A sharp upper bound on the spectral radius of $\theta(1,3,3)$-free graphs with given size*

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Abstract

A graph G is F-free if G does not contain F as a subgraph. Let $\rho(G)$ be the spectral radius of a graph G. Let $\theta(1,p,q)$ denote the theta graph, which is obtained by connecting two distinct vertices with three internally disjoint paths with lengths $1,p,q$, where $p \leq q$. Let $S_{n,k}$ denote the graph obtained by joining every vertex of K_k to $n-k$ isolated vertices and $S^-_{n,k}$ denote the graph obtained from $S_{n,k}$ by deleting an edge incident to a vertex of degree k, respectively. In this paper, we show that if $\rho(G) \geq \rho(S^-_{\frac{m+4}{2},2})$ for a graph G with even size $m \geq 92$, then G contains a $\theta(1,3,3)$ unless $G \cong S^-_{\frac{m+4}{2},2}$.

Key Words: Spectral Turán type problem, Spectral radius, theta graph

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1 Introduction

Throughout this paper, we consider all graphs are always undirected and simple. We follow the traditional notation and terminology [1]. Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and size m with edge set $E(G)$. For a vertex $u \in V(G)$, let $N_G(u)$ be the neighborhood set of a vertex u, $N_G[u] = N_G(u) \cup \{u\}$ and $N^2_G(u)$ be the set of vertices of distance two to u in G. In particular, $N_S(v) = N(v) \cap S$ and $d_S(v) = |N_S(v)|$ for a subset $S \subseteq V(G)$. Let $d_G(u) = |N_G(u)|$ be the degree of a vertex u. For the sake of simplicity, we omit all the subscripts if G is clear from the context, for example, $N(u)$, $N[u]$, $N^2(u)$ and $d(u)$. For a graph G and a subset $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S. For two vertex subsets S and T of $V(G)$ (where $S \cap T$ may not be empty), let $e(S,T)$ denote the number of edges with one endpoint in S and the other in T. $e(S,S)$ is simplified by $e(S)$. Given two vertex-disjoint graphs G_1 and G_2, we denote by $G_1 \cup G_2$ the disjoint union of the two graphs, and by $G_1 \vee G_2$ the joint graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with each vertex of G_2. The adjacency matrix of a graph G is an $n \times n$ matrix $A(G)$

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whose (i,j)-entry is 1 if v_i is adjacent to v_j and 0 otherwise. The spectral radius $\rho(G)$ of a graph G is the largest eigenvalue of its adjacency matrix $A(G)$.

Let $P_n, C_n, K_{1,n-1}$ and $K_{a,b}$ be the path of order n, the cycle of order n, the star graph of order n and the complete bipartite graph with two parts of sizes a, b, respectively. Let S_n^k be the graph obtained from $K_{1,n-1}$ by adding k disjoint edges within its independent sets. Let $S_{n,k}$ be the graph obtained by joining every vertex of K_k to $n-k$ isolated vertices. Let $S_{n,k}^-$ be the graph obtained from $S_{n,k}$ by deleting an edge incident to a vertex of degree k. Let $\theta(1,p,q)$ denote the theta graph, which is obtained by connecting two distinct vertices with three internally disjoint paths with lengths $1, p, q$, where $q \leq p$.

Given a graph F, a graph G is F-free if it does not contain F as a subgraph. Let $\mathcal{G}(m,F)$ denote the family of F-free graphs with m edges and without isolated vertices. The classic Turán type problem asks what is the maximum number of edges in an F-free graph of order n. In spectral graph theory, Nikiforov [16] posed a spectral Turán type problem which asks to determine the maximum spectral radius of an F-free graphs of n vertices, which is known as the Brualdi-Solheid-Turán type problem. In the past decades, this problem has received much attention. For more details, we suggest the reader to see surveys [3, 6, 10, 17], and references therein. In addition, Brualdi and Hoffman raised another spectral Turán type problem: What is the maximal spectral radius of an F-free graph with given size m? This problem is called the Brualdi-Hoffman-Turán type problem. This problem has been studied for various families of graphs. For example, K_3 [18], K_{r+1} [14, 15], $K_{2,r+1}$ [20], F_{2k+2} [8] (where $F_k = K_1 \vee P_{k-1}$), $F_{k,3}$ [8] (where $F_{k,3}$ is the friendship graph obtained from k triangles by sharing a common vertex).

For theta graphs, Sun, Li and Wei [19] characterized the extremal graph with maximum spectral radius of $\theta(1,2,3)$-free and $\theta(1,2,4)$-free graphs with odd size. Fang and You [5] characterized the extremal graph with maximum spectral radius of $\theta(1,2,3)$-free graphs with even size. Liu and Wang [12] characterized the extremal graph with maximum spectral radius of $\theta(1,2,4)$-free graphs with even size. Lu, Lu and Li [11] characterized the extremal graph with maximum spectral radius of $\theta(1,2,5)$-free with given size. Li, Zhai and Shu [9] characterized the extremal graph with maximum spectral radius of $\theta(1,2,2k-1)$-free or $\theta(1,2,2k)$-free with given size.

Recently, Li, Zhao and Zou [8] characterized the extremal graph with maximum spectral radius of $\theta(1,p,q)$-free with size m for $q \geq p \geq 3$ and $p+q \geq 7$.

Theorem 1.1. ([8]) *Let $k \geq 3$ and $m \geq \frac{9}{4}k^6 + 6k^5 + 46k^4 + 56k^3 + 196k^2$. If $G \in \mathcal{G}(m,\theta(1,p,q)) \cup \mathcal{G}(m,\theta(1,r,s))$ with $q \geq p \geq 3$, $s \geq r \geq 3$, $p+q = 2k+1$ and $r+s = 2k+2$, then*

$$\rho(G) \leq \frac{k-1+\sqrt{4m-k^2+1}}{2},$$

and equality holds if and only if $G \cong K_k \vee (\frac{m}{k} - \frac{k-1}{2})K_1$.

At the same time, they [8] proposed the following problem.

Problem 1.2. ([8]) *How can we characterize the graphs among $\mathcal{G}(m,\theta(1,3,3))$ having the largest spectral radius?*

Theorem 1.3. ([7]) *Let $G \in \mathcal{G}(m,\theta(1,3,3))$ be a graph of size $m \geq 43$. Then*

$$\rho(G) \leq \frac{1+\sqrt{4m-3}}{2},$$

and equality holds if and only if $G \cong S_{\frac{m+3}{2},2}$.

