ONE-POINT RESTRICTED CONFORMAL BLOCKS AND THE FUSION RULES

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Abstract. We investigate the one-point restriction of the conformal blocks on $(\mathbb{P}^1, \infty, w, 0)$ defined by modules over a vertex operator algebra. By restricting the module attached to the point ∞ to its bottom degree, we obtain a new formula to calculate fusion rules using a module over the degree-zero Borcherds' Lie algebra. This formula holds under more general assumptions than Frenkel-Zhu's fusion rules theorem. By restricting the module attached to the point w to its bottom degree, we obtain a more general version of Li's nuclear democracy theorem for vertex operator algebras.

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1. Introduction

The space of conformal blocks on the three-pointed genus-zero smooth curve defined by modules over a vertex operator algebra (VOA) V is isomorphic to the vector space of intertwining operators among these modules, whose dimension is the fusion rule [NT05, FBZ04]. Using the restriction technique of conformal blocks [GLZ24], we obtain a new hom-space identification of the space of intertwining operators $I\binom{M^3}{M^1\ M^2}$ by a module $M^1 \odot M^2$ over the degree-zero Borcherds Lie algebra $L(V)_0$ [B86] or the Zhu's algebra $A(V)$ [Z96].

Motivated by finding a mathematical rigorous definition of the WZNW-conformal field theory, Tsuchiya, Ueno, and Yamada introduced the notions of coinvariants (vacua) and conformal blocks (covacua) on stable algebraic curves defined by highest-weight representations of affine Kac-Moody algebras $\widehat{\mathfrak{g}}$ of non-generic level $k \in \mathbb{Z}_{>0}$ [TUY89]. From the VOA point of view, the representation theory of affine Lie algebras is in parallel with the representation theory of affine VOAs of the same level [FZ92]. With this key observation, the notions of coinvariants and conformal blocks were generalized to the VOA case by Zhu [Z94], Frenkel-Ben-Zvi [FBZ04], and Nagatomo-Tsuchiya [NT05] for smooth curves, and by Damiolini-Gibney-Tarasca [DGT24] for general stable curves. The space of three-pointed conformal blocks associated to the projective line $(\mathbb{P}^1, \infty, 1, 0, (M^3)', M^1, M^2)$ is canonically isomorphic to the space of intertwining operators $I\binom{M^3}{M^1\ M^2}$ among these V-modules. These spaces are the building blocks of the space of conformal blocks on higher-genus algebraic curves via the factorization theorem [DGT24]. The dimension of these spaces, namely the fusion rules, are not only one of the central subjects in the conformal field theory (CFT), but also carry important information about the rank of the vector bundle on the moduli space $\overline{\mathcal{M}}_{g,n}$, parametrizing n-pointed genus-g stable curves, defined by the VOA-conformal blocks [DGT24, DG23].

The structure of the space $I\binom{M^3}{M^1\ M^2}$ has been studied extensively in the theory of both VOAs and CFTs. For instance, on the CFT side, Tsuchiya and Kanie proved that for the WZNW-model of type A_1, an intertwining operator of type $\binom{j_3}{j\ j_2}$ can be uniquely determined by an element $\Phi(u, z) \in \mathrm{Hom}_{\mathbb{C}}(L_{\widehat{sl_2}}(k, j_2), L_{\widehat{sl_2}}(k, j_3))\{z\}$ associated to $u \in L(j)$ satisfying certain bracket equalities with respect to the Sugawara operator $L(m)$ and the affine Lie algebra operator $a(m) \in \widehat{sl_2}$ [TK87]. This is the so-called nuclear democracy theorem, which was generalized by Li to the rational VOA case in [Li98]. On the VOA side, Frenkel and Zhu proposed a theorem which states that $I\binom{M^3}{M^1\ M^2} \cong \mathrm{Hom}_{A(V)}(A(M^1) \otimes_{A(V)} M^2(0), M^3(0))$ [FZ92], where $A(M^1)$ is a bimodule over Zhu's algebra $A(V)$. Li improved this theorem by adding the assumptions that M^2 and $(M^3)'$ are generalized Verma modules [DLM98] associated to their bottom degrees $M^2(0)$ and $M^3(0)^*$, respectively [Li99]. The author proved a variant of this theorem using the technique of three-pointed correlation functions defined by intertwining operators [Liu23], which was further generalized to the g-twisted case by Gao, the author, and Zhu by developing the theory of twisted correlation functions [GLZ23]. The fusion rules theorem had generalizations from various aspects in the theory of VOAs. For instance, Dong and Ren generalized it to the higher-level and modular representation case in [DR13, DR14] using a bimodule over the higher-level Zhu's algebra $A_N(V)$ [DLM98(2)]; Huang and Yang generalized it to the logarithmic intertwining operator case in [HY12]; Huang also gave an interpretation of the space of logarithmic intertwining operators case using a bimodule over his associative algebra $A^\infty(V)$ in [H22, H24]. All of these generalizations were influenced by the idea of using a bimodule over certain variants of the associative algebra $A(V)$ to describe the space of intertwining operators. Recently, Gao, the author, and Zhu introduced a notion of (twisted) restricted conformal blocks in [GLZ24]. With this new notion, we noticed that the hom space $\mathrm{Hom}_{A(V)}(A(M^1) \otimes_{A(V)} M^2(0), M^3(0))$ can be identified with the space of two-pointed restricted conformal blocks defined on the projective line $(\mathbb{P}^1, \infty, 1, 0, M^3(0)^*, M^1, M^2(0))$, where the V-modules $(M^3)'$ and M^2 attached to ∞ and 0 are restricted to their bottom degrees $M^3(0)^*$ and $M^2(0)$, respectively, and the fusion rules theorem can be interpreted as a theorem about extending a restricted conformal block to a regular conformal block.

In this paper, by examining the space of conformal blocks on the one-point restricted projective line $(\mathbb{P}^1, \infty, 1, 0, M^3(0)^*, M^1, M^2)$, we give a new formula to describe the space of intertwining operators $I\binom{M^3}{M^1\ M^2}$ without using the $A(V)$-bimodule $A(M)$. Our formula also leads to a sharper upper bound of the fusion rules for certain irrational VOAs.

To state our results precisely, and describe how they are proved, we set a small amount of notation. Let $(C, P_\bullet)$ be a stable n-pointed curve with marked points $P_\bullet = (P_1, \dots, P_n)$, and let $M^1, \dots, M^n$ be irreducible admissible modules over a VOA V. A conformal block associated to the datum $(C, P_\bullet, M^\bullet)$ is a linear functional $f : M^1 \otimes \ldots \otimes M^n \to \mathbb{C}$ that is invariant under the action of the *chiral Lie algebra* $\mathcal{L}_{C\backslash P_\bullet}(V) = H^0(C\backslash P_\bullet, \mathscr{V}_C \otimes \Omega_C/\mathrm{Im}\nabla)$ on the tensor product $M^1 \otimes \ldots \otimes M^n$, where $\mathscr{V}_C$ is a vertex algebra bundle on C, ∇ is a flat connection of $\mathscr{V}_C$, and Ω_C is the dualizing sheaf on C [FBZ04, DGT24]. In this paper, we will mainly be focusing on the case when $(C, P_\bullet) = (\mathbb{P}^1, \infty, 1, 0)$, with $(M^3)'$, M^1, and M^2 attached to the points ∞, 1, and 0, respectively. In this case, elements in the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ can be represented by $a \otimes f(z)$, where $f(z)$ is a rational function in z and $z-1$. The chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ has natural actions on the V-modules $(M^3)'$, M^1, and M^2 with given by the trivialization of the vertex algebra bundle $\mathscr{V}_{\mathbb{P}^1}$ around the points ∞, 1, and 0, respectively. The space of *three-pointed conformal blocks on* $\mathbb{P}^1$, denoted by $\mathscr{C}\left(\Sigma_1((M^3)', M^1, M^2)\right)$, consists of linear functionals $f \in (M^3)' \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2 \to \mathbb{C}$ that is invariant under the action of the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ on the tensor product.

We define the ∞*-restricted chiral Lie algebra* $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$ to be the Lie subalgebra of $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ spanned by elements that leave the subspace $M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2$ invariant, then define a ∞*-restricted conformal block* to be a linear functional $f : M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2 \to \mathbb{C}$ that is invariant under the action of $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$ (see Definition 4.1). The Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$ has an ideal $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$ consisting of elements whose action vanishes on $M^3(0)^*$, and such that the quotient algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}/\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$ is isomorphic to the degree zero Borcherds Lie algebra $L(V)_0$ (see Lemma 3.5). Let

$$M^1 \odot M^2 := M^1 \otimes_{\mathbb{C}} M^2/\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}.(M^1 \otimes_{\mathbb{C}} M^2),$$

which is a module over the Lie algebra $L(V)_0$. Using a set of spanning elements of $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$, we can show that $M^1 \odot M^2$ is spanned by the equivalent classes $v_1 \odot v_2$ of the elements $v_1 \otimes v_2 \in M^1 \otimes_{\mathbb{C}} M^2$, subject to the following relations (see Definition 4.3):

$$\sum_{j\geq 0}\binom{\mathrm{wt}a-1}{j}a(j-1)v_1 \odot v_2 = v_1 \odot \sum_{j\geq 0} a(\mathrm{wt}a - 1 + j)v_2,$$

$$\sum_{j\geq 0}\binom{\mathrm{wt}a-k}{j}a(j)v_1 \odot v_2 = -v_1 \odot a(\mathrm{wt}a-k)v_2,$$

where $a \in V$, $v_1 \in M^1, v_2 \in M^2$, $k \geq 2$. The following is our main theorem (see Theorem 5.6).

Theorem A. Let M^1, M^2, and M^3 be ordinary V-modules of conformal weights h_1, h_2, and h_3, respectively. Suppose the contragredient module $(M^3)'$ is isomorphic to the generalized Verma module $\bar{M}(M^3(0)^*)$. Then we have an isomorphism of vector spaces

$$I\binom{M^3}{M^1\ M^2} \cong \mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, M^3(0)). \tag{1.1}$$

In particular, $N\binom{M^3}{M^1\ M^2} = \dim \mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, M^3(0))$. If V is rational, then (1.1) holds for any irreducible V-modules M^1, M^2, and M^3.

One advantage of Theorem A in comparison with Frenkel-Zhu's fusion rules theorem is that we do not need M^2 to be a generalized Verma module, which makes an essential difference when the VOA V is not rational. Moreover, if we only assume $(M^3)'$ is generated by its bottom degree $M^3(0)^*$, then we have an estimate of the fusion rule (see Lemma 5.1):

$$N\binom{M^3}{M^1\ M^2} \leq \dim \mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, M^3(0)). \tag{1.2}$$

Another advantage of our hom-space description (1.1) is that the estimate (1.2) is in general sharper than the estimate $N\binom{M^3}{M^1\ M^2} \leq \dim \mathrm{Hom}_{A(V)}(A(M^1) \otimes_{A(V)} M^2(0), M^3(0))$. We use Li's example of modules over the universal Virasoro VOA [Li99] to illustrate this fact in Section 5.4.2. If the V-module M^2 is generated by $M^2(0)$, we can show that $M^1 \odot M^2$ is a left module over $A(V)$, and it is a quotient module of $A(M^1)\otimes_{A(V)} M^2(0)$ (see Theorem 4.7). In other words, $M^1\odot M^2$ in general has more relations than $A(M^1)\otimes_{A(V)} M^2(0)$, which explains why the estimate (1.2) leads to a sharper upper bound of the fusion rules.

Instead of using the language of correlation functions, we will use the language of conformal blocks to prove Theorem A, with the assistance of chiral Lie algebra. The proof is much shorter than the proof of the fusion rules theorem [Liu23, GLZ23]. It is well-known that the space of intertwining operators is isomorphic to the space of conformal blocks $\mathscr{C}\left(\Sigma_1((M^3)', M^1, M^2)\right)$ [FBZ04, NT05]. Using some basic facts about the representation theory of Lie algebras, we can also show that the space $\mathrm{Hom}_{L(V)_0}(M^1\odot M^2, M^3(0))$ can be identified with the space of ∞-restricted conformal blocks $\mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right)$ (see Proposition 4.4). Then Theorem A is equivalent to showing that $\mathscr{C}\left(\Sigma_1((M^3)', M^1, M^2)\right)\cong \mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right)$. This can be done using an explicit set of spanning elements of $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ (see Theorem 5.5). There is an alternative proof of a more general version of the isomorphism between restricted and unrestricted conformal blocks in [GLZ24] using the Riemann-Roch theorem of algebraic curves. The method we used in this paper is purely algebraic.

The idea of restricting one module in the three-pointed conformal blocks to its bottom degree has further applications. Instead of restricting the module $(M^3)'$ attached to ∞ to its bottom degree $M^3(0)^*$, one can also restrict the module M^2 attached to a point $w\in\mathbb{P}^1\backslash\{0,\infty\}$ to its bottom degree $M^2(0)$ and obtain another space of one-point restricted conformal blocks $\mathscr{C}\left(\Sigma_w((M^3)', M^1(0), M^2)\right)$. It turns out that this space is isomorphic to the space of *generalized intertwining operators* introduced by Li in [Li98] (see Proposition 6.7). The extension theorem of restricted conformal blocks then lead to an alternative proof of Li's *generalized nuclear democracy theorem (GNDT)* [Li98] (see Theorem 6.10). We also find a hom-space description of the GNDT using the w-restricted chiral Lie algebras (see Theorem 6.11).

This paper is organized as follows: we first recall the basics of VOAs and related constructions and then recall the definition of three-pointed conformal blocks on $\mathbb{P}^1$ in Section 2. We introduce the notion of ∞-restricted chiral Lie algebras in Section 3 and discuss its basic properties and spanning elements. In Section 4, we introduce the notions of ∞-restricted conformal blocks and the $L(V)_0$-module $M^1\odot M^2$ using the results in the previous Sections. We prove Theorem A in Section 5 and discuss some of its consequences. Finally, in Section 6, we reinterpret Li's generalized nuclear democracy theorem using the one-point restricted conformal blocks and prove some of its variants.

Conventions. Throughout this paper, we adopt the following conventions:

- All vector spaces and algebraic curves are defined over $\mathbb{C}$, the complex number field.
- $\mathbb{N}$ represents the set of all natural numbers, including 0.

2. Space of conformal blocks associated to representation of VOAs on three-pointed $\mathbb{P}^1$

In this Section, we first review vertex operator algebras and related constructions, then introduce the notions of chiral Lie algebra ancillary to the three-pointed projective line $(\mathbb{P}^1,\infty,1,0)$ and the space of three-pointed conformal blocks using an algebraic language.

2.1. **Preliminaries of VOAs.**

We recall the definitions of vertex operator algebras (VOAs) and the related notions like modules over VOAs, intertwining operators and fusion rules, contragredient modules, Borcherds' Lie algebra and Zhu's algebra, and the generalized Verma modules over VOAs. These notions will be used later in this paper. We refer to [B86, FLM88, FHL93, FZ92, Z96, DLM98, LL04] for more details about these notions.

2.1.1. *Vertex operator algebras and modules.*

Definition 2.1. A *vertex operator algebra (VOA)* is a quadruple $V=(V,Y,\mathbf{1},\omega)$, where $V=\bigoplus_{n\in\mathbb{Z}}V_n$ is a $\mathbb{Z}$-graded vector space, with $\dim V_n<\infty$ for all n and $V_n=0$ if $n\ll 0$;

$\mathbf{1} \in V_0$ is called the *vacuum element*; $\omega \in V_2$ is called the *Virasoro element*; and $Y : V \to \mathrm{End}(V)[\![z, z^{-1}]\!], a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$ is a linear map called the *state-field correspondence*, subject to the following axioms:

(1) (truncation property) $Y(a, z)b \in V((z))$. i.e., for any $a, b \in V$, $a_n b = 0$ for $n \gg 0$.
(2) (vacuum and creation property) $Y(\mathbf{1}, z) = \mathrm{Id}_V$ and $Y(a, z)\mathbf{1} = a + a_{-2}\mathbf{1}z + a_{-3}\mathbf{1}z^2 + \dots$,
(3) (weak commutativity) For $a, b \in V$, there exists $k \in \mathbb{N}$ such that

$$(z_1 - z_2)^k Y(a, z_1)Y(b, z_2) = (z_1 - z_2)^k Y(b, z_2)Y(a, z_1). \tag{2.1}$$

(4) (weak associativity) For $a, b, c \in V$, there exists $k \in \mathbb{N}$ (depending on a and c) such that

$$(z_0 + z_2)^k Y(Y(a, z_0)b, z_2)c = (z_0 + z_2)^k Y(a, z_0 + z_2)Y(b, z_2)c. \tag{2.2}$$

(5) (Virasoro relation) Let $Y(\omega, z) = \sum_{\in\mathbb{Z}} L(n)z^{-n-2}$, then $[L(m), L(n)] = (m - n)L(m + n) + \frac{(m^3-m)}{12}\delta_{m+n,0}c\mathrm{Id}_V$, for any $m, n \in \mathbb{Z}$, where c is called the *central charge* of V.
(6) ($L(-1)$-derivative property) $Y(L(-1)a, z) = \frac{d}{dz}Y(a, z)$ for any $a \in V$.
(7) ($L(0)$-eigenspace property) For any $n \in \mathbb{Z}$ and $a \in V_n$, $L(0)a = na$.

We write $\mathrm{wt}a = n$ if $n \in V_n$. We abbreviate the quadruple $(V, Y, \mathbf{1}, \omega)$ by V. A VOA V is said to be of *CFT-type* if $V = V_0 \oplus V_+$, where $V_0 = \mathbb{C}\mathbf{1}$ and $V_+ = \bigoplus_{n=1}^{\infty} V_n$.

The axioms of a V-module are similar to the VOA V itself [FHL93, DLM98]:

Definition 2.2. Let V be a VOA. An *admissible V-module* is a $\mathbb{N}$-graded vector space $M = \bigoplus_{n=0}^{\infty} M(n)$, equipped with a linear map $Y_M : V \to \mathrm{End}(M)[\![z, z^{-1}]\!]$, $Y_M(a, z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$ called the *module vertex operator*, satisfying

(1) (truncation property) For any $a \in V$ and $u \in M$, $Y_M(a, z)u \in M((z))$.
(2) (vacuum property) $Y_M(\mathbf{1}, z) = \mathrm{Id}_M$.
(3) (Jacobi identity for Y_M) for any $a, b \in V$ and $u \in M$,

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_M(a, z_1)Y_M(b, z_2)u - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)Y_M(b, z_2)Y_M(a, z_1)u \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_M(Y(a, z_0)b, z_2)u. \end{aligned} \tag{2.3}$$

(4) ($L(-1)$-derivative property) $Y_M(L(-1)a, z) = \frac{d}{dz}Y_M(a, z)$ for any $a \in V$.
(5) (grading property) For any $a \in V$, $m \in \mathbb{Z}$, and $n \in \mathbb{N}$, $a(m)M(n) \subseteq M(n + \mathrm{wt}a - m - 1)$. i.e., $\mathrm{wt}(a(m)) = \mathrm{wt}a - m - 1$.

We write $\deg v = n$ if $v \in M(n)$, and call it the *degree of v*.

An admissible V-module M is called *ordinary* if each degree-n subspace $M(n)$ is a finite-dimensional eigenspace of $L(0)$ of eigenvalue $h + n$, where $h \in \mathbb{C}$ is called the *conformal weight* of M. In particular, if we write $L(0)v = (\mathrm{wt}v) \cdot v$ for $v \in M(n)$, then $\mathrm{wt}v = \deg v + h$.

We abbreviate an *ordinary V-module* simply by a *V-module*. Submodules, quotient modules, and irreducible modules are defined in the usual categorical sense. V is called *rational* if the category of V-modules is semisimple.

Remark 2.3. We remark the following well-known facts about the Jacobi identity:

(1) One can replace the weak commutativity (2.1) and the weak associativity (2.2) in the definition of VOAs by the formal variable Jacobi identity (2.3) for Y [LL04].
(2) Using the Cauchy's integral (or residue) theorem, one can rewrite the formal variable Jacobi identity (2.3) into the *residue form* [FLM88, FZ92, Z96]:

$$\begin{aligned} &\mathrm{Res}_{z=0}Y_M(a, z)Y_M(b, w)\iota_{z,w}(F(z, w)) - \mathrm{Res}_{z=0}Y_M(b, w)Y_M(a, z)\iota_{w,z}(F(z, w)) \\ &= \mathrm{Res}_{z-w=0}Y_M(Y(a, z - w)b, w)\iota_{w,z-w}(F(z, w)), \end{aligned} \tag{2.4}$$

where $F(z,w)=z^nw^m(z-w)^l$, with $m,n,l\in\mathbb{Z}$, and $\iota_{z,w}$, $\iota_{w,z}$, and $\iota_{w,z-w}$ are the expansion operations of a rational function in complex variables z and w on the domains $|z|>|w|$, $|w|>|z|$, and $|w|>|z-w|$, respectively.

(3) The Jacobi identity (2.4) has the following component form:

$$\begin{aligned}&\sum_{i=0}^{\infty}\binom{l}{i}a(m+l-i)b(n+i)-\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}b(n+l-i)a(m+i)\\&=\sum_{i=0}^{\infty}(a(l+i)b)(m+n-i),\end{aligned}\tag{2.5}$$

where $a,b\in V$ and $m,n,l\in\mathbb{Z}$, which is also called the *Borcherds identity* [B86].

