

# Node Reliability: Approximation, Upper Bounds, and Applications to Network Robustness

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**Abstract**—This paper discusses the reliability of a graph in which the links are perfectly reliable but the nodes may fail with certain probability  $p$ . Calculating graph node reliability is an NP-Hard problem. We introduce an efficient and accurate Monte Carlo method and a stochastic approximation for the node reliability polynomial based solely on the degree distribution. We provide the formulas for the node reliability polynomial of both Erdős–Rényi graphs and Random Geometric graphs. The phase transition in the node reliability of Erdős–Rényi graphs such as are also discussed. Additionally, we propose two increasingly accurate upper bounds for the node reliability polynomial solely based on the graph’s degree distributions. The advantages and disadvantages of these two upper bounds are thoroughly compared. Beyond the computation of node reliability polynomials, we also estimate the number of cut sets and present a solution to the reliability-based network enhancement problem.

**Index Terms**—network robustness, node failure, probabilistic graph, reliability polynomial

## I. INTRODUCTION

**R**ELIABILITY research in network science is concerned with the estimation of the probability that the residual network remains operational after the failure of some components [3]. In 1956, Moore and Shannon [2] proposed a probabilistic model for network reliability. Based on the types of component that can fail, network reliability can be classified into two categories:

- **Network reliability w.r.t. link failures:** defined as the probability that the nodes of graph  $G$  remain connected if each link is operational with probability  $p$ , assuming the nodes of the graph are perfectly reliable [4]. This type of network reliability can be expressed as a so-called reliability polynomial:

$$\text{Rel}_G(p) = \sum_{j=0}^L F_j(G) (1-p)^j p_1^{L-j}, \quad (1)$$

where  $F_j(G)$  is the number of sets of  $j$  links whose removal leaves  $G$  connected, and  $F_0(G) = 1$ .

- **Network reliability w.r.t. node failures:** defined as the probability that the operational nodes of graph  $G$  remain

connected if each node is operational with probability  $p$ , assuming the links of the graph are perfectly reliable [4]. This type of network reliability can be expressed as the node reliability polynomial:

$$\text{nRel}_G(p) = \sum_{k=0}^N S_k(G) p^k (1-p)^{N-k}, \quad (2)$$

where  $S_k(G)$  denotes the number of induced connected subgraphs with  $k$  nodes.

Most studies on network reliability focus on link failures. This paper will focus on node failures. The problems of computing the reliability polynomial  $\text{Rel}_G(p)$  and node reliability polynomial  $\text{nRel}_G(p)$  are NP-hard [1], [5], [7]. Closed-form analytic expressions for the node reliability polynomial only exist for some specific graph topologies [6]. We give examples in Appendix A. Various Monte Carlo methods give accurate estimations for the node reliability polynomial, but suffer from a high computational complexity [8]–[10]. The reliability polynomial is a useful tool in network theory, used to characterize network structures and to guide optimal design. It captures crucial information about a network’s connectivity by encoding all possible cut sets—the sets of links or nodes whose removal would disconnect the network. The reliability polynomial thus serves as a comprehensive measure of a network’s global robustness. Networks with higher values of the reliability polynomial, under the same operational probability  $p$ , tend to be more resilient to disconnection, allowing comparisons between different network topologies. In addition to a structural analysis, the reliability polynomial plays a key role in network design [3], [19], [20]. The reliability polynomial can be used to optimize network reliability by identifying critical nodes or links whose addition or removal significantly affects the overall reliability [20]. For instance, adding links can enhance reliability in communication networks by increasing redundancy, whereas removing specific links can effectively contain the spread of diseases [20]. The reliability polynomial finds practical applications in fields such as communication networks, infrastructure systems, and public health [20], [24], [25]. The reliability polynomial helps to optimize network performance and resilience [21].

This paper first introduces a Laplace approximation for the node reliability polynomial in Section II. In Section III, we propose a Monte Carlo method for node reliability polynomials. The proposed Monte Carlo method is inspired by a recent fast approach designed for network reliability polynomials [18]. Additionally, the Monte Carlo method is combined with the Laplace approximation to develop a new hybrid approach,

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referred to as the Laplace Monte Carlo method. Section IV introduces a stochastic approximation for the node reliability polynomial. The relation between the reliability polynomial  $\text{Rel}_G(p)$  and the node reliability polynomial  $\text{nRel}_G(p)$  is also analyzed in Section IV. In Section V, we give the formulas of the node reliability polynomial for the Erdős-Rényi graph and random geometric graph. The intersection of node reliability polynomials for different Erdős-Rényi graphs and the phase transition of the node reliability for Erdős-Rényi graphs are also analyzed. Two different kinds of upper bounds for the node reliability polynomial are given in Section VI. Section VII discusses the practical applications of the network reliability.

## II. THE LAPLACE APPROXIMATION FOR THE NODE RELIABILITY POLYNOMIAL $\text{nRel}_G(p)$

The number of combinations of  $k$  different nodes out of  $N$  nodes is the binomial coefficient  $\binom{N}{k} = \frac{N!}{k!(N-k)!}$ . We define the number  $C_j(G)$  as the number of subsets of  $j$  nodes whose removal disconnect the graph  $G$ . Every subset of  $k$  nodes in  $G$  must either be connected or disconnected, leading to the following relationship:

$$S_k(G) + C_{N-k}(G) = \binom{N}{k} \quad (3)$$

After substituting  $S_k(G) = \binom{N}{k} - C_{N-k}(G)$  into (2), and applying Newton's binomial theorem,  $(a+b)^N = \sum_{k=0}^N \binom{N}{k} a^{N-k} b^k$ , we obtain:

$$\text{nRel}_G(p) = 1 - \sum_{k=0}^N C_{N-k}(G) p^k (1-p)^{N-k} \quad (4)$$

Hence, the (all-terminal) node reliability polynomial can be expressed both in the “ $S$ -form” and in the “ $C$ -form” as

$$\begin{aligned} \text{nRel}_G(p) &= \sum_{k=0}^N S_k(G) p^k (1-p)^{N-k} \\ &= 1 - \sum_{j=0}^N C_j(G) p^{N-j} (1-p)^j \end{aligned} \quad (5)$$

where  $S_k(G)$  counts the number of induced connected subgraphs on  $k$  nodes and  $C_j(G)$  counts the number of vertex cut sets with  $j$  nodes, which is the number of subsets of  $j$  nodes whose removal disconnect the graph.

We can express the node reliability polynomials in the “ $S$ -form” and “ $C$ -form” in *binomial forms*:

$$\begin{aligned} \text{nRel}_G(p) &= \sum_{k=0}^N \binom{N}{k} s_k(G) p^k (1-p)^{N-k} \\ &= 1 - \sum_{j=0}^N \binom{N}{j} c_j(G) p^{N-j} (1-p)^j \end{aligned} \quad (6)$$

where  $s_k(G) = \frac{S_k(G)}{\binom{N}{k}}$  and  $c_j(G) = \frac{C_j(G)}{\binom{N}{j}}$  are the fractions of induced connected subgraphs with  $k$  nodes and vertex cut sets

of graph  $G$  with  $j$  nodes in all possible node combinations of  $k$  nodes and  $j$  nodes from  $N$  nodes. In other words,  $s_k(G)$  equals the probability that the residual network remains connected after removing  $N-k$  nodes and  $c_j(G)$  is the probability that the residual part of graph  $G$  is disconnected after removing  $j$  nodes.

Although computing  $S_k(G)$  and  $C_j(G)$  is NP-hard, the node reliability polynomials  $\text{nRel}_G(p)$  can still be approximated by estimating the probabilities  $s_k(G)$  and  $c_j(G)$ . In this paper, we propose a Laplace Monte-Carlo approximation of  $\text{nRel}_G(p)$  based on the “ $C$ -form” of the node reliability polynomial and a stochastic approximation of  $\text{nRel}_G(p)$  based on the “ $S$ -form” node reliability polynomial.

The term  $\binom{N}{k} p^k (1-p)^{N-k}$  represents the probability density function (pdf) of the binomial distribution. The Central Limit theorem states [11], [12] that the binomial distribution approaches the Gaussian distribution for large  $N$ . For large  $N$ , the “ $S$ -” and “ $C$ -form” of the node reliability polynomials can then be approximated as:

$$\begin{aligned} \text{nRel}_G(p) &= \sum_{k=0}^N \binom{N}{k} s_k(G) p^k (1-p)^{N-k} \\ &\simeq \int_0^N s_k(G) \frac{\exp\left(-\frac{(Np-k)^2}{2Np(1-p)}\right)}{\sqrt{2\pi Np(1-p)}} dk \end{aligned} \quad (7)$$

Substituting  $x = \frac{k}{N}$  transforms the integral in (7) into

$$\int_0^1 s_{Nx}(G) \frac{1}{\sqrt{2\pi} \sqrt{\frac{p(1-p)}{N}}} \exp\left(-\frac{(p-x)^2}{2\frac{p(1-p)}{N}}\right) dx \quad (8)$$

If we define  $\tilde{\mu} = p$ ,  $\tilde{\sigma} = \sqrt{\frac{p(1-p)}{N}}$ , then:

$$\text{nRel}_G(p) \simeq \int_0^1 s_{Nx}(G) \frac{1}{\sqrt{2\pi}\tilde{\sigma}} \exp\left(-\frac{(\tilde{\mu}-x)^2}{2\tilde{\sigma}^2}\right) dx \quad (9)$$

The Gaussian pdf  $\frac{1}{\sqrt{2\pi}\tilde{\sigma}} \exp\left(-\frac{(\tilde{\mu}-x)^2}{2\tilde{\sigma}^2}\right)$  serves as an approximation of the Dirac delta function as [13, Sec. 7.1]:

$$\delta(x - \tilde{\mu}) = \lim_{\tilde{\sigma} \rightarrow 0} \frac{1}{\sqrt{2\pi}\tilde{\sigma}} \exp\left(-\frac{(\tilde{\mu}-x)^2}{2\tilde{\sigma}^2}\right) \quad (10)$$

Assuming that  $\tilde{\sigma}$  tend to 0, the node reliability polynomial can be approximated by the following expression:

$$\text{nRel}_G(p) \simeq \int_0^1 s_{Nx}(G) \delta(x - \tilde{\mu}) dx = s_{N\tilde{\mu}}(G) \quad (11)$$

where  $\tilde{\mu} = p$ .