However, for a $\theta(1,3,3)$-free graph G with size m, the bound $\rho(G) \le \frac{1+\sqrt{4m-3}}{2}$ is sharp only for odd m. Motivated by this, we want to obtain a sharp upper bound of $\rho(G)$ for even m. Our result is presented as follows.

Theorem 1.4. *Let* $G \in \mathcal{G}(m, \theta(1,3,3))$ *be a graph of even size* $m \ge 92$*, then* $\rho(G) \le \rho(S^-_{\frac{m+4}{2},2})$*, and equality holds if and only if* $G \cong S^-_{\frac{m+4}{2},2}$.

2 Preliminary

In this section, we introduce some lemmas which are used to prove our result.

Lemma 2.1. ([21]) *Let* u, v *be two distinct vertices of the connected graph* G*,* $\{v_i | i = 1,2,\ldots,s\} \subseteq N(v) \setminus N(u)$*, and* $X = (x_1, x_2, \ldots, x_n)^T$ *be the Perron vector of* G*. Let* $G' = G - \sum_{i=1}^{s} v_i v + \sum_{i=1}^{s} v_i u$*. If* $x_u \ge x_v$*, then* $\rho(G) < \rho(G')$.

Definition 2.2. ([4]) *Given a graph* G*, the vertex partition* Π*:* $V(G) = V_1 \cup V_2 \cup \ldots \cup V_k$ *is said to be an equitable partition if, for each* $u \in V_i$*,* $|V_j \cap N(u)| = b_{ij}$ *is a constant depending only on* i, j $(1 \le i, j \le k)$*. The matrix* $B_\Pi = (b_{ij})$ *is called the quotient matrix of* G *with respect to* Π.

Lemma 2.3. ([4]) *Let* Π*:* $V(G) = V_1 \cup V_2 \ldots \cup V_k$ *be an equitable partition of a graph* G *with quotient matrix* B_Π*. Then* $det(xI - B_\Pi) \mid det(xI - A(G))$*. Furthermore, the largest eigenvalue of* B_Π *is just the spectral radius of* G.

Lemma 2.4. ([20]) *Let* G^* *be the extremal graph with the maximum spectral radius in* $\mathcal{G}(m, F)$*. Let* $X = (x_1, x_2, \ldots, x_n)^T$ *be the Perron vector of the graph* G^**. If* F *is a 2-connected graph and* $x_{u^*} = max\{x_v \mid v \in V(G^*)\}$*, then the following statements hold.*

(i) G^* *is connected.*

(ii) There exists no cut vertex in $V(G^*) \setminus \{u^*\}$*, and hence* $d(u) \ge 2$ *for any* $u \in V(G^*) \setminus N[u^*]$.

Lemma 2.5. ([2]) *Let* G *be a bipartite graph of size* m*. Then* $\rho(G) \le \sqrt{m}$*, and equality holds if and only if* G *is a disjoint union of a complete bipartite graph and isolated vertices.*

Lemma 2.6. ([13]) $\rho(S^-_{\frac{m+4}{2},2}) > \frac{1+\sqrt{4m-5}}{2}$ *for* $m \ge 6$.

Lemma 2.7. ([13]) *Let* $X = (x_1, x_2, \ldots, x_n)^T$ *be the Perron vector of a connected graph* G *of size* m *and let* $x_{u^\star} = max\{x_v | v \in V(G)\}$ *and* $W = V(G) \setminus N[u^\star]$*. If* $\rho(G) > \frac{1+\sqrt{4m-5}}{2}$ *and there exists a vertex* v *of* G *such that* $x_v < (1-\beta) x_{u^\star}$*, where* $0 < \beta < 1$*, then*

$$e(W) < e(N(u^\star)) - |N(u^\star) \setminus N_0(u^\star)| + \frac{3}{2} - \beta d_{N(u^\star)}(v),$$

for $v \in N^2(u^\star)$.

3 Proof of Theorem 1.4.

Let G^* be the extremal graph with maximum spectral radius in $\mathcal{G}(m, F)$. Let $\rho = \rho(G^*)$ and $X^* = (x_1, x_2, \dots, x_n)^T$ be the Perron vector of G^* with coordinate x_v corresponding to the vertex $v \in V(G^*)$. A vertex u^* in G^* is said to be an extremal vertex if $x_{u^*} = max\{x_v \mid v \in V(G^*)\}$. Let $W = V(G^*)\backslash N[u^*]$ and $N_+(u^*) = N(u^*) \setminus N_0(u^*)$, where $N_0(u^*)$ denotes the set of isolated vertices of $G^*[N(u^*)]$. Let $W_H = \cup_{u \in V(H)} N_W(u)$ for any component H of $G^*[N(u^*)]$.

Lemma 3.1. *Let $G^* \in \mathcal{G}(m, \theta(1,3,3))$. Then $G^*[N(u^*)]$ is P_5-free, that is, each component H of $G^*[N(u^*)]$ is one of the following.*

(i) a graph with C_4 as its spanning subgraph, that is, C_4, $\theta(1,2,2)$ or K_4;

(ii) a copy of S^1_{r+1} for $r \geq 2$, where S^1_3 is a triangle for $r = 2$;

(iii) a double star $D_{a,b}$ for $a, b \geq 1$, which is obtained from two stars $K_{1,a}$ and $K_{1,b}$ by joining a new edge between their centers;

(iv) a star $K_{1,r}$ for $r \geq 0$, where $K_{1,0}$ is a singleton component.

Lemma 3.2. ([7]) *For any non-trivial component H in $G^*[N(u^*)]$, if H contains a cycle of length four, then $N_W(u) \cap N_W(v) = \emptyset$ for any two vertices u and v in the cycle of length four.*

Since $A(G^*)X = \rho X$, we have

$$\rho x_{u^*} = \sum_{u \in N_+(u^*)} x_u + \sum_{u \in N_0(u^*)} x_u.$$

Furthermore, we have

$$\begin{aligned} \rho^2 x_{u^*} &= \sum_{u \in N_+(u^*)} \rho x_u + \sum_{u \in N_0(u^*)} \rho x_u \\ &= |N_+(u^*)| x_{u^*} + \sum_{u \in N_+(u^*)} d_{N(u^*)}(u) x_u + \sum_{w \in N_W(u)} d_W(u) x_w + |N_0(u^*)| x_{u^*} \\ &= d(u^*) x_{u^*} + \sum_{u \in N_+(u^*)} d_{N(u^*)}(u) x_u + \sum_{w \in N^2(u^*)} d_{N(u^*)}(w) x_w. \end{aligned}$$