2.1.2. *Intertwining operators and fusion rules.* Intertwining operators among V-modules are generalizations of intertwining operators among modules over Lie algebras [FHL93, FZ92, Li98]:

Definition 2.4. Let M^1,M^2,M^3 be ordinary V-modules of conformal weights $h_1,h_2,h_3\in\mathbb{C}$, respectively. Let $h:=h_1+h_2-h_3$. An *intertwining operator of type* $\binom{M^3}{M^1\ M^2}$ is a linear map

$$I(\cdot,z):M^1\to\mathrm{Hom}(M^2,M^3)[\![z,z^{-1}]\!]z^{-h},\quad I(v_1,z)=\sum_{n\in\mathbb{Z}}v_1(n)z^{-n-1-h},$$

satisfying the following axioms:

(1) (truncation property) For any $v_1\in M^1$ and $v_2\in M^2$, $v_1(n)v_2=0$ when $n\gg 0$.
(2) ($L(-1)$-derivative property) For any $v_1\in M^1$, $I(L(-1)v_1,z)=\frac{d}{dz}I(v_1,z)$.
(3) (Jacobi identity) For any $v_1\in M^1$, $v_2\in M^2$, and $a\in V$, one has

$$\begin{aligned}&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_{M^3}(a,z_1)I(v_1,z_2)v_2-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)I(v_1,z_2)Y_{M^2}(a,z_1)v_2\\&=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)I(Y_{M^1}(a,z_0)v_1,z_2)v_2.\end{aligned}$$

The vector space of intertwining operators of type $\binom{M^3}{M^1\ M^2}$ is denoted by $I\binom{M^3}{M^1\ M^2}$. Its dimension, denoted by $N\binom{M^3}{M^1\ M^2}$, is called the *fusion rule among* M^1*,* M^2*, and* M^3.

Using the Jacobi identity and the $L(0)$-eigenspace property for $M^i(n)$, one can easily show that $v_1(n)M^2(m)\subseteq M^3(\deg v_1-n-1+m)$, for any $v_1\in M^1$, $n\in\mathbb{Z}$, and $m\in\mathbb{N}$ [FZ92].

2.1.3. *Contragredient modules.*

Definition 2.5. [FHL93] Let M be an ordinary V-module. Its *contragredient module* is the graded dual space $M'=\bigoplus_{n=0}^{\infty}(M(n))^*$, with $Y_{M'}:V\to\mathrm{End}(M')[\![z,z^{-1}]\!]$ given by

$$\langle Y_{M'}(a,z)v',v\rangle:=\langle v',Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a,z^{-1})v\rangle=\langle v',Y_M'(a,z)v\rangle,\quad v'\in M',v\in M,\tag{2.6}$$

where $\langle\cdot,\cdot\rangle:M'\times M\to\mathbb{C}$ is the natural pair between graded vector spaces.

It was proved in [FHL93, Section 5] that $Y_{M'}$ defined by (2.6) satisfies the Jacobi identity (2.3), and $Y_M''(a,z)=Y_M(a,z)$.

Moreover, if we write $Y_M'(a,z)=\sum_{n\in\mathbb{Z}}a'(n)z^{-n-1}$, then by taking the formal residue,

$$a'(n)=\sum_{j\geq 0}\frac{(-1)^{\mathrm{wt}a}}{j!}(L(1)^ja)(2\mathrm{wt}a-n-j-2).\tag{2.7}$$

It follows that $a'(n)M(m)\subseteq M(-\mathrm{wt}a+n+1+m)$. i.e., $\mathrm{wt}(a'(n))=-\mathrm{wt}a+n+1$.

2.1.4. *Borcherd's Lie algebra and Zhu's algebra associated to a VOA.* The commutative unital differential algebra $(\mathbb{C}[t,t^{-1}], \frac{d}{dt}, 1)$ is a vertex algebra, with $Y(t^n, z)t^m = (e^{z\frac{d}{dt}}t^n)\cdot t^m$. Let V be a VOA, then $\widehat{V} = V\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]$ is the tensor product vertex algebra [FHL93], with vacuum element $\mathbf{1}\otimes 1$ and differential $\nabla = L(-1)\otimes \mathrm{Id} + \mathrm{Id}\otimes \frac{d}{dt}$, see [B86] for more details.

Definition 2.6. [B86] Let V be a VOA. The Borcherds Lie algebra $L(V)$ is defined by

$$L(V) = \widehat{V}/\nabla(\widehat{V}) = \mathrm{span}\{a_{[n]} = a\otimes t^n + \nabla(\widehat{V}) : a\in V, n\in\mathbb{Z}\}, \tag{2.8}$$

with $(L(-1)a)_{[n]} = -na_{[n-1]}$, and the Lie bracket

$$[a_{[m]}, b_{[n]}] = \sum_{j\geq 0}\binom{m}{j}(a(j)b)_{[m+n-j]}, \quad a,b\in V, m,n\in\mathbb{Z}. \tag{2.9}$$

Let $\deg(a_{[n]}) := \mathrm{wt}a - n - 1$, then $L(V)$ has a triangular decomposition $L(V) = L(V)_-\oplus L(V)_0\oplus L(V)_+$, where $L(V)_\pm = \mathrm{span}\{a_{[n]} : \deg(a_{[n]})\in\mathbb{Z}_\pm\}$ and $L(V)_0 = \mathrm{span}\{a_{[\mathrm{wt}a-1]} : a\in V\}$ are Lie subalgebras of $L(V)$.

Definition 2.7. [Z96] Let V be a VOA, *Zhu's algebra* $A(V)$ is defined by $A(V) = V/O(V)$, where

$$O(V) = \mathrm{span}\left\{a\circ b = \mathrm{Res}_{z=1} Y(a, z-1)b\iota_{1,z-1}\left(\frac{z^{\mathrm{wt}a}}{(z-1)^2}\right) : a,b\in V\right\}. \tag{2.10}$$

$A(V) = \mathrm{span}\{[a] = a + O(V) : a\in V\}$ is an associative algebra with respect to

$$[a]*[b] = \mathrm{Res}_{z=1}[Y(a,z-1)b]\iota_{1,z-1}\left(\frac{z^{\mathrm{wt}a}}{z-1}\right) = \sum_{j\geq 0}\binom{\mathrm{wt}a}{j}[a(j-1)b], \quad a,b\in V. \tag{2.11}$$

We remark the following facts about Zhu's algebra and Borcherd's Lie algebra, see [B86, Z96, DLM98] for more details.

(1) There is an anti-involution $\theta : A(V)\to A(V)$ defined by $\theta([a]) = [e^{L(1)}(-1)^{L(0)}a]$, with $\theta([a]*[b]) = \theta([b])*\theta([a])$.
(2) There is a similar anti-involution $\theta : L(V)\to L(V)$ defined by $\theta(a_{[n]}) = a'_{[n]}$ (2.7), with $\theta([a_{[m]}, b_{[n]}]) = [\theta(b_n), \theta(a_{[m]})]$.
(3) Let $M = \bigoplus_{n=0}^\infty M(n)$ be an admissible V-module. Then the bottom-level $M(0)$ is a left $A(V)$-module via $A(V)\to \mathrm{End}(M(0)), [a]\mapsto o(a) := a(\mathrm{wt}a-1)$. If M is an irreducible V-module, then $M(0)$ is an irreducible $A(V)$-module.
(4) For the contragredient module M', the bottom-level $M'(0) = M(0)^*$ is naturally a right $A(V)$-module. It is a left $A(V)$-module via θ. i.e.,

$$\langle[\theta(a)].v'\,|\,v\rangle := \langle v'.[a]\,|\,v\rangle = \langle v'\,|\,[a].v\rangle = \langle v'\,|\,o(a)v\rangle, \quad v'\in M(0)^*,\ v\in M(0). \tag{2.12}$$

(5) There is an epimorphism of Lie algebras

$$L(V)_0\to A(V)_{\mathrm{Lie}}, \quad a_{[\mathrm{wt}a-1]}\mapsto [a], \tag{2.13}$$

2.1.5. *Generalized Verma module associated to an $A(V)$-module.* Let U be an irreducible left module over $A(V)$. Then U is a module over the Lie algebra $L(V)_0$ via (2.13), which can be lifted to a module over $L(V)_0\oplus L(V)_+$ by letting $(L(V)_+).U := 0$. Consider the following induced module over $L(V)$:

$$M(U) := U(L(V))\otimes_{U(L(V)_0\oplus L(V)_+)} U = U(L(V)_-)\otimes_{\mathbb{C}} U.$$

Let J be the $U(L(V))$-submodule of $M(U)$ generated by the coefficients of the weak associativity (2.2). It was proved in [DLM98] that

$$\bar{M}(U) := M(U)/J$$

is an admissible V-module generated by U, with bottom-level $\bar{M}(U)(0) = U$. The module vertex operator is given by $Y_{\bar{M}(U)}(a,z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} = \sum_{n\in\mathbb{Z}} a_{[n]}z^{-n-1}$, for any $a\in V$. Moreover, any admissible V-module W generated by $W(0) = U$ is a quotient module of $\bar{M}(U)$.

Let M be an irreducible V-moule, then $M(0)^*$ is an irreducible left $A(V)$-module via (2.12). The anti-involution $\theta : L(V) \to L(V)$ of Lie algebras induces an isomorphism of associative algebras

$$\theta : U(L(V)) \to U(L(V))^{\mathrm{op}}.$$

Since $\theta(b_r(n_r)\dots b_1(n_1)) = b'_1(n_1)\dots b'_r(n_r)$, where $b'(n)$ is given by (2.7), and $\bar{M}(M(0)^*) = U(L(V)).M(0)^* = \theta(U(L(V))).M(0)^*$. It follows that

$$\bar{M}(M(0)^*) = \mathrm{span}\{b'_1(n_1)\dots b'_r(n_r)v' : r \geq 0, b_i \in V, n_i \in \mathbb{Z}, \forall i, v' \in M(0)^*\}. \tag{2.14}$$

Moreover, by carefully choosing the coefficients in the weak associativity (2.2) for the vertex operator $Y_{\bar{M}(U)}$, one can also show that $\bar{M}(M(0)^*) = \mathrm{span}\{b'(n)v' : b \in V, n \in \mathbb{Z}, v' \in M(0)^*\}$ [LL04]. These facts will be used in Section 5

2.2. **The chiral Lie algebra ancillary to** $(\mathbb{P}^1, \infty, 1, 0)$**.** The chiral Lie algebra $\mathcal{L}_{C\backslash P_\bullet}(V)$ ancillary to a VOA V and a stable n-pointed curve $(C, P_\bullet)$ was defined as $H^0(C\backslash P_\bullet, \mathscr{V}_C \otimes \Omega_C/\mathrm{Im}\nabla)$, see [FBZ04, BD04, DGT24]. Instead of a family of stable curves, in this paper we are only interested in one smooth curve $C = \mathbb{P}^1$ with three marked points $P_\bullet = (\infty, 1, 0)$. We can reinterpret the trivialization of the vector bundle $\mathscr{V}_C \otimes \Omega_C/\mathrm{Im}\nabla$ on $\mathbb{P}^1$ using a purely algebraic language, and define the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ ancillary to $\mathbb{P}^1\backslash\{\infty, 1, 0\}$ as follows:

Consider the localization $\mathbb{C}[z^{\pm 1}, (z-1)^{\pm 1}]$ of the Laurent polynomial ring $\mathbb{C}[z^{\pm 1}]$ with respect to the element $(z-1)$. Since d/dz preserves the set $\{(z-1)^{-n} : n \in \mathbb{N}\}$, it induces a differential operator on the localization, which makes the pair $(\mathbb{C}[z^{\pm 1}, (z-1)^{\pm 1}], d/dz, 1)$ a commutative differential unital algebra. Then $V \otimes \mathbb{C}[z^{\pm 1}, (z-1)^{\pm 1}]$ is a vertex algebra.

Definition 2.8. The chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ ancillary to $\mathbb{P}^1\backslash\{\infty, 1, 0\}$ is defined by

$$\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V) := \left(V \otimes \mathbb{C}[z^{\pm 1}, (z-1)^{\pm 1}]\right)/\mathrm{Im}\nabla, \tag{2.15}$$

where $\nabla = L(-1) \otimes \mathrm{Id} + \mathrm{Id} \otimes \frac{d}{dz}$. Then as a vector space we have

$$\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V) = \mathrm{span}\left\{a \otimes \frac{z^n}{(z-1)^m} : a \in V, m, n \in \mathbb{Z}\right\}, \tag{2.16}$$

with $(L(-1)a) \otimes \frac{z^n}{(z-1)^m} = -a \otimes \frac{d}{dz}\left(\frac{z^n}{(z-1)^m}\right)$, where we use the same symbol for the equivalent class of $a \otimes \frac{z^n}{(z-1)^m} \in V \otimes \mathbb{C}[z^{\pm 1}, (z-1)^{\pm 1}]$ in the quotient space. The Lie bracket on $\mathcal{L}(V)$ is given by

$$[a \otimes \frac{z^n}{(z-1)^m}, b \otimes \frac{z^s}{(z-1)^t}] = \sum_{i\geq 0}\sum_{j\geq 0}\binom{n}{i}\binom{-m}{j} a_{i+j}b \otimes \frac{z^{n+s-i}}{(z-1)^{m+t+j}}, \tag{2.17}$$

where $a, b \in V$, and $m, n, s, t \in \mathbb{Z}$, see Section 3.9 in [GLZ24].

The following Proposition is a purely algebraic version of the chiral Lie algebra action on the space of coinvariants [FBZ04, DGT24]. The proof is an immediate consequence of the Jacobi identity of VOAs, together with (2.17), we omit it.

Proposition 2.9. *Let M^1, M^2, and M^3 be V-modules, and let $a \otimes \frac{z^n}{(z-1)^m} \in \mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$. Then*

(1) $(M^3)'$ is a module over the Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ via $\rho_\infty : \mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V) \to \mathfrak{gl}((M^3)')$,

$$\begin{aligned}\rho_\infty\left(a \otimes \frac{z^n}{(z-1)^m}\right)(v'_3) &= \mathrm{Res}_{z=\infty} Y_{(M^3)'}(\vartheta(a), z^{-1})v'_3\iota_{z,1}\left(\frac{z^n}{(z-1)^m}\right)\\ &= -\sum_{j\geq 0}\binom{-m}{j}(-1)^j a'(n-m-j)v'_3, \quad v'_3 \in (M^3)',\end{aligned} \tag{2.18}$$

where $\vartheta(a) = -e^{zL(1)}(-z^{-2})^{L(0)}(a)$, and $a'(k)$ is given by (2.7).

(2) M^1 is a module over the Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ via $\rho_1 : \mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V) \to \mathfrak{gl}(M^1)$,

$$\begin{aligned}\rho_1\left(a\otimes\frac{z^n}{(z-1)^m}\right)(v_1) &= \mathrm{Res}_{z=1}Y_{M^1}(a,z-1)v_1\iota_{1,z-1}\left(\frac{z^n}{(z-1)^m}\right)\\ &= \sum_{j\geq 0}\binom{n}{j}a(j-m)v_1, \quad v_1\in M^1.\end{aligned} \tag{2.19}$$

(3) M^2 is a module over the Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ via $\rho_0 : \mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V) \to \mathfrak{gl}(M^2)$,

$$\begin{aligned}\rho_0\left(a\otimes\frac{z^n}{(z-1)^m}\right)(v_2) &= \mathrm{Res}_{z=0}Y_{M^1}(a,z)v_2\iota_{1,z}\left(\frac{z^n}{(z-1)^m}\right)\\ &= \sum_{j\geq 0}\binom{-m}{j}(-1)^{-m-j}a(n+j)v_2, \quad v_2\in M^2.\end{aligned} \tag{2.20}$$

In particular, $(M^3)'\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2$ is a tensor product module over the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$, with the module action given as follows:

$$\begin{aligned}&\left(a\otimes\frac{z^n}{(z-1)^m}\right).(v_3'\otimes v_1\otimes v_2)\\ &= -\sum_{j\geq 0}\binom{-m}{j}(-1)^j a'(n-m-j)v_3'\otimes v_1\otimes v_2 + \sum_{j\geq 0}\binom{n}{j}v_3'\otimes a(j-m)v_1\otimes v_2\\ &\quad + \sum_{j\geq 0}\binom{-m}{j}(-1)^{-m-j}v_3'\otimes v_1\otimes a(n+j)v_2,\end{aligned} \tag{2.21}$$

where $v_3'\otimes v_1\otimes v_2\in (M^3)'\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2$.

2.3. **Space of three-pointed conformal blocks on $\mathbb{P}^1$.** Consider the following datum:

$$\Sigma_1((M^3)',M^1,M^2) = \left(\mathbb{P}^1,\infty,1,0,1/z,z-1,z,(M^3)',M^1,M^2\right), \tag{2.22}$$

where $1/z$, $z-1$, and z are the local coordinate around the points ∞, 1, and 0 on $\mathbb{P}^1$, respectively. The contragredient module $(M^3)'$ is attached to ∞, and the V-modules M^1 and M^2 are attached to the points 1 and 0, respectively. Recall that $(M^3)'\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2$ is a module over the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ via (2.21).

Definition 2.10. [NT05, FBZ04, DGT24] Let V be a VOA, and M^1, M^2, and M^3 be V-modules. The quotient space

$$\mathbb{V}\left(\Sigma_1((M^3)',M^1,M^2)\right) := \frac{(M^3)'\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2}{\mathcal{L}_{\mathbb{P}^1\backslash\{0,1,\infty\}}(V).\left((M^3)'\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2\right)}. \tag{2.23}$$

is called the **space of coinvariants** associated to the datum $\Sigma_1((M^3)',M^1,M^2)$. The dual space

$$\mathscr{C}\left(\Sigma_1((M^3)',M^1,M^2)\right) := \left(\frac{(M^3)'\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2}{\mathcal{L}_{\mathbb{P}^1\backslash\{0,1,\infty\}}(V).\left((M^3)'\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2\right)}\right)^* \tag{2.24}$$

is called the **space of conformal blocks** associated to $\Sigma_1((M^3)',M^1,M^2)$. We refer to an element $\varphi\in\mathscr{C}\left(\Sigma_1((M^3)',M^1,M^2)\right)$ as a **(three-pointed) conformal block** associated to the datum $\Sigma_1((M^3)',M^1,M^2)$.

One can replace the marked point 1 on $\mathbb{P}^1$ by another point $w\in\mathbb{C}^\times$ on the same chart containing 0, and define the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,w,0\}}(V)$ and its actions in similar ways as Definition 2.8 and Proposition 2.9, with 1 replaced by w. In particular,

$$\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,w,0\}}(V) = \mathrm{span}\left\{a\otimes\frac{z^n}{(z-w)^m} : a\in V, m,n\in\mathbb{Z}\right\}, \tag{2.25}$$

with $(L(-1)a)\otimes\frac{z^n}{(z-w)^m}=-a\otimes\frac{d}{dz}\left(\frac{z^n}{(z-w)^m}\right)$, and the module action of the Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,w,0\}}(V)$ on $(M^3)'\otimes_{\mathbb{C}}M^1\otimes_{\mathbb{C}}M^2$ is given by

$$\begin{aligned}&\left(a\otimes\frac{z^n}{(z-w)^m}\right).(v_3'\otimes v_1\otimes v_2)\\&=-\sum_{j\geq 0}\binom{-m}{j}(-1)^jw^ja'(n-m-j)v_3'\otimes v_1\otimes v_2+\sum_{j\geq 0}\binom{n}{j}w^{n-j}v_3'\otimes a(j-m)v_1\otimes v_2\\&\quad+\sum_{j\geq 0}\binom{-m}{j}(-1)^{-m-j}w^{-m-j}v_3'\otimes v_1\otimes a(n+j)v_2.\end{aligned}\tag{2.26}$$

Similar to (2.24), for the datum

$$\Sigma_w((M^3)',M^1,M^2):=\left(\mathbb{P}^1,\infty,w,0,1/z,z-w,z,(M^3)',M^1,M^2\right),$$

we can define the space conformal blocks $\mathscr{C}\left(\Sigma_w((M^3)',M^1,M^2)\right)$ as the vector space of linear functionals on $(M^3)'\otimes_{\mathbb{C}}M^1\otimes_{\mathbb{C}}M^2$ that are invariant under the actions of the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,w,0\}}(V)$. The conformal blocks associated to $\Sigma_1((M^3)',M^1,M^2)$ and $\Sigma_w((M^3)',M^1,M^2)$ are related by the following formula, see [GLZ24, (eq. 4.11)]:

$$\begin{aligned}&\mathscr{C}\left(\Sigma_1((M^3)',M^1,M^2)\right)\xrightarrow{\cong}\mathscr{C}\left(\Sigma_w((M^3)',M^1,M^2)\right),\quad\varphi_1\mapsto\varphi_w,\\&\langle\varphi_w\,|\,v_3'\otimes v_1\otimes v_2\rangle=\langle\varphi_1\,|\,w^{L(0)-h_3}v_3'\otimes w^{-L(0)+h_1}v_1\otimes w^{-L(0)+h_2}v_2\rangle.\end{aligned}\tag{2.27}$$

It is well-known that there is a one-to-one correspondence between three-pointed conformal blocks and intertwining operators of VOAs, see [TUY89, FBZ04, NT05, GLZ24]:

Proposition 2.11. *Let M^1, M^2, and M^3 be V-modules. Then there is an isomorphism of vector spaces $\mathscr{C}\left(\Sigma_1((M^3)',M^1,M^2)\right)\cong I\binom{M^3}{M^1\ M^2}$. In particular, the fusion rule $N\binom{M^3}{M^1\ M^2}$ is equal to the dimension of the space of three-pointed conformal blocks on $\mathbb{P}^1$.*

3. Restriction of the chiral Lie algebra at ∞

In this Section, we introduce the notion of ∞-restricted chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$ and its augmented ideal $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$. We discuss their basic properties and give a short list of spanning elements of these Lie algebras. These properties will used to define the ∞-restricted conformal blocks in the next Section.

