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SPECTRAL GAP FOR PRODUCTS AND A STRONG NORMAL SUBGROUP THEOREM

URI BADER, TSACHIK GELANDER AND ARIE LEVIT

ABSTRACT. We establish a general spectral gap theorem for actions of products of groups which may replace Kazhdan's property (T) in various situations. As a main application, we prove that a confined subgroup of an irreducible lattice in a higher rank semisimple Lie group is of finite index. This significantly strengthens the classical normal subgroup theorem of Margulis and removes the property (T) assumption from the recent counterpart result of Fraczyk and Gelander. We further show that any confined discrete subgroup of a higher rank semisimple Lie group satisfying a certain irreducibility condition is an irreducible lattice. This implies a variant of the Stuck–Zimmer conjecture under a strong irreducibility assumption of the action.

Dedicated to Gregory Margulis with great admiration and affection.

1. Introduction

A subgroup Λ of a given discrete group Γ is called *confined* if there exists a finite subset $F \subset \Gamma \setminus \{e\}$ such that $\Lambda^\gamma \cap F \neq \emptyset$ holds true for every element $\gamma \in \Gamma$. One of our main results is the following:

Theorem 1.1. *Let G be a connected semisimple Lie group of real rank at least two and with trivial center. Let Γ be an irreducible lattice in G. Then any confined subgroup of Γ has finite index.*

Note that every non-trivial normal subgroup is confined. So Theorem 1.1 vastly extends the celebrated normal subgroup theorem of Margulis [Mar78, Mar79], which says in turn that every non-trivial normal subgroup of a higher-rank irreducible lattice has finite index. Loosely speaking, this theorem and its generalizations [SZ94, BM00, Bek07, Cre17, FG23] are consequences of the conflict between two incompatible analytic properties, namely "amenability" and "spectral gap". Hence these results are easier to obtain if the semisimple group G, or least one of its factors, has Kazhdan's property (T). The general case of the normal subgroup theorem where the group G does not have property (T) required special attention, which Margulis carried out in [Mar79]. In the classical context of lattices in higher rank semisimple Lie groups, this has been the state of the art until today; no improvement has been made in the absence of property (T). [1]

Here is another way to think about Theorem 1.1. Given a discrete group Γ we consider the space of its subgroups $\mathrm{Sub}(\Gamma)$, called its *Chabauty space*. Put a compact topology on $\mathrm{Sub}(\Gamma)$ by identifying it with a closed subset of the Cantor space $\{0,1\}^\Gamma$.

[1]The works [BM00, Sha00, BS06] do not rely on property (T). However, they improve the normal subgroup theorem in a different direction than the current paper, i.e. they extend the class of groups to which the theorem applies, while here we obtain a stronger result in the classical case of semisimple Lie groups.

The group Γ acts on its Chabauty space by homeomorphisms via conjugation. In this language, a subgroup $\Lambda \in \mathrm{Sub}(\Gamma)$ is *unconfined* (i.e. not confined) if and only if the trivial subgroup of Γ belongs to the closure of the Γ-orbit of the subgroup Λ in the Chabauty space.

Theorem 1.1 (Reformulation)**.** *Let Γ be an irreducible lattice in a connected, center-free semisimple Lie group of real rank at least two. Then any infinite index subgroup of Γ is unconfined, i.e. admits a sequence of conjugates converging to the trivial subgroup in the Chabauty topology.*

A *uniformly recurrent subgroup* (URS) of Γ is a closed minimal Γ-subsystem of $\mathrm{Sub}(\Gamma)$ [GW15]. Note that if $X \subset \mathrm{Sub}(\Gamma)$ is a non-trivial uniformly recurrent subgroup then any subgroup $\Lambda \in X$ is confined.

Corollary 1.2. *Let Γ be an irreducible lattice as in Theorem 1.1. Any non-trivial uniformly recurrent subgroup $X \subset \mathrm{Sub}(\Gamma)$ is of the form $X = \{\Lambda^\gamma : \gamma \in \Gamma\}$ where Λ is some finite-index subgroup of Γ.*

The above results hold true more generally for S-arithmetic subgroups of semisimple algebraic groups over local fields of zero characteristic. For a rigorous statement of the results in that generality see Theorem 8.4 below.

In the special case where G has a simple factor with Kazhdan's property (T), these results follow from the recent work of Fraczyk–Gelander [FG23] as well as the work of Bader–Boutonnet–Houdayer–Peterson [BBHP22]. The breakthrough of the current paper is that it applies to all higher rank groups regardless of Property (T).

The preceding discussion concerns subgroups which are a priori contained in a given lattice. We now turn to consider more general discrete subgroups. A suitable adaptation of the notion of confined subgroups is required here. A subgroup Λ of a second countable locally compact group G is called *confined* if the trivial subgroup is not in the Chabauty closure of the orbit Λ^G via conjugation. When G is a semisimple Lie group, a discrete subgroup $\Lambda \leq G$ is confined if and only if the corresponding locally symmetric space $\Lambda \backslash G/K$ has a uniform upper bound on its injectivity radius at all points (where K is a maximal compact subgroup of G). It is proven in [FG23] that confined discrete subgroups of simple center-free Lie groups of real rank at least two are lattices; see [FG23] for a more refined statements and for analogs in the semisimple setup where property (T) is assumed.

To state our next result we will need a sharpening of the confined condition. It is used to ensure that a discrete subgroup of a product does not degenerate into any proper factor when taking conjugates. The term *conjugate limit* in Definition 1.3 stands for any subgroup in the Chabauty orbit closure under conjugation of the given subgroup.

Definition 1.3. A subgroup Λ of a locally compact second countable group G is *strongly confined* if no conjugate limit of Λ is contained in a proper normal subgroup of G. It is *irreducibly confined* if it is strongly confined and the intersection $\Lambda \cap H$ is trivial for any proper normal subgroup $H \lhd G$.

Certainly, every strongly confined subgroup is confined, and every confined subgroup of a simple Lie group is strongly (and irreducibly) confined.

Theorem 1.4. *Let G be a connected semisimple Lie group of real rank at least two and with trivial center. A discrete subgroup Λ of G is irreducibly confined if and only if Λ is an irreducible lattice.*

We refer to Theorem 10.1 below for a stronger but somewhat more involved version of Theorem 1.4 (relying on an assumption weaker than irreducibly confined).

Moving beyond Lie groups we have fewer techniques at our disposal. However, it is possible to get around this by requiring a stronger irreducibility condition.

Theorem 1.5. *Let $G = G_1 \times G_2$ be a product of two second countable locally compact groups. Let $\Lambda \leq G$ be a discrete coamenable subgroup. Assume that G_2 has a compact abelianization and that there are no G_2-invariant vectors in $L^2_0(G/\Lambda)$. If every conjugate limit of Λ projects densely to the factor G_2 then Λ is a lattice in G.*

The novelty of the current work is to rely on the product structure of the group G as a replacement for property (T). The main breakthrough is the following general spectral gap result for actions of product groups, under certain assumptions on the stabilizer structure of the action.

Theorem 1.6 (Spectral gap for actions of products)**.** *Let G_1 and G_2 be a pair of locally compact second countable compactly generated groups such that G_2 has compact abelianization. Set $G = G_1 \times G_2$. Let X be a locally compact topological G-space endowed with a G-invariant (finite or infinite) measure m. Assume that*

- $L^2_0(X, m)^{G_2} = 0$, *and*
- *there is a G-invariant closed subset of* $\mathrm{Sub}(G)$ *containing* $\mathrm{Stab}_G(x)$ *for m-almost every point $x \in X$ such that every subgroup H in this subset satisfies $\overline{G_1 H} = G$.*

Then the unitary Koopman G-representation $L^2_0(X, m)$ has a spectral gap.

See Theorem 4.8 below for a more technically demanding but sharper statement. Additionally, in the context of products of semisimple real or p-adic Lie groups we obtain the stronger Theorem 6.10.

Recall that a probability measure preserving action of a center-free semisimple Lie group G is irreducible if every simple factor of G acts ergodically. The rigidity theorem of Stuck and Zimmer [SZ94] says that if G has higher rank and property (T) then every irreducible probability measure preserving action of G is either essentially free or essentially transitive. Hartman and Tamuz [HT16] showed that it is enough to suppose that one of the simple factors of G has property (T). The famous Stuck–Zimmer conjecture says that the rigidity theorem should apply to all higher rank semisimple Lie groups regardless of property (T).

Let us say that a discrete subgroup of a semisimple Lie group G is *irreducible* if it projects densely to every proper factor of G. An ergodic action of G with almost surely irreducible stabilizers is irreducible [FG23, §7]. Let us say that a discrete subgroup $\Lambda \leq G$ is *strongly irreducible* if every discrete conjugate limit of Λ is irreducible, and that a probability measure preserving action is strongly irreducible if almost every stabilizer is strongly irreducible.

Corollary 1.7 (A weak version of the Stuck–Zimmer conjecture)**.** *Let G be a connected center-free semisimple Lie group of real rank at least two. Then every strongly irreducible probability measure preserving action of G is essentially transitive.*

See Theorem 10.14 below for another weak version of the Stuck–Zimmer conjecture. It says that an irreducible invariant random subgroup of a connected center-free higher rank semisimple Lie group is supported on lattices if and only if it is almost surely irreducibly confined (in the sense of Definition 1.3).

Structure of the paper. In §2 we address various preliminaries such as unitary representations, spectral gap and asymptotically invariant vectors as well as the Chabauty topology. In §3 we make some remarks regarding the representation theory of direct product of groups with compact abeliazniations. The technical heart of this paper is §4 in which we prove Theorem 1.6, as well as its generalization, Theorem 4.8. In §5 we introduce our working notion of standard semisimple groups. In §6 we study geometric properties of discrete subgroups of standard semisimple groups and establish our spectral gap theorem for actions of products of semisimple groups. In §7 we introduce confined and strongly confined subgroups. In §8 and §10 we deal with confined subgroups of lattices and strongly confined subgroups of semisimple Lie groups, respectively. Theorem 1.1 and Corollary 1.2 are proven in §8 and Theorem 1.4 and Theorem 1.5 are proven in §10. Lastly, the standalone §9 takes up the notion of Margulis functions needed to ensure discreteness in a certain argument in §10.

2. Preliminaries

We set up some basic notions and terminology to be used throughout the entire paper.

The Chabauty space. Let G be a locally compact second countable group. The Haar measure on G will be denoted by m_G. Typically, the group G will have a compact abelianization, and in particular it will be unimodular.

A discrete subgroup of the group G will typically be denoted by Λ. The Haar measure on the quotient G/Λ will be denoted by $m_{G/\Lambda}$. If the measure $m_{G/\Lambda}$ is finite we say that Λ is a *lattice* in G. A lattice will typically be denoted by Γ. In case the quotient G/Γ is compact the lattice Γ is called *uniform*.

We denote by $\mathrm{Sub}(G)$ the space of all closed subgroups of the group G endowed with the Chabauty topology [Cha50]. Recall that the space $\mathrm{Sub}(G)$ is compact, and the group G acts on it by homeomorphisms via conjugation. A non-empty minimal closed G-invariant subset of $\mathrm{Sub}(G)$ is called an *uniformly recurrent subgroup (URS)* of G, see [GW15]. We will use the notation

$$\Lambda^G = \{\Lambda^g \ : \ g \in G\} \subset \mathrm{Sub}(G)$$

for the G-orbit under conjugation of a given subgroup $\Lambda \in \mathrm{Sub}(G)$.

Definition 2.1. A *conjugate limit* of a subgroup $\Lambda \le G$ is any subgroup $\Delta \in \overline{\Lambda^G}$.

We denote by $\mathrm{Prob}(\mathrm{Sub}(G))$ the space of all probability measures on $\mathrm{Sub}(G)$ endowed with the weak-$*$ topology. This makes $\mathrm{Prob}(\mathrm{Sub}(G))$ a compact convex space. It is regarded with the natural G-action. A G-fixed point in this space is called an *invariant random subgroup (IRS)* of G. We denote the space $\mathrm{Prob}(\mathrm{Sub}(G))^G$ of all invariant random subgroups by $\mathrm{IRS}(G)$. See [AGV14, ABB$^+$17, ABB$^+$20, Gel18a, GL18] or the surveys [Gel18c, Gel18b].

Say that Γ is a lattice in G and $\nu \in \mathrm{IRS}(\Gamma)$ is an invariant random subgroup of Γ. It is possible to *induce* ν to obtain an invariant random subgroup $\overline{\nu} \in \mathrm{IRS}(G)$. This is done as follows. View $\mathrm{Sub}(\Gamma)$ as a subset of $\mathrm{Sub}(G)$ and regard ν as a probability measure on the space $\mathrm{Sub}(G)$. Fix an arbitrary Borel fundamental domain $\mathcal{F} \subset G$ for the lattice Γ and normalize the Haar measure so that $m_G(\mathcal{F}) = 1$. Finally take $\overline{\nu} = \int_{\mathcal{F}} g_* \nu \, \mathrm{d}m_G(g)$.

Recall that a probability measure preserving Borel G-space is called *irreducible* if every non-trivial normal subgroup of G is acting ergodically. An invariant random subgroup ν is *irreducible* if the Borel G-space $(\mathrm{Sub}(G), \nu)$ is irreducible.

Let μ be probability measure on the group G. We will typically assume that μ is absolutely continuous with respect to the Haar measure and that its support generates the group G. A *μ-stationary random subgroup* of the group G is a probability measure $\nu \in \mathrm{Prob}(\mathrm{Sub}(G))$ satisfying $\mu * \nu = \nu$. For some recent works dealing with stationary random subgroups see [GLM22, FG23, GL23].

Algebras of functions. We denote by $C_c(G)$ the $\mathbb{C}$-algebra of all compactly supported continuous complex-valued functions on the group G with the algebra product given by convolution. We regard $C_c(G)$ as a normed algebra with respect to the topology of uniform convergence on compact subsets. We also endow $C_c(G)$ with the supremum norm $\|\cdot\|_\infty$. For a function f on the group G we write $\check{f} = f \circ \iota$, where $\iota : G \to G$ is the inversion. We set

$$\mathcal{A}(G) = \{f \in C_c(G) \,:\, f \geq 0,\ f = \check{f} \text{ and } m_G(f) = 1\}.$$

A function $f \in C_c(G)$ is *generating* if its support $\mathrm{supp}(f)$ generates the group G. Note that if the group G is connected then any non-zero function is generating.

Unitary representations. Vector spaces are taken over the complex numbers. In particular, the Banach algebra $L^1(G)$ and the Hilbert spaces $L^2(G/\Lambda)$ where Λ is some discrete subgroup of G are taken with complex coefficients. It is tacitly assumed that these spaces are taken with respect to the Haar measures m_G and $m_{G/\Lambda}$ correspondingly. The corresponding norms are denoted by $\|\cdot\|_1$ and $\|\cdot\|_2$.

Hilbert spaces will typically be denoted by V and assumed to be separable. An unindexed norm $\|\cdot\|$ is typically associated with a Hilbert space which should be clear from the context. We will denote by $\mathrm{B}(V)$ the algebra of bounded operators on the Hilbert space V and by $\|\cdot\|_{\mathrm{op}}$ the operator norm on $\mathrm{B}(V)$. We denote by $\mathrm{U}(V)$ the group of unitary operators in $\mathrm{B}(V)$ endowed with the strong operator topology. This is a Polish topological group.

By a *unitary representation* we mean a continuous homomorphism $G \to \mathrm{U}(V)$. By an obvious abuse of notation, given such a unitary representation, an element $g \in G$ and a vector $v \in V$, we denote by gv the image of v under the unitary operator associated with g. Such a unitary representation extends to a representation of the algebra of complex-valued measures of bounded total variation on the group G. In particular, probability measures on the group G act on the Hilbert space V via averaging operators. Symmetric probability measures give rise to self-adjoint operators of norm at most one.

Regarding elements of the Banach algebra $L^1(G)$ as densities of m_G-absolutely continuous measures on the group G, we get a representation $L^1(G) \to \mathrm{B}(V)$. In particular, an element $f \in \mathcal{A}(G)$ gives rise to a self-adjoint operator of norm at most 1, which we regard as a smooth averaging operator. By an abuse of notation, given any vector $v \in V$ we denote by fv the image of v under this operator.

Lemma 2.2. *Let X be a space endowed with a positive measure (either finite or infinite). The map*

$$\{f \in L^2(X) \,:\, \|f\|_2 = 1,\ f \geq 0\} \to \{f \in L^1(X) \,:\, \|f\|_1 = 1,\ f \geq 0\}, \quad f \mapsto f^2$$

is a uniform homeomorphism, namely this map and its inverse are uniformly continuous.

The domain and the range of the map $f \mapsto f^2$ are endowed with the $\|\cdot\|_2$-metric and the $\|\cdot\|_1$-metric, respectively.

Proof of Lemma 2.2. Consider any pair of functions $f, g \in L^2(X)$ with $\|f\|_2 = \|g\|_2 = 1$ and $f, g \geq 0$. Observe that the pair f, g satisfies

$$\|f^2 - g^2\|_1 = \langle |f+g|, |f-g| \rangle \leq \|f+g\|_2 \cdot \|f-g\|_2 \leq 2\|f-g\|_2.$$

Therefore the map $f \mapsto f^2$ is uniformly continuous. To show that its inverse is uniformly continuous, we use the inequality $|a-b|^2 \leq |a^2 - b^2|$ which is valid for any pair of real numbers $a, b \geq 0$. This implies that the pair of functions f, g satisfies

$$\|f - g\|_2 \leq \|f^2 - g^2\|_1^{1/2}.$$

This concludes the proof. □

Asymptotically invariant vectors. Let $G \to \mathrm{U}(V)$ be a unitary representation of the group G on the Hilbert space V.

Definition 2.3. A sequence of non-zero vectors $v_n \in V$ is called *asymptotically G-invariant* if for every compact subset $K \subset G$

$$\lim_n \sup_{k \in K} \|(1-k)v_n\| / \|v_n\| = 0.$$

Fix a generating function $\phi \in \mathcal{A}(G)$. It is well known that a sequence of non-zero vectors $v_n \in V$ is asymptotically G-invariant if and only if

$$\lim_n \|(1-\phi)v_n\| / \|v_n\| = 0.$$

In case such an asymptotically G-invariant sequence exists, we say that the representation V *almost has G-invariant vectors*. Otherwise, we say that it has a *spectral gap*. Indeed, spectral gap is equivalent to saying that $\|\phi\|_{\mathrm{op}} < 1$ or to the fact that 0 is not contained in the spectrum of the positive operator $1 - \phi$.

Definition 2.4 ([Mar91, Definition IV.3.5])**.** A subset $A \subset V \setminus \{0\}$ is said to be *G-uniform* if for every $\varepsilon > 0$ there exists an identity neighborhood $U \subset G$ such that

$$\sup_{v \in A} \sup_{g \in U} \|(1-g)v\| / \|v\| \leq \varepsilon.$$

A sequence of vectors $v_n \in V$ is said to be *G-uniform* if the set $\{v_n\}$ is.

Another way to think about this definition is to say that orbit maps of vectors from the subset A mapping into the Hilbert space V are uniformly equicontinuous. It is easy to see that every asymptotically G-invariant sequence in V is G-uniform.

Here are some elementary lemmas concerning the above notions.

Lemma 2.5. *Let $v_n, u_n \in V$ be two sequences of vectors with $\liminf_n \|v_n\| > 0$ and $\limsup_n \|u_n\| = 0$. If the sequence v_n is asymptotically G-invariant (respectively G-uniform) then the sequence $v_n + u_n$ has the same property.*

Proof. Fix a generating function $\varphi \in \mathcal{A}(G)$. Assume to begin with that the sequence v_n is asymptotically G-invariant. For all n sufficiently large so that $\|u_n\| \leq \frac{1}{2}\|v_n\|$ we have $\|v_n + u_n\| \geq \frac{1}{2}\|v_n\|$ and

$$\begin{aligned} \frac{\|(1-\varphi)(v_n+u_n)\|}{\|v_n+u_n\|} &\leq 2\frac{\|\varphi v_n - v_n\| + \|\varphi u_n\| + \|u_n\|}{\|v_n\|} \leq \\ &\leq 2\frac{\|(1-\varphi)v_n\| + 2\|u_n\|}{\|v_n\|}. \end{aligned} \tag{2.1}$$

We have used the fact that φ gives rise to a contracting operator so that $\|\varphi\|_{\text{op}} \leq 1$. Letting $n \to \infty$ shows that the sequence $v_n + u_n$ is indeed asymptotically G-invariant.

Next, assume that the sequence v_n is G-uniform. The verification of the fact that the sequence $v_n + u_n$ is also G-uniform is very similar to the above computation, up to considering the operator g for some sufficiently small element $g \in G$ instead of the averaging operator φ. □

Lemma 2.6. *Let $v_n \in V$ be a G-uniform sequence of non-zero vectors. If the sequence v_n is* not *asymptotically G-invariant then there is some element $g \in G$ such that*

$$\limsup_n \|(1-g)v_n\|/\|v_n\| > 0.$$

Proof. Assume that the sequence v_n is not asymptotically G-invariant. This means that there is some compact subset $K \subset G$ and some $\varepsilon > 0$ such that

$$\limsup_n \sup_{g \in K} \|(1-g)v_n\|/\|v_n\| > \varepsilon.$$

Since the sequence v_n is G-uniform we may find a symmetric identity neighborhood $U \subset G$ such that

$$\sup_n \sup_{g \in U} \|(1-g)v_n\|/\|v_n\| \leq \varepsilon/2.$$

Let $Ug_1, \ldots, Ug_N$ be a finite cover of the compact subset K for some choice of elements $g_1, \ldots, g_N \in G$. It follows that one of these elements g_i is as required. □

It is useful to note that both properties of being asymptotically G-invariant as well as that of being G-uniform are preserved under rescaling (i.e. $v_n \mapsto c_n v_n$ for some arbitrary scaling constants $c_n > 0$).

