Double groupoids and 2-groupoids in regular Mal'tsev categories

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Abstract

We prove that the category 2-Grpd($\mathscr{C}$) of internal 2-groupoids is a Birkhoff subcategory of the category Grpd^2($\mathscr{C}$) of double groupoids in a regular Mal'tsev category $\mathscr{C}$ with finite colimits. In particular, when $\mathscr{C}$ is a Mal'tsev variety of universal algebras, the category 2-Grpd($\mathscr{C}$) is also a Mal'tsev variety, of which we describe the corresponding algebraic theory. When $\mathscr{C}$ is a naturally Mal'tsev category, the reflector from Grpd^2($\mathscr{C}$) to 2-Grpd($\mathscr{C}$) has an additional property related to the commutator of equivalence relations. We prove that the category 2-Grpd($\mathscr{C}$) is semi-abelian when $\mathscr{C}$ is semi-abelian, and then provide sufficient conditions for 2-Grpd($\mathscr{C}$) to be action representable.

Keywords: internal 2-groupoid, double groupoid, Mal'tsev variety, Mal'tsev category, semi-abelian category, action representable category, reflective subcategory.
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Introduction

During the last four decades the internal groupoids in the categories of groups, Lie algebras, commutative algebras and more general varieties of universal algebras have been shown to form very rich algebraic categories. They have strong connections with commutator theory (see [7, 18, 22, 14, 5], for instance) and are also of interest in homotopical algebra (see [20, 23, 9, 11], for instance). An internal groupoid in the category Grp of groups can be equivalently presented as a diagram

$$C_1 \xrightarrow[\xleftarrow{e}]{\overset{d}{\underset{c}{\longrightarrow}}} C_0 \tag{1}$$

in the category of groups such that $de = 1_{C_0} = ce$, so that it is a reflexive graph, and the kernels $\mathsf{Ker}(d)$ and $\mathsf{Ker}(c)$ of the "domain" and "codomain" homomorphisms d and c have trivial commutator: $[\mathsf{Ker}(d), \mathsf{Ker}(c)] = \{0\}$ [20]. The "composition" of the internal groupoid is then a homomorphism $m : C_1 \times_{C_0} C_1 \to C_1$, where $C_1 \times_{C_0} C_1$ represents the group of pairs of "composable arrows", and is defined, for any pair $(f, g) \in C_1 \times_{C_0} C_1$, by $m(f, g) = f \cdot 1_Y^{-1} \cdot g$, where $\cdot$ represents the multiplication in the "group of arrows" C_1. A similar description of the category of internal groupoids is given in Mal'tsev varieties [18], which are the varieties of universal algebras whose theory contains a ternary term $p(x, y, z)$ satisying the identities $p(x, y, y) = x$ and $p(x, x, y) = y$ [24]. In particular any variety whose theory contains the operations and the identities of the theory of groups has such a Mal'tsev term, since it contains the ternary term $p(x, y, z) = x \cdot y^{-1} \cdot z$. Also quasigroups, loops, Heyting algebras and MV-algebras are Mal'tsev varieties, and there are then plenty of interesting motivating examples coming both from classical algebra and from algebraic logic.

Many exactness properties of the category Grpd($\mathscr{C}$) of internal groupoids in a Mal'tsev variety $\mathscr{C}$, and in more general algebraic categories, have been established in various articles (see [12, 22, 14, 15, 13], for instance). The notion of regular Mal'tsev category [6] generalizes the one of Mal'tsev variety, in the sense that the *syntactic* property of the existence of the ternary Mal'tsev term $p(x, y, z)$ is replaced by the *semantic* property that any internal reflexive relation in it is necessarily an equivalence relation. Besides all the varietal examples recalled above, among regular Mal'tsev categories we find the categories of topological groups, Banach spaces, $\mathbb{C}^*$-algebras, torsion-free (abelian) groups, cocommutative Hopf algebras over a field, the dual of the category of sets, any abelian category, and many more (see [5] and the references therein).

In the present work we explore the properties of the categories $\mathrm{Grpd}^2(\mathscr{C})$ of internal double groupoids and $2\text{-}\mathrm{Grpd}(\mathscr{C})$ of 2-groupoids in any finitely cocomplete regular Mal'tsev category $\mathscr{C}$. Recall that a double groupoid is simply an internal groupoid in the category $\mathrm{Grpd}(\mathscr{C})$. It has an underlying double reflexive graph that we depict as

$$
\begin{array}{ccc}
C_1^1 & \underset{\underset{e^1}{\xleftarrow{}}}{\overset{\overset{d^1}{\xrightarrow{}}}{\xrightarrow{}}} & C_0^1 \\[2pt]
{\scriptstyle d_1}\Big\downarrow\!{\scriptstyle e_1}\!\Big\uparrow\!\Big\downarrow{\scriptstyle c_1} & {\scriptstyle c^1}\,\underset{\underset{d^0}{\xrightarrow{}}}{\xrightarrow{}}\,{\scriptstyle d_0} & \Big\uparrow{\scriptstyle e_0}\,\Big\downarrow{\scriptstyle c_0} \\[2pt]
C_1^0 & \underset{\underset{c^0}{\xrightarrow{}}}{\overset{\overset{}{\xleftarrow{\;e^0\;}}}{}} & C_0^0.
\end{array}
\tag{2}
$$

Such an internal groupoid is a 2-groupoid precisely when the morphism e^0 is an isomorphism, this property expressing the fact that the "vertical structure" of the double groupoid is discrete.