3.1. **Spanning elements of ∞-restricted chiral Lie algebra.** Observe that if $\mathfrak{g}$ is a Lie algebra, M is a $\mathfrak{g}$-module, and $V\subset M$ is a subspace, then the stabilizer $\mathrm{Stab}_M(V)=\{X\in\mathfrak{g}:X.V\subseteq V\}$ is clearly a Lie subalgebra of $\mathfrak{g}$. In particular,

$$\begin{aligned}&\mathrm{Stab}_{(M^3)'\otimes_{\mathbb{C}}M^1\otimes_{\mathbb{C}}M^2}(M^3(0)^*\otimes_{\mathbb{C}}M^1\otimes_{\mathbb{C}}M^2)\\&=\{X\in\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V):X.(M^3(0)^*\otimes_{\mathbb{C}}M^1\otimes_{\mathbb{C}}M^2)\subseteq M^3(0)^*\otimes_{\mathbb{C}}M^1\otimes_{\mathbb{C}}M^2\}\end{aligned}\tag{3.1}$$

is a Lie subalgebra of the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$.

Definition 3.1. We call the following subspace

$$\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}=\mathrm{span}\left\{a\otimes\frac{z^n}{(z-1)^m}\in\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V):a\in V,\ n-m\leq\mathrm{wt}a-1\right\}\tag{3.2}$$

of the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ the **∞-restricted chiral Lie algebra**.

Lemma 3.2. *The following properties hold for $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$ given by (3.2):*

(1) $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$ is a Lie subalgebra of $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$.
(2) $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}\subseteq\mathrm{Stab}_{(M^3)'\otimes_{\mathbb{C}}M^1\otimes_{\mathbb{C}}M^2}(M^3(0)^*\otimes_{\mathbb{C}}M^1\otimes_{\mathbb{C}}M^2)$.

Proof. Let $a\otimes \frac{z^n}{(z-1)^m}$ and $b\otimes \frac{z^s}{(z-1)^t} \in \mathcal{L}_{\mathbb{P}^1\setminus\{1,0\}}(V)_{\leq 0}$, with $n-m\leq \mathrm{wt}a-1$ and $s-t\leq \mathrm{wt}b-1$. By (2.17), $[a\otimes \frac{z^n}{(z-1)^m}, b\otimes \frac{z^s}{(z-1)^t}] = \sum_{i\geq 0}\sum_{j\geq 0}\binom{n}{i}\binom{-m}{j} a_{i+j}b\otimes \frac{z^{n+s-i}}{(z-1)^{m+t+j}}$, with

$$\begin{aligned}(n+s-i)-(m+t+j) &= (n-m)+(s-t)-i-j\leq \mathrm{wt}a-1+\mathrm{wt}b-1-i-j\\ &= \mathrm{wt}(a_{i+j}b)-1,\end{aligned}\tag{3.3}$$

for all $i,j\geq 0$. Thus, $[a\otimes \frac{z^n}{(z-1)^m}, b\otimes \frac{z^s}{(z-1)^t}] \in \mathcal{L}_{\mathbb{P}^1\setminus\{1,0\}}(V)_{\leq 0}$, and $\mathcal{L}_{\mathbb{P}^1\setminus\{1,0\}}(V)_{\leq 0}$ is a Lie subalgebra.

Let $v_3'\otimes v_1\otimes v_2\in M^3(0)^*\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2$, and $a\otimes \frac{z^n}{(z-1)^m}\in \mathcal{L}_{\mathbb{P}^1\setminus\{1,0\}}(V)_{\leq 0}$. Since $\deg(a'(n-m-j)v_3') = -\mathrm{wt}a+n-m-j+1\leq -j\leq 0$ for any $j\geq 0$, in view of (3.2), it follows that $\sum_{j\geq 0}\binom{-m}{j}(-1)^j a'(n-m-j)v_3'\in M^3(0)^*$. Then by (2.21) we have

$$\begin{aligned}&\left(a\otimes \frac{z^n}{(z-1)^m}\right).(v_3'\otimes v_1\otimes v_2)\\ &= -\sum_{j\geq 0}\binom{-m}{j}(-1)^j a'(n-m-j)v_3'\otimes v_1\otimes v_2+\sum_{j\geq 0}\binom{n}{j}v_3'\otimes a(j-m)v_1\otimes v_2\\ &\quad +\sum_{j\geq 0}\binom{-m}{j}(-1)^{-m-j}v_3'\otimes v_1\otimes a(n+j)v_2\in M^3(0)^*\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2.\end{aligned}$$

This shows $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}\subseteq \mathrm{Stab}_{(M^3)'\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2}(M^3(0)^*\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2)$. □

Proposition 3.3. *The Lie algebra $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}$ is spanned by the following elements:*

$$a\otimes \frac{z^{\mathrm{wt}a}}{z-1},\quad a\otimes z^{\mathrm{wt}a-k},\quad a\in V \text{ homogeneous},\ k\geq 1.\tag{3.4}$$

We make a table for the pairs (n,m) such that $n-m\leq \mathrm{wt}a-1$:

$$\begin{array}{ccccccc} \dots & (\mathrm{wt}a-3,-2) & (\mathrm{wt}a-2,-1) & (\mathrm{wt}a-1,0) & (\mathrm{wt}a,1) & (\mathrm{wt}a+1,2) & \dots\\ \dots & (\mathrm{wt}a-3,-1) & (\mathrm{wt}a-2,0) & (\mathrm{wt}a-1,1) & (\mathrm{wt}a,2) & (\mathrm{wt}a+1,3) & \dots\\ \dots & (\mathrm{wt}a-3,0) & (\mathrm{wt}a-2,1) & (\mathrm{wt}a-1,2) & (\mathrm{wt}a,3) & (\mathrm{wt}a+1,4) & \dots\\ \dots & (\mathrm{wt}a-3,1) & (\mathrm{wt}a-2,2) & (\mathrm{wt}a-1,3) & (\mathrm{wt}a,4) & (\mathrm{wt}a+1,5) & \dots\\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots\\ \dots & (-2) & (-1) & (0) & (1) & (2) & \dots,\end{array}\tag{3.5}$$

wherein the columns are labeled by the indices (i), with $i\in\mathbb{Z}$. The pairs $(n,m)=(\mathrm{wt}a,1),(\mathrm{wt}a-k,0)$, with $k\geq 1$, corresponding to elements (3.4) are marked in red in table (3.5).

proof of Proposition 3.3. Let $\mathfrak{g}$ be the subspace of $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}$ spanned by the elements (3.4). We need to show that $a\otimes\frac{z^n}{(z-1)^m}$, with (n,m) given by non-red pairs in (3.5), are contained in $\mathfrak{g}$, for any $a\in V$. By abuse of language, we also say that the pair (n,m) belongs to $\mathfrak{g}$ if the corresponding element $a\otimes \frac{z^n}{(z-1)^m}$ belongs to $\mathfrak{g}$.

Fix a $l\geq 0$. We claim that

$$\text{If } a\otimes \frac{z^{\mathrm{wt}a-l}}{z-1}\in\mathfrak{g},\ \forall a\in V,\quad \text{then}\quad a\otimes\frac{z^{\mathrm{wt}a-l}}{(z-1)^m}\in\mathfrak{g},\ \forall a\in V,\ m\geq 1.\tag{3.6}$$

i.e., if the pair $(\mathrm{wt}a-l,1)$ belongs to $\mathfrak{g}$, then all the pairs lying on the same column (l) that are below $(\mathrm{wt}a-l,1)$ are contained in $\mathfrak{g}$.

Indeed, since $a\otimes\frac{z^{\mathrm{wt}a-l}}{z-1}\in\mathfrak{g}$ for all $a\in V$ and $\mathrm{wt}(L(-1)a)=\mathrm{wt}a+1$, by (2.15) we have

$$\begin{aligned}0\equiv L(-1)a\otimes\frac{z^{\mathrm{wt}a+1-l}}{z-1} &= -a\otimes\frac{d}{dz}\left(\frac{z^{\mathrm{wt}a+1-l}}{z-1}\right) = -a\otimes\frac{(\mathrm{wt}a-l)z^{\mathrm{wt}a-l}}{z-1}+a\otimes\frac{z^{\mathrm{wt}a-l}}{(z-1)^2}\\ &\equiv 0+a\otimes\frac{z^{\mathrm{wt}a-l}}{(z-1)^2}\pmod{\mathfrak{g}}.\end{aligned}$$

Hence $(\mathrm{wt}a - l, 2)$ belongs to $\mathfrak{g}$ for all $a \in V$. Proceed like this, using induction on m, we can easily show that $(\mathrm{wt}a - l, m)$ belongs to $\mathfrak{g}$, for all $m \geq 2$ and $a \in V$. This proves our claim (3.6). In particular, all the pairs on the column (1) are contained in $\mathfrak{g}$.

On the other hand, we observe that for any $a \in V$ and $l, m \in \mathbb{Z}$, we have

$$a \otimes \frac{z^{\mathrm{wt}a - l}}{(z-1)^m} + a \otimes \frac{z^{\mathrm{wt}a - l}}{(z-1)^{m+1}} = a \otimes \frac{z^{\mathrm{wt}a - l + 1}}{(z-1)^{m+1}}.$$

We use the following graph for the pairs (n, m) to illustrate this property:

$$\begin{array}{ccc} (\mathrm{wt}a - l, m) & & (\mathrm{wt}a - l + 1, m + 1) \\ \big\downarrow + & \nearrow = & \\ (\mathrm{wt}a - l, m + 1) & & \end{array} \tag{3.7}$$

Using (3.7), it is easy to see that all the pairs on column (i), with $i \geq 2$, are contained in $\mathfrak{g}$. Furthermore, apply (3.7) to the triple

$$\begin{array}{cc} (\mathrm{wt}a - 1, 0) & (\mathrm{wt}a, 1) \\ (\mathrm{wt}a - 1, 1) & \end{array}$$

We have $(\mathrm{wt}a - 1, 1) \in \mathfrak{g}$. Then by Claim (3.6), all the pairs on column (0) are contained in $\mathfrak{g}$. Now apply (3.7) to the triple

$$\begin{array}{cc} (\mathrm{wt}a - 2, 0) & (\mathrm{wt}a - 1, 1) \\ (\mathrm{wt}a - 2, 1) & \end{array}$$

We have $(\mathrm{wt}a - 2, 1) \in \mathfrak{g}$. By the Claim (3.6) again, all the pairs on column (-1) that are lying below $(\mathrm{wt}a - 2, 0)$ are contained in $\mathfrak{g}$. Proceed like this, we can show that all the pairs below the ones marked in red in table (3.5) are contained in $\mathfrak{g}$. By applying (3.7) successively, starting with the triple

$$\begin{array}{cc} (\mathrm{wt}a - 2, -1) & (\mathrm{wt}a - 1, 0) \\ (\mathrm{wt}a - 2, 0) & \end{array}$$

we can easily show that all the pairs on top of the red ones are contained in $\mathfrak{g}$ as well. Hence all the pairs in (3.5) are contained in $\mathfrak{g}$. $\square$

3.2. The augmented ideal of the ∞-restricted chiral Lie algebra. Inspired by the definition of $O(V)$ in the Zhu's algebra $A(V)$ [Z96], we let

$$\mathrm{Ann}_{\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}}(M^3(0)^*) := \left\{X \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0} : \rho_\infty(X)(M^3(0)^*) = 0\right\}, \tag{3.8}$$

which is clearly an ideal of the Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$.

Definition 3.4. We call the subspace

$$\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0} = \mathrm{span}\left\{a \otimes \frac{z^n}{(z-1)^m} \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0} : a \in V,\ n - m < \mathrm{wt}a - 1\right\} \tag{3.9}$$

an **augmented ideal** of the ∞-restricted chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$.

The fact that $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$ is an ideal of $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$ follows from a similar estimate as (3.3), we omit the details. In the following table, spanning elements of $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$ correspond to the pairs (n, m) that are lying below the horizontal line:

...	$(\mathrm{wt}a - 3, -2)$	$(\mathrm{wt}a - 2, -1)$	$(\mathrm{wt}a - 1, 0)$	$(\mathrm{wt}a, 1)$	$(\mathrm{wt}a + 1, 2)$	...
...	$(\mathrm{wt}a - 3, -1)$	$(\mathrm{wt}a - 2, 0)$	$(\mathrm{wt}a - 1, 1)$	$(\mathrm{wt}a, 2)$	$(\mathrm{wt}a + 1, 3)$	...
...	$(\mathrm{wt}a - 3, 0)$	$(\mathrm{wt}a - 2, 1)$	$(\mathrm{wt}a - 1, 2)$	$(\mathrm{wt}a, 3)$	$(\mathrm{wt}a + 1, 4)$	...
...	$(\mathrm{wt}a - 3, 1)$	$(\mathrm{wt}a - 2, 2)$	$(\mathrm{wt}a - 1, 3)$	$(\mathrm{wt}a, 4)$	$(\mathrm{wt}a + 1, 5)$	...
...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...

(3.10)

Lemma 3.5. *The ideal $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$ satisfies the following properties:*

(1) $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0} \subseteq \mathrm{Ann}_{\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}}(M^3(0)^*)$. *In particular, we have*

$$\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}.\left(M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2\right) = M^3(0)^* \otimes_{\mathbb{C}} \left(\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}.(M^1 \otimes M^2)\right). \tag{3.11}$$

(2) The quotient Lie algebra $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}/\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$ *is isomorphic to the subalgebra* $L(V)_0$ *of the Borcherd's Lie algebra in Definition 2.6.*

Proof. Let $a \otimes \frac{z^n}{(z-1)^m}, \in \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$. By (2.18), for any $v_3' \in M^3(0)^*$,

$$\rho_\infty\left(a \otimes \frac{z^n}{(z-1)^m}\right)(v_3') = -\sum_{j\geq 0}\binom{-m}{j}(-1)^j a'(n-m-j)v_3' = 0,$$

since $\deg(a'(n-m-j)v_3') = -\mathrm{wt} a + n - m - j + 1 < -j \leq 0$ for any $j \geq 0$. Thus, $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$ annihilates $M^3(0)^*$.

Next, we show that $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}/\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0} = \mathrm{span}\{a \otimes z^{\mathrm{wt} a-1} + \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0} : a \in V\}$. The pair $(\mathrm{wt} a - 1, 0)$ that corresponds to $a \otimes z^{\mathrm{wt} a-1}$ is marked in red in (3.10). Apply (3.7) to the following triple in table (3.10):

$$\frac{(\mathrm{wt} a - 1, 0) \quad (\mathrm{wt} a, 1)}{(\mathrm{wt} a - 1, 1)}$$

we see that $a \otimes \frac{z^{\mathrm{wt} a}}{z-1} \equiv a \otimes z^{\mathrm{wt} a-1} \pmod{\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}}$ since the pair $(\mathrm{wt} a - 1, 1)$ belongs to $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$. Then apply (3.7) to the triple:

$$\frac{(\mathrm{wt} a, 1) \quad (\mathrm{wt} a + 1, 2)}{(\mathrm{wt} a, 2)}$$

we have $a \otimes \frac{z^{\mathrm{wt} a+1}}{(z-1)^2} \equiv a \otimes \frac{z^{\mathrm{wt} a}}{z-1} \equiv a \otimes z^{\mathrm{wt} a-1} \pmod{\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}}$ since the pair $(\mathrm{wt} a, 2)$ belongs to $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$. Proceed like this, we can show that the elements $a \otimes \frac{z^n}{(z-1)^m}$ corresponding to all the pairs lying on the first line of (3.10) are congruent to $a \otimes z^{\mathrm{wt} a-1}$ modulo $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$.

Furthermore, by (2.15) we have $L(-1)a \otimes z^{\mathrm{wt} a} + \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0} = -(\mathrm{wt} a)a \otimes z^{\mathrm{wt} a-1} + \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$, and it follows from (2.17) that

$$\begin{aligned}&[a \otimes z^{\mathrm{wt} a-1} + \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}, b \otimes z^{\mathrm{wt} b-1} + \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}] \\ &= \sum_{i\geq 0}\binom{\mathrm{wt} a - 1}{i} a_i b \otimes z^{\mathrm{wt} a-1+\mathrm{wt} b-1-i} + +\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}.\end{aligned} \tag{3.12}$$

On the other hand, by Definition 2.6, the Lie algebra $L(V)_0 = \mathrm{span}\{a_{[\mathrm{wt} a-1]} : a \in V\}$, subject to the relations $(L(-1)a)_{[\mathrm{wt} a]} = -(\mathrm{wt} a)a_{[\mathrm{wt} a-1]}$ and

$$[a_{[\mathrm{wt} a-1]}, b_{[\mathrm{wt} b-1]}] = \sum_{j\geq 0}\binom{\mathrm{wt} a - 1}{j}(a_j b)_{[\mathrm{wt} a-1+\mathrm{wt} b-1-j]}. \tag{3.13}$$

Comparing (3.12) and (3.13), it is easy to see that $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}/\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0} \cong L(V)_0$ as Lie algebras. □

By adopting a similar argument as the proof of Proposition 3.3, we can easily prove the following fact about the spanning elements of $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$.

Lemma 3.6. *The augmented ideal* $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$ *is spanned by the following elements*

$$a \otimes \frac{z^{\mathrm{wt} a-1}}{z-1}, \quad a \otimes z^{\mathrm{wt} a-k}, \quad a \in V \text{ homogeneous}, \ k \geq 2 \tag{3.14}$$

(3.15)

...	$(\mathrm{wt} a-3,-2)$	$(\mathrm{wt} a-2,-1)$	$(\mathrm{wt} a-1,0)$	$(\mathrm{wt} a,1)$	$(\mathrm{wt} a+1,2)$	...
...	$(\mathrm{wt} a-3,-1)$	$(\mathrm{wt} a-2,0)$	$(\mathrm{wt} a-1,1)$	$(\mathrm{wt} a,2)$	$(\mathrm{wt} a+1,3)$	...
...	$(\mathrm{wt} a-3,0)$	$(\mathrm{wt} a-2,1)$	$(\mathrm{wt} a-1,2)$	$(\mathrm{wt} a,3)$	$(\mathrm{wt} a+1,4)$	...
...	$(\mathrm{wt} a-3,1)$	$(\mathrm{wt} a-2,2)$	$(\mathrm{wt} a-1,3)$	$(\mathrm{wt} a,4)$	$(\mathrm{wt} a+1,5)$	...
...	⋮	⋮	⋮	⋮	⋮	...

The pairs (n, m) corresponding to the spanning elements (3.14) are marked in red in table (3.15).

4. Space of ∞-restricted three-pointed conformal blocks

We introduce the notion of ∞-restricted three-pointed conformal blocks on $\mathbb{P}^1$ in this Section using the ∞-restricted chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$. We then use the ideal $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$ to define the contracted tensor product $M^1 \odot M^2$. This space is closely related to the left $A(V)$-module $A(M^1) \otimes_{A(V)} M^2(0)$ [FZ92], we discuss their relations by the end of this Section.

4.1. ∞-restricted three-pointed conformal blocks on $\mathbb{P}^1$.

We restrict the module $(M^3)'$ in the datum (2.22) to its bottom degree and obtain the following datum:

$$\Sigma_1(M^3(0)^*, M^1, M^2) = \left(\mathbb{P}^1, \infty, 1, 0, 1/z, z-1, z, M^3(0)^*, M^1, M^2\right). \tag{4.1}$$

Note that $M^3(0)^*$ is naturally a right module over the Zhu's algebra $A(V)$ and a left module over $A(V)$ via the involution $\theta : A(V) \to A(V), [a] \mapsto [e^{L(1)}(-1)^{L(0)}a]$.

Consider the ∞-restricted chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$ in the previous Section. It follows from Lemma 3.2 that $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}.(M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2) \subseteq (M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2)$

Definition 4.1. The vector space

$$\mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right) := \left(\frac{M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2}{\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}.\,(M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2)}\right)^* \tag{4.2}$$

is called the **space of (three-pointed) ∞-restricted conformal blocks on $\mathbb{P}^1$**. An element $\varphi \in \mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right)$ is called a **(three-pointed) ∞-restricted conformal block** associated to the datum $\Sigma_1(M^3(0)^*, M^1, M^2)$.

We want to express the right hand side of (4.2) in terms of a hom space. We observe the following facts in the representation theory of Lie algebras. The proof is standard argument, we omit it. See also the proof of Theorem 6.11.

Lemma 4.2. *Let $\mathfrak{g}$ be a Lie algebra, W and M be $\mathfrak{g}$-modules, V be a finite-dimensional $\mathfrak{g}$-module, and V^* be the dual module of V. Then*

(1) There is an isomorphism of vector spaces:

$$\mathrm{Hom}_{\mathbb{C}}(V^* \otimes_{\mathbb{C}} W/\mathfrak{g}.(V^* \otimes_{\mathbb{C}} W), \mathbb{C}) \cong \mathrm{Hom}_{\mathfrak{g}}(W, V), \tag{4.3}$$

where $V^* \otimes W$ is the tensor product of $\mathfrak{g}$-modules.

(2) Let $O(\mathfrak{g}) \leq \mathfrak{g}$ be an ideal of $\mathfrak{g}$. Then $M/O(\mathfrak{g}).M$ is a $\mathfrak{g}/O(\mathfrak{g})$-module, and we have an isomorphism of vector spaces:

$$M/\mathfrak{g}.M \cong (M/O(\mathfrak{g}).M)/(\mathfrak{g}/O(\mathfrak{g})).(M/O(\mathfrak{g}).M) \tag{4.4}$$

We will apply Lemma 4.2 to the following datum:

$$\begin{aligned} &\mathfrak{g} = \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}, \quad W = M^1 \otimes_{\mathbb{C}} M^2, \qquad M = M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2, \\ &V = M^3(0), \qquad\quad O(\mathfrak{g}) = \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}. \end{aligned} \tag{4.5}$$

First, we introduce the following notion based on Lemma 3.5:

Definition 4.3. Let M^1 and M^2 be V-modules. Define

$$M^1 \odot M^2 : = (M^1 \otimes_{\mathbb{C}} M^2)/\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}.(M^1 \otimes_{\mathbb{C}} M^2). \tag{4.6}$$

We call it the **contracted tensor product of M^1 and M^2**.