In summary, we call the approximation

$$\text{nRel}_G(p) \simeq s_{Np}(G) \quad (12)$$

*the Laplace approximation*. Following a similar derivation, the Laplace approximation of the “ $C$ -form” node reliability polynomial is given by

$$\text{nRel}_G(p) \simeq 1 - c_{N(1-p)}(G) \quad (13)$$

TABLE I: Network reliability polynomials for some simple networks

Network (on $N$ nodes)	$\text{nRel}_G(p)$
Complete graph $K_N$	$\text{nRel}_G(p) = 1 - (1-p)^N$
Complete graph $K_{N-1}$ with a pendant node $K_N^*$	$\text{nRel}_G(p) = p^2 + p(1-p)^{N-1} + (1-p)(1-(1-p)^{N-1})$
Cycle graph $C_N$	$\text{nRel}_G(p) = \frac{Np(p^N - (1-p)^N)}{2^{2p-1}} - (N-1)p^N$
Path graphs $P_N$	$\text{nRel}_G(p) = \frac{Np(1-p)^{N+1} - (N+1)p^2(1-p)^N + p^{N+2}}{(1-2p)^2}$
Star graph $S_N$	$\text{nRel}_G(p) = p + (N-1)p(1-p)^{N-1}$
Star graph $S_{N-1}$ with a pendant node $K_N^*$	$\text{nRel}_G(p) = p^3 + p^2(1-p)^{N-2} + p(1-p)^{N-1} + (1-p)(p + (N-2)p(1-p)^{N-2})$

The Laplace approximation (12) holds when  $\tilde{\sigma} = \sqrt{\frac{p(1-p)}{N}}$  is very small. Since for a fixed  $p$ ,  $\tilde{\mu}$  decrease with  $\sqrt{N}$ , the Laplace approximation becomes increasingly accurate as the number of nodes  $N$  in the network grows larger.

### III. MONTE CARLO METHOD OF NODE RELIABILITY POLYNOMIAL $\text{nREL}_G(p)$

The Monte Carlo method for estimating node reliability polynomials is based on a node deletion process, where at each time step a randomly selected node is removed. In a given graph, nodes are removed one by one until all nodes are eliminated. After each removal, the residual network is checked to determine whether it remains connected. By repeating the node deletion process  $M$  times, the number of cases  $R_j$  in which the removal of  $j$  nodes disconnects the residual graph is obtained for each node  $j \in [1, N]$ . When the number of realizations  $M$  is large, the probability that the removal of  $j$  nodes disconnects the graph is approximately  $\tilde{c}_j(G) = \frac{R_j}{M}$ . Thus,

$$c_j(G) = \frac{C_j(G)}{\binom{N}{j}} \simeq \tilde{c}_j(G) \quad (14)$$

The  $C$ -form 4 of the node reliability polynomial  $\text{nRel}_G(p) = 1 - \sum_{j=1}^N \binom{N}{j} c_j(G) p^{N-j} (1-p)^j$  can be approximated by:

$$\text{nRel}_{G,MC}(p) \simeq 1 - \sum_{j=1}^N \binom{N}{j} \tilde{c}_j(G) p^{N-j} (1-p)^j \quad (15)$$

where we denote the Monte Carlo approximation of  $\text{nRel}_G(p)$  by  $\text{nRel}_{G,MC}(p)$ . The Monte Carlo method is applied to several simple graphs, for which explicit closed-form analytical expressions of node reliability polynomials are known, see Appendix A. The graphs are the complete graph  $K_N$ , the complete graph  $K_{N-1}$  with a pendant node, denoted by  $K_N^*$  ( $N$  nodes), the cycle graph  $C_N$ , the path graph  $P_N$ , the star graph  $S_N$  ( $N$  nodes) and the the star graph  $S_{N-1}$  with a pendant node, denoted by  $S_N^*$  ( $N$  nodes). The node reliability polynomials and the result of the Monte Carlo simulations are depicted in Fig.I, which demonstrates that the Monte Carlo approximation is accurate for the considered graphs. In the remainder of this paper, the Monte Carlo method is used as a benchmark to evaluate other approximations of the node reliability polynomial.

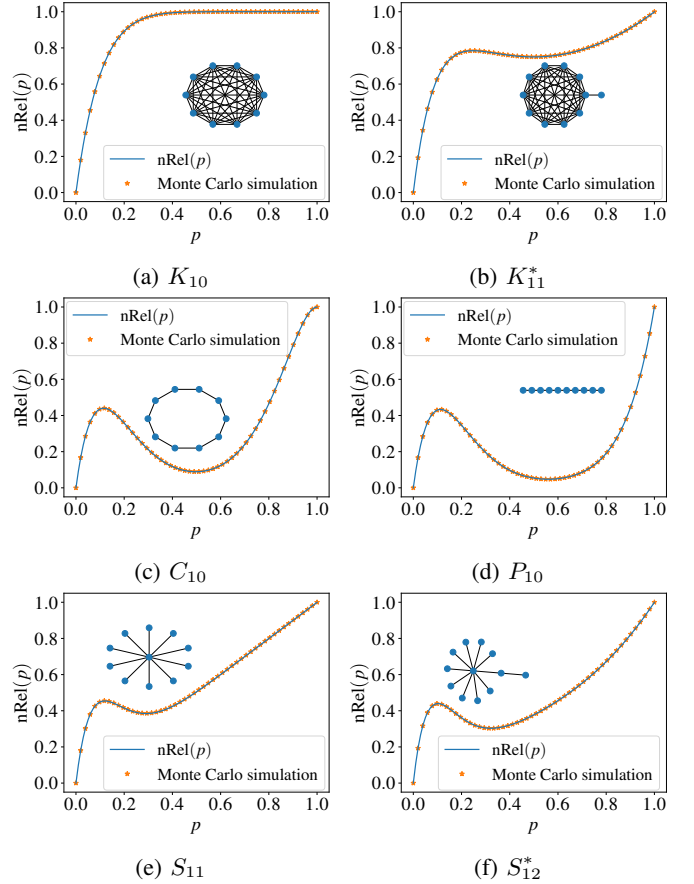
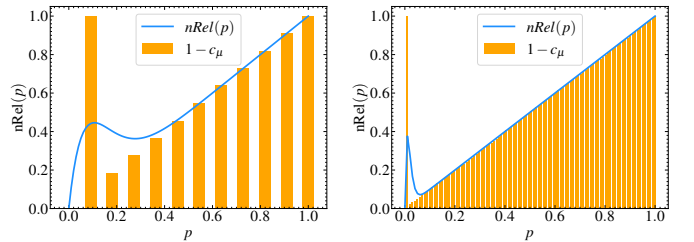


Fig. 1: Monte Carlo simulations and exact value of node reliability polynomials for different graphs



(a) Star graph  $S_{10}$  with  $N = 11$  (b) Star graph  $S_{100}$  with  $N = 101$

Fig. 2: The analytical expressions and the Laplace Monte Carlo simulation result  $1 - c_N(1-p)$  of node reliability polynomial for star graphs  $S_{N-1}$  with different number of nodes

The analysis in Section II has shown that, if the number of nodes  $N$  in the graph  $G$  is large, the node reliability polynomial can be approximated by (13)  $\text{nRel}_G(p) \simeq 1 - c_{N(1-p)}$ . The combination of the Monte Carlo method and the Laplace approximation is referred to as the Laplace Monte Carlo method. Figure 2 presents a comparison between the theoretical values of the node reliability polynomial  $\text{nRel}_G(p)$  and the results from the Laplace Monte Carlo approximation, represented as  $1 - c_{N(1-p)}$ . The results indicate that the Laplace Monte Carlo approximation fits the theoretical values more closely as the size of the graph increases.

#### IV. STOCHASTIC APPROXIMATION $\overline{\text{nRel}}_G(p)$ FOR THE NODE RELIABILITY POLYNOMIAL

Our previous work [23] has introduced a stochastic approximation for the reliability polynomial  $\overline{\text{Rel}}_G(p)$ . In this paper, we extend that approach towards a new stochastic approximation for the node reliability polynomial  $\overline{\text{nRel}}_G(p)$ .

We denote the random residual graph with the failure of  $N - k$  randomly selected nodes in  $G$  as  $\widehat{G}_k$ . The probability that the residual graph  $\widehat{G}_k$  is still connected equals  $\Pr[\widehat{G}_k \text{ is connected}] = s_k(G)$ .