Therefore,

$$\begin{aligned} (\rho^2 - \rho) x_{u^*} &= d(u^*) x_{u^*} + \sum_{v \in N_+(u^*)} (d_{N(u^*)}(v) - 1) x_v + \sum_{w \in N^2(u^*)} d_{N(u^*)}(w) x_w - \sum_{v \in N_0(u^*)} x_v \\ &\leq |N(u^*)| x_{u^*} + \sum_{u \in N_+(u^*)} (d_{N(u^*)}(u) - 1) x_u + e(N(u^*), W) x_{u^*} - \sum_{u \in N_0(u^*)} x_u. \end{aligned} \tag{1}$$

Note that $S^-_{\frac{m+4}{2},2}$ is $\theta(1,3,3)$-free, we have $\rho \geq \rho(S^-_{\frac{m+4}{2},2}) > \frac{1+\sqrt{4m-5}}{2} > 10$ for $m \geq 92$. By Lemma 2.6, we have

$$(\rho^2 - \rho) x_{u^*} > (m - \frac{3}{2}) x_{u^*} = (|N(u^*)| + e(N_+(u^*)) + e(N(u^*), W) + e(W) - \frac{3}{2}) x_{u^*}. \tag{2}$$

Combining with (1) and (2), we get

$$\sum_{u\in N_+(u^*)}(d_{N(u^*)}(u)-1)x_u > \left(e(N_+(u^*))+e(W)+\sum_{u\in N_0(u^*)}\frac{x_u}{x_{u^*}}-\frac{3}{2}\right)x_{u^*}.$$

Let $\mathcal{H}$ be the set of all non-trivial components in $G^*[N(u^*)]$. For each non-trivial connected component H of $\mathcal{H}$, we denote $\zeta(H)=\sum_{u\in V(H)}(d_H(u)-1)x_u$. Obviously,

$$\sum_{H\in\mathcal{H}}\zeta(H) > \left(e(N_+(u^*))+e(W)+\sum_{u\in N_0(u^*)}\frac{x_u}{x_{u^*}}-\frac{3}{2}\right)x_{u^*}. \tag{3}$$

Lemma 3.3. $G^*[N(u^*)]$ *contains no any cycle of length four.*

Proof. Let $\mathcal{H}'$ be the family of components of $G^*[N(u^*)]$ each of which contains C_4 as a spanning subgraph and $\mathcal{H}\setminus\mathcal{H}'$ be the family of other non-trivial components of $G^*[N(u^*)]$ each of which contains no C_4 as a spanning subgraph. By Lemma 3.1 (*ii*)-(*iv*), for each $H\in\mathcal{H}\setminus\mathcal{H}'$, we have

$$\zeta(H)=\sum_{v\in V(H)}(d_H(v)-1)x_v\leq(2e(H)-|V(H)|)x_{u^*}\leq e(H)x_{u^*}.$$

Next we show that

$$\zeta(H)<(e(H)-\frac{3}{2})x_{u^*}+\frac{2\sum_{w\in W_H}x_w}{\rho-3}$$

for each $H\in\mathcal{H}'$. Let $H^*\in\mathcal{H}'$ with $V(H^*)=\{u_1,u_2,u_3,u_4\}$ and the cycle of length four be $u_1u_2u_3u_4u_1$.

First, we consider $W_{H^*}=\emptyset$. Assume that $x_{u_1}=max\{x_{u_i}|1\leq i\leq 4\}$. Then

$$\rho x_{u_1}=\sum_{u\in N(u_1)}x_u\leq x_{u^*}+x_{u_2}+x_{u_3}+x_{u_4}\leq x_{u^*}+3x_{u_1}.$$

Hence, $x_{u_1}\leq\frac{1}{\rho-3}x_{u^*}<\frac{1}{4}x_{u^*}$ for $\rho\geq\rho(S^-_{\frac{m+4}{2},2})>\frac{1+\sqrt{4m-5}}{2}>10$ due to $m\geq 92$. Furthermore,

$$\zeta(H^*)\leq(2e(H^*)-|V(H^*)|)x_{u_1}<(e(H^*)-\frac{3}{2})x_{u^*}+\frac{2\sum_{w\in W_{H^*}}x_w}{\rho-3},$$

as desired.

In the following, we assume that $W_{H^*}\neq\emptyset$, we consider the following two cases.

Case 1. All vertices in W_{H^*} have a unique common neighbor in $V(H^*)$.

Without loss of generality, let $u_1\in V(H^*)$ be the unique common neighbor. Thus, $N_W(u_i)=\emptyset$ for $i\in\{2,3,4\}$. Assume that $x_{u_2}=max\{x_{u_i}|2\leq i\leq 4\}$, we have

$$\rho x_{u_2}\leq x_{u^*}+x_{u_1}+x_{u_3}+x_{u_4}\leq 2x_{u_2}+2x_{u^*}.$$

Thus, $x_{u_2} \leq \frac{2}{\rho-2} x_{u^*} < \frac{1}{4} x_{u^*}$ due to $\rho > 10$. Thus,

$$\begin{aligned}\zeta(H^*) &= \sum_{u \in V(H^*)} (d_{H^*}(u) - 1) x_u \\ &\leq (d_{H^*}(u_1) - 1) x_{u_1} + (2e(H^*) - d_{H^*}(u_1) - 3) x_{u_2} \\ &< \left(d_{H^*}(u_1) - 1 + \frac{1}{2} e(H^*) - \frac{1}{4} d_{H^*}(u_1) - \frac{3}{4} \right) x_{u^*} \\ &\leq (\frac{3}{4} d_{H^*}(u_1) + \frac{1}{2} e(H^*) - \frac{7}{4}) x_{u^*} \\ &\leq (\frac{1}{2} e(H^*) + \frac{1}{2}) x_{u^*} \\ &< (e(H^*) - \frac{3}{2}) x_{u^*} + \frac{2 \sum_{w \in W_{H^*}} x_w}{\rho - 3}\end{aligned}$$

due to $d_{H^*}(u_1) \leq 3$, as desired.