We remark the following facts for the contracted tensor product $M^1 \odot M^2$:

(1) By Lemma 3.6, together with (2.21), $M^1 \odot M^2$ is spanned by the symbols $v_1 \odot v_2$ which is bilinear in v^1 and v^2, with $v_1 \in M^1$ and $v_2 \in M^2$, subject to the following relations:

$$\sum_{j\geq 0} \binom{\mathrm{wt} a - 1}{j} a(j-1)v_1 \odot v_2 = v_1 \odot \sum_{j\geq 0} a(\mathrm{wt} a - 1 + j)v_2, \tag{4.7}$$

$$\sum_{j\geq 0} \binom{\mathrm{wt} a - k}{j} a(j)v_1 \odot v_2 = -v_1 \odot a(\mathrm{wt} a - k)v_2, \quad k \geq 2, \tag{4.8}$$

for any $a \in V$, $v_1 \in M^1$ and $v_2 \in M^2$.

(2) By Lemmas 3.5 and 4.2, $M^1 \odot M^2$ is a module over the Lie algebra $L(V)_0$, with the module action given by

$$a_{[\mathrm{wt} a-1]}.(v_1 \odot v_2) = \sum_{j\geq 0} \binom{\mathrm{wt} a - 1}{j} a(j)v_1 \odot v_2 + v_1 \odot o(a)v_2, \tag{4.9}$$

for any $a_{[\mathrm{wt} a-1]} \in L(V)_0$, $v_1 \in M^1$ and $v_2 \in M^2$, where $o(a) = a(\mathrm{wt} a - 1)$.

(3) Since $a \otimes \frac{z^{\mathrm{wt} a-1}}{z-1}$ and $a \otimes z^{\mathrm{wt} a-k}$, with $k \geq 2$, are spanning elements of $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$, it follows that the relations (4.7) and (4.8) lead to the following general relations in $M^1 \odot M^2$, which corresponds to the action of $a \otimes \frac{z^{\mathrm{wt} a-s}}{(z-1)^t} \in \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}$ on $v_1 \otimes v_2$:

$$\sum_{j\geq 0} \binom{\mathrm{wt} a - s}{j} a(j-t)v_1 \odot v_2 = -v_1 \odot \sum_{j\geq 0} \binom{-t}{j} (-1)^{t+j} a(\mathrm{wt} a - s + j)v_2, \tag{4.10}$$

where $s, t \in \mathbb{Z}$ such that $s + t > 1$, in view of (3.9).

Note that the left $A(V)$-module $M^3(0)$ is a also module over the Lie algebra $L(V)_0$ via the Lie algebra homomorphism $L(V)_0 \to A(V)_{\mathrm{Lie}}$, $a_{[\mathrm{wt} a-1]} \mapsto [a]$, see [DLM98].

Proposition 4.4. *There is an isomorphism of vector spaces:*

$$\mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right) \cong \mathrm{Hom}_{L(V)_0}\left(M^1 \odot M^2, M^3(0)\right). \tag{4.11}$$

Proof. Apply Lemma 4.2 to the datum (4.5), we have

$$\mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right) = \left(\frac{M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2}{\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}.\,(M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2)}\right)^*$$

$$\cong \mathrm{Hom}_{\mathbb{C}}\left(\frac{(M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2)/\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}.(M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2)}{(\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}/\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}).(\text{The Numerator})}, \mathbb{C}\right) \quad \text{(by (4.4))}$$

$$\cong \mathrm{Hom}_{\mathbb{C}}\left(\frac{(M^3(0)^* \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2)/\left(M^3(0)^* \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}.(M^1 \otimes M^2)\right)}{(L(V)_0).(\text{The Numerator})}, \mathbb{C}\right) \quad \text{(by (3.11))}$$

$$\cong \mathrm{Hom}_{\mathbb{C}}\left(\frac{M^3(0)^* \otimes_{\mathbb{C}} \left(M^1 \odot M^2\right)}{(L(V)_0).\,(M^3(0)^* \otimes_{\mathbb{C}} (M^1 \odot M^2))}, \mathbb{C}\right) \quad \text{(by (4.6))}$$

$$\cong \mathrm{Hom}_{L(V)_0}\left(M^1 \odot M^2, M^3(0)\right) \quad \text{(by (4.3))},$$

where we used the isomorphism $\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}/\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0} \cong L(V)_0$ in Lemma 3.5. □

4.2. **The $L(V)_0$-module $M^1 \odot M^2$ and the $A(V)$-action.** We discuss some basic properties of the $L(V)_0$-module $M^1 \odot M^2$ in Definition 4.3. In particular, we show that $M^1 \odot M^2$ is also a left $A(V)$-module when M^2 is generated by $M^2(0)$.

Proposition 4.5. *Assume the V-module M^2 is generated by $M^2(0)$. Then we have*

$$M^1 \odot M^2 = \mathrm{span}\{v_1 \odot v^2 : v_1 \in M^1, v^2 \in M^2(0)\}. \tag{4.12}$$

In particular, (4.12) holds if M^2 is an irreducible V-module.

Proof. By assumption, we can write $M^2 = \mathrm{span}\{a^1(n_1)\dots a^r(n_r)v^2 : a^i \in V, n_i \in \mathbb{Z}, v^2 \in M^2(0)\}$. Note that $a^i(\mathrm{wt}a^i - k_i)v^2 \in M^2(k_i-1) = M^2(0)$ or 0 if $k_i \leq 1$. It follows that M^2 is spanned by the following elements:

$$a^1(\mathrm{wt}a^1 - k_1)\dots a^r(\mathrm{wt}a^r - k_r)v^2, \quad r \geq 0, a^i \in V,\ k_i \geq 2,\ v^2 \in M^2(0). \tag{4.13}$$

Then by (4.8) we have

$$\begin{aligned}
&v_1 \odot a^1(\mathrm{wt}a^1 - k_1)a^2(\mathrm{wt}a^2 - k_2)\dots a^r(\mathrm{wt}a^r - k_r)v^2\\
&= -\sum_{j_1\geq 0}\binom{\mathrm{wt}a^1 - k_1}{j_1}a^1(j_1)v_1 \odot a^2(\mathrm{wt}a^2 - k_2)\dots a^r(\mathrm{wt}a^r - k_r)v^2\\
&\vdots\\
&= (-1)^r\sum_{j_1\geq 0}\cdots\sum_{j_r\geq 0}\binom{\mathrm{wt}a^1 - k_1}{j_1}\dots\binom{\mathrm{wt}a^r - k_r}{j_r}a^r(j_r)\dots a^1(j_1)v_1 \odot v^2,
\end{aligned}$$

where $v_1 \in M^1$, $a^i \in V$ and $k_i \geq 2$ for all i, $v^2 \in M^2(0)$. This proves (4.12). ☐

Let M be an admissible V-module. We recall the construction of $A(V)$-bimodule $A(M)$ in [FZ92]. By [FZ92, Definition 1.5.2], there are left and right $*$-actions of V on M:

$$V \times M \to M, \quad (a, v) \mapsto a * v := \mathrm{Res}_{z=1} Y_M(a, z-1)v\frac{z^{\mathrm{wt}a}}{z-1} = \sum_{j\geq 0}\binom{\mathrm{wt}a}{j}a(j-1)v, \tag{4.14}$$

$$M \times V \to M, \quad (v, a) \mapsto v * a = \mathrm{Res}_{z=1} Y_M(a, z-1)v\frac{z^{\mathrm{wt}a-1}}{z-1} = \sum_{j\geq 0}\binom{\mathrm{wt}a - 1}{j}a(j-1)v. \tag{4.15}$$

Using the notations in (4.14) and (4.15), we can rewrite formulas (4.7) and (4.9) as follows:

$$(v_1 * a) \odot v_2 = v_1 \odot \sum_{j\geq 0} a(\mathrm{wt}a - 1 + j)v_2, \tag{4.16}$$

$$a_{[\mathrm{wt}a-1]}.(v_1 \odot v_2) = (a * v_1 - v_1 * a) \odot v_2 + v_1 \odot o(a)v_2. \tag{4.17}$$

The following formulas are (1.5.11) – (1.5.15) in [FZ92], which shows that $A(M) = M/O(M)$ is a bimodule over $A(V)$ 2.7:

$$\begin{aligned}
&O(V) * v \subseteq O(M), && v * O(V) \subseteq O(M),\\
&a * O(M) \subseteq O(M), && O(M) * a \subseteq O(M),\\
&(a * b) * v - a * (b * v) \in O(M), && (v * a) * b - v * (a * b) \in O(M),\\
&(a * v) * b - a * (v * b) \in O(M). &&
\end{aligned} \tag{4.18}$$

Now we define an $*$-action of V on $M^1 \otimes_{\mathbb{C}} M^2$ as follows:

$$V \times \left(M^1 \otimes_{\mathbb{C}} M^2\right) \to M^1 \otimes_{\mathbb{C}} M^2, \quad a * (v_1 \otimes v_2) := (a * v_1 - v_1 * a) \otimes v_2 + v_1 \otimes o(a)v_2, \tag{4.19}$$

where $a * v_1$ and $v_1 * a$ are given by (4.14) and (4.15), respectively, with $v_1 \in M^1$ and $a \in V$.

Lemma 4.6. *The action* (4.19) *satisfies*

$$a * \left(\mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}.(M^1 \otimes_{\mathbb{C}} M^2)\right) \subseteq \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{<0}.(M^1 \otimes_{\mathbb{C}} M^2).$$

In particular, the $*$-action induces an action of V on $M^1 \odot M^2$:

$$V \times \left(M^1 \odot M^2\right) \to M^1 \odot M^2, \quad a * (v_1 \odot v_2) = (a * v_1 - v_1 * a) \odot v_2 + v_1 \odot o(a)v_2. \tag{4.20}$$

i.e., $a * (v_1 \odot v_2) = a_{[\mathrm{wt}a-1]}.(v_1 \odot v_2)$, in view of (4.17).

Proof. Let $X \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$. Using the notations in Proposition 2.9, we have $\rho_0(a\otimes z^{\mathrm{wt}a-1})(v_2) = o(a)v_2$ for any $v_2 \in M^2$. Then by (4.20) and (4.17) we have

$$\begin{aligned} a * (X.(v_1 \otimes v_2)) &= a * (\rho_1(X)(v_1) \otimes v_2 + v_1 \otimes \rho_0(X)(v_2)) \\ &= \rho_1(a \otimes z^{\mathrm{wt}a-1}) (\rho_1(X)(v_1)) \otimes v_2 + \rho_1(X)(v_1) \otimes o(a)v_2 \\ &\quad + \rho_1(a \otimes z^{\mathrm{wt}a-1})(v_1) \otimes \rho_0(X)(v_2) + v_1 \otimes o(a) (\rho_0(X)v_2) \\ &= \rho_1(X) \left(\rho_1(a \otimes z^{\mathrm{wt}a-1})(v_1)\right) \otimes v_2 + \rho_1([a \otimes z^{\mathrm{wt}a-1}, X])(v_1) \otimes v_2 \\ &\quad + \rho_1(a \otimes z^{\mathrm{wt}a-1})(v_1) \otimes \rho_0(X)(v_2) + \rho_1(X)(v_1) \otimes o(a)v_2 \\ &\quad + v_1 \otimes \rho_0(X) (o(a)v_2) + v_1 \otimes \rho_0([a \otimes z^{\mathrm{wt}a-1}, X])(v_2). \end{aligned}$$

It follows from (2.21) that

$$\begin{aligned} &\rho_1(X) \left(\rho_1(a \otimes z^{\mathrm{wt}a-1})(v_1)\right) \otimes v_2 + \rho_1(a \otimes z^{\mathrm{wt}a-1})(v_1) \otimes \rho_0(X)(v_2) \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}.(M^1 \otimes_{\mathbb{C}} M^2), \\ &\rho_1(X)(v_1) \otimes o(a)v_2 + v_1 \otimes \rho_0(X) (o(a)v_2) \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}.(M^1 \otimes_{\mathbb{C}} M^2). \end{aligned}$$

Since $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$ is an ideal of the ∞-restricted chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$ and $a\otimes z^{\mathrm{wt}a-1} \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{\leq 0}$, we have $[a \otimes z^{\mathrm{wt}a-1}, X] \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$, and so

$$\rho_1([a \otimes z^{\mathrm{wt}a-1}, X])(v_1) \otimes v_2 + v_1 \otimes \rho_0([a \otimes z^{\mathrm{wt}a-1}, X])(v_2) \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}.(M^1 \otimes_{\mathbb{C}} M^2).$$

Hence we have $a * (X.(v_1 \otimes v_2)) \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}.(M^1 \otimes_{\mathbb{C}} M^2)$. Finally, since $M^1 \odot M^2 = (M^1 \otimes_{\mathbb{C}} M^2)/\mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}.(M^1 \otimes_{\mathbb{C}} M^2)$ in view of (4.6), the action (4.19) induces an action (4.20). □

We have the following Theorem by Proposition 4.5 and Lemma 4.6.

Theorem 4.7. *Assume the V-module M^2 is generated by $M^2(0)$, then (4.20) induces an action*

$$A(V) \times \left(M^1 \odot M^2\right) \to M^1 \odot M^2, \quad [a].(v_1 \odot v^2) = (a * v_1) \odot v^2, \tag{4.21}$$

where $[a] \in A(V)$, $v_1 \in M^1$ and $v^2 \in M^2(0)$, which makes $M^1 \odot M^2$ a left $A(V)$-module.

Proof. Let $v^2 \in M^2(0)$. We claim that

$$O(M^1) \odot v^2 = 0. \tag{4.22}$$

Indeed, recall that $O(M^1) = \mathrm{span}\{a \circ v_1 = \sum_{j\geq 0} \binom{\mathrm{wt}a}{j} a(j-2)v_1 : a \in V, v_1 \in M^1\}$ [FZ92]. Since $a \otimes \frac{z^{\mathrm{wt}a}}{(z-1)^2} \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,1\}}(V)_{<0}$ in view of table (3.15), we have

$$\begin{aligned} (a \circ v_1) \odot v^2 &= \mathrm{Res}_{z=1} Y_{M^1}(a, z-1) v_1 \iota_{1,z-1} \left(\frac{z^{\mathrm{wt}a-1}}{(z-1)^2}\right) \odot v^2 = \rho_1 \left(a \otimes \frac{z^{\mathrm{wt}a}}{(z-1)^2}\right)(v_1) \odot v^2 \\ &= -v_1 \odot \rho_0 \left(a \otimes \frac{z^{\mathrm{wt}a}}{(z-1)^2}\right) v^2 = -v_1 \odot \left(\sum_{j\geq 0} j a(\mathrm{wt}a + j - 1)v^2\right) = 0. \end{aligned}$$

This proves (4.22). We want to show that $O(V) * (M^1 \odot M^2) = 0$, where the $*$-action of V is given by (4.20). By Lemma 4.5, it suffices to show $O(V) * (v_1 \odot v^2) = 0$ for any $v^2 \in M^2(0)$ and $v_1 \in M^1$. Recall that $o(O(V))M^2(0) = 0$ [Z96], then by (4.22) and (4.18), we have

$$O(V) * (v_1 \otimes v^2) = (O(V) * v_1 - v_1 * O(V)) \odot v^2 + v_1 \odot o(O(V))v^2 \subseteq O(M^1) \odot v^2 + 0 = 0.$$

Hence (4.20) descends to a well-defined bilinear map

$$A(V) \times \left(M^1 \odot M^2\right) \to M^1 \odot M^2, \quad [a].(v_1 \odot v_2) = (a * v_1 - v_1 * a) \odot v_2 + v_1 \odot o(a)v_2, \tag{4.23}$$

where $[a] \in A(V)$, $v_1 \in M^1$ and $v_2 \in M^2$.

In particular, for $v_1 \in M^1$ and $v^2 \in M^2(0)$, by (4.16) we have $[a].(v_1 \odot v^2) = (a * v_1 - v_1 * a) \odot v^2 + v_1 \odot o(a)v^2 = (a * v_1) \odot v^2 - v_1 \odot o(a)v^2 + v_1 \odot o(a)v^2 = (a * v_1) \odot v^2$. This proves (4.21). Moreover, for any $a, b \in V$, by (4.18) and (4.22) we have

$$\begin{aligned}&([a] * [b]).(v_1 \odot v^2) - [a].\big([b].(v_1 \odot v^2)\big)\\ &= ((a * b) * v_1 - a * (b * v_1)) \odot v^2 \in O(M^1) \odot v^2 = 0\end{aligned}$$

Thus, (4.21) makes $M^1 \odot M^2$ a left $A(V)$-module. □

Remark 4.8. Note that (4.23) agrees with the Lie algebra $L(V)_0$-action (4.17), which makes $M^1 \odot M^2$ an $A(V)_{\rm Lie}$-module. Since the $A(V)$-action (4.21) is a specialization of the action (4.23) on the bottom level of $M^2(0)$, it follows that if M^2 is generated by $M^2(0)$, then the $L(V)_0$-module homomorphism $\rho : L(V)_0 \to \mathfrak{gl}(M^1 \odot M^2)$ factors through the Lie algebra epimorphism $L(V)_0 \to A(V)_{\rm Lie}$, $a_{[{\rm wt} a-1]} \mapsto [a]$. i.e., we have a commutative diagram:

$$\begin{array}{ccc} L(V)_0 & \xrightarrow{\ \rho\ } & \mathfrak{gl}(M^1 \odot M^2), \\ \big\downarrow & \nearrow_{\varphi} & \\ A(V)_{\rm Lie} & & \end{array}$$

where ρ is given by (4.9) and φ is given by (4.21). In particular, we have

$$\mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, M^3(0)) = \mathrm{Hom}_{A(V)}(M^1 \odot M^2, M^3(0)), \tag{4.24}$$

where M^1, M^2, M^3 are V-modules, and M^2 is generated by $M^2(0)$.

Corollary 4.9. *Assume that the V-module M^2 is generated by $M^2(0)$. The following is an epimorphism of left $A(V)$-modules or $L(V)_0$-modules:*

$$\pi : A(M^1) \otimes_{A(V)} M^2(0) \twoheadrightarrow M^1 \odot M^2, \quad [v_1] \otimes v^2 \mapsto v_1 \odot v^2. \tag{4.25}$$

In particular, we have an estimate:

$$\dim \mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, M^3(0)) \leq \dim \mathrm{Hom}_{A(V)}(A(M^1) \otimes_{A(V)} M^2(0), M^3(0)). \tag{4.26}$$

Proof. To show π is well-defined, we need to show $\pi([O(M^1)] \otimes v^2) = 0$ and $\pi([v_1] * [a] \otimes v^2) = \pi(v_1 \otimes o(a)v^2)$. Indeed, by (4.22) we have $\pi([O(M^1)] \otimes v^2) = O(M^1) \odot v^2 = 0$. Moreover, by (4.16) and (4.21) we have

$$\begin{aligned}\pi([v_1] * [a] \otimes v^2) &= \pi([v_1 * a] \otimes v^2) = v_1 * a \odot v^2 = v_1 \odot o(a)v^2 = \pi(v_1 \otimes o(a)v^2),\\ \pi([a].([v_1] \otimes v^2)) &= \pi([a * v_1] \otimes v^2) = (a * v_1) \odot v^2 = [a].(v_1 \odot v^2) = [a].\pi([v_1] \otimes v^2).\end{aligned}$$

Hence π in (4.25) is a well-defined $A(V)$-module homomorphism. By Lemma 4.5, it is surjective. Moreover, $A(M^1) \otimes_{A(V)} M^2(0)$ is also a module over the Lie algebra $L(V)_0$ via (2.13), with the Lie algebra action given by

$$\begin{aligned}a_{[{\rm wt} a-1]}.([v_1] \otimes v^2) &= [a * v_1] \otimes v^2 = [a * v_1 - v_1 * a] \otimes v^2 + [v_1] \otimes o(a)v^2\\ &= \sum_{j \geq 0} \binom{{\rm wt} a - 1}{j} [a(j)v_1] \otimes v^2 + [v_1] \otimes o(a)v_2,\end{aligned}$$

which agrees with the action of $a_{[{\rm wt} a-1]}$ on $v_1 \odot v_2$ in view of (4.9). Thus π is a homomorphism of $L(V)_0$-modules. Finally, the estimate (4.26) follows from (4.24). □

Remark 4.10. In Section 5.4, we will see that the epimorphism π (4.25) is not necessarily injective, and so the estimate (4.26) is sharp for certain examples of VOAs. On the other hand, if the VOA V is rational and C_2-cofinite, then π is in fact an isomorphism of left $A(V)$-modules.

5. Extension of the ∞-restricted conformal blocks

In this Section, we prove that an ∞-restricted conformal block φ in $\mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right)$ can be extended to a regular three-pointed conformal block $\widetilde{\varphi}$ in $\mathscr{C}\left(\Sigma_1((M^3)', M^1, M^2)\right)$, where $(M^3)' \cong \bar{M}(M^3(0)^*)$, which leads to our hom-space description $\mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, M^3(0))$ of the space of intertwining operators $I\binom{M^3}{M^1\ M^2}$.

5.1. Construction of the extended conformal blocks.

The following lemma shows that any regular conformal block can be restricted to an ∞-restricted conformal block, which also gives an estimate of the fusion rule generalizing the estimate in [Li99, Proposition 2.10].