Lemma 2.7. *Let $D \subset G$ be any subset. For each vector $v \in V$ and for all $n \in \mathbb{N}$ we have*

$$\sup_{g \in D^n} \|(1-g)v\| \leq n \sup_{g \in D} \|(1-g)v\|.$$

Proof. Fix a vector $v \in V$. By the triangle inequality, any pair of elements $g, h \in G$ satisfies

$$\|(1-gh)v\| \leq \|(1-g)v\| + \|g(1-h)v\| = \|(1-g)v\| + \|(1-h)v\|.$$

The desired conclusion follows by induction on n. □

We will require the following lemma dealing with asymptotic invariance in the L^1-sense.

Lemma 2.8. *Let G be a locally compact group admitting a measure preserving action on a probability measure space (X, m). Assume that the representation $L^2_0(X)$ has spectral gap. If $v_i \in L^1(X)$ is an asymptotically G-invariant sequence of vectors with $\|v_i\|_1 = 1$ and $v_i \geq 0$ then it converges in $L^1(X)$ to the constant function* 1.

In the above statement, the notion of an asymptotically G-invariant sequence is understood in the L^1-sense.

Proof. The lemma is a direct consequence of Lemma 2.2. Indeed, the sequence $v_i^{\frac{1}{2}} \in L^2(X)$ is asymptotically invariant (in the L^2-sense). Therefore $\|v_i^{\frac{1}{2}} - 1\|_2 \to 0$. Another application of Lemma 2.2 gives $\|v_i - 1\|_1 \to 0$, as required. □

3. Unitary representations of product groups

In this section we establish the following special property of the representation theory of product groups.

Lemma 3.1. *Let $G = G_1 \times G_2$ where G_1 and G_2 are compactly generated locally compact groups. Let $G \to \mathrm{U}(V)$ be a unitary representation without spectral gap and with $V^{G_2} = 0$. If G_2 has compact abelianization then there exists a sequence of unit vectors in V which is G-uniform, asymptotically G_1-invariant and* not *asymptotically G_2-invariant.*

This lemma is a version of [Mar91, Lemma IV.3.7]. Margulis proves this result for groups which form a Gelfand pair with respect to a compact subgroup. He refers to this condition as *property (Q)*. Our version is more general, as we replace this assumption by compact abelianization. We will call a sequence of vectors with the peculiar properties provided by Lemma 3.1 a *discordant sequence*. The lemma is proved at the end of this section.

Remark 3.2. *The assumption that the representation $G \to \mathrm{U}(V)$ has no spectral gap implicitly implies that the Hilbert space V is non-zero. Therefore the additional assumption $V^{G_2} = 0$ forces the group G_2 to be non-trivial. We allow G_1 to be trivial.*

Compact abelianization and unitary representations. Let G a compactly generated locally compact group with compact abelianization. Let $K \subset G$ be a compact, symmetric and generating identity neighborhood. The following lemma shows that the compact abelianization property can be tracked on a compact subset.

Lemma 3.3. *There is a symmetric compact subset $Q \subset G$ and a constant $\delta > 0$ with the following property — for every continuous function $\phi : Q \to \mathbb{C}$ satisfying $\|\phi|_K\|_\infty = 1$ there is a pair of elements $k_1 \in K$ and $k_2 \in Q$ with*

$$|\phi(k_1) + \phi(k_2) - \phi(k_1 k_2)| \geq \delta.$$

Proof. We set inductively $K_1 = K$ and $K_n = K_1 \cdot K_{n-1}$ for all $n \in \mathbb{N}$. Assume by contradiction that there exists a sequence of continuous functions $\phi_n : K_n \to \mathbb{C}$ satisfying $\|\phi_n|_K\|_\infty = 1$ and such that

$$\sup_{(k_1,k_2)\in K\times K_{n-1}} |\phi_n(k_1) + \phi_n(k_2) - \phi_n(k_1 k_2)| \leq 1/n$$

for every $n \in \mathbb{N}$. Fix an index $i \in \mathbb{N}$. We claim that:

- The sequence $(\phi_n|_{K_i})_{n>i}$ is uniformly bounded on K_i. Indeed for each n, using $\|\phi_n|_K\|_\infty = 1$ and the triangle inequality we see that

$$\|\phi_n|_{K_i}\|_\infty \leq i + (i-1)/n < 2i.$$

- The sequence $(\phi_n|_{K_i})_{n>i}$ is equicontinuous on K_{i-1}. For each $m \in \mathbb{N}$ let $U_m \subset G$ be a sufficiently small identity neighborhood so that $U_m^m \subset K$. Provided that $n \geq m$, the triangle inequality implies that every element $h \in U_m$ satisfies

$$|m\phi_n(h) - \phi_n(h^m)| \leq (m-1)/n < 1.$$

  Thus using $\|\phi_n|_K\|_\infty = 1$ we get

$$|m\phi_n(h)| \leq |m\phi_n(h) - \phi_n(h^m)| + |\phi_n(h^m)| < 2$$

for every element $h \in U_m$. We conclude that $\|\phi_n|_{U_m}\|_\infty \leq 2/m$ for all $n \geq m$. Therefore every pair of elements $g \in K_i$ and $h \in U_m$ satisfies

$$|\phi_n(g) - \phi_n(hg)| < 1/n + |\phi_n(h)| \leq 1/n + 2/m \leq 3/m$$

provided that $n \geq m$. The desired equicontinuity follows.

We use the Arzela–Ascoli theorem to conclude that for each fixed $i \in \mathbb{N}$, the sequence $(\phi_n|_{K_i})_{n>i}$ has a uniformly convergent subsequence in $C(K_i)$. Hence, by a standard diagonal argument, the sequence ϕ_n admits a convergent subsequence with respect to the topology of uniform convergence on compact subsets of G. The triangle inequality shows that the limit of this subsequence is a continuous homomorphism from G to the additive group of $\mathbb{C}$. This homomorphism is non-trivial as $\|\phi_n|_K\|_\infty = 1$ for all n. This is a contradiction.

To finish the proof, we may take $Q = K_n$ and $\delta = \frac{1}{n}$ for some suitable index n where our assumption toward contradiction fails. □

The following proposition shows that in any unitary representation of a group with compact abelianization, every non-invariant vector can be transformed in a uniform fashion into another non-invariant vector by "differentiating", i.e applying an operator of the form $1 - k$ where the element k is taken from a compact subset.

Proposition 3.4. *There exists a constant $\alpha = \alpha(G) > 0$ with the following property. Let $G \to \mathrm{U}(V)$ be any unitary representation and $v \in V$ any vector. If $k_0 \in K$ is an element satisfying*

$$\|(1 - k_0)v\| = \sup_{k \in K} \|(1 - k)v\|$$

then

$$\sup_{k \in K} \|(1 - k)(1 - k_0)v\| \geq \alpha \|(1 - k_0)v\|.$$

Proof. Let $Q \subset G$ be the compact symmetric identity neighborhood and $\delta > 0$ be the constant provided by Lemma 3.3. Let $n \in \mathbb{N}$ be such that $Q \subset K^n$. Take $\alpha = \delta/n$. Fix any element $k_0 \in K$ such that

$$c = \|(1 - k_0)v\| = \sup_{k \in K} \|(1 - k)v\|.$$

We set

$$u = (1 - k_0)v \quad \text{and} \quad d = \sup_{k \in K} \|(1 - k)u\|.$$

Note that $\|u\| = c$. Our goal is to show that $d \geq \alpha c$. We will assume as we may that $c > 0$.

Consider the complex-valued continuous function

$$\phi \in C(G), \quad \phi(g) = \langle (1 - g)v, u \rangle / c^2 \quad \forall g \in G.$$

Notice that $\|\phi|_K\|_\infty = \phi(k_0) = 1$. At this point we use the compact abelianization assumption together with Lemma 3.3 to find a pair of elements $k_1 \in K$ and $k_2 \in Q$ satisfying

$$|\phi(k_1) + \phi(k_2) - \phi(k_1 k_2)| \geq \delta.$$

Denote $h_1 = k_1^{-1} \in K$. We get

$$\begin{aligned} n\alpha = \delta \leq |\phi(k_1) + \phi(k_2) - \phi(k_1 k_2)| &= |\langle (1 - k_1)(1 - k_2)v, u \rangle| / c^2 \\ &= |\langle (1 - k_2)v, (1 - h_1)u \rangle| / c^2 \leq \|(1 - k_2)v\| \cdot \|(1 - h_1)u\| / c^2. \end{aligned}$$

It follows from Lemma 2.7 that

$$\|(1-k_2)v\| \leq nc \quad \text{and} \quad \|(1-h_1)u\| \leq d.$$

Putting everything together gives $n\alpha \leq nd/c$. This inequality is equivalent to the desired conclusion. □

Constructing a discordant sequence of vectors. Let $f \in \mathcal{A}(G)$ be any function. Denote $K = \operatorname{supp}(f) = \overline{\{g \in G \,:\, f(g) \neq 0\}}$. The following lemma, which is quite technical, will be used in the proof of Lemma 3.1 below to allow for a smoothing procedure to be applied.

Lemma 3.5. *Consider the constants $\alpha_n = 1 - 2^{-n}$ for all $n \in \mathbb{N}$. Let $G \to \mathrm{U}(V)$ be any unitary representation. Denote $p_n = P([\alpha_n, \alpha_{n+1}])$ where P is the projection-valued measure associated to f regarded as a self-adjoint operator on V. Then every vector $v \in V$ satisfies for all $n \geq 4$ that*

$$\sup_{k \in K} \|(1-k)p_n v\| \leq 2 \sup_{k \in K} \|(1-k) f p_n v\|.$$

Proof. Let $v \in V$ be an arbitrary vector. We assume without loss of generality that $v \in p_n V$ and that $v \neq 0$. Set $u = fv$ and observe that $\|u\| \geq \alpha_n \|v\| > 0$. We normalize these vectors so that $\|u\| = 1$. Note that $u \in p_n V$ so that

$$\langle fu, u\rangle \leq \|fu\| \leq \alpha_{n+1} \quad \text{and} \quad \|(1-f)u\| \leq 1 - \alpha_n = 2^{-n}.$$

Denote

$$c = \sup_{k \in K} \|(1-k)u\|.$$

Every group element $k \in K$ satisfies

$$2\operatorname{Re}\langle (1-k)u, u\rangle = 2 - 2\operatorname{Re}\langle ku, u\rangle = \|(1-k)u\|^2 \leq c^2.$$

As $K = \operatorname{supp}(f)$, we may act with the positive averaging operator f and obtain

$$2\langle (1-f)u, u\rangle = 2\operatorname{Re}\langle (1-f)u, u\rangle \leq c^2.$$

Putting all of the above information together gives

$$\|(1-f)u\| \leq 2^{-n} = 2 \cdot (1 - \alpha_{n+1}) \leq 2 \cdot (\langle u, u\rangle - \langle fu, u\rangle) = 2 \cdot \langle (1-f)u, u\rangle \leq c^2.$$

Provided that $n \geq 4$ we certainly have $\|(1-f)u\| \leq 2^{-n} \leq 1/16$. These last two inequalities give

$$\|(1-f)u\|^2 \leq c^2/16 \quad \text{so that} \quad 4\|(1-f)u\| \leq c.$$

Consider the vector $w = (1-f)v \in p_n V$. We obtain

$$\|w\| \leq \alpha_n^{-1} \|fw\| \leq 2\|fw\| = 2\|f(1-f)v\| = 2\|(1-f)u\| \leq c/2.$$

This means that every element $k \in K$ satisfies

$$\|(1-k)v\| \leq \|(1-k)fv\| + \|(1-k)(1-f)v\| \leq c + 2\|w\| \leq 2c.$$

The above inequality is the desired conclusion. □

We are ready to construct a discordant sequence of vectors, that is, a sequence of vectors with the particular properties demanded in Lemma 3.1, thereby extending Lemma IV.3.7 of [Mar91] to products of groups with compact abelianization.

Proof of Lemma 3.1. Let $f_i \in \mathcal{A}(G_i)$ be a pair of continuous functions such that their supports $K_i = \operatorname{supp}(f_i)$ contain a neighborhood of the identity and generate the group G_i for $i \in \{1, 2\}$. We take $\alpha > 0$ to be the constant given in Proposition 3.4 with respect to the compact, symmetric and generating subset K_2 of the group G_2.

We regard f_1 and f_2 as a pair of commuting self-adjoint operators on the Hilbert space V. Let P and Q be the respective projection-valued measures. We consider the spectral projections

$$p_n = P([\alpha_n, 1]) \quad \text{and} \quad q_n = Q([\alpha_n, \alpha_{n+1}])$$

defined in terms of the constants $\alpha_n = 1 - 2^{-n}$ introduced in Lemma 3.5. The asymmetry is intentional. Note that all of these projections pairwise commute. In addition, the p_n's commute with G_2 and the q_n's commute with G_1. By the assumption that there are asymptotically G-invariant vectors and $V^{G_2} = 0$, we find two strictly increasing sequences $n_i, m_i \in \mathbb{N}$ with $m_1 \geq 4$ such that $p_{n_i} q_{m_i} \neq 0$.

We turn to constructing the desired sequence of vectors $u_i \in V$. For each $i \in \mathbb{N}$ take an arbitrary unit vector $v_i \in p_{n_i,m_i} V$ and some element $k_i \in K_2$ such that

$$\|(1 - k_i) f_2 v_i\| = \sup_{k \in K_2} \|(1 - k) f_2 v_i\|.$$

We set $u_i = (1 - k_i) f_2 v_i$. Note that $u_i \neq 0$ for otherwise the non-zero vector $f_2 v_i$ would have been G_2-invariant. By Proposition 3.4 and the choice of the constant α we get

$$\sup_{k \in K_2} \|(1 - k) u_i\| \geq \alpha \|u_i\|.$$

This means that the sequence u_i is not asymptotically G_2-invariant, as required. As $v_i \in p_{n_i} V$ and as f_1 commutes with both f_2 and k_i, we see that $u_i \in p_{n_i} V$. Hence the sequence u_i is asymptotically G_1-invariant. The fact that it is G_1-uniform follows. We are left to show that the sequence u_i is G_2-uniform, which we now proceed to do.

Fix $\varepsilon > 0$. We will show that there exists an identity neighborhood $U_2 \subset G_2$ such that for every element $g \in U_2$ and every i we have

$$\|(1 - g) u_i\| \leq \varepsilon \|u_i\|.$$

The uniform continuity of the function f_2 together with the compactness of the subset K_2 allow us to find an identity neighborhood $U_2 \subset K_2$ such that every pair of elements $g \in U_2$ and $k \in K_2$ satisfy

$$\|(1 - g)(1 - k) f_2\|_\infty \leq \varepsilon / 6.$$

Note that for every element $g \in U_2$ and all $i \in \mathbb{N}$ we have

$$\operatorname{supp}(1 - g)(1 - k_i) f_2 \subset K_2^3 = K_2 \cdot K_2 \cdot K_2.$$

For every element $x \in K_2^3$, the two Lemmas 2.7 and 3.5 respectively imply the first and second inequality in the following equation

$$\|(1 - x) v_i\| \leq 3 \sup_{k \in K_2} \|(1 - k) v_i\| \leq 6 \|u_i\|.$$

Since $m_G((1 - g)(1 - k_i) f_2) = 0$, we have for every element $g \in U_2$ and all $i \in \mathbb{N}$

$$(1 - g) u_i = \int_G (1 - g)(1 - k_i) f_2(x)(x - 1) v_i \, \mathrm{d}m_G(x).$$

Hence every element $g \in U_2$ satisfies

$$\|(1-g)u_i\| \leq \|(1-g)(1-k_i)f_2\|_\infty \cdot \sup_{x \in K_2^3} \|(x-1)v_i\| \leq \frac{\varepsilon}{6} \cdot 6\|u_i\| = \varepsilon\|u_i\|$$

for all $i \in \mathbb{N}$. This means that the sequence u_i is G_2-uniform, as required. □

We mention one additional lemma of Margulis, showing that an averaging operator can be applied to the "discordant sequence" constructed in Lemma 3.1 without losing its particular properties.

Lemma 3.6. *Let $G = G_1 \times G_2$ where G_1 and G_2 are second countable locally compact groups. Let $G \to \mathrm{U}(V)$ be a unitary representation. Let $v_n \in \mathcal{H}$ be a G-uniform, asymptotically G_1-invariant and* not *asymptotically G_2-invariant sequence of vectors.*

Then there is an open identity neighborhood $U \subset G$ such that for every function $\psi \in \mathcal{A}(G)$ with $\mathrm{supp}(\psi) \subset U$ the sequence ψv_n has the same three properties and satisfies $\|\psi v_n\| > \|v_n\|/2$ for all $n \in \mathbb{N}$.

Proof. By [Mar91, Lemma IV.3.6] we may find an identity neighborhood $U_1 \subset G$ such that for every function $\psi \in \mathcal{A}(G)$ with $\mathrm{supp}(\psi) \subset U_1$, the sequence ψv_n has the same above-mentioned three properties as the sequence v_n. As the sequence v_n is G-uniform, there is another identity neighborhood $U_2 \subset G$ such that for every function $\psi \in \mathcal{A}(G)$ with $\mathrm{supp}(\psi) \subset U_2$ we have $\|\psi v_n\| > \|v_n\|/2$ for all $n \in \mathbb{N}$. The desired conclusion follows by taking $U = U_1 \cap U_2$. □

4. Koopman representations over actions with stabilizers

Let G be a second countable locally compact group acting continuously on a locally compact topological space X. Let m be a G-invariant measure on the space X, either finite or infinite. We consider the corresponding Koopman representation $L^2(X)$, which is taken implicitly with respect to the measure m. The stabilizer map is the Borel measurable map given by

$$\mathrm{Stab} : X \to \mathrm{Sub}(G), \quad \mathrm{Stab} : x \mapsto \mathrm{Stab}_G(x) \quad \forall x \in X.$$

The following is the main result of this section. Roughly speaking, it says that under the right conditions, asymptotic invariance carries from one factor to the other. See Example 4.7 below regarding the necessity of some of its assumptions.

Proposition 4.1. *Assume that $G = G_1 \times G_2$. Fix a function $\varphi \in \mathcal{A}(G)$. Let $u_n \in L^2(X)$ be a G-uniform and asymptotically G_1-invariant sequence of unit vectors of the form $u_n = \varphi v_n$ for some $v_n \in L^2(X)$ with $\|v_n\| \leq 2$. If the sequence of probability measures $\mathrm{Stab}_*(|u_n|^2 \cdot m)$ converges to $\mu \in \mathrm{Prob}(\mathrm{Sub}(G))$ and μ-almost every subgroup has dense projections to the factor G_2 then the sequence u_n is asymptotically G_2-invariant.*

The proof of Proposition 4.1 will be given after a bit of preliminary work. It will be very useful to introduce some notations first.

Notation 4.2. For every set $W \subset G$ we define the subset $\Omega(W) \subset X$ given by

$$\begin{aligned}\Omega(W) &= \{x \in X \,:\, \mathrm{Stab}(x) \cap W \neq \emptyset\} = \\ &= \{x \in X \,:\, \exists g \in W \text{ such that } gx = x\}.\end{aligned}$$

Here are some elementary properties of the operation Ω, whose proof is immediate. For every subset $W \subset G$, family of subsets $W_i \subset G$ and element $g \in G$ we have

- $\Omega(W) = \Omega(W^{-1})$,
- $\Omega(\bigcup W_i) = \bigcup \Omega(W_i)$,
- $\Omega(\bigcap W_i) \subset \bigcap_i \Omega(W_i)$,
- $g\Omega(W) = \Omega(gWg^{-1})$, and
- if $e \in W$ then $\Omega(W) = X$.