Case 2. There are at least two distinct vertices of W_{H^*} such that they have distinct neighbors in $V(H^*)$. Since

$$\begin{cases} \rho x_{u_1} \leq x_{u^*} + x_{u_2} + x_{u_3} + x_{u_4} + \sum_{w \in N_{W_{H^*}}(u_1)} x_w, \\ \rho x_{u_2} \leq x_{u^*} + x_{u_1} + x_{u_3} + x_{u_4} + \sum_{w \in N_{W_{H^*}}(u_2)} x_w, \\ \rho x_{u_3} \leq x_{u^*} + x_{u_1} + x_{u_2} + x_{u_4} + \sum_{w \in N_{W_{H^*}}(u_3)} x_w, \\ \rho x_{u_4} \leq x_{u^*} + x_{u_1} + x_{u_2} + x_{u_3} + \sum_{w \in N_{W_{H^*}}(u_4)} x_w, \end{cases}$$

we obtain

$$\rho(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4}) \leq 3(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4}) + 4x_{u^*} + \sum_{i=1}^{4} \sum_{w \in N_{W_{H^*}}(u_i)} x_w.$$

By Lemma 3.2, we get $N_{W_{H^*}}(u_i) \cap N_{W_{H^*}}(u_j) = \emptyset$ for arbitrary two distinct vertices $u_i, u_j \in V(H^*)$. Thus, $\sum_{w \in N_{W_{H^*}}} x_w = \sum_{w \in N_W(V(H^*))} x_w = \sum_{i=1}^{4} \sum_{w \in N_{W_{H^*}}(u_i)} x_w$. Combining $\rho > 10$, we obtain

$$\begin{aligned}x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4} &\leq \frac{4x_{u^*}}{\rho - 3} + \frac{\sum_{w \in W_{H^*}} x_w}{\rho - 3} \\ &< \frac{4}{7} x_{u^*} + \frac{\sum_{w \in W_{H^*}} x_w}{\rho - 3}.\end{aligned}$$

Hence, by the definition of $\zeta(H^*)$,

$$\begin{aligned}\zeta(H^*) &\leq 2(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4}) \\ &< \frac{8}{7} x_{u^*} + \frac{2 \sum_{w \in W_{H^*}} x_w}{\rho - 3} \\ &< (e(H^*) - \frac{3}{2}) x_{u^*} + \frac{2 \sum_{w \in W_{H^*}} x_w}{\rho - 3}.\end{aligned}$$

Thus, we obtain $\zeta(H) < (e(H) - \frac{3}{2})x_{u^*} + \frac{2\sum_{w\in W_H} x_w}{\rho-3}$ for each $H \in \mathcal{H}'$. Recall that $\zeta(H) \le e(H)x_{u^*}$ for each $H \in \mathcal{H} \setminus \mathcal{H}'$. Furthermore,

$$\begin{aligned}\sum_{H\in\mathcal{H}} \zeta(H) &= \sum_{H\in\mathcal{H}'} \zeta(H) + \sum_{H\in\mathcal{H}\setminus\mathcal{H}'} \zeta(H)\\ &< \sum_{H\in\mathcal{H}'} (e(H) - \frac{3}{2})x_{u^*} + \sum_{H\in\mathcal{H}'} \frac{2\sum_{w\in W_H} x_w}{\rho - 3} + \sum_{H\in\mathcal{H}\setminus\mathcal{H}'} e(H)x_{u^*}\\ &= e(N_+(u^*))x_{u^*} - \frac{3}{2}\sum_{H\in\mathcal{H}'} x_{u^*} + \sum_{H\in\mathcal{H}'} \frac{2\sum_{w\in W_H} x_w}{\rho - 3}.\end{aligned}$$

For any $H \in \mathcal{H}'$ satisfying $W_H = \emptyset$, we have $\sum_{w\in W_H} x_w = 0$. For any $H \in \mathcal{H}'$ satisfying $W_H \neq \emptyset$ and any $w \in W_H$, since G^* is $\theta(1, 3, 3)$-free, we get $W_H \cap W_{N(u^*)\setminus H} = \emptyset$. Then $d_{N(u^*)\setminus H}(w) = 0$. By Lemma 3.2, we have $d_H(w) = 1$. Furthermore, $d_{N(u^*)}(w) = 1$ and $d_W(w) \ge 1$ from Lemma 2.4. Thus, $\sum_{H\in\mathcal{H}'}\sum_{w\in W_H} x_w \le \sum_{H\in\mathcal{H}'}\sum_{w\in W_H} d_W(w)x_w \le \sum_{H\in\mathcal{H}'}\sum_{w\in W_H} d_W(w)x_{u^*} \le 2e(W)x_{u^*}$. Note that $\rho > 10$. Thus,

$$\begin{aligned}\sum_{H\in\mathcal{H}} \zeta(H) &< e(N_+(u^*))x_{u^*} - \frac{3}{2}\sum_{H\in\mathcal{H}'} x_{u^*} + \sum_{H\in\mathcal{H}'} \frac{2\sum_{w\in W_H} x_w}{\rho - 3}\\ &< (e(N_+(u^*)) + \frac{4}{7}e(W) - \sum_{H\in\mathcal{H}'} \frac{3}{2})x_{u^*}\\ &< (e(N_+(u^*)) + e(W) - \sum_{H\in\mathcal{H}'} \frac{3}{2})x_{u^*},\end{aligned}$$

which contradicts (3). Furthermore, $G^*[N(u^*)]$ contains no C_4. This completes the proof of Lemma 3.3. □

By Lemma 3.1, we know that each non-trivial component of $G^*[N(u^*)]$ is either a tree or a unicyclic graph S^1_{r+1} with $r \ge 2$. Let c be the number of non-trivial tree components of $G^*[N(u^*)]$. Then

$$\sum_{H\in\mathcal{H}} \zeta(H) \le \sum_{H\in\mathcal{H}}\sum_{u\in V(H)} (d_H(u) - 1)x_{u^*} = \sum_{H\in\mathcal{H}} (2e(H) - |H|)x_{u^*} = (e(N_+(u^*)) - c)x_{u^*}.$$

Combining (1), we get

$$e(W) < \frac{3}{2} - c - \sum_{u\in N_0(u^*)} \frac{x_u}{x_{u^*}}. \tag{4}$$

Thus, $e(W) \le 1$ and $c \le 1$. In addition, if $e(W) = 1$, then $c = 0$ and $\sum_{u\in N_0(u^*)} \frac{x_u}{x_{u^*}} < \frac{1}{2}$.

Lemma 3.4. $e(W) = 0$.

