4.1 can be used to characterise the approximation spaces.

Theorem 4.5. *Assume that the assumptions of Theorem 4.3 hold. Then, for $0 < s < r$ and $0 < \kappa \leq \infty$, the approximation spaces with respect to tensor-product wavelets are given by*

$$\mathfrak{A}^s_\kappa\big(H^q(\square)\big) = \big(H^q(\square),\ \mathfrak{B}^{q,r,\tau}_\tau(\square)\big)_{\frac{s}{r},\kappa}. \tag{24}$$

5. Comparison with Isotropic Nonlinear Approximation

As stated in Section 4, for nonlinear approximation with isotropic wavelet bases in arbitrary dimensions, the Jackson and Bernstein inequalities hold for the isotropic Besov space $B^{q+rn,\tau}_\tau(\square)$. Intuitively, this space requires more regularity on the functions, since, in comparison to Theorem 4.5, rn additional isotropic derivatives are needed instead of only r mixed derivatives. However, the classical function space $B^{q+rn,\tau}_\tau(\square)$ cannot be characterised by tensor-product wavelets unless $\tau = 2$, cf. [25] and the references therein. In this case, the space $B^{q+rn,2}_2(\square) = H^{q+rn}(\square)$

is a Sobolev space and this identity can also be directly concluded from the norm equivalences [19].

To the authors' best knowledge, it is not known yet whether the classical Besov spaces $B_\tau^{q+sn,p}(\square)$ are included in the Besov space of hybrid regularity $\mathfrak{B}_\tau^{q,s,p}(\square)$. In this section, we will therefore address this question up to a certain point.

5.1. **Change of Bases.** First, we need to find a way to express the isotropic basis functions in terms of tensor-product functions. For simplicity, we only treat the case $n = 2$ explicitly again, but we emphasise that the main results, Theorems 5.4 and 5.6, carry over to the n-dimensional case as well, if $2s$ is replaced by sn.

Therefore, we consider a set

$$\Theta_m^{\mathbf{e}} = \Theta_m^{e_1} \otimes \Theta_m^{e_2}.$$

If $e = 1$, then $\Theta_m^e = \Psi_m$, so in this case, the corresponding tensor factor is already a one-dimensional wavelet. On the other hand, (4) implies that

$$\big[\Theta_{m-1}^0, \Theta_m^1\big] = \Theta_m^0 \big[\mathbf{M}_{m,0}, \mathbf{M}_{m,1}\big].$$

Hence, there holds

$$\begin{aligned}
\big[\Psi_{j_0}, \ldots, \Psi_{m-1}\big] \otimes \Psi_m &= \Theta_m^{(0,1)}\big(\mathbf{T}_{m-1} \otimes \mathbf{I}\big), \\
\Psi_m \otimes \big[\Psi_{j_0}, \ldots, \Psi_{m-1}\big] &= \Theta_m^{(1,0)}\big(\mathbf{I} \otimes \mathbf{T}_{m-1}\big), \\
\Psi_m \otimes \Psi_m &= \Theta_m^{(1,1)}\big(\mathbf{I} \otimes \mathbf{I}\big).
\end{aligned}$$

To estimate the classical Sobolev or Besov regularity of a function u which is discretised by tensor-product wavelets, we can therefore write

$$\begin{aligned}
u = \Psi\mathbf{u} &= \sum_{m=j_0}^{\infty} \left[\sum_{j=0}^{m-1} \Psi_{(j,m)}\mathbf{u}\big|_{\boldsymbol{\nabla}_{(j,m)}} + \Psi_{(m,j)}\mathbf{u}\big|_{\boldsymbol{\nabla}_{(m,j)}}\right] + \Psi_{(m,m)}\mathbf{u}\big|_{\boldsymbol{\nabla}_{(m,m)}} \\
&= \sum_{m=j_0}^{\infty} \Theta_m^{(0,1)}\left(\mathbf{T}_{m-1} \otimes \mathbf{I}\right)\mathbf{u}\big|_{\bigcup_{j<m} \nabla_j \times \nabla_m} \\
&\qquad + \Theta_m^{(1,0)}\left(\mathbf{I} \otimes \mathbf{T}_{m-1}\right)\mathbf{u}\big|_{\bigcup_{j<m} \nabla_m \times \nabla_j} + \Theta_m^{(1,1)}\mathbf{u}\big|_{\nabla_m \times \nabla_m}.
\end{aligned}$$

