Asymptotic Properties of Generalized Shortfall Risk Measures for Heavy-tailed Risks

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May 4, 2023

Abstract

We study a general risk measure called the generalized shortfall risk measure, which was first introduced in Mao and Cai (2018). It is proposed under the rank-dependent expected utility framework, or equivalently induced from the cumulative prospect theory. This risk measure can be flexibly designed to capture the decision maker's behavior toward risks and wealth when measuring risk. In this paper, we derive the first- and second-order asymptotic expansions for the generalized shortfall risk measure. Our asymptotic results can be viewed as unifying theory for, among others, distortion risk measures and utility-based shortfall risk measures. They also provide a blueprint for the estimation of these measures at extreme levels, and we illustrate this principle by constructing and studying a quantile-based estimator in a special case. The accuracy of the asymptotic expansions and of the estimator is assessed on several numerical examples.

Keywords: Generalized shortfall risk measure, Asymptotic expansions, Heavy tails, Estimation

1 Introduction

In this paper we study extreme value properties of a general risk measure, called the generalized shortfall risk measure and defined as follows. Let u_1, u_2 be (strictly) increasing functions on $\mathbb{R}_+$ with $u_1(0) = u_2(0) = 0$, and h_1 and h_2 be two distortion functions on $[0, 1]$, supposed to be right-continuous and increasing throughout, such that $h_i(0) = 0$ and $h_i(1) = 1$ with no jumps at 0 and

1. For a random variable X with distribution function F, the generalized shortfall risk measure, denoted by $x_\tau = x_\tau(X, u_1, h_1, u_2, h_2)$, is defined as the solution to the following equation:

$$\tau \,\mathrm{H}_{u_1,h_1}((X-x)_+) = (1-\tau)\,\mathrm{H}_{u_2,h_2}((X-x)_-), \tag{1.1}$$
$$\text{where } \mathrm{H}_{u_1,h_1}((X-x)_+) = \int_x^{\infty} u_1(y-x)\,\mathrm{d}h_1(F(y)),$$
$$\text{and } \mathrm{H}_{u_2,h_2}((X-x)_-) = \int_{-\infty}^{x} u_2(x-y)\,\mathrm{d}h_2(F(y)).$$

This problem is written under the appropriate regularity and integrability assumptions making both sides in (1.1) finite and ensuring that the solution is indeed unique; see Section 3 below for a discussion. The generalized shortfall risk measure was first introduced in Mao and Cai (2018) as an extension of the generalized quantile risk measure. The quantile of random variable X, or generalized (left-continuous) inverse function at level $\tau \in (0,1)$, or Value-at-Risk (VaR), is defined as $F^{\leftarrow}(\tau) = \inf\{x \in \mathbb{R}, F(x) \geqslant \tau\}$. It is well known that $F^{\leftarrow}(\tau)$ can also be represented as

$$F^{\leftarrow}(\tau) = \underset{x\in\mathbb{R}}{\arg\min}\left\{\tau\mathbb{E}[(X-x)_+] + (1-\tau)\mathbb{E}[(X-x)_-]\right\},$$

where $x_+ = \max\{x, 0\}$ and $x_- = \max\{-x, 0\} = -\min\{x, 0\}$, provided $\mathbb{E}|X| < \infty$. By transforming the shortfall risk $(X-x)_+$ to $\phi_1((X-x)_+)$ and the (in the terminology of Mao and Cai, 2018) over-required capital risk $(X-x)_-$ to $\phi_2((X-x)_-)$, where ϕ_1, ϕ_2 are increasing convex functions, the generalized quantile was proposed in Bellini et al. (2014) as

$$\underset{x\in\mathbb{R}}{\arg\min}\left\{\tau\mathbb{E}[\phi_1((X-x)_+)] + (1-\tau)\mathbb{E}[\phi_2((X-x)_-)]\right\}. \tag{1.2}$$

When $\phi_1(x) = \phi_2(x) = x^2$, the generalized quantile (1.2) reduces to the well-known expectile, which was proposed in Newey and Powell (1987). Since both the shortfall risk and over-required capital risk are evaluated under the original probability measure, the generalized quantile is defined in the sense of the classical expected utility. Mao and Cai (2018) further generalized it by using the rank-dependent expected utility (RDEU) to evaluate risks and wealth (see e.g. Quiggin, 1993). To be more specific, letting ϕ_1 and ϕ_2 be two nondegenerate increasing convex functions on $[0,\infty)$, the so-called generalized quantile based on RDEU theory is defined as

$$\underset{x\in\mathbb{R}}{\arg\min}\left\{\tau\mathrm{H}_{\phi_1,h_1}((X-x)_+) + (1-\tau)\mathrm{H}_{\phi_2,h_2}((X-x)_-)\right\}. \tag{1.3}$$

In Mao and Cai (2018), Proposition 2.2 (ii) shows that if $u_1(x) = \phi_1'(x)$ and $u_2(x) = \phi_2'(x)$, then the generalized quantile based on RDEU theory defined in (1.3) coincides with the generalized shortfall risk measure in (1.1). This shows that besides the utility functions that can be selected, the generalized shortfall risk measure allows decision makers to choose the appropriate distorted probability measure to describe their behavior towards risks and wealth. This brings great flexibility in measuring the risk.

Further, Theorem 3.1 of Mao and Cai (2018) showed that the generalized shortfall risk measure is equivalent to the so-called generalized shortfall induced by cumulative prospect theory (CPT) defined in (1.4) below when v, h_1 and h_2 are chosen properly. CPT was proposed by Tversky and Kahneman (1992) and has been applied in various areas such as portfolio selection and pricing insurance contracts; see *e.g.* Schmidt and Zank (2007), Kaluszka and Krzeszowiec (2012a), Kaluszka and Krzeszowiec (2012b) and Jin and Zhou (2013). For an increasing continuous function v on $\mathbb{R}$, the generalized shortfall induced by CPT is defined as

$$\inf\{x \in \mathbb{R}, H_{v,h_1,h_2}(X - x) \leqslant 0\}, \tag{1.4}$$

where

$$H_{v,h_1,h_2}(X) = \int_{-\infty}^{0} v(y)\,\mathrm{d}h_1(F(y)) + \int_{0}^{\infty} v(y)\,\mathrm{d}h_2(F(y)).$$

The generalized shortfall risk measure understood in the form of (1.4) contains utility-based shortfall risk measures (Föllmer and Schied, 2016) as special cases.

The study of tail risks and their disastrous consequences in finance has attracted substantial attention and many empirical studies have shown that asset returns in finance and large losses in insurance exhibit heavy tails: see, for example, Loretan and Phillips (1994), Gabaix et al. (2003), and Gabaix (2009). Moreover, regulators such as Basel III have recommended to estimate VaR with a confidence level very close to 1. In the same spirit, we are interested in the behavior of the generalized shortfall risk measure for heavy-tailed risks when the confidence level τ is close to 1. However, since the generalized shortfall risk measure extends in particular the expectile, for which no closed form is available in general, no simple explicit expression of x_τ is available, which makes the study of the risk measure for heavy-tailed risks difficult. Asymptotic expansions of risk measures, in terms of the quantile of the random variable of interest (viewed as a well-understood risk measure), provide an intuitive way to study extreme risk measures for heavy-tailed risks; see for example, the asymptotic expansions of the Haezendonck–Goovaerts risk measure in Tang and Yang (2012) and Mao and Hu (2012), the conditional tail expectation in Hua and Joe (2011) and Hua and Joe (2014), the expectiles in Bellini et al. (2014) and Mao et al. (2015), the risk concentration based on expectiles in Mao and Yang (2015).

It is precisely the objective of this paper to study the first- and second-order asymptotic expansions of x_τ for a heavy-tailed random variable X as the confidence level τ converges to 1. From the technical point of view, the methodology used in this paper to derive the expansions is very general, in the sense that it can be applied to derive the asymptotic expansions of other quantile-based risk measures. A potential statistical benefit of such results is that, while the lack of a simple explicit expression of x_τ makes the estimation and practical use of x_τ difficult, an asymptotic expansion in terms of extreme quantiles is helpful in studying the asymptotic behavior of simple plug-in estimators of x_τ at extreme levels (see the so-called indirect estimator of Daouia et al., 2018). Such expansions also allow to quantify bias terms and are fundamental in the derivation of asymptotic normality results for estimators at extreme levels. This estimation approach of an extreme risk measure has been adopted for, among others, the estimation of the marginal expected shortfall in Cai et al. (2015), expectiles in Daouia et al. (2018), M-quantiles in Daouia et al. (2019) and the Haezendonck–Goovaerts risk measure in Zhao et al. (2021). We illustrate that in this article by studying the asymptotic properties of an estimator of x_τ (with $\tau = \tau_n \uparrow 1$ as the size n of the available sample of data tends to infinity) based on extreme quantiles of a distorted version of the underlying distribution. Our high-level result may be valid even when serial dependence is present in the data, as long as the observations come from a strictly stationary sequence.

The rest of the paper is organized as follows. Section 2 provides necessary technical background on regular variation. In Sections 3 and 4, we derive the first- and second-order expansions of the generalized shortfall risk measure, respectively. Section 5 discusses the estimation of the generalized shortfall risk measure at extreme levels. In Section 6, we give a couple of examples where our theory applies and we discuss a small-scale simulation study illustrating the performance of our estimator. All the proofs are relegated to Section 7.

2 Regular variation

We start by introducing regular variation conditions that will be the backbone of our model on risk variables.

Definition 2.1. An eventually nonnegative measurable function $f(\cdot)$ is said to be *regularly varying* at ∞ with index $\alpha \in \mathbb{R}$, if for all $x > 0$,

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\alpha. \tag{2.1}$$

We write $f(\cdot) \in \mathrm{RV}_\alpha$.

For a random variable X with distribution function F, we say that X is regularly varying with extreme value index $\gamma > 0$ if its survival function $\overline{F} = 1 - F$ is regularly varying with index $-1/\gamma$. This is also denoted by $X \in \mathrm{RV}_{-1/\gamma}$.

For a distribution function F, its (left-continuous inverse) quantile function is defined as $F^{\leftarrow}(p) = \inf\{x \in \mathbb{R} : F(x) \geqslant p\}$ for $p \in (0,1)$. The *tail quantile function* $U(\cdot)$ of F is defined as

$$U(t) = \left(\frac{1}{\overline{F}}\right)^{\leftarrow}(t) = F^{\leftarrow}\left(1 - \frac{1}{t}\right), \quad t > 1.$$

The RV definition of a survival function $\overline{F} = 1 - F$ can be equivalently presented in terms of the tail quantile function U by requiring that $U \in \mathrm{RV}_{\gamma}$, that is,

$$\forall x > 0, \ \lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^{\gamma}.$$

This assumption connects tail quantiles to arbitrarily extreme quantiles further away in the right tail through the approximation $F^{\leftarrow}(p') \approx [(1-p')/(1-p)]^{-\gamma} F^{\leftarrow}(p)$, for $0 < p < p' < 1$ both close to 1.

In practice it is necessary to quantify the bias incurred through the use of this approximation. This is typically done thanks to a second-order regular variation condition, itself most easily written using the concept of extended regular variation, which we recall below.

Definition 2.2. A measurable function $f(\cdot)$ on $(0, \infty)$ is said to be *extended regularly varying* at ∞, with an index $\gamma \in \mathbb{R}$ and an auxiliary function $a(\cdot)$ having constant sign, if for all $x > 0$,

$$\lim_{t \to \infty} \frac{f(tx) - f(t)}{a(t)} = \frac{x^{\gamma} - 1}{\gamma}. \tag{2.2}$$

When $\gamma = 0$, the limit $(x^{\gamma} - 1)/\gamma$ is understood as $\log x$. Denote this by $f(\cdot) \in \mathrm{ERV}_{\gamma}$.

Compared to the definition of extended regular variation in Section B.2 of de Haan and Ferreira (2006), we absorb the potential multiplicative constant appearing in the right-hand side into the function a. This results in a simpler limit, but the auxiliary function is allowed to be negative.

This allows us to introduce second-order regular variation as a special case of extended regular variation through the following definition.

Definition 2.3. A regularly varying function $f(\cdot)$ is said to be *second-order regularly varying* at ∞ with first-order index $\gamma \in \mathbb{R}$ and second-order index $\rho \leqslant 0$, if there exists a measurable function $A(\cdot)$, which does not change sign eventually and converges to 0, such that $t \mapsto t^{-\gamma} f(t) \in \mathrm{ERV}_{\rho}$

with auxiliary function $a : t \mapsto t^{-\gamma} f(t)A(t)$. In other words,

$$\lim_{t\to\infty} \frac{f(tx)/f(t) - x^\gamma}{A(t)} = x^\gamma \frac{x^\rho - 1}{\rho} =: J_{\gamma,\rho}(x). \tag{2.3}$$

When $\rho = 0$, $J_{\gamma,\rho}(x)$ is understood as $x^\gamma \log x$. We write $f(\cdot) \in 2\mathrm{RV}_{\gamma,\rho}$ and A is called (second-order) auxiliary function.

It is worth noting that each of the convergences in (2.1), (2.2) and (2.3) is uniform with respect to x in any compact subset of $(0, \infty)$: see, for example, Theorem B.1.4 and Theorem B.2.9 of de Haan and Ferreira (2006). In our specific context of heavy-tailed distributions, Theorem 2.3.9 of de Haan and Ferreira (2006) shows that for $\gamma > 0$ and $\rho \leqslant 0$, $U(\cdot) \in 2\mathrm{RV}_{\gamma,\rho}$ with an auxiliary function $A(\cdot)$ if and only if $\overline{F}(\cdot) \in 2\mathrm{RV}_{-1/\gamma,\rho/\gamma}$ with an auxiliary function $A(1/\overline{F}(\cdot))$. In this case necessarily $A(\cdot) \in \mathrm{RV}_\rho$.

Lastly, we present two useful expansions of a 2RV function and its inverse function, when the second-order parameter is negative; see the proof of Lemma 3 in Hua and Joe (2011) for Lemma 2.1 (i) and Proposition 2.5 of Mao and Hu (2012) for Lemma 2.1 (ii).

Lemma 2.1. *Let $\gamma \in \mathbb{R}$, $\rho < 0$ and A be a measurable function having constant sign.*

(i) Then $h \in 2\mathrm{RV}_{\gamma,\rho}$ with auxiliary function $A(\cdot)$ if and only if there exists a constant $c > 0$ such that

$$h(t) = ct^\gamma \left[1 + \frac{1}{\rho} A(t) + o(A(t))\right],\ t \to \infty.$$

(ii) Then, when $\gamma > 0$ and with the notation of (i), $h^\leftarrow$ has the following representation:

$$h^\leftarrow(t) = c^{-1/\gamma} t^{1/\gamma} \left[1 - \frac{1}{\gamma\rho} A(h^\leftarrow(t)) + o(A(h^\leftarrow(t)))\right],\ t \to \infty.$$

In particular $h^\leftarrow \in 2\mathrm{RV}_{1/\gamma,\rho/\gamma}$.

3 First-order expansions of generalized shortfall risk measures

In this section, we study the first-order asymptotics of generalized shortfall risk measures. All the proofs are relegated to Section 7.

The first key observation is that, in the heavy-tailed setting, the risk measure x_τ is an increasing function of τ and diverges to $+\infty$. Along with mild conditions for existence and uniqueness of x_τ as a solution of (1.1), this is the essential message of the following result, in which we say that a random variable X is nondegenerate if it is not constant.