For a given graph, the implication  $\{G \text{ is connected}\} \Rightarrow \{D_{\min} \geq 1\}$ , where the minimum degree is  $D_{\min} = \min_{\text{all nodes } \in G} D$ , is always true. However, the opposite implication does not generally hold, because it is possible for a network to be composed of several disconnected clusters where each node has a minimum degree greater than 1. Van der Hofstad [14] proves that for an Erdős–Rényi (ER) graph with large  $N$  and certain link density  $p_l$  which depends on  $N$ , the opposite implication  $\{D_{\min} \geq 1\} \Rightarrow \{G \text{ is connected}\}$  holds almost for sure. For other network models with large  $N$  and high link density  $p_l$ , the equation  $\{D_{\min} \geq 1\} \Leftrightarrow \{G \text{ is connected}\}$  also holds [15]–[17]. The main hypothesis of the stochastic approximation is that

$$\Pr[\widehat{G}_k \text{ is connected}] = \Pr[\widehat{D}_{\min} \geq 1] + o(1) \quad (16)$$

where  $\widehat{D}_{\min} = \min_{\text{all nodes } \in \widehat{G}_k} \widehat{D}$ .

Let  $\Pr[D = k]$  be the probability that a randomly chosen node in the graph  $G$  has degree  $k$ . The probability generating function (pgf) of the node degree  $D$  in the graph  $G$  is defined [11] as:

$$\varphi_D(z) = E[z^D] = \sum_{j=0}^{N-1} \Pr[D = j] z^j \quad (17)$$

If the number of operational nodes is  $n$ , the probability that all neighbors of a node with degree  $j$  fail independently of each other equals  $(1 - \frac{k}{N})^j$ . Consequently, the probability that a randomly chosen residual node  $i$  in  $\widehat{G}$  is isolated  $\Pr[d_i = 0]$  equals  $\varphi_D(1 - \frac{k}{N})$  [23]. The probability that the minimum degree  $D_{\min}$  is larger than 0 is approximated by

$$\Pr[\widehat{D}_{\min} \geq 1] = \prod_{i=1}^k (1 - \Pr[d_i = 0]) \simeq (1 - \Pr[d_i = 0])^k \quad (18)$$

The probability that the residual graph  $\widehat{G}_n$  remains connected is approximated as  $\Pr[\widehat{G}_k \text{ is connected}] \simeq \Pr[\widehat{D}_{\min} \geq 1]$

$$\Pr[\widehat{G}_k \text{ is connected}] \simeq \left(1 - \varphi_D \left(1 - \frac{k}{N}\right)\right)^k \quad (19)$$

The definition of the coefficients  $S_k(G)$  in the “ $S$ -form” in 6 of the node reliability polynomial indicates that

$$S_k(G) = \binom{N}{k} s_k(G) \simeq \binom{N}{k} \left(1 - \varphi_D \left(1 - \frac{k}{N}\right)\right)^k \quad (20)$$

Substituting (20) to the “ $S$ -form” of node reliability polynomial (2), leads to the approximation of the “ $S$ -form” node reliability polynomial  $\text{nRel}_G(p)$  as

$$\text{nRel}_G(p) \simeq \sum_{n=0}^N \binom{N}{n} \left(1 - \varphi_D \left(1 - \frac{k}{N}\right)\right)^k p^k (1-p)^{N-k} \quad (21)$$

When  $N$  is large, the value of node reliability polynomial  $\text{nRel}_G(p)$  can be approximated by  $s_{Np}(G)$  as:

$$\begin{aligned} \text{nRel}_G(p) &\simeq s_{Np}(G) = \Pr[\widehat{G}_{Np} \text{ is connected}] \\ &\simeq (1 - \varphi_D(1-p))^{Np} \end{aligned} \quad (22)$$

Following the approach in our previous work [23], the stochastic approximation of the node reliability polynomial is denoted as:

$$\overline{\text{nRel}}_G(p) = (1 - \varphi_D(1-p))^{Np} \quad (23)$$

The implication  $\{G \text{ is connected}\} \Leftrightarrow \{D_{\min} \geq 1\}$ , where the minimum degree is  $D_{\min} = \min_{\text{all nodes } \in G} D$  holds when the number of nodes  $N$  and the link density  $p_l$  is large, thus the accuracy of stochastic approximation  $\overline{\text{nRel}}_G(p)$  increases with the increase of  $N$  and  $p_l$ .

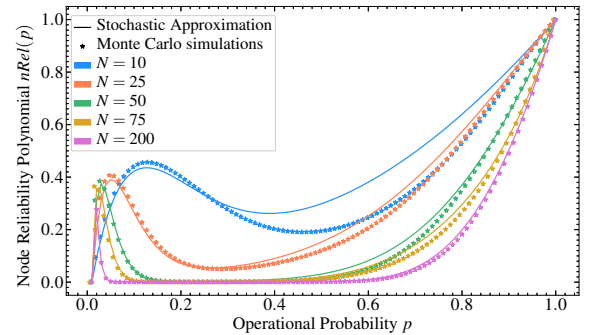


Fig. 3: Node reliability polynomial  $\text{nRel}_G(p)$  obtained by the stochastic approximation and Monte Carlo simulations for Erdős–Rényi graphs with different number of nodes  $N$  and link probability  $p_l = \frac{\log N}{N}$  depending on the node number.

We first perform the simulations on Erdős–Rényi graphs with different number of nodes  $N$  and link probability  $p_l = \frac{\log N}{N}$ . Fig. 3 demonstrates that the accuracy of the stochastic approximation increases with the size of the network.

Fig. 4, 5, 6, and 7 depict the node reliability polynomial  $\text{nRel}_G(p)$  obtained by stochastic approximation and Monte Carlo simulation for Barabási–Albert graphs, Erdős–Rényi

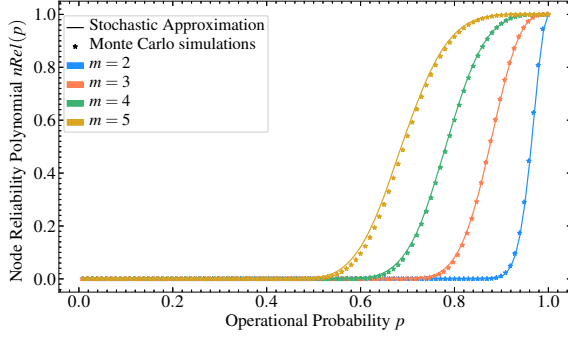


Fig. 4: Stochastic approximation and Monte Carlo simulations of Barabási–Albert graphs with  $N = 1000$  and different number of edges added per step  $m$

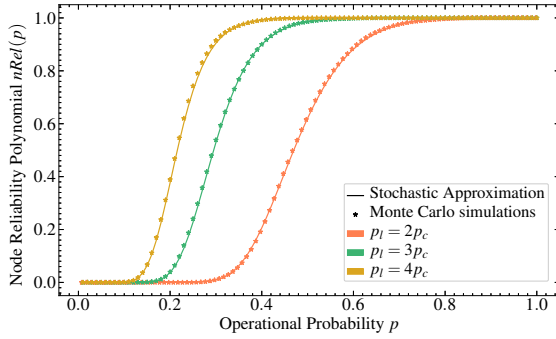


Fig. 5: Stochastic approximation and Monte Carlo simulations for the Erdős–Rényi graphs with  $N = 1000$  and critical link density  $p_c \sim \frac{\log N}{N} = 0.0069$ .

graphs, 2D-lattice graphs and 3D-lattice graphs. We find that the accuracy of the stochastic approximation increases with link density  $p_l$  and the number of nodes  $N$ .

Fig. 21 in Appendix B shows the node reliability polynomial  $nRel_G(p)$  obtained by stochastic approximation and Monte Carlo simulation for some real-world networks. The corresponding parameters for these networks are provided in Table II in Appendix B. We find that the stochastic approximation demonstrate a high accuracy in approximating the node reliability polynomial when the network is larger and dense.

We [23] show that for graphs with a large number of nodes

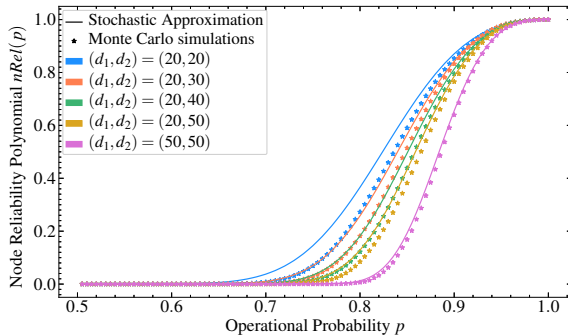


Fig. 6: Stochastic approximation and Monte Carlo simulations for 2D-lattices with width  $d_1$  and height  $d_2$

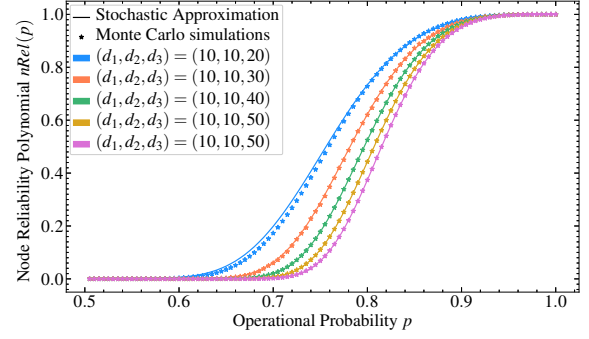


Fig. 7: The stochastic approximation and Monte Carlo simulations for 3D-lattices with width  $d_1$ , length  $d_2$  and height  $d_3$

and high link density, the link reliability polynomial can be accurately approximated by:

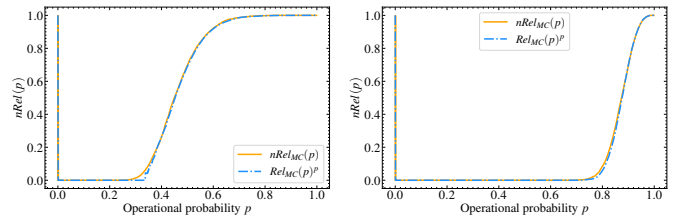
$$\overline{Rel}_G(\tilde{p}) \simeq (1 - \varphi_D(1 - \tilde{p}))^N \quad (24)$$

where the variable  $\tilde{p}$  denotes the probability of links being operational.