On the other hand, we can also express the tensor-product coefficients in terms of the isotropic ones, meaning that

$$\begin{aligned}
u = \Theta\mathbf{v} &= \sum_{m=j_0}^{\infty} \Theta_m^{(0,1)}\mathbf{v}\big|_{\hexagon_m^{(0,1)}} + \Theta_m^{(1,0)}\mathbf{v}\big|_{\hexagon_m^{(1,0)}} + \Theta_m^{(1,1)}\mathbf{v}\big|_{\hexagon_m^{(1,1)}} \\
&= \sum_{m=j_0}^{\infty} \big([\Psi_{j_0} \ldots \Psi_{m-1}] \otimes \Psi_m\big)\big(\tilde{\mathbf{T}}_{m-1}^{\intercal} \otimes \mathbf{I}\big)\mathbf{v}\big|_{\hexagon_m^{(0,1)}} \\
&\qquad + \big(\Psi_m \otimes [\Psi_{j_0} \ldots \Psi_{m-1}]\big)\big(\mathbf{I} \otimes \tilde{\mathbf{T}}_{m-1}^{\intercal}\big)\mathbf{v}\big|_{\hexagon_m^{(1,0)}} + \big(\Psi_m \otimes \Psi_m\big)\mathbf{v}\big|_{\hexagon_m^{(1,1)}}.
\end{aligned}$$

Hence, if $\frac{1}{\tau} = s + \frac{1}{2}$, any function $u = \Psi\mathbf{u} = \Theta\mathbf{v}$ satisfies

$$\begin{aligned}
|u|^\tau_{\mathfrak{B}^{q,s,\tau}_\tau(\square)} &= |\mathbf{u}^{(\tau)}|^\tau_{\mathfrak{b}^{q,s,\tau}_\tau} = \sum_{\mathbf{j}\geq\mathbf{j}_0} 2^{\tau q|\mathbf{j}|_\infty + \tau s|\mathbf{j}|_1} \|\mathbf{u}^{(\tau)}\|^\tau_{\ell^\tau(\boldsymbol{\nabla}_\mathbf{j})} \\
&= \sum_{m=j_0}^{\infty} \sum_{j=j_0}^{m-1} 2^{\tau q m + \tau(s+\frac{1}{2}-\frac{1}{\tau})(j+m)} \left(\|\mathbf{u}\|^\tau_{\ell^\tau(\boldsymbol{\nabla}_{(j,m)})} + \|\mathbf{u}\|^\tau_{\ell^\tau(\boldsymbol{\nabla}_{(j,k)})} \right) \\
&\quad + 2^{\tau m q + 2\tau m(s+\frac{1}{2}-\frac{1}{\tau})} \|\mathbf{u}\|^\tau_{\ell^\tau(\boldsymbol{\nabla}_{(m,m)})} \\
&= \sum_{m=j_0}^{\infty} 2^{\tau q m} \Big(\big\| (\tilde{\mathbf{T}}^\intercal_{m-1} \otimes \mathbf{I})\mathbf{v}|_{\bigcirc^{(0,1)}_m} \big\|^\tau_{\ell^\tau(\bigcup_{j<m} \nabla_j \times \nabla_m)} \\
&\quad + \big\| (\mathbf{I} \otimes \tilde{\mathbf{T}}^\intercal_{m-1})\mathbf{v}|_{\bigcirc^{(1,0)}_m} \big\|^\tau_{\ell^\tau(\bigcup_{j<m} \nabla_m \times \nabla_j)} \Big) + 2^{\tau m q} \|\mathbf{v}\|^\tau_{\ell^\tau(\bigcirc^{(1,1)}_m)},
\end{aligned} \tag{25}$$