Proposition 3.1. *Let* $x_\star = \inf\{x \in \mathbb{R}, F(x) > 0\}$ *and* $x^\star = \inf\{x \in \mathbb{R}, F(x) \geqslant 1\}$ *denote the left and right endpoints of* X*, assumed to be a nondegenerate random variable, so that* $x_\star < x^\star$*.*

(i) If the quantities $\mathrm{H}_{u_1,h_1}((X-x)_+)$ *and* $\mathrm{H}_{u_2,h_2}((X-x)_-)$ *define continuous finite functions of* $x \in (x_\star, x^\star)$ *then, for any* $\tau \in (0,1)$*, Equation* (1.1) *has a unique and finite solution* x_τ*.*

(ii) If the quantities $\mathrm{H}_{u_1,h_1}((X-x)_+)$ *and* $\mathrm{H}_{u_2,h_2}((X-x)_-)$ *are finite when* $x \in (x_\star, x^\star)$ *and Equation* (1.1) *has a unique and finite solution* x_τ*, then* $x_\tau < x^\star$ *when* $\tau < 1$*,* $\tau \in (0,1) \mapsto x_\tau \in \mathbb{R}$ *is nondecreasing, and* $\lim_{\tau\uparrow 1} x_\tau = x^\star$*.*

(iii) Suppose that:

- $\overline{F} \in \mathrm{RV}_{-1/\gamma}$ *with* $\gamma > 0$*,*
- u_1 *is continuous on* $[0,\infty)$ *and* $u_1 \in \mathrm{RV}_{\alpha_1}$ *with* $\alpha_1 > 0$*,*
- $1 - h_1(1 - 1/\cdot) \in \mathrm{RV}_{-\beta_1}$ *with* $\beta_1 > 0$*,*
- u_2 *is continuous on* $[0,\infty)$ *and* $u_2 \in \mathrm{RV}_{\alpha_2}$ *with* $\alpha_2 > 0$*.*

Assume further that $\beta_1/\gamma > \alpha_1$ *and* $\int_{-\infty}^{\infty} |z|^{\alpha_2+\delta}\, \mathrm{d}h_2(F(z)) < \infty$ *for some* $\delta > 0$*. Then* x_τ *exists and is unique for any* $\tau \in (0,1)$*,* $\tau \in (0,1) \mapsto x_\tau \in \mathbb{R}$ *is nondecreasing, and* $\lim_{\tau\uparrow 1} x_\tau = +\infty$*.*

Remark 3.1. Under the regular variation conditions in (iii) above, $\beta_1/\gamma \geqslant \alpha_1$ is a necessary condition for the existence of $\mathrm{H}_{u_1,h_1}((X-x)_+)$. Indeed

$$\mathrm{H}_{u_1,h_1}((X-x)_+) = -\int_{z=0}^{\infty} u_1(z)\, \mathrm{d}(1 - h_1(1 - 1/(1/\overline{F}(x+z)))).$$

Since $u_1 \in \mathrm{RV}_{\alpha_1}$ and $1 - h_1(1 - 1/(1/\overline{F}(x+\cdot))) \in \mathrm{RV}_{-\beta_1/\gamma}$, it follows that $\mathrm{H}_{u_1,h_1}((X-x)_+)$ can only be finite if $\alpha_1 - \beta_1/\gamma \leqslant 0$, that is, $\beta_1/\gamma \geqslant \alpha_1$.

Besides, the assumption that $\int_{-\infty}^{\infty} |z|^{\alpha_2+\delta}\, \mathrm{d}h_2(F(z)) < \infty$ is required to control the left tail behavior of the function F. An inspection of the proof of Lemma 3.1(ii) reveals that this assumption can be weakened to $\int_{-\infty}^{\infty} |z|^{\alpha_2}\, \mathrm{d}h_2(F(z)) < \infty$ if $u_2(y)$ is asymptotically equivalent to a multiple of y^{α_2} as $y \to \infty$. For $u_2(y) = y$ and $h_2(x) = x$, we find the condition $\mathbb{E}(|X|) < \infty$, which is exactly the condition necessary and sufficient for the existence of expectiles.

In the remainder of this paper we implicitly assume that the problem of which x_τ is solution is indeed well-defined and has a unique finite solution; by Proposition 3.1, if appropriate regular variation conditions on the functions involved are met, this will be guaranteed by simply assuming that u_1 and u_2 are continuous on $[0,\infty)$. We proceed by deriving, under this regularity assumption, the asymptotic expansions of each side of (1.1), as $x \to \infty$.

Lemma 3.1. *Assume that $\overline{F} \in \mathrm{RV}_{-1/\gamma}$ with $\gamma > 0$ and denote by $\mathrm{B}(\cdot,\cdot)$ the Beta function, that is, $\mathrm{B}(a,b) = \int_0^1 z^{a-1}(1-z)^{b-1}\,\mathrm{d}z$ for any $a,b>0$.*

(i) Assume $u_1 \in \mathrm{RV}_{\alpha_1}$ for $\alpha_1 > 0$, $1 - h_1(1-1/\cdot) \in \mathrm{RV}_{-\beta_1}$ with $\beta_1 > 0$, and $\beta_1/\gamma > \alpha_1$. Then

$$\lim_{x\to\infty} \frac{\mathrm{H}_{u_1,h_1}((X-x)_+)}{(1-h_1(F(x)))\,u_1(x)} = \frac{\beta_1}{\gamma}\mathrm{B}(\beta_1/\gamma - \alpha_1, \alpha_1 + 1) =: \Delta_0.$$

(ii) Assume $u_2 \in \mathrm{RV}_{\alpha_2}$ with $\alpha_2 > 0$, and $\int_{-\infty}^{\infty} |z|^{\alpha_2+\delta}\,\mathrm{d}h_2(F(z)) < \infty$ for some $\delta > 0$. Then

$$\lim_{x\to\infty} \frac{\mathrm{H}_{u_2,h_2}((X-x)_-)}{u_2(x)} = 1.$$

We are now in position to state the first-order asymptotic expansion of the generalized shortfall risk measure.

Theorem 3.1. *Assume that $u_1 \in \mathrm{RV}_{\alpha_1}$ with $\alpha_1 > 0$, $1-h_1(1-1/\cdot) \in \mathrm{RV}_{-\beta_1}$ with $\beta_1 > 0$, $u_2 \in \mathrm{RV}_{\alpha_2}$ with $\alpha_2 > 0$, $\overline{F} \in \mathrm{RV}_{-1/\gamma}$ with $\gamma > 0$. Define a function φ as*

$$\varphi(x) = \frac{u_2(x)}{u_1(x)(1-h_1(F(x)))}.$$

Then $\varphi \in \mathrm{RV}_s$, with $s = \alpha_2 - \alpha_1 + \beta_1/\gamma$. Assume henceforth that $s > 0$. Then $\varphi(x)$ diverges to $+\infty$ as $x \to +\infty$ and its generalized inverse function

$$\varphi^{\leftarrow} : q \mapsto \inf\{x : \varphi(x) \geqslant q\}, \quad q \in (0,1) \tag{3.1}$$

is well-defined. Further assume that $\beta_1/\gamma > \alpha_1$ and $\int_{-\infty}^{\infty} |z|^{\alpha_2+\delta}\,\mathrm{d}h_2(F(z)) < \infty$ for some $\delta > 0$. Then the first-order expansion of the shortfall risk measure is

$$x_\tau = \left[\frac{\beta_1}{\gamma}\mathrm{B}(\beta_1/\gamma - \alpha_1, \alpha_1+1)\right]^{1/s} \varphi^{\leftarrow}((1-\tau)^{-1})(1+o(1)) = \Delta_1 \varphi^{\leftarrow}((1-\tau)^{-1})(1+o(1)).$$

[In other words, $\Delta_1 = \Delta_0^{1/s}$, with Δ_0 defined in Lemma 3.1(i).]

Remark 3.2. When $u_1 = u_2$ and $h_1(x) = x$, the function $\varphi^{\leftarrow}$ is nothing but the tail quantile function U. In this case Theorem 3.1 directly connects x_τ to extreme quantiles of X. This is reminiscent of the kind of asymptotic proportionality relationships obtained for L^p-quantiles, see *e.g.* Daouia et al. (2019).

4 Second-order expansions of generalized shortfall risk measures

In this section, we study the second-order asymptotics of generalized shortfall risk measures. Again, all the proofs are relegated to Section 7. We first prepare a few assumptions and lemmas. The first lemma is regarding a set of uniform inequalities for 2RV functions. It plays a key role in the later proofs. Moreover, it is also an interesting result on its own as it is a complement to the usual inequalities on 2RV by providing uniform inequalities in a neighborhood of 0.

Lemma 4.1. *Assume that $g \in 2\mathrm{RV}_{\gamma,\rho}$, with $\gamma > 0$, $\rho < 0$ and auxiliary function B, is such that $t^{-\gamma} g(t)$ is bounded on intervals of the form $(0, t_0]$, with $t_0 > 0$. There exists $\widetilde{B} \sim B$ such that for any $\varepsilon > 0$ and $\delta > 0$, there exists $c > 0$ and t_0 such that for all $t \geqslant t_0$ and $0 < v < \delta$,*

$$\left| \frac{\frac{g(vt)}{g(t)} - v^\gamma}{\widetilde{B}(t)} \right| \leqslant -\frac{v^\gamma}{\rho}(1 + cv^{\rho-\varepsilon}).$$

[A fixed choice of $c > 0$ is possible for $\delta \in (0, 1)$.]

We next present the second-order conditions about U, u_1, u_2 and h_1 that we require to obtain the second-order asymptotics of x_τ.

Assumption 4.1. *$U \in 2\mathrm{RV}_{\gamma,\rho}$ for $\gamma > 0$ and $\rho < 0$ with auxiliary function $A(t)$.*

Assumption 4.2. *For $i = 1, 2$, $u_i \in 2\mathrm{RV}_{\alpha_i,\eta_i}$ for $\alpha_i > 0$ and $\eta_i < 0$ with auxiliary function $B_i(t)$, and $t^{-\alpha_i} u_i(t)$ is bounded on intervals of the form $(0, t_0]$, for $t_0 > 0$.*

Assumption 4.3. *$1 - h_1(1 - 1/\cdot) \in 2\mathrm{RV}_{-\beta_1,\varsigma}$ for $\beta_1 > 0$ and $\varsigma < 0$ with auxiliary function $C(t)$, and $1 - h_2(1 - 1/\cdot) \in \mathrm{RV}_{-\beta_2}$ for $\beta_2 > 0$.*

Under Assumptions 4.1, 4.2, and 4.3, by Lemma 2.1, and Propositions 2.6 and 2.9 in Lv et al. (2012), we immediately obtain the following useful results.

Lemma 4.2. *Under Assumptions 4.1, 4.2, and 4.3,*

(i) U has the representation, as $x \to \infty$,

$$U(x) = cx^\gamma \left[1 + \frac{1}{\rho} A(x) + o(A(x))\right], \text{ where } c > 0.$$

Consequently, $\overline{F}(\cdot) \in 2\mathrm{RV}_{-1/\gamma,\rho/\gamma}$ with auxiliary function $A_F(t) = \gamma^{-2} A(1/\overline{F}(t))$, and $\overline{F}(\cdot)$ has the representation, as $x \to \infty$,

$$\overline{F}(x) = c^{1/\gamma} x^{-1/\gamma} \left[1 + \frac{1}{\gamma\rho} A(1/\overline{F}(x)) + o(A(1/\overline{F}(x)))\right].$$

(ii) For $i = 1, 2$, u_i has the representation, as $x \to \infty$,

$$u_i(x) = a_i x^{\alpha_i} \left[1 + \frac{1}{\eta_i} B_i(x) + o(B_i(x))\right], \text{ where } a_i > 0.$$

(iii) $1 - h_1(1 - 1/\cdot)$ has the representation, as $x \to \infty$,

$$1 - h_1\left(1 - \frac{1}{x}\right) = b x^{-\beta_1} \left[1 + \frac{1}{\varsigma} C(x) + o(C(x))\right], \text{ where } b > 0.$$

(iv) $1 - h_1(F(\cdot))$ has the representation, as $x \to \infty$,

$$1 - h_1(F(x)) = b c^{\beta_1/\gamma} x^{-\beta_1/\gamma} \left[1 + \frac{\beta_1}{\gamma\rho} A(1/\overline{F}(x))(1 + o(1)) + \frac{1}{\varsigma} C(1/\overline{F}(x))(1 + o(1))\right].$$

In particular, if $C(x)/A(x) \to \kappa \in [-\infty, +\infty]$ as $x \to \infty$ with $\kappa \neq -\beta_1/\gamma$, then $A_h(\cdot) = \gamma^{-1}((\beta_1/\gamma)A(1/\overline{F}(\cdot)) + C(1/\overline{F}(\cdot)))$ is nonzero and has constant sign in a neighborhood of infinity, $|A_h(\cdot)|$ is regularly varying with index $\rho_h = \max\{\rho, \varsigma\}/\gamma$, and $1 - h_1(F(\cdot)) \in 2\mathrm{RV}_{-\beta_1/\gamma, \rho_h}$ for $\rho_h = \max\{\rho, \varsigma\}/\gamma$ with auxiliary function A_h.

(v) $1 - h_2(F(\cdot)) \in \mathrm{RV}_{-\beta_2/\gamma}$.

Remark 4.1. Condition $C(x)/A(x) \to \kappa \in [-\infty, +\infty]$ as $x \to \infty$ with $\kappa \neq -\beta_1/\gamma$ in Lemma 4.2(iv) is very mild. It is in particular satisfied as soon as $\rho \neq \varsigma$, corresponding to the case when either A or C dominates in A_h. When $\rho = \varsigma$, in typical second-order regularly varying models A and C will be proportional to the same negative power function $t \mapsto t^{\rho}$, and the condition simply says that the proportionality constants should not cancel in the calculation of A_h. If this condition is not satisfied, then $1 - h_1(F(\cdot))$ would typically still be second-order regularly varying, but the second-order parameter and auxiliary function would depend on the *third-order* regular variation properties of $\overline{F}$ and $1 - h_1(1 - 1/\cdot)$.

The next lemma analyzes the second-order regular variation properties of the left-continuous inverse function $\varphi^{\leftarrow}$ defined in (3.1) and its connection with extreme quantiles of the distribution function F. It will be used in the proof of the main result of this section.