Notation 4.3. For each measurable subset $Y \subset X$ and $f \in L^2(X)$ we denote

$$\|f\|_Y^2 = \int_Y |f(x)|^2 \, \mathrm{d}m(x).$$

Lemma 4.4. *For every $\varphi \in C_c(G)$ and for every $\varepsilon > 0$ there exist an identity neighborhood $U \subset G$ and a function $\psi \in L^1(G)$ satisfying $\psi \geq 0$ and $\|\psi\|_1 = \varepsilon$ with the following property — every vector $f \in L^2(X)$ and element $g \in G$ satisfy*

$$|(1-g)\varphi f(x)| \leq \psi \, |f|(x).$$

at m-almost every point $x \in \Omega(Ug^{-1})$, and in particular, for every $Y \subset \Omega(Ug^{-1})$, we have

$$\|(1-g)\varphi f\|_Y \leq \|\, \psi |f| \,\|_Y.$$

Proof. Fix a function $\varphi \in C_c(G)$ and a constant $\varepsilon > 0$. Fix an arbitrary relatively compact identity neighborhood $U_0 \subset G$ and set $W = U_0 \cdot \mathrm{supp}(\varphi)$. By the uniform continuity of the function φ, it is possible to fix an identity neighborhood U contained in U_0 such that for every pair of elements $h, h' \in G$ with $h'h^{-1} \in U$,

$$|\varphi(h') - \varphi(h)| < \varepsilon / m_G(W).$$

Set $\psi = \frac{\varepsilon}{m_G(W)} \cdot \chi_W$ and note that $\|\psi\|_1 = \varepsilon$. For every $u \in U$, since the function $(1-u)\varphi$ vanishes outside $U \cdot \mathrm{supp}(\varphi) \subset W$, we get

$$|(1-u)\varphi| \leq \psi.$$

To conclude the proof, consider some vector $f \in L^2(X)$ and some group element $g \in G$. For every point $x \in \Omega(Ug^{-1})$ there exists an element $u \in U$ such that $g^{-1}x = u^{-1}x$. Therefore

$$|(1-g)\varphi f(x)| = |(1-u)\varphi f(x)| \leq |(1-u)\varphi| \, |f|(x) \leq \psi \, |f|(x).$$

The last conclusion follows by integration over Y. □

The following lemma is essentially an immediate consequence of the Portmanteau theorem on weak-$*$ convergence of probability measures.

Lemma 4.5. *Assume that $G = G_1 \times G_2$. Let $u_n \in L^2(X)$ be any sequence of unit vectors. Assume that the sequence of probability measures $\mathrm{Stab}_*(|u_n|^2 \cdot m)$ converges to $\mu \in \mathrm{Prob}(\mathrm{Sub}(G))$ and μ-almost every subgroup has dense projections to G_2. Then for every open subset $U \subset G_2$ and every $\varepsilon > 0$ there is an open relatively compact subset $V \subset G_1$ such that the subset $V \times U$ satisfies*

$$\|u_n\|^2_{\Omega(V \times U)} \geq 1 - \varepsilon$$

for all n sufficiently large.

Proof. Let $U \subset G_2$ be any open subset and let $\varepsilon > 0$. Since μ-almost every subgroup projects densely to G_2, there is some open relatively compact subset $V \subset G_1$ such that the Chabauty open subset

$$\widetilde{\Omega}(V \times U) = \{\Gamma \leq G \, : \, \Gamma \cap (V \times U) \neq \emptyset\} \subset \mathrm{Sub}(G)$$

satisfies

$$\mu(\widetilde{\Omega}(V \times U)) \geq 1 - \frac{\varepsilon}{2}.$$

By the Portmanteau theorem

$$\mathrm{Stab}_*(|u_n|^2 \cdot m)(\widetilde{\Omega}(V \times U)) \geq 1 - \varepsilon$$

for all n sufficiently large. This is equivalent to the desired conclusion. □

We are ready to complete the main technical result of the current section §4.

Proof of Proposition 4.1. Assume towards contradiction that the sequence u_n is not asymptotically G_2-invariant. Using Lemma 2.6 we fix an element $g_2 \in G_2$ such that

$$c = \limsup_n \|(1 - g_2)u_n\| > 0.$$

We will arrive at a contradiction by showing that for every n large enough

$$\|(1 - g_2)u_n\|^2 \leq \frac{c}{2}. \tag{4.1}$$

We apply Lemma 4.4 with respect to the constant $\varepsilon = \sqrt{c/32}$ to get an identity neighborhood $U = U_1 \times U_2 \subset G_1 \times G_2$ and a function $\psi \in L^1(G)$ such that

$$\|\psi\|_1 = \sqrt{c/32} \tag{4.2}$$

and such that for every function $f \in L^2(X)$, element $g \in G$ and measurable subset $Y \subset \Omega((U_1 \times U_2) \cdot g^{-1})$,

$$\|(1 - g)\varphi f\|_Y \leq \|\, \psi \, |f| \, \|_Y. \tag{4.3}$$

Set $V_2 = g_2 U_2 \cap U_2 g_2$. This is an open neighborhood of g_2 in G_2. By Lemma 4.5, applied to the sequence u_n and the open set $V_2 \subset G_2$, there exists an open relatively compact subset $V_1 \subset G_1$ such that the subset $\Omega(V_1 \times V_2)$ satisfies

$$\|u_n\|^2_{\Omega(V_1 \times V_2)} \geq 1 - \frac{c}{16}$$

for all n sufficiently large. Note that both subsets $\Omega(V_1 \times g_2 U_2)$ and $\Omega(V_1 \times U_2 g_2)$ contain the subset $\Omega(V_1 \times V_2)$. Thus we have

$$\|u_n\|^2_{\Omega(V_1 \times g_2 U_2)} \geq 1 - \frac{c}{16}.$$

Using the left-invariance of the Haar measure and the equation

$$g_2^{-1}\Omega(V_1 \times g_2 U_2) = \Omega(V_1 \times U_2 g_2)$$

we also have

$$\|g_2 u_n\|^2_{\Omega(V_1 \times g_2 U_2)} = \|u_n\|^2_{\Omega(V_1 \times U_2 g_2)} \geq 1 - \frac{c}{16}.$$

Denote by F the complement of $\Omega(V_1 \times g_2 U_2)$ in X. We conclude that $\|u_n\|^2_F \leq c/16$ as well as $\|g u_n\|^2_F \leq c/16$ for all n sufficiently large. Therefore

$$\|(1 - g_2)u_n\|^2_F = \|u_n - g_2 u_n\|^2_F \leq (\|u_n\|_F + \|g_2 u_n\|_F)^2 \leq (2\sqrt{\frac{c}{16}})^2 = \frac{c}{4}$$

for all n sufficiently large. Note that

$$\begin{aligned}\|(1 - g_2)u_n\|^2 &= \|(1 - g_2)u_n\|^2_{\Omega(V_1 \times g_2 U_2)} + \|(1 - g_2)u_n\|^2_F \\ &\leq \|(1 - g_2)u_n\|^2_{\Omega(V_1 \times g_2 U_2)} + \frac{c}{4}.\end{aligned}$$

Towards establishing Equation (4.1), we are therefore left to deal with the remaining summand and show that

$$\|(1-g_2)u_n\|^2_{\Omega(V_1\times g_2U_2)} \le \frac{c}{4} \tag{4.4}$$

for all n large enough. This is what we proceed to do.

We use the relative compactness of V_1 to find a finite collection of elements $h_1,\dots,h_m \in G_1$ for some $m\in\mathbb{N}$ such that $V_1 \subset \bigcup_{i=1}^m U_1h_i$. Using the inclusion

$$\Omega(V_1\times g_2U_2) \subset \Omega(\bigcup_{i=1}^m U_1h_i \times g_2U_2) = \bigcup_{i=1}^m \Omega(U_1h_i\times g_2U_2),$$

we fix an arbitrary measurable partition

$$\Omega(V_1\times g_2U_2) = \coprod_{i=1}^m A_i, \quad A_i \subset \Omega(U_1h_i\times g_2U_2).$$

For every $i\in\{1,\dots,m\}$ there is the inclusion

$$g_2^{-1}A_i \subset g_2^{-1}\Omega(U_1h_i\times g_2U_2) = \Omega(U_1h_i\times U_2g_2) = \Omega((U_1\times U_2)\cdot(h_ig_2)).$$

We apply Equation (4.3) for $f=v_n$, $g=(h_ig_2)^{-1}$ and $Y=g_2^{-1}A_i$. This gives

$$\begin{aligned}\|(h_i^{-1}-g_2)u_n\|_{A_i} &= \|(g_2^{-1}h_i^{-1}-1)u_n\|_{g_2^{-1}A_i} = \|((h_ig_2)^{-1}-1)\varphi v_n\|_{g_2^{-1}A_i}\\ &\le \|\,\psi\,|v_n|\,\|_{g_2^{-1}A_i}.\end{aligned}$$

Combining this with Equation (4.2) and the assumption that $\|v_n\|\le 2$ gives

$$\begin{aligned}\sum_{i=1}^m \|(h_i^{-1}-g_2)u_n\|^2_{A_i} &\le \sum_{i=1}^m \|\,\psi\,|v_n|\,\|^2_{g_2^{-1}A_i} \le \|\,\psi\,|v_n|\,\|^2\\ &\le \|\psi\|_1^2\cdot\|v_n\|^2 \le \frac{c}{32}\cdot 4 = \frac{c}{8}.\end{aligned} \tag{4.5}$$

Making use of the elementary Lemma 4.6 for each $i\in\{1,\dots,m\}$ with respect to the vectors $s=(1-h_i^{-1})u_n|_{A_i}$ and $t=(h_i^{-1}-g_2)u_n|_{A_i}$ we get the estimates

$$\begin{aligned}\|(1-g_2)u_n\|^2_{A_i} &= \|(1-h_i^{-1})u_n + (h_i^{-1}-g_2)u_n\|^2_{A_i}\\ &\le 6\cdot\|(1-h_i^{-1})u_n\|_{A_i} + \|(h_i^{-1}-g_2)u_n\|^2_{A_i}.\end{aligned} \tag{4.6}$$

Finally, we use the asymptotic G_1-invariance of the sequence u_n to deduce that for every n large enough

$$\|(1-h_i^{-1})u_n\|_{A_i} \le \|(1-h_i^{-1})u_n\| \le \frac{c}{48m}$$

holds true for every $i\in\{1,\dots,m\}$. Combining Equations (4.6) and (4.5) we deduce that for sufficiently large n we have

$$\begin{aligned}\|(1-g_2)u_n\|^2_{\Omega(V_1\times g_2U_2)} &= \sum_{i=1}^m \|(1-g_2)u_n\|^2_{A_i}\\ &\le \sum_{i=1}^m \left(6\cdot\|(1-h_i^{-1})u_n\|_{A_i} + \|(h_i^{-1}-g_2)u_n\|^2_{A_i}\right)\\ &\le \sum_{i=1}^m 6\cdot\frac{c}{48m} + \sum_{i=1}^m \|(h_i^{-1}-g_2)u_n\|^2_{A_i} \le \frac{c}{8}+\frac{c}{8} = \frac{c}{4}.\end{aligned}$$

This gives Equation (4.4). The proof of the proposition is complete. □

Lemma 4.6. *Let V be a Hilbert space. If $s, t \in V$ is a pair of vectors with $\|s\|, \|t\| \leq 2$ then $\|s+t\|^2 \leq 6\|s\| + \|t\|^2$.*

Proof. Indeed,

$$\|s+t\|^2 \leq (\|s\| + \|t\|)^2 = (\|s\| + 2\|t\|) \cdot \|s\| + \|t\|^2 \leq 6\|s\| + \|t\|^2.$$

□

On the necessity of the assumptions. Proposition 4.1 has several technical assumptions. Ignoring these, it is tempting think of it loosely as the statement "an asymptotically G_1-invariant sequence that converges (in the sense of measures) to one with dense projections to G_2 is also asymptotically G_2-invariant". However, this would be an oversimplification, as the following example shows.

Example 4.7. Take $G_1 = \mathbb{Z}$. Let $G_2 = \widehat{\mathbb{Z}}$ be the profinite completion of G_1 and set $G = G_1 \times G_2$. Consider the group $\Gamma \cong \mathbb{Z}$ diagonally embedded in G. Thus Γ is a uniform lattice in G admitting dense projections to G_2. Let m be the Haar probability measure on G/Γ. Consider the Hilbert space $V = L^2_0(G/\Gamma) \cong L^2_0(\widehat{\mathbb{Z}})$. As the group $\mathbb{Z}$ has no property (τ) it acts on its profinite completion without spectral gap [LZ05]. Hence there exists an asymptotically G_1-invariant sequence of unit vectors $u_n \in V$. As the group G is abelian, we have $\operatorname{Stab}_*(|u_n|^2 \cdot m) = \delta_\Gamma$ for all n. However, the sequence u_n is not asymptotically G_2-invariant, as the group G_2 is compact, and as such has property (T).

The group $\mathbb{Z}$ in Example 4.7 can be replaced with any other group not having property (τ).

Spectral gap for actions of products. By putting together our results on unitary representations of product groups, we obtain a quite general spectral gap theorem. Recall that G is a second countable locally compact group. Suppose in addition that G is compactly generated.

Theorem 4.8 (Spectral gap for actions of products)**.** *Assume that $G = G_1 \times G_2$ and that G_2 has a compact abelianization. Let X be a locally compact topological G-space endowed with a G-invariant measure m, either finite or infinite. Assume that*

- *$L^2_0(X,m)^{G_2} = 0$, and*
- *For any asymptotically G_1-invariant sequence of unit vectors $f_n \in L^2(X,m)$, every accumulation point $\mu \in \operatorname{Prob}(\operatorname{Sub}(G))$ of the sequence of probability measures $\operatorname{Stab}_*(|f_n|^2 \cdot m)$ satisfies $\overline{G_1 H} = G$ for μ-almost every subgroup $H \in \operatorname{Sub}(G)$.*

Then the Koopman G-representation $L^2_0(X,m)$ has spectral gap.

In a nutshell, Theorem 4.8 follows from the tension between Lemma 3.1 and Proposition 4.1. But of course, there are details to provide.

Proof of Theorem 4.8. Assume towards contradiction that the Koopman representation $L^2_0(X,m)$ has no spectral gap. This representation has no G_2-invariant vectors by assumption. According to Lemma 3.1 we may find a "discordant" sequence of non-zero vectors $f_i \in L^2_0(X,m)$, namely, a sequence with the following three properties:

(1) The sequence is G-uniform.

(2) The sequence is asymptotically G_1-invariant.
(3) The sequence is not asymptotically G_2-invariant.

We fix a smoothing operator $\psi \in \mathcal{A}(G)$ as provided by Lemma 3.6 with respect to the sequence f_i. Consider the sequence $g_i = \psi f_i$ which we may suppose (by scaling f_i) to be unit vectors. It continues to satisfy the above properties (1)-(3). Moreover it has the following two properties:

(4) The sequence g_i consists of unit vectors.
(5) For each i we have $g_i = \psi f_i$ and $\|f_i\|_2 \leq 2$.

Consider the sequence of probability measures $\mu_i = \mathrm{Stab}_*(|g_i|^2 \cdot m)$ on the Chabauty space $\mathrm{Sub}(G)$. Up to passing to a subsequence, we may maintain all of the properties (1)-(5) and further assume that the probability measures μ_i converge in the weak-$*$ topology to some probability measure $\mu \in \mathrm{Prob}(\mathrm{Sub}(G))$ satisfying:

(6) The probability measure μ is G_1-invariant.
(7) μ-almost every subgroup has dense projections to G_2.

Finally, relying on the above properties, we are in a position to apply Proposition 4.1 and deduce that the sequence g_i must be asymptotically G_2-invariant. This is a contradiction. □

Proof of Theorem 1.6 of the introduction. The statement in the introduction is a special case of Theorem 4.8. □

Remark 4.9. *Strictly speaking, a representation on the zero Hilbert space has no asymptotically invariant vectors, and as such has spectral gap. For this reason, Theorem 4.8 is non-void only provided that the Hilbert space $L^2_0(X, m)$ is non-zero. In which case, it follows implicitly from the assumptions that both groups G_1 and G_2 are not trivial (so that G is a direct product in a non-trivial manner).*

5. Standard semisimple groups and irreducible lattices

In this section we set up our terminology, conventions and notations regarding semisimple groups and their lattices. For brevity, we will use the non-standard notion of "standard semisimple groups", which we now introduce.

A *standard simple group* is a topological group G of the form $\mathbf{G}(k)^+$, where k is a local field of zero characteristic and $\mathbf{G}$ is an isotropic adjoint connected absolutely simple k-algebraic group. The topology on G is the one induced from its k-analytic structure. Such groups are discussed in [Mar91, Chapter I, §1.5, §1.8 and §2.3]. In particular, the group G is compactly generated, non-compact, and simple as an abstract group. Moreover, the local field k as well as the algebraic group $\mathbf{G}$ are canonically associated to G, in the sense that they can be recovered from its topological group structure.

Remark 5.1. *The requirement that $\mathbf{G}$ is absolutely simple can be relaxed to simple, via a restriction of scalars. In particular, letting $k = \mathbb{R}$ or $k = \mathbb{Q}_p$ for some prime number p, and letting $\mathbf{G}$ be an isotropic adjoint connected simple k-algebraic group, we have that $\mathbf{G}(k)^+$ is a standard simple group. Every simple real Lie group with trivial center is a standard simple group* [Zim13, 3.1.6]. *However, we emphasise that upon allowing this relaxation, we lose the possibility to recover k and $\mathbf{G}$ from G, and also we will need to make several unpleasant adjustments to the discussion of arithmeticity and spectral gap below. So we keep with absolutely simple.*

The closure of the subfield $\mathbb{Q}$ in k is a local field isomorphic to $\mathbb{Q}_p$ for some prime number p or for $p = \infty$ (we use the convention $\mathbb{Q}_\infty = \mathbb{R}$). We will say that G is a *standard simple group of type* p. We denote $\mathrm{rank}(G) = \mathrm{rank}_k(\mathbf{G})$. This is a positive integer, by the assumption saying that $\mathbf{G}$ is isotropic.

A *standard semisimple group* is a topological group of the form $\prod_{i=1}^n G_i$ for some $n \in \mathbb{N}$, where each G_i is a standard simple group. The groups G_i are said to be the *simple factors* of G. We denote

$$\mathrm{rank}(G) = \sum_{i=1}^n \mathrm{rank}(G_i).$$

If $\mathrm{rank}(G) = 1$ then G it is said to be of *rank one*. Otherwise G is said to be of *higher rank*. For each prime number p and for $p = \infty$, we define $G^{(p)}$ to be the subgroup of G consisting of the product of all simple factors of type p. This is the *p-component* of G. It is said to be a *standard semisimple group of type* p. We view $G^{(p)}$ as a subgroup of a $\mathbb{Q}_p$-algebraic group, by restricting the scalars of $\mathbf{G}_i$ for each simple factor G_i of $G^{(p)}$ and taking the corresponding direct product. We restrict to $G^{(p)}$ the corresponding Zariski topology and denote it the *p-Zariski topology* of $G^{(p)}$. In this way, we may also endow $G^{(p)}$ with a $\mathbb{Q}_p$-analytic structure, arising from it being a closed subgroup of the group of $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-algebraic group.

Definition 5.2. A subgroup $H \leq G$ is *fully Zariski dense* if the projection of H to each p-component $G^{(p)}$ is p-Zariski dense, for each prime number p and for $p = \infty$.

Lattices and invariant random subgroups. Recall that any lattice in a standard semisimple group G is fully Zariski dense by the Borel density theorem. This classical fact has been extended to invariant random subgroups, first for real Lie groups in [ABB+17], and then for standard semisimple groups over a single local field in [GL18]. Relying on those results and ideas, we provide a statement applicable to any standard semisimple group.

Proposition 5.3. *Let G be a standard semisimple group and ν an ergodic invariant random subgroup of G. Then there are two semisimple factors (i.e. two normal subgroups) M and N of G with $N \cap M = \{e\}$ such that ν-almost every subgroup*

1. *projects densely to each p-component $N^{(p)}$,*
2. *projects discretely and fully Zariski densely to M, and*
3. *is contained in $N \times M$ (i.e. projects trivially to $G/(N \times M)$).*

Proof. Consider the p-component $G^{(p)}$ of the standard semisimple group G for each prime number p as well as for $p = \infty$. Let $\nu^{(p)}$ the the invariant random subgroup of the p-component $G^{(p)}$ obtained by considering the natural projection $\pi^{(p)} : G \to G^{(p)}$ and pushing forward ν via the map $H \mapsto \overline{\pi^{(p)}(H)}$. We may apply the Borel density theorem for invariant random subgroups of standard semisimple groups over a single local field [GL18, Theorem 1.9] to find a pair of normal subgroups $N^{(p)}, M^{(p)} \leq G^{(p)}$ with $N^{(p)} \cap M^{(p)} = \{e\}$ and such that $\nu^{(p)}$-almost every subgroup contains $N^{(p)}$, projects discretely and p-Zariski densely to $M^{(p)}$ and is contained in $N^{(p)} \times M^{(p)}$. The two desired normal subgroups are constructed by taking $N = \prod_p N^{(p)}$ and $M = \prod_p M^{(p)}$ where p runs over $p = \infty$ and all prime numbers involved in G. $\square$

It will be useful for us to know the following property of lattices in standard semisimple groups.

Lemma 5.4. *Let G be a standard semisimple group and Γ a lattice in G. Then the conjugacy class of every non-trivial element $\gamma \in \Gamma$ is infinite.*

Proof. Let $\gamma \in \Gamma$ be some non-trivial element and assume towards contradiction that its centralizer $\Delta = C_\Gamma(\gamma)$ has a finite index in Γ. It follows that Δ is a lattice in G. Therefore the projection of Δ to each p-component $G^{(p)}$ is p-Zariski-dense by the Borel density theorem. It follows that the projection of the element γ to each p-component $G^{(p)}$ must be central. We arrive at a contradiction to the fact that the standard semisimple group G is center-free. □

Lastly, we briefly recall the notion of irreducible lattices.

Definition 5.5. A lattice Γ in a standard semisimple group G is said to be *irreducible* if the projection of Γ to G/H has a dense image for each simple factor H of G.