and conversely, also

$$\begin{aligned}
|u|^\tau_{B^{\alpha,\tau}_\tau(\square)} &= |\mathbf{v}^{(\tau)}|^\tau_{\mathbf{b}^{\alpha,\tau}_\tau} = \sum_{m=j_0}^{\infty} 2^{\tau \alpha m} \|\mathbf{v}^{(\tau)}\|^\tau_{\ell^\tau(\bigcirc_m)} \\
&= \sum_{m=j_0}^{\infty} 2^{\tau m(\alpha + 2(\frac{1}{2}-\frac{1}{\tau}))} \left(\|\mathbf{v}\|_{\ell^\tau(\bigcirc^{(0,1)}_m)} + \|\mathbf{v}\|_{\ell^\tau(\bigcirc^{(1,0)}_m)} + \|\mathbf{v}\|_{\ell^\tau(\bigcirc^{(1,1)}_m)} \right) \\
&= \sum_{m=j_0}^{\infty} 2^{\tau m(\alpha - 2s)} \Big(\big\| (\mathbf{T}_{m-1} \otimes \mathbf{I})\mathbf{u}\big|_{\bigcup_{j<m} \nabla_j \times \nabla_m} \big\|^\tau_{\ell^\tau(\bigcirc_m)^{(0,1)}} \\
&\quad + \big\| (\mathbf{I} \otimes \mathbf{T}_{m-1})\mathbf{u}\big|_{\bigcup_{j<m} \nabla_m \times \nabla_j} \big\|^\tau_{\ell^\tau(\bigcirc_m)^{(1,0)}} + \|\mathbf{u}\|^\tau_{\ell^\tau(\boldsymbol{\nabla}_{(m,m)})} \Big).
\end{aligned} \tag{26}$$

5.2. **Comparison of Approximation Spaces.** The goal of this section is to compare the classical approximation spaces $B^{q+2s,\tau}_\tau(\square)$ with the approximation spaces for tensor-product wavelets $\mathfrak{B}^{q,s,\tau}_\tau(\square)$. We will see that there is a range of regularity spaces whose elements can be approximated by tensor-product wavelets but not by isotropic wavelets.

If a bit of additional smoothness is available, with the help of [1], we can immediately conclude that the approximation with tensor-product wavelets performs at least as well.

Theorem 5.1. *For any $\varepsilon > 0$, $s \geq 0$, and $u = \Psi\mathbf{u}$, there holds with $\frac{1}{\tau} = s + \frac{1}{2}$,*

$$|u|_{\mathfrak{B}^{s,0,\tau}_\tau(\square)} \lesssim \|u\|_{B^{s+\varepsilon,\tau}_\tau(\square)}.$$

Proof. We remark that we need to consider Besov spaces with logarithmic correction, cf. [1]. Due to [1, Theorem 2.6] (after rescaling the coefficients such that they correspond to wavelets normalised in $L^\tau(\square)$), there holds $u = \Psi\mathbf{u} \in B^{s,\tau}_{\tau,|\log|^\beta}$ if

$$\overline{\sigma}^s(\mathbf{u}) := \left[\sum_{\mathbf{j}\geq\mathbf{j}_0} |\mathbf{j}|_\infty^{\overline{\varrho}-\beta\tau} 2^{\tau s|\mathbf{j}|_\infty} \|\mathbf{u}^{(\tau)}\|^\tau_{\ell^\tau(\boldsymbol{\nabla}_\mathbf{j})} \right]^{\frac{1}{\tau}} < \infty.$$

On the other hand, if $u = \Psi\mathbf{u} \in B^{s,\tau}_{\tau,|\log|^\beta}$, then

$$\underline{\sigma}^s(\mathbf{u}) := \left[\sum_{\mathbf{j}\geq\mathbf{j}_0} |\mathbf{j}|_\infty^{-\underline{\varrho}-\beta\tau-1} 2^{\tau s|\mathbf{j}|_\infty} |\mathbf{u}^{(\tau)}|^\tau_{\ell^\tau(\boldsymbol{\nabla}_\mathbf{j})} \right]^{\frac{1}{\tau}} < \infty.$$

Herein,

$$\beta = \left|\frac{1}{\tau} - 1\right|, \qquad \overline{\varrho} = \begin{cases} 1-\tau, & s \geq \frac{1}{2}, \\ 0, & s < \frac{1}{2}, \end{cases} \qquad \underline{\varrho} = \begin{cases} 1, & s > \frac{1}{2}, \\ \max\{2-\tau, 0\}, & s \leq \frac{1}{2}. \end{cases}$$