We prove that the category $2\text{-}\mathrm{Grpd}(\mathscr{C})$ of 2-groupoids in $\mathscr{C}$ is a reflective subcategory of the category $\mathrm{Grpd}^2(\mathscr{C})$ of double groupoids in $\mathscr{C}$, whenever $\mathscr{C}$ is a regular Mal'tsev category with finite colimits. The left adjoint $\mathsf{F} : \mathrm{Grpd}^2(\mathscr{C}) \to 2\text{-}\mathrm{Grpd}(\mathscr{C})$ of the forgetful functor $\mathsf{U} : 2\text{-}\mathrm{Grpd} \to \mathrm{Grpd}^2(\mathscr{C})$ is quite simple to describe, thanks to the property that the category $\mathrm{Grpd}(\mathscr{C})$ is closed in the category $\mathrm{RG}(\mathscr{C})$ of reflexive graphs in $\mathscr{C}$ under regular quotients. It is actually a Birkhoff subcategory (Theorem 2.1), this latter property meaning that $2\text{-}\mathrm{Grpd}(\mathscr{C})$ is also closed in $\mathrm{Grpd}^2(\mathscr{C})$ under subobjects and regular quotients. In particular, when $\mathscr{C}$ is a Mal'tsev variety, this result implies that $2\text{-}\mathrm{Grpd}(\mathscr{C})$ is a subvariety of the Mal'tsev variety $\mathrm{Grpd}^2(\mathscr{C})$, for which Corollary 2.3 provides a precise description of the algebraic theory. In the next section we investigate the same adjunction in the context of naturally Mal'tsev categories [19], where the unit components of the previously considered adjunction turn out to have an additional property (Proposition 3.2). In Section 4 we restrict ourselves to the case where the base category $\mathscr{C}$ is semi-abelian [17], and we investigate the problem of establishing a sufficient condition for the category $2\text{-}\mathrm{Grpd}(\mathscr{C})$ to be action representable in the sense of [2]. By using the recent result in [13] concerning the action representability of the category $\mathrm{Grpd}^2(\mathscr{C})$, we deduce that $2\text{-}\mathrm{Grpd}(\mathscr{C})$ is semi-abelian, action representable, algebraically coherent [8] with normalizers [16] whenever the base category $\mathscr{C}$ has all these properties (Theorem 4.5). This essentially depends on the fact that $2\text{-}\mathrm{Grpd}(\mathscr{C})$ is not only a Birkhoff subcategory of $\mathrm{Grpd}^2(\mathscr{C})$, but it is also coreflective (Lemma 4.3).

1   Preliminaries

In this section, we will recall the notions of internal groupoid, Mal'tsev category, semi-abelian category and Birkhoff subcategory, and some of their properties.

Internal groupoids

In the following, $\mathscr{C}$ will always be a finitely complete category. An internal reflexive graph (1) in $\mathscr{C}$ has an internal category structure if there exists a "composition" morphism $m : C_1 \times_{C_0} C_1 \to C_1$, where $C_1 \times_{C_0} C_1$ is the object part of the pullback of the "codomain" morphism c along the "domain" morphism d as displayed in the diagram

$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\;p_2\;} & C_1 \\
{\scriptstyle p_1}\Big\downarrow & & \Big\downarrow{\scriptstyle d} \\
C_1 & \xrightarrow{\;c\;} & C_0,
\end{array}
$$

that satisfies the usual axioms expressing the incidence, unitality and associativity properties of a category (see [21]). An internal category, i.e. a reflexive graph with a fixed internal category structure, is called an internal groupoid if there exists also an "inverse" morphism $i : C_1 \to C_1$ satisfying the usual incidence and inverse axioms. One can show that i, if it exists, is necessarily unique.

A morphism from a reflexive graph $\mathbb{C}$ to another reflexive graph $\mathbb{D}$ is given by a pair (f_0, f_1) of morphisms in $\mathscr{C}$ as displayed in the diagram

$$
\begin{array}{ccc}
C_1 & \xrightarrow[\substack{\longleftarrow\, e\, \longrightarrow \\ c}]{d} & C_0 \\
f_1 \downarrow & & \downarrow f_0 \\
D_1 & \xrightarrow[\substack{\longleftarrow\, e\, \longrightarrow \\ c}]{d} & D_0,
\end{array}
\tag{3}
$$

such that $f_0 d = d f_1$, $f_0 c = c f_1$ and $f_1 e = e f_0$. An internal functor from an internal category $\mathbb{C}$ to another internal category $\mathbb{D}$ is a morphism of the underlying reflexive graphs of $\mathbb{C}$ and $\mathbb{D}$ that also respects the composition maps in the usual sense.

We denote by $\mathrm{RG}(\mathscr{C})$ the category whose objects are the reflexive graphs in $\mathscr{C}$ and whose morphisms are the morphisms of reflexive graphs just defined. We denote by $\mathrm{Cat}(\mathscr{C})$ the category whose objects are the internal categories in $\mathscr{C}$ and whose morphisms are the internal functors. The full subcategory of $\mathrm{Cat}(\mathscr{C})$ whose objects are the internal groupoids in $\mathscr{C}$ is denoted by $\mathrm{Grpd}(\mathscr{C})$. It is well-known that an internal functor between two internal groupoids automatically preserves the inverse morphisms i.

When $\mathscr{C}$ is finitely complete, so is the category $\mathrm{Grpd}(\mathscr{C})$, and its limits are computed "levelwise". One then defines the category $\mathrm{Grpd}^2(\mathscr{C}) := \mathrm{Grpd}(\mathrm{Grpd}(\mathscr{C}))$ of internal double groupoids in $\mathscr{C}$. Every double groupoid $\mathbb{C}$ in $\mathscr{C}$ has an underlying double reflexive graph (2). We denote by $2\text{-}\mathrm{Grpd}(\mathscr{C})$ the full subcategory of $\mathrm{Grpd}^2(\mathscr{C})$ whose objects are the internal 2-groupoids, these latter being characterized by the property that e^0 is an isomorphism. When this is the case, one clearly has that $d^0 = c^0 = (e^0)^{-1}$.

Mal'tsev categories and semi-abelian categories

As already mentioned in the Introduction, a Mal'tsev category is a finitely complete category such that any internal reflexive relation in it is an equivalence relation. One of their interesting properties is that any reflexive graph in a Mal'tsev category admits at most one internal category structure, and that any internal category is automatically an internal groupoid [7]. This means that the categories $\mathrm{Cat}(\mathscr{C})$ and $\mathrm{Grpd}(\mathscr{C})$ are isomorphic. Moreover, the category $\mathrm{Grpd}(\mathscr{C})$ of internal groupoids in a Mal'tsev category $\mathscr{C}$ is a full subcategory of $\mathrm{RG}(\mathscr{C})$ since any morphism of reflexive graphs between two internal groupoids automatically preserves the composition [7].

Let us recall that a regular category $\mathscr{C}$ is a finitely complete category that has coequalizers of kernel pairs and whose regular epimorphisms are pullback stable. This implies that any morphism in $\mathscr{C}$ can be factorized as a regular epimorphism followed by a monomorphism, and that these factorizations are pullback stable. If $\mathscr{C}$ is a regular Mal'tsev category, then $\mathrm{Grpd}(\mathscr{C})$ is a regular Mal'tsev category [12]. Indeed, this mainly follows from the observation that a morphism (f_0, f_1) as in diagram (3), where $\mathbb{C}$ and $\mathbb{D}$ now represent the reflexive graphs underlying two internal groupoids, is a regular epimorphism in $\mathrm{Grpd}(\mathscr{C})$ if and only if it is so in $\mathrm{RG}(\mathscr{C})$, i.e., both f_0 and f_1 are regular epimorphisms in $\mathscr{C}$. As a consequence, the category $\mathrm{Grpd}(\mathscr{C})$ is closed in $\mathrm{RG}(\mathscr{C})$ under regular quotients, meaning that, if (f_0, f_1) is a morphism of reflexive graphs whose domain is an internal groupoid, then its codomain is also an internal groupoid. Recall that an exact category in the sense of Barr [1] is a regular category in which any equivalence relation is effective, i.e., it is a kernel pair. If $\mathscr{C}$ is an exact Mal'tsev category, then so is the category $\mathrm{Grpd}(\mathscr{C})$.