Indeed, a lattice Γ in a standard semisimple group G is irreducible if and only if the Borel G-space G/Γ with the normalized probability measure is irreducible in the sense of §2. If G is a standard simple group, then all lattices in G are irreducible, as the condition in the above Definition 5.5 is satisfied trivially. The classical notion of an irreducible lattice corresponds with the notion of an irreducible subgroup defined in the introduction.

Arithmetic groups. We now describe the construction of the standard semisimple group associated with a given algebraic group defined over a number field.

Example 5.6. Fix a number field K and an adjoint connected absolutely simple K-group $\mathbf{H}$. Recall that a *place* $s : K \to k_s$ is a dense embedding of the field K in a local field k_s, defined up to a natural equivalence. Corresponding to s we get the k_s-group $\mathbf{H}_s$ obtained by an extension of scalars. This is an adjoint connected absolutely simple k_s-algebraic group. We say that the place s is isotropic if the group $\mathbf{H}_s$ is k_s-isotropic, and in this case we denote $H_s = \mathbf{H}_s(k_s)^+$. This is the standard simple group associated with $\mathbf{H}$ at the isotropic place s. Fixing a finite set S of isotropic places, we obtain the standard semisimple group associated with $\mathbf{H}$ at S, namely $H_S = \prod_{s\in S} H_s$. Note that the group $\mathbf{H}(K)$ embeds diagonally in $\prod_{s\in S} \mathbf{H}_s(k_s)$. Accordingly, as H_s has finite index in $\mathbf{H}_s(k_s)$ for each $s \in S$, an appropriate finite-index subgroup of $\mathbf{H}(K)$ is embedded in the finite-index subgroup H_S of $\prod_{s\in S} \mathbf{H}_s(k_s)$. This embedding is dense by the strong approximation theorem.

Recall that a pair of subgroups Δ_1 and Δ_2 of a standard semisimple group G are called *commensurable* if their intersection $\Delta_1 \cap \Delta_2$ has finite index in both Δ_1 and Δ_2. Commensurability is an equivalence relation. Being a lattice is obviously a commensurability invariant. Likewise, being an irreducible lattice in a standard semisimple group is also a commensurability invariant.

The groups discussed in Example 5.6 admit a canonical commensurability class of irreducible lattices, described in the following paragraph.

Example 5.7. Fix a number field K and an adjoint connected absolutely simple K-group $\mathbf{H}$. A place $s : K \to k_s$ is called Archimedean (or non-Archimedean) if the local field k_s is. Let $\mathcal{O} < K$ be the ring of integers. For every non-Archimedean place s denote by $\pi_s \lhd \mathcal{O}$ the corresponding prime ideal, that is, the preimage of unique maximal ideal in the closure of $s(\mathcal{O})$. Fix a finite set of isotropic places S which includes all Archimedean isotropic places. Let $\mathcal{O}_S$ be the localization of $\mathcal{O}$ by the ideals π_s ranging over all the non-Archimedean places $s \in S$. Consider the standard semisimple group H_S discussed in Example 5.6. Upon choosing a non-trivial K-representation $\rho : \mathbf{H} \to \mathrm{GL}_n$, we set $\Lambda_S = \rho^{-1}(\mathrm{GL}_n(\mathcal{O}_S))$. Note that the commensurability class of Λ_S does not depend on the choice of ρ. We consider the finite-index subgroup of $\mathbf{H}(K)$ which embeds densely in H_S, as discussed at the end of Example 5.6, and let Λ_S^+ be its intersection with Λ_S. This is a finite-index subgroup of Λ_S which embeds in H_S. We let Γ_S be the corresponding image of Λ_S^+ is H_S. Then Γ_S is an irreducible lattice in the standard semisimple group H_S. Its commensurability class is independent on the chosen representation ρ.

Definition 5.8. An irreducible lattice Γ in a standard semisimple group G is said to be *arithmetic* if there exist a number field K, an adjoint connected absolutely simple K-group $\mathbf{H}$ and a finite set of isotropic places S which includes all Archimedean isotropic places, such that G is isomorphic as a topological group to the group H_S discussed in Example 5.6 and the image of Γ under this isomorphism can be conjugated to a lattice in the commensurability class of the lattice Γ_S described in Example 5.7.

The following is a fundamental theorem due to Margulis.

Theorem 5.9 (Margulis Arithmeticity, [Mar91, Chapter IX, Theorem 1.11])**.** *Let G be a standard semisimple group of higher rank and $\Gamma < G$ be an irreducible lattice. Then Γ is arithmetic.*

Spectral gap. We end this section by stating an important corollary of Clozel's theorem on spectral gap, namely [Clo03, Theorem 3.1].

Theorem 5.10. *Let G be a standard semisimple group and $\Gamma < G$ be an irreducible lattice. Then for each simple factor F of G, the unitary F-representation $L^2_0(G/\Gamma)$ has a spectral gap.*

Proof. If the standard semisimple group G is simple then it certainly has no proper simple factors. Therefore the desired conclusion follows from [Bek98, Lemma 3] for Lie groups, [BL11, Theorem 1] for standard simple groups over non-Archimedean local fields, [Mar91, Chapter III, Corollary 1.10] for uniform lattices and [GLM22, Theorem 1.8] in general.

From now on, assume that the standard semisimple group G is not simple, and fix a simple factor F. In particular G is of higher rank. By Margulis Arithmeticity (Theorem 5.9) the lattice Γ is arithmetic. We adopt below the notation introduced in Example 5.7. In particular, there is a number field K, an adjoint connected absolutely simple K-group $\mathbf{H}$ and a finite set S of isotropic places containing all the Archimedean isotropic ones. We view Γ as a lattice in the standard semisimple group H_S. A conjugate of Γ is commensurable to the lattice Γ_S. The simple factor F corresponds to the factor H_s for some place $s \in S$, as discussed in Example 5.6. Up to replacing Γ by its conjugate, we will assume that Γ is actually commensurable to Γ_S. This does not change the unitary H_s-representation $L^2_0(H_S/\Gamma)$ (up to an isomorphism). By [KM99, Lemma 3.1], we know that the H_s-representation $L^2_0(H_S/\Gamma)$ has spectral gap if and only if the H_s-representation $L^2_0(H_S/\Gamma_S)$ does. It remains for us to prove the latter statement.

Let $\tilde{\mathbf{H}}$ be the simply connected covering of $\mathbf{H}$ and $\pi : \tilde{\mathbf{H}} \to \mathbf{H}$ be the associated K-central isogeny. By [Clo03, Theorem 3.1] the trivial representation is isolated in the automorphic dual of $\tilde{H}_s = \tilde{\mathbf{H}}_s(k_s)$. This means that the unitary $\tilde{H}_s$-representation $L^2_0(\tilde{\mathbf{H}}(\mathbb{A}_K)/\tilde{\mathbf{H}}(K))$ has a spectral gap, where $\mathbb{A}_K$ is the K-adele ring. Write $\tilde{\mathbf{H}}(\mathbb{A}_K) = \tilde{H}_S \times \tilde{H}_{S^c}$, where $\tilde{H}_S$ is the product of groups corresponding to the places in S and $\tilde{H}_{S^c}$ the restricted product of groups corresponding to the complement set of S. Since S contains all isotropic Archimedean factors, we can find a compact open subgroup $C < \tilde{H}_{S^c}$. Then $\tilde{H}_S \times C$ is an open subgroup of $\tilde{\mathbf{H}}(\mathbb{A}_K)$ and $\Lambda = \tilde{\mathbf{H}}(K) \cap (\tilde{H}_S \times C)$ is a lattice in $\tilde{H}_S \times C$. We identify $(\tilde{H}_S \times C)/\Lambda$ with an $\tilde{H}_s$-invariant open subset of $\tilde{\mathbf{H}}(\mathbb{A}_K)/\tilde{\mathbf{H}}(K)$, and view $L^2_0((\tilde{H}_S \times C)/\Lambda)$ as an $\tilde{H}_s$-subrepresentation of $L^2_0(\tilde{\mathbf{H}}(\mathbb{A}_K)/\tilde{\mathbf{H}}(K))$. We conclude that $L^2_0((\tilde{H}_S \times C)/\Lambda)$ has a spectral gap regarded as a $\tilde{H}_s$-representation, and so does the $\tilde{H}_s$-subrepresentation of C-invariants, denoted $L^2_0((\tilde{H}_S \times C)/\Lambda)^C$. We identify this representation with $L^2_0(\tilde{H}_S/\Lambda_1)$, where Λ_1 denotes the image of Λ under the proper projection map $\tilde{H}_S \times C \to \tilde{H}_S$.

Next, by [Mar91, Chapter I, Proposition 1.5.5 and Theorem 2.3.1] the central isogeny π gives a surjective map $\tilde{H}_S \to H_S$ with a finite central kernel, which we denote by $Z = \ker \pi$. Let $\Lambda_2 < H_S$ be the image of Λ_1 under this map and identify the unitary representation $L^2_0(H_S/\Lambda_2)$ with the space of Z-invariants in $L^2_0(\tilde{H}_S/\Lambda_1)$. We conclude that the H_s-representation $L^2_0(H_S/\Lambda_2)$ has a spectral gap. Since, by construction, Λ_2 is commensurable with Γ_S, and using [KM99, Lemma 3.1] once more, we conclude that the H_s-representation $L^2_0(H_S/\Gamma_S)$ has a spectral gap. This finishes the proof. □

6. Discrete subgroups of standard semisimple groups

Throughout this section G will be a standard semisimple group, a notion introduced and discussed in detail in §5. Terminology and notation from §5 will be used freely in the current §6.

Deformations and Zariski density. Recall that $\mathbb{Q}_p$ stands either for the field of p-adic numbers when p is a prime number or for the field $\mathbb{R}$ when p is ∞. We use Ad to denote the adjoint representation.

Lemma 6.1. *Let G be a standard semisimple group of type p, where p is either ∞ or a prime number. A closed subgroup $H \le G$ is p-Zariski-dense if and only if the projection of H to each simple factor of G is infinite and*

$$\mathrm{span}_{\mathbb{Q}_p} \mathrm{Ad}(H) = \mathrm{span}_{\mathbb{Q}_p} \mathrm{Ad}(G).$$

Proof. To ease our notation denote $k = \mathbb{Q}_p$. Let $\mathfrak{g}$ denote the semisimple k-Lie algebra of G regarded as a k-analytic group. Set $A = \mathrm{span}_k \mathrm{Ad}(G)$, which is an associative k-subalgebra of $\mathrm{End}_k(\mathfrak{g})$. Consider a closed subgroup $H \le G$ such that $\mathrm{Ad}(H)$ spans the k-algebra A and such that H has an infinite projection to each simple factor of G. Let $\mathfrak{h}$ denote the Lie algebra of the p-Zariski closure of H. Certainly $\mathfrak{h}$ is an $\mathrm{Ad}(H)$-invariant subalgebra of $\mathfrak{g}$. The projection of the subalgebra $\mathfrak{h}$ to each simple subalgebra of $\mathfrak{g}$ is non-zero. Hence $\mathfrak{h} = \mathfrak{g}$ and so the p-Zariski closure of the subgroup H is open (in the Hausdorff topology arising from the k-analytic structure). Therefore H is p-Zariski dense by [PR93, Lemma 3.2], as required. The converse direction of the lemma is immediate. □

Lemma 6.2. *Let G be a standard semisimple group of type p, where p is either ∞ or a prime number. The subset of* $\mathrm{Sub}(G)$ *consisting of p-Zariski-dense subgroups is open in the Chabauty topology.*

Proof. Denote $k = \mathbb{Q}_p$ and set $A = \mathrm{span}_k \mathrm{Ad}(G)$ as in the proof of Lemma 6.1. Let $H \leq G$ be a closed p-Zariski-dense subgroup. We know by Lemma 6.1 that $\mathrm{Ad}(H)$ spans A over k. Thus we may find a finite set of elements $h_1, \ldots, h_m \in H$ for some $m \in \mathbb{N}$ such that $\mathrm{span}_k\{\mathrm{Ad}(h_i)\} = A$. It follows that there is some Chabauty neighborhood of H such that any group H' in that neighborhood has $\mathrm{span}_k \mathrm{Ad}(H') = A$.

In order to deduce that every such subgroup H' is p-Zariski-dense, it is enough by Lemma 6.1 to show that the projection of H' to any simple factor of G is infinite. In doing so, we may pass to smaller Chabauty neighborhood of the subgroup H, and assume that H' lies in that neighborhood.

In the Archimedean case one may argue as follows. By the Jordan–Schur theorem [Jor78], there is some $n \in \mathbb{N}$ such that every finite subgroup of G admits a normal abelian subgroup of index at most n (for a more recent treatment see e.g. [Rag72, Theorem 8.29]). Since H is real-Zariski-dense, there are elements $\alpha, \beta \in H$ such that the projection of $[\alpha^{n!}, \beta^{n!}]$ to every simple factor of G is non-trivial. This property is preserved in a small Chabauty neighborhood of H, and guarantees that the projection to each factor is infinite.

In the non-Archimedean case the above claim is straightforward, since there is an upper bound on the order of finite subgroups [Ser09, Theorem 1 on p. 124]. □

Using the above Lemma 6.2 combined with the definition of the Chabauty topology we obtain the following.

Corollary 6.3. *Let G be a standard semisimple group. The subset of* $\mathrm{Sub}(G)$ *consisting of fully Zariski dense subgroups (in the sense of Definition 5.2) is open in the Chabauty topology.*

Subgroups of product groups. We assume for the remainder of §6 that the group G is of the form $G = G_1 \times G_2$ where G_1 and G_2 are both non-trivial standard semisimple subgroups. Let $\pi_i : G \to G_i$ for $i \in \{1, 2\}$ denote the natural projections.

Lemma 6.4. *Let Δ be a discrete subgroup of G. If $\Lambda \in \overline{\Delta^G}$ is a discrete conjugate limit whose intersection with G_2 is fully Zariski dense then $\Lambda \cap G_1$ is a subgroup of some conjugate limit of $\Delta \cap G_1$.*

The terminology *conjugate limit* was introduced in Definition 2.1.

Proof of Lemma 6.4. Let Λ be a discrete conjugate limit of Δ, say $\Lambda = \lim_j \Delta^{g_j}$ in the Chabauty topology for some sequence of elements $g_j \in G$. Since Λ is discrete there is some identity neighborhood $U \subset G$ such that $\Delta^{g_j} \cap U = \{e\}$ for all $j \in \mathbb{N}$.

Assume that the subgroup $\Lambda_2 = \Lambda \cap G_2$ is fully Zariski dense in G_2. Corollary 6.3 allows us to find a finite subset $S \subset \Lambda_2$ such that any sufficiently small deformation of the set S inside G_2 generates a subgroup which is still fully Zariski dense in G_2. For all sufficiently large j we find finite subsets $R_j \subset \Delta^{g_j}$ such that the projection to G_2 of the group generated by R_j is fully Zariski-dense and R_j converge to S in the obvious sense.

Consider any particular element $h \in \Lambda \cap G_1$. There are elements $h_j \in \Delta^{g_j}$ such that the sequence h_j converges to the element h. Note that for all j sufficiently large we have $[R_j, h_j] \subset U \cap \Delta^{g_j} = \{e\}$. This implies that $h_j \in C_G(\langle R_j \rangle)$. Since the projection of the subgroup $\langle R_j \rangle$ to G_2 is fully Zariski-dense, it follows that $h_j \in (\Delta^{g_j} \cap G_1) = (\Delta \cap G_1)^{g_j}$ for all j sufficiently large. This shows that the intersection $\Lambda \cap G_1$ is contained in any conjugate limit of the intersection $\Delta \cap G_1$ obtained as an accumulation point of the sequence $(\Delta \cap G_1)^{g_j}$, as required. $\square$

Corollary 6.5. *Let Δ be a discrete subgroup of G. Let $\Lambda \in \overline{\Delta^G}$ be a discrete conjugate limit of Δ whose intersection with G_2 is fully Zariski dense.*

(1) If $\Delta \cap G_1$ is trivial then $\Lambda \cap G_1$ is trivial.
(2) If $\Delta \cap G_1$ is not fully Zariski dense in G_1 then $\Lambda \cap G_1$ is not fully Zariski dense in G_1.

Proof. Both parts of the statement follow from the previous Lemma 6.4. Indeed, part (1) follows immediately. To deduce part (2) we need to further rely on the fact that being fully Zariski dense is a Chabauty open condition; see Corollary 6.3. $\square$

Random subgroups invariant for a single factor. Recall that G is a standard semisimple group which is assumed to be a direct product $G = G_1 \times G_2$ of two standard semisimple factors with projections $\pi_i : G \to G_i$.

Lemma 6.6. *Let $\Delta_1 \leq G_1$ be a closed, not relatively compact and fully Zariski dense subgroup. Let ν be a Δ_1-invariant probability measure on* $\mathrm{Sub}(G)$ *such that ν-almost every subgroup intersects G_1 trivially and projects to G_2 discretely. Then ν-almost every subgroup is contained in G_2.*

This is a variant of [FG23, Lemma 3.14]. The proof is very short and we reproduce it here.

Proof of Lemma 6.6. We may suppose without loss generality that the measure ν is Δ_1-ergodic. Recall that π_i denotes the projection from the group G to each semisimple factor G_i. Assume towards contradiction that not ν-almost every subgroup is contained in G_2. Then one can find an open subset $O_2 \subset G_2 \setminus \{e_2\}$ with respect to which

$$\nu(\{H \leq G \,:\, |\pi_1(H \cap \pi_2^{-1}(O_2)) \setminus \{e_1\}| = 1\}) > 0.$$

This set is ν-conull by Δ_1-ergodicity. The push forward of ν via the map taking a subgroup H to the unique point in $\pi_1(H \cap \pi_2^{-1}(O_2))$ gives a probability measure on $G_1 \setminus \{e_1\}$ which is invariant under conjugation by Δ_1. By projecting to one of the simple factors of the group G_1, we may assume without loss of generality that G_1 is simple. The existence of such a probability measure stands in contradiction to [BDL17, Proposition 1.9] (see also the main result of [Sha99]). $\square$

In the context of semisimple Lie groups, another way to arrive at a contradiction would be to borrow the argument from [Fur01, p. 38]. Yet another such way is the closely related to [FG23, Corollary 3.11] dealing with stationary random subgroups.

Remark 6.7. The assumption that a Zariski-dense subgroup is not relatively compact is only relevant in the non-Archimedean case. Indeed, in the real case, any Zariski-dense subgroup of an isotropic simple algebraic group is automatically not relatively compact.

Proposition 6.8. *Let G be a standard semisimple group of the form $G = G_1 \times G_2$ where G_1 is standard semisimple and G_2 is standard* simple *of type p_0 for $p_0 = \infty$ or for some prime number p_0. Let ν be a G_1-invariant Borel probability measure on* $\mathrm{Sub}(G)$. *If ν-almost every subgroup Γ satisfies that*

(1) Γ is discrete,

(2) Γ *is not contained in the factor* G_2*,*
(3) Γ *has* p_0*-Zariski-dense and not relatively compact projection to* G_2*, and*
(4) if $\Gamma \cap G_2$ *is* p_0*-Zariski-dense then* $\Gamma \cap G_1$ *is trivial*

then ν*-almost every subgroup* Γ *has a dense projection to* G_2*.*

Proof. By passing to ergodic components we may assume without loss of generality that the measure ν is G_1-ergodic. The G_1-invariant map $\mathrm{Sub}(G) \to \mathrm{Sub}(G_2)$ given by $\Gamma \mapsto \overline{\mathrm{pr}_2(\Gamma)}$ is therefore ν-essentially constant. We denote its essential image by $\Delta_2 \in \mathrm{Sub}(G_2)$. Note that Δ_2 is closed, non-compact and p_0-Zariski-dense subgroup of G_2. Since the group G_2 is simple, it must either be the case that $\Delta_2 = G_2$ or that Δ_2 is discrete; see e.g. [GM13, §3]. In the first case we are done. In what follows we assume towards contradiction that the subgroup Δ_2 is discrete.

We apply the Borel density theorem for invariant random subgroups (Proposition 5.3) with respect to the pushforward of ν via the map $\mathrm{Sub}(G) \to \mathrm{Sub}(G_1)$ given by $\Gamma \mapsto \overline{\mathrm{pr}_1(\Gamma)}$. This provides a pair of normal subgroups $N, M \leq G_1$ such that $N \cap M = \{e\}$ and such that ν-almost every subgroup Γ projects densely to each p-component $N^{(p)}$, projects discretely and fully Zariski-densely to M and is contained in $N \times M$. From this point onward, we assume as we may that $G_1 = M \times N$.

The measure ν is Δ_2-invariant by [FG23, Lemma 7.2]. We fix a generic Δ_2-ergodic component ν_0 of the measure ν. By Δ_2-ergodicty, there is some fixed non-trivial closed subgroup $\Delta_1 \leq G_1$ such that $\overline{\mathrm{pr}_1(\Gamma)} = \Delta_1$ for ν_0-almost every subgroup Γ. As ν_0 is generic, we have that Δ_1 is fully Zariski-dense in G_1 and not relatively compact (for it is infinite and discrete). The measure ν_0 is Δ_1-invariant by [FG23, Lemma 7.2] and Δ_1-ergodic by [FG23, Corollary 7.3]. Thus, ν_0 can be regarded as a discrete Δ_1-ergodic and Δ_2-ergodic invariant random subgroup of the product group $\Delta_1 \times \Delta_2$.