Moreover, as $\beta \geq 0$, there holds $B_\tau^{s+\varepsilon,\tau}(\square) \hookrightarrow B_{\tau,|\log|^\beta}^{s+\varepsilon,\tau}(\square) \hookrightarrow B_\tau^{s,\tau}(\square)$. Hence, if we can show that $|\mathbf{u}^{(\tau)}|_{\mathfrak{b}_\tau^{s,0,\tau}} \lesssim \underline{\sigma}^{s+\varepsilon}(\mathbf{u})$, we have the embeddings

$$B_\tau^{s+\varepsilon,\tau}(\square) \hookrightarrow B_{\tau,|\log|^\beta}^{s+\varepsilon,\tau}(\square) \hookrightarrow \mathfrak{B}_\tau^{s,0,\tau}(\square),$$

which is what we want to show.

To this end, let $\mathbf{u}^{(\tau)} \in \mathfrak{b}_\tau^{s,0,\tau}$. By standard estimates, there holds

$$\begin{aligned} |\mathbf{u}^{(\tau)}|_{\mathfrak{b}_\tau^{s,0,\tau}}^\tau &= \sum_{\mathbf{j}\geq\mathbf{j}_0} 2^{\tau s|\mathbf{j}|_\infty} \|\mathbf{u}^{(\tau)}\|_{\ell^\tau(\boldsymbol{\nabla}_\mathbf{j})}^\tau \lesssim \sum_{\mathbf{j}\geq\mathbf{j}_0} |\mathbf{j}|_\infty^{-\underline{\varrho}-\beta\tau-1} 2^{\tau(s+\varepsilon)|\mathbf{j}|_\infty} \|\mathbf{u}^{(\tau)}\|_{\ell^\tau(\boldsymbol{\nabla}_\mathbf{j})}^\tau \\ &= \underline{\sigma}^{s+\varepsilon}(\mathbf{u})^\tau, \end{aligned}$$

by which this theorem is proven. □

Remark 5.2. *By the same arguments, since $\overline{\varrho} - \beta\tau < 0$, we also see that we have*

$$\|u\|_{B_{\tau,|\log|^\beta}^{s,\tau}(\square)} \lesssim |u|_{\mathfrak{B}_\tau^{s,0,\tau}(\square)}.$$

Corollary 5.3. *For any $\varepsilon > 0$, $s \geq 0$, and $u = \Psi\mathbf{u}$, there holds with $\frac{1}{\tau} = s + \frac{1}{2}$,*

$$|u|_{\mathfrak{B}_\tau^{q,s,\tau}(\square)} \lesssim |u|_{\mathfrak{B}_\tau^{q+2s,0,\tau}(\square)} \lesssim \|u\|_{B_\tau^{q+2s+\varepsilon,\tau}(\square)}.$$

On the other hand, there also holds

$$\|u\|_{B_\tau^{q+s-\varepsilon,\tau}(\square)} \lesssim |u|_{\mathfrak{B}_\tau^{q+s,0,\tau}(\square)} \lesssim |u|_{\mathfrak{B}_\tau^{q,s,\tau}(\square)}.$$

Proof. The first embedding is due to Theorem 5.1, whereas the second embedding follows trivially from the definition of the seminorm $|\cdot|_{\mathfrak{b}_\tau^{q,s,\tau}}$ in (18) and the estimate $|\mathbf{j}|_\infty \leq |\mathbf{j}|_1 \leq 2|\mathbf{j}|_\infty$. □

This corollary immediately implies that there are a lot of functions which satisfy $u \in \mathfrak{A}^s\big(H^q(\square)\big)$ but $u \notin A^s\big(H^q(\square)\big)$, where $\mathfrak{A}^s$ and A^s denote the approximation spaces with respect to the tensor-product and the isotropic wavelet bases, respectively. In particular, all functions which admit slightly more regularity than minimally required to be in $A^s\big(H^q(\square)\big)$ are also in $\mathfrak{A}^s\big(H^q(\square)\big)$. Nevertheless, there might also exist functions $u \in A^s\big(H^q(\square)\big)$, with $u \notin \mathfrak{A}^s\big(H^q(\square)\big)$, i.e., functions which can be approximated better by isotropic wavelets.