Proof. Suppose on the contrary that $e(W) = 1$. In this case, we have $c = 0$ and $\sum_{u\in N_0(u^*)} \frac{x_u}{x_{u^*}} < \frac{1}{2}$. It follows that each component of $G^*[N(u^*)]$ is isomorphic to a unicyclic graph S^1_{r+1} with $r \ge 2$. That is, each component of $G^*[N(u^*)]$ contains a triangle. Let H^* be a component of $G^*[N(u^*)]$ and $u_1u_2u_3$ is a C_3 of H^*. Let w_1w_2 be the unique edge of $e(W)$. By Lemma 2.4, we obtain that $d_{N(u^*)}(w_i) \geq 1$ for each $i \in \{1,2\}$. If $H^* \cong S_3^1$, then we obtain $d_{S_3^1}(w) \leq 3$. If $H^* \cong S_{r+1}^1$ with $r \geq 3$, then let $d_{H^*}(u_2) = d_{H^*}(u_3) = 2$ and $u_4, u_5, \cdots, u_{r+1}$ be the neighbors of u_1. For $w \in W_{H^*}$, we claim $d_{H^*}(w) \leq 1$. Otherwise, we consider the following five cases in the sense of symmetry. If $\{u_1, u_2\} \subseteq N_{H^*}(w)$, then u_1u_3,$u_1wu_2u_3$,$u_1u_4u^*u_3$ are three internally disjoint paths of lengths $1,3,3$ between u_1 and u_3, a contradiction. If $\{u_2, u_3\} \subseteq N_{H^*}(w)$, then u_1u_3,$u_1u_2wu_3$,$u_1u_4u^*u_3$ are three internally disjoint paths of lengths $1,3,3$ between u_1 and u_3, a contradiction. If $\{u_1, u_4\} \subseteq N_{H^*}(w)$, then u_1u^*,$u_1u_2u_3u^*$,$u_1wu_4u^*$ are three internally disjoint paths of lengths $1,3,3$ between u_1 and u^*, a contradiction. If $\{u_2, u_4\} \subseteq N_{H^*}(w)$, then u^*u_2,$u_2wu_4u^*$,$u_2u_1u_3u^*$ are three internally disjoint paths of lengths $1,3,3$ between u^* and u_2, a contradiction. If $\{u_4, u_5\} \subseteq N_{H^*}(w)$, then u^*u_4,$u_4wu_5u^*$,$u_4u_1u_3u^*$ are three internally disjoint paths of lengths $1,3,3$ between u^* and u_4, a contradiction. Thus, $d_{H^*}(w) \leq 1$ for $w \in W_{H^*}$ and $H^* \cong S_{r+1}^1$ with $r \geq 3$. In addition, we can check that $W_{H^*} \cap W_{N(u^*)\setminus H^*} = \emptyset$ for $H^* \cong S_{r+1}^1$ with $r \geq 2$. Consequently, $d(w) = d_{H^*}(w)$ for $w \in W_{H^*}$. Furthermore, let $x_{w_1} \geq x_{w_2}$, we consider the following two cases.

Case 1. At least a vertex $w_i \in \cup_{j=1}^3 N_W(u_j)$ for some $i \in \{1,2\}$.

Subcase 1.1. $w_1, w_2 \in \cup_{j=1}^3 N_W(u_j)$.

In this case, we obtain $(\cup_{j=1}^3 N_W(u_j)) \cap (W_{N(u^*)} \setminus \{\cup_{j=1}^3 N_W(u_j)\}) = \emptyset$. Thus, $N_{N(u^*)}(w) \subseteq \cup_{j=1}^3 N_W(u_j)$ for each $w \in \cup_{j=1}^3 N_W(u_j)$. If $N_{C_3}(w_1) \cap N_{C_3}(w_2) = \emptyset$, then there exists a $\theta(1,3,3)$, a contradiction. If $|N_{C_3}(w_1) \cap N_{C_3}(w_2)| \geq 2$, then there exists a $\theta(1,3,3)$, a contradiction. Thus, $|N_{C_3}(w_1) \cap N_{C_3}(w_2)| = 1$ and hence there exists a cut vertex, which contradicts Lemma 2.4 (ii).

Subcase 1.2. $w_1 \in \cup_{j=1}^3 N_W(u_j)$ and $w_2 \notin \cup_{j=1}^3 N_W(u_j)$.

In this case, suppose that $w_2 \in W_{H^*}$, where $H^* \cong S_{r+1}^1$ with $r \geq 3$. Then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u_1} + x_{u_2} + x_{u_3} \leq x_{w_2} + 3x_{u^*}, \\ \rho x_{w_2} = x_{w_1} + x_{u_k} \leq x_{w_1} + x_{u^*}, \end{cases}$$

where $u_k \in V(H^*) \setminus \{u_1, u_2, u_3\}$. Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{3\rho+1}{\rho^2-1} x_{u^*}$. Since $f(x) = \frac{3x+1}{x^2-1}$ is decreasing with $x > 10$, we have $x_{w_1} \leq \frac{3\rho+1}{\rho^2-1} x_{u^*} < \frac{31}{99} x_{u^*}$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{68}{99} = \frac{161}{198}$, a contradiction. Suppose that $w_2 \in W_{H^\star}$, where $H^\star \subseteq N_+(u^*) \setminus H^*$ and $H^\star \cong S_{r'+1}^1$ with $r' \geq 2$. Furthermore, if $H^* \cong S_3^1 : u_1u_2u_3$ and $H^\star \cong S_3^1 : u_1'u_2'u_3'$, then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u_1} + x_{u_2} + x_{u_3} \leq x_{w_2} + 3x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + x_{u_1'} + x_{u_2'} + x_{u_3'} \leq x_{w_1} + 3x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{3}{\rho-1} x_{u^*} < \frac{1}{3} x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$, a contradiction. If $H^* \cong S_3^1$ and $H^\star \cong S_{r'+1}^1$ with $r' \geq 3$, then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u_1} + x_{u_2} + x_{u_3} \leq x_{w_2} + 3x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{3\rho+1}{\rho^2-1} x_{u^*}$. Since $f(x) = \frac{3x+1}{x^2-1}$ is decreasing with $x > 10$, we have $x_{w_1} \leq \frac{3\rho+1}{\rho^2-1} x_{u^*} < \frac{31}{99} x_{u^*}$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{68}{99} = \frac{161}{198}$, a contradiction. If $H^* \cong S_{r+1}^1$ with $r \geq 3$ and $H^\star \cong S_{r'+1}^1$ with $r' \geq 3$, then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{1}{\rho-1}x_{u^*} < \frac{1}{9}x_{u^*}$ due to $x > 10$. By Lemma 2.7, we get $1 = e(W) < 0+\frac{3}{2}-\frac{8}{9} = \frac{11}{18}$, a contradiction. Suppose that $w_2 \in W_{N_0(u^*)}$ and $H^* \cong S_3^1$, then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u_1} + x_{u_2} + x_{u_3} \leq x_{w_2} + 3x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + \sum_{u \in N_0(u^*)} x_u < x_{w_1} + \frac{1}{2}x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} < \frac{3\rho+\frac{1}{2}}{\rho^2-1}x_{u^*} < \frac{30\frac{1}{2}}{99}$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{68\frac{1}{2}}{99} = \frac{80}{99}$, a contradiction. Suppose that $w_2 \in W_{N_0(u^*)}$ and $H^* \cong S_{r+1}^1$ with $r \geq 3$, then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + \sum_{u \in N_0(u^*)} x_u < x_{w_1} + \frac{1}{2}x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} < \frac{2\rho-1}{2\rho^2-\rho-2}x_{u^*} < \frac{19}{188}$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{169}{188} = \frac{113}{188}$, a contradiction.