A semi-abelian category is a category that is pointed, exact, protomodular and that has binary coproducts. In this context protomodularity is equivalent to the validity of the Split Short Five Lemma, i.e., given a diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{\ k\ } & B & \underset{s}{\overset{p}{\rightleftarrows}} & C \\
\parallel & & b\downarrow & & \parallel \\
A & \xrightarrow[k']{} & B' & \underset{s'}{\overset{p'}{\rightleftarrows}} & C,
\end{array}
$$

where $ps = 1_C$, $p's' = 1_C$, $k = \ker(p)$, $k' = \ker(p')$, , $bk = k'$, $p'b = b$ and $bs = s'$, then b is an isomorphism. Any semi-abelian category is a finitely cocomplete and Mal'tsev category. Examples of semi-abelian categories are given by the categories of groups, rings, Lie algebras, cocommutative Hopf algebras over a field, or $\mathbb{C}^*$-algebras. If $\mathscr{C}$ is semi-abelian, then $\mathrm{Grpd}(\mathscr{C})$ is also semi-abelian.

Birkhoff subcategories

In the following, let $\mathscr{X}$ be a replete, i.e. closed under isomorphisms, full reflective subcategory of a category $\mathscr{C}$. It is called a Birkhoff subcategory of $\mathscr{C}$ if it is closed both under subobjects and regular quotients. If $\mathscr{C}$ is a regular category, then $\mathscr{X}$ is closed under subobjects in $\mathscr{C}$ if and only if it is regular epi-reflective, i.e. each component of the unit of the reflection is a regular epimorphism in $\mathscr{C}$. Whereas this latter condition suffices to guarantee that $\mathscr{X}$ is a regular category whenever $\mathscr{C}$ is so, and in this case a morphism in $\mathscr{X}$ is a regular epimorphism (monomorphism) if and only if it is a regular epimorphism (monomorphism) in $\mathscr{C}$, this is not true for exactness. However, any Birkhoff subcategory $\mathscr{X}$ is exact whenever $\mathscr{C}$ is so.

The terminology "Birkhoff subcategory" is justified by the classical theorem due to G. Birkhoff, asserting that a class of universal algebras $\mathscr{A}$ is a subvariety of a variety $\mathscr{B}$ of universal algebras if and only if it is closed in $\mathscr{B}$ under products, subalgebras and homomorphic images. Accordingly, the Birkhoff subcategories of a given variety $\mathscr{B}$ are exactly its subvarieties.

We note that a subcategory $\mathscr{X}$ as above is a regular Mal'tsev category whenever $\mathscr{C}$ is so and $\mathscr{X}$ is closed under subobjects in $\mathscr{C}$. It is exact Mal'tsev (resp., semi-abelian) whenever $\mathscr{C}$ is so and $\mathscr{X}$ is a Birkhoff subcategory of $\mathscr{C}$. Moreover, if $\mathscr{C}$ is a regular Mal'tsev category with coequalizers, then $\mathrm{Grpd}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{RG}(\mathscr{C})$.

2 The Birkhoff subcategory of internal 2-groupoids

Theorem 2.1. Let $\mathscr{C}$ be a finitely cocomplete regular Mal'tsev category. Then $2\text{-}\mathrm{Grpd}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{Grpd}^2(\mathscr{C})$.

Proof. Let us define the reflector $\mathsf{F} : \mathrm{Grpd}^2(\mathscr{C}) \to 2\text{-}\mathrm{Grpd}(\mathscr{C})$ as follows. For a double groupoid $\mathbb{C}$ in $\mathscr{C}$, we define $\mathsf{F}(\mathbb{C})$ as the 2-groupoid $\mathbb{F}$ in the front part of the following diagram:

$$(4)$$

Here $(F_0^0, (\eta_{\mathbb{C}})_0^0)$ is the coequalizer of d^0 and c^0. We set $(\eta_{\mathbb{C}})_1^0 := (\eta_{\mathbb{C}})_0^0 d^0 = (\eta_{\mathbb{C}})_0^0 c^0$. $(F_0^1, (\eta_{\mathbb{C}})_0^1, \varepsilon_0)$ is the pushout of $(\eta_{\mathbb{C}})_0^0$ along e_0. The morphisms δ_0 and γ_0 are the unique ones such that $\delta_0 (\eta_{\mathbb{C}})_0^0 = (\eta_{\mathbb{C}})_0^0 d_0$ and $\delta_0 \varepsilon_0 = 1_{F_0^0}$, and $\gamma_0 (\eta_{\mathbb{C}})_0^0 = (\eta_{\mathbb{C}})_0^0 c_0$ and $\gamma_0 \varepsilon_0 = 1_{F_0^0}$, respectively, which are induced by the universal property of the pushout F_0^1. Similarly, $(F_1^1, (\eta_{\mathbb{C}})_1^1, \varepsilon_1)$ is the pushout of $(\eta_{\mathbb{C}})_1^0$ along e_1. The morphism ε^1 is the unique one such that $\varepsilon^1 (\eta_{\mathbb{C}})_0^1 = (\eta_{\mathbb{C}})_1^1 e^1$ and $\varepsilon^1 \varepsilon_0 = \varepsilon_1$, induced by the universal property of the pushout F_1^1. The morphisms δ^1 and γ^1 are the unique ones such that $\delta^1 (\eta_{\mathbb{C}})_1^1 = (\eta_{\mathbb{C}})_0^1 d^1$ and $\delta^1 \varepsilon_1 = \varepsilon_0$, and $\gamma^1 (\eta_{\mathbb{C}})_1^1 = (\eta_{\mathbb{C}})_0^1 c^1$ and $\gamma^1 \varepsilon_1 = \varepsilon_0$, respectively, induced by the universal property of the pushout defining F_1^1. We set $\delta_1 := \delta_0 \delta^1$, $\gamma_1 := \gamma_0 \gamma^1$ and $\varepsilon_1 := \varepsilon^1 \varepsilon_0$.