Lemma 4.3. *Under Assumptions 4.1, 4.2, and 4.3, and if there is a regularly varying function D such that $A(1/\overline{F}(x))/D(x) \to a \in \mathbb{R}$, $B_i(x)/D(x) \to b_i \in \mathbb{R}$ and $C(x)/D(x) \to \kappa \in \mathbb{R}$ as $x \to \infty$, with $b_2/\eta_2 - b_1/\eta_1 - (a\beta_1/\gamma + \kappa)/(\gamma\rho_h) \neq 0$, we have, as $\tau \to 1$,*

$$\varphi^{\leftarrow}((1-\tau)^{-1}) = c^*(1-\tau)^{-1/s}\left(1 - \frac{1}{s} A^*(\varphi^{\leftarrow}((1-\tau)^{-1}))(1 + o(1))\right)$$

where $s = \alpha_2 - \alpha_1 + \beta_1/\gamma$ *as in Theorem 3.1, and* $c^* = \left(\frac{a_2}{bc^{\beta_1/\gamma}a_1}\right)^{-1/s}$ *and* $A^*(t) = \frac{1}{\eta_2}B_2(t) - \frac{1}{\eta_1}B_1(t) - \frac{1}{\rho_h}A_h(t)$ *is regularly varying with index* $\eta^* = \max\{\eta_1, \rho_h, \eta_2\}$*, with the notation of Lemma 4.2. In particular,* $\varphi^{\leftarrow} \in 2\mathrm{RV}_{1/s,\eta^*/s}$ *and*

$$\frac{\varphi^{\leftarrow}((1-\tau)^{-1})}{(F^{\leftarrow}(\tau))^{1/(\gamma s)}} = c_0\left(1 - \frac{c_0^{\eta^*}}{s}A^*((F^{\leftarrow}(\tau))^{1/(\gamma s)})(1+o(1)) - \frac{1}{\gamma s\rho}A((1-\tau)^{-1})(1+o(1))\right)$$

where $c_0 = c^*c^{-1/(\gamma s)}$*. In the specific setting when* $u_1 = u_2$*, the condition linking* A*, the* B_i*,* C *and* a*,* b_1*,* b_2 *and* κ *can be replaced by supposing that* $C(x)/A(x) \to \kappa \in [-\infty, +\infty]$ *as* $x \to \infty$ *with* $\kappa \neq -\beta_1/\gamma$*, in which case* $A^* = -\frac{1}{\rho_h}A_h$*.*

To derive the second-order asymptotic expansions for the generalized shortfall risk measure, we proceed by analyzing the two sides of (1.1) separately.

Lemma 4.4. *Under Assumptions 4.1, 4.2, and 4.3, further assume that* $C(x)/A(x) \to \kappa \in [-\infty, +\infty]$ *as* $x \to \infty$ *with* $\kappa \neq -\beta_1/\gamma$*, as well as* $\beta_1/\gamma > \alpha_1$ *and* $\alpha_1 + \eta_1 > 0$*. Then as* $x \to \infty$*,*

$$\frac{\mathrm{H}_{u_1,h_1}((X-x)_+)}{(1-h_1(F(x)))\,u_1(x)} = \Delta_0 + \Gamma_1 B_1(x)(1+o(1)) + \Gamma_2 A_h(x)(1+o(1))$$

with

$$\Gamma_1 = \frac{\beta_1}{\gamma} \times \frac{1}{\eta_1}(\mathrm{B}(\beta_1/\gamma - \alpha_1 - \eta_1, \alpha_1 + \eta_1 + 1) - \mathrm{B}(\beta_1/\gamma - \alpha_1, \alpha_1 + 1))$$

and

$$\Gamma_2 = \frac{1}{\rho_h}\left(\left(\frac{\beta_1}{\gamma} - \rho_h\right)\mathrm{B}(\beta_1/\gamma - \alpha_1 - \rho_h, \alpha_1 + 1) - \frac{\beta_1}{\gamma}\mathrm{B}(\beta_1/\gamma - \alpha_1, \alpha_1 + 1)\right).$$

Now we turn to the right-hand side of (1.1).

Lemma 4.5. *Assume that* u_2 *is differentiable and* $u_2' \in \mathrm{RV}_{\alpha_2 - 1}$ *is bounded on finite intervals of the form* $(0, t_0]$ $(t_0 > 0)$*, with either* $\alpha_2 > 1$ *or* $\alpha_2 = 1$ *and* u_2' *nondecreasing, and* $\overline{F} \in \mathrm{RV}_{-1/\gamma}$ *with* $\gamma > 0$*. Suppose* $\int_{-\infty}^{\infty} |z|^{\alpha_2+\delta}\,\mathrm{d}h_2(F(z)) < \infty$ *for some* $\delta > 0$*. Then as* $x \to \infty$*,*

$$\frac{\mathrm{H}_{u_2,h_2}((X-x)_-)}{u_2(x)} = 1 - (1 - h_2(F(x))) - x^{-1}(\alpha_2\mathbb{E}[Z] + o(1)),$$

where $\mathbb{E}[Z] = \int_{-\infty}^{\infty} z\,\mathrm{d}h_2(F(z))$*. [The random variable* Z *has distribution function* $h_2(F(\cdot))$*.]*

Remark 4.2. In Lemma 4.5, the tail index α_2 is restricted to be greater than 1. This is because if $\alpha_2 < 1$, then additional conditions are needed to ensure $h_2(F(\cdot))$ is regularly varying at 0. For simplicity, we omit this case. The assumption that u_2' is bounded on finite intervals of the form $(0, t_0]$ essentially amounts to assuming that $t \mapsto t^{-\alpha_2}u_2(t) = L_2(t)$ is smooth in a neighborhood of

0 and $t \mapsto L_2(t)/t$ is bounded. It therefore intuitively represents a strengthened version of part of Assumption 4.2.

Next, we present the second-order expansion of x_τ in terms of $\varphi^{\leftarrow}((1-\tau)^{-1})$, obtained essentially by combining Lemmas 4.4 and 4.5.

Theorem 4.1. *Under Assumptions 4.1, 4.2, and 4.3, further assume that there is a regularly varying function D such that $A(1/\overline{F}(x))/D(x) \to a \in \mathbb{R}$, $B_i(x)/D(x) \to b_i \in \mathbb{R}$ and $C(x)/D(x) \to \kappa \in \mathbb{R}$ as $x \to \infty$, with $b_2/\eta_2 - b_1/\eta_1 - (a\beta_1/\gamma + \kappa)/(\gamma\rho_h) \neq 0$. Suppose also that u_2 is differentiable and $u_2' \in \mathrm{RV}_{\alpha_2 - 1}$ is bounded on finite intervals of the form $(0, t_0]$, with either $\alpha_2 > 1$ or $\alpha_2 = 1$ and u_2' nondecreasing. Suppose $\beta_1/\gamma > \alpha_1$, $\alpha_1 + \eta_1 > 0$ and $\int_{-\infty}^{\infty} |z|^{\alpha_2+\delta}\,\mathrm{d}h_2(F(z)) < \infty$ for some $\delta > 0$. We have, as $\tau \to 1$,*

$$
\begin{aligned}
&\frac{x_\tau}{\Delta_1 \varphi^{\leftarrow}((1-\tau)^{-1})} - 1 \\
&= \frac{1}{s}\Bigg(\frac{\Gamma_1}{\Delta_0^{1-\eta_1/s}} B_1(\varphi^{\leftarrow}((1-\tau)^{-1}))(1+o(1)) + \frac{\Gamma_2}{\Delta_0^{1-\rho_h/s}} A_h(\varphi^{\leftarrow}((1-\tau)^{-1}))(1+o(1)) \\
&+ \Delta_0^{-\beta_2/(\gamma s)}(1 - h_2(F(\varphi^{\leftarrow}((1-\tau)^{-1}))))(1+o(1)) + \frac{\alpha_2 \Delta_0^{-1/s}}{\varphi^{\leftarrow}((1-\tau)^{-1})}(\mathbb{E}[Z] + o(1)) \\
&-(\Delta_0^{\eta^*/s} - 1)A^*(\varphi^{\leftarrow}((1-\tau)^{-1}))(1+o(1)) - (1-\tau)(1+o(1))\Bigg)
\end{aligned}
$$

with the notation of the above lemmas.

Combining Lemma 4.3 and Theorem 4.1, we finally obtain the desired second-order expansion of x_τ in terms of $F^{\leftarrow}(\tau)$.

Theorem 4.2. *Under the conditions of Theorem 4.1, we have, as $\tau \to 1$ and with c_0 as in Lemma 4.3,*

$$
\begin{aligned}
&\frac{x_\tau}{c_0 \Delta_1 (F^{\leftarrow}(\tau))^{1/(\gamma s)}} - 1 \\
&= \Delta_2 B_1((F^{\leftarrow}(\tau))^{1/(\gamma s)})(1+o(1)) + \Delta_3 A_h((F^{\leftarrow}(\tau))^{1/(\gamma s)})(1+o(1)) \\
&+ \Delta_4(1 - h_2(F((F^{\leftarrow}(\tau))^{1/(\gamma s)})))(1+o(1)) + (F^{\leftarrow}(\tau))^{-1/(\gamma s)}(\Delta_5 + o(1)) \\
&- \Delta_6 A^*((F^{\leftarrow}(\tau))^{1/(\gamma s)})(1+o(1)) - \frac{1}{\gamma s \rho} A((1-\tau)^{-1})(1+o(1)) - \frac{1}{s}(1-\tau)(1+o(1)),
\end{aligned}
$$

with $\Delta_2 = s^{-1}\Gamma_1 c_0^{\eta_1} \Delta_0^{\eta_1/s - 1}$, $\Delta_3 = s^{-1}\Gamma_2 c_0^{\rho_h} \Delta_0^{\rho_h/s-1}$, $\Delta_4 = s^{-1} c_0^{-\beta_2/\gamma} \Delta_0^{-\beta_2/(\gamma s)}$, $\Delta_5 = s^{-1} c_0^{-1} \alpha_2 \mathbb{E}[Z] \Delta_0^{-1/s}$, and $\Delta_6 = s^{-1} c_0^{\eta^*} \Delta_0^{\eta^*/s}$.

Remark 4.3. The auxiliary function $A(t)$ in (2.3) of Definition 2.3 is of course only unique up to asymptotic equivalence. Given a distribution function F or tail quantile function U of a random variable X, a reasonable choice of auxiliary function, readily computed, would be the function A_0 in Theorem 2.3.9 of de Haan and Ferreira (2006), which guarantees a uniform kind of second-order regular variation. That being said, the asymptotic expansions in Theorems 4.1 and 4.2 hold true for any other choice of A asymptotically equivalent to this function A_0, and similarly for the choices of B_1, B_2 and C.

Corollary 4.1. *Under the conditions of Theorem 4.1, we have, as* $\tau \to 1$,

$$x_\tau = \Delta_1 \varphi^{\leftarrow}((1-\tau)^{-1})\big(1 + O((1-\tau)^{1/\max(s,1)}) + O(1 - h_2(F((1-\tau)^{-1/s}))) \\ + O(A((1-\tau)^{-1/(\gamma s)})) + O(B_1((1-\tau)^{-1/s})) + O(B_2((1-\tau)^{-1/s})) + O(C((1-\tau)^{-1/(\gamma s)}))\big).$$

Remark 4.4. Theorems 4.1 and 4.2 and Corollary 4.1 also hold if either of the functions U, u_i or $1 - h_i(1 - 1/\cdot)$ is a multiple of a pure power function, with corresponding conditions on the second-order parameter(s) dropped and the corresponding auxiliary function(s) involved taken identically equal to 0. Such examples are considered in Section 6 below. In Corollary 4.1, the first term $O((1-\tau)^{1/\max(s,1)})$ should in practice be understood as $O(1-\tau) + O(1/\varphi^{\leftarrow}((1-\tau)^{-1}))$; when $u_1 = u_2$, corresponding to the *a priori* reasonable setting in risk management when the (non-distorted) cost of a deviation of the predictor from below or above X is the same, then $\varphi^{\leftarrow}$ is nothing but the tail quantile function of the (distorted) distribution function $h_1(F(\cdot))$. Terms proportional to the reciprocal of a tail quantile function are standard in asymptotic expansions of risk measures, see *e.g.* Daouia et al. (2018) and Daouia et al. (2019) in the expectile and L^p-quantile setting. In this case, note that, as in Lemma 4.3, the condition linking A, the B_i, C and a, b_1, b_2 and κ can be replaced by supposing that $C(x)/A(x) \to \kappa \in [-\infty, +\infty]$ as $x \to \infty$ with $\kappa \neq -\beta_1/\gamma$.

5 Estimation

Theorem 3.1 provides an asymptotic equivalent of the non-explicit shortfall risk measure x_τ (at extreme levels) in terms of the generalized inverse of the function φ, which is obtained by simple operations on the functions u_1, u_2 and h_1 chosen by the user, and the unknown distribution function F. An estimator of x_τ at extreme levels can thus essentially be constructed by estimating the function F at extreme levels and inverting the resulting estimated version of φ. Since the main statistical difficulty resides in the estimation of F, we illustrate this principle in the particular situation when $u_1 = u_2 = u$, and h_1, h_2 are continuous and strictly increasing functions with $1 - h_1(1 - 1/\cdot) \in \mathrm{RV}_{-1}$. This results in the simpler setting when $\varphi(\cdot) = 1/(1 - h_1(F(\cdot)))$, making it possible to avoid technicalities due to the (different) regular variation properties of u_1, u_2, h_1

and h_2, and contains not only the case when $h_1 = h_2$ is the identity function, for which $\varphi^{\leftarrow}$ is nothing but the tail quantile function of X, but also the interesting case when $1 - h_1(1 - 1/x)$ and $1 - h_2(1 - 1/x)$ are equivalent to a multiple of $1/x$ as $x \to \infty$. The former situation contains the example of L^p-quantiles and the latter encompasses the example of generalized expectiles, both of which will be considered in Section 6. The general case is of course handled in much the same way, at the price of further burdensome calculations.

When $u_1 = u_2 = u$ is regularly varying with index $\alpha > 0$ and h_1 is such that $1 - h_1(1 - 1/\cdot) \in \mathrm{RV}_{-1}$, Theorem 3.1 suggests that

$$x_\tau = \left(\frac{1}{\gamma}\mathrm{B}(1/\gamma - \alpha, \alpha + 1)\right)^{\gamma} \varphi^{\leftarrow}((1-\tau)^{-1})(1 + o(1)) \text{ as } \tau \uparrow 1.$$

Since $\varphi(\cdot) = 1/(1 - h_1(F(\cdot)))$, $\varphi^{\leftarrow}((1-\tau)^{-1})$ is nothing but the quantile of level τ of the random variable having distribution function $h_1(F(\cdot))$, that is,

$$\begin{aligned} x_\tau &= \left(\frac{1}{\gamma}\mathrm{B}(1/\gamma - \alpha, \alpha + 1)\right)^{\gamma} F^{\leftarrow}(h_1^{-1}(\tau))(1 + o(1)) \\ &= \left(\frac{\gamma}{\mathrm{B}(1/\gamma - \alpha, \alpha + 1)}\right)^{-\gamma} F^{\leftarrow}(h_1^{-1}(\tau))(1 + o(1)) \text{ as } \tau \uparrow 1. \end{aligned}$$

Since $h_1^{-1}(\tau) \uparrow 1$ as $\tau \uparrow 1$, the above identity shows that the problem of estimating x_τ for τ large reduces to estimating γ and extreme quantiles of F.