Comparing Eq.(23) and Eq.(24) we find for large and dense graphs that there is a relation between the reliability polynomial and the node reliability polynomial, namely:

$$nRel_G(p) \simeq (Rel_G(p))^P \quad (25)$$

Fig 8 shows the Monte Carlo simulation results of  $nRel_G(p)$  and  $(Rel_G(p))^P$  as a function of  $p$  for Erdős–Rényi graphs and Barabási–Albert graphs and demonstrates that the curves of  $nRel_{G,MC}(p)$  and  $(Rel_{G,MC}(p))^P$  are close to each other, which implies the approximation in Eq.(25) holds.



(a) Erdős–Rényi graph with  $N = 1000$  and  $p_l = 0.014$  (b) Barabási–Albert model with  $N = 1000$  and  $k_{\min} = 3$

Fig. 8: Monte Carlo simulations of node reliability polynomial  $nRel_{G,MC}(p)$  and the  $p$ -th power of Monte Carlo simulations of reliability polynomial  $Rel_{G,MC}(p)^P$

## V. NODE RELIABILITY POLYNOMIALS $nRel_G(p)$ OF ERDŐS–RÉNYI GRAPHS AND RANDOM GEOMETRIC GRAPHS

### A. The Node Reliability Polynomial $nRel_G(p)$ of Erdős–Rényi graphs

In an Erdős–Rényi graph  $G_{ER}$ , a node connects to another node with a probability  $p_l$ . The node reliability polynomial of an Erdős–Rényi graph can be approximated using a stochastic approach. The detailed derivation of the approximation is provided in Appendix C.



The general form of the node reliability polynomial  $\overline{\text{nRel}}_{G,ER}(p)$  for an Erdős-Rényi graph is given by:

$$\overline{\text{nRel}}_{G,ER}(p) \simeq \exp\left(-\frac{\bar{k}}{e^{\bar{k} \cdot p_l}}\right) \quad (26)$$

where  $\bar{k} = Np$  is the average degree of the graph.

The degree distribution of the Erdős-Rényi graph follows a binomial distribution. The general approximation form 63 of the node reliability polynomial for an Erdős-Rényi graph ignores the specific degree distribution of the network; its value depends only on the number of nodes and the connection probability. The Law of Large Numbers states that, as the number of nodes  $N$  in an Erdős-Rényi graph increases, the degree of all nodes will converge to the expected value,  $E[D_{ER}](N-1)p$ . The Central Limit Theorem further implies that as  $N$  grows larger, the degree distribution of the graph will increasingly resemble a Poisson distribution centered around the expected value  $(N-1)p$ . Therefore, with a larger  $N$ , the degree distribution of a specific realization of an Erdős-Rényi graph will more closely match the theoretical degree distribution [11, p. 34-40]. Consequently, as the size of the Erdős-Rényi graph increases, the general form of its node reliability polynomial becomes more accurate, as demonstrated in Fig. 9.

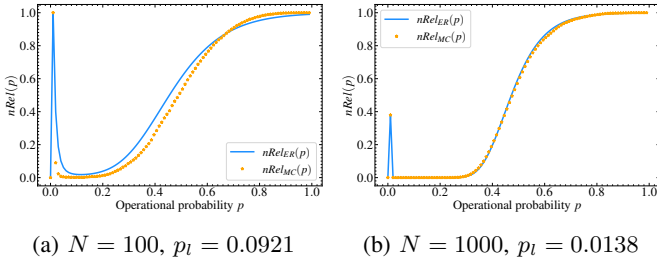


Fig. 9: Monte Carlo simulations  $\text{nRel}_{G,MC}$  and general form of node reliability polynomials  $\text{nRel}_{G,ER}(p)$  for Erdős-Rényi graphs with different number of nodes  $N$  and link probability  $p_l$ .

Studying the intersections of reliability polynomials helps evaluate and compare the reliability of different networks. Here we give an analytical solution of intersections  $(p_i, \text{nRel}_G(p))$  of different Erdős-Rényi graphs. Given two different Erdős-Rényi graph  $G_1$  (with  $N_1$  nodes and link probability  $p_{l1}$ ) and  $G_2$  (with  $N_2$  nodes and link probability  $p_{l2}$ ), the intersection of the node reliability polynomials of these two graph is

$$p_i = \exp\left(\frac{\bar{k}_1 \log N_2 - \bar{k}_2 \log N_1}{\bar{k}_2 - \bar{k}_1}\right) \quad (27)$$

The detailed derivation is given in Appendix C-A.

From (27), the condition for  $p_i$  to have a solution in the range  $(0, 1)$  is:

- If  $\bar{k}_2 > \bar{k}_1$ , then  $\frac{\log N_1}{\log N_2} > \frac{\bar{k}_1}{\bar{k}_2}$ .
- If  $\bar{k}_2 < \bar{k}_1$ , then  $\frac{\log N_1}{\log N_2} < \frac{\bar{k}_1}{\bar{k}_2}$ .

which means that for two Erdős-Rényi graphs, the node reliability polynomials will only intersect if the graph with

the smaller average degree has more nodes. For Erdős-Rényi graphs with same number of nodes  $N$ , the larger the average degree  $\bar{k}$ , the larger the node reliability polynomial  $\text{nRel}_G(p)$  as is shown in Fig. 10. Fig. 10 also demonstrates that the larger the average degree  $\bar{k}$ , the faster the  $\text{nRel}_G(p)$  curve changes from  $0 + \epsilon$  to  $1 - \epsilon$ , where  $\epsilon$  is a very small value. It is expected that for Erdős-Rényi graphs with very large average degree  $\bar{k}$ , the  $\text{nRel}_G(p)$  will exhibit a sharp transition from 0 to 1, resembling a phase transition. We prove in the Appendix C-B that the length of the transition region of  $\text{nRel}_G(p)$  is  $O(1/\bar{k})$ .

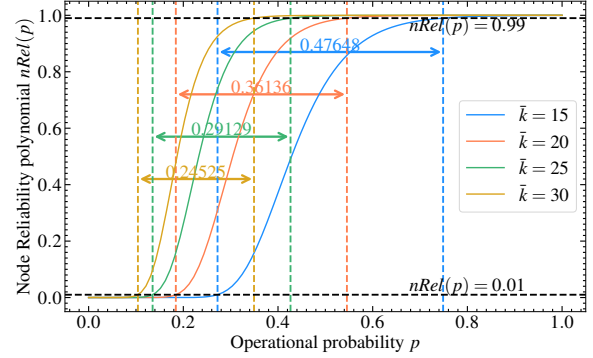


Fig. 10: the Monte Carlo simulation results of the node reliability polynomial  $\text{nRel}_G(p)$  of Erdős-Rényi graphs with  $N = 1000$  and different average degrees  $\bar{k}$ . The numbers in different colors indicate the length of the corresponding polynomials change from  $\text{nRel}_G(p) = 0.01$  to  $\text{nRel}_G(p) = 0.99$ .

### B. The Node Reliability Polynomial $\text{nRel}_G(p)$ of Random Geometric Graphs

Random Geometric Graphs (RGGs) are formed by placing nodes at random positions within a given metric space, such as the Euclidean plane. Each pair of nodes is connected by a link if the distance between them is less than a specified threshold  $r$ , known as the connection radius. In a RGG, a node will connect all the nodes in a circle centered on it with radius  $r$ . Because the nodes in RGG are randomly placed, the probability that a randomly chosen node  $b$  is located in the circle of certain node  $a$  is the area of node  $a$ 's circle divided by the area of the metric space, which is  $p_{a,b} = \pi r^2$ . The probability that a node in RGG has no neighbor is  $\Pr_{D=0} = (1 - \pi r^2)^{N-1}$ . Neglecting the boundary conditions, the probability that there are no isolated nodes in the graph is  $\Pr[D_{\min} \geq 1] = (1 - p_{D=0})^N$ . The general form of node reliability polynomial  $\text{nRel}_G(p)$  of a RGG can therefore be approximated by

$$\begin{aligned} \text{nRel}_G(p) &\approx s_\mu = \Pr[\widehat{G}_\mu \text{ is connected}] \approx \Pr[\widehat{D}_{\min} \geq 1] \\ &= \left(1 - (1 - \pi r^2)^{Np-1}\right)^{Np} \end{aligned} \quad (28)$$