We now claim that the factor N must be trivial, so that $G_1 = M$ and the subgroup Δ_1 is discrete. We consider the map $\mathrm{Sub}(G) \to \mathrm{Sub}(N)$ given by $\Gamma \mapsto \Gamma \cap N$. By Δ_2-ergodicity this map is ν_0-essentially constant. Its essential image is a certain discrete subgroup of N normalized by Δ_1, as the measure ν_0 is Δ_1-invariant. Since Δ_1 projects densely to each p-component $N^{(p)}$, this discrete subgroup must be trivial. Consider the product decomposition $G = N \times (M \times G_2)$. We have that ν_0-almost every subgroup intersects N trivially and projects to $M \times G_2$ discretely. Since ν_0 is Δ_1-invariant, we may apply Lemma 6.6 with respect to this product decomposition, and deduce that ν_0-almost every subgroup is contained in $M \times G_2$. This proves the claim.

We consider the map $\mathrm{Sub}(G) \to \mathrm{Sub}(G_1)$ given by $\Gamma \mapsto \Gamma \cap G_1$. By Δ_2-ergodicity this map is ν_0-essentially constant and we denote its essential image by Λ_1. Note that $\Lambda_1 \lhd \Delta_1$ as the measure ν_0 is Δ_1-invariant. By Lemma 6.6 applied with respect to the product decomposition $G = G_1 \times G_2 = M \times G_2$ we have that Λ_1 is non-trivial.

Now we invert the roles of the two factors G_1 and G_2 and consider the map $\mathrm{Sub}(G) \to \mathrm{Sub}(G_2)$ given by $\Gamma \mapsto \Gamma \cap G_2$. This map ν_0-essentially constant and its essential image is some non-trivial normal subgroup $\Lambda_2 \lhd \Delta_2$. The subgroup Λ_2 is p_0-Zariski dense in G_2, being normal in the p_0-Zariski dense subgroup Δ_2. This contradicts assumption (4). □

Corollary 6.9. *Let* ν *be a non-trivial discrete ergodic invariant random subgroup of* G*. If* ν*-almost every subgroup intersects trivially each proper semisimple factor* *of G then ν is irreducible. Furthermore, any ergodic probability measure preserving G-space $(X, \overline{\nu})$ satisfying $(\mathrm{Stab}_G)_* \overline{\nu} = \nu$ is irreducible as well.*

Proof. We claim that ν-almost every subgroup projects densely to any *proper* semisimple factor H of the standard semisimple group G. The claim will be established by induction on the number of types (i.e. either ∞ or prime numbers) involved in the factor H.

For the base of the induction, assume that H is a proper semisimple factor of a single type p (where p is either ∞ or some prime number). Write $G = H \times L$ for some non-trivial factor L. Let ν_H denote the pushforward of ν via the map $\mathrm{Sub}(G) \to \mathrm{Sub}(H)$ given by $\Lambda \mapsto \overline{\mathrm{pr}_H(\Lambda)}$ so that ν_H is an invariant random subgroup of the group H. By Proposition 5.3 applied to ν_H there is a pair of normal subgroups $N, M \leq H$ with $N \cap M = \{e\}$ such that ν_H-almost every subgroup projects densely to N, discretely and p-Zariski-densely to M and is contained in $N \times M$. Lemma 6.6 applied with respect to the direct product decomposition $G = (N \times L) \times M$ implies that the subgroup M is trivial so that $N = H$. This means that $\nu_H = \delta_H$. The claim in the base of the induction is established.

For the induction step, assume that H is a proper semisimple factor involving more than a single type, and that the claim has already been established with respect to factors with fewer types. Write $G = H \times L$ for some non-trivial factor L. Let ν_H denote the pushforward of ν via the map $\mathrm{Sub}(G) \to \mathrm{Sub}(H)$ given by $\Lambda \mapsto \overline{\mathrm{pr}_H(\Lambda)}$ so that ν_H is an invariant random subgroup of the group H. Let p be any *finite prime* such that $H^{(p)}$ is non-trivial, and write $H = H^{(p)} \times R$ where $R = \prod_{q \neq p} H^{(q)}$ is the complement to $H^{(p)}$ in H. As the local field $\mathbb{Q}_p$ is non-Archimedean the standard semisimple group $H^{(p)}$ admits some compact open subgroup O. Consider the map

$$\mathrm{Sub}(G) \to \mathrm{Sub}(L \times R), \quad \Gamma \mapsto \mathrm{pr}_{L \times R}(\Gamma \cap \mathrm{pr}^{-1}_{H^{(p)}}(O)) \quad \forall \Gamma \in \mathrm{Sub}(G).$$

Let λ be the resulting pushforward invariant random subgroup of the standard semisimple group $L \times R$. By the induction hypothesis (applied to the subgroup R of the group $L \times R$) we know that λ-almost every subgroup projects densely to the proper factor R. It follows that ν_H-almost every subgroup contains the factor R. So, we may conclude by applying the base case of the induction to the invariant random subgroup of the factor $L \times H^{(p)}$ obtained by intersecting ν-almost every subgroup with that factor.

We know that ν projects densely to each proper semisimple factor of G. Hence ν is irreducible by [FG23, Corollary 7.3]. The additional clause in the statement concerning the irreducibility of $\overline{\nu}$ follows from the same [FG23, Corollary 7.3]. □

Spectral gap. We specialize our spectral gap result (Theorem 4.8) for actions of standard semisimple groups.

Theorem 6.10 (Spectral gap for actions of products — semisimple group case)**.** *Let G be a standard semisimple group with $G = G_1 \times G_2$ where G_1 is standard semisimple and G_2 is standard* simple *of type p_0. Let X be a locally compact topological G-space endowed with a G-invariant measure m, either finite or infinite. Assume that*

- $L^2_0(X, m)^{G_2} = 0$,
- *m-almost every point x has $\mathrm{Stab}(x) \cap G_1 = \mathrm{Stab}(x) \cap G_2 = \{e\}$ and*

- *if $f_n \in L^2(X,m)$ is an asymptotically G_1-invariant sequence of unit vectors then every accumulation point $\nu \in \mathrm{Prob}(\mathrm{Sub}(G))$ of the sequence of probability measures $\mathrm{Stab}_*(|f_n|^2 \cdot m)$ is such that ν-almost every subgroup is discrete, not contained in the factor G_2 and admits p_0-Zariski-dense and not relatively compact projections to G_2.*

Then the unitary Koopman G-representation $L^2_0(X,m)$ has spectral gap.

Proof. The idea is to deduce the spectral gap for the Koopman representation from our general spectral gap theorem for products, namely Theorem 4.8. With that goal in mind, let us verify the two assumptions of that theorem. The first assumption that $L^2_0(X,m)^{G_2} = 0$ is maintained in the current statement as well. As for the second assumption, consider some asymptotically G_1-invariant sequence of unit vectors $f_n \in L^2_0(X,m)$. Denote $\nu_n = \mathrm{Stab}_*(|f_n|^2 \cdot m) \in \mathrm{Prob}(\mathrm{Sub}(G))$ and let $\nu \in \mathrm{Prob}(\mathrm{Sub}(G))$ be any weak-$*$ accumulation point of the sequence of probability measures ν_n. We are required to show that ν-almost every subgroup has dense projections to the factor G_2. Note that the probability measure ν is G_1-invariant. By our assumptions, ν-almost every subgroup Γ satisfies

(1) Γ is discrete,
(2) Γ is not contained in the factor G_2, and
(3) Γ has p_0-Zariski-dense and not relatively compact projections to G_2.

Moreover, by applying part (1) of Corollary 6.5 for the subgroup $\Delta = \mathrm{Stab}_G(x)$ with respect to a m-generic point $x \in X$ we get that

(4) if $\Gamma \cap G_2$ is p_0-Zariski-dense then $\Gamma \cap G_1$ is trivial.

Applying Proposition 6.8 we deduce that ν-almost every subgroup has dense projections to the factor G_2. This concludes the reduction of the current proof to the statement of Theorem 4.8. □

Remark 6.11. *If the action of a standard semisimple group $G = G_1 \times G_2$ on a probability measure space (X,m) is faithful, irreducible and measure preserving then the stabilizer of m-almost every point has trivial intersection with both factors, so that this assumption in Theorem 6.10 becomes redundant.*

Remark 6.12. *Provided m is not supported on a singleton, the assumptions of Theorem 6.10 imply that both semisimple factors G_1 and G_2 are not trivial.*

7. Confined and strongly confined subgroups

In this section we study the notion of confined subgroups, which is a crucial ingredient in this work.

We consider this notion for discrete groups first. Recall that a subgroup Λ of a discrete group Γ is called *confined* if there is a finite subset $F \subset \Gamma \setminus \{e\}$ such that the condition $\Lambda^\gamma \cap F \neq \emptyset$ holds true for every element $\gamma \in \Gamma$. Here is an equivalent definition for the negation of this notion. A subgroup Λ of a discrete group Γ is called *unconfined* (i.e. not confined) if the trivial subgroup of Γ is a conjugate limit of Λ. Thus, being unconfined and non-trivial is a vast strengthening of being non-normal. Indeed, non-trivial normal subgroups are precisely the fixed points for the conjugation action of Γ on the space $\mathrm{Sub}(\Gamma) \setminus \{\{e\}\}$, while unconfined subgroups are those with unbounded (namely, non-relatively compact) orbits.

We shall require a generalization of this definition to the context of locally compact groups.

Definition 7.1. A closed subgroup H of a locally compact second countable group G is called *confined* if the trivial subgroup $\{e\} \leq G$ is not a conjugate limit of H.

Equivalently, a subgroup H is confined in G if and only if there is a compact set $C \subset G \setminus \{e\}$ which intersects non-trivially every conjugate of H. It follows immediately from the definition that any conjugate limit of a confined subgroup is confined.

Example 7.2. Any non-trivial normal closed subgroup is confined. Any closed subgroup containing a confined subgroup is confined. In particular, a closed subgroup containing a non-trivial normal closed subgroup is confined.

In the general context of locally compact groups, it is natural to consider the following variant of the definition.

Definition 7.3. Let G be a locally compact second countable group. A closed subgroup H of G is *weakly confined* if there is a compact subset $C \subset G$ satisfying

$$(C \cap H^g) \setminus \{e\} \neq \emptyset$$

for every element $g \in G$.

Confined subgroups are obviously weakly confined. Every non-discrete subgroup of G is weakly confined. In the p-adic Lie group $\mathrm{SL}_2(\mathbb{Q}_p)$, the compact upper-triangular unipotent subgroup with coefficients in $\mathbb{Z}_p$ gives an example of a weakly confined subgroup which is unconfined. However, for groups with NSS (no small subgroups) property[2] the two notions coincide [GL23, Proposition 10.2]. In particular, if G is a real Lie group (possibly with infinitely many connected components) then every weakly confined subgroup is confined.

The property of being weakly confined generalizes normal subgroups and also lattices (just as the notion of invariant random subgroups generalizes those).

Lemma 7.4. *Let G be a locally compact second countable group. Any non-trivial lattice in G is weakly confined.*

Note that the trivial subgroup is a lattice in G if and only if the group G is compact.

Proof of Lemma 7.4. Let Γ be a non-trivial lattice in G. In view of [Rag72, Theorem 1.12] or [Gel14, Lemma 3.1], there is a compact subset $K \subset G$ such that for every element $g \notin K$ the intersection $\Gamma^g \cap K$ contains a non-trivial element. Let $\gamma \in \Gamma$ be any non-trivial element. Then the compact set $C = K \cup \gamma^K$ intersects every conjugate of the lattice Γ in a non-trivial element. $\square$

Thus, lattices in non-compact real Lie groups as well as finite-index subgroups of infinite discrete groups are confined.

Following [Gel18a] we will say that the locally compact group G has the *NDSS property* (i.e. no discrete small subgroups) if there is an identity neighborhood $U \subset G$ which contains no non-trivial finite subgroups. Obviously, if G has NDSS then a discrete subgroup of G is confined if and only if it is weakly confined.

Corollary 7.5. *Let G be a locally compact second countable group with NDSS. Then all lattices in G are confined.*

[2]A topological group has the *NSS property* (no small subgroups) it is admits an identity neighborhood containing no non-trivial closed subgroups.

The class of NDSS groups contains all real and p-adic Lie groups and is closed under products. Therefore we obtain the following.

Lemma 7.6. *Lattices in standard semisimple groups are confined.*

In center-free semisimple real Lie groups there is a geometric criterion for a discrete subgroup to be confined. Let G be such a Lie group with associated symmetric space $X = G/K$, where K is a maximal compact subgroup of G. A discrete subgroup $\Lambda \leq G$ is confined if and only if there is some $R > 0$ such that the injectivity radius of the orbifold $M_\Lambda = \Lambda \backslash X$ is upper bounded by R at all points of M_Λ. Such orbifolds are called *uniformly slim* in [FG23].

Generally speaking, being a confined subgroup is not a transitive notion. The following example shows that even a normal subgroup Λ of a lattice Γ in a simple Lie group G may not in itself be confined in G. Note that this normal subgroup Λ is even co-amenable in Γ and therefore also in G.

Example 7.7 (Being confined is not transitive)**.** Consider the simple Lie group $G = \mathrm{PSL}(2, \mathbb{R})$ and the lattice $\mathrm{PSL}(2, \mathbb{Z}) \leq G$. Recall that the lattice $\mathrm{PSL}(2, \mathbb{Z})$ is isomorphic to the free product

$$\mathrm{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$$

Let Γ be given by the following exact sequence

$$1 \to \Gamma \to \mathrm{PSL}(2, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \to 1.$$

We have $[\mathrm{PSL}(2, \mathbb{Z}) : \Gamma] = 6$ so that Γ is a lattice in G. The group Γ is isomorphic to the free group F_2. We may view Γ as the fundamental group of a three-holed sphere. The holes are cusps in the corresponding finite-volume hyperbolic metric. Let a, b represent the primitive loops around two of the cusps. Then $\Gamma = \langle a, b \rangle$ and ab is a loop around the third cusp. Every unipotent element of Γ is conjugate to either a power of a, a power of b or a power of ab.

Let $\Lambda = [\Gamma, \Gamma]$ be the commutator group of Γ. Then Λ is normal in Γ and co-amenable in G. The subgroup Λ contains no non-trivial unipotent elements. Indeed, the images of a, b precisely generate the abelianization $\Gamma/\Lambda \cong \mathbb{Z}^2$.

In this situation, if a sequence of elements $g_n \in G$ is such that $g_n\Gamma \to \infty$ in the quotient G/Γ then the sequence of conjugates $g_n \Lambda g_n^{-1}$ tends to the trivial subgroup in the Chabauty topology on $\mathrm{Sub}(G)$. Equivalently, if a sequence of points x_n tends to infinity in one of the cusps in the hyperbolic surface $\Gamma \backslash G/K$ then the injectivity radius at any lift $\widetilde{x}_n$ of the point x_n in $\Lambda \backslash G/K$ tends to infinity. This is because any loop through x_n in $\Gamma \backslash G/K$ represented by a non-unipotent element must wind around another hole, hence pass through the thick part.

Confined subgroups in ICC groups. Recall that a group Γ is called *ICC (i.e. infinite conjugacy classes)* if the conjugacy class of every non-trivial element of Γ is infinite. We consider the properties of confined subgroups in ICC groups.

Lemma 7.8. *Let Γ be a discrete ICC group. If F is a finite subgroup of Γ then F is not confined in Γ.*

Proof. Let F be a finite subgroup of Γ. Assume that $|F| > 1$, for otherwise there is nothing to prove. So the subgroup F admits some non-trivial element $h \in F$. Since the group Γ is ICC, there is a sequence of elements $\gamma_i \in \Gamma$ such that the conjugates h^{γ_i} are pairwise distinct. Up to passing to a subsequence, we may assume that the limit $F' = \lim_i F^{\gamma_i}$ exists in the Chabauty topology. Note that the conjuguate limit F' satisfies $|F'| < |F|$. We conclude that the trivial subgroup is a conjugate limit of the given subgroup F arguing by induction on $|F|$. Namely F is not confined. □

Lemma 7.9. *Let Γ be a discrete group without finite confined subgroups. Then the notion of being confined for subgroups of Γ is a commensurability invariant.*

Proof. It is enough to check that the property is preserved by passing to finite-index subgroups. Let Δ be a subgroup of Γ and $\Delta_1 \leq \Delta$ be a subgroup of finite index. Assume towards contradiction that Δ is confined in Γ but Δ_1 is not. Let $\gamma_i \in \Gamma$ be a sequence of elements such that the subgroups $\Delta_1^{\gamma_i}$ converge to the trivial subgroup in the Chabauty topology. Up to passing to a subsequence, we may assume that the Chabauty limit $\Lambda = \lim_i \Delta^{\gamma_i}$ exists. We claim that Λ is finite. Consider a pair of distinct non-trivial elements $\delta_1, \delta_2 \in \Lambda$. Then $\delta_1, \delta_2 \in (\Delta \setminus \Delta_1)^{\gamma_i}$ for all i sufficiently large. Note that δ_1 and δ_2 belong to different cosets in $\Delta^{\gamma_i}/\Delta_1^{\gamma_i}$ for all i sufficiently large, for otherwise $\delta_1^{-1}\delta_2 \in \Delta_1^{\gamma_i}$ for arbitrarily large i, which is impossible. Therefore $|\Lambda| \leq [\Delta : \Delta_1] < \infty$. On the other hand, note that the subgroup Λ is confined, being a conjugate limit of the confined subgroup Δ. A contradiction. □

This lemma applies in particular to torsion-free groups as well as to ICC groups in view of Lemma 7.8. We obtain the following.

Corollary 7.10. *Let Γ be a discrete group. Assume that Γ is either torsion-free or ICC. Then the notion of being confined for subgroups of Γ is invariant under commensurability.*

Confined actions. Let G be a locally compact second countable group. For every Borel G-space X there is a Borel map $\mathrm{Stab}_G : X \to \mathrm{Sub}(G)$ given by $x \mapsto \mathrm{Stab}_G(x)$.

Definition 7.11. We introduce several notions related to confined actions.

(a) A uniformly recurrent subgroup X, or more generally a closed G-invariant subset X of $\mathrm{Sub}(G)$, is called *confined* if $\{e\} \notin X$.
(b) An invariant random subgroup $\nu \in \mathrm{IRS}(G)$ is called *confined* if $\mathrm{supp}(\nu)$ is confined.
(c) A topological G-space X has *confined stabilizers* if the closed G-invariant subset $\overline{\mathrm{Stab}_G(X)} \subset \mathrm{Sub}(G)$ is confined.
(d) A probability measure preserving Borel G-space (X, μ) has *confined stabilizers* if the invariant random subgroup $(\mathrm{Stab}_G)_*\mu$ is confined.

We caution the reader that, say, the non-confined invariant random subgroup $\delta_{\{e\}}$ has confined stabilizers as a probability measure preserving space.

Example 7.12. *Let H be a closed subgroup of G. The following statements are equivalent:*

(1) The closed subgroup $H \leq G$ is confined in the sense of Definition 7.1.
(2) The orbit closure $\overline{H^G}$ is a confined subset of $\mathrm{Sub}(G)$ in the sense of (a) in Definition 7.11.
(3) The topological G-space G/H has confined stabilizers in the sense of (c) in Definition 7.11.

Lemma 7.13. *Assume that the group G is discrete. Let ν be a confined invariant random subgroup of G. Then there is a compact G-space Y with confined stabilizers admitting a G-invariant probability measure μ such that $(\mathrm{Stab}_G)_*\mu = \nu$.*

Proof. According to [AGV14, Proposition 13] there is a Borel probability measure preserving G-space (Z, η) satisfying $(\mathrm{Stab}_G)_* \eta = \nu$. By Varadarajan's compact model theorem, there is a compact G-space (Y, μ) with $\mathrm{supp}(\mu) = Y$ and such that (Z, η) and (Y, μ) are isomorphic as Borel G-spaces. In particular $(\mathrm{Stab}_G)_* \mu = \nu$ so that (Y, μ) has confined stabilizers as a probability measure preserving space.

To conclude the proof it remains to show that the compact G-space Y has confined stabilizers as a topological space. Note that μ-almost every point $y \in Y$ has $\mathrm{Stab}_G(y) \in \mathrm{supp}(\nu)$. As the group G is discrete, map $\mathrm{Stab}_G : Y \to \mathrm{Sub}(G)$ is upper semi-continuous, in the sense that any point $x \in X$ has a neighborhood $x \in O \subset X$ such that any point $y \in O$ satisfies $\mathrm{Stab}_G(y) \leq \mathrm{Stab}_G(x)$. Put together, these two facts imply that $\overline{\mathrm{Stab}_G(Y)}$ is confined, as required. □

Confined subgroups of arithmetic lattices. We study confined subgroups of irreducible lattices and their actions factoring through rank-one simple factors.

