An improved multiplicity bound for eigenvalues of the clamped disk

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Abstract

We prove that no eigenvalue of the clamped disk has multiplicity greater than four. This improves upon a previous bound. Exploiting a linear recursion formula of order two for cross-product Bessel functions in which the coefficients are non-rational functions satisfying a non-linear algebraic recursion, we show that higher multiplicity eigenvalues must be algebraic, in contradiction with the Siegel-Shidlovskii theory.

1 Introduction and background

1.1 The vibrating clamped plate

In this paper we are concerned with the vibrating clamped circular plate [1, ch. V§6], that is, the fourth order eigenvalue problem

$$\text{(VP)} \qquad \begin{cases} \Delta^2 u &= \lambda u \qquad \text{in } \mathbb{D}, \\ u &= 0 \qquad \text{on } \partial\mathbb{D}, \\ \partial_n u &= 0 \qquad \text{on } \partial\mathbb{D}. \end{cases}$$

where $\Delta = \operatorname{div} \circ \operatorname{grad}$ is the Laplacian. A basis of eigenfunctions is given in polar coordinates by the family

$$u_{m.k}(r, \phi) = \big(I_m(w_{m,k})J_m(w_{m,k}r) - J_m(w_{m,k})I_m(w_{m,k}r)\big)e^{im\phi}$$

with corresponding eignevalues $\lambda = w_{m,k}^4$, where J_m, I_m denote the Bessel and modified Bessel functions respectively of the first kind, and $w_{m,k}$ is a zero of the cross product

$$W_m := I_{m+1}J_m + I_mJ_{m+1} = I_{m-1}J_m - I_mJ_{m-1}\ . \tag{1}$$

We are interested to know whether non-trivial multiplicities occur in the spectrum. The analogous problem for the vibrating circular membrane was solved in [3] (see also [4, 5, ch. 15.28]). The problem amounts to the question whether there exist $m_1, m_2 \in \mathbb{N}_0$ distinct and $x_0 > 0$ with $W_{m_1}(x_0) = W_{m_2}(x_0) = 0$. However, this seems to be a difficult open problem. In [2] it was shown that there do not exist $m_1, m_2, m_3, m_4 \in \mathbb{N}_0$ pairwise distinct and $x_0 > 0$ for which $W_{m_1}(x_0) = W_{m_2}(x_0) = W_{m_3}(x_0) = W_{m_4}(x_0) = 0$. The proof was based on a *fourth* order recursion formula for the sequence W_m with rational functions as coefficients. The aim of this paper is to prove

Theorem 1.1. *There do not exist $m_1, m_2, m_3 \in \mathbb{N}_0$ pairwise distinct and $x_0 > 0$ such that $W_{m_1}(x_0) = W_{m_2}(x_0) = W_{m_3}(x_0) = 0$.*

As a corollary we obtain

Corollary 1.2. *Let λ be an eigenvalue of the vibrating clamped disk problem. Then, λ is of multiplicity four at most.*

To prove Theorem 1.1 we show that if $m_1, m_2, m_3 \in \mathbb{N}_0$ pairwise distinct and $x_0 > 0$ with $W_{m_1}(x_0) = W_{m_2}(x_0) = W_{m_3}(x_0) = 0$ exist, then x_0 must be algebraic. However, an immediate application of the Siegel-Shidlovskii theory shows that any positive root of the equation $W_m(x_0) = 0$ is transcendental. The main new ingredient in our proof with respect to [2], is a *second* order linear recursion for the sequence W_m, whose coefficients, while not rational, satisfy an algebraic *non-linear* recursion of degree two. At a first step we show that each joint zero x_0 of W_m and $W_{m'}$ leads to an equation of the form

$$P_{m,m'}(x_0, f(x_0)) = 0 \tag{2}$$

where $P_{m,m'}(x, y)$ is a polynomial of degree two with respect to y and f is a transcendental function. At a second step we prove that it is possible to eliminate f from a system of any two such equations, leading to a non-trivial polynomial equation for x_0.

1.2 Acknowledgments

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2 Bessel functions and their quotients

Let m be an integer. The Bessel function J_m can be defined as the power series

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2k}$$

The modified Bessel function I_m is the power series

$$I_m(x) = \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2k}$$

Proposition 2.1. *[5, chs. 2.12, 3.71] The following recursions are satisfied.*

$$J_{m+1} = \frac{2m}{x} J_m - J_{m-1}$$
$$I_{m+1} = -\frac{2m}{x} I_m + I_{m-1}$$

We will consider quotients of successive modified Bessel functions.