In the case of wavelets with compactly supported duals, however, as the following theorem shows, this is not possible.

Theorem 5.4. *For $u = \Psi\mathbf{u} = \Theta\mathbf{v}$, where Ψ and Θ are the tensor-product and isotropic wavelet bases defined in Section 2.2, there holds*

$$|\mathbf{v}^{(\tau)}|_{\mathbf{b}_\tau^{q+s,\tau}} \lesssim |\mathbf{u}^{(\tau)}|_{\mathfrak{b}_\tau^{q,s,\tau}} \lesssim |\mathbf{v}^{(\tau)}|_{\mathbf{b}_\tau^{q+2s,\tau}},$$

with $\frac{1}{\tau} = s + \frac{1}{2}$. In particular, we have $|u|_{B_\tau^{q+s,\tau}(\square)} \lesssim |u|_{\mathfrak{B}_\tau^{q,s,\tau}(\square)} \lesssim |u|_{B_\tau^{q+2s,\tau}(\square)}$.

Proof. Let $u = \Psi\mathbf{u} = \Theta\mathbf{v} \in B_\tau^{q+2s,\tau}(\square)$. We first want to apply the suitable coordinate transforms to derive the estimate

$$|\mathbf{u}^{(\tau)}|_{\mathfrak{b}_\tau^{q,s,\tau}} \lesssim |\mathbf{v}^{(\tau)}|_{\mathbf{b}_\tau^{q+2s,\tau}} = \left[\sum_{m=0}^\infty 2^{\tau m(q+2s)} \sum_{\mu\in\bigcirc_m} |v_\mu^{(\tau)}|^\tau\right]^{\frac{1}{\tau}}.$$

In view of (25), there holds

$$|\mathbf{u}^{(\tau)}|^{\tau}_{\mathfrak{b}^{q,s,\tau}_{\tau}} = \sum_{m=j_0}^{\infty} 2^{\tau mq} \Big(\|(\tilde{\mathbf{T}}^{\intercal}_{m-1} \otimes \mathbf{I})\mathbf{v}|_{\bigcirc^{(0,1)}_m}\|^{\tau}_{\ell^{\tau}(\bigcup_{j<m} \nabla_j \times \nabla_m)} + \|(\mathbf{I} \otimes \tilde{\mathbf{T}}^{\intercal}_{m-1})\mathbf{v}|_{\bigcirc^{(1,0)}_m}\|^{\tau}_{\ell^{\tau}(\bigcup_{j<m} \nabla_m \times \nabla_j)} \Big) + 2^{\tau mq}\|\mathbf{v}\|^{\tau}_{\ell^{\tau}(\bigcirc^{(1,1)}_m)}.$$

If $\tau < 1$, we have due to Lemma 2.1 and Lemma 2.5

$$\big\|\tilde{\mathbf{T}}^{\intercal}_{m-1} \otimes \mathbf{I}\big\|^{\tau}_{\tau} \le \big\|(\tilde{\mathbf{T}}^{\odot\tau}_{m-1} \otimes \mathbf{I})^{\intercal}\big\|_1 \le \big\|(\tilde{\mathbf{T}}^{\odot\tau}_{m-1})^{\intercal}\big\|_1,$$

which can be shown to be uniformly bounded using the arguments of the proof of Lemma 2.3. If $1 \le \tau \le 2$, and $\frac{1}{\tau} = \frac{1-\theta}{1} + \frac{\theta}{2}$, interpolation and Lemma 2.5 yield

$$\begin{aligned} \big\|\tilde{\mathbf{T}}^{\intercal}_{m-1} \otimes \mathbf{I}\big\|^{\tau}_{\tau} &\le \big\|\tilde{\mathbf{T}}^{\intercal}_{m-1} \otimes \mathbf{I}\big\|^{(1-\theta)\tau}_1 \cdot \big\|\tilde{\mathbf{T}}^{\intercal}_{m-1} \otimes \mathbf{I}\big\|^{\theta\tau}_2 \\ &\le \big\|\tilde{\mathbf{T}}^{\intercal}_{m-1}\big\|^{(1-\theta)\tau}_1 \cdot \big\|\tilde{\mathbf{T}}^{\intercal}_{m-1}\big\|^{\theta\tau}_2, \end{aligned}$$