### VI. ARITHMETIC FIRST-ORDER UPPER BOUND AND GEOMETRIC FIRST-ORDER UPPER BOUND

In addition to the stochastic approximation, we provide two upper bounds for the node reliability polynomials, the

arithmetic first-order upper bound  $\overline{\text{nRel}}_{G,\text{arith}}(p)$  and geometric first-order upper bound  $\overline{\text{nRel}}_{G,\text{geom}}(p)$ . Given a node  $i$  with degree  $d_i$ , we define the probability that node  $i$  is active while all its neighbor nodes have failed by  $f_i$ ,

$$\begin{aligned} f_i &= \Pr[\text{Node } i \text{ is active and } \widehat{d}_i = 0] \\ &= \Pr[\text{Node } i \text{ is active}] \Pr[\widehat{d}_i = 0] \\ &= p(1-p)^{d_i} \end{aligned} \quad (29)$$

Since the absence of isolated nodes is a necessary, but not sufficient condition for graph connectivity, the node reliability polynomial can be upper bounded by calculating the probability that no isolated nodes exist in the graph. The resulting geometric upper bound for the node reliability polynomial is

$$\overline{\text{nRel}}_{G,\text{geom}}(p) = \Pr[\text{There are no isolated nodes in the graph}] \quad (30)$$

For graph  $G$ , the absence of isolated nodes is equivalent to each node being active and having a degree greater than zero. Assuming independence between these events, the following holds:

$$\begin{aligned} \overline{\text{nRel}}_{G,\text{geom}}(p) &= \prod_{i=1}^N \left( \Pr[\text{Node } i \text{ is active and } \widehat{d}_i > 0] \right) \\ &= \prod_{i=1}^N \left( 1 - \Pr[\text{Node } i \text{ is active and } \widehat{d}_i = 0] \right) \end{aligned} \quad (31)$$

Substituting the definition  $\Pr[\text{Node } i \text{ is active and } \widehat{d}_i = 0] = f_i = p(1-p)^{d_i}$  (29) into (31)

$$\overline{\text{nRel}}_{G,\text{geom}}(p) = \prod_{i=1}^N (1 - f_i) = \prod_{i=1}^N (1 - p(1-p)^{d_i}) \quad (32)$$

When calculating the node reliability of large networks, concatenating the probabilities that each node is not isolated becomes computationally demanding. We also propose an arithmetic first-order upper bound by computing the probability that a randomly selected node is not isolated. The arithmetic mean probability that a randomly selected node is not isolated is

$$P_{AM} = \frac{1}{N} \sum_{i=1}^N \left( 1 - \Pr[\text{Node } i \text{ is active and } \widehat{d}_i = 0] \right) \quad (33)$$

Substituting  $\Pr[\text{Node } i \text{ is active and } \widehat{d}_i = 0] = f_i = p(1-p)^{d_i}$  (29) into (33), we obtain

$$P_{AM} = \frac{1}{N} \sum_{i=1}^N (1 - f_i) = \frac{1}{N} \sum_{i=1}^N \left( 1 - p(1-p)^{d_i} \right) \quad (34)$$

which we rewrite as a sum over the nodal degrees by denoting  $n_j$  as the number of nodes with degree  $j$  and realizing that  $\Pr[D = j] = \frac{n_j}{N}$ ,

$$\begin{aligned} P_{AM} &= \frac{1}{N} \sum_{j=0}^{N-1} n_{d=j} \left( 1 - p(1-p)^j \right) \\ &= \sum_{j=0}^{N-1} \Pr[d = j] \left( 1 - p(1-p)^j \right) \end{aligned} \quad (35)$$

Using the definition of pgf of the node degree 17, we obtain:

$$P_{AM} = \sum_{j=0}^{N-1} \Pr[d = j] \left( 1 - p(1-p)^j \right) = 1 - p\varphi_D(1-p) \quad (36)$$

By raising  $P_{AM}$  to the  $N$ th power, the arithmetic first-order upper bound is derived as:

$$\overline{\text{nRel}}_{G,\text{arith}}(p) = (1 - p\varphi_D(1-p))^N \quad (37)$$

The geometric mean of  $1 - f_i$  over all nodes is  $P_{GM} = \sqrt[N]{\prod_{i=1}^N (1 - f_i)}$ , and by the definition of  $f_i$ :

$$P_{GM} = \sqrt[N]{\prod_{i=1}^N \left( 1 - p(1-p)^{d_i} \right)}, \quad (38)$$

which is the  $N$ th root of  $\overline{\text{nRel}}_{G,\text{geom}}(p)$ . Because the arithmetic mean is always larger than or equal to the geometric mean, the arithmetic first-order upper bound  $\overline{\text{nRel}}_{G,\text{arith}}(p)$  is larger than or equal to the geometric first-order upper bound  $\overline{\text{nRel}}_{G,\text{geom}}(p)$ .

The computational complexity of the arithmetic first-order upper bound,

$$\overline{\text{nRel}}_{G,\text{arith}} = \exp(N \log(1 - p\varphi_D(1-p))),$$

depends on the computation of  $\varphi_D(1-p)$ , which has a complexity of  $O(\mathcal{N}_D)$ , where  $\mathcal{N}_D$  represents the number of distinct degrees in the graph. In contrast, the geometric first-order upper bound,  $\overline{\text{nRel}}_{G,\text{geom}}(p)$ , involves calculating the term  $\left( 1 - p(1-p)^{d_i} \right)$  for each of the  $N$  nodes, resulting in a computational complexity of  $O(N)$ . Therefore, in networks where  $\mathcal{N}_D \ll N$ , the arithmetic first-order upper bound requires significantly fewer computational resources, leading to a notable reduction in computation time and increased efficiency. As the network size increases, this advantage becomes even more significant, making the arithmetic first-order upper bound a practical choice for analyzing large-scale networks with a limited number of distinct degrees. For cases requiring higher precision, the geometric first-order upper bound  $\overline{\text{nRel}}_{G,\text{geom}}(p)$  can be used, while the arithmetic first-order upper bound  $\overline{\text{nRel}}_{G,\text{arith}}$  is preferred for its computational efficiency.

## VII. APPLICATIONS OF NODE RELIABILITY POLYNOMIALS $\overline{\text{nRel}}_G(p)$

The reliability polynomial captures essential information about the structure of a network, especially concerning its

*cut sets* — the sets of links or nodes whose removal would disconnect the network. As a result, the reliability polynomial serves as a comprehensive measure of a network's global connectivity. In this section, we introduce a way to estimate the number of cut set (vertex cut set) of a graph, based on the reliability polynomial (node reliability polynomial).

The reliability is an important robustness measure of graph  $G(N, L)$ . Given a graph  $G = (N, L)$  and a budget of  $k$  links to be added, a set  $\mathcal{S} \subset \binom{N}{2} \setminus L$  of size  $k$  that optimizes the robustness of  $G$  is a common problem. Predari *et al.* refer to this optimisation problem as  $k$ -GRIP problem, short for graph robustness improvement problem [22]. We select reliability polynomial  $\text{Rel}_G(p)$  (node reliability polynomial  $\text{nRel}_G(p)$ ) as robustness measure for  $k$ -GRIP. Within the stochastic approximations of reliability polynomial  $\widetilde{\text{Rel}}_G(p)$  and node reliability polynomial  $\widetilde{\text{nRel}}_G(p)$ , we give optimal solutions  $\mathcal{S}$  of  $\widetilde{\text{Rel}}_G(p)$  and  $\widetilde{\text{nRel}}_G(p)$ . The optimal solution  $\mathcal{S}$  are approximate optimal solutions for reliability-based  $k$ -GRIP and node-reliability-based  $k$ -GRIP problems.

#### A. Estimation of the number of vertex cut set

Given any approximation  $\widetilde{\text{nRel}}_G(p)$  of the reliability polynomial  $\text{nRel}_G(p)$ , we estimate the number of vertex cut set by the following method.

Substituting  $N$  different node operational probabilities  $p = \{p_1, p_2, \dots, p_N\}$  into the C-form node reliability polynomial, we obtain  $N$  different equations:

$$\begin{aligned} \text{nRel}_G(p_i) &= 1 - \sum_{j=0}^N C_j (1-p_i)^j p_i^{N-j} \\ &\approx \widetilde{\text{nRel}}_G(p_i), \quad i = 0, 1, \dots, N \end{aligned} \quad (39)$$

In matrix form, these  $N$  equations can be written as:

$$\mathbf{1} - \mathbf{PC} = \widetilde{\mathbf{nRel}}_G \quad (40)$$

where

$$\mathbf{P} = \begin{bmatrix} (1-p_0)^0 p_0^N & (1-p_0)^1 p_0^{N-1} & \cdots & (1-p_0)^N p_0^0 \\ (1-p_1)^0 p_1^N & (1-p_1)^1 p_1^{N-1} & \cdots & (1-p_1)^N p_1^0 \\ \vdots & \vdots & \ddots & \vdots \\ (1-p_N)^0 p_N^N & (1-p_N)^1 p_N^{N-1} & \cdots & (1-p_N)^N p_N^0 \end{bmatrix} \quad (41)$$

is the probability matrix which represents the contribution of different vertex cut sets to the node reliability for each probability value  $p_i$ ,

$$\mathbf{C} = [C_0, C_1, \dots, C_N] \quad (42)$$

the vertex cut set coefficient vector which contains the coefficients  $C_n$  that represent the number of vertex cut sets of sizes  $n$ , and

$$\widetilde{\mathbf{nRel}}_G = [\widetilde{\text{nRel}}_G(p_0), \widetilde{\text{nRel}}_G(p_1), \dots, \widetilde{\text{nRel}}_G(p_N)] \quad (43)$$

the node reliability approximation vector, which holds the approximate values of the node reliability polynomial at different probabilities.