Case 2. $w_i \notin \cup_{j=1}^3 N_W(u_j)$ for any $i \in \{1, 2\}$.

Suppose that $w_1, w_2 \in W_{H^*}$, where $H^* \cong S_{r+1}^1$ for $r \geq 3$. Then $N_{H^*}(w_1) \cap N_{H^*}(w_2) = \emptyset$. Otherwise there exists a cut vertex, which contradicts Lemma 2.4 (ii). By lemma 2.4, we get $|N_{H^*}(w_i)| = 1$ for each $i \in \{1, 2\}$. Moreover,

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{1}{\rho-1}x_{u^*} < \frac{1}{9}$ due to $\rho > 10$. By Lemma 2.7, we get $1 = e(W) < 0+\frac{3}{2}-\frac{8}{9} = \frac{11}{18}$, a contradiction. Suppose that $w_1 \in W_{H^*}$, where $H^* \cong S_{r+1}^1$ for $r \geq 3$ and $w_2 \in W_{H^\star}$, where $H^\star \cong S_{r'+1}^1$ for $r' \geq 3$ and $H^\star \in N_+(u^*) \backslash H^*$. Then $|N_{H^\star}(w_2)| = 1$. Similar with above, we get a contradiction. Suppose that $w_1 \in W_{H^*}$, where $H^* \cong S_{r+1}^1$ for $r \geq 3$ and $w_2 \in W_{N_0(u^*)}$. Then $|N_{H^*}(w_1)| = 1$ and $|N_{N_0(u^*)}(w_2)| \leq |N_0(u^*)|$. Thus, we get

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + \sum_{u \in N_0(u^*)} x_u < x_{w_1} + \frac{1}{2}x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{\rho+\frac{1}{2}}{\rho^2-1}x_{u^*} < \frac{10\frac{1}{2}}{99}$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{88\frac{1}{2}}{99} = \frac{60}{99}$, a contradiction. Suppose that $w_1, w_2 \in W_{N_0(u^*)}$ for each $i \in \{1, 2\}$. Then $|N_{N_0(u^*)}(w_i)| \leq |N_0(u^*)|$ for each $i \in \{1, 2\}$. Thus, we get

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + \sum_{u \in N_0(u^*)} x_u < x_{w_1} + \frac{1}{2}x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + \sum_{u \in N_0(u^*)} x_u < x_{w_1} + \frac{1}{2}x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} < \frac{1}{2(\rho-1)}x_{u^*} < \frac{1}{18}x_{u^*}$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{17}{18} = \frac{5}{9}$, a contradiction. This completes the proof of Lemma 3.4. □

Lemma 3.5. *$G^*[N(u^*)]$ contains no triangle.*

Proof. Suppose on the contrary that $G^*[N(u^*)]$ contains triangles. Then $G^*[N(u^*)]$ contains a component which is isomorphic to S^1_{r+1} for $r \geq 2$. Let $H^* \cong S^1_{r+1}$ be a component of $G^*[N(u^*)]$. Then $e(H^*) = r+1$. Let $u_1u_2u_3u_1$ be the triangle of H^* and $d_{H^*}(u_1) = d_{H^*}(u_2)$. If $W_{H^*} = \emptyset$, then $x_{u_1} = x_{u_2}$. Furthermore,

$$\rho x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} \leq x_{u_1} + 2x_{u^*}.$$

Thus, $x_{u_1} \leq \frac{2}{\rho-1}x_{u^*} < \frac{2}{9}x_{u^*}$ due to $\rho > 10$.

$$\begin{aligned}\zeta(H^*) &= x_{u_1} + x_{u_2} + (r-1)x_{u_3} \\ &< (\frac{4}{9} + r - 1)x_{u^*} \\ &= (e(H^*) - \frac{14}{9})x_{u^*}.\end{aligned}$$

Recall that $\zeta(H) \leq e(H)x_{u^*}$ for $H \in \mathcal{H} \setminus H^*$. Thus,

$$\begin{aligned}\sum_{H\in\mathcal{H}} \zeta(H) &= \zeta(H^*) + \sum_{H\in\mathcal{H}\setminus H^*} \zeta(H) \\ &< (e(H^*) - \frac{14}{9})x_{u^*} + \sum_{H\in\mathcal{H}\setminus H^*} e(H)x_{u^*}, \\ &= (e(N_+(u^*)) - \frac{14}{9})x_{u^*}\end{aligned}$$

which contradicts (3). Thus, $W_{H^*} \neq \emptyset$.

Since $e(W) = 0$, combining Lemma 2.4, we have $d_{N(u^*)}(w) \geq 2$. If $r \geq 3$, then $W_{H^*} \cap W_{N(u^*)\setminus H^*} = \emptyset$, and hence $d(w) = d_{H^*}(w) \leq 1$ for $w \in W_{H^*}$, a contradiction. Thus, $r = 2$, that is H^* is a triangle $u_1u_2u_3$. Since G^* is $\theta(1,3,3)$-free, we obtain that $W_{H^*} \cap W_{N(u^*)\setminus H^*} = \emptyset$, and hence $d(w) = d_{H^*}(w) \leq 3$. First, we assume that $|W_{H^*}| = 1$. Let $W_{H^*} = \{w\}$. As H^* is a triangle, we obtain $2 \leq d(w) = d_{H^*}(w) \leq 3$. We consider two cases as follows.

Case 1. $d_{H^*}(w) = 2$.