Lemma 7.14. *Let G be a standard semisimple group of higher rank and Γ be an irreducible lattice in G. Let F be a rank one simple factor of G of type p, where p is either ∞ or a prime number. Let Δ be a confined subgroup of Γ. Then the projection of Δ to the factor F is p-Zariski-dense and not relatively compact.*

Proof. The irreducible lattice Γ is arithmetic by Margulis' arithmeticity (Theorem 5.9). We will use the notations introduced in Examples 5.6 and 5.7. Namely K is a number field, $\mathbf{H}$ is an adjoint connected absolutely simple K-group and S is a finite set of isotropic places on K containing all Archimedean isotropic ones. We identify the standard semisimple group G with the group H_S and the rank one simple factor F with the factor H_s for some place $s \in S$. Let k_s be the local field associated to the place s.

The lattice Γ is ICC according to Lemma 5.4. Therefore Corollary 7.10 applies, and we may replace Γ and Δ by finite-index subgroups preserving the assumptions of the lemma. Upon doing so and conjugating, we assume as we may that $\Gamma = \Gamma_S$. In particular, the lattice Γ is contained in the group of rational points $\mathbf{H}(K)$.

Let X be the rank-one symmetric space or Bruhat–Tits building associated to the standard simple group F, depending on whether p is ∞ or a prime. In both cases X is a proper Gromov hyperbolic metric space. Its Gromov boundary ∂X can be identified with $(\mathbf{H}_s/\mathbf{P})(k_s)$ where $\mathbf{P}$ is a minimal parabolic k_s-subgroup of $\mathbf{H}_s$ [Bor12, Propoisiton 20.5]. The variety $\mathbf{H}_s/\mathbf{P}$ is in fact defined over some finite extension L of the field $s(K)$ [Bor12, Corollary 18.8]. We consider Γ as a subgroup of $\mathbf{H}(L)$.

Let μ be any symmetric probability measure on the lattice Γ whose support generates Γ. Let ν be any μ-stationary random subgroup supported on the conjugate closure $\overline{\Delta^\Gamma}$. The assumption that the subgroup Δ is confined ensures that ν-almost every subgroup of Γ is not trivial.

Consider the action of the lattice Γ on the Gromov boundary $(\mathbf{H}_s/\mathbf{P})(k_s)$. We conclude that the fixed point set of every element in Γ is defined over L, hence so is the fixed point set of every subgroup of Γ. We know by [GL23, Proposotion 4.7] that ν-almost every subgroup fixes at most a single point of the boundary ∂X. Every such fixed point must belong to the countable set $(\mathbf{H}_s/\mathbf{P})(L)$ by the preceding remarks. However, any ν-stationary probability measure on a countable set is finitely supported and Γ-invariant [BQ11, Lemma 8.3]. We conclude that ν-almost every subgroup fixes no point of the boundary ∂X. Therefore its projection to F is acting on X without proper closed convex invariant subsets [GL23, Corollary 7.6]. This implies that ν-almost every subgroup projects p-Zariski-densely to F [CM09, Proposition 2.8]. The fact that the projection of the group Δ itself (rather than its conjugate limit) to the factor F is p-Zariski-dense follows from Lemma 6.2.

Similarly, we know that ν-almost every subgroup acts on X with unbounded orbits [GL23, Proposition 4.8 or 7.8]. By this fact and by the previous paragraph, this action cannot be bounded or horocyclic (in the sense outlined e.g. in [GL23, §3]). Hence the action admits hyperbolic elements. Being hyperbolic is an open condition in the group F [GL19, Proposition 3.1]. We conclude that the group Δ itself admits an element whose projection to F is hyperbolic. As such, the projection of the subgroup Δ to the factor F is not relatively compact. □

Strongly confined subgroups. We require a more refined notion than just being confined, which takes into account degeneration of conjugate limits into proper factors.

Definition 7.15. A closed subgroup H of a locally compact second countable group G is *strongly confined* if no conjugate limit of H is contained in a proper normal subgroup.

Certainly, a strongly confined subgroup is confined. Moreover, any conjugate limit of a strongly confined subgroup is still strongly confined.

The following observation is needed to obtain the converse (easier) direction to one of our main results.

Lemma 7.16. *Let G be a standard semisimple group and Γ be an irreducible lattice in G. Then any subgroup of Γ which is confined regarded as a subgroup of G is strongly confined in G.*

Proof. We assume as we may that the group G is semisimple but not simple, for otherwise the two notions of confined and strongly confined are equivalent. In particular G is of higher rank and the lattice Γ is arithmetic (see Theorem 5.9).

We use the notation introduced in Examples 5.6 and 5.7 and identify G with the group $H_S = \prod_{i=1}^m H_{s_i}$ where $S = \{s_1, \ldots, s_m\}$ is a set of places. The lattice Γ is ICC according to Lemma 5.4. Up to conjugating the lattice Γ and using Corollary 7.10 to replace Γ by a finite-index subgroup if necessary, we assume as we may that $\Gamma = \Gamma_S$. In particular Γ contained in $\mathbf{H}(K)$ for some number field K.

Let $\Lambda \leq \Gamma$ be a subgroup. Assume towards contradiction that Λ is confined but not strongly confined regarded as a subgroup of G. Upon reordering S, we get that there is a sequence of elements $g_n \in G$ such that $\Lambda^{g_n} \to \Delta$ in the Chabauty topology with $\Delta \leq L$ and $L = \prod_{i=1}^{m-1} H_{s_i}$. Note that $\Delta \neq \{e\}$ since Λ is confined. This means that for each element $h \in \Delta$ there is a sequence of elements $\gamma_n = (\gamma_{1,n}, \ldots, \gamma_{m,n}) \in \Delta$ where $\gamma_{i,n} \in H_{s_i}$ so that $(\gamma_{1,n}, \ldots, \gamma_{m-1,n})^{g_n} \to h$ and at the same time $\gamma_{m,n}^{g_n} \to e$. The coefficients of the characteristic polynomials of the transformations $\mathrm{Ad}(\gamma_{1,n}), \ldots, \mathrm{Ad}(\gamma_{m,n})$ are all uniformly bounded and the characteristic polynomials of $\mathrm{Ad}(\gamma_{m,n})$ tend to the polynomial $p(x) = (x-1)^{\dim_K \mathbf{H}}$ as n tends to infinity. As the algebraic S-integers form a lattice in the product of local fields $\prod_{s \in S} k_s$, the set of all possible coefficients of such characteristic polynomials is finite. Hence the projection of the element γ_n to each p-component must be unipotent for all n sufficiently large. It follows that every element belonging to the projection of the subgroup Δ to each p-component is unipotent. This implies that the p-Zariski closure U_p of the projection of Δ to each p-component is a connected unipotent subgroup.

In each p-component of the group L, there is some horospherical subgroup V_p containing the unipotent subgroup U_p [BT71, Corollary 3.7]. Let V be the direct product of these horospherical subgroups. Let $a(t)$ be a suitable one-parameter subgroup of L which expands the subgroup V, namely $v^{a(t)} \xrightarrow{t\to\infty} \infty$ holds true for every element $v \in V$.

For a given radius $R > 0$ let $B_e(R)$ denote the ball at the identity of radius R in the group G with respect to some fixed proper and continuous metric. As the subgroup Δ is discrete, for each $i \in \mathbb{N}$ there is some sufficiently large $t_i > 0$ such that

$$((\Delta \cap B_e(i)) \setminus \{e\})^{a_1(t_i)} \subset G \setminus B_e(2i)$$

as well as $d(x^{a_1(t_i)}, e) \geq 2d(x, e)$ for all $x \in V$. Let $R_i > i$ be a sufficiently large radius such that

$$(G \setminus B_e(R_i))^{a_1(t_i)} \subset G \setminus B_e(i)$$

for each i. Let $j = j(i)$ be a sufficiently large index, such that every element of the intersection $\Lambda^{g_{j(i)}} \cap B_e(R_i)$ is "sufficiently close" to some element of $\Delta \cap B_e(R_i)$ for each i. More precisely, we require as we may for each i that

$$((\Lambda^{g_{j(i)}} \cap B_e(R_i)) \setminus \{e\})^{a_1(t_i)} \subset G \setminus B_e(i).$$

It follows that the sequence of conjugates $\Lambda_i = \Lambda^{g_{j(i)}a_1(t_i)}$ converges to the trivial subgroup in the Chabauty topology on $\mathrm{Sub}(G)$. This is a contradiction to the assumption that Λ is confined when regarded as a subgroup of G. □

Confined discrete subgroups of rank one simple Lie groups are Zariski dense, see [FG23, Lemma 9.14]. We show that projections of irrreducibly confined discrete subgroups to simple rank one factors are also Zariski dense, under certain conditions.

Lemma 7.17. *Let G be a standard semisimple group of type p, where p is either ∞ or a prime number. Let H be a simple factor of G with $\mathrm{rank}(H) = 1$. Let Δ be a strongly confined subgroup of G. Then the projection of Δ to the factor H is p-Zariski-dense and not relatively compact.*

Proof. Write $G = H \times H'$ where H' is a suitable standard semisimple subgroup. Let μ and μ' be a pair of probability measures on the groups H and H' respectively, specifically the particular ones considered in [GLM22]. Denote $\mu_0 = \mu \otimes \mu'$ and let ν be any μ_0-stationary limit of the subgroup Δ (in the sense of Definition 10.4). In particular ν is a μ-stationary random subgroup supported on the conjugate closure $\overline{\Delta^G}$. In the real case, we know that ν-almost every subgroup of G is discrete by [FG23, Theorem 1.6]. In the p-adic case, ν-almost every subgroup is certainly discrete, as G has a compact open subgroup containing no no-trivial discrete subgroups [Ser09, Theorem 1 on p. 124]. In any case ν-almost every subgroup is not contained in the factor H' by the strongly confined assumption. The fact that ν-almost every subgroup fixes no point in its action on the Gromov boundary of the rank one symmetric space or Bruhat–Tits building associated to the rank one group H follows immediately from [GL23, Theorem 6.1]. From this point onward, we conclude the proof exactly as in Lemma 7.14. □

The above two Lemmas 7.14 and 7.17 are very much closely related. However, a priori a confined subgroup of the lattice Λ may not be confined regarded as a subgroup of the enveloping group G, as was the case in Example 7.7.

8. Confined subgroups of irreducible lattices

The current section is devoted to the proof of Theorem 8.4 saying that any confined subgroup of an irreducible lattice of a higher rank standard semisimple group has finite index. This is the generalization of Theorem 1.1 as well as Corollary 1.2 of the introduction to the setting of standard semisimple groups over local fields (rather than Lie groups).

If Γ is a cocompact lattice in the locally compact group G then every confined subgroup of Γ is confined in G. However, this need not be the case if Γ is a non-cocompact lattice, see Example 7.7. The next two lemmas are here to remedy this failure. They are not needed in the cocompact case.

For a Borel subset $M \subset \mathrm{Sub}(G)$, we denote

$$M^G = \{H^g \mid H \in M,\ g \in G\}$$

and

$$\mathrm{Prob}(M) = \{\nu \in \mathrm{Prob}(\mathrm{Sub}(G)) \ :\ \nu(M) = 1\}.$$

Lemma 8.1. *Let Γ be a lattice in the locally compact group G. Let X be a topological Γ-space and $Y = G \times_\Gamma X$ be the induced topological G-space*[3] *with projection $\pi : Y \to G/\Gamma$. Denote*

$$\mathcal{C}^G_\Gamma(X) = \{\mu \in \mathrm{Prob}(Y) \ :\ \pi_*\mu = m_{G/\Gamma}\}.$$

Then

$$\overline{(\mathrm{Stab}_G)_*(\mathcal{C}^G_\Gamma(X))} \subset \mathrm{Prob}(\overline{\mathrm{Stab}_\Gamma(X)}^G).$$

Proof. We consider $\mathrm{Sub}(\Gamma)$ as a closed subset of $\mathrm{Sub}(G)$. Denote $S = \overline{\mathrm{Stab}_\Gamma(X)}$ so that $S \subset \mathrm{Sub}(\Gamma)$ is a Γ-invariant closed subset. For each point $g\Gamma \in G/\Gamma$ consider the space of probability measures

$$A_{g\Gamma} = \mathrm{Prob}\left(S^g\right)$$

regarded as a compact convex subset of $\mathrm{Prob}(\mathrm{Sub}(G))$. Consider the compact convex space

$$Q = F(G/\Gamma, \{A_{g\Gamma}\})$$

consisting of all $m_{G/\Gamma}$-measurable functions $f : G/\Gamma \to \mathrm{Prob}(\mathrm{Sub}(G))$ satisfying $m_{G/\Gamma}$-almost surely $f(g\Gamma) \in A_{g\Gamma}$, see [Zim13, p. 78]. The barycenter map with respect to the normalized Haar measure $m_{G/\Gamma}$ sets up a continuous affine map

$$\mathrm{bar} : Q \to \mathrm{Prob}(S^G) \subset \mathrm{Prob}(\mathrm{Sub}(G)).$$

To conclude the proof it will suffice to construct an affine map $\varphi : \mathcal{C}^G_\Gamma(X) \to Q$ such that $\mathrm{Stab}_* = \mathrm{bar} \circ \varphi$ on the space $\mathcal{C}^G_\Gamma(X)$, which is what we proceed to do.

Given a probability measure $\mu \in \mathcal{C}^G_\Gamma(X) \subset \mathrm{Prob}(Y)$, consider its disintegration d_μ over the projection π to the probability space $(G/\Gamma, m_{G/\Gamma})$, given by a $m_{G/\Gamma}$-measurable map

$$d_\mu : G/\Gamma \to \mathrm{Prob}(Y)$$

[3]For the notion of induced actions we refer e.g. to [Zim13, p. 75].

such that $m_{G/\Gamma}$-almost surely $d_\mu(g\Gamma)$ gives full measure to the fiber $\pi^{-1}(g\Gamma)$. In particular $\mathrm{Stab}_* d_\mu(g\Gamma) \subset A_{g\Gamma}$ almost surely. The affine map

$$\varphi : \mathcal{C}^G_\Gamma(X) \to Q, \quad \varphi(\mu)(g\Gamma) = \mathrm{Stab}_* d_\mu(g\Gamma) \quad \forall \mu \in \mathcal{C}^G_\Gamma(X),\ \forall g\Gamma \in G/\Gamma$$

is as required. □

Lemma 8.2. *Let G be a standard semisimple group and $\Gamma < G$ an irreducible lattice. Let X be a topological Γ-space and Y be the induced topological G-space equipped with the projection $\pi : Y \to G/\Gamma$. Fix a G-invariant probability measure $\mu \in \mathrm{Prob}(Y)$ such that $\pi_*\mu = m_{G/\Gamma}$. Let $f_i \in L^2(Y,\mu)$ be an asymptotically H-invariant sequence of unit vectors for some non-trivial normal subgroup $H \lhd G$. Then any accumulation point of the sequence of probability measures $\mathrm{Stab}_*(|f_i|^2 \cdot \mu)$ in the space $\mathrm{Prob}(\mathrm{Sub}(G))$ belongs to $\mathrm{Prob}(\overline{\mathrm{Stab}_\Gamma(X)}^G)$.*

Proof. Consider the natural linear map $T : L^1(Y,\mu) \to L^1(G/\Gamma)$ corresponding to the projection π. It has operator norm $\|T\|_{\mathrm{op}} = 1$. Furthermore $\|Tf\|_1 = \|f\|_1$ for non-negative functions (i.e. provided $f \geq 0$).

We consider the sequence of elements $|f_i|^2 \in L^1(Y,\mu)$. By Lemma 2.2 this is an asymptotically H-invariant sequence of non-negative unit vectors in $L^1(Y,\mu)$. We consider the probability measures $\nu_i = |f_i|^2 \cdot \mu \in \mathrm{Prob}(Y)$ and their push forward to $\mathrm{Sub}(G)$ under the map $\mathrm{Stab} : Y \to \mathrm{Sub}(G)$. Up to passing to a subsequence, we assume that the probability measures $\mathrm{Stab}_*\nu_i$ converge to some probability measure $\zeta \in \mathrm{Prob}(\mathrm{Sub}(G))$. The asymptotic H-invariance of the sequence $|f_i|^2$ implies that the measure ζ is H-invariant.

Consider the sequence of functions $g_i \in L^1(G/\Gamma)$ given by $g_i = T(|f_i|^2)$. The sequence g_i is asymptotically H-invariant, as T is contracting. By Clozel's theorem (essentially, see Theorem 5.10 for details), the group H has spectral gap in its representation on $L^2_0(G/\Gamma)$. It follows by Lemma 2.8 that $\|g_i - 1_{G/\Gamma}\|_1 \to 0$.

Our strategy is to construct a sequence of probability measures η_i all satisfying $\pi_*\eta_i = m_{G/\Gamma}$, such that $\mathrm{Stab}_*\nu_i$ and $\mathrm{Stab}_*\eta_i$ have the same asymptotic behaviour. Here are the details. Look at the positive and negative pairs of the difference $1 - g_i$. Namely

$$1 - g_i = h_i^+ - h_i^-$$

where $h_i^+, h_i^- \in L^1(G/\Gamma)$, $h_i^+, h_i^- \geq 0$ and $\|h_i\|_1 = \|h_i^+\|_1 + \|h_i^-\|_1$. Let $l_i^+ \in L^1(Y,\mu)$ be an arbitrary lift satisfying $T(l_i^+) = h_i^+$ and $l_i^+ \geq 0$. We note that $T(|f_i|^2 + l_i^+) - h_i^- = 1$, thus $T(|f_i|^2 + l_i^+) \geq h_i^-$. Let $l_i^- \in L^1(Y,\mu)$ be an arbitrary lift with $T(l_i^-) = h_i^-$ and $|f_i|^2 + l_i^+ \geq l_i^- \geq 0$. We set $l_i = l_i^+ - l_i^-$. Observe that

$$\|l_i\|_1 \leq \|l_i^+\|_1 + \|l_i^-\|_1 = \|h_i^+\|_1 + \|h_i^-\|_1 = \|1 - g_i\|_1 \xrightarrow{i\to\infty} 0.$$

We note that

$$|f_i|^2 + l_i = |f_i|^2 + l_i^+ - l_i^- \geq 0 \quad \text{and} \quad T(|f_i|^2 + l_i) = T(|f_i|^2) + 1 - g_i = 1.$$

In particular, each $\eta_i = (|f_i|^2 + l_i) \cdot m_{G/\Lambda}$ is a probability measure on Y. Since $T(|f_i|^2 + l_i) = 1$, the measures η_i all satisfy $\pi_*\eta_i = m_{G/\Gamma}$. In particular

$$\lim \mathrm{Stab}_*\eta_i \in \mathrm{Prob}(\overline{\mathrm{Stab}_\Gamma(X)}^G)$$

according to Lemma 8.1. On the other hand, the fact that $\|l_i\|_1 \to 0$ implies

$$\zeta = \lim \mathrm{Stab}_*\,\nu_i = \lim \mathrm{Stab}_*\,\eta_i.$$

Since these two limits coincide, the desired conclusion follows. □

Theorem 8.3. *Let G be a standard semisimple group with* $\operatorname{rank}(G) \geq 2$ *and* Γ *an irreducible lattice in G. Then every ergodic confined invariant random subgroup of* Γ *almost surely has finite index.*

Proof. Let ν be an ergodic confined invariant random subgroup of the group Γ. According to Lemma 7.13 there is a compact confined Γ-space X and a Γ-invariant probability measure $\mu \in \operatorname{Prob}(X)$ such that $(\operatorname{Stab}_\Gamma)_*\mu = \nu$. Let $(Y, \overline{\mu})$ be the probability measure preserving topological G-space induced from the Γ-space (X, μ). The invariant random subgroup $\overline{\nu} = (\operatorname{Stab}_G)_*\overline{\mu}$ coincides with the invariant random subgroup of the group G induced from ν as defined in §2. Note that the G-space $(Y, \overline{\mu})$ is ergodic [Zim13, Remark (1) on p. 75]. Hence the invariant random subgroup $\overline{\nu}$ is ergodic as well. Therefore the G-space $(Y, \overline{\mu})$ (as well as the invariant random subgroup $\overline{\nu}$) is irreducible by Corollary 6.9.

We now claim that the unitary G-representation $L^2_0(Y, \overline{\mu})$ has spectral gap. If the group G has Kazhdan's property (T) then the statement is immediate. Assume from now on that G has no Kazhdan's property (T). As such, the group G splits as a non-trivial direct product of standard semisimple groups $G = G_1 \times G_2$ where G_1 is a standard semisimple group and G_2 is a standard simple group of type p_0 and with $\operatorname{rank}(G_2) = 1$.

We would like to apply our Theorem 6.10 establishing spectral gap for actions of products with respect to the unitary representation $L^2_0(Y, \mu)$. The assumptions of that theorem are verified as follows:

- The fact that $L^2_0(Y, \overline{\mu})^{G_2} = 0$ follows from the irreducibility of $\overline{\mu}$.
- The stabilizer of $\overline{\mu}$-almost every point is distributed according to $\overline{\nu}$, and as such, is conjugate to a subgroup of the irreducible lattice Γ and has trivial intersections with every proper normal subgroup of G.
- Let $f_n \in L^2(Y, \overline{\mu})$ be any asymptotically G_1-invariant sequence of vectors. Consider any weak-$*$ accumulation point $\zeta \in \operatorname{Prob}(\operatorname{Sub}(G))$ of the sequence of probability measures $\operatorname{Stab}_*(|f_n|^2 \cdot \overline{\mu})$. Then according to Lemma 8.2 $\zeta \in \operatorname{Prob}(\overline{\operatorname{Stab}_\Gamma(X)}^G)$. This means that ζ-almost every subgroup is discrete and not contained in any proper normal factor. Further, this implies that ζ-almost every subgroup is conjugated in G to a confined subgroup of Γ, thus by Lemma 7.14, ζ-almost every subgroup has p-Zariski-dense and not relatively compact projections to G_2.