Coefficient vector  $\mathbf{C}$  can be obtained by

$$\widetilde{\mathbf{C}} = \mathbf{P}^{-1}(\mathbf{1} - \widetilde{\mathbf{Rel}}_G) \quad (44)$$

Given any approximation of the reliability polynomial  $\widetilde{\text{Rel}}_G(p)$ , the number of cut set is found via (44). Using the similar method on the approximation of reliability polynomial  $\widetilde{\text{Rel}}_G(p)$ , we obtain the number of link cut set with different number of links.

#### B. Analytical Optimization Strategy for the Node-Reliability-Based $k$ -GRIP Problem

By using the reliability polynomial as an objective measure, we design networks that optimize for specific reliability criteria. For example, to enhance a network's reliability, we might seek to add or reinforce connections (links) in such a way that the reliability polynomial achieves its highest possible values under expected operational probabilities. Since computing the exact expression of the reliability polynomial is NP-hard, it is not possible to provide an analytical solution to the reliability-based  $k$ -GRIP problem. However, a stochastic approximation of the reliability polynomial can be optimized analytically, which offers a practical approach to solving the reliability-based  $k$ -GRIP problem.

We recall the stochastic approximation of the reliability polynomial in (24),

$$\overline{\text{Rel}}_G(p) = (1 - \varphi_D(1-p))^N,$$

and the stochastic approximation of the node reliability polynomial in (23),

$$\overline{\text{nRel}}_G(p) = (1 - \varphi_D(1-p))^{Np},$$

that both depend on the value of term  $1 - \varphi_D(1-p)$ . The function  $f(x) = x^c$ , where  $c$  is a positive number, is monotonically increasing for  $x$  in domain  $[0, 1]$ . Thus a larger  $1 - \varphi_D(1-p)$  contribute to higher reliability and node reliability. Consequently, the problem of optimizing the stochastic approximations  $\overline{\text{Rel}}_G(p)$  and  $\overline{\text{nRel}}_G(p)$  reduces to maximizing the value of  $1 - \varphi_D(1-p)$ . We denote the graph obtained by adding the links of  $\mathcal{S}$  into  $G$  as  $G' := G \cup \mathcal{S}$  and degree distribution of graph  $G'$  as  $D'$ , where  $\mathcal{S} \subset \binom{N}{2} \setminus L$ . Based on the analysis in our previous work [23], the term  $1 - \varphi_D(1-p)$  can be expressed as

$$1 - \varphi_D(1-p) = \frac{1}{N} \sum_{i=1}^N \left(1 - (1-p)^{d_i}\right) \quad (45)$$

where  $d_i$  is the degree of node  $i$ . Here we denote the degree vector of graph  $G$  as  $\mathbf{d} = [d_1, d_2, \dots, d_N]$  and the degree change vector after  $k$  links added into  $G$  as  $\mathbf{a} = [a_1, a_2, \dots, a_N]$ , where  $a_i \geq 0$ . Then the degree vector of graph  $G'$  becomes  $\mathbf{d} = [d_1 + a_1, d_2 + a_2, \dots, d_N + a_N]$ . The reliability-based  $k$ -GRIP problem is then transformed into:

**Objective:**

$$\begin{aligned} &\max_A 1 - \varphi_{D+A}(1-p) \\ &= \max_{A=[a_1, a_2, \dots, a_N]} \sum_{i=1}^N \left(1 - (1-p)^{d_i+a_i}\right) \end{aligned} \quad (46)$$



**Subject to:**

$$s.t. \sum_{i=1}^N a_i = 2k, a_i \geq 0, a_i \in \mathbb{Z} \quad (47)$$

Suppose here are two sets

$$\begin{aligned} A_1 &= [a_1, a_2, \dots, a_m, \dots, a_n, \dots, a_N] \\ A_2 &= [a_1, a_2, \dots, a_m + 1, \dots, a_n - 1, \dots, a_N] \end{aligned} \quad (48)$$

where  $a_m, a_n \geq 1$ .

The only difference between  $A_1$  and  $A_2$  is  $m$ th element of  $A_2$  is  $m$ th element of  $A_1$  plus one, and the  $n$ th element of  $A_2$  is  $n$ th element of  $A_1$  minus one. The difference of  $\sum_{i=1}^N (1 - (1-p)^{d_i+a_i})|_{A_2}$  and  $\sum_{i=1}^N (1 - (1-p)^{d_i+a_i})|_{A_1}$  is

$$\begin{aligned} \Delta &= \sum_{i=1}^N (1 - (1-p)^{d_i+a_i})|_{A_2} - \sum_{i=1}^N (1 - (1-p)^{d_i+a_i})|_{A_1} \\ &= (1 - (1-p)^{d_m+a_m+1}) + (1 - (1-p)^{d_n+a_n-1}) \\ &\quad - (1 - (1-p)^{d_m+a_m}) - (1 - (1-p)^{d_n+a_n}) \\ &= p \left( (1-p)^{d_n+a_n-1} - (1-p)^{d_m+a_m} \right) \end{aligned} \quad (49)$$

The value of  $\Delta$  is larger than 0 only when  $d_n + a_n - 1 > d_m + a_m$ .  $d_m + a_m$  and  $d_n + a_n$  are the degree of node  $m$  and node  $n$  after  $k$  links are added to the graph according to set  $A_1$ . When the degree  $d_n + a_n$  of the node  $n$  is larger than  $d_m + a_m$  of node  $m$  plus one, reconnecting the end of one link connect to node  $n$  to node  $m$  contribute to a better link-adding set  $\mathcal{S}$ . Here we define a  $k$ -GRIP descending restructuring, which disconnects the end of an added link to a node  $n$  with degree  $d_n + a_n$  in the graph  $G'$  and reconnects that end of the link to another node  $m$  with degree  $d_m + a_m$ , where  $d_n + a_n - 1 > d_m + a_m$  and  $a_n > 0$ , like the case in 11 shows. The ascending restructuring is defined as the inverse of the descending restructuring. The above analysis shows that  $1 - \varphi_{D+A}(1-p)$  after descending restructuring is always larger than that before descending restructuring. Figures in Fig. 12 illustrate that both for reliability and the node reliability are improved by descending restructuring.

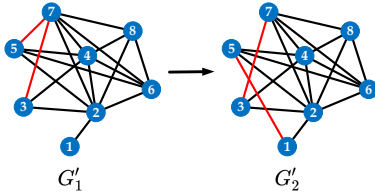


Fig. 11: Schematic of the  $k$ -GRIP descending restructuring: Graph  $G'_1$  is obtained by adding two links  $l_{3,7}$  and  $l_{5,7}$  into a graph  $G$  with  $N = 8$  nodes and  $L = 16$  links. Graph  $G'_2$  is obtained by reconnecting the link  $l_{5,7}$  to  $l_{1,5}$ .

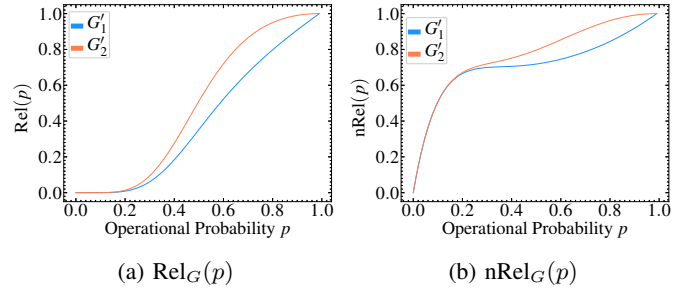


Fig. 12: Monte Carlo simulations results of the reliability polynomial  $\text{Rel}_G(p)$  and the node reliability polynomial  $\text{nRel}_G(p)$  of graphs  $G'_1$  and  $G'_2$  (see Fig. 11). (a) The reliability polynomial  $\text{Rel}_G(p)$ . (b) The node reliability polynomial  $\text{nRel}_G(p)$ .

We denote the set of all possible graphs, where  $k$  links are added into graph  $G$ , as  $\langle G \rangle$ . Start from a random graph  $G'$  in  $\langle G \rangle$ , any graph in  $\langle G \rangle$  can be obtained by multiple descending and ascending restructurings. Since descending restructuring always contribute to a higher  $\sum_{i=1}^N (1 - (1-p)^{d_i+a_i})$ , the optimal graph  $G^*$  is a graph where no descending restructuring could occur. To construct the optimal graph  $G^*$ , links can be greedily added by connecting pairs of nodes with the lowest degrees, provided the link does not already exist. In this paper, the algorithm of greedily adding  $k$  links between pairs of nodes with the lowest degrees that are not already connected is referred to as the **Greedy Lowest-Degree Pairing Edge Addition Algorithm**.