Suppose then that $X_1, \ldots, X_n$ is a sample of data from a distribution function F such that $\overline{F}(\cdot) \in 2\mathrm{RV}_{-1/\gamma,\rho/\gamma}$. The data $X_1, \ldots, X_n$ are allowed to be serially dependent. Let also $\tau_n \uparrow 1$ be an extreme level: typical interesting cases are those when $n(1-\tau_n)$ is bounded in n, such as $\tau_n = 1 - 1/n$. A standard way to estimate the extreme quantile $q_{\tau_n} \equiv F^{\leftarrow}(\tau_n)$ is to use the estimator due to Weissman (1978), defined as

$$\widehat{q}_{\tau_n} \equiv \widehat{q}_{\tau_n}(k_n) = \left(\frac{k_n}{n(1-\tau_n)}\right)^{\widehat{\gamma}_n} X_{n-k_n,n}$$

where (k_n) is a sequence of integers tending to infinity, with $k_n/n \to 0$ and $n(1-\tau_n)/k_n \to 0$, $X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$ are the order statistics of the sample $(X_1, \ldots, X_n)$ arranged in increasing order, and $\widehat{\gamma}_n$ is an estimator of the parameter γ. A reasonable choice of $\widehat{\gamma}_n$ is the estimator of Hill (1975):

$$\widehat{\gamma}_n = \frac{1}{k_n}\sum_{i=1}^{k_n} \log X_{n-i+1,n} - \log X_{n-k_n,n}.$$

We may then define the following estimator of x_{τ_n}:

$$\widehat{x}_{\tau_n} \equiv \widehat{x}_{\tau_n}(k_n) = \left(\frac{1}{\widehat{\gamma}_n} \mathrm{B}(1/\widehat{\gamma}_n - \alpha, \alpha + 1) \right)^{\widehat{\gamma}_n} \widehat{q}_{h_1^{-1}(\tau_n)}(k_n)$$
$$= \left(\frac{k_n}{n(1 - h_1^{-1}(\tau_n))} \right)^{\widehat{\gamma}_n} \left\{ \left(\frac{1}{\widehat{\gamma}_n} \mathrm{B}(1/\widehat{\gamma}_n - \alpha, \alpha + 1) \right)^{\widehat{\gamma}_n} X_{n-k_n,n} \right\}.$$

This is also a Weissman-type estimator of x_{τ_n}. We have the following convergence result for $\widehat{x}_{\tau_n}$.

Theorem 5.1. *Assume that:*

- $U \in 2\mathrm{RV}_{\gamma,\rho}$ *for* $\gamma > 0$ *and* $\rho < 0$ *with auxiliary function* A*,*
- $u_1 = u_2 = u \in 2\mathrm{RV}_{\alpha,\eta}$ *for* $\alpha > 0$ *and* $\eta < 0$ *with auxiliary function* B*, and* $t^{-\alpha}u(t)$ *is bounded on intervals of the form* $(0, t_0]$*, for* $t_0 > 0$*,*
- u *is differentiable and* $u' \in \mathrm{RV}_{\alpha-1}$ *is bounded on finite intervals of the form* $(0, t_0]$*, with either* $\alpha > 1$*, or* $\alpha = 1$ *and* u' *nondecreasing,*
- $1 - h_1(1 - 1/\cdot) \in 2\mathrm{RV}_{-1,\varsigma}$ *for* $\varsigma < 0$ *with auxiliary function* C*.*

Assume also that $C(x)/A(x) \to \kappa \in [-\infty, +\infty]$ *as* $x \to \infty$ *with* $\kappa \neq -1/\gamma$*, and that* $1/\gamma > \alpha$*,* $\alpha + \eta > 0$ *and* $\int_{-\infty}^{\infty} |z|^{\alpha+\delta} \, \mathrm{d}h_2(F(z)) < \infty$ *for some* $\delta > 0$*. Let* (k_n) *be a sequence of integers and* (τ_n) *be a sequence converging to 1 such that* $k_n \to \infty$*,* $k_n/n \to 0$*,* $n(1 - \tau_n)/k_n \to 0$*,* $\log(k_n/(n(1 - \tau_n)))/\sqrt{k_n} \to 0$ *and* $\sqrt{k_n}(k_n/n + |A(n/k_n)| + |B(q_{1-k_n/n})| + |C(n/k_n)| + 1/q_{1-k_n/n}) = \mathrm{O}(1)$ *as* $n \to \infty$*. If*

$$\sqrt{k_n}(\widehat{\gamma}_n - \gamma) \xrightarrow{d} N \quad \textit{and} \quad \sqrt{k_n} \left(\frac{X_{n-k_n,n}}{q_{1-k_n/n}} - 1 \right) \xrightarrow{d} N'$$

where N *and* N' *are nondegenerate distributions, then*

$$\frac{\sqrt{k_n}}{\log(k_n/(n(1 - \tau_n)))} \left(\frac{\widehat{x}_{\tau_n}}{x_{\tau_n}} - 1 \right) \xrightarrow{d} N.$$

Note that, following Remark 4.4, if either of the functions U, u or $1 - h_1(1 - 1/\cdot)$ is a multiple of a pure power function, then Theorem 5.1 holds with corresponding conditions on the second-order parameter(s) dropped and the corresponding auxiliary function(s) involved taken identically equal to 0. For instance, if $u(x)$ is proportional to x^α, then B can be taken equal to 0 and condition $\alpha + \eta > 0$ disappears.

An important subcase in which Theorem 5.1 applies is when the X_i are independent. In this setting, it is known that when $\sqrt{k_n}A(n/k_n) \to \lambda \in \mathbb{R}$,

$$\sqrt{k_n}\left(\widehat{\gamma}_n - \gamma, \frac{X_{n-k_n,n}}{q_{1-k_n/n}} - 1\right) \xrightarrow{d} \left(\frac{\lambda}{1-\rho}, 0\right) + \gamma(\Theta, \Psi) \tag{5.1}$$

where Θ and Ψ are independent standard normal random variables, as can be seen by combining Lemma 3.2.3 and Theorem 3.2.5 in de Haan and Ferreira (2006). Then, by Theorem 5.1,

$$\frac{\sqrt{k_n}}{\log(k_n/(n(1-\tau_n)))}\left(\frac{\widehat{x}_{\tau_n}}{x_{\tau_n}} - 1\right) \xrightarrow{d} \mathcal{N}\left(\frac{\lambda}{1-\rho}, \gamma^2\right).$$

Extensions of convergence (5.1) to the case when the X_i are serially dependent, such as when the X_i are strongly mixing (namely, $\alpha-$mixing) or absolutely regular (namely, $\beta-$mixing), thus covering standard linear time series or conditionally heteroskedastic random processes, are examined in *e.g.* Hsing (1991) and Drees (2003). In such models, just like $\widehat{\gamma}_n$, the estimator $\widehat{x}_{\tau_n}$ will still be asymptotically Gaussian but with an enlarged variance, due to the loss of information entailed by the presence of serial dependence in the data.

6 Examples and numerical illustrations

In this section, we discuss two interesting examples of generalized shortfall risk measures, and we briefly examine the finite-sample performance of the estimator presented in Section 5.

Example 6.1. (L^p-quantiles) Let $u_1(x) = u_2(x) = px^{p-1}$, $p \geqslant 1$, and $h_1(x) = h_2(x) = x$. Then x_τ is reduced to the L^p-quantile in Daouia et al. (2019), denoted by x_τ^{Lp}. We examine the first- and second-order expansions of x_τ^{Lp} arising from our results when $\overline{F} \in 2\text{RV}_{-1/\gamma,\rho/\gamma}$ with $\gamma > 0$ and $\rho < 0$.

Clearly $u_i \in \text{RV}_{p-1}$ and $1 - h_i(1 - 1/\cdot) \in \text{RV}_{-1}$ for $i = 1, 2$. Conditions $1/\gamma > p - 1$ and $\int_{-\infty}^{\infty} |z|^{p-1+\delta} \mathrm{d}h_2(F(z)) < \infty$ for some $\delta > 0$ reduce to $\gamma < 1/(p-1)$ and $\mathbb{E}(|\min(X,0)|^{p-1+\delta}) < \infty$ (the latter can be replaced by $\mathbb{E}(|\min(X,0)|^{p-1}) < \infty$, see Remark 3.1). The function φ is nothing but $1/\overline{F}$, so $\varphi((1-\tau)^{-1}) = F^{\leftarrow}(\tau)$. By Theorem 3.1, the first-order asymptotic expansion of x_τ^{Lp} is

$$x_\tau^{Lp} = \Delta_1 F^{\leftarrow}(\tau)(1 + o(1)) = \left(\frac{1}{\gamma}\mathrm{B}(1/\gamma - p + 1, p)\right)^{\gamma} F^{\leftarrow}(\tau)(1 + o(1)) \quad \text{as} \quad \tau \uparrow 1.$$

This recovers Corollary 1 of Daouia et al. (2019).

We now analyze the second-order expansion provided by Theorem 4.1 when $p \geqslant 2$, in which case X has a finite moment of order 1 under our assumptions. Obviously the u_i and $1 - h_i(1 - 1/\cdot)$ are multiples of pure power functions, and $1 - h_1(F(\cdot)) = 1 - F(\cdot) \in 2\text{RV}_{-1/\gamma,\rho/\gamma}$ with auxiliary function $\gamma^{-2}A(1/\overline{F}(\cdot))$. In other words, with the notation of Theorem 4.1, $B_1 = B_2 \equiv 0$, $A_h(\cdot) = \gamma^{-2}A(1/\overline{F}(\cdot))$ is regularly varying with index $\rho_h = \rho/\gamma$, $\alpha_1 = \alpha_2 = p-1$, $\beta_1 = \beta_2 = 1$, $\mathbb{E}[Z] = \mathbb{E}[X]$, $\eta^* = \rho_h = \rho/\gamma$, $A^*(\cdot) = -(\gamma\rho)^{-1}A(1/\overline{F}(\cdot))$, and

$$\Gamma_2 = \frac{1}{\rho}((1-\rho)\mathrm{B}((1-\rho)/\gamma - p + 1, p) - \mathrm{B}(1/\gamma - p + 1, p)).$$

It follows that the second-order expansion of x_τ^{Lp} is

$$\begin{aligned}\frac{x_\tau^{Lp}}{\Delta_1 F^{\leftarrow}(\tau)} &= 1 + \frac{\gamma(p-1)}{\Delta_1 F^{\leftarrow}(\tau)}(\mathbb{E}[X] + o(1)) + \gamma\left(\left(\frac{1}{\gamma}\mathrm{B}(1/\gamma - p + 1, p)\right)^{-1} - 1\right)(1-\tau)(1+o(1)) \\ &\quad + \frac{1}{\rho}\left(((1-\rho)\mathrm{B}((1-\rho)/\gamma - p + 1, p) - \mathrm{B}(1/\gamma - p + 1, p)) \times \frac{1}{\gamma}\left(\frac{1}{\gamma}\mathrm{B}(1/\gamma - p + 1, p)\right)^{\rho-1}\right. \\ &\qquad \left. + \left(\frac{1}{\gamma}\mathrm{B}(1/\gamma - p + 1, p)\right)^{\rho} - 1 + o(1)\right) A((1-\tau)^{-1})\end{aligned}$$

This matches expansion (A14) of Stupfler and Usseglio-Carleve (2022), itself a corrected version of Proposition 3 of Daouia et al. (2019): note that, despite the fact that the term in $(1-\tau)$ is not reported as being the same in this Proposition 3, the proof of their Proposition 2 indeed shows that there are two contributions proportional to $(1-\tau)$, due to (with the notation therein) a term called $I_1(q;p)$ in their Equation (A.10) and the $\overline{F}(q)$ term in their Equation (A.11). In the current setting $\gamma < 1/(p-1) < 1$, so any term in $(1-\tau)$ is negligible with respect to $1/F^{\leftarrow}(\tau)$. It follows that

$$\begin{aligned}&\frac{x_\tau^{Lp}}{\Delta_1 F^{\leftarrow}(\tau)} \\ &= 1 + \frac{1}{\rho}\left(((1-\rho)\mathrm{B}((1-\rho)/\gamma - p + 1, p) - \mathrm{B}(1/\gamma - p + 1, p)) \times \frac{1}{\gamma}\left(\frac{1}{\gamma}\mathrm{B}(1/\gamma - p + 1, p)\right)^{\rho-1}\right. \\ &\qquad \left. + \left(\frac{1}{\gamma}\mathrm{B}(1/\gamma - p + 1, p)\right)^{\rho} - 1 + o(1)\right) A((1-\tau)^{-1}) + \frac{\gamma(p-1)}{\Delta_1 F^{\leftarrow}(\tau)}(\mathbb{E}[X] + o(1)).\end{aligned}$$

Example 6.2. (Generalized expectiles) Recall Example 3.4 of Mao and Cai (2018) in which the coherent generalized expectile is defined as the unique solution to the equation

$$\tau\,\mathrm{TVaR}_p((X-x)_+) + (1-\tau)\mathrm{TVaR}_q(-(X-x)_-) = 0, \; x \in \mathbb{R}, \tag{6.1}$$

satisfying $p \leqslant q$ and $\tau/(1-\tau) \geqslant (1-p)/(1-q)$. This is an example of the generalized Dutch type II risk measures introduced in Cai and Mao (2020). In fact, when $u_i(x) = 2x$ for $x > 0$, and $h_1(x) = (x-p)_+/(1-p)$ and $h_2(x) = (x-q)_+/(1-q)$ with $p, q \in [0, 1)$, the generalized shortfall risk measure coincides with the coherent generalized expectile in (6.1), denoted by x^e_τ. Next, we make explicit the first- and second-order expansions of x^e_τ when $\overline{F} \in 2\mathrm{RV}_{-1/\gamma,\rho/\gamma}$ with $\gamma > 0$ and $\rho < 0$. For the sake of simplicity, we assume that F *is continuous*.

Obviously $u_i \in \mathrm{RV}_1$ and, for x large enough, $1 - h_1(1 - 1/x) = x^{-1}/(1-p)$, so $1 - h_1(1 - 1/\cdot) \in \mathrm{RV}_{-1}$ and similarly $1 - h_2(1 - 1/\cdot) \in \mathrm{RV}_{-1}$. Again the conditions of Theorem 3.1 reduce to $\gamma < 1$ and $\mathbb{E}(|\min(X, 0)|) < \infty$, and since $\varphi^{\leftarrow}(t) = U(t/(1-p))$ for large t, the first-order expansion of x^e_τ reads

$$\begin{aligned} x^e_\tau = \Delta_1(1-p)^{-\gamma} F^{\leftarrow}(\tau)(1+o(1)) &= (\gamma^{-1}\mathrm{B}(\gamma^{-1}-1, 2))^{\gamma}(1-p)^{-\gamma} F^{\leftarrow}(\tau)(1+o(1)) \\ &= (\gamma^{-1}-1)^{-\gamma}(1-p)^{-\gamma} F^{\leftarrow}(\tau)(1+o(1)) \ \text{ as } \tau \uparrow 1. \end{aligned}$$

This can be viewed as an extension of the standard asymptotic equivalent for expectiles in terms of their quantile counterparts.