Lemma 5.1. *Assume the V-module $(M^3)'$ is generated by its bottom degree $M^3(0)^*$, then the restriction map*

$$G : \mathscr{C}\left(\Sigma_1((M^3)', M^1, M^2)\right) \to \mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right), \quad \psi \mapsto \psi|_{M^3(0)^*\otimes M^1\otimes M^2}, \tag{5.1}$$

is injective. In particular, $N\binom{M^3}{M^1\ M^2} \leq \dim \mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, M^3(0))$.

Proof. By the remark in Section 2.1.5, $(M^3)'$ is spanned by $\{b'(n)v_3' : b \in V, n \in \mathbb{Z}, v_3' \in M^3(0)^*\}$, where $b'(n) = \sum_{j\geq 0} \frac{(-1)^{\mathrm{wt} b}}{j!}(L(1)^j b)(2\mathrm{wt} b - n - j - 2)$. Now assume $\psi \in \mathscr{C}\left(\Sigma_1((M^3)', M^1, M^2)\right)$ such that $\psi|_{M^3(0)^*\otimes M^1\otimes M^2} = 0$. Then by (2.21),

$$\begin{aligned}
&\psi(b'(n)v_3' \otimes v_1 \otimes v_2)\\
&= -\psi\left((b \otimes z^n).(v_3' \otimes v_1 \otimes v_2)\right) + \psi(v_3' \otimes \mathrm{Res}_{z=1} Y_{M^1}(a, z-1)v_1 \otimes v_2)\iota_{1,z-1}(z^n)\\
&\quad + \psi(v_3' \otimes v_1 \otimes \mathrm{Res}_{z=0} Y_{M^2}(b, z)v_2)z^n\\
&= 0,
\end{aligned}$$

where the last equality follows from the facts that ψ is invariant under the action of $b \otimes z^n$, and $\psi|_{M^3(0)^*\otimes M^1\otimes M^2} = 0$. Hence $\psi = 0$ on $(M^3)' \otimes M^1 \otimes M^2$. i.e., G is injective. The estimate of the fusion rule follows from Proposition 2.11 and Proposition 4.4. □

Let M^3 be a V-module with bottom degree $M^3(0)$, and let $\bar{M}(M^3(0)^*)$ be the generalized Verma module associated to the left $A(V)$-module $M^3(0)^*$ via the contragredient action (2.12). Note that $\bar{M}(M^3(0)^*) = \mathrm{span}\{b'(n)v_3' : b \in V, n \in \mathbb{Z}, v_3' \in M^3(0)^*\}$.

Construction 5.2. Given an ∞-restricted conformal block $\varphi \in \mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right)$, define a linear map $\widetilde{\varphi} : \bar{M}(M^3(0)^*) \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2 \to \mathbb{C}$ by letting

$$\begin{aligned}
\langle \widetilde{\varphi} \,|\, b'(n)v_3' \otimes v_1 \otimes v_2 \rangle :&= \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z=1} Y_{M^1}(b, z-1)v_1 \otimes v_2 \rangle \iota_{1,z-1}(z^n)\\
&\quad + \langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z=0} Y_{M^2}(b, z)v_2 \rangle z^n,
\end{aligned} \tag{5.2}$$

where $b \in V$, $n \in Z$, and $v_3' \in M^3(0)^*$.

Moreover, for a general spanning element $b_1'(n_1)b_2'(n_2)\dots b_r'(n_r)v_3'$ of $\bar{M}(M^3(0)^*)$, where $b_i \in V$ and $n_i \in \mathbb{Z}$, with $-\mathrm{wt} b_i + n_i + 1 \geq 0$ for all i, we define $\widetilde{\varphi}$ on the element $b_1'(n_1)b_2'(n_2)\dots b_r'(n_r)v_3' \otimes v_1 \otimes v_2$ inductively by the following formula:

$$\begin{aligned}
&\langle \widetilde{\varphi} \,|\, b_1'(n_1)b_2'(n_2)\dots b_r'(n_r)v_3' \otimes v_1 \otimes v_2 \rangle\\
&= \langle \widetilde{\varphi} \,|\, b_2'(n_2)\dots b_r'(n_r)v_3' \otimes \mathrm{Res}_{z=1} Y_{M^1}(b_1, z-1)v_1 \otimes v_2 \rangle \iota_{1,z-1}(z^{n_1})\\
&\quad + \langle \widetilde{\varphi} \,|\, b_2'(n_2)\dots b_r'(n_r)v_3' \otimes v_1 \otimes \mathrm{Res}_{z=0} Y_{M^2}(b_1, z)v_2 \rangle z^{n_1}.
\end{aligned} \tag{5.3}$$

Remark 5.3. Observe that if φ is an element in $\mathscr{C}\left(\Sigma_1(\bar{M}(M^3(0)^*), M^1, M^2)\right)$, then it must satisfy the following equality by (2.21):

$$\begin{aligned}&\left\langle\varphi\,\middle|\,\mathrm{Res}_{z=\infty}Y_{\bar{M}(M^3(0)^*)}(\iota(b),z^{-1})v_3'\otimes v_1\otimes v_2\right\rangle\iota_{z,1}\left(\frac{z^n}{(z-1)^m}\right)\\&=\langle\varphi\,|\,v_3'\otimes\mathrm{Res}_{z=1}Y_{M^1}(b,z-1)v_1\otimes v_2\rangle\iota_{1,z-1}\left(\frac{z^n}{(z-1)^m}\right)\\&\quad+\langle\varphi\,|\,v_3'\otimes v_1\otimes\mathrm{Res}_{z=0}Y_{M^2}(b,z)v_2\rangle\iota_{1,z}\left(\frac{z^n}{(z-1)^m}\right),\end{aligned}\tag{5.4}$$

where $b\in V$, $v_3'\in\bar{M}(M^3(0)^*)$, and $m,n\in\mathbb{Z}$. We construct $\widetilde{\varphi}$ in (5.2) and (5.3) in such a way to make equality hold.

The only relations among the spanning elements $b_1'(n_1)b_2'(n_2)\dots b_r'(n_r)v_3'$ of the generalized Verma module $\bar{M}(M^3(0)^*)$ is the Jacobi identity [DLM98]. Note that (5.4) essentially a variation of the Jacobi identity which on the other hand defines $\widetilde{\varphi}$ in (5.2) and (5.3). Hence $\widetilde{\varphi}$ is a well-defined element in $\left(\bar{M}(M^3(0)^*)\otimes_{\mathbb{C}}M^1\otimes_{\mathbb{C}}M^2\right)^*$.

We want to show that $\widetilde{\varphi}$ is an element in the space of conformal blocks associated to the datum $\Sigma_1(\bar{M}(M^3(0)^*), M^1, M^2)$. i.e., it is also invariant under the action of the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$, see Definition 2.10. Using a similar combinatorial argument as in the proof of Proposition 3.3, we can give a short list of spanning elements of $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$, which is essential for the proof of the invariance of $\widetilde{\varphi}$.

Lemma 5.4. *The chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$ is spanned by the following elements:*

$$a\otimes\frac{z^{\mathrm{wt}a-1}}{z-1},\quad a\otimes z^{\mathrm{wt}a-1},\quad a\otimes z^{\mathrm{wt}a-k},\quad a\otimes z^{\mathrm{wt}a+l},\tag{5.5}$$

where $a\in V$ is homogeneous, $k\geq 2$ and $l\geq 0$.

The following table for the pairs (n,m) illustrates the spanning elements of $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$:

$$\begin{array}{ccccccc}\dots&\vdots&\vdots&\vdots&\vdots&\vdots&\dots\\\dots&(\mathrm{wt}a-3,-4)&(\mathrm{wt}a-2,-3)&(\mathrm{wt}a-1,-2)&(\mathrm{wt}a,-1)&(\mathrm{wt}a+1,0)&\dots\\\dots&(\mathrm{wt}a-3,-3)&(\mathrm{wt}a-2,-2)&(\mathrm{wt}a-1,-1)&(\mathrm{wt}a,0)&(\mathrm{wt}a+1,1)&\dots\\\dots&(\mathrm{wt}a-3,-2)&(\mathrm{wt}a-2,-1)&(\mathrm{wt}a-1,0)&(\mathrm{wt}a,1)&(\mathrm{wt}a+1,2)&\dots\\\dots&(\mathrm{wt}a-3,-1)&(\mathrm{wt}a-2,0)&(\mathrm{wt}a-1,1)&(\mathrm{wt}a,2)&(\mathrm{wt}a+1,3)&\dots\\\dots&(\mathrm{wt}a-3,0)&(\mathrm{wt}a-2,1)&(\mathrm{wt}a-1,2)&(\mathrm{wt}a,3)&(\mathrm{wt}a+1,4)&\dots\\\dots&\vdots&\vdots&\vdots&\vdots&\vdots&\dots\end{array}\tag{5.6}$$

5.2. **The extension theorem.** Now we prove our main theorem about the extension of ∞-restricted conformal blocks. A more general twisted version was proved in [GLZ24, Theorems 5.18, 5.19] using the Riemann-Roch theorem of algebraic curves. Here we give a purely algebraic proof.

Theorem 5.5. *Let $\varphi\in\mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right)$, then $\widetilde{\varphi}$ given by (5.2) and (5.3) is invariant under the action of the three-pointed chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,1,0\}}(V)$. In particular, we have an isomorphism of vector spaces*

$$F:\mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right)\cong\mathscr{C}\left(\Sigma_1(\bar{M}(M^3(0)^*), M^1, M^2)\right),\quad\varphi\mapsto\widetilde{\varphi},\tag{5.7}$$

whose inverse is the restriction map G in (5.1).

Proof. We need to show $\left\langle\widetilde{\varphi}\,\middle|\,\mathcal{L}_{\mathbb{P}^1\setminus\{\infty,1,0\}}(V).(\bar{M}(M^3(0)^*)\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2)\right\rangle=0$. By Lemma 5.4, it is suffices to show that

$$\left\langle\widetilde{\varphi}\,\middle|\left(a\otimes\frac{z^{\mathrm{wt}a-1}}{z-1}\right).(b'(n)v_3'\otimes v_1\otimes v_2)\right\rangle=0, \tag{5.8}$$

$$\left\langle\widetilde{\varphi}\,\middle|\left(a\otimes z^{\mathrm{wt}a-1}\right).(b'(n)v_3'\otimes v_1\otimes v_2)\right\rangle=0, \tag{5.9}$$

$$\left\langle\widetilde{\varphi}\,\middle|\left(a\otimes z^{\mathrm{wt}a-k}\right).(b'(n)v_3'\otimes v_1\otimes v_2)\right\rangle=0,\quad k\geq 2, \tag{5.10}$$

$$\left\langle\widetilde{\varphi}\,\middle|\left(a\otimes z^{\mathrm{wt}a+l}\right).(b'(n)v_3'\otimes v_1\otimes v_2)\right\rangle=0,\quad l\geq 0, \tag{5.11}$$

for any homogeneous $a\in V$, and $b'(n)v_3'\otimes v_1\otimes v_2\in\bar{M}(M^3(0)^*)\otimes_{\mathbb{C}} M^1\otimes_{\mathbb{C}} M^2$.

Case I. Proof of (5.8).

We state some easy facts first. For any $v_3'\in M^3(0)^*$, since $\deg(a'(\mathrm{wt}a-j-2))=-\mathrm{wt}a+\mathrm{wt}a-j-2+1=-j-1<0$ for any $j\geq 0$, we have

$$\mathrm{Res}_{z=\infty}Y_{\bar{M}(M^3(0)^*)}(\vartheta(a),z^{-1})v_3'\iota_{z,1}\left(\frac{z^{\mathrm{wt}a-1}}{z-1}\right)=-\sum_{j\geq 0}a'(\mathrm{wt}a-j-2)v_3'=0.$$

Moreover, let M be a V-module, and M' be the contragredient module of M. For any $v'\in M'$, $v\in M$, $a,b\in V$, and $m,n\in\mathbb{Z}$, by the Jacobi identity for Y_M, we have

$$\begin{aligned}\langle[b'(n),a'(m)]v',v\rangle&=\langle v',[a(m),b(n)v]\rangle=-\sum_{i\geq 0}\binom{n}{i}\langle v',(b(i)a)(m+n-i)v\rangle\\&=-\sum_{i\geq 0}\langle(b(i)a)'(m+n-i)v',v\rangle.\end{aligned} \tag{5.12}$$

It follows that for any $v_3'\in M^3(0)^*$,

$$\begin{aligned}&\mathrm{Res}_{z=\infty}Y_{\bar{M}(M^3(0)^*)}(\vartheta(a),z_2^{-1})b'(n)v_3'\iota_{z,1}\left(\frac{z^{\mathrm{wt}a-1}}{z-1}\right)\\&=\mathrm{Res}_{z=\infty}\left(b'(n)Y_{\bar{M}(M^3(0)^*)}(\vartheta(a),z_2^{-1})v_3'-[b'(n),Y_{\bar{M}(M^3(0)^*)}(\vartheta(a),z_2^{-1})]v_3'\right)\iota_{z,1}\left(\frac{z^{\mathrm{wt}a-1}}{z-1}\right)\\&=\mathrm{Res}_{z=\infty}Y_{\bar{M}(M^3(0)^*)}(\vartheta(b(i)a),z_2^{-1})v_3'\iota_{z,1}\left(\frac{z^{\mathrm{wt}a-1}}{z-1}\right).\end{aligned}$$

Then by (5.4), (2.21), together with the equation above, we have

$$\begin{aligned}&\left\langle\widetilde{\varphi}\,\middle|\left(a\otimes\frac{z^{\mathrm{wt}a-1}}{z-1}\right).(b'(n)v_3'\otimes v_1\otimes v_2)\right\rangle\\&=\left\langle\widetilde{\varphi}\,\middle|\,\mathrm{Res}_{z=\infty}Y_{\bar{M}(M^3(0)^*)}(\vartheta(a),z_2^{-1})b'(n)v_3'\otimes v_1\otimes v_2\right\rangle\iota_{z_2,1}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\&\quad+\left\langle\widetilde{\varphi}\,\middle|\,b'(n)v_3'\otimes\mathrm{Res}_{z_2=1}Y_{M^1}(a,z_2-1)v_1\otimes v_2\right\rangle\iota_{1,z_2-1}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\&\quad+\left\langle\widetilde{\varphi}\,\middle|\,b'(n)v_3'\otimes v_1\otimes\mathrm{Res}_{z_2=0}Y_{M^2}(a,z_2)v_2\right\rangle\iota_{1,z_2}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\&=\sum_{i\geq 0}\binom{n}{i}\left\langle\widetilde{\varphi}\,\middle|\,\mathrm{Res}_{z=\infty}Y_{\bar{M}(M^3(0)^*)}(\vartheta(b(i)a),z_2^{-1})v_3'\otimes v_1\otimes v_2\right\rangle z_2^{n-i}\iota_{z_2,1}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\&\quad+\left\langle\varphi\,\middle|\,v_3'\otimes\mathrm{Res}_{z_1=1}\mathrm{Res}_{z_2=1}Y_{M^1}(b,z_1-1)Y_{M^1}(a,z_2-1)v_1\otimes v_2\right\rangle\iota_{1,z_1-1}(z_1^n)\iota_{1,z_2-1}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\end{aligned}$$

$$
\begin{aligned}
&+ \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1} Y_{M^1}(a, z_2-1)v_1 \otimes \mathrm{Res}_{z_1=0} Y_{M^2}(b, z_1)v_2 \rangle z_1^n \iota_{1,z_2-1}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\
&+ \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_1=1} Y_{M^1}(b, z_1-1)v_1 \otimes \mathrm{Res}_{z_2=0} Y_{M^2}(a, z_2)v_2 \rangle \iota_{1,z_1-1}(z_1^n)\iota_{1,z_2}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\
&+ \langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} Y_{M^2}(b, z_1)Y_{M^2}(a, z_2)v_2 \rangle z_1^n \iota_{1,z_2}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\
=& -\underbrace{\sum_{i\geq 0}\binom{n}{i}\langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1} Y_{M^1}(b(i)a, z_2-1)v_1 \otimes v_2 \rangle \iota_{1,z_2-1}\left(\frac{z_2^{\mathrm{wt}a-1+n-i}}{z_2-1}\right)}_{(A)}\\
&-\underbrace{\sum_{i\geq 0}\binom{n}{i}\langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0} Y_{M^2}(b(i)a, z_2)v_2 \rangle \iota_{1,z_2}\left(\frac{z_2^{\mathrm{wt}a-1+n-i}}{z_2-1}\right)}_{(B)}\\
&+\underbrace{\langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_1=1}\mathrm{Res}_{z_2=1} Y_{M^1}(b, z_1-1)Y_{M^1}(a, z_2-1)v_1 \otimes v_2 \rangle \iota_{1,z_1-1}(z_1^n)\iota_{1,z_2-1}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)}_{(C)}\\
&-\underbrace{\langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} Y_{M^2}(a, z_2)Y_{M^2}(b, z_1)v_2 \rangle z_1^n \iota_{1,z_2}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)}_{(D)}\\
&-\underbrace{\langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1}\mathrm{Res}_{z_1=1} Y_{M^1}(a, z_2-1)Y_{M^1}(b, z_1-1)v_1 \otimes v_2 \rangle \iota_{1,z_1-1}(z_1^n)\iota_{1,z_2-1}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)}_{(E)}\\
&+\underbrace{\langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} Y_{M^2}(b, z_1)Y_{M^2}(a, z_2)v_2 \rangle z_1^n \iota_{1,z_2}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)}_{(F)}\\
=& -(A)-(B)+(C)-(D)-(E)+(F),
\end{aligned}
$$

where the last equality follows from the fact that $\varphi \in \mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right)$ is invariant under the action of $a \otimes \frac{z^{\mathrm{wt}a-1}}{z-1} \in \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}$. Using the Jacobi identity for Y_{M^1}, we have

$$
\begin{aligned}
&(C)-(E)\\
=& \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_1=1}\mathrm{Res}_{z_2=1} Y_{M^1}(b, z_1-1)Y_{M^1}(a, z_2-1)v_1 \otimes v_2 \rangle \iota_{1,z_1-1}(z_1^n)\iota_{1,z_2-1}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\
&- \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1}\mathrm{Res}_{z_1=1} Y_{M^1}(a, z_2-1)Y_{M^1}(b, z_1-1)v_1 \otimes v_2 \rangle \iota_{1,z_1-1}(z_1^n)\iota_{1,z_2-1}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\
=& \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1}\mathrm{Res}_{z_1-z_2=0} Y_{M^1}(Y(b, z_1-z_2)a, z_2-1)v_1 \otimes v_2 \rangle \iota_{z_2,z_1-z_2}(z_1^n)\iota_{1,z_2-1}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\
=& \sum_{i\geq 0}\binom{n}{i}\langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1} Y_{M^1}(b(i)a, z_2-1)v_1 \otimes v_2 \rangle \iota_{1,z_2-1}\left(\frac{z_2^{\mathrm{wt}a-1+n-i}}{z_2-1}\right)\\
=& (A).
\end{aligned}
$$

Similarly, using the Jacobi identity for Y_{M^2}, we have

$$
-(D)+(F)
$$

$$
\begin{aligned}
&= -\langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} Y_{M^2}(a,z_2)Y_{M^2}(b,z_1)v_2\rangle z_1^n \iota_{1,z_2}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\
&\quad + \langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} Y_{M^2}(b,z_1)Y_{M^2}(a,z_2)v_2\rangle z_1^n \iota_{1,z_2}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\
&= \langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1-z_2=0} Y_{M^2}(Y(b,z_1-z_2)a,z_2)v_2\rangle \iota_{z_2,z_1-z_2}(z_1^n) \iota_{1,z_2}\left(\frac{z_2^{\mathrm{wt}a-1}}{z_2-1}\right)\\
&= \sum_{i\geq 0}\binom{n}{i}\langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0} Y_{M^2}(b(i)a,z_2)v_2\rangle \iota_{1,z_2}\left(\frac{z_2^{\mathrm{wt}a-1+n-i}}{z_2-1}\right)\\
&= (B).
\end{aligned}
$$

This shows (5.8) because

$$
\left\langle \widetilde{\varphi} \,\middle|\, \left(a \otimes \frac{z^{\mathrm{wt}a-1}}{z-1}\right).(b'(n)v_3' \otimes v_1 \otimes v_2)\right\rangle = -(A)-(B)+(C)-(D)-(E)+(F) = 0.
$$

Case II. Proof of (5.9). It follows from (5.2) that

$$
\begin{aligned}
&\left\langle \widetilde{\varphi} \,\middle|\, \left(a \otimes z^{\mathrm{wt}a-1}\right).(b'(n)v_3' \otimes v_1 \otimes v_2)\right\rangle\\
&= -\underbrace{\langle \widetilde{\varphi} \,|\, a'(\mathrm{wt}a-1)b'(n)v_3' \otimes v_1 \otimes v_2\rangle}_{(G)} + \underbrace{\langle \widetilde{\varphi} \,|\, b'(n)v_3' \otimes \mathrm{Res}_{z_2=1} Y_{M^1}(a,z_2-1)v_1 \otimes v_2\rangle \iota_{1,z_2-1}(z_2^{\mathrm{wt}a-1})}_{(H)}\\
&\quad + \underbrace{\langle \widetilde{\varphi} \,|\, b'(n)v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0} Y_{M^2}(a,z_2)v_2\rangle z_2^{\mathrm{wt}a-1}}_{(I)}\\
&= -(G)+(H)+(I).
\end{aligned}
$$

