NONCOMMUTATIVE COMPLEX STRUCTURES FOR THE FULL QUANTUM FLAG MANIFOLD OF $\mathcal{O}_q(\mathrm{SU}_3)$

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ABSTRACT. In recent work, Lusztig's positive root vectors (with respect to a distinguished choice of reduced decomposition of the longest element of the Weyl group) were shown to give a quantum tangent space for every A-series Drinfeld–Jimbo full quantum flag manifold $\mathcal{O}_q(\mathrm{F}_n)$. Moreover, the associated differential calculus $\Omega_q^{(0,\bullet)}(\mathrm{F}_n)$ was shown to have classical dimension, giving a direct q-deformation of the classical anti-holomorphic Dolbeault complex of F_n. Here we examine in detail the rank two case, namely the full quantum flag manifold of $\mathcal{O}_q(\mathrm{SU}_3)$. In particular, we examine the $*$-differential calculus associated to $\Omega_q^{(0,\bullet)}(\mathrm{F}_3)$ and its non-commutative complex geometry. We find that the number of almost-complex structures reduces from 8 (that is 2 to the power of the number of positive roots of $\mathfrak{sl}_3$) to 4 (that is 2 to the power of the number of simple roots of $\mathfrak{sl}_3$). Moreover, we show that each of these almost-complex structures is integrable, which is to say, each of them is a complex structure. Finally, we observe that, due to non-centrality of all the non-degenerate coinvariant 2-forms, none of these complex structures admits a left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant noncommutative Kähler structure.

1. INTRODUCTION

Constructing a theory of noncommutative geometry for Drinfeld–Jimbo quantum groups is a very important but very challenging problem. Despite numerous significant contributions over the past three decades, this field remains largely under development. Throughout the literature, the essential example has been the celebrated Podleś sphere $\mathcal{O}_q(S^2)$, which serves as a fundamental test for evaluating new ideas. While many important questions remain, the Podleś sphere stands out for its relatively well understood noncommutative geometry. This is in sharp contrast to the quantum group $\mathcal{O}_q(\mathrm{SU}_2)$, where the obstruction posed by the non-existence of a bicovariant differential calculus of classical dimension remains unresolved.

The Podleś sphere is the simplest example of a quantum flag manifold. For the last two decades this class of quantum homogeneous spaces has been the focus of intense study, as the noncommutative geometry community has tried to extend its understanding of the Podleś sphere to this general class of examples. In particular, attention has

RÓB is supported by the GAČR/NCN grant *Quantum Geometric Representation Theory and Noncommutative Fibrations* 24-11728K. All three authors acknowledge support from COST Action 21109 CaLISTA, supported by COST (European Cooperation in Science and Technology) and HORIZON-MSCA-2022-SE-01-01 CaLIGOLA. AC acknowledges support from MSCA-DN CaLiForNIA - 101119552. AC and JR are grateful for the hospitality offered by Charles University and by the Institute of Mathematics of the Czech Academy of Sciences.

focused on those quantum flag manifolds of irreducible type, a special, more tractable, subfamily of the general quantum flag manifolds. This has seen many successes, the most notable being Heckenberger and Kolb's proof that the irreducible quantum flag manifolds admit an essentially unique q-deformed covariant de Rham complex. This result directly generalises Podleś' construction and classification of differential calculi for $\mathcal{O}_q(S^2)$. It has been shown that the noncommutative complex and Kähler geometry of the Podleś sphere [30] extends to the irreducible quantum flag setting [39, 31], as does the Bott–Borel–Weil theorem for $\mathcal{O}_q(S^2)$ [30, 27, 28, 26, 12, 11]. Following Connes' C^*-algebraic approach to noncommutative geometry, spectral triples have been constructed for many irreducible quantum flag manifolds [15, 14, 16, 43, 32, 18], extending the construction of Dąbrowski–Sitarz for the Podleś sphere [17]. Recently, the family of quantum projective spaces have been endowed with the structure of a compact quantum metric space [35], extending the same construction for the Podleś sphere [1].

While many interesting and challenging problems remain in the irreducible setting, the time has now come to start examining the non-irreducible situation. The first steps in this direction have already been taken. For example, there is the work of Yuncken and Voigt on the noncommutative geometry of the full quantum flag manifold of $\mathcal{O}_q(\mathrm{SU}_2)$, which uses a quantum BGG sequence to verify the Baum–Connes conjecture for $U_q(\mathfrak{sl}_3)$ [42]. More recently, Somberg and the second author constructed an anti-holomorhic Dolbeault complex for the A-series full quantum flag manifolds using Lusztig's root vectors and extended the Borel–Weil theorem to this setting [40]. Subsequently, in [33] Matassa introduced an alternative construction of first-order differential calculi for all quantum flag manifolds.