Definition 2.2. *For $x \in \mathbb{R}$ we set*

$$F_m(x) := \frac{I_m(x)}{xI_{m-1}(x)} \ .$$

The following identity, which can be viewed as a discrete Riccati equation will be important in the sequel.

Key Identity 2.3.

$$x^2 F_{m+1}(x) F_m(x) = 1 - 2mF_m(x)$$

Proof. From the definition of F_m and Proposition 2.1 we have

$$x^2 F_{m+1} F_m = x^2 \cdot \frac{I_{m+1}}{xI_m} \cdot \frac{I_m}{xI_{m-1}} = \frac{1}{I_{m-1}} \left(I_{m-1} - \frac{2m}{x} I_m \right) = 1 - 2mF_m \ .$$

□

A similar computation, which we omit, shows also

Lemma 2.4. $x^2 F_{-m+1} = x^2 F_{m+1} + 2m$

3 Second order recursion for cross products of Bessel functions

The sequence $(W_m)_{m \in \mathbb{Z}}$ satisfies a fourth order linear recurrence with non-constant coefficients in $\mathbb{Q}(x)$ (see [2]). However, it also satisfies a second order linear recurrence whose coefficients, while not in $\mathbb{Q}(x)$, satisfy themselves a quadratic recursion. We prove

Theorem 3.1. *The following recursion formula holds.*

$$W_{m+1} = 2mF_m W_m + (2mF_m - 1) W_{m-1}$$

Proof. On the one hand we have by (1) and Proposition 2.1

$$\begin{aligned} W_{m-1} + W_{m+1} &= (I_m J_{m-1} + I_{m-1} J_m) + (I_m J_{m+1} - I_{m+1} J_m) \\ &= I_m \frac{2m}{x} J_m + \frac{2m}{x} I_m J_m = \frac{4m}{x} I_m J_m \ . \end{aligned}$$

On the other hand,

$$2mF_m(W_m + W_{m-1}) = 2m \frac{I_m}{xI_{m-1}} (I_{m-1} J_m - I_m J_{m-1} + I_m J_{m-1} + I_{m-1} J_m) = \frac{4m}{x} I_m J_m \ .$$

Comparing the preceding expressions gives the desired identity. □

4 Rolling out the recursion

In this section we use the second order recursion for W_m (Theorem 3.1) in order to express any element in the sequence in terms of two initial consecutive terms.

Proposition 4.1. *Let $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$. There exist polynomials $A_{m,n}$, $B_{m,n}$, $\tilde{B}_{m,n}$, $C_{m,n} \in \mathbb{Q}[x]$ such that:*

$$x^{2n} W_{m+n+1} = \big(A_{m,n} F_m + x^2 B_{m,n} + C_{m,n} F_m^{-1}\big) W_m + \Big(A_{m,n} F_m + \tilde{B}_{m,n} - C_{m,n} F_m^{-1}\Big) W_{m-1} \tag{3}$$

Remark. *Note that the coefficients in the preceding formula are linear in F_m and F_m^{-1}.*

Proof. The case $n = 0$ follows from Theorem 3.1. For $n \geq 1$

$$\begin{aligned} x^{2n} W_{m+n+1} &= x^2 x^{2n-2} W_{(m+1)+(n-1)+1} = \\ &\big(x^2 A_{m+1,n-1} F_{m+1} + x^4 B_{m+1,n-1} + x^2 C_{m+1,n-1} F_{m+1}^{-1}\big) W_{m+1} \\ &+ \Big(x^2 A_{m+1,n-1} F_{m+1} + x^2 \tilde{B}_{m+1,n-1} - x^2 C_{m+1,n-1} F_{m+1}^{-1}\Big) W_m \end{aligned}$$

We substitute W_{m+1} using Theorem 3.1 and Key Identity 2.3.