In this case, without loss of generality, we suppose $N(w) = \{u_1, u_2\}$. Then $x_{u_1} = x_{u_2}$. Since

$$\rho x_{u_3} = x_{u_1} + x_{u_2} + x_{u^*} \leq 3x_{u^*}.$$

Furthermore, $x_{u_3} \leq \frac{3}{\rho}x_{u^*}$. Similarly,

$$\rho x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} + x_w \leq x_{u_1} + \frac{3}{\rho}x_{u^*} + 2x_{u^*},$$

we obtain that $x_{u_1} \leq \frac{2\rho+3}{\rho(\rho-1)}x_{u^*}$. Thus,

$$\zeta(H^*) = x_{u_1} + x_{u_2} + x_{u_3} \leq \frac{7\rho+3}{\rho(\rho-1)}x_{u^*}.$$

Since $\frac{7x+3}{x(x-1)}$ is decreasing in variable $x > 10$, we get

$$\zeta(H^*) < \frac{73}{90}x_{u^*} < (e(H^*) - \frac{3}{2})x_{u^*}.$$

Recall that $\zeta(H) \leq e(H)x_{u^*}$ for $H \in \mathcal{H} \setminus H^*$. Thus,

$$\begin{aligned}\sum_{H\in\mathcal{H}} \zeta(H) = \zeta(H^*) + &\sum_{H\in\mathcal{H}\setminus H^*} \zeta(H) \\ &< (e(H^*) - \frac{3}{2})x_{u^*} + \sum_{H\in\mathcal{H}\setminus H^*} \zeta(H), \\ &= (e(N_+(u^*)) - \frac{3}{2})x_{u^*}\end{aligned}$$

which contradicts (3).

Case 2. $d_{H^*}(w) = 3$.

In this case, let $N_{H^*}(w) = \{u_1, u_2, u_3\}$. Then $x_{u_1} = x_{u_2} = x_{u_3}$. Since

$$\rho x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} + x_w \leq 2x_{u_1} + 2x_{u^*},$$

we obtain $x_{u_1} \leq \frac{2}{\rho-2}x_{u^*}$. Furthermore,

$$\zeta(H^*) = x_{u_1} + x_{u_2} + x_{u_3} \leq \frac{6}{\rho - 2}x_{u^*} < \frac{3}{4}x_{u^*} < (e(H^*) - \frac{3}{2})x_{u^*}.$$

Thus,

$$\sum_{H\in\mathcal{H}} \zeta(H) < (e(N_+(u^*)) - \frac{3}{2})x_{u^*},$$

a contradiction. Thus, $|W_{H^*}| \geq 2$.

Recall that $W_{H^*} \cap W_{N(u^*)\setminus H^*}(w) = \emptyset$ for any $w \in W_{H^*}$ and $2 \leq d(w) = d_{H^*}(w) \leq 3$. Let $w_1 \in W_{H^*}$ such that $N(w_1) = \{u_1, u_2, u_3\}$ and $w_2 \neq w_1$ in W_{H^*} satisfying $d(w_2) \geq 2$. Suppose that $u_1, u_2 \in N(w_2)$. Then $u^*u_1, u^*u_3w_1u_1$ and $u^*u_2w_2u_1$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and u_1, a contradiction. Hence, $d(w) = d_{H^*}(w) = 2$ for any $w \in W_{H^*}$. This implies $1 \leq |N(w_1) \cap N(w_2)| \leq 2$ for any two vertices $w_1, w_2 \in W_{H^*}$. If $|N(w_1) \cap N(w_2)| = 1$, without loss of generality, let $N(w_1) = \{u_1, u_2\}$ and $N(w_2) = \{u_1, u_3\}$, then $u^*u_1, u^*u_3w_2u_1$ and $u^*u_2w_1u_1$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and u_1, a contradiction. Thus, $|N(w_1) \cap N(w_2)| = 2$. That is $N(w_1) = N(w_2)$ for any two vertices $w_1, w_2 \in W_{H^*}$. Without loss of generality, suppose that $N(w) = \{u_1, u_2\}$ for any $w \in W_{H^*}$. Let $G_1 = G^* - \{u_1w | w \in N_W(u_1)\} + \{u^*w | w \in N_W(u_1)\}$. Obviously, G_1 is $\theta(1, 3, 3)$-free. By Lemma 2.1, we get $\rho(G_1) > \rho$, a contradiction. This completes the proof of Lemma 3.5. □

Proof of Theorem 1.4. By Lemmas 3.1,3.3 and 3.5, we obtain that each component of $G^*[N_+(u^*)]$ is a non-trivial tree. If $c = 0$, then G^* is bipartite. By Lemma 2.5, we have $\rho \leq \sqrt{m} < \frac{1+\sqrt{4m-3}}{2}$ for $m \geq 92$, a contradiction. Thus $c = 1$ and $\sum_{u\in N_0(u^*)} \frac{x_u}{x_{u^*}} < \frac{1}{2}$ from Inequality (4). Let H be the unique component of $G^*[N_+(u^*)]$, where H is a non-trivial tree. By Lemma 3.1 (iii) and (iv), we have $diam(H) \leq 3$.

If $diam(H) = 3$, then H is a double star. Let u_1 and u_2 be the two center vertices of H. Let $\{v_1, v_2, \cdots, v_a\} \in N_H(u_1) \setminus u_2$ and let $\{z_1, z_2, \cdots, z_b\} \in N_H(u_2) \setminus u_1$ for $a, b \geq 1$. If $W_H = \emptyset$, without loss of generality, assume that $x_{u_1} \geq x_{u_2}$, then let $G_2 = G^* - \{u_2v | v \in N_H(u_2) \setminus \{u_1\}\} + \{u_1v | v \in N_H(u_2) \setminus \{u_2\}\}$. We can verify that G_2 is $\theta(1, 3, 3)$-free. By Lemma 2.1, we get $\rho(G_2) > \rho$, a contradiction. Thus, $W_H \neq \emptyset$. It is easily checked $W_H \cap W_{N(u^*)\setminus H} = \emptyset$ for any $w \in W_H$. Hence, $N(w) \subseteq V(H)$ and by Lemma 2.4, $d(w) = d_H(w) \geq 2$. We claim