which is uniformly bounded by Lemma 2.3. After applying the same estimate to $\mathbf{I} \otimes \tilde{\mathbf{T}}^{\intercal}_{m-1}$, we can conclude that

$$|\mathbf{u}^{(\tau)}|^{\tau}_{\mathfrak{b}^{q,s,\tau}_{\tau}} \lesssim \sum_{m=j_0}^{\infty} 2^{\tau mq}\|\mathbf{v}\|^{\tau}_{\ell^{\tau}(\bigcirc_m)} = \sum_{m=j_0}^{\infty} 2^{\tau m(q+2s)}\|\mathbf{v}^{(\tau)}\|^{\tau}_{\ell^{\tau}(\bigcirc_m)},$$

which is what we wanted to show.

For the other estimate, we use (26) to see that

$$|\mathbf{v}^{(\tau)}|_{\mathbf{b}^{q+s,\tau}_{\tau}} = \sum_{m=j_0}^{\infty} 2^{\tau m(q-s)} \Big(\big\|(\mathbf{T}_{m-1} \otimes \mathbf{I})\mathbf{u}\big|_{\bigcup_{j<m} \nabla_j \times \nabla_m}\big\|^{\tau}_{\ell^{\tau}(\bigcirc_m)^{(0,1)}} + \big\|(\mathbf{I} \otimes \mathbf{T}_{m-1})\mathbf{u}\big|_{\bigcup_{j<m} \nabla_m \times \nabla_j}\big\|^{\tau}_{\ell^{\tau}(\bigcirc_m)^{(1,0)}} + \|\mathbf{u}\|^{\tau}_{\ell^{\tau}(\boldsymbol{\nabla}_{(m,m)})} \Big). \quad (27)$$

Also here, by using the arguments of the Lemmata 2.1, 2.5, and 2.3, we see that

$$\big\|\mathbf{T}_{m-1} \otimes \mathbf{I}\big\|^{\tau}_{\tau} \le \big\|\mathbf{T}^{\odot\tau}_{m-1} \otimes \mathbf{I}\big\|_1 \le \big\|\mathbf{T}^{\odot\tau}_{m-1}\big\|_1 \lesssim 2^{\tau m(\frac{1}{\tau}-\frac{1}{2})} = 2^{\tau ms},$$

if $0 < \tau \le 1$. If $1 \le \tau \le 2$ and $\frac{1}{\tau} = \frac{1-\theta}{1} + \frac{\theta}{2}$, meaning that $1 - \theta = 2s$, we may interpolate again to deduce that

$$\big\|\mathbf{T}_{m-1} \otimes \mathbf{I}\big\|^{\tau}_{\tau} \le \big\|\mathbf{T}_{m-1} \otimes \mathbf{I}\big\|^{2s\tau}_1 \cdot \big\|\mathbf{T}_{m-1} \otimes \mathbf{I}\big\|^{\theta\tau}_2 \lesssim \big\|\mathbf{T}_{m-1}\big\|^{2s\tau}_1 \lesssim 2^{\tau ms}.$$

Since this applies for $\mathbf{I} \otimes \mathbf{T}_{m-1}$ in the same style, there finally holds

$$|\mathbf{v}^{(\tau)}|_{\mathbf{b}^{q+s,\tau}_{\tau}} \lesssim \sum_{m=j_0}^{\infty} 2^{\tau mq} \sum_{|\mathbf{j}|_\infty = m} \|\mathbf{u}\|^{\tau}_{\ell^{\tau}(\boldsymbol{\nabla}_{\mathbf{j}})} = \sum_{\mathbf{j} \ge \mathbf{j}_0} 2^{\tau q|\mathbf{j}|_\infty + \tau s|\mathbf{j}|_1}\|\mathbf{u}^{(\tau)}\|^{\tau}_{\ell^{\tau}(\boldsymbol{\nabla}_{\mathbf{j}})}.$$