$$\begin{aligned} &x^{2n} W_{m+n+1} = \\ &\quad \big(2mx^2 A_{m+1,n-1} F_m F_{m+1} + 2mx^4 B_{m+1,n-1} F_m + 2mx^2 C_{m+1,n-1} F_m F_{m+1}^{-1}\big) W_m \\ &\quad - \big(x^2 A_{m+1,n-1} F_{m+1} + x^4 B_{m+1,n-1} + x^2 C_{m+1,n-1} F_{m+1}^{-1}\big) x^2 F_m F_{m+1} W_{m-1} \\ &\quad + \Big(x^2 A_{m+1,n-1} F_{m+1} + x^2 \tilde{B}_{m+1,n-1} - x^2 C_{m+1,n-1} F_{m+1}^{-1}\Big) W_m \end{aligned}$$

Applying Key Identity 2.3 and collecting terms gives

$$\begin{aligned} &x^{2n} W_{m+n+1} = \\ &\quad \Big(2mA_{m+1,n-1}(1 - 2mF_m) + 2mx^4 B_{m+1,n-1} F_m \\ &\qquad - x^2 C_{m+1,n-1}(1 - 2mF_m) F_{m+1}^{-1} + A_{m+1,n-1}(F_m^{-1} - 2m) + x^2 \tilde{B}_{m+1,n-1}\Big) W_m \\ &\quad - \big(x^2 A_{m+1,n-1} F_{m+1}(1 - 2mF_m) + x^4 B_{m+1,n-1}(1 - 2mF_m) + x^4 C_{m+1,n-1} F_m\big) W_{m-1} \end{aligned}$$

Applying once more Key Identity 2.3 and collecting terms gives

$$\begin{aligned} &x^{2n} W_{m+n+1} = \\ &\quad \Big(\big(-4m^2 A_{m+1,n-1} + 2mx^4 B_{m+1,n-1} - x^4 C_{m+1,n-1}\big) F_m \\ &\qquad + x^2 \tilde{B}_{m+1,n-1} + A_{m+1,n-1} F_m^{-1}\Big) W_m \\ &+ \Big(2mA_{m+1,n-1}(1 - 2mF_m) - A_{m+1,n-1}(F_m^{-1} - 2m) + \big(2mx^4 B_{m+1,n-1} - x^4 C_{m+1,n-1}\big) F_m \\ &\qquad - x^4 B_{m+1,n-1}\Big) W_{m-1} \end{aligned}$$

and finally,

$$
\begin{aligned}
&x^{2n}W_{m+n+1} = \\
&\quad\Big(\big(-4m^2A_{m+1,n-1} + 2mx^4B_{m+1,n-1} - x^4C_{m+1,n-1}\big)F_m \\
&\qquad + x^2\tilde{B}_{m+1,n-1} + A_{m+1,n-1}F_m^{-1}\Big)W_m \\
&\quad+\Big(\big(-4m^2A_{m+1,n-1} + 2mx^4B_{m+1,n-1} - x^4C_{m+1,n-1}\big)F_m \\
&\qquad + 4mA_{m+1,n-1} - x^4B_{m+1,n-1} - A_{m+1,n-1}F_m^{-1}\Big)W_{m-1}
\end{aligned}
$$

which is of the desired form. □

As an immediate consequence of the above computation we obtain the following Lemma.

Lemma 4.2. *Let $A_{m,n}$, $B_{m,n}$, $\tilde{B}_{m,n}$, and $C_{m,n}$ be as in Proposition 4.1. Then, the following recursive relations hold.*

(i) $A_{m,0} = 2m$, $A_{m,n} = -4m^2A_{m+1,n-1} + 2mx^4B_{m+1,n-1} - x^4C_{m+1,n-1}$

(ii) $B_{m,0} = 0$, $B_{m,n} = \tilde{B}_{m+1,n-1}$

(iii) $\tilde{B}_{m,0} = -1$, $\tilde{B}_{m,n} = 4mA_{m+1,n-1} - x^4B_{m+1,n-1}$

(iv) $C_{m,0} = 0$, $C_{m,n} = A_{m+1,n-1}$

As a corollary we have

Lemma 4.3. *Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then,*

$$A_{m,n} \equiv 2(-4)^n(m+n)\prod_{k=0}^{n-1}(m+k)^2 \mod x^4 \ .$$

In particular, if $m > 0$ or $m < -n$, then $A_{m,n} \not\equiv 0 \mod x$.