Then clearly u_i are multiples of pure power functions, so $B_1 = B_2 \equiv 0$, and $1 - h_1(F(x)) = \overline{F}(x)/(1-p)$, $1 - h_2(F(x)) = \overline{F}(x)/(1-q)$ for x large enough, so $1 - h_1(F(\cdot)) \in 2\mathrm{RV}_{-1/\gamma,\rho/\gamma}$ with auxiliary function $\gamma^{-2}A(1/\overline{F}(\cdot))$. Recall also that $\varphi^{\leftarrow}(t) = U(t/(1-p))$ for large t (and then $\varphi^{\leftarrow}((1-\tau)^{-1}) = F^{\leftarrow}(p+(1-p)\tau)$). This means that with the notation of Theorem 4.1, $B_1 = B_2 \equiv 0$, $A_h(\cdot) = \gamma^{-2}A(1/\overline{F}(\cdot))$ is regularly varying with index $\rho_h = \rho/\gamma$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $\mathbb{E}[Z] = \mathbb{E}[X\mathbb{1}\{X > F^{\leftarrow}(q)\}]/(1-q) = \mathbb{E}[X|X > F^{\leftarrow}(q)]$, $\eta^* = \rho_h = \rho/\gamma$, $A^*(\cdot) = -(\gamma\rho)^{-1}A(1/\overline{F}(\cdot))$, and

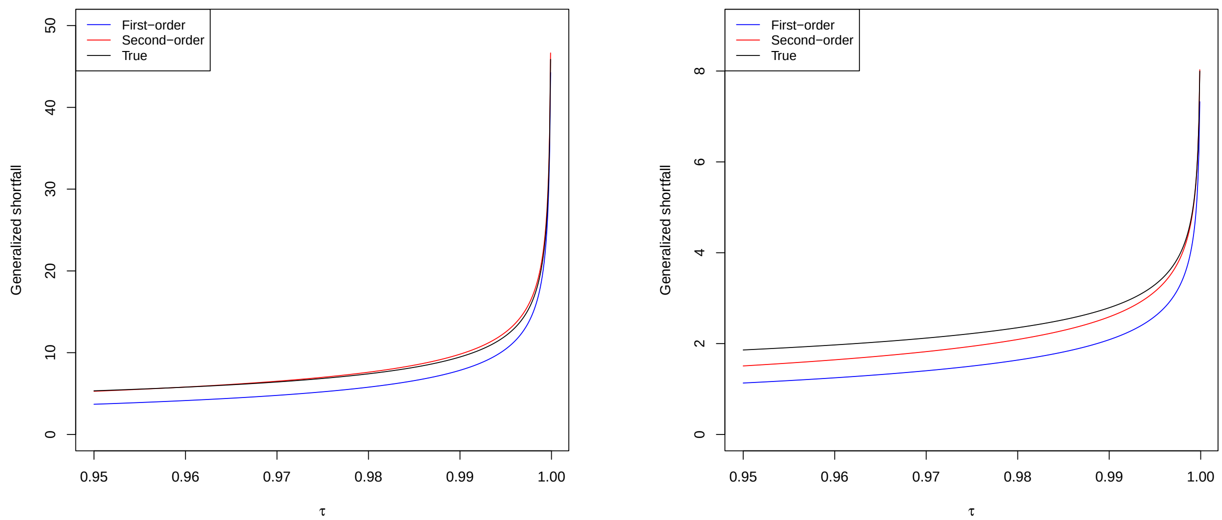
$$\Gamma_2 = \frac{1}{\rho}((1-\rho)\mathrm{B}((1-\rho)/\gamma - 1, 2) - \mathrm{B}(1/\gamma - 1, 2)) = \frac{\gamma^2}{(1-\gamma)(1-\gamma-\rho)}.$$

Then, after straightforward calculations

$$\begin{aligned} &\frac{x^e_\tau}{\Delta_1 F^{\leftarrow}(p+(1-p)\tau)} \\ &= 1 + (1-p)^{\gamma}\frac{\gamma(\gamma^{-1}-1)^{\gamma}}{F^{\leftarrow}(\tau)}(\mathbb{E}[X|X > F^{\leftarrow}(q)] + o(1)) + \left((1-\gamma)\frac{1-p}{1-q} - 1\right)(1-\tau)(1+o(1)) \\ &+ (1-p)^{-\rho}\left(\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho} + \frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho} + o(1)\right)A((1-\tau)^{-1}). \end{aligned}$$

Since

$$\frac{F^{\leftarrow}(p+(1-p)\tau)}{F^{\leftarrow}(\tau)} = \frac{U((1-\tau)^{-1}/(1-p))}{U((1-\tau)^{-1})} = (1-p)^{-\gamma}\left(1 + \frac{(1-p)^{-\rho}-1}{\rho}A((1-\tau)^{-1})(1+o(1))\right)$$



(a) $\gamma = 1/3$ (b) $\gamma = 1/5$

Figure 1: Comparison of first- and second-order expansions with the true values of the generalized shortfall risk measure of a generalized Pareto distribution.

and $\gamma < 1$, one may finally conclude that

$$
\begin{aligned}
&\frac{x_\tau^e}{\Delta_1(1-p)^{-\gamma}F^{\leftarrow}(\tau)} \\
&= 1 + (1-p)^{\gamma}\frac{\gamma(\gamma^{-1}-1)^{\gamma}}{F^{\leftarrow}(\tau)}(\mathbb{E}[X|X > F^{\leftarrow}(q)] + o(1)) \\
&+ \left((1-p)^{-\rho}\left(\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho} + \frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho}\right) + \frac{(1-p)^{-\rho}-1}{\rho} + o(1)\right) A((1-\tau)^{-1}).
\end{aligned}
$$

This coincides with the second-order asymptotic expansion of expectiles when $p = q = 0$, see Proposition 1 in Daouia et al. (2020).

We examine the accuracy of this expansion when F is the generalized Pareto distribution function $F(x) = 1 - (\theta/(x+\theta))^{1/\gamma}$ for $x > 0$, where $\gamma, \theta > 0$. In this setting, $U(t) = \theta(t^{\gamma} - 1)$, and then $U \in 2\text{RV}_{\gamma,-\gamma}$ with auxiliary function $A(t) = \gamma t^{-\gamma}$. We take $\gamma = 1/3, 1/5$ and $\theta = 1$, $p = q = 0.95$. In Figure 1, by varying τ from 0.95 to 0.9999, we plot the values obtained through the use of the first- and second-order expansions of x_τ^e. For comparison, the true values of x_τ^e are also plotted, which are calculated using the `uniroot` function in `R`. From Figure 1, it can be seen that the second-order expansion improves the first-order expansion significantly, especially in the lighter-tailed case.

We now examine the finite-sample performance of the estimator $\widehat{x}_{\tau_n}$ of Section 5 in this last example. We consider the following distributions:

- The pure Pareto distribution with distribution function $F(x) = 1 - x^{-1/\gamma}$, $x > 1$,
- The Fréchet distribution with distribution function $F(x) = \exp(-x^{-1/\gamma})$, $x > 0$,
- The Burr distribution with distribution function $F(x) = 1 - (1 + x^{-\rho/\gamma})^{1/\rho}$, $x > 0$ (here ρ is the negative second-order parameter of the distribution).

For each of these three distributions we take $\gamma = 1/5$ or $1/3$, and for the Burr distribution we use $\rho = -2$. In each case, we simulate $N = 10{,}000$ replications of an independent sample of size $n \in \{500, 1{,}000\}$, for which the true generalized expectile risk measure x_{τ_n} with $p = q = 0.95$ and $\tau_n = 1 - 1/n \in \{0.998, 0.999\}$ has been calculated numerically. This was done using the R function `uniroot` in order to find the solution of (6.1), where the function `cubintegrate` from the R package `cubature` has been used beforehand in order to calculate the distorted Tail-Value-at-Risk. In each replication we estimate this risk measure with the estimator introduced in Section 5, where the intermediate level $k = k_n$ is allowed to vary between $n/50$ and $2n/3$ (corresponding respectively to 2% and 66.7% of the total sample size). This produces estimates $\widehat{x}^{(j)}_{\tau_n}(k)$, $j = 1, 2, \ldots, N$, which are used to calculate the Monte-Carlo approximation to the relative Mean Squared Error (relative MSE) of the estimator $\widehat{x}_{\tau_n}$, that is,

$$\mathrm{rMSE}(k) = \frac{1}{N} \sum_{j=1}^{N} \left(\frac{\widehat{x}^{(j)}_{\tau_n}(k)}{x_{\tau_n}} - 1 \right)^2 .$$

These errors are represented as a function of k in Figures 2 and 3 in the twelve situations considered. The MSE tends to be high when k is low, due to the variance of the extreme value estimators dominating in that region, and it also tends to be high when k is large because their bias then dominates, except in the Pareto example for which bias due to the extreme value procedures is exactly 0. Bias is lower in the Burr example than in the Fréchet example: this is due to the second-order parameter $\rho = -2$ being further away from 0 in the Burr example than it is in the Fréchet example, where it is equal to -1. Solving the bias-variance tradeoff produces a stability region for moderately large values of k where MSE is comparatively lower, and this stability region tends to be larger as the second-order parameter gets away from 0. Since the Hill estimator is used in the extrapolation, the asymptotic variance of $\widehat{x}_{\tau_n}$ is asymptotically proportional to γ^2 by Theorem 5.1, so the higher the extreme value index, the higher the MSE should be, just as can be observed by comparing the top and bottom rows in each figure.

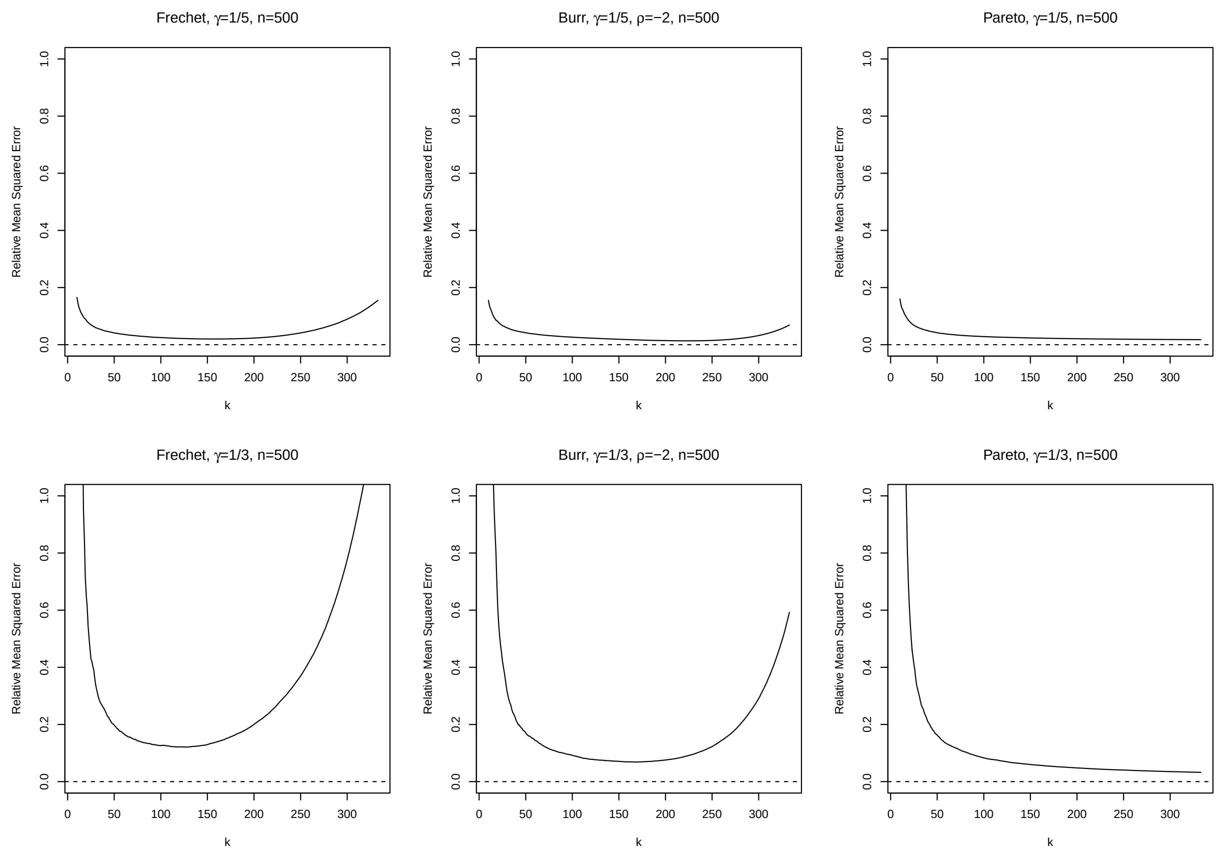


Figure 2: Relative Mean Squared Error of the estimator $\widehat{x}_{\tau_n}$, for $n = 500$ and $\tau_n = 1-1/n = 0.998$. Left panels: Fréchet distribution, middle panels: Burr distribution with $\rho = -2$, right panels: Pareto distribution. Top panels: $\gamma = 1/5$, bottom panels: $\gamma = 1/3$.

7 Proofs

Proof of Proposition 3.1. Note first that the quantities

$$\mathrm{H}_{u_1,h_1}((X-x)_+) = \int_x^\infty u_1(y-x)\,\mathrm{d}h_1(F(y)) \text{ and } \mathrm{H}_{u_2,h_2}((X-x)_-) = \int_{-\infty}^x u_2(x-y)\,\mathrm{d}h_2(F(y))$$

are always well-defined, because u_1, u_2 are positive on $[0,\infty)$ and $h_1 \circ F$, $h_2 \circ F$ are distribution functions, so that $\mathrm{H}_{u_1,h_1}((X-x)_+)$ and $\mathrm{H}_{u_2,h_2}((X-x)_-)$ are integrals of a positive measurable function with respect to a probability measure.

(i) We note that $x \mapsto \mathrm{H}_{u_1,h_1}((X-x)_+)$ and $x \mapsto \mathrm{H}_{u_2,h_2}((X-x)_-)$ are (strictly) decreasing and increasing positive functions on $(x_\star, x^\star)$, respectively. Indeed, if $x \in (x_\star, x^\star)$ and $\varepsilon > 0$ is such that

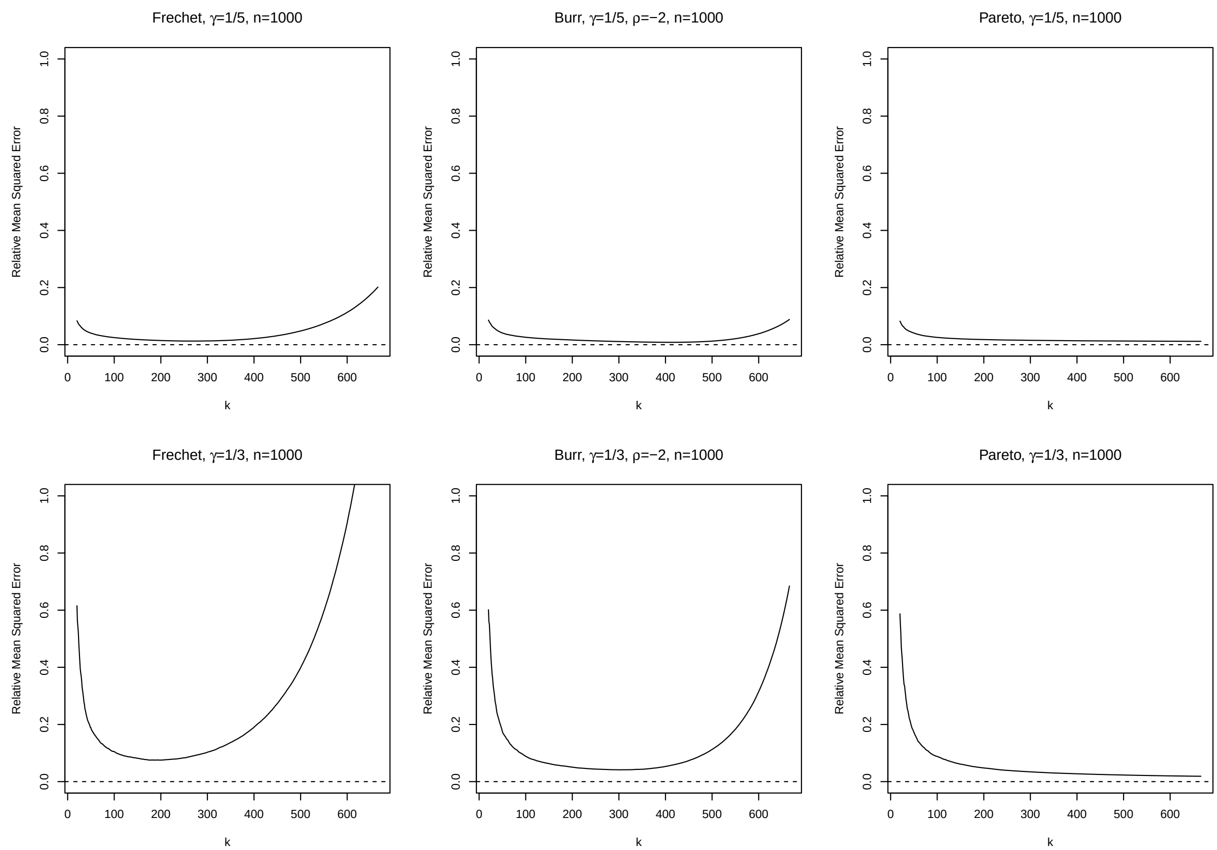


Figure 3: Relative Mean Squared Error of the estimator $\widehat{x}_{\tau_n}$, for $n = 1000$ and $\tau_n = 1-1/n = 0.999$. Left panels: Fréchet distribution, middle panels: Burr distribution with $\rho = -2$, right panels: Pareto distribution. Top panels: $\gamma = 1/5$, bottom panels: $\gamma = 1/3$.