Having verified all assumptions of Theorem 6.10 we conclude that $L^2_0(Y, \overline{\mu})$ has spectral gap. By relying on the methods of Stuck–Zimmer [SZ94] and Bader–Shalom [BS06], and making use of the product structure of the group G, this implies that the G-space $(Y, \overline{\mu})$ is essentially transitive, see e.g. [Cre17, Proposition 7.6] or [Lev20, Theorem 3] for details. Therefore $\overline{\nu}$-almost every subgroup is a lattice in G. We conclude that ν-almost every subgroup has finite index in Γ, as required. □

Theorem 8.4. *Let G be a standard semisimple group with* $\operatorname{rank}(G) \geq 2$ *and* Γ *an irreducible lattice in G. Then any confined subgroup of* Γ *has finite index. Further, every non-trivial uniformly recurrent subgroup X of* Γ *is the set of conjugates of some finite-index subgroup of* Γ*.*

Proof. Recall that the lattice Γ is an ICC group by Lemma 5.4. In particular, we may replace Γ by any of its finite-index subgroups without a loss of generality, see Corollary 7.10. Specifically, using Margulis' arithmeticity (Theorem 5.9) and the notation introduced in Examples 5.6 and 5.7, we thus assume that the lattice Γ is S-arithmetic in the sense that $\Gamma = \Gamma_S$. The significance of this fact for our purpose is that the arithmetic lattice Γ is *charmenable* by [BBH23, Theorem B]. Further, note that Γ has a trivial amenable radical.

Let Λ be any confined subgroup of the lattice Γ. Consider any uniformly recurrent subgroup X contained in the Γ-orbit closure $\overline{\Lambda^\Gamma}$. Note that X is non-trivial by the assumption that Λ is confined. Therefore X carries an ergodic Γ-invariant Borel probability measure ν by [BBHP22, Proposition 3.5]. Theorem 8.3 implies that ν-almost every subgroup has finite index in Γ. This certainly implies that *some* subgroup $\Delta \in \overline{\Lambda^\Gamma}$ has finite index in Γ. Since finite-index subgroups are isolated points of $\mathrm{Sub}(\Gamma)$, we conclude that the subgroup Λ itself has finite index in Γ to begin with.

The second statement of the theorem follows just as in the previous paragraph. □

Remark 8.5. *Consider the special case where G is a standard* simple *group. Then G has property (T), hence the lattice Γ is* charfinite *by* [BBH23, Theorem A]*. Every uniformly recurrent subgroup of a charfinite group with trivial amenable radical is finite* [BBHP22, Proposition 3.5]*. In that case, we may invoke the classical Margulis normal subgroup theorem to conclude that X is the set of conjugates of some finite-index subgroup.*

Proof of Theorem 1.1 and Corollary 1.2. These two statements from the introduction follow as special cases of Theorem 8.4, by noting that connected center-free semisimple Lie groups are standard semisimple groups over $\mathbb{R}$ [Zim13, 3.1.6]. □

9. Margulis functions on factors

The current section deals with the notion of Margulis functions [EMM98, EM22] (also called Foster–Lyapunov functions, see e.g. [MT12]). We first discuss this idea in the abstract. We then apply these functions to study sequences of vectors which are asymptotically invariant with respect to a single factor of a semisimple real Lie group, relying on methods from the work [GLM22]. This is instrumental towards dealing with general strongly confined subgroups of products in §10.

Abstract properties of Margulis functions. *All throughout this first part of §9, let G be a second countable locally compact group admitting a continuous action on a locally compact σ-compact topological space X.* Let μ be a compactly supported symmetric probability measure on the group G whose support generates G.

Definition 9.1. Let $\Phi : X \to [a, \infty)$ be a proper continuous map for some $a > 0$. The map Φ is a (μ, c, b)-*Margulis function* for some $0 < c < 1$ and $b > 0$ if

$$\mu * \Phi(x) = \int_G \Phi(gx)\, \mathrm{d}\mu(g) < c\Phi(x) + b \tag{9.1}$$

for every point $x \in X$.

The above Definition 9.1 coincides with the authoritative [EM22, Definition 1.1], with an additional assumption that the Margulis function Φ is not allowed to take the value ∞. We will sometimes drop the constants c and b from our notations and refer to Φ as a μ-Margulis function, or simply as a Margulis function.

It is useful to note that by reiterating the convolution operator, it is possible to "improve" a given Margulis function.

Lemma 9.2. *Let Φ be a (μ, c, b)-Margulis function. There is a sequence $0 < L_1 < L_2 < L_3 < \cdots$ such that for each $i \in \mathbb{N}$ we have*

$$\mu^{*i} * \Phi(z) < \left(\frac{1+c}{2}\right)^i \Phi(z) \tag{9.2}$$

for all points $z \in X$ satisfying $\Phi(z) \geq L_i$.

Proof. We will construct the sequence L_i inductively. Taking $L_1 = \frac{2b}{1-c}$ handles the base of the induction, as follows directly from Equation 9.1. Next, arguing by induction, assume that the constant L_i for some $i \in \mathbb{N}$ has been defined in such a way that Equation 9.2 is satisfied. Consider the closed subset $X_i = \Phi^{-1}([a, L_i])$. The subset X_i is compact as the Margulis function Φ is proper. Take $L_{i+1} > L_i$ to be any sufficiently large constant so that $\Phi(\text{supp}(\mu)X_i) \subset [a, L_{i+1})$. This implies that any point $z \in X$ with $\Phi(z) \geq L_{i+1}$ also satisfies $\mu * \Phi(z) \geq L_i$. The validity of Equation 9.2 with respect to $i+1$ readily follows from the induction assumption. □

We now consider the situation where the space X is equipped with a μ-stationary probability measure.

Lemma 9.3. *Let Φ be a (μ, c, b)-Margulis function. Assume that X admits a μ-stationary probability measure ν.*

(1) Set $B = \frac{b}{1-c}$, then

$$\nu(\{x \in X \,:\, \Phi(x) \geq M\}) < \frac{B}{M}$$

for all $M > 0$.

(2) The Margulis function Φ satisfies $\Phi^{\frac{1}{2}} \in L^1(X, \nu)$.

Proof. (1) is [GLM22, Lemma 2.1]. We will now present[4] a proof of (2).

Let $D : [0, 1] \to \mathbb{R}_{\geq 0}$ be the inverse cumulative distribution function corresponding to $\Phi^{\frac{1}{2}}$. It is defined as

$$D(t) = \inf\{s \in \mathbb{R}_{\geq 0} \,:\, \nu(\{x \in X \,:\, \Phi^{\frac{1}{2}}(x) \leq s\}) \geq t\}.$$

In other words, the function D satisfies

$$\nu(\{x \in X \,:\, \Phi^{\frac{1}{2}}(x) \leq D(t)\}) = t \quad \forall t \in [0, 1]\,.$$

Substituting $s = M^{-1}$ in Part (1) of the current lemma, we get that for every $s > 0$

$$\nu(\{x \in X \,:\, \Phi^{\frac{1}{2}}(x) \leq s^{-\frac{1}{2}}\}) \geq 1 - Bs$$

with the above constant B. It follows that $D(1 - Bs) \leq s^{-\frac{1}{2}}$ for all $s \in [0, B^{-1}]$. In other words

$$D(t) \leq B^{\frac{1}{2}}(1 - t)^{-\frac{1}{2}}$$

for all $t \in [0, 1]$. It follows from Fubini's theorem applied to the "area under the graph" of the function $\Phi^{\frac{1}{2}}$ regarded as a subset of $X \times \mathbb{R}_{\geq 0}$ that

$$\int_X \Phi^{\frac{1}{2}}(x)\, \mathrm{d}\nu(x) = \int_{[0,1]} D(t)\, \mathrm{d}\lambda(t) \tag{9.3}$$

[4] The proof of part (2) in Lemma 9.3 is taken from the proof of [GLM22, Proposition 9.2]. We have chosen to reproduce it here, for the convenience of the reader and for self-containedness. The difference in notations makes it difficult to quote that proposition from [GLM22] as-is.

where λ is the Lebesgue measure. As the integral $B^{\frac{1}{2}} \int_0^1 (1-t)^{-\frac{1}{2}} \mathrm{d}t$ converges, we conclude from Equation (9.3) that the function $\Phi^{\frac{1}{2}}$ is indeed ν-integrable. □

The concave variant of Jensen's inequality implies that if Φ is a (μ, c, b)-Margulis function then Φ^α is a $(\mu, c^\alpha, b^\alpha)$-Margulis function for every exponent $0 < \alpha < 1$. This observation coupled Lemma 9.3 allows us to assume without loss of generality that we are working with an L^1 (or L^2) Margulis function to begin with.

We shall now consider the more restrictive situation, where the space X admits an invariant probability measure.

Lemma 9.4. *Let ν be a G-invariant probability measure on X. Let Φ be a (μ, c, b)-Margulis function with $\Phi \in L^2(X, \nu)$. Let $0 < L_1 < L_2 < \cdots$ be the constants provided by Lemma 9.2 and denote $\Omega_i = \Phi^{-1}([L_i, \infty))$ for each $i \in \mathbb{N}$. Let P_i be the orthogonal projection operator in $L^2(X, \nu)$ given by*

$$\mathrm{P}_i : f \mapsto f \cdot 1_{\Omega_i} \quad \forall f \in L^2(X, \nu)$$

Then $\|\mathrm{P}_i \mu \mathrm{P}_i\|_{\mathrm{op}} \leq \left(\frac{1+c}{2}\right)^i$ for each $i \in \mathbb{N}$.

This is a reminiscent of [GLM22, §9], see the proof of Theorem 9.3 there.

Proof of Lemma 9.4. Let an arbitrary $i \in \mathbb{N}$ be fixed. To ease our notations, set $\mathrm{P} = \mathrm{P}_i$ and $C = \left(\frac{1+c}{2}\right)^i$. Our goal will be to show that $\|\mathrm{P}\mu\mathrm{P}\|_{\mathrm{op}} \leq C$. The spectrum of the operator $\mathrm{P}\mu\mathrm{P}$ is a closed subset of the real interval $[-1, 1]$. Assume towards contradiction that this spectrum admits a point $\lambda \in [-1, 1]$ with $|\lambda| > C > 0$. We can find a sequence of functions $f_n \in L^2(X, \nu)$ with $\|f_n\| = 1$ and a sequence of real numbers $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$\|\mathrm{P}\mu\mathrm{P} f_n - \lambda f_n\| < \varepsilon_n.$$

The fact that Φ is a (μ, c, b)-Margulis function means that the inequality

$$\mathrm{P}\mu\mathrm{P}\Phi \leq C\Phi$$

holds true ν-almost everywhere, see Lemma 9.2. Since the convolution operator μ is symmetric, the composition $\mathrm{P}\mu\mathrm{P}$ is a self-adjoint operator. For each n

$$\begin{aligned} C\langle \Phi, |f_n| \rangle &\geq \langle \mathrm{P}\mu\mathrm{P}\Phi, |f_n| \rangle = \langle \Phi, \mathrm{P}\mu\mathrm{P}|f_n| \rangle \geq \\ &\geq \langle \Phi, |\mathrm{P}\mu\mathrm{P} f_n| \rangle = \langle \Phi, |\lambda f_n + (\mathrm{P}\mu\mathrm{P} - \lambda) f_n| \rangle \geq \\ &\geq |\lambda| \langle \Phi, |f_n| \rangle - \langle \Phi, |(\mathrm{P}\mu\mathrm{P} - \lambda) f_n| \rangle \geq \\ &\geq |\lambda| \langle \Phi, |f_n| \rangle - \|\Phi\| \cdot \|(\mathrm{P}\mu\mathrm{P} - \lambda) f_n\| \geq |\lambda| \langle \Phi, |f_n| \rangle - \varepsilon_n \|\Phi\|. \end{aligned}$$

Note that $\langle \Phi, |f_n| \rangle > 0$ for otherwise $\mathrm{P} f_n = 0$. We arrive at a contradiction to the assumption that $|\lambda| > C$. □

Lemma 9.5. *Maintain all the assumptions and notations of Lemma 9.4. In addition, let $f_n \in L^2(X, \nu)$ be an asymptotically G-invariant sequence of unit vectors. Consider the sequence of probability measures $m_n = |f_n|^2 \cdot \nu$ on the space X. Then*

$$\limsup_n m_n(\Omega_{i+1}) \leq 4 \left(\frac{1+c}{2}\right)^{2i}$$

holds true for each $i \in \mathbb{N}$.

Proof. Let an arbitrary $i \in \mathbb{N}$ be fixed. To ease notation set $C = \left(\frac{1+c}{2}\right)^i$. Let P and P′ denote the two orthogonal projections in $L^2(X,\nu)$ given by $f \mapsto f \cdot 1_{\Omega_i}$ and $f \mapsto f \cdot 1_{\Omega_{i+1}}$ respectively. Recall that $\operatorname{supp}(\mu)\Omega_{i+1} \subset \Omega_i$. Hence $\mathrm{P}\mu\mathrm{P}' = \mu\mathrm{P}'$ as well as $\mathrm{P}'\mathrm{P} = \mathrm{P}\mathrm{P}' = \mathrm{P}'$. On the one hand, the asymptotic G-invariance of the sequence f_n means that $\|\mu * f_n\| \to 1$ as $n \to \infty$. On the other hand, Lemma 9.4 implies

$$\begin{aligned}\|\mu * f_n\| &= \|\mu * (\mathrm{P}'f_n + (1-\mathrm{P}')f_n)\| \leq \\ &\leq \|(\mathrm{P}\mu\mathrm{P})\mathrm{P}'f_n\| + \|\mu(1-\mathrm{P}')f_n\| \leq C\|\mathrm{P}'f_n\| + \|(1-\mathrm{P}')f_n\|.\end{aligned}$$

Let $x_n = \|\mathrm{P}'f_n\|$ so that $\sqrt{1-x_n^2} = \|(1-\mathrm{P}')f_n\|$. We get

$$\liminf_n (Cx_n + \sqrt{1-x_n^2}) \geq 1.$$

Note that $x_n^2 = \|\mathrm{P}'f_n\|^2 = m_n(\Omega_{i+1})$ for all n. Solving the above inequality for x_n^2 gives

$$\limsup x_n^2 \leq \left(\frac{2C}{C^2+1}\right)^2 \leq 4C^2$$

as required. □

Corollary 9.6. *In the situation of Lemma 9.5, any accumulation point m of the sequence of probability measures m_n satisfies $m(X) = 1$.*

In other words, the probability measures m_n do not have "escape of mass at infinity".

A Margulis function on discrete subsets with the Zassenhaus property. In [GLM22, Theorem 1.5] it is shown that a certain function depending on the injectivity radius is a Margulis function on the space of discrete subgroups of a standard semisimple algebraic group. That function is denoted $\mathcal{I}^{-\delta}$ and the result is termed *the key inequality*. An inspection of that proof shows that the domain of definition of the Margulis function $\mathcal{I}^{-\delta}$ can be slightly extended. In this subsection we recall the setting of [GLM22] and present this extension.

In the discussion of the key inequality in [GLM22] the authors regard an algebraic group over an arbitrary local field (of good characteristic). However, in the present work we are only interested in characteristic zero. Moreover, since the function $\mathcal{I}^{-\delta}$ is actually constant when working over a zero characteristic non-Archimedean local field, we focus here only on the Archimedean case. By restricting scalars we may assume that we are working over the reals.

Let G be a standard semisimple group of type ∞. Let K be a maximal compact subgroup of G and endow the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ with an $\mathrm{Ad}(K)$-invariant norm. Denote by m_K the normalized Haar probability measure on the group K. Fix a semisimple group element $s = s(G) \in G$ as defined in [GLM22, Equation (6.22)]. Let

$$\mu_s = \frac{1}{2} m_K * (\delta_s + \delta_{s^{-1}}) * m_K$$

be the corresponding probability measure on the group G. Note that μ_s is symmetric[5] and compactly supported.

[5]In some sections of [GLM22] the authors work with the non-symmetric measure $m_K * \delta_s * m_K$. However, as $\mu_s = \frac{1}{2}(m_K * \delta_s * m_K + m_K * \delta_{s^{-1}} * m_K)$, the results easily carry over.

Next, we will fix three positive real parameters (δ, R and ρ) and two identity neighborhoods (V and V_0) associated with the group G. Let $\delta = \delta(G)$ be the parameter given in [GLM22, Equation (6.20)]. Let $R = R(G)$ be the radius given by the Zassenhaus lemma.

Lemma 9.7 (Zassenhaus lemma [Zas37] or [KM68, Lemma 2])**.** *There exists a radius $R = R(G) > 0$ such that for every discrete subgroup $\Gamma < G$, the subset*

$$\{\gamma \in \Gamma \,:\, \gamma = \exp(X) \text{ for some } X \in \mathfrak{g} \text{ with } \|X\| \leq R\}$$

is contained in some connected nilpotent Lie subgroup of G.

We consider the identity neighborhood

$$V = \exp\{X \in \mathfrak{g} \,:\, \|X\| \leq R\}$$

in the group G. Lastly, fix a sufficiently small radius $0 < \rho = \rho(G) < R$ such that the identity neighborhood

$$V_0 = \exp\{X \in \mathfrak{g} \,:\, \|X\| \leq \rho\}$$

satisfies

$$V_0 \subset V \cap V^s \cap V^{s^{-1}}.$$

For all this, we refer to [GLM22, Equations (7.3), (7.4) and (7.5)].

Consider the space of discrete subgroups of the Lie group G. This space will be denoted $\mathrm{Sub}_d(G)$ and regarded as an open subset of the Chabauty space $\mathrm{Sub}(G)$. We define on the space $\mathrm{Sub}_d(G)$ the function $\mathcal{I} : \mathrm{Sub}_d(G) \to (0, \rho]$ given by

$$\mathcal{I}(\Gamma) = \sup\{0 < r \leq \rho \,:\, \Gamma \cap \exp\{X \in \mathfrak{g} \,:\, \|X\| \leq r\} = \{e\}\} \quad \forall \Gamma \in \mathrm{Sub}_d(G).$$

Theorem 9.8. *$\mathcal{I}^{-\delta} : \mathrm{Sub}_d(G) \to [\rho^{-\delta}, \infty)$ is a μ_s-Margulis function.*

Proof. We will see in Theorem 9.10 below that the function $\mathcal{I}^{-\delta}$ is continuous and proper (in fact, it has these properties over an extended domain of definition). The function $\mathcal{I}^{-\delta}$ satisfies the inequality in Equation (9.1) with respect to the probability measure $m_K * \delta_s * m_K$ by [GLM22, Theorem 1.5]. The same is true with respect to the probability measure $m_K * \delta_{s^{-1}} * m_K$. Therefore $\mathcal{I}^{-\delta}$ is a μ_s-Margulis function, as the symmetric probability measure μ_s is a convex combination of these two probability measures. □

We are now ready to extend the domain of definition of the Margulis function $\mathcal{I}^{-\delta}$. Consider the space $\mathrm{Cl}(G)$ consisting of all closed subsets of the group G. The space $\mathrm{Cl}(G)$ is endowed with the Fell topology, and the Chabauty space $\mathrm{Sub}(G)$ is a closed subspace of $\mathrm{Cl}(G)$.

Definition 9.9. A closed subset $A \in \mathrm{Cl}(G)$ has the *Zassenhaus property* if for every element $g \in G$, the subset $A^g \cap V$ is contained in some connected nilpotent Lie subgroup of G and satisfies

$$(A^g \cap V)^2 \subset A^g. \tag{9.4}$$

We denote by $\mathrm{Cl}^Z(G)$ the subset of $\mathrm{Cl}(G)$ consisting of all sets with the Zassenhaus property. This is a closed and G-invariant subset of $\mathrm{Cl}(G)$. We denote by $\mathrm{Cl}^Z_d(G)$ the subset of $\mathrm{Cl}^Z(G)$ consisting of sets containing the identity element of the group G as an isolated point. This is an open and G-invariant subset of $\mathrm{Cl}^Z(G)$, which contains $\mathrm{Sub}_d(G)$ by the Zassenhaus lemma (namely Lemma 9.7). The function $\mathcal{I}$

extends naturally to the space $\mathrm{Cl}^Z_d(G)$. We set a function $\mathcal{J} : \mathrm{Cl}^Z_d(G) \to (0, \rho]$ given by

$$\mathcal{J}(A) = \sup\{0 < r \le \rho \, : \, A \cap \exp\{X \in \mathfrak{g} \, : \, \|X\| \le r\} = \{e\}\} \quad \forall A \in \mathrm{Cl}^Z_d(G).$$

Theorem 9.10. *$\mathcal{J}^{-\delta} : \mathrm{Cl}^Z_d(G) \to [\rho^{-\delta}, \infty)$ is a μ_s-Margulis function.*

Proof. The continuity of the function $\mathcal{J}$ follows from the definition of the Chabauty topology, from the property given in Equation (9.4) and from the fact that the Lie group G has no small subgroups. The fact that the function $\mathcal{J}^{-\delta}$ is proper is equivalent to saying that $\mathcal{J}(A_n) \to 0$ for every sequence of subsets $A_n \in \mathrm{Cl}^Z_d(G)$ converging to a subset $A \in \mathrm{Cl}^Z(G)$ in which the identity element is not an isolated point. This later statement holds true.