Proof. The proof follows immediately from Lemma 4.2, part (i). □

5 Proof of Theorem 1.1

We recall

Proposition 5.1 ([2])**.** *The functions W_m and W_{m+1} have no joint positive zeros.*

Proof. Assume $W_m(x_0) = W_{m+1}(x_0) = 0$ for some $x_0 > 0$. Observe that

$$W_{m+1} + W_m = I_mJ_{m+1} - I_{m+1}J_m + I_{m+1}J_m + I_mJ_{m+1} = 2I_mJ_{m+1}$$

and

$$W_{m+1} - W_m = I_mJ_{m+1} - I_{m+1}J_m - I_{m+1}J_m - I_mJ_{m+1} = -2I_{m+1}J_m$$

It follows that $J_m(x_0) = J_{m+1}(x_0) = 0$. However, this is impossible since it would imply that $J_m'(x_0) = (m/x_0)J_m(x_0) - J_{m+1}(x_0)$ is also zero, while J_m satisfies a second order linear ODE. □

A direct consequence of the preceding proposition and Proposition 4.1 is

Corollary 5.2. *Let $m \in \mathbb{Z}$, and $n \in \mathbb{N}_0$.*

(a) If x_0 is a joint zero of W_m, and W_{m+n+2}, then

$$A_{m+1,n}(x_0)F_{m+1}(x_0)^2 + x_0^2 B_{m+1,n}(x_0)F_{m+1}(x_0) + C_{m+1,n}(x_0) = 0$$

(b) If x_0 is a joint zero of W_m, and W_{m-n-2}, then

$$\begin{aligned} A_{-m+1,n}(x_0)\big(x_0^2 F_{m+1}(x_0) + 2m\big)^2 \\ + x_0^4 B_{-m+1,n}(x_0)\left(x_0^2 F_{m+1}(x_0) + 2m\right) + x_0^4 C_{-m+1,n}(x_0) = 0 \end{aligned}$$

Proof. Part (a) follows from Proposition 4.1 with m replaced by $m+1$, taking into account Proposition 5.1. Part (b) follows from Part (a) with m replaced by $-m$, taking into account Lemma 2.4. □

Proof of Theorem 1.1. Assume $0 \le m_1 < m_2 < m_3$ and $x_0 > 0$ are such that $W_{m_1}(x_0) = W_{m_2}(x_0) = W_{m_3}(x_0) = 0$. By Proposition 5.1 we can write $m_1 = m_2 - l - 2$, $m_2 = m$ and $m_3 = m_2 + n + 2$ with $l, m, n \in \mathbb{N}_0$. By Corollary 5.2 setting $x = x_0$ solves a system

$$\begin{cases} A_{m+1,n}(x)F_{m+1}(x)^2 + x^2 B_{m+1,n}(x)F_{m+1}(x) + C_{m+1,n}(x) = 0 \\ A_{-m+1,l}(x)\big(x^2F_{m+1}(x) + 2m\big)^2 + x^4 B_{-m+1,n}(x)(x^2F_{m+1}(x) + 2m) + x^4 C_{-m+1,n}(x) = 0 \end{cases} \tag{4}$$

Eliminating F_{m+1}^2 from the preceding system we obtain that x_0 is a root of an equation of the form

$$\big(4mA_{m+1,n}(x)A_{-m+1,l}(x) + x^4 P_1(x)\big)x^2 F_{m+1}(x) + 4m^2 A_{m+1,n}(x)A_{-m+1,l}(x) + x^4 P_2(x) = 0$$

for some polynomials $P_1, P_2 \in \mathbb{Q}[x]$, depending on l, m, n.

By Lemma 4.3 (and the fact that $m > l+1$) the polynomial $4mA_{m+1,n}A_{-m+1,l} + x^4 P_1$ is not zero. Hence, in case it vanishes at the point x_0 we get that x_0 is algebraic. Otherwise, using the preceding equation to eliminate F_{m+1} from the first equation in (4) leads to an equation of the form

$$16m^4 A_{m+1,n}(x_0)^3 A_{-m+1,l}(x_0)^2 + x_0^4 P_3(x_0) = 0$$

with $P_3 \in \mathbb{Q}[x]$ depending on l, m, n. From Lemma 4.3 it follows that x_0 is algebraic in this case too.

We have shown that x_0 is algebraic. However, this is impossible, as by the Siegel-Shidlovskii theory all positive roots of $W_m(x) = 0$ are transcendental (see [2, cor. 6.4]). □

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