$x + \varepsilon < x^\star$,

$$\begin{aligned} &\mathrm{H}_{u_1,h_1}((X-x)_+) - \mathrm{H}_{u_1,h_1}((X-(x+\varepsilon))_+) \\ &= \int_x^{x+\varepsilon} u_1(y-x)\,\mathrm{d}h_1(F(y)) + \int_{x+\varepsilon}^{\infty} (u_1(y-x) - u_1(y-x-\varepsilon))\,\mathrm{d}h_1(F(y)) > 0 \end{aligned} \tag{7.1}$$

since u_1 is increasing, $u_1(0) = 0$ (meaning that the first integral is nonnegative), and $x + \varepsilon < x^\star$ (meaning that the second integral is positive). The proof that $x \mapsto \mathrm{H}_{u_2,h_2}((X-x)_-)$ is increasing is similar; the above identity also shows that $x \mapsto \mathrm{H}_{u_1,h_1}((X-x)_+)$ and $x \mapsto \mathrm{H}_{u_2,h_2}((X-x)_-)$ define nonincreasing and nondecreasing functions on $\mathbb{R}$, respectively. Moreover, in the specific case when $x^\star < \infty$, one has, for any $x \geqslant x^\star$,

$$\mathrm{H}_{u_1,h_1}((X-x)_+) = \int_x^{\infty} u_1(y-x)\,\mathrm{d}h_1(F(y)) = 0$$

due to the fact that F is constant equal to 1 on $[x^\star, \infty)$ and h_1 does not have a jump at 1. Similarly $\mathrm{H}_{u_2,h_2}((X-x)_-) = 0$ for any $x \leqslant x_\star$ when $x^\star > -\infty$. Conclude by the intermediate value theorem that, since $x \mapsto \mathrm{H}_{u_1,h_1}((X-x)_+)$ and $x \mapsto \mathrm{H}_{u_2,h_2}((X-x)_-)$ are continuous on $(x_\star, x^\star)$, the equation

$$\tau\, \mathrm{H}_{u_1,h_1}((X-x)_+) - (1-\tau)\, \mathrm{H}_{u_2,h_2}((X-x)_-) = 0$$

has a unique solution which necessarily lies in the interval $(x_\star, x^\star)$.

(ii) Recall that $x \mapsto \mathrm{H}_{u_1,h_1}((X-x)_+)$ and $x \mapsto \mathrm{H}_{u_2,h_2}((X-x)_-)$ are nonincreasing and nondecreasing, respectively, and that $x_\tau < x^\star$ for any $\tau < 1$. Suppose now that there are $0 < \tau < \tau' < 1$ such that $x_\tau > x_{\tau'}$. Then

$$(1-\tau')\mathrm{H}_{u_2,h_2}((X-x_{\tau'})_-) < (1-\tau)\mathrm{H}_{u_2,h_2}((X-x_\tau)_-) = \tau\mathrm{H}_{u_1,h_1}((X-x_\tau)_+) < \tau'\mathrm{H}_{u_1,h_1}((X-x_{\tau'})_+).$$

This is a contradiction because the left- and right-most terms are equal. Hence $x_\tau \leqslant x_{\tau'}$ and $\tau \in (0,1) \mapsto x_\tau \in \mathbb{R}$ is nondecreasing, and in particular $x_1 = \lim_{\tau\uparrow 1} x_\tau$ is well-defined. If $x_1 = +\infty$ then obviously $x^\star$ is infinite too and $x_1 = x^\star$; otherwise, we clearly have, for any $\tau \in (0,1)$,

$$\tau\mathrm{H}_{u_1,h_1}((X-x_1)_+) \leqslant \tau\mathrm{H}_{u_1,h_1}((X-x_\tau)_+) = (1-\tau)\mathrm{H}_{u_2,h_2}((X-x_\tau)_-) \leqslant (1-\tau)\mathrm{H}_{u_2,h_2}((X-x_1)_-).$$

Let $\tau \uparrow 1$ to find $\mathrm{H}_{u_1,h_1}((X-x_1)_+) \leqslant 0$ and therefore $\mathrm{H}_{u_1,h_1}((X-x_1)_+) = 0$. This implies that $x_1 \geqslant x^\star$ and then $x_1 = x^\star$.

(iii) Clearly $x^\star = +\infty$ because $\overline{F} \in \mathrm{RV}_{-1/\gamma}$ with $\gamma > 0$. Combine (i) and (ii) to find that it is enough to show the continuity and finiteness of $x \mapsto \mathrm{H}_{u_1,h_1}((X-x)_+)$ and $x \mapsto \mathrm{H}_{u_2,h_2}((X-x)_-)$ on $(x_\star, +\infty)$. We start by finiteness. Fix $x \in (x_\star, +\infty)$. Then

$$\mathrm{H}_{u_1,h_1}((X-x)_+) = \lim_{T\to+\infty} -\int_{z=0}^{T} u_1(z)\,\mathrm{d}(1 - h_1(1 - 1/(1/\overline{F}(x+z)))).$$

Recall that $u_1 \in \mathrm{RV}_{\alpha_1}$ and u_1 is bounded on finite intervals of $[0,\infty)$. Then, for any arbitrary $\delta > 0$ we have, if z is chosen large enough, that $u_1(z)$ is bounded from above by a multiple of $z^{\alpha_1+\delta}$ using Potter bounds (see, e.g. Proposition B.1.9.5 of de Haan and Ferreira (2006)). Moreover, $1-h_1(1-1/(1/\overline{F}(x+\cdot))) \in \mathrm{RV}_{-\beta_1/\gamma}$. Use the assumption $\beta_1/\gamma > \alpha_1$ and Theorem 1.6.5 of Bingham et al. (1987) to find that $\mathrm{H}_{u_1,h_1}((X-x)_+)$ is indeed finite. The argument for $\mathrm{H}_{u_2,h_2}((X-x)_-)$ is slightly different: write

$$\mathrm{H}_{u_2,h_2}((X-x)_-) = \int_{-\infty}^{x/2} u_2(x-z)\,\mathrm{d}h_2(F(z)) + \int_{x/2}^{x} u_2(x-z)\,\mathrm{d}h_2(F(z)).$$

The first term is shown to be finite using Potter bounds and the assumption $\int_{-\infty}^{\infty} |z|^{\alpha_2+\delta}\,\mathrm{d}h_2(F(z)) < \infty$ for some $\delta > 0$. The second term is obviously finite because it is bounded from above by $u_2(x/2)$. Hence the finiteness of $\mathrm{H}_{u_2,h_2}((X-x)_-)$. We turn to continuity. Recall (7.1): the first term therein clearly converges to 0 as $\varepsilon \downarrow 0$ because $h_1 \circ F$ is a distribution function and is therefore right-continuous. The second term, meanwhile, converges to 0 by the dominated convergence theorem, because of the continuity of u_1 and $0 \leqslant (u_1(y-x) - u_1(y-x-\varepsilon))\mathbb{1}\{y \geqslant x+\varepsilon\} \leqslant u_1(y-x)\mathbb{1}\{y \geqslant x\}$ with the right-hand side being integrable with respect to $\mathrm{d}h_1(F(y))$. The continuity of $x \mapsto \mathrm{H}_{u_2,h_2}((X-x)_-)$ is shown similarly. □

Proof of Lemma 3.1. (i) First note that for any $\varepsilon_0 > 0$,

$$\begin{aligned}
\frac{\mathrm{H}_{u_1,h_1}((X-x)_+)}{u_1(x)\,(1-h_1(F(x)))} &= \frac{\int_x^{\infty} u_1(y-x)\,\mathrm{d}h_1(F(y))}{u_1(x)\,(1-h_1(F(x)))} \\
&= \int_1^{\infty} \frac{u_1(xy-x)}{u_1(x)}\,\mathrm{d}\frac{h_1(F(xy))}{1-h_1(F(x))} \\
&= \left(\int_{1+\varepsilon_0}^{\infty} + \int_1^{1+\varepsilon_0}\right) \frac{u_1(xy-x)}{u_1(x)}\,\mathrm{d}\frac{h_1(F(xy))}{1-h_1(F(x))} \\
&:= I_1(x) + I_2(x).
\end{aligned} \tag{7.2}$$

Since $u_1 \in \mathrm{RV}_{\alpha_1}$ with $\alpha_1 > 0$ and $1 - h_1(1-1/\cdot) \in \mathrm{RV}_{-\beta_1}$ with $\beta_1 > 0$, and $\beta_1/\gamma > \alpha_1$, using Potter bounds (see, e.g. Proposition B.1.9.5 of de Haan and Ferreira (2006)), for any $\varepsilon_1, \delta_1 > 0$, one has, for x large enough,

$$\begin{aligned}
I_1(x) &\leqslant \int_{1+\varepsilon_0}^{\infty} (1+\varepsilon_1)\,(y-1)^{\alpha_1 \pm \delta_1}\;\mathrm{d}\frac{h_1(F(xy))}{1-h_1(F(x))} \\
&= (1+\varepsilon_1)\left(\varepsilon_0^{\alpha_1\pm\delta_1}\frac{1-h_1(F(x(1+\varepsilon_0))}{1-h_1(F(x))} + \int_{1+\varepsilon_0}^{\infty} \frac{1-h_1(F(xy))}{1-h_1(F(x))}\,\mathrm{d}\,(y-1)^{\alpha_1\pm\delta_1}\right).
\end{aligned}$$

Now $1-h_1(F(\cdot)) = 1-h_1(1-1/(1/\overline{F}(\cdot))) \in \mathrm{RV}_{-\beta_1/\gamma}$ and therefore, by Proposition B.1.10 in de Haan and Ferreira (2006),

$$\limsup_{x\to\infty} I_1(x) \leqslant (1+\varepsilon_1)\left(\varepsilon_0^{\alpha_1\pm\delta_1}\,(1+\varepsilon_0)^{-\beta_1/\gamma} + \int_{1+\varepsilon_0}^{\infty} y^{-\beta_1/\gamma}\,\mathrm{d}\,(y-1)^{\alpha_1\pm\delta_1}\right)$$

A similar lower bound applies with ε_1 replaced by $-\varepsilon_1$. Conclude, since ε_1 and δ_1 are arbitrarily small, that

$$\lim_{x\to\infty} I_1(x) = \varepsilon_0^{\alpha_1}\,(1+\varepsilon_0)^{-\beta_1/\gamma} + \int_{1+\varepsilon_0}^{\infty} y^{-\beta_1/\gamma}\,\mathrm{d}\,(y-1)^{\alpha_1}\,.$$

Now we turn to I_2. Since $u_1 \in \mathrm{RV}_{\alpha_1}$ with $\alpha_1 > 0$, by Proposition B.1.9.6 of de Haan and Ferreira

(2006), there exists $c > 0$ such that for large enough x,

$$0 \leqslant I_2(x) \leqslant \int_1^{1+\varepsilon_0} c\,\mathrm{d}\frac{h_1(F(xy))}{1-h_1(F(x))} = c\left(1 - \frac{1-h_1(F(x(1+\varepsilon_0)))}{1-h_1(F(x))}\right).$$

[The constant c can be chosen sufficiently large so that it is universal for small values of ε_0, see the proof of Proposition B.1.9.6 of de Haan and Ferreira (2006).] Hence the bound

$$0 \leqslant \liminf_{x\to\infty} I_2(x) \leqslant \limsup_{x\to\infty} I_2(x) \leqslant c(1-(1+\varepsilon_0)^{-\beta_1/\gamma}).$$

Taking limits as $x \to \infty$ and letting $\varepsilon_0 \to 0$, the desired result follows as

$$\begin{aligned}\lim_{x\to\infty} \frac{\mathrm{H}_{u_1,h_1}((X-x)_+)}{u_1(x)\,(1-h_1(F(x)))} &= \int_1^\infty y^{-\beta_1/\gamma}\,\mathrm{d}\,(y-1)^{\alpha_1} \\ &= \alpha_1 \mathrm{B}(\beta_1/\gamma - \alpha_1, \alpha_1) = \frac{\beta_1}{\gamma}\mathrm{B}(\beta_1/\gamma-\alpha_1,\alpha_1+1).\end{aligned} \tag{7.3}$$

[Recall the recurrence formula $(x+y)\mathrm{B}(x,y+1) = y\mathrm{B}(x,y)$ valid for any $x, y > 0$.]