It is easily seen that the front part of the above diagram is a double reflexive graph $\mathbb{F}$ in $\mathscr{C}$. Since $\mathrm{Grpd}(\mathscr{C})$ is closed under regular quotients in $\mathrm{RG}(\mathscr{C})$ when $\mathscr{C}$ is a regular Mal'tsev category [12], it follows that $\mathbb{F}$ is actually a double groupoid in $\mathscr{C}$. Furthermore, one checks that $\eta_{\mathbb{C}} : \mathbb{C} \to \mathbb{F}$ is a morphism of double reflexive graphs, hence of double groupoids, since $\mathrm{Grpd}(\mathscr{C})$ is a full subcategory of $\mathrm{RG}(\mathscr{C})$ when $\mathscr{C}$ is a Mal'tsev category.

Let now

$$
\begin{array}{c}
C_1^1 \underset{\substack{e^1 \\ c^1}}{\overset{d^1}{\rightleftarrows\!\!\!\rightleftarrows}} C_0^1 \\[2mm]
\left.{\scriptstyle d_1\ e_1\ c_1}\right\downarrow\uparrow\downarrow \qquad \left.{\scriptstyle d_0\ e_0\ c_0}\right\downarrow\uparrow\downarrow \\[2mm]
C_1^0 \underset{\substack{e^0 \\ c^0}}{\overset{d^0}{\rightleftarrows\!\!\!\rightleftarrows}} C_0^0
\end{array}
$$

with maps $f_0^1,\ f_1^1,\ f_0^0,\ f_1^0$ to

$$
\begin{array}{c}
D_1^1 \underset{\substack{e^1 \\ c^1}}{\overset{d^1}{\rightleftarrows\!\!\!\rightleftarrows}} D_0^1 \\[2mm]
\left.{\scriptstyle d_1\ e_1\ d_1}\right\downarrow\uparrow\downarrow \qquad \left.{\scriptstyle d_0\ e_0\ c_0}\right\downarrow\uparrow\downarrow \\[2mm]
D_1^0 \underset{\substack{e^0 \\ c^0}}{\overset{d^0}{\rightleftarrows\!\!\!\rightleftarrows}} D_0^0
\end{array}
$$

represent a double functor $f : \mathbb{C} \to \mathbb{D}$ between double groupoids $\mathbb{C}$ and $\mathbb{D}$. We define $\mathsf{F}(f)$ as φ in the following diagram:

$$
\begin{array}{c}
F_1^1 \underset{\substack{\varepsilon^1 \\ \gamma^1}}{\overset{\delta^1}{\rightleftarrows\!\!\!\rightleftarrows}} F_0^1 \\[2mm]
\left.{\scriptstyle \delta_1\ \varepsilon_1\ \gamma_1}\right\downarrow\uparrow\downarrow \qquad \left.{\scriptstyle \delta_0\ \varepsilon_0\ \gamma_0}\right\downarrow\uparrow\downarrow \\[2mm]
F_1^0 \underset{\substack{\varepsilon^0 \\ \gamma^0}}{\overset{\delta^0}{\rightleftarrows\!\!\!\rightleftarrows}} F_0^0
\end{array}
$$

with maps $\varphi_0^1,\ \varphi_1^1,\ \varphi_0^0,\ \varphi_1^0$ to

$$
\begin{array}{c}
G_1^1 \underset{\substack{\varepsilon^1 \\ \gamma^1}}{\overset{\delta^1}{\rightleftarrows\!\!\!\rightleftarrows}} G_0^1 \\[2mm]
\left.{\scriptstyle \delta_1\ \varepsilon_1\ \delta_1}\right\downarrow\uparrow\downarrow \qquad \left.{\scriptstyle \delta_0\ \varepsilon_0\ \gamma_0}\right\downarrow\uparrow\downarrow \\[2mm]
G_1^0 \underset{\substack{\varepsilon^0 \\ \gamma^0}}{\overset{\delta^0}{\rightleftarrows\!\!\!\rightleftarrows}} G_0^0
\end{array}
$$

Here $\mathbb{F}$ and $\mathbb{G}$ are $\mathsf{F}(\mathbb{C})$ and $\mathsf{F}(\mathbb{D})$, respectively, that are both constructed as explained above. We define φ_0^0 to be the unique morphism such that $\varphi_0^0(\eta_{\mathbb{C}})_0^0 = (\eta_{\mathbb{D}})_0^0 f^0$ induced by the universal property of the coequalizer $(\eta_{\mathbb{C}})_0^0 : C_0^0 \to F_0^0$. We set $\varphi_1^0 := \varphi_0^0$. We define φ_0^1 to be the unique morphism such that $\varphi_0^1 \varepsilon_0 = \varepsilon_0 \varphi_0^0$ and $\varphi_0^1(\eta_{\mathbb{C}})_0^1 = (\eta_{\mathbb{D}})_0^1 f_0^1$ induced by the universal property of the pushout defining G_0^1. Finally, we define φ_1^1 to be the unique morphism such that $\varphi_1^1 \varepsilon_1 = \varepsilon_1 \varphi_1^0$ and $\varphi_1^1(\eta_{\mathbb{C}})_1^1 = (\eta_{\mathbb{D}})_1^1 f_1^1$ induced by the universal property of the pushout defining G_1^1.

It is easily seen that φ is a double functor from $\mathbb{F}$ to $\mathbb{G}$. Moreover, η is a natural transformation from $1_{\mathrm{Grpd}^2(\mathscr{C})}$ to $\mathsf{U}\mathsf{F}$, where U is the inclusion functor of 2-$\mathrm{Grpd}(\mathscr{C})$ into $\mathrm{Grpd}^2(\mathscr{C})$. Note that $\eta_{\mathbb{C}}$ is a regular epimorphism in $\mathrm{Grpd}^2(\mathscr{C})$ since $(\eta_{\mathbb{C}})_j^i$ is a regular epimorphism in $\mathscr{C}$ for all $i, j \in \{0, 1\}$.