Note that $a'(\mathrm{wt}a-1)v_3' = o(a)v_3'$ for $v_3' \in M^3(0)^*$. Then by (5.12) and (5.2), we have

$$
\begin{aligned}
(G) &= \langle \widetilde{\varphi} \,|\, b'(n)o(a)v_3' \otimes v_1 \otimes v_2\rangle + \sum_{i\geq 0}\binom{n}{i}\langle \widetilde{\varphi} \,|\, (b(i)a)'(\mathrm{wt}a-1+n-i)v_3' \otimes v_1 \otimes v_2\rangle\\
&= \underbrace{\langle \varphi \,|\, o(a)v_3' \otimes \mathrm{Res}_{z_1=1} Y_{M^1}(b,z_1-1)v_1 \otimes v_2\rangle \iota_{1,z_1-1}(z_1^n)}_{(G1)}\\
&\quad + \underbrace{\langle \varphi \,|\, o(a)v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0} Y(b,z_1)v_2\rangle z_1^n}_{(G2)}\\
&\quad + \underbrace{\langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1}\mathrm{Res}_{z_1-z_2=0} Y_{M^1}(Y(b,z_1-z_2)a,z_2-1)v_1 \otimes v_2\rangle \iota_{z_2,z_1-z_2}(z_1^n) z_2^{\mathrm{wt}a-1}}_{(G3)}\\
&\quad + \underbrace{\langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1-z_2=0} Y_{M^2}(Y(b,z_1-z_2)a,z_2)v_2\rangle \iota_{z_2,z_1-z_2}(z_1^n) z_2^{\mathrm{wt}a-1}}_{(G4)}\\
&= (G1)+(G2)+(G3)+(G4).
\end{aligned}
$$

On the other hand, using the invariance of φ under the action of $a \otimes z^{\mathrm{wt}a-1} \in \mathcal{L}_{\mathbb{P}^1\setminus\{0,1\}}(V)_{\leq 0}$, together with (5.2), we have

$$
\begin{aligned}
(H) &= \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_1=1}\mathrm{Res}_{z_2=1} Y_{M^1}(b,z_1-1)Y_{M^1}(a,z_2-1)v_1 \otimes v_2\rangle \iota_{1,z_2-1}(z_2^{\mathrm{wt}a-1})\iota_{1,z_1-1}(z_1^n)\\
&\quad + \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1} Y_{M^1}(a,z_2-1)v_1 \otimes \mathrm{Res}_{z_1=0} Y_{M^2}(b,z_1)v_2\rangle \iota_{1,z_2-1}(z_2^{\mathrm{wt}a-1}) z_1^n\\
&= \underbrace{\langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_1=1}\mathrm{Res}_{z_2=1} Y_{M^1}(b,z_1-1)Y_{M^1}(a,z_2-1)v_1 \otimes v_2\rangle \iota_{1,z_2-1}(z_2^{\mathrm{wt}a-1})\iota_{1,z_1-1}(z_1^n)}_{(H1)}\\
&\quad + \underbrace{\langle \varphi \,|\, o(a)v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0} Y_{M^2}(b,z_1)v_2\rangle z_1^n}_{(H2)}
\end{aligned}
$$

$$
\begin{aligned}
&- \underbrace{\langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1=0} Y_{M^2}(a,z_2)Y_{M^1}(b,z_1)v_2\rangle z_1^n z_2^{\mathrm{wt}a-1}}_{(H3)}\\
&= (H1) + (H2) - (H3).
\end{aligned}
$$

$$
\begin{aligned}
(I) &= \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_1=1} Y_{M^1}(b,z_1-1)v_1 \otimes \mathrm{Res}_{z_2=0} Y_{M^2}(a,z_2)v_2\rangle z_2^{\mathrm{wt}a-1}\iota_{1,z_1-1}(z_1^n)\\
&\quad + \langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} Y_{M^2}(b,z_1)Y_{M^1}(a,z_2)v_2\rangle z_2^{\mathrm{wt}a-1} z_1^n\\
&= \underbrace{\langle \varphi \,|\, o(a)v_3' \otimes \mathrm{Res}_{z_1=1} Y_{M^1}(b,z_1-1)v_1 \otimes v_2\rangle \iota_{1,z_1-1}(z_1^n)}_{(I1)}\\
&\quad - \underbrace{\langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1}\mathrm{Res}_{z_1=1} Y_{M^1}(a,z_2-1)Y_{M^1}(b,z_1-1)v_1 \otimes v_2\rangle \iota_{1,z_2-1}(z_2^{\mathrm{wt}a-1})\iota_{1,z_1-1}(z_1^n)}_{(I2)}\\
&\quad + \underbrace{\langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} Y_{M^2}(b,z_1)Y_{M^1}(a,z_2)v_2\rangle z_2^{\mathrm{wt}a-1} z_1^n}_{(I3)}\\
&= (I1) - (I2) + (I3).
\end{aligned}
$$

Note that $-(G1) + (I1) = 0$ and $-(G2) + (H2) = 0$. By the Jacobi identity for Y_{M^1} and Y_{M^2}, we have $(H1) - (I2) = (G3)$ and $-(H3) + (I3) = (G4)$.

$$
\begin{aligned}
&\left\langle \widetilde{\varphi} \,\middle|\, \left(a \otimes z^{\mathrm{wt}a-1}\right).(b'(n)v_3' \otimes v_1 \otimes v_2)\right\rangle = -(G) + (H) + (I)\\
&\quad = -(G1) - (G2) - (G3) - (G4) + (H1) + (H2) - (H3) + (I1) - (I2) + (I3) = 0.
\end{aligned}
$$

Case III. Proof of (5.10).

Let $k \geq 2$. Note that $a'(\mathrm{wt}a - k)v_3' = 0$ for $v_3' \in M^3(0)^*$. Similar to the argument above, using (5.12), we have

$$
\begin{aligned}
&\left\langle \widetilde{\varphi} \,\middle|\, \left(a \otimes z^{\mathrm{wt}a-k}\right).(b'(n)v_3' \otimes v_1 \otimes v_2)\right\rangle\\
&= -\left\langle \widetilde{\varphi} \,\middle|\, \sum_{i\geq 0}\binom{n}{i}(b(i)a)'(\mathrm{wt}a - k + n - i)v_3' \otimes v_1 \otimes v_2\right\rangle\\
&\quad + \langle \widetilde{\varphi} \,|\, b'(n)v_3' \otimes \mathrm{Res}_{z_2=1} Y_{M^1}(a,z_2-1)v_1 \otimes v_2\rangle \iota_{1,z_2-1}(z_2^{\mathrm{wt}a-k})\\
&\quad + \langle \widetilde{\varphi} \,|\, b'(n)v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0} Y_{M^2}(a,z_2)v_2\rangle z_2^{\mathrm{wt}a-k}\\
&= -\sum_{i\geq 0}\binom{n}{i}\langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1} Y_{M^1}(b(i)a,z_2-1)v_1 \otimes v_2\rangle \iota_{1,z_2-1}(z_2^{\mathrm{wt}a-k+n-i})\\
&\quad - \sum_{i\geq 0}\binom{n}{i}\langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0} Y_{M^2}(b(i)a,z_2)v_2\rangle z_2^{\mathrm{wt}a-k+n-i}\\
&\quad + \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_1=1}\mathrm{Res}_{z_2=1} Y_{M^1}(b,z_1-1)Y_{M^1}(a,z_2-1)v_1 \otimes v_2\rangle \iota_{1,z_1-1}(z_1^n)\iota_{1,z_2-1}(z_2^{\mathrm{wt}a-k})\\
&\quad + \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1} Y_{M^1}(a,z_2-1)v_1 \otimes \mathrm{Res}_{z_1=0} Y_{M^2}(b,z_1)v_2\rangle z_1^n \iota_{1,z_2-1}(z_2^{\mathrm{wt}a-k})\\
&\quad + \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_1=1} Y_{M^1}(b,z_1-1)v_1 \otimes \mathrm{Res}_{z_2=0} Y_{M^2}(a,z_2)v_2\rangle \iota_{1,z_1-1}(z_1^n) z_2^{\mathrm{wt}a-k}\\
&\quad + \langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} Y_{M^2}(b,z_1)Y_{M^2}(a,z_2)v_2\rangle z_1^n z_2^{\mathrm{wt}a-k}\\
&= -\langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1}\mathrm{Res}_{z_1-z_2=0} Y_{M^1}(Y(b,z_1-z_2)a,z_2-1)v_1 \otimes v_2\rangle \iota_{z_2,z_1-z_2}(z_1^n)\iota_{1,z_2-1}(z_2^{\mathrm{wt}a-k})\\
&\quad - \langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1-z_2=0} Y_{M^1}(Y(b,z_1-z_2)a,z_2)v_2\rangle \iota_{z_2,z_1-z_2}(z_1^n) z_2^{\mathrm{wt}a-k}\\
&\quad + \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_1=1}\mathrm{Res}_{z_2=1} Y_{M^1}(b,z_1-1)Y_{M^1}(a,z_2-1)v_1 \otimes v_2\rangle \iota_{1,z_1-1}(z_1^n)\iota_{1,z_2-1}(z_2^{\mathrm{wt}a-k})\\
&\quad - \langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1=0} Y_{M^2}(a,z_2)Y_{M^2}(b,z_1)v_2\rangle z_1^n z_2^{\mathrm{wt}a-k}\\
&\quad - \langle \varphi \,|\, v_3' \otimes \mathrm{Res}_{z_2=1}\mathrm{Res}_{z_1=1} Y_{M^1}(a,z_2-1)Y_{M^1}(b,z_2-1)v_1 \otimes v_2\rangle \iota_{1,z_1-1}(z_1^n)\iota_{1,z_2-1}(z_2^{\mathrm{wt}a-k})\\
&\quad + \langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} Y_{M^2}(b,z_1)Y_{M^2}(a,z_2)v_2\rangle z_1^n z_2^{\mathrm{wt}a-k}
\end{aligned}
$$

$= 0.$

Case IV. Proof of (5.11). Since $\deg(a'(\mathrm{wt}a + l)) = l + 1 \geq 1$, we have $a'(\mathrm{wt}a + l)b'(n)v_3' \in \bar{M}(M^3(0)^*)$. Then it follows from (5.3) that

$$\begin{aligned}
&\left\langle \widetilde{\varphi} \,\middle|\, \left(a \otimes z^{\mathrm{wt}a+l}\right).(b'(n)v_3' \otimes v_1 \otimes v_2)\right\rangle \\
&= -\langle \widetilde{\varphi} \,|\, a'(\mathrm{wt}a + l)b'(n)v_3' \otimes v_1 \otimes v_2\rangle + \langle \widetilde{\varphi} \,|\, b'(n)v_3' \otimes \mathrm{Res}_{z_2=1} Y_{M^1}(a, z_2 - 1)v_1 \otimes v_2\rangle \iota_{1,z_2-1}(z_2^{\mathrm{wt}a+l}) \\
&\quad + \langle \widetilde{\varphi} \,|\, b'(n)v_3' \otimes v_1 \otimes \mathrm{Res}_{z_2=0} Y_{M^2}(a, z_2)v_2\rangle z_2^{\mathrm{wt}a+l} \\
&= 0.
\end{aligned}$$

Thus, $\left\langle \widetilde{\varphi} \,\middle|\, \mathcal{L}_{\mathbb{P}^1\setminus\{0,1,\infty\}}(V).(\bar{M}(M^3(0)^*) \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2)\right\rangle = 0$ and $\widetilde{\varphi} \in \mathscr{C}\left(\Sigma_1(\bar{M}(M^3(0)^*), M^1, M^2)\right)$.

It remains to show that the induced map F in (5.7) is an isomorphism of vector spaces. Choose $b'(n) = \mathbf{1}'(-1)$ in (5.2), we have

$$\begin{aligned}
\langle \widetilde{\varphi} \,|\, v_3' \otimes v_1 \otimes v_2\rangle &= \langle \widetilde{\varphi} \,|\, \mathbf{1}'(-1)v_3' \otimes v_1 \otimes v_2\rangle \\
&= \sum_{j\geq 0} \binom{-1}{j} \langle \varphi \,|\, v_3' \otimes \mathbf{1}(j)v_1 \otimes v_2\rangle + \langle \varphi \,|\, v_3' \otimes v_1 \otimes \mathbf{1}(-1)v_2\rangle \\
&= \langle \varphi \,|\, v_3' \otimes v_1 \otimes v_2\rangle.
\end{aligned}$$

Hence $\widetilde{\varphi}|_{M^3(0)^*\otimes M^1 \otimes M^2} = \varphi$, and so $G \circ F = \mathrm{Id}$. Conversely, let $\psi \in \mathscr{C}\left(\Sigma_1(\bar{M}(M^3(0)^*), M^1, M^2)\right)$, using the invariance of ψ under $b \otimes z^n \in \mathcal{L}_{\mathbb{P}^1\setminus\{0,1,\infty\}}(V)$, with $-\mathrm{wt}b + n + 1 \geq 0$, we have

$$\begin{aligned}
&\psi(b'(n)v_3' \otimes v_1 \otimes v_2) \\
&= -\psi\left((b \otimes z^n).(v_3' \otimes v_1 \otimes v_2)\right) + \psi(v_3' \otimes \mathrm{Res}_{z=1} Y_{M^1}(a, z - 1)v_1 \otimes v_2)\iota_{1,z-1}(z^n) \\
&\quad + \psi(v_3' \otimes v_1 \otimes \mathrm{Res}_{z=0} Y_{M^2}(b, z)v_2)z^n \\
&= 0 + G(\psi)(v_3' \otimes \mathrm{Res}_{z=1} Y_{M^1}(a, z - 1)v_1 \otimes v_2)\iota_{1,z-1}(z^n) + G(\psi)(v_3' \otimes v_1 \otimes \mathrm{Res}_{z=0} Y_{M^2}(b, z)v_2)z^n \\
&= (F \circ G)(\psi)(b'(n)v_3' \otimes v_1 \otimes v_2).
\end{aligned}$$

Since $\bar{M}(M^3(0)^*)$ is spanned by $b'(n)v_3'$, with $-\mathrm{wt}b + n + 1 \geq 0$, it follows that $F \circ G = \mathrm{Id}$. Thus, F in (5.7) is an isomorphism of vector spaces. □

5.3. **Fusion rules determined by ∞-restricted conformal blocks.** The following theorem allows us to determine the fusion rule among V-modules in terms of the hom-space that involves $M^1 \odot M^2$ in Definition 4.3.

Theorem 5.6. *Let M^1, M^2, M^3 be ordinary V-modules of conformal weights h_1, h_2, h_3, respectively. Suppose the contragredient module $(M^3)'$ is isomorphic to the generalized Verma module $\bar{M}(M^3(0)^*)$. Then we have $I\binom{M^3}{M^1\ M^2} \cong \mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, M^3(0)) \cong \mathrm{Hom}_{A(V)}(M^1 \odot M^2, M^3(0))$. In particular,*

$$N\binom{M^3}{M^1\ M^2} = \dim \mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, M^3(0)), \tag{5.13}$$

where $M^1 \odot M^2$ is given by Definition 4.3. In particular, if V is rational, then (5.13) holds for any irreducible V-modules M^1, M^2, and M^3.

Proof. (5.13) follows from the following commutative diagram:

$$\begin{array}{ccc}
I\binom{(\bar{M}(M^3(0)^*))'}{M^1\ M^2} & \xrightarrow[\cong]{\text{Prop.2.11}} & \mathscr{C}\left(\Sigma_1(\bar{M}(M^3(0)^*), M^1, M^2)\right) \\
\Big\downarrow & & \Big\downarrow{\cong\ \text{Thm.5.5}} \\
\mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, M^3(0)) & \xleftarrow[\cong]{\text{Prop.4.4}} & \mathscr{C}\left(\Sigma_1(M^3(0)^*, M^1, M^2)\right).
\end{array}$$

If V is rational, and M^3 is an irreducible V-module, then $M^3(0)^*$ is an irreducible left $A(V)$-module. Recall that the left module action is given by the involution $\theta : A(V) \to A(V)$, where $\langle [\theta(a)].v_3' \,|\, v_3\rangle := \langle v_3'.[a] \,|\, v_3\rangle = \langle v_3' \,|\, [a].v_3\rangle = \langle v_3' \,|\, o(a)v_3\rangle$, see (2.12). On the other hand, the bottom degree $(M^3)'(0) = M^3(0)^*$ of the V-module $(M^3)'$ is a left $A(V)$-module with $\langle [a] * v_3' \,|\, v_3\rangle = \langle o(a)v_3' \,|\, v_3\rangle = \langle v_3' \,|\, o(\theta(a))v_3\rangle$. Thus, $\langle [\theta(a)] * v_3' \,|\, v_3\rangle = \langle [\theta(a)].v_3' \,|\, v_3\rangle$, and so $(M^3)'(0)$ is isomorphic to $M^3(0)^*$ as a left $A(V)$-modules. But the generalized Verma module $\bar{M}(M^3(0)^*)$ is an irreducible V-module, see [DLM98, Theorem 7.2]. Hence $\bar{M}(M^3(0)^*) \cong (M^3)'$ as V-modules. □

Corollary 5.7. *Let V be a rational VOA, and let M^1, M^2 be irreducible V-modules. Then*

$$A(M^1) \otimes_{A(V)} M^2(0) \cong M^1 \odot M^2$$

as both $L(V)_0$-modules and left $A(V)$-modules.

Proof. Let M^3 be an irreducible V-module. Since V is rational, Frenkel-Zhu's fusion rules theorem holds [FZ92, Li99, Liu23]:

$$I\binom{M^3}{M^1\ M^2} \cong \mathrm{Hom}_{A(V)}(A(M^1) \otimes_{A(V)} M^2(0), M^3(0)). \tag{5.14}$$

It suffices to show $A(M^1) \otimes_{A(V)} M^2(0) \cong M^1 \odot M^2$ as left $A(V)$-modules. Since V is rational, $A(V)$ is a semisimple associative algebra [Z96, DLM98]. By (4.25), there is an epimorphism of left $A(V)$-modules $A(M^1) \otimes_{A(V)} M^2(0) \twoheadrightarrow M^1 \odot M^2$. Then by Schur's lemma, it suffices to show that the multiplicities of an irreducible $A(V)$-module $M^3(0)$ in $A(M^1) \otimes_{A(V)} M^2(0)$ and $M^1 \odot M^2$ are the same. Indeed, let M^3 be the irreducible V-module with bottom level $M^3(0)$, it follows from Theorem 5.6 and (5.14) that

$$\mathrm{Hom}_{A(V)}(M^1 \odot M^2, M^3(0)) \cong I\binom{M^3}{M^1\ M^2} \cong \mathrm{Hom}_{A(V)}(A(M^1) \otimes_{A(V)} M^2(0), M^3(0)).$$

Noting that the proof of (5.14) did not use the isomorphism $A(M^1) \otimes_{A(V)} M^2(0) \cong M^1 \odot M^2$, see [Li99, Liu23]. □

Corollary 5.8. *Let V be a rational VOA such that the fusion rules among irreducible modules are all finite, and let M^1, M^2 be irreducible V-modules. Then the generalized Verma module $\bar{M}(M^1 \odot M^2)$ associated to the left $A(V)$-module $M^1 \odot M^2$ is isomorphic to the tensor product module $M^1 \boxtimes_{P(z)} M^2$ in* [HL95].

Proof. Let $\mathscr{W}$ be the set of irreducible V-modules. Then $\mathscr{W}$ is finite [Z96, DLM98]. Moreover, by [HL95, Proposition 4.13], $M^1 \boxtimes_{P(z)} M^2 \cong \bigoplus_{W\in\mathscr{W}} N\binom{W}{M^1\ M^2} W$ as a V-module. Since $A(V)$ is semisimple, and $\{W(0) : W \in \mathscr{W}\}$ is the complete list of irreducible left $A(V)$-modules, it follows from (5.13) and Schur's lemma that

$$M^1 \odot M^2 = \bigoplus_{W\in\mathscr{W}} \dim \mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, W(0))W(0) = \bigoplus_{W\in\mathscr{W}} N\binom{W}{M^1\ M^2} W(0).$$

It is clear that from the construction that the functor $\bar{M}(\cdot)$ preserves direct sum and satisfies $\bar{M}(W(0)) \cong W$ when V is rational, see [DLM98, Section 5]. Then we have $\bar{M}(M^1 \odot M^2) \cong \bigoplus_{W\in\mathscr{W}} N\binom{W}{M^1\ M^2} \bar{W(0)} \cong M^1 \boxtimes_{P(z)} M^2$ as V-modules. □

5.4. **The contracted tensor product $M^1 \odot M^2$ for some irrational VOAs.** We examine the contracted tensor product $M^1 \odot M^2$ in Definition 4.3 for some irrational VOAs. It turns out that the left $A(V)$-module $A(M^1) \otimes_{A(V)} M^2(0)$ may or may not be isomorphic to $M^1 \odot M^2$, unlike the rational case in Corollary 5.7.

5.4.1. *Examples in the Heisenberg and vacuum module VOAs.* Let $V = V_{\hat{\mathfrak{g}}}(k, 0)$ be the Heisenberg or vacuum module VOA of level $k \in \mathbb{C}$, where $\mathfrak{g}$ is an abelian or simple Lie algebra, respectively. Recall that $A(V) \cong U(\mathfrak{g})$ [FZ92]. Let U be an irreducible $A(V)$-module. The **generalized Verma module associated to** U [DLM98, LL04] is given by an induced module

$$V_{\hat{\mathfrak{g}}}(k, U) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0)} U \cong U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} U, \tag{5.15}$$

where $K.u = ku$ for some fixed $k \in \mathbb{C}$, $a(0).u = a.u$, and $a(n).u = 0$, for any $a \in \mathfrak{g}$, $n > 0$, and $u \in U$. It was proved in [LL04] that $V_{\hat{\mathfrak{g}}}(k, U)$ is a V-module.