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**Algorithm 1** Greedy Lowest-Degree Pairing Edge Addition Algorithm

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**Input:** a graph  $G$ , number of links to add  $k$

**Output:** a new graph  $G^*$

---

- 1: Generate the degree vector  $\mathbf{d}$  for graph  $G$
  - 2: **for**  $t = 1$  to  $k$  **do**
  - 3:     Sort nodes by their degree in ascending order
  - 4:     Find node  $i$  with the smallest degree
  - 5:     Find node  $j$  with the smallest degree that is not connected to  $i$
  - 6:     Add link between nodes  $i$  and  $j$  in the graph
  - 7:     Update the graph  $G$  and the degree vector  $\mathbf{d}$  after adding the new link
  - 8: **end for**
  - 9: Return the new graph  $G^*$
- 

To evaluate the performance of the **Greedy Lowest-Degree Pairing Edge Addition Algorithm** in improving network reliability, we applied **Greedy Lowest-Degree Pairing Edge Addition Algorithm** and other two link-adding strategies to real-world networks from Network Repository [26] and compared their effectiveness. The first strategy, **Random Pairing Strategy**, adds links randomly between node pairs that are not already linked. The second strategy, **Greedy Highest-Degree Pairing Strategy**, focuses on adding links between the highest-degree nodes that are not yet connected, aiming to strengthen the already well-connected nodes. Figures in Fig. 13 and 14 illustrate that **Greedy Lowest-Degree Pairing**

**Edge Addition Algorithm** has the most significant effect in enhancing network reliability and node reliability, far outperforming random link addition and adding links between high-degree nodes. Adding links between high-degree nodes shows the worst performance, with almost no noticeable impact on network reliability. The simulation demonstrates that one of the most effective ways to enhance network robustness from the perspective of network connectivity is to add links between low-degree nodes.

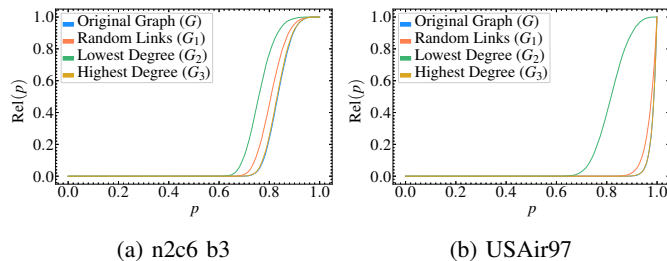


Fig. 13: Monte Carlo simulations results of reliability polynomial  $\text{Rel}_G(p)$  in two real-world graphs and graphs constructed by adding links to these graphs in three different strategies. (a) The original graph is a simplicial complex network with  $N = 1365$  nodes and  $L = 5263$  links [26].  $l = 500$  links are added into the original graph in three different strategies. (b) The original graph is 1997 U.S. flight network with  $N = 332$  nodes and  $L = 2126$  links [26].  $l = 100$  links are added into the original graph in three different strategies.

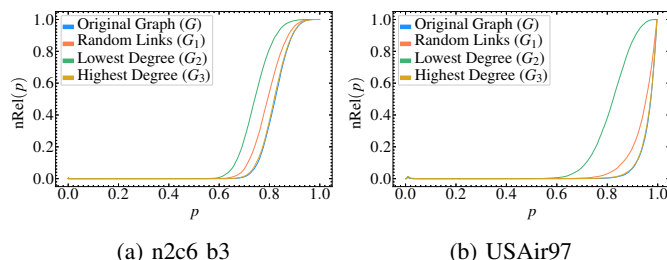


Fig. 14: Monte Carlo simulations results of node reliability polynomial  $\text{nRel}_G(p)$  in two real-world graphs and graphs constructed by adding links to these graphs in three different strategies. (a) The original graph is a simplicial complex network with  $N = 1365$  nodes and  $L = 5263$  links [26].  $l = 500$  links are added into the original graph in three different strategies. (b) The original graph is 1997 U.S. flight network with  $N = 332$  nodes and  $L = 2126$  links [26].  $l = 100$  links are added into the original graph in three different strategies.

## VIII. CONCLUSION

Calculating the exact node reliability polynomial is known to be NP-Hard, prompting the need for efficient approximation techniques. We have approximated the node reliability polynomial of a graph, which is a key measure of the robustness of networks against node failures.

First, we presented a Laplace approximation method for the node reliability polynomial, utilizing “the  $C$ -form” and “ $S$ -form” representations. We demonstrated that by approximating these forms with probabilistic methods, significant computational efficiency can be achieved while maintaining reasonable accuracy on approximating the node reliability polynomial  $\text{nRel}_G(p)$ .

We then proposed the Laplace Monte Carlo method that offers a practical approach for estimating the node reliability polynomial. This method involves simulating node failures and observing the resulting network connectivity, allowing for accurate estimation even for complex graph structures. The effectiveness of Monte Carlo method was validated against known node reliability polynomials for several well-structured networks.

Additionally, we introduced a stochastic approximation method, leveraging the probability generating function of node degrees. The stochastic approximation provides a quick and reasonably accurate estimation of node reliability polynomials, particularly effective for large and dense networks. For Erdős–Rényi graphs, we derived a general form of the node reliability polynomial and analyzed its behavior under various conditions. We found that as the size of the network increases, the reliability polynomial exhibits a sharp transition, resembling a step function. We also extended our analysis to Random Geometric Graphs, providing an approximation for their node reliability polynomials.

Moreover, we proposed for the node reliability polynomial the arithmetic first-order upper bound and the geometric first-order upper bound. These bounds provide useful benchmarks for evaluating the reliability of a network without the need for exhaustive computation.

Finally, we discussed applications of reliability polynomials: estimating the number of vertex cut sets and enhancing network reliability through link additions.

**Acknowledgements** Van Mieghem has been funded by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 101019718).

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APPENDIX A  
NODE RELIABILITY POLYNOMIALS FOR SOME GRAPH  
FAMILIES

A. Complete Graph  $K_N$

For a complete graph  $K_N$  with  $N$  nodes, the only way to disconnect the graph is when all the nodes fail. For a node operational probability  $p$ , the probability that all nodes are failed is  $(1-p)^N$ . Thus we obtain the following node reliability polynomial:

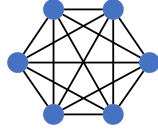


Fig. 15: Complete graph  $K_N$

$$\text{nRel}_G(p) = \Pr[\widehat{G} \text{ is connected}] = 1 - (1-p)^N \quad (50)$$

B. Complete Graph  $K_{N-1}$  with a Pendant Node

We will denote the complete graph  $K_{N-1}$  with a pendant node as  $K_N^*$ . The node reliability polynomial of  $K_N^*$  is given by:

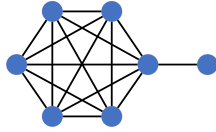


Fig. 16: The graph  $K_{N-1} + P_1$

$$\begin{aligned} \text{nRel}_G(p) = \Pr[\widehat{G} \text{ is connected}] &= (1-p)(1 - (1-p)^{N-1}) + p^2 \\ &+ p(1-p)^{N-1} \end{aligned} \quad (51)$$

C. Cycle Graph  $C_N$

For a cycle graph  $C_N$  with  $N$  nodes, after  $k$  nodes are removed, the remaining  $N-k$  nodes should be connected. Thus, the node reliability polynomial of the cycle graph is:

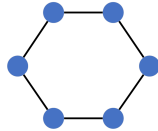


Fig. 17: Cycle graph  $C_N$

$$\begin{aligned} \text{nRel}_G(p) = \Pr[\widehat{G} \text{ is connected}] &= p^N + N \sum_{k=1}^{N-1} p^{N-k} (1-p)^k \\ &= \frac{Np(p^N - (1-p)^N)}{2p-1} - (N-1)p^N \end{aligned} \quad (52)$$

D. Path Graph  $P_N$

For a path graph  $P_N$ , the node reliability polynomial is:



Fig. 18: Path graph  $P_N$

$$\begin{aligned} \text{nRel}_G(p) &= 1 - \sum_{j=0}^N \left( \binom{N}{j} - j - 1 \right) p^{N-j} (1-p)^j \\ &= \sum_{j=0}^N (j+1) p^{N-j} (1-p)^j \\ &= \frac{Np(1-p)^{N+1} - (N+1)p^2(1-p)^N + p^{N+2}}{(1-2p)^2} \end{aligned} \quad (53)$$

E. Star Graph  $S_N$

For a star graph  $S_N$  with one central node and  $N-1$  edge nodes, the node reliability polynomial is:

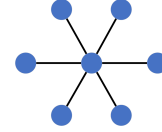


Fig. 19: Star graph  $S_N$

$$\text{nRel}_G(p) = p + (N-1)p(1-p)^{N-1} \quad (54)$$

F. Star Graph  $S_{N-1}$  with a Pendant Node

For a star graph  $S_{N-1}$  with a pendant node, denoted as  $S_N^*$ , the node reliability polynomial is:

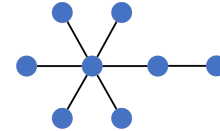
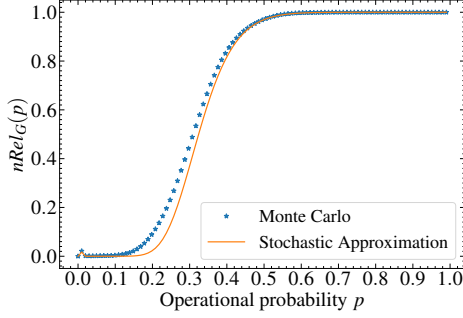