□

Remark 5.5. *The most important step in the proof of Theorem 5.4 is the estimate of the transformation matrices. Although this was only carried out for the two-dimensional case, the same can be concluded in n dimensions. Indeed, since if Lemma* $\|(\tilde{\mathbf{T}}^{\odot\tau}_m)^{\intercal}\|_1$ *is uniformly bounded, Lemma 2.5 can be applied recursively.*

For the opposite estimate, we need to be slightly more careful. Indeed, the matrices arising from the transforms may be given by $(n-1)$ *tensor factors* $\mathbf{T}_{m-1}$ *and only one identity, resulting in a norm of* $2^{\tau m(n-1)s}$*. However, by rescaling the coefficients from* $L^{\tau}(\square)$ *to* $L^2(\square)$*, the weight in* (27) *is now given by* $2^{\tau m(q-(n-1)s)}$*, which multiplies to* $2^{\tau mq}$*, as desired.*

With this remark, we can immediately deduce the following result.

Theorem 5.6. *Let* $q, s > 0$*. Then, for* $\frac{1}{\tau} = s + \frac{1}{2}$*, the embeddings* $B^{q+sn,\tau}_{\tau}(\square) \hookrightarrow \mathfrak{B}^{q,s,\tau}_{\tau}(\square) \hookrightarrow B^{q+s,\tau}_{\tau}(\square)$ *are continuous.*

Proof. Since $q > 0$, there holds with Theorem 5.4, that

$$\|u\|_{\mathfrak{B}_\tau^{q,s,\tau}(\square)} = \|u\|_{L^p(\square)} + |\mathbf{u}^{(\tau)}|_{\mathfrak{b}_\tau^{q,s,\tau}} \lesssim \|u\|_{L^p(\square)} + |\mathbf{v}^{(\tau)}|_{\mathbf{b}_\tau^{q+sn,\tau}} = \|u\|_{B_\tau^{q+sn,\tau}(\square)}.$$

Similarly,

$$\|u\|_{B_\tau^{q+s,\tau}(\square)} = \|u\|_{L^p(\square)} + |\mathbf{v}^{(\tau)}|_{\mathbf{b}_\tau^{q+s,\tau}} \lesssim \|u\|_{L^p(\square)} + |\mathbf{u}^{(\tau)}|_{\mathfrak{b}_\tau^{q,s,\tau}} = \|u\|_{\mathfrak{B}_\tau^{q,s,\tau}(\square)},$$

which implies the claim. □

6. Conclusion

We have extended the wavelet characterisations of the hybrid regularity Sobolev spaces $\mathfrak{H}^{q,s}(\square)$. Therefrom outgoing, we have shown that the approximation spaces $\mathfrak{A}^s(H^q(\square))$, with respect to tensor-product wavelets, correspond to sequences in $\mathfrak{b}_\tau^{q,s,\tau}$ with $\frac{1}{\tau} = s + \frac{1}{2}$. These sequence spaces characterise the seminorms of the Besov spaces of hybrid regularity $\mathfrak{B}_\tau^{q,s,\tau}(\square)$. Finally, we have shown by elementary coordinate transforms that all functions in $B_\tau^{q+sn,\tau}(\square) \subseteq A^s(H^q(\square))$ can also be approximated at least at the same rate N^{-s} by N-term tensor-product wavelets. Although this seems natural, this was not known up to now. Moreover, for positive regularity, we have shown the embedding $B_\tau^{q+sn,\tau}(\square) \hookrightarrow \mathfrak{B}_\tau^{q,s,\tau}(\square)$, meaning that an isotropic space is included in a space of dominating mixed smoothness.

On the other hand, also the other natural embedding $\mathfrak{B}_\tau^{q,s,\tau} \hookrightarrow B_\tau^{q+s,\tau}(\square)$ was shown to be continuous. In all these proofs, merely estimates on the coordinate transforms between tenor-product wavelets and isotropic wavelets have been used. To the authors' best knowledge, this technique has, up to now, not been applied to investigate Besov spaces. Therefore, it might provide new insight into a whole range of function spaces.

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