Let now $f : \mathbb{C} \to \mathbb{D}$ be a morphism in $\mathrm{Grpd}^2(\mathscr{C})$, where $\mathbb{D} \in$ 2-$\mathrm{Grpd}(\mathscr{C})$, as depicted in the left-hand side

of the following diagram:
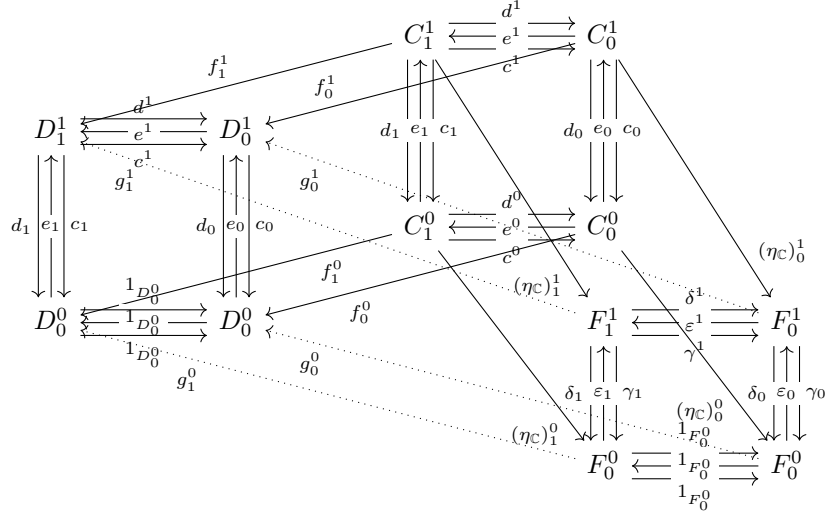
We show that there exists a unique double functor $g: \mathbb{F} \to \mathbb{D}$ such that $g\eta_{\mathbb{C}} = f$. Since $f_0^0 d^0 = f_1^0 = f_0^0 c^0$, there exists a unique morphism g_0^0 such $g_0^0 (\eta_{\mathbb{C}})_0^0 = f_0^0$ by the universal property of the coequalizer F_0^0. We set $g_1^0 := g_0^0$. Since $e_0 g_0^0 (\eta_{\mathbb{C}})_0^0 = e_0 f_0^0 = f_0^1 e_0$, there exists a unique morphism g_0^1 such that $g_0^1 (\eta_{\mathbb{C}})_0^1 = f_0^1$ and $g_0^1 \varepsilon_0 = e_0 g_0^0$ by the universal property of the pushout F_0^1. Since

$$ e_1 g_1^0 (\eta_{\mathbb{C}})_1^0 = e_1 g_0^0 (\eta_{\mathbb{C}})_0^0 d^0 = e_1 f_0^0 d^0 = e_1 f_1^0 = f_1^1 e_1, $$

by the universal property of the pushout F_1^1 there exists a unique morphism $g_1^1: F_1^1 \to D_1^1$ such that $g_1^1 (\eta_{\mathbb{C}})_1^1 = f_1^1$ and $g_1^1 \varepsilon_1 = e_1 g_1^0$. It is easily seen that the constructed g is an internal double functor. Its uniqueness follows from the fact that $\eta_{\mathbb{C}}$ is a regular epimorphism.

To complete the proof, it suffices to observe that 2-Grpd($\mathscr{C}$) is closed under regular quotients in Grpd^2($\mathscr{C}$). Given any regular epimorphism $f: \mathbb{C} \to \mathbb{D}$ in Grpd^2($\mathscr{C}$)
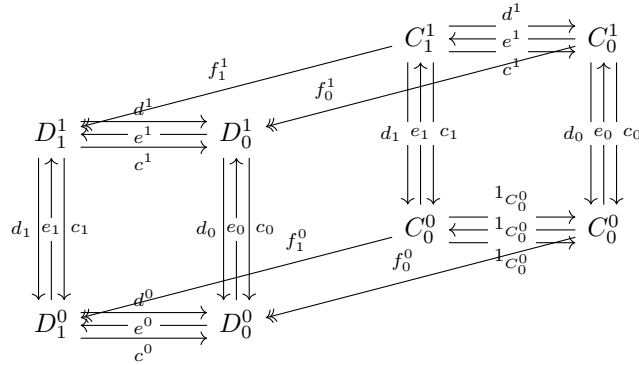
where $\mathbb{C}$ is a 2-groupoid, the fact that f_1^0 is a regular epimorphism in $\mathscr{C}$ easily implies that e^0 is an isomorphism, so that $\mathbb{D} \in$ 2-Grpd($\mathscr{C}$), as desired. $\square$

Corollary 2.2. Let $\mathscr{C}$ be a finitely cocomplete regular Mal'tsev category. Then 2-Grpd($\mathscr{C}$) is also a finitely cocomplete regular Mal'tsev category. Moreover, when $\mathscr{C}$ is also exact, then the category 2-Grpd($\mathscr{C}$) is also exact.

Proof. As recalled before a regular epi-reflective subcategory $\mathscr{X}$ of a regular Mal'tsev category $\mathscr{C}$ is again a regular Mal'tsev category. If $\mathscr{C}$ is a finitely cocomplete regular Mal'tsev category, then Grpd^2($\mathscr{C}$) satisfies the same properties. Hence by Proposition 2.1 it follows that the same holds for 2-Grpd($\mathscr{C}$). Finally, when $\mathscr{C}$ is exact, the Birkhoff assumption implies that 2-Grpd($\mathscr{C}$) is exact. $\square$

It is well-known that the category Grpd($\mathscr{C}$) of internal groupoids in a Mal'tsev variety $\mathscr{C}$ can be presented as a finitary variety of universal algebras, since it is a Birkhoff subcategory of the variety RG($\mathscr{C}$) of reflexive

graphs in $\mathscr{C}$ (see Corollary 2.4 in [14], which is based on some results in [18]). There is a simple relationship between the algebraic theory of the Mal'tsev variety $\mathscr{C}$ and the theory of the variety $\mathrm{Grpd}(\mathscr{C})$. Indeed, let us assume that $\mathscr{C}$ is a Mal'tsev variety whose theory contains a Mal'tsev term $p(x, y, z)$, some n_i-ary terms $\omega_i(x_1, \cdots, x_{n_i})$ for $i \in I$ and $n_i \in \mathbb{N}$, satisfying some identities $\tau_j = \sigma_j$, for some terms τ_j and σ_j (with $j \in J$). Then the category $\mathrm{Grpd}(\mathscr{C})$ is equivalent to a Mal'tsev variety whose algebraic theory contains all the terms and the identities above, plus two additional unary operations s and t, such that s and t are homomorphisms in $\mathscr{C}$, $st = t$ and $ts = s$, and the universal algebraic commutator $[Eq(s), Eq(t)]$ of the kernel congruences $Eq(s)$ and $Eq(t)$ of s and t is trivial. This algebraic description of the category of internal groupoids is an extension to Mal'tsev varieties of the one originally given by J.-L. Loday in [20] in the special case of the variety of groups. Starting from a groupoid whose underlying reflexive graph is diagram (1), the unary operations s and t are the composite homomorphisms $s = ed$ and $t = ec$, that clearly satisfy the identities $st = t$ and $ts = s$. The commutator $[Eq(s), Eq(t)]$ vanishes if and only if there is a unique groupoid structure on the reflexive graph (1) [18].