Proposition 5.9. *Let $V = V_{\hat{\mathfrak{g}}}(k, 0)$ be the Heisenberg or vacuum module VOA of level $k \in \mathbb{C}$, and let $M^1 = V_{\hat{\mathfrak{g}}}(k, \lambda)$ and $M^2 = V_{\hat{\mathfrak{g}}}(k, \mu)$. Then $M^1 \odot M^2 \cong A(M^1) \otimes_{A(V)} M^2(0) \cong M^1(0) \otimes_{\mathbb{C}} M^2(0)$.*

Proof. By [FZ92, Theorem 3.2.1], $A(M^1) \otimes_{A(V)} M^2(0) \cong (M^1(0) \otimes_{\mathbb{C}} A(V)) \otimes_{A(V)} M^2(0) \cong M^1(0) \otimes_{\mathbb{C}} M^2(0)$. First, consider the case when $\mathfrak{g} = \mathfrak{h}$ is an abelian Lie algebra. i.e., when $V_{\hat{\mathfrak{g}}}(k, 0) = M_{\hat{\mathfrak{h}}}(k, 0)$ is the Heisenberg VOA. Then $A(M^1) \otimes_{A(V)} M^2(0) \cong \mathbb{C}e^{\lambda} \otimes \mathbb{C}e^{\mu}$ which surjects onto $M^1 \odot M^2$ (4.25). Moreover, $M^1 \odot M^2 \neq 0$, since $\mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, \mathbb{C}e^{\lambda+\mu}) \neq 0$ in view of (5.13). Hence $A(M^1) \otimes_{A(V)} M^2(0) \cong \mathbb{C}e^{\lambda} \otimes \mathbb{C}e^{\mu} \cong M^1 \odot M^2$.

Now consider the case when $\mathfrak{g}$ is a semi-simple Lie algebra and $V = V_{\hat{\mathfrak{g}}}(k, 0)$ is the vacuum module VOA. Since $A(M^1) \otimes_{A(V)} M^2(0)$ is finite-dimensional and surjects onto $M^1 \odot M^2$, then by Weyl's decomposition theorem, both $A(M^1) \otimes_{A(V)} M^2(0)$ and $M^1(0) \otimes_{\mathbb{C}} M^2(0)$ are direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules.

Let $M^3(0)^* = U$ be a finite-dimensional irreducible $A(V_{\hat{\mathfrak{g}}}(k, 0)) \cong U(\mathfrak{g})$-module. Then by (4.24) and Theorem 5.6, we have $\mathrm{Hom}_{U(\mathfrak{g})}(M^1 \odot M^2, U) \cong \mathrm{Hom}_{L(V)_0}(M^1 \odot M^2, U) \cong I\binom{(\bar{M}(U^*))'}{M^1\ M^2}$. On the other hand, $M^2 = V_{\hat{\mathfrak{g}}}(k, \mu)$ is also a generalized Verma module as a module over the VOA $V_{\hat{\mathfrak{g}}}(k, 0)$. Then by [Liu23, Theorem 4.20] and [GLZ23, Proposition 6.3], we have

$$\mathrm{Hom}_{U(\mathfrak{g})}(A(M^1) \otimes_{A(V)} M^2(0), U) \cong I\binom{(\bar{M}(U^*))'}{M^1\ M^2}.$$

Thus the multiplicities of the irreducible $\mathfrak{g}$-module U in $M^1 \odot M^2$ and $A(M^1) \otimes_{A(V)} M^2(0)$ are the same. Hence $A(M^1) \otimes_{A(V)} M^2(0) \cong M^1(0) \otimes_{\mathbb{C}} M^2(0)$. □

5.4.2. *Examples in the Virasoro VOAs and Li's example.* Let $V = M_c$ be the (universal) Virasoro VOA [FZ92] of central charge c. Li gave an example in [Li99, Section 2] that shows Frenkel-Zhu's fusion rules theorem 5.14 does not hold if M^2 and M^3 are not generalized Verma modules.

Let $M(c, h)$ be the Verma module over the Virasoro algebra of highest weight h and central charge c. Recall that $M_c = M(c, 0)/\langle L(-1)v_{c,0} \rangle$, where $v_{c,0}$ is the highest-weight vector. Then $M(c, h)$ is a module over the VOA M_c. Li noticed that if $h_1 \neq h_2$, then

$$N\binom{M(c, h_2)}{M(c, h_1)\ M_c} = 0, \quad \text{but} \quad \dim \mathrm{Hom}_{A(M_c)}(A(M(c, h_1)) \otimes_{A(M_c)} M_c(0), M(c, h_2)(0)) = 1.$$

This is due to the fact that $M^2 = M_c$ as a module over itself is not a generalized Verma module. But the formula in Theorem 5.6 holds for this example since $M^3 = M(c, h_2)$ is a generalized Verma module. We can also see it from the following Proposition.

Proposition 5.10. *The contracted tensor product $M(c, h_1) \odot M_c = \mathrm{span}\{v_{c,h_1} \odot \overline{v_{c,0}}\}$, with the $L(M_c)_0$ module action given by*

$$\omega_{[\mathrm{wt}\omega-1]}.(v_{c,h_1} \odot \overline{v_{c,0}}) = h_1 \cdot (v_{c,h_1} \odot \overline{v_{c,0}}). \tag{5.16}$$

In particular, we have $\mathrm{Hom}_{L(M_c)_0}(M(c, h_1) \odot M_c, M(c, h_2)(0)) = 0 = I\binom{M(c,h_2)}{M(c,h_1)\ M_c}$.

Proof. By Proposition 4.5, $M(c,h_1)\odot M_c = \mathrm{span}\{L(-n_1)\dots L(-n_r)v_{c,h_1}\odot\overline{v_{c,0}} : n_1\geq\cdots\geq n_r\geq 1\}$. Given $v_1\in M(c,h_1)$, by (4.22) and (4.7) we have

$$(L(-n-3)+2L(-n-2)+L(-n-1))\,v_1\odot\overline{v_{c,0}}=0,\quad n\geq 0,$$
$$L(-2)v_1\odot\overline{v_{c,0}}+L(-1)v_1\odot\overline{v_{c,0}}=v_1\odot L(0)\overline{v_{c,0}}=0.$$

Hence $M(c,h_1)\odot M_c$ is spanned by elements of the form $L(-1)^m v_{c,h_1}\odot\overline{v_{c,0}}$, with $m\geq 0$. Moreover, apply (4.8) to $a=\omega$ and $k=2$, we have

$$L(-1)v_1\odot\overline{v_{c,0}}=-v_1\odot L(-1)\overline{v_{c,0}}=-v_1\odot\overline{L(-1)v_{c,0}}=0.$$

This shows $M(c,h_1)\odot M_c=\mathrm{span}\{v_{c,h_1}\odot\overline{v_{c,0}}\}$. Finally, by (4.9) we have

$$\omega_{[\mathrm{wt}\omega-1]}.(v_{c,h_1}\odot\overline{v_{c,0}})=L(-1)v_{c,h_1}\odot\overline{v_{c,0}}+L(0)v_{c,h_1}\odot\overline{v_{c,0}}+v_{c,h_1}\odot L(0)\overline{v_{c,0}}=h_1\cdot(v_{c,h_1}\odot\overline{v_{c,0}}).$$

Since $\omega_{[\mathrm{wt}\omega-1]}.v_{c,h_2}=h_2\cdot v_{c,h_2}$, we have $\mathrm{Hom}_{L(M_c)_0}(M(c,h_1)\odot M_c, M(c,h_2)(0))=0$. □

Remark 5.11. It was proved by Li that $A(M(c,h_1))\otimes_{A(M_c)}M_c(0)\cong\mathbb{C}[t_1]$, see [Li99, Section 2]. Proposition 5.10 shows that $A(M^1)\otimes_{A(V)}M^2(0)$ is **not** isomorphic to $M^1\odot M^2$ in general if V is not a rational VOA. We also have a sharp estimate

$$\dim\mathrm{Hom}_{L(M_c)_0}(M(c,h_1)\odot M_c,\mathbb{C}v_{c,h_2})<\dim\mathrm{Hom}_{A(M_c)}(A(M(c,h_1))\otimes_{A(M_c)}M_c(0),\mathbb{C}v_{c,h_2}),$$

in view of Corollary 4.9. Thus, the formula 5.13 holds under more general assumptions than the fusion rules theorem (5.14).

6. Space of w-restricted conformal blocks and the nuclear democracy theorem

In this Section, instead of ∞-restricted datum, we consider the following restricted datum:

$$\Sigma_w((M^3)',M^1(0),M^2):=\left(\mathbb{P}^1,\infty,w,0,1/z,z-w,z,(M^3)',M^1(0),M^2\right),$$

where $w\in\mathbb{P}^1\backslash\{0,\infty\}$, and we restrict the module M^1 attached to the point w to its bottom degree $M^1(0)$. We show that the space of conformal blocks associated to $\Sigma_w((M^3)',M^1(0),M^2)$ is isomorphic to the space of generalized intertwining operators defined by Li [Li98], and the extension theorem of conformal blocks leads to a new proof of a variant of the generalized nuclear democracy theorem.

6.1. w-restricted three-pointed conformal blocks on $\mathbb{P}^1$.

Similar to the ∞-restricted chiral Lie algebra (3.2), we define the w-restricted chiral Lie algebra to be the subalgebra of $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,w,0\}}(V)$ (2.25) spanned by elements that leave the subspace $(M^3)'\otimes_{\mathbb{C}}M^1(0)\otimes_{\mathbb{C}}M^2$ invariant in $(M^3)'\otimes_{\mathbb{C}}M^1\otimes_{\mathbb{C}}M^2$.

Definition 6.1. We call the following subspace

$$\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}=\mathrm{span}\left\{a\otimes\frac{z^n}{(z-w)^m}\in\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,w,0\}}(V):a\in V,\ n\in\mathbb{Z},\ -m\geq\mathrm{wt}a-1\right\}\tag{6.1}$$

of the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,w,0\}}(V)$ the w**-restricted (three-pointed) chiral Lie algebra**, and call the subspace

$$\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{<0}=\mathrm{span}\left\{a\otimes\frac{z^n}{(z-w)^m}\in\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}:a\in V,\ -m>\mathrm{wt}a-1\right\}\tag{6.2}$$

the **augmented ideal** of $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}$.

Similar to Lemma 3.2, Proposition 3.3, and Lemma 3.5, we have

Lemma 6.2. *The following properties hold for $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}$ and $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{<0}$:*

(1) *$\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}$ is a Lie subalgebra of the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,w,0\}}(V)$, $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{<0}$ is an ideal of $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}$.*

(2) $\rho_0(\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0})(M^1(0)) \subseteq M^1(0)$ *and* $\rho_0(\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{<0})(M^1(0)) = 0$. *In particular, we have* $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}.((M^3)' \otimes_{\mathbb{C}} M^1(0) \otimes_{\mathbb{C}} M^2) \subseteq (M^3)' \otimes_{\mathbb{C}} M^1(0) \otimes_{\mathbb{C}} M^2$.

(3) *The spanning elements of these subalgebras are given as follows:*

$$\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\prec 0} = \operatorname{span}\left\{a \otimes \frac{(z-w)^{\mathrm{wt}a}}{z},\ a \otimes (z-w)^{\mathrm{wt}a-1+k} : a \in V, k \succ 0\right\}, \tag{6.3}$$

where $\prec$ *represents* $\leq$ *or* $<$.

(4) $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}/\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{<0} \cong L(V)_0$.

We make the following table for the pairs $(n, -m)$ in (6.1), the ideal $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{<0}$ corresponds to the pairs lying above the horizontal line. The spanning elements in (6.3) are marked in red.

$$\begin{array}{ccccccc}
\dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\
\dots & (-2, \mathrm{wt}a+2) & (-1, \mathrm{wt}a+2) & (0, \mathrm{wt}a+2) & (1, \mathrm{wt}a+2) & (2, \mathrm{wt}a+2) & \dots \\
\dots & (-2, \mathrm{wt}a+1) & (-1, \mathrm{wt}a+1) & (0, \mathrm{wt}a+1) & (1, \mathrm{wt}a+1) & (2, \mathrm{wt}a+1) & \dots \\
\dots & (-2, \mathrm{wt}a) & (-1, \mathrm{wt}a) & (0, \mathrm{wt}a) & (1, \mathrm{wt}a) & (2, \mathrm{wt}a) & \dots \\
\hline
\dots & (-2, \mathrm{wt}a-1) & (-1, \mathrm{wt}a-1) & (0, \mathrm{wt}a-1) & (1, \mathrm{wt}a-1) & (2, \mathrm{wt}a-1) & \dots
\end{array} \tag{6.4}$$

proof of Lemma 6.2. The proof of parts (1) and (2) are similar to the proof of Lemma 3.2 and Lemma 3.5, we omit the details.

We adopt a similar combinatorial argument as the proof of Proposition 3.3 to show part (3). Denote the subspace on the right hand side of (6.3) by $\mathfrak{g}$. Again, for a given $a \in V$, we say that the pair $(n, -m) \in \mathfrak{g}$ if the corresponding term $a \otimes z^n(z-w)^{-m} \in \mathfrak{g}$.

First, note that for any $a \in V$, $n \in \mathbb{Z}$, and $k \geq 0$, we have

$$a \otimes z^n(z-w)^{\mathrm{wt}a-1+k+1} + a \otimes wz^n(z-w)^{\mathrm{wt}a-1+k} = a \otimes z^{n+1}(z-w)^{\mathrm{wt}a-1+k}.$$

We use the following graph for the paris $(n, -m)$ to illustrate this property:

$$\begin{array}{ccc}
(n, \mathrm{wt}a - 1 + (k+1)) & & \\
\Big|{+} & & \\
w \cdot (n, \mathrm{wt}a - 1 + k) & \xrightarrow{=} & ((n+1), \mathrm{wt}a - 1 + k).
\end{array} \tag{6.5}$$

Using (6.5), it is easy to see that all the pairs on the right side of the middle red column of (6.4) are contained in $\mathfrak{g}$. Since $(-1, \mathrm{wt}a) \in \mathfrak{g}$, it is easy to see that the first column on the left side of the middle red column are contained in $\mathfrak{g}$.

Moreover, given $n \neq 0$, suppose $a \otimes z^{-n}(z-w)^{\mathrm{wt}a-1+k} \in \mathfrak{g}$ for all $a \in V$, we claim that $a \otimes z^{-n-1}(z-w)^{\mathrm{wt}a-1+k} \in \mathfrak{g}$ for all $a \in V$. Indeed, replace a by $L(-1)a$ in the assumption, we have

$$\begin{aligned}
0 &\equiv L(-1)a \otimes z^{-n}(z-w)^{\mathrm{wt}a+k} = -a \otimes (-n)z^{-n-1}(z-w)^{\mathrm{wt}a+k} - a \otimes z^{-n}(\mathrm{wt}a+k)(z-w)^{\mathrm{wt}a+k-1} \\
&= (n - \mathrm{wt}a - k)a \otimes z^{-n}(z-w)^{\mathrm{wt}a+k-1} + (nw)a \otimes z^{-n-1}(z-w)^{\mathrm{wt}a+k-1} \\
&\equiv (nw)a \otimes z^{-n-1}(z-w)^{\mathrm{wt}a+k-1} \pmod{\mathfrak{g}}.
\end{aligned}$$

Then it follows from induction on $n < 0$ that all the pairs on the left side of the middle red column of (6.4) are contained in $\mathfrak{g}$.

Finally, from table (6.4) and (6.5), it is easy to see that $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}/\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{<0}$ is spanned by the equivalent classes $a \otimes (z-w)^{\mathrm{wt}a-1} + \mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{<0}$. Then by (2.17), we have

$$[a \otimes (z-w)^{\mathrm{wt}a-1}, b \otimes (z-w)^{\mathrm{wt}b-1}] = \sum_{j\geq 0}\binom{\mathrm{wt}a-1}{j}a(j)b \otimes (z-w)^{\mathrm{wt}a-1+\mathrm{wt}b-1-j},$$

for any $a, b \in V$. Hence $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}/\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{<0} \cong L(V)_0$, in view of (2.9). □

Definition 6.3. We define the space of *w*-**restricted conformal blocks** associated to the datum

$$\Sigma_w((M^3)', M^1(0), M^2) := \left(\mathbb{P}^1, \infty, w, 0, 1/z, z-w, z, (M^3)', M^1(0), M^2\right)$$

to be the following vector space:

$$\mathscr{C}\left(\Sigma_w((M^3)', M^1(0), M^2)\right) := \left(\frac{(M^3)' \otimes_{\mathbb{C}} M^1(0) \otimes_{\mathbb{C}} M^2}{\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}.((M^3)' \otimes_{\mathbb{C}} M^1(0) \otimes_{\mathbb{C}} M^2)}\right)^*. \tag{6.6}$$

By part (2) of Lemma 6.2, the space $\mathscr{C}\left(\Sigma_w((M^3)', M^1(0), M^2)\right)$ is well-defined.

Using Lemma 4.2 again, we have the following Hom-space description of the space of w-restricted conformal blocks similar to Proposition 4.4.

Proposition 6.4. *There is an isomorphism of vector spaces:*

$$\mathscr{C}\left(\Sigma_w((M^3)', M^1(0), M^2)\right) \cong \mathrm{Hom}_{L(V)_0}((M^3)' \boxdot M^2, M^1(0)^*), \tag{6.7}$$

where $(M^3)' \boxdot M^2 := (M^3)' \otimes_{\mathbb{C}} M^2 / \mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{<0}.((M^3)' \otimes_{\mathbb{C}} M^2)$ *is spanned by the elements* $v_3' \boxdot v_2 = \overline{v_3' \otimes v_2}$, *subject to the following relations:*

$$-\sum_{j\geq 0}\binom{\mathrm{wt} a}{j}(-w)^j a'(\mathrm{wt} a - 1 - j)v_3' \boxdot v_2 + \sum_{j\geq 0}\binom{\mathrm{wt} a}{j}(-w)^{-\mathrm{wt} a - j} v_3' \boxdot a(-1+j)v_2, \tag{6.8}$$

$$\begin{aligned}
&-\sum_{j\geq 0}\binom{\mathrm{wt} a - 1 + k}{j}(-w)^j a'(\mathrm{wt} a - 1 + k - j)v_3' \boxdot v_2 \\
&+\sum_{j\geq 0}\binom{\mathrm{wt} a - 1 + k}{j}(-w)^{-\mathrm{wt} a - 1 + k - j} v_3' \boxdot a(j)v_2,
\end{aligned} \tag{6.9}$$

where $v_3' \in (M^3)'$, $v_2 \in M^2$, $a \in V$, *and* $k \geq 0$.

Proof. Similar to Proposition 4.4, we omit the details. □

6.2. ***w*-restricted conformal blocks and generalized intertwining operators.** The following notion of generalized intertwining operators is a slight modification of [Li98, Definition 4.1]. It fits better into the conformal block picture.

Definition 6.5. Let M^1, M^2, M^3 be ordinary V-modules of conformal weights h_1, h_2, and h_3, respectively. Let $h = h_1 + h_2 - h_3$ as in Definition 2.4. Regard $M^1(0)$ as a module over the Borcherd's Lie algebra $L(V)_0$. A **generalized intertwining operator** is a linear map

$$\Phi(\cdot, w) : M^1(0) \to \mathrm{Hom}_{\mathbb{C}}(M^2, M^3)[\![w, w^{-1}]\!]w^{-h}, \quad \Phi(v_1, w) = \sum_{n\in\mathbb{Z}} v_1(n)w^{-n-1-h},$$

satisfying the following conditions:

(1) (truncation property) For any $v_1 \in M^1(0)$ and $v_2 \in M^2$, we have $v_1(n)v_2 = 0$ for $n \gg 0$;
(2) ($L(-1)$-bracket derivative property) $[L(-1), \Phi(v_1, w)] = \frac{d}{dw}\Phi(v_1, w)$;
(3) (restricted Jacobi identity) [Li98, eq. (4.36)]:

$$\begin{aligned}
&(z-w)^{\mathrm{wt} a - 1} Y_{M^3}(a, z)\Phi(v_1, w) - (-w+z)^{\mathrm{wt} a - 1}\Phi(v_1, w)Y_{M^2}(a, z) \\
&= z^{-1}\delta\left(\frac{w}{z}\right)\Phi(a_{[\mathrm{wt} a - 1]}v_1, w),
\end{aligned} \tag{6.10}$$

for any $a \in V$ and $v_1 \in M^1(0)$.

We denote the vector space of generalized intertwining operators by $I\binom{M^3}{M^1(0)\ M^2}$.

Remark 6.6. For $v_1 \in M^1(0)$, Li called the element $\Phi(v_1, w) \in \mathrm{Hom}_{\mathbb{C}}(M^2, M^3))[\![w, w^{-1}]\!]w^{-h}$ a generalized intertwining operator. The axioms we impose on $\Phi(\cdot, w)$ are based on the axioms of $\Phi(v_1, w)$ in [Li98, Definition 4.1].

The isomorphism (2.27) of conformal blocks restricts to an isomorphism of restricted conformal blocks $\mathscr{C}\left(\Sigma_1((M^3)', M^1(0), M^2)\right) \cong \mathscr{C}\left(\Sigma_w((M^3)', M^1(0), M^2)\right), \varphi_1 \mapsto \varphi_w$, with

$$\begin{aligned}\langle \varphi_w \,|\, v_3' \otimes v_1 \otimes v_2 \rangle &= \langle \varphi_1 \,|\, w^{L(0)-h_3} v_3' \otimes w^{-L(0)+h_1} v_1 \otimes w^{-L(0)+h_2} v_2 \rangle \\ &= \langle \varphi_1 \,|\, v_3' \otimes v_1 \otimes v_2 \rangle w^{\deg v_3' - \deg v_2},\end{aligned}$$

where $v_3' \otimes v_1 \otimes v_2 \in (M^3)' \otimes_{\mathbb{C}} M^1(0) \otimes_{\mathbb{C}} M^2$.