For (ii), let

$$U_{h_2}(x) = \left(\frac{1}{1-h_2(F)}\right)^{\leftarrow}(x),\ \ x > 1.$$

Then $U_{h_2} \in \mathrm{RV}_{\gamma/\beta_2}$. Again, if $W \sim \mathrm{Uniform}[0,1]$ then $U_{h_2}(1/W) \stackrel{\mathrm{d}}{=} Z \sim h_2(F)$. Then we have

$$\mathrm{H}_{u_2,h_2}((X-x)_-) = \int_{-\infty}^{x} u_2(x-z)\,\mathrm{d}h_2(F(y)) = \mathbb{E}[u_2((Z-x)_-)].$$

Consider the split

$$\frac{\mathbb{E}[u_2((Z-x)_-)]}{u_2(x)} = \int_{-\infty}^{x/2} \frac{u_2(x-z)}{u_2(x)}\,\mathrm{d}h_2(F(z)) + \int_{x/2}^{x} \frac{u_2(x-z)}{u_2(x)}\,\mathrm{d}h_2(F(z)) := I_1(x) + I_2(x).$$

To control $I_1(x)$, write

$$I_1(x) = \int_{\mathbb{R}} \frac{u_2(x-z)}{u_2(x)}\mathbf{1}\{z \leqslant x/2\}\,\mathrm{d}h_2(F(z))$$

where we extend the definition of u_2 on $\mathbb{R}$ by deciding that $u_2(y) = 0$ for $y < 0$. Clearly, since u_2 is regularly varying, $(u_2(x-z)/u_2(x))\mathbf{1}\{z \leqslant x/2\} \to 1$ pointwise in z as $x \to \infty$. Pick now $\delta > 0$ with $\int_{-\infty}^{\infty} |z|^{\alpha_2+\delta}\,\mathrm{d}h_2(F(z)) < \infty$. Since $u_2 \in \mathrm{RV}_{\alpha_2}$, Potter bounds yield, for x large enough,

$$\frac{u_2(x-z)}{u_2(x)}\mathbf{1}\{z \leqslant x/2\} \leqslant (1+\delta)\left(\frac{x-z}{x}\right)^{\alpha_2+\delta}\mathbf{1}\{z \leqslant x/2\} \leqslant C_\delta(1+|z|)^{\alpha_2+\delta}$$

where C_δ is a positive constant. This is an integrable function with respect to the measure

$\mathrm{d}h_2(F(z))$, so the dominated convergence theorem yields

$$\lim_{x\to\infty} I_1(x) = 1.$$

To control $I_2(x)$, note that $u_2(x) \geqslant u_2(x-z) \geqslant 0$ when $x/2 \leqslant z \leqslant x$ because u_2 is increasing. Therefore

$$0 \leqslant I_2(x) \leqslant \int_{x/2}^{x} \mathrm{d}h_2(F(z)) = h_2(F(x)) - h_2(F(x/2)) \to 0$$

as $x \to \infty$. Thus,

$$\lim_{x\to\infty} \frac{\mathbb{E}[u_2((Z-x)_-)]}{u_2(x)} = 1.$$

The desired result follows. □

Proof of Theorem 3.1. The assertions on the regular variation property of φ and the existence of the generalized inverse $\varphi^{\leftarrow}$ are immediate, see for example Definition B.1.8 p.366 of de Haan and Ferreira (2006). Combine Equation (1.1) and the first-order expansions in Lemma 3.1 to get

$$\Delta_0(1 - h_1(F(x_\tau)))u_1(x_\tau) \sim (1-\tau)u_2(x_\tau).$$

as $\tau \to 1$. This is readily seen to be equivalent to $\varphi(x_\tau) \sim \Delta_0(1-\tau)^{-1}$. When $s > 0$, $\varphi^{\leftarrow} \in \mathrm{RV}_{1/s}$, see Proposition B.1.9.9 p.367 of de Haan and Ferreira (2006). It immediately follows, by this same proposition, that

$$x_\tau \sim \varphi^{\leftarrow}(\varphi(x_\tau)) \sim \varphi^{\leftarrow}(\Delta_0(1-\tau)^{-1}) \sim \Delta_0^{1/s}\varphi^{\leftarrow}((1-\tau)^{-1})$$

as $\tau \to 1$. This is the required result. □

Proof of Lemma 4.1. Note that for any $v > 0$,

$$\lim_{t\to\infty} v^{-\gamma}\frac{\frac{g(vt)}{g(t)} - v^{\gamma}}{B(t)} = \lim_{t\to\infty} \frac{(tv)^{-\gamma}g(vt) - t^{-\gamma}g(t)}{t^{-\gamma}g(t)B(t)} = \frac{v^{\rho}-1}{\rho}.$$

This implies that $t^{-\gamma}g(t) \in \mathrm{ERV}_\rho$. Since $\rho < 0$, by Theorem B.2.2 of de Haan and Ferreira (2006), $g_0 = \lim_{t\to\infty} t^{-\gamma}g(t)$ exists, and $h(t) := g_0 - t^{-\gamma}g(t) \in \mathrm{RV}_\rho$. Besides, by Theorem B.2.18 of de Haan and Ferreira (2006), there exists $\widetilde{B}(t) \sim B(t)$ (which may be chosen bounded on intervals of the form $(0, t_0]$) such that $t^{-\gamma}g(t)\widetilde{B}(t) = -\rho h(t)$. We have

$$v^{-\gamma}\frac{\frac{g(vt)}{g(t)} - v^{\gamma}}{\widetilde{B}(t)} = \frac{h(t) - h(tv)}{-\rho h(t)} = -\frac{1}{\rho}\left(1 - \frac{h(tv)}{h(t)}\right).$$

Conclude, by Proposition B.1.9.7 of de Haan and Ferreira (2006), that for any $\varepsilon, \delta > 0$, there exist $c > 0$ and t_0 such that for all $t \geqslant t_0$ and $0 < v < \delta$,

$$\left| \frac{\frac{g(vt)}{g(t)} - v^\gamma}{\widetilde{B}(t)} \right| = -\frac{v^\gamma}{\rho} \left| \frac{h(tv)}{h(t)} - 1 \right| \leqslant -\frac{v^\gamma}{\rho} \left(1 + \left| \frac{h(tv)}{h(t)} \right| \right) \leqslant -\frac{v^\gamma}{\rho} (1 + cv^{\rho - \varepsilon}).$$

This is the desired result. □

Proof of Lemma 4.3. Set $c_h = bc^{\beta_1/\gamma}$. By Lemma 4.2 (ii) and (iv), we have

$$\begin{aligned}
\varphi(x) &= \frac{u_2(x)}{u_1(x)(1 - h_1(F(x)))} \\
&= \frac{a_2 x^{\alpha_2} \left[1 + \frac{1}{\eta_2} B_2(x) + o(B_2(x)) \right]}{a_1 x^{\alpha_1} \left[1 + \frac{1}{\eta_1} B_1(x) + o(B_1(x)) \right] \; c_h x^{-\beta_1/\gamma} \left[1 + \frac{1}{\rho_h} A_h(x) + o(A_h(x)) \right]} \\
&= \frac{a_2}{a_1 c_h} \; x^s \left[1 + \frac{1}{\eta_2} B_2(x)(1 + o(1)) - \frac{1}{\eta_1} B_1(x)(1 + o(1)) - \frac{1}{\rho_h} A_h(x)(1 + o(1)) \right] \\
&= \frac{a_2}{a_1 c_h} \; x^s \left[1 + \left(\frac{1}{\eta_2} B_2(x) - \frac{1}{\eta_1} B_1(x) - \frac{1}{\rho_h} A_h(x) \right) (1 + o(1)) \right] \\
&=: \frac{a_2}{a_1 c_h} \; x^s \left[1 + A^*(x)(1 + o(1)) \right].
\end{aligned}$$

[In the penultimate line the condition linking a, the b_i and κ was used to "merge" the $o(1)$ terms.] By Lemma 2.1 (ii), we have

$$\varphi^{\leftarrow}(x) = \left(\frac{a_2}{c_h a_1} \right)^{-1/s} x^{1/s} \left(1 - \frac{1}{s} A^*(\varphi^{\leftarrow}(x))(1 + o(1)) \right)$$

and $\varphi^{\leftarrow} \in 2\mathrm{RV}_{1/s, \eta^*/s}$, where $\eta^* = \max\{\eta_1, \rho_h, \eta_2\}$. The desired representation of $\varphi^{\leftarrow}((1-\tau)^{-1})$ follows.

Then, from the representation of $F^{\leftarrow}(\tau) = U((1-\tau)^{-1})$ in Lemma 4.2, we have

$$\begin{aligned}
\frac{\varphi^{\leftarrow}((1-\tau)^{-1})}{(F^{\leftarrow}(\tau))^{1/(\gamma s)}} &= \frac{c^*(1-\tau)^{-1/s} \left(1 - \frac{1}{s} A^*(\varphi^{\leftarrow}((1-\tau)^{-1}))(1 + o(1)) \right)}{\left(c(1-\tau)^{-\gamma} \left[1 + \frac{1}{\rho} A\left(\frac{1}{1-\tau} \right)(1 + o(1)) \right] \right)^{1/(\gamma s)}} \\
&= \frac{c^*}{c^{1/(\gamma s)}} \left(1 - \frac{1}{s} A^*(\varphi^{\leftarrow}((1-\tau)^{-1}))(1 + o(1)) - \frac{1}{\gamma s \rho} A\left(\frac{1}{1-\tau} \right)(1 + o(1)) \right).
\end{aligned}$$

It follows that $\varphi^{\leftarrow}((1-\tau)^{-1})$ is asymptotically equivalent to $c_0(F^{\leftarrow}(\tau))^{1/(\gamma s)}$ and then

$$\frac{\varphi^{\leftarrow}((1-\tau)^{-1})}{(F^{\leftarrow}(\tau))^{1/(\gamma s)}} = c_0 \left(1 - \frac{1}{s} A^*(c_0 \, (F^{\leftarrow}(\tau))^{1/(\gamma s)})(1 + o(1)) - \frac{1}{\gamma s \rho} A\left(\frac{1}{1-\tau} \right)(1 + o(1)) \right).$$

The proof is complete. $\square$

Proof of Lemma 4.4. Recall (7.2) and write, for $x > 0$,

$$\begin{aligned}
\frac{\mathrm{H}_{u_1,h_1}((X-x)_+)}{u_1(x)\,(1-h_1(F(x)))} - \Delta_0 &= -\left(\int_1^\infty \frac{u_1(xy-x)}{u_1(x)}\,\mathrm{d}\frac{1-h_1(F(xy))}{1-h_1(F(x))} - \int_1^\infty (y-1)^{\alpha_1}\ \mathrm{d}y^{-\beta_1/\gamma}\right) \\
&= -\int_1^\infty \left(\frac{u_1(xy-x)}{u_1(x)} - (y-1)^{\alpha_1}\right)\mathrm{d}\frac{1-h_1(F(xy))}{1-h_1(F(x))} \\
&\quad - \left(\int_1^\infty (y-1)^{\alpha_1}\ \mathrm{d}\frac{1-h_1(F(xy))}{1-h_1(F(x))} - \int_1^\infty (y-1)^{\alpha_1}\ \mathrm{d}y^{-\beta_1/\gamma}\right) \\
&= -\int_1^\infty \left(\frac{u_1(xy-x)}{u_1(x)} - (y-1)^{\alpha_1}\right)\mathrm{d}\frac{1-h_1(F(xy))}{1-h_1(F(x))} \\
&\quad + \int_1^\infty \left(\frac{1-h_1(F(xy))}{1-h_1(F(x))} - y^{-\beta_1/\gamma}\right)\mathrm{d}\,(y-1)^{\alpha_1} \\
&:= -\int_1^\infty I_1(x,y)\,\mathrm{d}\frac{1-h_1(F(xy))}{1-h_1(F(x))} + \int_1^\infty I_2(x,y)\,\mathrm{d}\,(y-1)^{\alpha_1}\,.
\end{aligned}$$

where in the third step we used integration by parts.

We first analyze $I_1(x,y)$. Since $u_1 \in 2\mathrm{RV}_{\alpha_1,\eta_1}$ with auxiliary function B_1, there is $\tilde{B}_1 \sim B_1$ such that for any (henceforth fixed) $\varepsilon, \delta > 0$, there is $x_0 > 0$ such that the following inequality holds for all $x > x_0$ and $xy > x_0$,

$$\left|\frac{\frac{u_1(x(y-1))}{u_1(x)} - (y-1)^{\alpha_1}}{\tilde{B}_1(x)} - J_{\alpha_1,\eta_1}(y-1)\right| \leqslant \varepsilon (y-1)^{\alpha_1+\eta_1\pm\delta},$$

where $y^{\alpha\pm\delta} = y^\alpha \max(y^\delta, y^{-\delta})$ (and recall that $J_{\gamma,\rho}(x) = x^\gamma \frac{x^\rho - 1}{\rho}$). In particular, if $\varepsilon_0 \in (0,1)$ is fixed, then for x large enough,

$$\forall y \geqslant 1+\varepsilon_0,\ J_{\alpha_1,\eta_1}(y-1) - \varepsilon(y-1)^{\alpha_1+\eta_1\pm\delta} \leqslant \frac{I_1(x,y)}{\tilde{B}_1(x)} \leqslant J_{\alpha_1,\eta_1}(y-1) + \varepsilon(y-1)^{\alpha_1+\eta_1\pm\delta}.$$

By integration by parts and since $1 - h_1(F(\cdot)) \in \mathrm{RV}_{-\beta_1/\gamma}$, a lim sup / lim inf argument similar to that used in Lemma 3.1 yields,

$$\lim_{x\to\infty} \frac{-\int_{1+\varepsilon_0}^\infty I_1(x,y)\,\mathrm{d}\frac{1-h_1(F(xy))}{1-h_1(F(x))}}{\tilde{B}_1(x)} = J_{\alpha_1,\eta_1}(\varepsilon_0)(1+\varepsilon_0)^{-\beta_1/\gamma} + \int_{1+\varepsilon_0}^\infty y^{-\beta_1/\gamma}\,\mathrm{d}J_{\alpha_1,\eta_1}(y-1).$$

Besides, by Lemma 4.1, there is a constant $C = C(\varepsilon) > 0$ such that for x large enough,

$$1 < y < 1+\varepsilon_0 \Rightarrow \left|\frac{I_1(x,y)}{\tilde{B}_1(x)}\right| \leqslant -\frac{(y-1)^{\alpha_1}}{\eta_1}\left(1 + C(y-1)^{\eta_1-\varepsilon}\right).$$

Then, as in the proof of Lemma 3.1, one finds

$$\limsup_{x\to\infty}\left|\frac{\int_1^{1+\varepsilon_0} I_1(x,y)\,\mathrm{d}\frac{1-h_1(F(xy))}{1-h_1(F(x))}}{\tilde{B}_1(x)}\right|$$
$$\leqslant -\frac{\varepsilon_0^{\alpha_1}}{\eta_1}\left(1+C\varepsilon_0^{\eta_1-\varepsilon}\right)(1+\varepsilon_0)^{-\beta_1/\gamma} - \int_1^{1+\varepsilon_0} y^{-\beta_1/\gamma}\,\mathrm{d}\left\{\frac{(y-1)^{\alpha_1}}{\eta_1}\left(1+C(y-1)^{\eta_1-\varepsilon}\right)\right\}.$$

Recall that $\alpha_1+\eta_1>0$, so that the right-hand side above is well-defined and finite for $\varepsilon>0$ small enough, and tends to 0 as $\varepsilon_0\to 0$. Adding up the contributions from 1 to $1+\varepsilon_0$ and beyond $1+\varepsilon_0$, and letting $\varepsilon_0\to 0$, we get

$$\begin{aligned}
\lim_{x\to\infty}\frac{-\int_1^{\infty} I_1(x,y)\,\mathrm{d}\frac{1-h_1(F(xy))}{1-h_1(F(x))}}{\tilde{B}_1(x)} &= \int_1^{\infty} y^{-\beta_1/\gamma}\,\mathrm{d}J_{\alpha_1,\eta_1}(y-1)\\
&= \frac{1}{\eta_1}\int_0^1 u^{\beta_1/\gamma}\left((\alpha_1+\eta_1)(u^{-1}-1)^{\alpha_1+\eta_1-1}-\alpha_1(u^{-1}-1)^{\alpha_1-1}\right)\frac{\mathrm{d}u}{u^2}\\
&= \frac{1}{\eta_1}\left((\alpha_1+\eta_1)\mathrm{B}(\beta_1/\gamma-\alpha_1-\eta_1,\alpha_1+\eta_1)-\alpha_1\mathrm{B}(\beta_1/\gamma-\alpha_1,\alpha_1)\right)\\
&= \frac{\beta_1}{\gamma}\times\frac{1}{\eta_1}(\mathrm{B}(\beta_1/\gamma-\alpha_1-\eta_1,\alpha_1+\eta_1+1)-\mathrm{B}(\beta_1/\gamma-\alpha_1,\alpha_1+1)).
\end{aligned}$$