Corollary 2.3. Let $\mathscr{C}$ be a Mal'tsev variety of universal algebras. Then 2-$\mathrm{Grpd}(\mathscr{C})$ is a Mal'tsev variety.

Proof. The category $\mathrm{Grpd}^2(\mathscr{C})$ of double groupoids in $\mathscr{C}$ is also a variety, since it is again a category of internal groupoids in the Mal'tsev variety $\mathrm{Grpd}(\mathscr{C})$. More explicitly, when $\mathscr{C}$ is a Mal'tsev variety as above, then $\mathrm{Grpd}^2(\mathscr{C})$ is equivalent to a Mal'tsev variety whose algebraic theory contain all the terms $p(x, y, z)$, $\omega_i(x_1, \cdots, x_{n_i})$ for $i \in I$ and $n_i \in \mathbb{N}$ satisfying the identities $\tau_j = \sigma_j$, for some terms τ_j and σ_j (with $j \in J$), together with four additional unary operations s, t, u and v satisfying the following conditions:

- $st = t, ts = s, uv = v, vu = u, su = us, sv = vs, tu = ut, tv = vt$;

- the unary operations s, t, u and v are algebra homomorphisms in $\mathscr{C}$;

- the commutators $[Eq(s), Eq(t)]$ and $[Eq(u), Eq(v)]$ are trivial.

By Proposition 2.1 we deduce that 2-$\mathrm{Grpd}(\mathscr{C})$ is a subvariety of the variety described here above. Indeed, with this presentation of the algebraic theory of $\mathrm{Grpd}^2(\mathscr{C})$ the additional identities determining the subvariety 2-$\mathrm{Grpd}(\mathscr{C})$ of 2-groupoids in $\mathscr{C}$ are the following:

$$u = us = ut, v = vs = vt.$$

$\square$

3   Naturally Mal'tsev categories

Recall that a category $\mathscr{C}$ with products is naturally Mal'tsev if there is a natural transformation $p \colon \mathsf{Id}_{\mathscr{C}} \times \mathsf{Id}_{\mathscr{C}} \times \mathsf{Id}_{\mathscr{C}} \to \mathsf{Id}_{\mathscr{C}}$ such that each component $p_A \colon A \times A \times A \to A$ of this natural transformation is an internal Mal'tsev operation for any $A \in \mathscr{C}$ [19]. Naturally Mal'tsev categories are characterized by the property that the forgetful functor from the category of internal groupoids to the category of reflexive graphs is an isomorphism. This also implies that the category of double groupoids is isomorphic to the category of double reflexive graphs.

   We are going to show that in the naturally Mal'tsev context the adjunction between the categories of internal 2-groupoids and of double groupoids described in the previous section has a remarkable additional property. For this, we shall use a characterization of naturally Mal'tsev categories due to D. Bourn (see Proposition 7.8.1 in [3]) that we shall reformulate using the following observation.

   Given a regular epimorphism $q \colon X \to Y$ and an arbitrary morphism $\delta \colon X \to S$ in a category $\mathscr{C}$ with kernel pairs and coequalizers, the pushout of q along δ exists and can be constructed in the following way. We consider the kernel pair $(\mathrm{Eq}(q), q_1, q_2)$ of q and construct the coequalizer (C, c) of δq_1 and δq_2. This yields a unique morphism $\bar{\delta}$ such that $\bar{\delta} q = c\delta$. Then the commutative square

$$
\begin{array}{ccc}
S & \xrightarrow{\;c\;} & C \\
{\scriptstyle \delta}\big\uparrow & & \big\uparrow{\scriptstyle \bar{\delta}} \\
X & \xrightarrow[\;q\;]{} & Y
\end{array}
\tag{5}
$$

is easily seen to be the pushout of q and δ.

   Now, if δ is a split monomorphism, so that there is an $f \colon S \to X$ such that $f\delta = 1_X$, its pushout $\bar{\delta}$ along q is a monomorphism such that there is an $\bar{f} \colon C \to Y$ with $\bar{f}\bar{\delta} = 1_Y$.

Proposition 3.1. [3] Let $\mathscr{C}$ be an exact Mal'tsev category with coequalizers. The following conditions are equivalent:

1. $\mathscr{C}$ is a naturally Mal'tsev category;

2. given a regular epimorphism $q\colon X \to Y$ and a monomorphism $\delta\colon X \to S$ such that there is $f\colon S \to X$ with $f\delta = 1_X$, the pushout (5) exists, and the induced commutative square

$$
\begin{array}{ccc}
S & \xrightarrow{\;c\;} & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\
X & \xrightarrow{\;q\;} & Y
\end{array}
$$

is a pullback.

Proposition 3.2. Let $\mathscr{C}$ be an exact naturally Mal'tsev category with coequalizers. Given a double groupoid $\mathbb{C}$ in $\mathscr{C}$, depicted as

$$
\begin{array}{ccc}
C_1^1 & \overset{d^1}{\underset{c^1}{\overset{\leftarrow e^1 -}{\rightrightarrows}}} & C_0^1 \\
{\scriptstyle d_1}\downarrow{\scriptstyle \uparrow e_1}\downarrow{\scriptstyle c_1} \quad {\scriptstyle c^1} & \overset{d^0}{\underset{c^0}{\overset{\leftarrow e^0 -}{\rightrightarrows}}} & {\scriptstyle d_0}\downarrow{\scriptstyle \uparrow e_0}\downarrow{\scriptstyle c_0} \\
C_1^0 & \overset{\leftarrow e^0 -}{\underset{c^0}{\rightrightarrows}} & C_0^0,
\end{array}
$$

by applying the reflector $F\colon \mathrm{Grpd}^2(\mathscr{C}) \to 2\text{-}\mathrm{Grpd}(\mathscr{C})$ to $\mathbb{C}$ the internal functors

$$
\begin{array}{ccc}
C_0^1 & \xrightarrow{(\eta_C)_0^1} & F_1^0 \\
{\scriptstyle d_0}\downarrow{\scriptstyle \uparrow e_0}\downarrow{\scriptstyle c_0} \quad {\scriptstyle \delta_0}\downarrow{\scriptstyle \uparrow \varepsilon_0}\downarrow{\scriptstyle \gamma_0} & & \\
C_0^0 & \xrightarrow{(\eta_C)_0^0} & F_0^0,
\end{array}
\qquad
\begin{array}{ccc}
C_1^1 & \xrightarrow{(\eta_C)_1^1} & F_1^0 \\
{\scriptstyle d_1}\downarrow{\scriptstyle \uparrow e_1}\downarrow{\scriptstyle c_1} \quad {\scriptstyle \delta_1}\downarrow{\scriptstyle \uparrow \varepsilon_1}\downarrow{\scriptstyle \gamma_1} & & \\
F_1^1 & \xrightarrow{(\eta_C)_1^0} & F_1^0,
\end{array}
$$

in diagram (4) are both discrete fibrations: all the downward directed commutative squares are pullbacks.