Proposition 6.7. *There is an isomorphism of vector spaces:*

$$\mathscr{C}\left(\Sigma_w((M^3)', M^1(0), M^2)\right) \to I\binom{M^3}{M^1(0)\ M^2}, \quad \varphi_w \mapsto \Phi(\cdot, w),$$

where $\Phi(v_1, w) = \sum_{n\in\mathbb{Z}} v_1(n) w^{-n-1-h}$, *with* $v_1(n) \in \mathrm{Hom}_{\mathbb{C}}(M^2, M^3)$ *defined by*

$$\langle v_3' | v_1(n) v_2 \rangle := \begin{cases} \langle \varphi_1 \,|\, v_3' \otimes v_1 \otimes v_2 \rangle & \text{if} \quad v_3' \in M^3(\deg v_2 - n - 1)^*, \\ 0 & \text{if} \quad v_3' \in M^3(m)^*,\ m \neq \deg v_2 - n - 1. \end{cases} \tag{6.11}$$

Equivalently, $\Phi(\cdot, w)$ *is defined by the following formula:*

$$\langle v_3' \,|\, \Phi(v_1, w) v_2 \rangle := \langle \varphi_w \,|\, v_3' \otimes v_1 \otimes v_2 \rangle w^{-h} = \langle \varphi_1 \,|\, v_3' \otimes v_1 \otimes v_2 \rangle w^{-h + \deg v_3' - \deg v_2}, \tag{6.12}$$

for any homogeneous elements $v_3' \in (M^3)'$, $v_2 \in M^2$, *and* $v_1 \in M^1(0)$.

Proof. We need to show $\Phi(\cdot, w)$ defined by (6.11) and (6.12) is a generalized intertwining operator. By (6.11), $v_1(n) v_2 \in M^3(\deg v_2 - n - 1)$, which is 0 when $n \gg 0$.

Note that $\omega \otimes (z - w) \in \mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}$ by (6.1), and φ_w is invariant under the action of $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}$. Then by (2.26), (2.7), and (6.12), we have

$$\begin{aligned}
0 &= \langle \varphi_w \,|\, (\omega \otimes (z-w)).(v_3' \otimes v_1 \otimes v_2) \rangle w^{-h} \\
&= -\sum_{j\geq 0} \binom{1}{j} (-w)^j \langle \varphi_w \,|\, \omega'(1-j) v_3' \otimes v_1 \otimes v_2 \rangle w^{-h} + \langle \varphi_w \,|\, v_3' \otimes \omega(1) v_1 \otimes v_2 \rangle w^{-h} \\
&\quad + \sum_{j\geq 0} \binom{1}{j} (-w)^{1-j} \langle \varphi_w \,|\, v_3' \otimes v_1 \otimes \omega(j) v_2 \rangle w^{-h} \\
&= -\langle \varphi_1 \,|\, \omega(1) v_3' \otimes v_1 \otimes v_2 \rangle w^{-h+\deg v_3' - \deg v_2} + w \langle \varphi_1 \,|\, \omega(2) v_3' \otimes v_1 \otimes v_2 \rangle w^{-h+\deg v_3' - 1 - \deg v_2} \\
&\quad + \langle \varphi_1 \,|\, v_3' \otimes \omega(1) v_1 \otimes v_2 \rangle w^{-h+\deg v_3' - \deg v_2} \\
&\quad - w \langle \varphi_1 \,|\, v_3' \otimes v_1 \otimes \omega(0) v_2 \rangle w^{-h+\deg v_3' - \deg v_2 - 1} + \langle \varphi_1 \,|\, v_3' \otimes v_1 \otimes \omega(1) v_2 \rangle w^{-h+\deg v_3' - \deg v_2} \\
&= (h_1 + h_2 - h_3 + \deg v_2 - \deg v_3') \langle \varphi_1 \,|\, v_3' \otimes v_1 \otimes v_2 \rangle w^{-h+\deg v_3' - \deg v_2} \\
&\quad + \langle \varphi_1 \,|\, (L(1) v_3' \otimes v_1 \otimes v_2 - v_3' \otimes v_1 \otimes L(-1) v_2) \rangle w^{-h+\deg v_3' - \deg v_2} \\
&= -w \frac{d}{dw} \langle v_3' \,|\, \Phi(v_1, w) v_2 \rangle + w \langle v_3' \,|\, L(-1) \Phi(v_1, w) v_2 \rangle - w \langle v_3' \,|\, \Phi(v_1, w) L(-1) v_2 \rangle.
\end{aligned}$$

Thus we have $[L(-1), \Phi(v_1, w)] = \frac{d}{dw} \Phi(v_1, w)$.

Furthermore, since φ_w is also invariant under the action of $a \otimes z^n (z-w)^{\mathrm{wt} a - 1}$ for any $n \in \mathbb{Z}$ (6.1), then by (2.26) and (6.12), we have

$$\begin{aligned}
0 &= \langle \varphi_w \,|\, (a \otimes z^n (z-w)^{\mathrm{wt} a - 1}).(v_3' \otimes v_1 \otimes v_2) \rangle w^{-h} \\
&= -\sum_{j\geq 0} \binom{\mathrm{wt} a - 1}{j} (-w)^j \langle \varphi_w \,|\, a'(n + \mathrm{wt} a - 1 - j) v_3' \otimes v_1 \otimes v_2 \rangle w^{-h} \\
&\quad + \sum_{j\geq 0} \binom{n}{j} w^{n-j} \langle \varphi_w \,|\, v_3' \otimes a(j + \mathrm{wt} a - 1) v_1 \otimes v_2 \rangle w^{-h}
\end{aligned}$$

$$+\sum_{j\geq 0}\binom{\mathrm{wt}a-1}{j}(-w)^{\mathrm{wt}a-1-j}\langle\varphi_w\,|\,v_3'\otimes v_1\otimes a(n+j)v_2\rangle w^{-h} \tag{6.13}$$
$$=-\sum_{j\geq 0}\binom{\mathrm{wt}a-1}{j}(-w)^{j}\langle v_3'\,|\,a(n+\mathrm{wt}a-1-j)\Phi(v_1,w)v_2\rangle+w^n\langle v_3'\,|\,\Phi(a(\mathrm{wt}a-1)v_1,w)v_2\rangle$$
$$+\sum_{j\geq 0}\binom{\mathrm{wt}a-1}{j}\langle v_3'\,|\,\Phi(v_1,w)a(n+j)v_2\rangle$$
$$=-\mathrm{Res}_z(z-w)^{\mathrm{wt}a-1}z^n\langle v_3'\,|\,Y_{M^3}(a,z)\Phi(v_1,w)v_2\rangle+\mathrm{Res}_z z^n z^{-1}\delta\left(\frac{w}{z}\right)\langle v_3'\,|\,\Phi(a_{[\mathrm{wt}a-1]}v_1,w)v_2\rangle$$
$$+\mathrm{Res}_z(-w+z)^{\mathrm{wt}a-1}z^n\langle v_3'\,|\,\Phi(v_1,w)Y_{M^2}(a,z)v_2\rangle.$$

Since v_3' and z^n are chosen arbitrarily, we have an identity of formal power series:

$$(z-w)^{\mathrm{wt}a-1}Y_{M^3}(a,z)\Phi(v_1,w)-(-w+z)^{\mathrm{wt}a-1}\Phi(v_1,w)Y_{M^2}(a,z)$$
$$=z^{-1}\delta\left(\frac{w}{z}\right)\Phi(a_{[\mathrm{wt}a-1]}v_1,w).$$

Thus, $\Phi(\cdot,w)$ defined by (6.12) is a generalized intertwining operator 6.5.

Conversely, given $\Phi(\cdot,w)\in I\binom{M^3}{M^1(0)\ M^2}$, we define $\varphi_w:(M^3)'\otimes_{\mathbb{C}}M^1(0)\otimes_{\mathbb{C}}M^2\to\mathbb{C}$ by the same formula (6.12). Then by reversing the argument in (6.13), we can easily show that φ_w is invariant under $a\otimes z^n(z-w)^{\mathrm{wt}a-1}\in\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}$ for any $n\in\mathbb{Z}$. Hence it is also invariant under the action of $\sum_{i\geq 0}\binom{k}{i}(-w)^i a\otimes z^{n+k-i}(z-w)^{\mathrm{wt}a-1}=a\otimes z^n(z-w)^{\mathrm{wt}a-1+k}$, which is a general spanning element of $\mathcal{L}_{\mathbb{P}^1\backslash\{0,\infty\}}(V)_{\leq 0}$, in view of Lemma 6.2. This shows $\mathscr{C}\left(\Sigma_w((M^3)',M^1(0),M^2)\right)\cong I\binom{M^3}{M^1(0)\ M^2}$. □

6.3. **Generalized nuclear democracy theorem for VOAs.** Let $\bar{M}(M^1(0))$ be the generalized Verma module associated to the $A(V)$-module $M^1(0)$ [DLM98].

Similar to the extension process in Section 5, given a w-restricted conformal block $\varphi_w\in\mathscr{C}\left(\Sigma_w((M^3)',M^1(0),M^2)\right)$, we may define a linear map $\widetilde{\varphi_w}:(M^3)'\otimes_{\mathbb{C}}\bar{M}(M^1(0))\otimes_{\mathbb{C}}M^2\to\mathbb{C}$ inductively by

$$\langle\widetilde{\varphi_w}\,|\,v_3'\otimes b_1(n_1)\dots b_r(n_r)v_1\otimes v_2\rangle$$
$$:=-\langle\varphi_w\,|\,\mathrm{Res}_{z=\infty}Y_{(M^3)'}(\vartheta(b_1),z^{-1})v_3'\otimes b_2(n_w)\dots b_r(n_r)v_1\otimes v_2\rangle\iota_{z,w}((z-w)^{n_1}) \tag{6.14}$$
$$-\langle\varphi_w\,|\,v_3'\otimes b_2(n_2)\dots b_r(n_r)v_1\otimes\mathrm{Res}_{z=0}Y_{M^2}(b_1,z)v_2\rangle\iota_{w,z}((z-w)^{n_1})$$

Proposition 6.8. *Let $\varphi\in\mathscr{C}\left(\Sigma_w((M^3)',M^1(0),M^2)\right)$. Then $\widetilde{\varphi_w}$ given by (6.14) is invariant under the action of the chiral Lie algebra $\mathcal{L}_{\mathbb{P}^1\backslash\{\infty,w,0\}}(V)$. In particular, we have an isomorphism of vector spaces:*

$$\mathscr{C}\left(\Sigma_w((M^3)',M^1(0),M^2)\right)\cong\mathscr{C}\left(\Sigma_w((M^3)',\bar{M}(M^1(0)),M^2)\right),\quad\varphi_w\mapsto\widetilde{\varphi_w}.$$

Proof. Simiar to the proof of theorem 5.5, we omit the details. See also [GLZ24, Theorems 5.18, 5.19]. □

The following is Li's generalized nuclear democracy theorem with our modified notion of generalized intertwining operator in Definition 6.5, see [Li98, Theorem 4.12].

Theorem 6.9. *Let U be a $L(V)_0$-module, and let $I_0(\cdot,w)\in I\binom{M^3}{U\ M^2}$ be a generalized intertwining operator which is injective as a linear map, then there exists a lowest-weight V-module W with U as its lowest-weight subspace generating W, and there is a unique intertwining operator $I\in\binom{M^3}{M^1\ M^2}$ extending I_0. In particular, if V is rational and U is an irreducible $L(V)_0$-module, then W is an irreducible V-module.*

The proof of Theorem 6.9 in [Li98] used a tensor product construction and an analog of the hom-functor for modules over VOAs. With the properties of w-resricted conformal blocks, we can give an alternative proof of a slightly different version of Li's generalized nuclear democracy theorem.

Theorem 6.10. *Let M^1, M^2, M^3 be ordinary V-modules of conformal weights h_1, h_2, and h_3, respectively. Assume that M^1 is the generalized Verma module associated to an $A(V)$-module $M^1(0)$ (viewed also as a $L(V)_0$-module). Then there is isomorphism of vector spaces:*

$$I\binom{M^3}{M^1\ M^2} \cong I\binom{M^3}{M^1(0)\ M^2} \cong \mathrm{Hom}_{L(V)_0}((M^3)' \boxdot M^2, (M^1(0))^*), \tag{6.15}$$

where $(M^3)' \boxdot M^2 := (M^3)' \otimes_{\mathbb{C}} M^2 / \mathscr{L}_{\mathbb{P}^1\setminus\{0,\infty\}}(V)_{<0}.((M^3)' \otimes_{\mathbb{C}} M^2)$. In particular, if V is rational, and M^1, M^2, M^3 are irreducible V-modules, then we have $I\binom{M^3}{M^1\ M^2} \cong I\binom{M^3}{M^1(0)\ M^2}$.

Proof. (6.15) follows from the following commutative diagram:

$$\begin{array}{ccccc} I\binom{M^3}{M^1\ M^2} & \xrightarrow[\cong]{\text{Prop.2.11}} & \mathscr{C}\left(\Sigma_1((M^3)', M^1, M^2)\right) & & \\ \Big\downarrow & & \cong \Big\downarrow \text{Prop.6.8} & & \\ I\binom{M^3}{U\ M^2} & \xrightarrow[\cong]{\text{Prop.6.7}} & \mathscr{C}\left(\Sigma_w((M^3)', M^1(0), M^2)\right) & \xrightarrow[\cong]{\text{Prop.6.4}} & \mathrm{Hom}_{L(V)_0}((M^3)' \boxdot M^2, U^*). \end{array} \tag{6.16}$$

If V is rational, and M^1 is an irreducible V-module, then $M^1(0)$ is an irreducible $A(V)$-module, and the generalized Verma module $\bar{M}(M^1(0))$ is an irreducible V-module which is isomorphic to M^1, see [DLM98, Theorem 7.2]. ◻

Using the w-restricted conformal blocks, we also have a variant of the generalized nuclear democracy theorem.

Theorem 6.11. *Let M^1, M^2, M^3 be ordinary V-modules of conformal weights h_1, h_2, and h_3, respectively. Assume that M^1 is the generalized Verma module associated to an $A(V)$-module $M^1(0)$. Then there is an isomorphism of vector spaces:*

$$I\binom{M^3}{M^1\ M^2} \cong I\binom{M^3}{M^1(0)\ M^2} \cong \mathrm{Hom}_{\mathscr{L}_{\mathbb{P}^1\setminus\{0,\infty\}}(V)_{\leq 0}}(M^1(0) \otimes_{\mathbb{C}} M^2, \overline{M^3}), \tag{6.17}$$

where $\overline{M^3} = \prod_{n=0}^{\infty} M^3(n) = \mathrm{Hom}_{\mathbb{C}}((M^3)', \mathbb{C})$, which is a module over the w-restricted chiral Lie algebra $\mathscr{L}_{\mathbb{P}^1\setminus\{0,\infty\}}(V)_{\leq 0}$ via ρ_0 (2.20).

Proof. In view of the diagram (6.16), we only need to show

$$\mathscr{C}\left(\Sigma_w((M^3)', M^1(0), M^2)\right) \cong \mathrm{Hom}_{\mathscr{L}_{\mathbb{P}^1\setminus\{0,\infty\}}(V)_{\leq 0}}(M^1(0) \otimes_{\mathbb{C}} M^2, \overline{M^3}), \tag{6.18}$$

which is a infinite-dimensional version of (4.3). To simplify our notations, we denote $M^1(0) \otimes_{\mathbb{C}} M^2$ by W, and denote the w-restricted chiral Lie algebra $\mathscr{L}_{\mathbb{P}^1\setminus\{0,\infty\}}(V)_{\leq 0}$ by $\mathfrak{g}$.

Let $\langle\cdot|\cdot\rangle : (M^3)' \times \overline{M^3} \to \mathbb{C}$ be the natural pair. We define the action of $\mathfrak{g}$ on $\overline{M^3}$ by the following formula:

$$\langle v_3' \,|\, \rho_0(X)(v_3)\rangle := -\langle \rho_\infty(X)(v_3') \,|\, v_3\rangle, \quad v_3' \in (M^3)',\ v_3 \in \overline{M^3}. \tag{6.19}$$

Since $(M^3)'$ is a module over the $\mathfrak{g}$ via ρ_∞, it follows that $\overline{M^3}$ is a module over $\mathfrak{g}$ via ρ_0. Then we need to show $\mathrm{Hom}_{\mathbb{C}}\left((M^3)' \otimes_{\mathbb{C}} W/\mathfrak{g}.((M^3)' \otimes_{\mathbb{C}} W), \mathbb{C}\right) \cong \mathrm{Hom}_{\mathfrak{g}}(W, \overline{M^3})$.

For each $n \in \mathbb{N}$, choose a basis $\{v_{(i,n)} : i = 1, \dots, N_n\}$ of $M^3(n)$, and a dual basis $\{v^{(i,n)} : i = 1, \dots, N_n\}$ of $M^3(n)^*$. Given $F \in \mathrm{Hom}_{\mathfrak{g}}(W, \overline{M^3})$, we define a linear map $\varphi_F : (M^3)' \otimes_{\mathbb{C}} W \to \mathbb{C}$ by $\langle \varphi_F \,|\, v_3' \otimes w\rangle := \langle v_3' \,|\, F(w)\rangle$. For any $X \in \mathfrak{g}$, by (6.19) we have

$$\langle \varphi_F \,|\, \rho_\infty(X)(v_3') \otimes w + v_3' \otimes \rho_0(X)(w)\rangle = -\langle v_3' \,|\, \rho_0(X)(F(w))\rangle + \langle v_3' \,|\, F(\rho_0(X)(w))\rangle = 0.$$

Hence φ_F reduces to an element in $\mathrm{Hom}_{\mathbb{C}}\left((M^3)'\otimes_{\mathbb{C}} W/\mathfrak{g}.((M^3)'\otimes_{\mathbb{C}} W),\mathbb{C}\right)$. On the other hand, given an element $\varphi\in\mathrm{Hom}_{\mathbb{C}}\left((M^3)'\otimes_{\mathbb{C}} W/\mathfrak{g}.((M^3)'\otimes_{\mathbb{C}} W),\mathbb{C}\right)$, define

$$F_\varphi : W\to\overline{M^3},\quad F_\varphi(w):=\sum_{n=0}^{\infty}\sum_{i=1}^{N_n}\langle\varphi\,|\,\overline{v^{(i,n)}\otimes w}\rangle v_{(i,n)},\quad w\in W.$$

To show F_φ is $\mathfrak{g}$-invariant, note that if we assume $\rho_0(X)(v_{(j,m)})=\sum_{k=0}^\infty\sum_s a_{(j,m),(s,k)}v_{(s,k)}$, then by (6.19), $\langle\rho_\infty(X)(v^{(i,n)})\,|\,v_{(j,m)}\rangle=-\langle v^{(i,n)}\,|\,\sum_{k=0}^\infty\sum_k a_{(j,m),(s,k)}v_{(s,k)}\rangle=-a_{(j,m),(i,n)}$. It follows that $\rho_\infty(X)(v^{(i,n)})=-\sum_{l=0}^\infty\sum_t a_{(t,l),(i,n)}v^{(t,l)}$. Thus we have

$$\begin{aligned}F_\varphi(X.w)&=\sum_{n=0}^{\infty}\sum_i\langle\varphi\,|\,\overline{v^{(i,n)}\otimes X.w}\rangle v_{(i,n)}=-\sum_{n=0}^{\infty}\sum_i\langle\varphi\,|\,\overline{\rho_\infty(X)(v^{(i,n)})\otimes w}\rangle v_{(i,n)}\\&=\sum_{n=0}^{\infty}\sum_i\sum_{l=0}^{\infty}\sum_t\langle\varphi\,|\,\overline{v^{(t,l)}\otimes w}\rangle a_{(t,l),(i,n)}v_{(i,n)}=\sum_{l=0}^{\infty}\sum_t\langle\varphi\,|\,\overline{v^{(t,l)}\otimes w}\rangle\rho_0(X)(v_{(t,l)})\\&=\rho_0(X)(F_\varphi(w)).\end{aligned}$$

Hence $F_\varphi\in\mathrm{Hom}_{\mathfrak{g}}(W,\overline{M^3})$. Finally, given $F\in\mathrm{Hom}_{\mathfrak{g}}(W,\overline{M^3})$ and $w\in W$, we have

$$F_{\varphi_F}(w)=\sum_{n=0}^{\infty}\sum_i\langle\varphi_F\,|\,\overline{v^{(i,n)}\otimes w}\rangle v_{(i,n)}=\sum_{n=0}^{\infty}\sum_i\langle v^{(i,n)}\,|\,F(w)\rangle v_{(i,n)}=F(w).$$

On the other hand, note that $v_3'=\sum_{n=0}^\infty\sum_i\langle v_3'\,|\,v_{(i,n)}\rangle v^{(i,n)}$ for any $v_3'\in(M^3)'$, we have

$$\langle\varphi_{F_\varphi}\,|\,\overline{v_3'\otimes w}\rangle=\langle v_3'\,|\,F_\varphi(w)\rangle=\sum_{n=0}^{\infty}\sum_i\langle\varphi\,|\,\overline{v^{(i,n)}\otimes w}\rangle\langle v_3'\,|\,v_{(i,n)}\rangle=\langle\varphi\,|\,\overline{v_3'\otimes w}\rangle.$$

This shows $\mathrm{Hom}_{\mathbb{C}}\left((M^3)'\otimes_{\mathbb{C}} W/\mathfrak{g}.((M^3)'\otimes_{\mathbb{C}} W),\mathbb{C}\right)\cong\mathrm{Hom}_{\mathfrak{g}}(W,\overline{M^3})$ and (6.18). □

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