We are left to show that the function $\mathcal{J}^{-\delta}$ satisfies Equation (9.1). Observe that Equation (7.8) as well as Propositions 7.3, 7.4 and 7.5 in [GLM22], originally formulated for the function $\mathcal{I}$, all apply mutatis mutandis to our new function $\mathcal{J}$. Indeed, the arguments of [GLM22] involve studying finite collections of elements all belonging to some connected nilpotent Lie subgroup, regardless on whether these elements come from any given envelopping discrete subgroup. Thus, the desired result follows by the same proof as that of the key inequality in [GLM22, §8]. □

Margulis functions on discrete subgroups of products. We let G_1 and G_2 be a pair of standard semisimple Lie groups of type ∞ and set $G = G_1 \times G_2$. Let $\mathfrak{g}_i = \mathrm{Lie}(G_i)$ be the corresponding semisimple Lie algebras. Consider the element $s_1 = s(G_1) \in G_1$, the probability measure $\mu_{s_1} \in \mathrm{Prob}(G_1)$ and the positive real parameters $\delta_1 = \delta(G_1)$, $R_1 = R(G_1)$ and $\rho_1 = \rho(G_1)$ as defined in the previous subsection. In addition, consider the radius $R_2 = R(G_2)$ and the corresponding identity neighborhoods

$$V_i = \exp\{X \in \mathfrak{g}_i \, : \, \|X\| \le R_i\} \subset G_i$$

for $i \in \{1, 2\}$ and $V = V_1 \times V_2 \subset G$. Consider the map

$$\Phi : \mathrm{Cl}(G) \to \mathrm{Cl}(G_1), \quad \Phi : A \mapsto \mathrm{pr}_1(A \cap (G_1 \times V_2)) \quad \forall A \in \mathrm{Cl}(G).$$

The map Φ is G_1-equivariant and Borel.

We consider the G-invariant Chabauty open subset $\mathrm{Sub}_d(G)$ of $\mathrm{Sub}(G)$ consisting of all discrete subgroups, and its G_1-invariant open subset

$$\mathcal{X}_1 = \{\Gamma \in \mathrm{Sub}_d(G) \, : \, \Gamma \cap (\{e_1\} \times V_2) = \{(e_1, e_2)\}\}.$$

As V is an identity neighborhood of the radius provided by Lemma 9.7 (i.e. V is a Zassenhaus neighborhood), we get that $\Phi(\mathcal{X}_1) \subset \mathrm{Cl}^Z_d(G_1)$. We can further apply the function $\mathcal{J}$ to this image. We obtain the function

$$\mathcal{L} : \mathcal{X}_1 \to (0, \rho_1], \quad \mathcal{L} = \mathcal{J} \circ \Phi : \mathcal{X}_1.$$

While the map Φ is not continuous, it turns out that the composed map $\mathcal{L}$ is. Moreover, we get the following.

Theorem 9.11. *$\mathcal{L}^{-\delta_1} : \mathcal{X}_1 \to [\rho_1^{-\delta_1}, \infty)$ is a μ_{s_1}-Margulis function.*

Proof. The fact that the map $\mathcal{L}^{-\delta_1}$ is continuous and proper follows is the same way as the proof of the same facts for the map $\mathcal{J}^{-\delta_1}$ in Theorem 9.10. The integral inequality in Equation (9.1) follows from Theorem 9.10 combined with the G_1-equivariance of the map Φ. □

Corollary 9.12. *Let ν be a discrete irreducible invariant random subgroup of the semisimple Lie group $G = G_1 \times G_2$. Let $f_n \in L^2(\mathrm{Sub}(G), \nu)$ be an asymptotically G_1-invariant sequence of* unit *vectors. Then any weak-$*$ accumulation point of the probability measures $|f_n|^2 \cdot \nu \in \mathrm{Prob}(\mathrm{Sub}(G))$ is almost surely discrete.*

Proof. We have that $\nu(\mathcal{X}_1) = 1$ by the assumption that the invariant random subgroup ν is irreducible. The desired conclusion follows immediately from Corollary 9.6 applied to $\mathcal{L}^{-\delta_1}$, which is a Margulis function by Theorem 9.11. ☐

10. Strongly confined subgroups of semisimple groups

Throughout this section, let G be a connected semisimple Lie group of real rank at least two, without compact factors and with trivial center. In terms of the terminology introduced in §5, this means that G is a standard semisimple group of type ∞ and of rank at least two. By Zariski topology on G we will refer to its real Zariski topology. Our main goal will be to prove the following result.

Theorem 10.1. *Let Λ be a discrete subgroup of G. Then Λ is an irreducible lattice in G if and only if the following two conditions hold:*

(1) the subgroup Λ is strongly confined, and
(2) no pair of non-trivial normal subgroups $H_1, H_2 \leq G$ has

$$H_1 \cap H_2 = \{e\}, \quad \overline{\Lambda \cap H_1}^{\mathrm{Z}} = H_1 \quad \text{and} \quad \overline{\Lambda \cap H_2}^{\mathrm{Z}} = H_2.$$

As will be explained below, Theorem 10.1 is stronger than and immediately implies Theorem 1.4 stated in the introduction. Note that if the Lie group G is simple (rather semisimple) then it has Kazhdan's property (T) so that Theorem 10.1 follows from the main result of [FG23].

Local rigidity of irreducible lattices.

Definition 10.2. A given subgroup $\Lambda \leq G$ is *Chabauty locally rigid* if it admits a Chabauty neighborhood $\Lambda \in \Omega \subset \mathrm{Sub}(G)$ such that any subgroup $\Lambda \in \Omega$ is in fact conjugate to Λ in G.

This notion was introduced and studied in [GL18], where "classical" local rigidity [Sel60, Cal61, Wei62] was leveraged to obtain the following.

Theorem 10.3 (Theorem 1.10 of [GL18])**.** *Every irreducible lattice Γ in the semisimple Lie group G is Chabauty locally rigid.*

Stationary limits and random subgroups. Let $\mu = \mu_s$ be the symmetric compactly supported probability measure on the Lie group G considered in [GLM22] and discussed explicitly in the above §9.

Definition 10.4. A *μ-stationary limit* of a given discrete subgroup $\Lambda \leq G$ is any weak-$*$ accumulation point ν of the sequence of Cesáro averages $\frac{1}{n}\sum_{1=1}^{n} \mu^{*n} * \delta_\Lambda$ in the space $\mathrm{Prob}(\mathrm{Sub}(G))$.

It is important to note that if ν is any μ-stationary limit of a given discrete subgroup Λ then $\mathrm{supp}(\nu) \subset \overline{\Lambda^G}$. In particular ν-almost every subgroup is a conjugate limit of Λ.

Any μ-stationary limit of a discrete subgroup of the semisimple Lie group G is almost surely discrete by [FG23, Theorem 2.2] (see also [GLM22, Corollary 1.6]).

The following deep stiffness result provides a detailed classification of discrete μ-stationary random subgroups of semisimple Lie groups. It relies on the celebrated structure theory of Nevo and Zimmer [NZ99, NZ02].

Theorem 10.5 (Theorem 6.5 of [FG23])**.** *Let ν be an ergodic discrete μ-stationary random subgroup of the semisimple Lie group G. Then the group G decomposes as a product of three semisimple factors $G = G_{\mathcal{I}} \times G_{\mathcal{H}} \times G_{\mathcal{T}}$ such that*

(1) ν projects to an invariant random subgroup in $G_{\mathcal{I}}$ for which all the irreducible factors are of rank at least 2,
(2) $G_{\mathcal{H}}$ is a product of rank one factors and ν projects discretely to every factor of $G_{\mathcal{H}}$, and
(3) ν projects trivially to $G_{\mathcal{T}}$.

Furthermore, the intersection of ν-almost every subgroup with every simple factor of $G_{\mathcal{H}}$ as well as with every irreducible factor of $G_{\mathcal{I}}$ is Zariski-dense in that factor.

The notion of an *irreducible factor* of an invariant random subgroup is introduced on [FG23, p. 401]. Namely, given an ergodic discrete invariant random subgroup ν of G, there is a factor decomposition $G = H_1 \times \cdots \times H_k$ such that ν-almost every subgroup projects to each H_i discretely but projects to each proper factor of H_i densely, see [FG23, Theorem 4.1]. These H_i's are the irreducible factors corresponding to ν.

Remark 10.6. *Unfortunately, we are not aware of a local field version of Nevo and Zimmer work* [NZ99, NZ02] *in the existing literature. While we do not expect this to be a significant obstacle, in the current state of affairs we consider only real Lie groups in §10.*

We mention a deep result which plays a crucial role in our analysis. Recall that an invariant random subgroup is called *irreducible* if every non-trivial normal subgroup is acting ergodically.

Theorem 10.7 (Stuck–Zimmer [SZ94], Hartman–Tamuz [HT16])**.** *Let ν be a non-trivial irreducible invariant random subgroup of the semisimple Lie group G. Then ν-almost every subgroup is coamenable.*

Here is a closely related statement.

Corollary 10.8. *Let ν be a non-trivial irreducible invariant random subgroup of the semisimple Lie group G. Assume that the G-space* $(\mathrm{Sub}(G), \nu)$ *has spectral gap. Then ν-almost every subgroup is a lattice.*

Proof. Since the action of G on $(\mathrm{Sub}(G), \nu)$ has spectral gap it cannot be properly ergodic. This fact is a consequence of Theorem 10.7. We refer to [Cre17, Proposition 7.6] or [Lev20, Theorem 3] for the full details concerning this implication. Therefore the action of G on $(\mathrm{Sub}(G), \nu)$ is essentially transitive. As such, the stabilizer (i.e. the normalizer) of ν-almost every subgroup Λ is a lattice [SZ94, Lemma 3.5]. In other words ν-almost every subgroup is a non-trivial normal subgroup of a lattice. We conclude by the Margulis normal subgroup theorem (or by our Theorem 1.1). $\square$

We remark that the standing higher rank assumption is crucial both in Theorem 10.7 and in its Corollary 10.8.

From strongly confined to coamenable. Recall that G is a connected, center-free semisimple Lie group without compact factors and of rank at least two. Let us recall the following notion (introduced in Definition 1.3 of the introduction in a more general setting).

Definition 10.9. A subgroup Λ of G is *irreducibly confined* if Λ is strongly confined (in the sense of Definition 7.15) and furthermore the intersection $\Lambda \cap H$ is trivial for any non-trivial proper normal subgroup $H \lhd G$.

Remark 10.10. *The two notions of confined and strongly confined are clearly closed with respect to the Chabauty topology. We do not know if the notion of irreducibly confined is closed in general. It is closed at least in the case where the group G is a direct product of rank one simple factors (as follows from Corollary 6.5 combined with Lemma 7.17).*

Theorem 10.11. *Let $\Delta \leq G$ be a strongly confined discrete subgroup. Assume that there is no pair of non-trivial normal subgroups $H_1, H_2 \leq G$ such that $H_1 \cap H_2 = \{e\}$, $\overline{\Delta \cap H_1}^{\mathrm{Z}} = H_1$ and $\overline{\Delta \cap H_2}^{\mathrm{Z}} = H_2$. Then there is a discrete irreducible invariant random subgroup ν of the group G with $\mathrm{supp}(\nu) \subset \overline{\Delta^G}$. Further ν-almost every subgroup is irreducibly confined.*

Proof. Let ν be any μ-stationary limit of the subgroup Δ. We know that ν-almost every subgroup is discrete by [FG23, Theorem 1.6]. Up to replacing ν by a generic ergodic component, we may assume that ν itself is ergodic.

We will now use the stiffness result (Theorem 10.5) and its notation to analyze the resulting discrete ergodic μ-stationary random subgroup ν. The fact that the factor $G_{\mathcal{T}}$ is trivial follows as the subgroup Δ is strongly confined.

We claim that $G = G_{\mathcal{I}}$ and that $G_{\mathcal{I}}$ does not have proper non-trivial irreducible factors. Indeed, in any other situation, either the factor $G_{\mathcal{H}}$ will be non-trivial or the factor $G_{\mathcal{I}}$ will have more than a single irreducible factor. In both cases, the group G itself must be semisimple but not simple, and it can we written as a non-trivial direct product $G = G_1 \times G_2$ in such a way that ν-almost every discrete subgroup Λ satisfies $\overline{\Lambda \cap G_1}^{\mathrm{Z}} = G_1$ as well as $\overline{\Lambda \cap G_2}^{\mathrm{Z}} = G_2$. Since ν-almost every subgroup is a conjugate limit of the subgroup Δ, this would lead to a contradiction to part (2) of Corollary 6.5.

To conclude, it follows from the stiffness result (Theorem 10.5) that ν is an irreducible invariant random subgroup. The fact that ν-almost every subgroup is strongly confined follows because ν-almost every subgroup is a conjugate limit of Δ. Lastly ν-almost every subgroup intersects trivially every proper semisimple factor by the irreducibility of ν, and as such is irreducibly confined. □

The above result has the following statement as an immediate special case.

Corollary 10.12. *Let $\Delta \leq G$ be an irreducibly confined discrete subgroup. Then there is a non-trivial discrete irreducible invariant random subgroup ν of the group G with $\mathrm{supp}(\nu) \subset \overline{\Delta^G}$.*

Further, we obtain the following result, which allows us to go from irreducibly confined subgroups to coamenable ones.

Corollary 10.13. *Let $\Delta \leq G$ be an irreducibly confined discrete subgroup. Then Δ admits a coamenable conjugate limit in $\mathrm{Sub}(G)$.*

Proof. According to Corollary 10.12 the group Δ admits a non-trivial discrete irreducible invariant random subgroup ν with $\operatorname{supp}(\nu) \subset \overline{\Delta^G}$. We know that ν-almost every subgroup is coamenable in G by Theorem 10.7. This concludes the proof. □

From a coamenable subgroup to a lattice. As before, recall that G is a connected, center-free higher rank semisimple Lie group without compact factors. The following result can be regarded as a variant of the Stuck–Zimmer theorem for higher rank semisimple Lie groups [SZ94] without assuming Kazhdan's property (T) but with the added assumption of "strongly confined".

Theorem 10.14. *Let ν be a discrete irreducible invariant random subgroup of the semisimple Lie group G. Assume that ν-almost every subgroup is strongly confined. Then ν-almost every subgroup is a lattice in G.*

Proof. In view of Corollary 10.8 it is enough to show that the G-action on $(\operatorname{Sub}(G), \nu)$ has a spectral gap, namely, the unitary representation $L^2_0(\operatorname{Sub}(G), \nu)$ does not almost have invariant vectors. We proceed to show that. If the group G has Kazhdan's property (T) then this is immediate, thus we assume as we may that it does not. As such, the group G can be written as a direct product $G = G_1 \times G_2$ where G_1 is some (non-trivial) semisimple Lie group and G_2 is a simple Lie group of real rank one. We will conclude by using our spectral gap Theorem 6.10 applied with respect to the space $(\operatorname{Sub}(G), \nu)$. It remains to verify its conditions.

- The fact that $L^2_0(\operatorname{Sub}(G), \nu)^{G_2} = 0$ follows from the irreducibility of ν.
- The stabilizer in G of any subgroup $\Lambda \in \operatorname{Sub}(G)$ is the normalizer $\mathrm{N}_G(\Lambda)$. By irreducibility the factor G_2 is ergodic. Hence $\mathrm{N}_G(\Lambda) \cap G_1 = \mathrm{N}_{G_1}(\Lambda)$ is ν-almost surely constant. This constant subgroup is G_1-invariant, i.e. a normal subgroup of G_1. Since Λ is not contained in any proper factor of G, it follows that $\mathrm{N}_{G_1}(\Lambda) = \{e\}$. Similar reasoning shows that $\mathrm{N}_{G_2}(\Lambda) = \{e\}$.

We are left to verify the assumption in the third bullet of Theorem 6.10. Consider any asymptotically G_1-invariant sequence of unit vectors $f_n \in L^2_0(\operatorname{Sub}(G), \nu)$ and let $\eta \in \operatorname{Prob}(\operatorname{Sub}(G))$ be an accumulation point of the sequence of probability measures $\operatorname{Stab}_*(|f_n|^2 \cdot \nu)$ (where the stabilizer map $\operatorname{Stab} : \operatorname{Sub}(G) \to \operatorname{Sub}(G)$ is just the normalizer map). We need to show that η-almost every subgroup is discrete, not contained in the factor G_2 and admits Zariski dense[6] projections to G_2. These three properties follow respectively from Corollary 9.12, the assumption that ν-almost every subgroup is strongly confined and Lemma 7.17. To be precise, Corollary 9.12 is to be applied with respect to the invariant random subgroup $\zeta = \operatorname{Stab}_*\nu$ and a corresponding asymptotically G_1-invariant sequence of unit vectors $g_n \in L^2(G, \zeta)$ such that $|g_n|^2 \cdot \zeta = \operatorname{Stab}_*(|f_n|^2 \cdot \nu)$. □

We are ready to prove the main result of §10.

Proof of Theorem 10.1. Every irreducible lattice is strongly confined by Lemma 7.16. In addition, every irreducible lattice is irreducibly confined, for it intersects trivially all proper factors. This conclusion is stronger than that in statement of Theorem 10.1.

[6]The statement of Theorem 6.10 requires these projections to G_2 to be Zariski-dense and and not relatively compact. However, Zariski density implies not relatively compact when working with real Lie groups.

We now show the converse direction, which is more interesting. Let Λ be a strongly confined discrete subgroup of G satisfying the conditions in the statement of the theorem, namely that there no pair of commuting non-trivial factors of G both admitting Zariski dense intersections with Λ. By Theorem 10.11 there exists a non-trivial discrete irreducible invariant random subgroup ν of the group G with $\mathrm{supp}(\nu) \subset \overline{\Lambda^G}$. In particular ν-almost every subgroup is irreducibly confined. According to Theorem 10.14 the invariant random subgroup ν arises from some irreducible lattice Γ in the group G (i.e. gives full measure to its conjugates). In particular, this irreducible lattice Γ is a conjugate limit of Λ. By the Chabauty local rigidity of irreducible lattices (Theorem 10.3) we conclude that the subgroup Λ itself must be an irreducible lattice, as required. □

Proof of Theorem 1.4 of the introduction. This follows immediately from Theorem 10.1. Indeed, note that any irreducibly confined discrete subgroup has trivial intersections with all proper factors, and as such certainly satisfies the (weaker) conditions of Theorem 10.1. □

Proof of Corollary 1.7 of the introduction. Let (X, m) be a strongly irreducible[7] probability measure preserving action of the semisimple Lie group G. Up to passing to a generic ergodic component, we may assume without loss of generality that (X, m) is ergodic. Strong irreducibility and ergodicity implies irreducibility by [FG23, Corollary 7.3], so that the action on (X, m) is irreducible. As in the first paragraph of the proof of Theorem 10.14, it will suffice to prove that the action (X, m) has spectral gap, and assume that G is written as $G = G_1 \times G_2$ where G_1 is a non-trivial semisimple Lie group and G_2 is a simple Lie group of real rank one. We will deduce spectral gap directly from Theorem 4.8. Let us verify the assumptions of that theorem.

- The fact that $L^2_0(X, m)^{G_2} = 0$ follows from the irreducibility of the action.
- Consider some asymptomatically G_1-invariant sequence of unit vectors $f_n \in L^2(X, m)$ and let $\mu \in \mathrm{Prob}(\mathrm{Sub}(G))$ be an accumulation point of the sequence of probability measures $\mathrm{Stab}_*(|f_n|^2 \cdot m) \in \mathrm{Prob}(\mathrm{Sub}(G))$. Note that μ-almost every subgroup is discrete by Corollary 9.12. As μ-almost every subgroup Λ is a conjugate limit of some m-generic subgroup, it satisfies $\overline{G_1 \Lambda} = G$ by the strong irreducibility assumption.

This verifies the assumptions of Theorem 4.8 and thereby concludes the proof. □

Products of general locally compact groups. In this final subsection, we deviate from the standing assumptions of §10, and let $G = G_1 \times G_2$ be a direct product of two locally compact second countable groups. Assume that G_2 has a compact abelianization. We prove Theorem 1.5 of the introduction. It says that under certain irreducibility conditions, a coamenable discrete subgroup of G must be a lattice.

Proof of Theorem 1.5. Let $\Lambda \leq G$ be a discrete coamenable subgroup, such that there are no G_2-invariant vectors in $L^2_0(G/\Lambda)$ and that every conjugate limit of Λ has dense projections to the factor G_2. It follows from Theorem 4.8 that the unitary G-representation $L^2_0(G/\Lambda)$ has a spectral gap. Since Λ is coamenable this must

[7]Recall that in the introduction we defined a discrete subgroup to be irreducible if it projects densely to each proper factor. We defined a (discrete) subgroup to be strongly irreducible if every discrete conjugate limit of it irreducible.

mean that Λ is a lattice (i.e. this situation is only possible provided $L^2_0(G/\Lambda)$ is a proper subrepresentation of $L^2(G/\Lambda)$. □

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