We turn to controlling $I_2(x,y)$. By Lemma 4.2 (iv),

$$\forall y>0,\ \lim_{x\to\infty}\frac{I_2(x,y)}{A_h(x)} = J_{-\beta_1/\gamma,\rho_h}(y)$$

and for any $\delta>0$, there exist $\tilde{A}_h\sim A_h$ and $x_0>0$ such that for all $x>x_0$ and $y\geqslant 1$,

$$\left|\frac{I_2(x,y)}{\tilde{A}_h(x)}\right|(y-1)^{\alpha_1-1}\leqslant (y-1)^{\alpha_1-1}(J_{-\beta_1/\gamma,\rho_h}(y)+y^{-\beta_1/\gamma+\rho_h+\delta}).$$

If $\delta>0$ is chosen sufficiently small then the right-hand side defines an integrable function on $(1,\infty)$. The dominated convergence theorem then entails

$$\begin{aligned}
\lim_{x\to\infty}\frac{\int_1^{\infty} I_2(x,y)\,\mathrm{d}\,(y-1)^{\alpha_1}}{\tilde{A}_h(x)} &= \int_1^{\infty} J_{-\beta_1/\gamma,\rho_h}(y)\,\mathrm{d}\,(y-1)^{\alpha_1}\\
&= \frac{\alpha_1}{\rho_h}\int_0^1\left(v^{-\rho_h}-1\right)(1-v)^{\alpha_1-1}v^{\beta_1/\gamma-\alpha_1-1}\,\mathrm{d}v\\
&= \frac{\alpha_1}{\rho_h}\left(\mathrm{B}(\beta_1/\gamma-\alpha_1-\rho_h,\alpha_1)-\mathrm{B}(\beta_1/\gamma-\alpha_1,\alpha_1)\right)\\
&= \frac{1}{\rho_h}\left(\left(\frac{\beta_1}{\gamma}-\rho_h\right)\mathrm{B}(\beta_1/\gamma-\alpha_1-\rho_h,\alpha_1+1)-\frac{\beta_1}{\gamma}\mathrm{B}(\beta_1/\gamma-\alpha_1,\alpha_1+1)\right).
\end{aligned}$$

The proof is complete. □

Proof of Lemma 4.5. Recall from the proof of Lemma 3.1 that if $Z \sim h_2(F)$,

$$\mathrm{H}_{u_2,h_2}((X-x)_-) = \int_{-\infty}^{x} u_2(x-z)\,\mathrm{d}h_2(F(z)) = \mathbb{E}[u_2((Z-x)_-)].$$

Note now that for any $z < x$,

$$\frac{u_2(x)-u_2(x-z)}{u_2'(x)} = \frac{u_2'(\xi)}{u_2'(x)}z, \tag{7.4}$$

where $\xi \in (x-z, x)$ if $0 \leqslant z < x$ and $\xi \in (x, x-z)$ if $z < 0$. Also, from (7.4),

$$\lim_{x\to\infty} \frac{u_2(x)-u_2(x-z)}{u_2'(x)} = z$$

holds for any $z \in \mathbb{R}$, because regular variation is locally uniform. Since

$$\mathrm{H}_{u_2,h_2}((X-x)_-) = u_2(x) - u_2(x)(1-h_2(F(x)) - u_2'(x)\int_{-\infty}^{x} \frac{u_2(x)-u_2(x-z)}{u_2'(x)}\,\mathrm{d}h_2(F(z)),$$

we are left to show that

$$\lim_{x\to\infty}\int_{-\infty}^{x} \frac{u_2(x)-u_2(x-z)}{u_2'(x)}\,\mathrm{d}h_2(F(z)) = \int_{-\infty}^{\infty} z\,\mathrm{d}h_2(F(z)), \tag{7.5}$$

that is, the integral and the limit are interchangeable. By Proposition B.1.9.6 of de Haan and Ferreira (2006), there exist $C > 0$, $x_0 > 0$ such that for $x \geqslant x_0$, $0 < \xi/x \leqslant 1$,

$$\frac{u_2'(\xi)}{u_2'(x)} \leqslant C. \tag{7.6}$$

Note that (7.6) holds for the case of $\alpha_2 = 1$ since in this case by assumption u_2' is nondecreasing. Moreover, by Proposition B.1.9.5 of de Haan and Ferreira (2006), for any $\delta > 0$, there is $x_1 > 0$ such that for $x \geqslant x_1$ and $\xi/x \geqslant 1$,

$$\frac{u_2'(\xi)}{u_2'(x)} \leqslant 2\left(\frac{\xi}{x}\right)^{\alpha_2-1+\delta}.$$

Conclude that, with ξ as in (7.4), that for x large enough,

$$\begin{aligned}\forall z < x,\ \left|\frac{u_2'(\xi)}{u_2'(x)}z\right| &\leqslant \left\{C\mathbb{1}\{0 \leqslant z < x\} + 2\left(\frac{x-z}{x}\right)^{\alpha_2-1+\delta}\mathbb{1}\{z<0\}\right\}|z| \\ &\leqslant \left\{C + 2(1-z)^{\alpha_2-1+\delta}\mathbb{1}\{z<0\}\right\}|z|.\end{aligned}$$

By the assumption that $\int_{-\infty}^{\infty} |z|^{\alpha_2+\delta}\,\mathrm{d}h_2(F(z)) < \infty$ for some $\delta > 0$ and the dominated convergence theorem, (7.5) holds and therefore, as $x \to \infty$,

$$\frac{\mathrm{H}_{u_2,h_2}((X-x)_-)}{u_2(x)} = 1 - (1 - h_2(F(x))) - \frac{u_2'(x)}{u_2(x)}(\mathbb{E}[Z] + o(1)).$$

The final identity is obtained by applying Theorem B.1.5 in de Haan and Ferreira (2006). □

Proof of Theorem 4.1 and Theorem 4.2. Combining Equation (1.1) with Lemmas 4.4 and 4.5, the shortfall risk measure x_τ satisfies

$$\begin{aligned}&(1 - h_1(F(x_\tau)))\,u_1(x_\tau)\,(\Delta_0 + \Gamma_1 B_1(x_\tau)(1+o(1)) + \Gamma_2 A_h(x_\tau)(1+o(1)))\\ &= (1-\tau)u_2(x_\tau)\left(1 - (1 - h_2(F(x_\tau))) - x_\tau^{-1}(\alpha_2\mathbb{E}[Z] + o(1)) + (1-\tau)(1+o(1))\right),\end{aligned}$$

where Z is a random variable having the distribution $h_2(F)$. After some rearrangements, taking $\varphi^{\leftarrow}$ on both sides above yields

$$\varphi^{\leftarrow}\left(\varphi(x_\tau)\frac{1 - (1 - h_2(F(x_\tau))) - x_\tau^{-1}(\alpha_2\mathbb{E}[Z] + o(1)) + (1-\tau)(1+o(1))}{\Delta_0 + \Gamma_1 B_1(x_\tau)(1+o(1)) + \Gamma_2 A_h(x_\tau)(1+o(1))}\right) = \varphi^{\leftarrow}((1-\tau)^{-1}). \tag{7.7}$$

The left-hand side in Equation (7.7) can be further rewritten as

$$\begin{aligned}\varphi^{\leftarrow}\Bigg(\Delta_0^{-1}\varphi(x_\tau)\Bigg[\Bigg(&1 - \frac{\Gamma_1}{\Delta_0}B_1(x_\tau)(1+o(1)) - \frac{\Gamma_2}{\Delta_0}A_h(x_\tau)(1+o(1))\\ &-(1 - h_2(F(x_\tau)))(1+o(1)) - \alpha_2\mathbb{E}[Z]\frac{1}{x_\tau}(1+o(1)) + (1-\tau)(1+o(1))\Bigg)\Bigg]\Bigg).\end{aligned}$$

Then by Lemma 2.1 and Lemma 4.3, with some calculations we obtain

$$\begin{aligned}\frac{\varphi^{\leftarrow}((1-\tau)^{-1})}{x_\tau} &= \frac{\varphi^{\leftarrow}((1-\tau)^{-1})}{\varphi^{\leftarrow}(\varphi(x_\tau))(1+o(A^*(x_\tau)))}\\ &= \Delta_0^{-1/s}\Bigg(1 - \Bigg(\frac{\Gamma_1}{s\Delta_0}B_1(x_\tau)(1+o(1)) + \frac{\Gamma_2}{s\Delta_0}A_h(x_\tau)(1+o(1)) + \frac{1}{s}(1 - h_2(F(x_\tau))(1+o(1))\\ &\quad + \frac{\alpha_2\mathbb{E}[Z]}{s}\frac{1}{x_\tau}(1+o(1)) - \frac{1 - \Delta_0^{-\eta^*/s}}{s}A^*(x_\tau)(1+o(1)) - \frac{1}{s}(1-\tau)(1+o(1))\Bigg)\Bigg).\end{aligned}$$

Applying Theorem 3.1, we have $x_\tau \sim \Delta_0^{1/s}\varphi^{\leftarrow}((1-\tau)^{-1})$, and therefore $B_1(x_\tau) \sim \Delta_0^{\eta_1/s}B_1(\varphi^{\leftarrow}((1-\tau)^{-1}))$, $A_h(x_\tau) \sim \Delta_0^{\rho_h/s}A_h(\varphi^{\leftarrow}((1-\tau)^{-1}))$, $1 - h_2(F(x_\tau)) \sim \Delta_0^{-\beta_2/(\gamma s)}(1 - h_2(F(\varphi^{\leftarrow}((1-\tau)^{-1}))))$ and $A^*(x_\tau) \sim \Delta_0^{\eta^*/s}A^*(\varphi^{\leftarrow}((1-\tau)^{-1}))$. The result of Theorem 4.1 follows. Theorem 4.2 is then obtained by applying Lemma 4.3. □

Proof of Theorem 5.1. By Lemma 2.1(ii), $1/(1-h_1^{-1}(1-1/\cdot))$, the inverse of $1/(h_1(1-1/\cdot))$, is $2\mathrm{RV}_{1,\varsigma}$, and therefore, by Lemma 2.1(i),

$$\frac{1-\tau_n}{1-h_1^{-1}(\tau_n)} = \frac{1-\tau_n}{1-h_1^{-1}(1-1/(1-\tau_n)^{-1})} \to K \in (0,\infty) \text{ as } n\to\infty.$$

We then break down $\log(\widehat{x}_{\tau_n}/x_{\tau_n})$ in the following fashion:

$$\log\frac{\widehat{x}_{\tau_n}}{x_{\tau_n}} = \log\left(\frac{1}{q_{\tau_n}}\left(\frac{k_n}{n(1-\tau_n)}\right)^{\widehat{\gamma}_n} X_{n-k_n,n}\right) + \log\frac{\Psi(\widehat{\gamma}_n)}{\Psi(\gamma)} + (\widehat{\gamma}_n-\gamma)\log\left(\frac{1-\tau_n}{1-h_1^{-1}(\tau_n)}\right)$$
$$+\log\left(\left(\frac{1-\tau_n}{1-h_1^{-1}(\tau_n)}\right)^{\gamma}\frac{q_{\tau_n}}{q_{h_1^{-1}(\tau_n)}}\right) + \log\left(\left(\frac{1}{\gamma}\mathrm{B}(1/\gamma-\alpha,\alpha+1)\right)^{\gamma}\frac{q_{h_1^{-1}(\tau_n)}}{x_{\tau_n}}\right)$$

with $\Psi(\gamma) = \left(\frac{1}{\gamma}\mathrm{B}(1/\gamma-\alpha,\alpha+1)\right)^{\gamma}$, a continuously differentiable function on the positive half-line. Now,

$$\frac{\sqrt{k_n}}{\log(k_n/(n(1-\tau_n)))}\log\left(\frac{1}{q_{\tau_n}}\left(\frac{k_n}{n(1-\tau_n)}\right)^{\widehat{\gamma}_n} X_{n-k_n,n}\right) \xrightarrow{d} N$$

by Theorem 4.3.8 p.138 of de Haan and Ferreira (2006). It only remains to show that the four other terms in the above decomposition of $\log(\widehat{x}_{\tau_n}/x_{\tau_n})$ are asymptotically negligible. We start by writing

$$\frac{\sqrt{k_n}}{\log(k_n/(n(1-\tau_n)))}\log\frac{\Psi(\widehat{\gamma}_n)}{\Psi(\gamma)} = o_{\mathbb{P}}\left(\sqrt{k_n}\log\frac{\Psi(\widehat{\gamma}_n)}{\Psi(\gamma)}\right) = o_{\mathbb{P}}(1)$$

by the delta-method. Likewise,

$$\frac{\sqrt{k_n}}{\log(k_n/(n(1-\tau_n)))}(\widehat{\gamma}_n-\gamma)\log\left(\frac{1-\tau_n}{1-h_1^{-1}(\tau_n)}\right) = O_{\mathbb{P}}\left(\frac{\sqrt{k_n}}{\log(k_n/(n(1-\tau_n)))}(\widehat{\gamma}_n-\gamma)\right) = o_{\mathbb{P}}(1).$$

The final two terms in the decomposition of $\log(\widehat{x}_{\tau_n}/x_{\tau_n})$ are bias terms. First of all, using assumption $U\in 2\mathrm{RV}_{\gamma,\rho}$,

$$\log\left(\left(\frac{1-\tau_n}{1-h_1^{-1}(\tau_n)}\right)^{\gamma}\frac{q_{\tau_n}}{q_{h_1^{-1}(\tau_n)}}\right) = O(A((1-\tau_n)^{-1})) = o(A(n/k_n))$$

so that

$$\frac{\sqrt{k_n}}{\log(k_n/(n(1-\tau_n)))}\log\left(\left(\frac{1-\tau_n}{1-h_1^{-1}(\tau_n)}\right)^{\gamma}\frac{q_{\tau_n}}{q_{h_1^{-1}(\tau_n)}}\right) = o(1).$$

Finally, since $q_{h_1^{-1}(\tau_n)} = \varphi^{\leftarrow}((1-\tau_n)^{-1})$,

$$\frac{\sqrt{k_n}}{\log(k_n/(n(1-\tau_n)))}\log\left(\left(\frac{1}{\gamma}\mathrm{B}(\alpha,1/\gamma-\alpha+1)\right)^{\gamma}\frac{q_{h_1^{-1}(\tau_n)}}{x_{\tau_n}}\right) = o(1)$$

by Corollary 4.1 and the assumption $\sqrt{k_n}(k_n/n+|A(n/k_n)|+|B(q_{1-k_n/n})|+|C(n/k_n)|+1/q_{1-k_n/n}) = O(1)$. The proof is complete. □

Acknowledgments

The authors gratefully acknowledge two anonymous referees for their helpful comments which resulted in a substantially improved version of this article, as well as Antoine Usseglio-Carleve for pointing us to the `cubature` package in R for numerical integration tasks. T. Mao gratefully acknowledges the financial support from Anhui Natural Science Foundation (No. 2208085MA07) and the National Natural Science Foundation of China (No. 71921001). G. Stupfler gratefully acknowledges support from the French ANR (grant ANR-19-CE40-0013, the EUR CHESS under grant ANR-17-EURE-0010, and the Centre Henri Lebesgue under grant ANR-11-LABX-0020-01), from an AXA Research Fund Award on "Mitigating risk in the wake of the COVID-19 pandemic", and from the TSE-HEC ACPR Chair. F. Yang gratefully acknowledges financial support from the Natural Sciences and Engineering Research Council of Canada (Grant Number: 04242).

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