Proof. This is a direct consequnce of the construction of the reflector $F\colon \mathrm{Grpd}^2(\mathscr{C}) \to 2\text{-}\mathrm{Grpd}(\mathscr{C})$ and Proposition 3.1. $\qquad\square$

Remark 3.3. By looking at the construction that M.C. Pedicchio gives in [22] of the commutator of two equivalence relations R and S in an exact Mal'tsev category with coequalizers, one sees that the construction of the smallest double equivalence relation Δ_R^S on R and S uses the same type of pushouts (5) we have considered here above. Given R and S, one first takes the coequalizer $q\colon X \to X/R$ and then the coequalizer of $\delta_S r_1$ and $\delta_S r_2$:

$$
\begin{array}{ccccc}
\Delta_{R,S} & \overset{\pi_1}{\underset{\pi_2}{\overset{\leftarrow \delta_\pi -}{\rightrightarrows}}} & S & \xrightarrow{\;c\;} & S/R \\
{\scriptstyle p_1}\downarrow{\scriptstyle \uparrow \delta_p}\downarrow{\scriptstyle p_2} \quad {\scriptstyle s_1} & & {\scriptstyle \uparrow \delta_S}\downarrow{\scriptstyle s_2} & & {\scriptstyle \bar{s}_1}\downarrow{\scriptstyle \uparrow \bar{\delta}_S}\downarrow{\scriptstyle \bar{s}_2} \\
R & \overset{\overset{r_1}{\leftarrow \delta_R -}}{\underset{r_2}{\rightrightarrows}} & X & \xrightarrow{\;q\;} & X/R
\end{array}
$$

Note that, as seen above, $(S/R, c, \bar{\delta}_S)$ is also the pushout of q and δ_S. The double equivalence relation Δ_R^S on R and S is obtained by taking the kernel pair $(\Delta_R^S, \pi_1, \pi_2)$ of c, and then it is easy to see that the induced maps p_1, p_2 and δ_P determine a reflexive relation, hence an equivalence relation, on R. Thanks to Proposition 3.1, when $\mathscr{C}$ is an exact naturally Mal'tsev category with coequalizers, this double equivalence relation is necessarily a *double centralizing relation* on R and S, which means that all the commutative squares on the left are pullbacks. This implies that the categorical commutator $[R, S]$ of R and S is trivial, i.e. the smallest equivalence relation on X. This shows an interesting and unexpected connection between commutator theory and the reflector F from the category of internal double groupoids to the one of 2-groupoids considered in the previous section.

4 Action representability of the category 2-Grpd($\mathscr{C}$)

The starting point of this section is the following

Theorem 4.1. [13, Theorem 2.10] For a category $\mathscr{C}$, the following conditions are equivalent:

1. $\mathscr{C}$ is a semi-abelian action representable algebraically coherent category with normalizers.

2. Grpd($\mathscr{C}$) is a semi-abelian action representable algebraically coherent category with normalizers.

We will show in this section that if $\mathscr{C}$ is semi-abelian action representable algebraically coherent with normalizers then also 2-Grpd($\mathscr{C}$) is so. Examples of such categories $\mathscr{C}$ are given by the categories Grp of groups, Lie$_R$ of Lie algebras over a commutative ring R, and Hopf$_{K,\mathrm{coc}}$ of cocommutative Hopf algebras over a field K.

We begin this section by proving

Corollary 4.2. Let $\mathscr{C}$ be a semi-abelian category. Then 2-Grpd($\mathscr{C}$) is a semi-abelian category.

Proof. If $\mathscr{C}$ is a semi-abelian category, then Grpd($\mathscr{C}$), and then Grpd^2($\mathscr{C}$), are again semi-abelian [4]. Any Birkhoff subcategory of a semi-abelian category is itself semi-abelian, hence so is the category 2-Grpd($\mathscr{C}$), by Proposition 2.1. $\qquad\square$

From now on we shall always assume that $\mathscr{C}$ is semi-abelian. Let us then recall the notion of action representable category, of algebraically coherent category, and of category with normalizers.

A split extension in $\mathscr{C}$ is a diagram

$$X \xrightarrow{\ \kappa\ } A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \tag{6}$$

in $\mathscr{C}$, where $\alpha\beta = 1_B$ and κ is the kernel of α. A morphism of split extensions in $\mathscr{C}$ is a diagram

$$
\begin{array}{ccc}
X \xrightarrow{\ \kappa\ } A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \\
\downarrow{\scriptstyle u} \qquad \downarrow{\scriptstyle v} \qquad \downarrow{\scriptstyle w} \\
X' \xrightarrow{\ \kappa'\ } A' \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} B
\end{array}
$$

in $\mathscr{C}$, where the top row is the domain split extension, the bottom row is the codomain split extension, $v\kappa = \kappa' u$, $\alpha'v = w\alpha$ and $v\beta = \beta'w$. Let us denote by SplExt($\mathscr{C}$) the category of split extensions in $\mathscr{C}$ and by $P : \mathrm{SplExt}(\mathscr{C}) \to \mathscr{C}$ and $K : \mathrm{SplExt}(\mathscr{C}) \to \mathscr{C}$ the functors sending a split extension (6) to B and to X, respectively. The category $\mathscr{C}$ is action representable [2] if and only if each fiber of the functor K has a terminal object. This means that for any X in $\mathscr{C}$ there is a split extension

$$X \xrightarrow{\ k\ } \overline{X} \underset{i}{\overset{p_1}{\rightleftarrows}} [X],$$

called the "generic split extension with kernel X", such that for any split extension (6) there is a unique morphism of split extensions

$$
\begin{array}{ccc}
X \xrightarrow{\ \kappa\ } A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \\
\| \qquad \downarrow{\scriptstyle v} \qquad \downarrow{\scriptstyle w} \\
X \xrightarrow{\ k\ } \overline{X} \underset{i}{\overset{p_1}{\rightleftarrows}} [X].
\end{array}
$$

When a generic split extension with kernel X as above exists, the object $[X]$ is usually called the actor of X. If $\mathscr{C}$ is the category of groups, the actor $[G]$ of a group G is given by its automorphism group Aut(G). In the category of Lie algebras over a commutative ring, the actor $[L]$ of a Lie algebra L is given by its Lie algebra of derivations Der(L).