$d(w) = d_H(w) = 2$. Suppose contrary that $d_H(w) \geq 3$. If $\{u_1, u_2, v_1\} \in W_H(w)$, then u_1u_2, $u_1v_1wu_2$, $u_1u^*z_1u_2$ are three internally disjoint paths of lengths $1, 3, 3$ between u_1 and u_2, a contradiction. If $\{u_1, v_1, v_2\} \in W_H(w)$, then u^*v_2, $u^*v_1wv_2$, $u^*u_2u_1v_2$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and v_2, a contradiction. If $\{u_2, v_1, v_2\} \in W_H(w)$, then u^*u_2, $u^*v_1wu_2$, $u^*v_2u_1u_2$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and u_2, a contradiction. If $\{v_1, v_2, z_1\} \in W_H(w)$, then v_1u^*, $v_1wv_2u^*$, $v_1u_1u_2u^*$ are three internally disjoint paths of lengths $1, 3, 3$ between v_1 and u^*, a contradiction. If $\{v_1, v_2, v_3\} \in W_H(w)$, then u^*v_1, $u^*v_3wv_1$, $u^*u_2u_1v_1$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and v_1, a contradiction. Thus, $d(w) = d_H(w) = 2$. We claim $N(w) = \{u_1, u_2\}$. Otherwise, we consider five cases in the sense of symmetry as follows. If $N(w) = \{u_1, v_1\}$, then u^*u_1, $u^*v_1wu_1$, $u^*z_1u_2u_1$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and u_1, a contradiction. If $N(w) = \{u_1, z_1\}$, then u_1u_2, $u_1wu^*z_1u_2$, $u_2u^*z_1u_1$ are three internally disjoint paths of lengths $1, 3, 3$ between u_1 and u_2, a contradiction. If $N(w) = \{v_1, v_2\}$, then u^*v_1, $u^*v_2wv_1$, $u^*u_2u_1v_1$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and v_1, a contradiction. If $N(w) = \{v_1, z_1\}$, then u^*v_1, $u^*z_1wv_1$, $u^*u_2u_1v_1$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and v_1, a contradiction. Thus, $N(w) = \{u_1, u_2\}$. Let $G_3 = G^* - \{u_2w | w \in N_W(u_2)\} + \{u^*w | w \in N_W(u_2)\}$. We can verify that G_3 is $\theta(1, 3, 3)$-free. By Lemma 2.1, we get $\rho(G_3) > \rho$, a contradiction.

If $diam(H) \leq 2$, then $H \cong K_{1,r}$ with $r \geq 1$. Let $V(H) = \{u_0, u_1, \cdots, u_r\}$ and u_0 be the center vertex of H with $r \geq 1$. We claim $r \geq 9$. By $\rho > 10$ and Inequality (4), we have

$$10x_{u^*} < \rho x_{u^*} = x_{u_0} + x_{u_1} + \cdots + x_{u_r} + \sum_{v \in N_0(u^*)} x_v < (r + 1 + \frac{1}{2})x_{u^*}.$$

Thus, $r \geq 9$. We claim $W_H = \emptyset$. Suppose on the contrary that $W_H \neq \emptyset$.

First, we assume $d_H(w) \geq 3$ for any vertex $w \in W_H$. If $\{u_0, u_1, u_2\} \in N_H(w)$, then $u^*u_1, u^*u_2wu_1$ and $u^*u_3u_0u_1$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and u_1, a contradiction. If $\{u_1, u_2, u_3\} \in N_H(w)$, then $u^*u_3, u^*u_4u_0u_3$ and $u^*u_1wu_3$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and u_3, a contradiction. Thus, $d_H(w) \leq 2$. If $d_H(w) = 1$, then we obtain that w is only adjacent to the center vertex u_0. Otherwise, let $N_H(w) = u_1$. By Lemma 2.4 and $e(W) = 0$, we obtain $|N_{N_0(u^*)}(w)| \geq 1$. Then u^*u_1, u^*vwu_1, $u^*u_2u_0u_1$ are three internally disjoint paths of lengths $1, 3, 3$ between u^* and u_1, where $v \in N_{N_0(u^*)}(w)$, a contradiction. Thus, $N_H(w) = \{u_0\}$. By Inequality (4), we have

$$\rho x_w \leq x_{u_0} + \sum_{v \in N_0(u^*)} x_v < \frac{3}{2}x_{u^*}.$$

Thus, $x_w < \frac{3}{2\rho}x_{u^*} < \frac{3}{20}x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $0 = e(W) < -1 + \frac{3}{2} - \frac{17}{20} = -\frac{7}{20}$, a contradiction. If $d_H(w) = 2$, then we have $N_{N_0(u^*)}(w) = \emptyset$. Otherwise, G^* contains a $\theta(1, 3, 3)$. Furthermore, we get

$$\rho x_w \leq x_{u_0} + x_{u_1} \leq 2x_{u^*}.$$

This implies that $x_w \leq \frac{1}{2\rho}x_{u^*} < \frac{1}{20}x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $0 = e(W) < -1 + \frac{3}{2} - \frac{19}{20} < 0$, a contradiction. Thus, $W_H = \emptyset$. If $W \neq \emptyset$, then by Lemma 2.1, we obtain $d(w) = d_{N_0(u^*)}(w)$ for any vertex $w \in W$. Furthermore, $N_0(u^*) \neq \emptyset$ and

$$\rho x_w \leq \sum_{v \in N_0(u^*)} x_v < \frac{1}{2}x_{u^*}.$$

This implies that $x_w < \frac{1}{2\rho}x_{u^*} < \frac{1}{20}x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $0 = e(W) < -1 + \frac{3}{2} - \frac{19}{20} = -\frac{9}{20}$, a contradiction. Thus, $W = \emptyset$ and hence $G^* \cong G_4$ (see Figure. 1). Let $|N_0(u^*)| = t$. Since m is even, we obtain that t is odd and $t \geq 1$. By Lemma 2.3, we obtain that ρ is the largest root of the equation $f(x,t) = 0$ where

$$f(x,t) = x^4 - mx^2 - (m-t-1)x + \frac{t(m-t-1)}{2}$$

for $m = t + 1 + 2r \geq 92$. Since

$$f(x,t) - f(x,1) = (t-1)x + \frac{(t-1)(m-t-2)}{2} > 0$$

for $x > 0$ and $t \geq 3$, which implies that $t = 1$ for the extremal graph G^*. By Lemma 2.6, we have $\rho(S^-_{\frac{m+4}{2},2}) > \frac{1+\sqrt{4m-5}}{2}$ for $m \geq 92$ and $G^* \cong S^-_{\frac{m+4}{2},2}$, as desired. This completes the proof of Theorem 1.4. ■

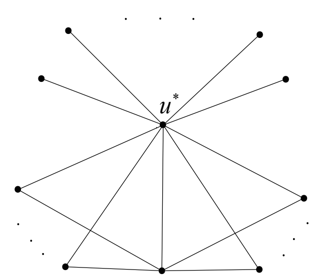

Figure 1: The graph G_4.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no conflict of interest.

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