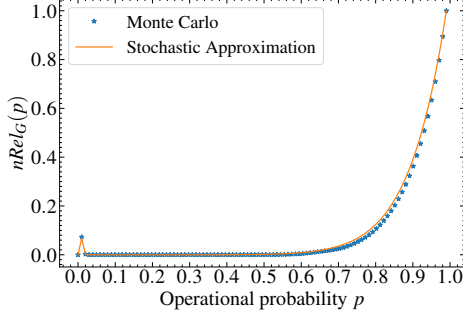
Fig. 20: The graph  $S_{N-1} + P_1$

$$\begin{aligned} \text{nRel}_G(p) &= (1-p) \left( p + (N-2)p(1-p)^{N-2} \right) + p^3 \\ &+ p^2(1-p)^{N-2} + p(1-p)^{N-1} \end{aligned} \quad (55)$$

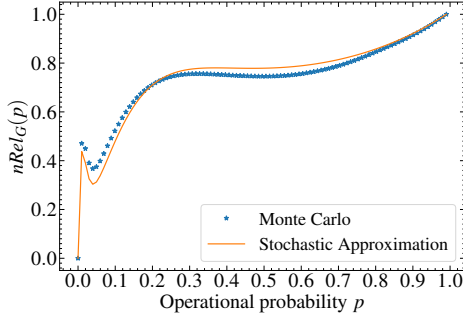
## APPENDIX B

PERFORMANCE OF STOCHASTIC APPROXIMATION IN  
REAL-WORLD NETWORKS

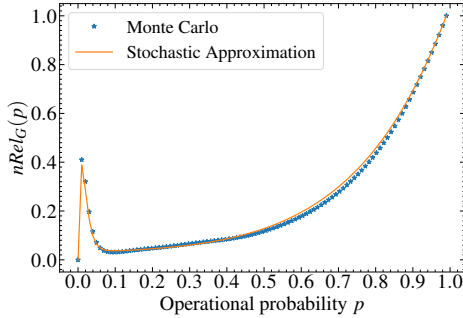
(a) 130bits



(b) Infect Dublin



(c) Infect Hyper



(d) Aves Songbird Social

Fig. 21: Monte Carlo simulations and stochastic approximations for different real-world graphs.

## APPENDIX C

DERIVATION OF THE NODE RELIABILITY POLYNOMIAL  
 $\overline{\text{nRel}}_G(p)$  OF ERDŐS-RÉNYI GRAPHS

In a Erdős-Rényi graph  $G_{ER}$ , a node connects to another node with a probability  $p_l$ . The random residual graph  $\hat{G}_n$  of an Erdős-Rényi graph  $G_{ER}$  retains the Erdős-Rényi property, where residual nodes remain connected with probability  $p_l$ . Thus, the probability that the degree  $\hat{D}_{\hat{G}_n}$  of a node in  $\hat{G}_n$  equals 0 is:

$$\Pr[\hat{D}_{\hat{G}_n} = 0] = (1 - p_l)^{n-1} \quad (56)$$

Invoking the analysis in Section IV (23), the stochastic approximation of the node reliability polynomial of Erdős-Rényi graph is:

$$\overline{\text{nRel}}_{G,ER}(p) \simeq \Pr[\hat{D}_{\min} \geq 1] = (1 - (1 - p_l)^{Np-1})^{Np} \quad (57)$$

. Rewriting the exponent, we obtain:

$$\begin{aligned} \overline{\text{nRel}}_{G,ER}(p) &= (1 - (1 - p_l)^{Np-1})^{Np} \\ &= \left(1 - (1 - p_l)^{\frac{1}{p_l} p_l (Np-1)}\right)^{Np} \end{aligned} \quad (58)$$

Using  $(1 - x)^{\frac{1}{x}} = e^{-1} + O(x)$  for  $x$  tends to 0 for small  $p_l$ , we have

$$\begin{aligned} \overline{\text{nRel}}_{G,ER}(p) &= \left(1 - (1 - p_l)^{\frac{1}{p_l} p_l (Np-1)}\right)^{Np} \\ &\simeq \left(1 - e^{-p_l(Np-1)}\right)^{Np} \end{aligned} \quad (59)$$

A new function of the node operational probability  $p$  is defined as:

$$b(p) = \frac{e^{N \cdot p \cdot p_l}}{N} \quad (60)$$

so that the link probability equals  $p_l = \frac{(\log b(p)N)}{pN}$ . Substituting  $p_l = \frac{(\log b(p)N)}{pN}$  to (59)

$$\begin{aligned} \overline{\text{nRel}}_{G,ER}(p) &\simeq \left(1 - \exp\left(-\frac{\log(b(p)N)}{pN} \cdot (Np - 1)\right)\right)^{Np} \\ &\simeq \left(1 - \frac{1}{b(p)N}\right)^{Np} \end{aligned} \quad (61)$$

which for large  $N$  lead to

$$\overline{\text{nRel}}_{G,ER}(p) \simeq e^{-\frac{p}{b(p)}} \quad (62)$$

After substituting the definition  $b(p) = \frac{e^{N \cdot p \cdot p_l}}{N}$  in (60) into (62), we obtain an approximation of the node reliability of Erdős-Rényi graph

$$\overline{\text{nRel}}_{G,ER}(p) \simeq \exp\left(-\frac{\bar{k}}{e^{\bar{k} \cdot p_l}}\right) \quad (63)$$

where  $\bar{k} = Np$  is the average degree of the graph.



TABLE II: Parameters for the networks shown in Figure 21

Network	Node Number	Link Density	Mean Squared Error	Mean Absolute Error
130bits	584	0.0358	1.328	1.862
Infect Dublin	410	0.0329	1.712	7.576
Infect Hyper	113	0.347	7.827	2.346
Aves Songbird Social	110	0.171	1.803	1.062

### A. Intersection of Node Reliability Polynomials $nRel_G(p)$ of Erdős-Rényi graphs

Invoking (62), the value of node reliability  $nRel_G(p)$  of Erdős-Rényi graph is determined by the term  $\frac{p}{b(p)}$ . For two different Erdős-Rényi graph  $G_1$  (with  $N_1$  nodes and link probability  $p_{l1}$ ) and  $G_2$  (with  $N_2$  nodes and link probability  $p_{l2}$ ), the intersection of two reliability polynomial can be obtained by solving

$$\frac{p_i}{b_1(p_i)} = \frac{p_i}{b_2(p_i)} \quad (64)$$

computing  $p_i$ ,  $b_1(p_i)$  and  $b_2(p_i)$  satisfy:

$$\begin{cases} p_i = \exp\left(\frac{\bar{k}_1 \log N_2 - \bar{k}_2 \log N_1}{\bar{k}_2 - \bar{k}_1}\right) \\ b_1(p_i) = \frac{e^{\bar{k}_1 p_i}}{N_1} \\ b_2(p_i) = \frac{e^{\bar{k}_2 p_i}}{N_2} \end{cases} \quad (65)$$

where  $\bar{k}_1 = N_1 \cdot p_{l1}$ ,  $\bar{k}_2 = N_2 \cdot p_{l2}$ . Based on these  $p_i$  and  $b_i(p_i)$ , we obtain the intersection  $(p_i, e^{-\frac{p_i}{b_1(p_i)}})$ . Fig. 22 illustrates that as in our derivation in Eq.(65), the reliability polynomials of Erdős-Rényi graphs with same  $p_i = 0.4$ ,  $b(p_i) = 0.6$  intersect at point  $(p_i, e^{-\frac{p_i}{b(p_i)}})$ .

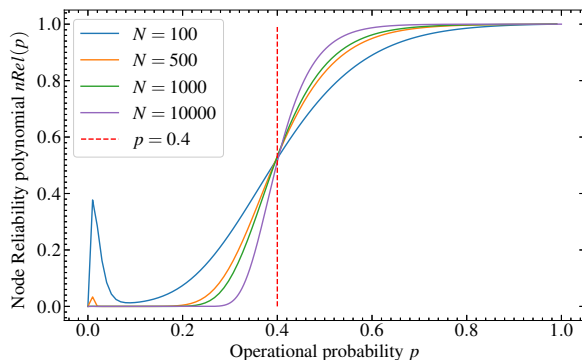


Fig. 22: The Monte Carlo simulation result of node reliability polynomials of Erdős-Rényi graphs with  $p_i = 0.4$ , different number of nodes  $N$  and auxiliary line  $p = 0.4$ .

### B. Asymptotic Behavior of Node Reliability in Erdős-Rényi Graphs with Respect to Link Density and Node Count

Equation (62) shows that the node reliability polynomial of an Erdős-Rényi graph depends on  $\frac{p}{b(p)}$ :

$$\frac{p}{b(p)} = \frac{pN}{e^{N \cdot p \cdot p_l}} = c' \quad (66)$$

Taking the logarithm of both sides:

$$\log p + \log N - N \cdot p \cdot p_l = \log c' \quad (67)$$

For two values  $p_1$  and  $p_2$ , we have:

$$\log \frac{p_1}{p_2} - (p_1 - p_2)Np_l = \log \frac{c'_1}{c'_2} \quad (68)$$

Rearranging the terms:

$$p_1 - p_2 \approx -\frac{\log c'_1 - \log c'_2}{Np_l} \quad (69)$$

Thus, as the average degree  $\bar{k} = Np_l$  becomes large, the range of  $p$  where  $nRel_{G,ER}(p)$  changes from  $c_2$  to  $c_1$  is approximately on the order of  $O\left(\frac{1}{Np_l}\right)$ .