A pointed protomodular category $\mathscr{C}$ is algebraically coherent [8] when, for each cospan of monomorphisms of split extensions

$$
\begin{array}{ccccc}
X_1 & \xrightarrow{\;\kappa_1\;} & A_1 & \overset{\alpha_1}{\underset{\beta_1}{\rightleftarrows}} & B \\
{\scriptstyle u_1}\downarrow & & {\scriptstyle v_1}\downarrow & & \| \\
X & \xrightarrow{\;\kappa\;} & A & \overset{\alpha}{\underset{\beta}{\rightleftarrows}} & B \\
{\scriptstyle u_2}\uparrow & & {\scriptstyle v_2}\uparrow & & \| \\
X_2 & \xrightarrow{\;\kappa_2\;} & A_2 & \overset{\alpha_2}{\underset{\beta_2}{\rightleftarrows}} & B
\end{array}
$$

if the morphisms v_1 and v_2 are jointly strongly epimorphic in $\mathscr{C}$, then so are the morphisms u_1 and u_2.

A category $\mathscr{C}$ has normalizers [16] if for any monomorphism $f : A \to B$ in $\mathscr{C}$, the category with objects triples (N, n, m) where n is a normal monomorphism, m is a monomorphism and $f = mn$, with expected morphisms, has a terminal object, the so-called *normalizer* of f.

We have seen before that 2-Grpd($\mathscr{C}$) is a Birkhoff subcategory of Grpd$^2(\mathscr{C})$ whenever $\mathscr{C}$ is a finitely cocomplete regular Mal'tsev category. In a more general context we have

Lemma 4.3. Let $\mathscr{C}$ be a finitely complete category. Then 2-Grpd($\mathscr{C}$) is a regular mono-coreflective subcategory of Grpd$^2(\mathscr{C})$.

Proof. Let us describe the coreflection $\mathsf{G} : \mathrm{Grpd}^2(\mathscr{C}) \to 2\text{-Grpd}(\mathscr{C})$. For a double groupoid $\mathbb{C}$ in $\mathscr{C}$, we define $\mathsf{G}(\mathbb{C})$ as $\mathbb{F}$ in the following diagram:

(7)

Here we set $F_0^0 := C_0^0$ and $(\varepsilon_C)_0^0 := 1_{C_0^0}$. Furthermore, we define $F_0^1 := C_0^1$, $\delta_0 := d_0$, $\gamma_0 := c_0$, $\varepsilon_0 := e_0$ and $(\varepsilon_C)_0^1 := 1_{C_0^1}$. Next we set $(\varepsilon_C)_1^0 := e^0$ and F_1^1, $(\varepsilon_C)_1^1$, δ_1 and γ_1 to be the object and morphisms, respectively, in the "joint pullback" of d_1 and c_1 along e^0 as displayed in the following diagram:

$$
\begin{array}{ccc}
F_1^1 & \xrightarrow{\;(\varepsilon_C)_1^1\;} & C_1^1 \\
{\scriptstyle (\delta_1, \gamma_1)}\downarrow & & \downarrow{\scriptstyle (d_1, c_1)} \\
C_0^0 \times C_0^0 & \xrightarrow[e^0 \times e^0]{} & C_1^0 \times C_1^0.
\end{array}
$$

Since $e^0 \times e^0$ is a split monomorphism in $\mathscr{C}$, $(\varepsilon_C)_1^1$ is a regular monomorphism in $\mathscr{C}$. The morphism ε_1 is the unique map such that $(\delta_1, \gamma_1)\varepsilon_1 = (1_{C_0^0}, 1_{C_0^0})$ and $(\varepsilon_C)_1^1 \varepsilon_1 = e_1 e^0$ induced by the universal property of the pullback F_1^1. We set $\delta^1 := d^1(\varepsilon_C)_1^1$ and $\gamma^1 := c^1(\varepsilon_C)_1^1$. Finally, ε^1 is the unique morphism such that $(\delta_1, \gamma_1)\varepsilon^1 = (d_0, c_0)$ and $(\varepsilon_C)_1^1 \varepsilon^1 = e^1$ induced by the universal property of the pullback F_1^1.

It is easily seen that the front part of the diagram (7) is a double reflexive graph $\mathbb{F}$ in $\mathscr{C}$, which is underlying a double groupoid structure. Furthermore, $\varepsilon_C : \mathbb{F} \to \mathbb{C}$ is a double functor, and it is not difficult to prove that it has the universal property of the counit of the adjunction. $\qquad\square$

Our proof of the main result of this section also relies on the following:

Proposition 4.4. [13, Proposition 2.9] Let $\mathscr{C}$ be a semi-abelian action representable category. Moreover, let $\mathscr{X}$ be a semi-abelian category and $\mathsf{H} : \mathscr{X} \to \mathscr{C}$ be a fully faithful left adjoint functor that is protoadditive [10], i.e. it preserves kernels of split epimorphisms. Then $\mathscr{X}$ is action representable.

Theorem 4.5. Let $\mathscr{C}$ be a semi-abelian action representable algebraically coherent category with normalizers. Then 2-Grpd($\mathscr{C}$) is a semi-abelian action representable algebraically coherent category with normalizers.

Proof. By Theorem 4.1, we know that $\mathrm{Grpd}^2(\mathscr{C})$ is a semi-abelian action representable algebraically coherent category with normalizers. By Proposition 2.1 and Lemma 4.3, 2-Grpd($\mathscr{C}$) is a coreflective Birkhoff subcategory of $\mathrm{Grpd}^2(\mathscr{C})$. We have already observed that 2-Grpd($\mathscr{C}$) is a semi-abelian category. It follows from [8, Proposition 3.7] that it is also algebraically coherent. By using the fact that $\mathrm{Grpd}^2(\mathscr{C})$ has normalizers one sees that the same is true for 2-Grpd($\mathscr{C}$). Proposition 4.4 shows that 2-Grpd($\mathscr{C}$) is action representable, since the inclusion functor 2-Grpd($\mathscr{C}$) $\to \mathrm{Grpd}^2(\mathscr{C})$ is clearly protoadditive (it has a left adjoint). $\qquad\square$

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