To a certain extent, the full quantum flag manifolds look closer to the Podleś sphere than the other irreducible quantum flag manifolds. For example, their relative Hopf modules are all direct sums of line bundles. However, their differential calculi $\Omega_q^{(0,\bullet)}(\mathrm{F}_n)$ have more noncommutative behaviour than the Heckenberger–Kolb calculi. In particular, their bimodule structure is more involved. This means that one cannot use the monoidal version of Takeuchi's categorical equivalence, a fact that has many important consequences. In the present paper we restrict to the simplest example of a full quantum flag manifold after the Podleś sphere, namely $\mathcal{O}_q(\mathrm{F}_3)$ the full quantum flag manifold of $\mathcal{O}_q(\mathrm{SU}_3)$. This offers an accessible and tractable example, making it an excellent starting point for future research in the non-irreducible setting. Just as Podleś' work advanced our understanding of the irreducible setting, $\mathcal{O}_q(\mathrm{F}_3)$ has the potential to do the same for the non-irreducible case. Indeed, recent work of Brzezinski and Szymanski [10] described $\mathcal{O}_q(\mathrm{F}_3)$ as the total space of a *quantum fibration* over the quantum projective plane, with a Podleś sphere fibre $\mathcal{O}_q(S^2)$. The authors put this non-principal quantum fibration forward as a motivating example for a proposed theory of noncommutative fibrations with quantum homogeneous fibers. The first steps in this direction were recently taken in [13], and a large family of new examples, extending the full flag manifold example, were produced.

At the first-order level, we observe that just as in the classical case, there exist covariant connections for the one-forms that are not torsion free. This contrasts with the

Heckenberger–Kolb case, where we have a unique covariant connection and this connection is torsion free. We then examine the maximal prolongation of the associated differential $*$-calculus, showing that it has classical dimension, that it is a Koszul and a Frobenius algebra, and we calculate the Nakayama automorphism σ. Notably, unlike for the special anti-holomorphic sub-calculus, σ is not of classical type. We next classify the left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant almost-complex structures on $\Omega^\bullet_q(\mathrm{F}_3)$. We find that the number of almost-complex structures reduces from $2^{|\Delta^+|}$ (where Δ^+ is a choice of positive roots for $\mathfrak{sl}_3$) to $2^{|\Pi|}$ (where Π is the set of associated simple roots). This is because certain almost-complex classical decompositions fail to be bimodule decompositions in the quantum setting, due to the involved bimodule structure of the differential calculus. An almost-complex structure admits a q-deformation only if it is integrable. When it does, integrabillity carries over to the quantum setting, meaning that we do not have any non-integrable noncommutative almost-complex structures. We contrast this with the irreducible quantum flag manifolds that have a unique complex structure, up to identification of opposite complex structures. It is conjectured that this situation generalises to all A-series full quantum flag manifolds.

An interesting observation is that the classical nearly Kähler structure of F_3 is associated to one of the non-integrable almost complex structures, meaning that we do not have a quantum nearly Kähler structure. Another very interesting feature is that the natural quantum analogue of the standard Kähler form and in fact all non-degenerate left $\mathcal{O}_q(\mathrm{SU}_3)$-coinvariant forms, is no longer central. This implies that the calculus does not admit a covariant non-commutative Kähler structure, nor a covariant metric in the sense of Beggs and Majid.

Summary of the Paper. The paper is organised as follows: In §2 we recall some necessary preliminaries about differential calculi, noncommutative complex structutres, and Drinfeld–Jimbo quantum groups.

In §3 we present the differential calculus $\Omega^1_q(\mathrm{F}_3)$ as the base of a homogeneous quantum principal bundle and observe that the zero map gives a principal connection. We then calculate the degree two relations of the maximal prolongation. Moreover, we present the associated quantum exterior algebra as a Frobenius algebra and calculate its Nakayama automorphism.

In §4 we discuss torsion for connections for covariant calculi over quantum homogeneous spaces. We show that, under the assumption that the quantum isotropy subgroup is cosemisimple, a covariant torsion free connection always exists. We also calculate the dimension of the affine space of covariant connections, and torsion free covariant connections, for $\Omega^1_q(\mathrm{F}_3)$.

In §5 we classify the left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant almost-complex structures of $\Omega^1_q(\mathrm{F}_3)$. We show that of the 8 classical almost-complex structures, only four pass to the quantum setting and that all of these are integrable. Finally, we examine how the standard classical Kähler form behaves in the quantum setting. We see that there is a three dimensional space of left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant 2-forms, and that none of these forms is both non-degenerate and central.

Acknowledgements: We would like to thank Edwin Beggs, Arnab Bhattacharjee, Andrey Krutov, and Ben McKay for many useful discussions.

2. Preliminaries

In this section we recall some basic material about covariant differential calculi over Hopf algebras and their associated tangent spaces. We use Sweedler notation, denote by Δ, ε and S the coproduct, counit and antipode of a Hopf algebra respectively. We write A° for the dual coalgebra (Hopf algebra) of a (Hopf) algebra A, and denote the pairing between A and A° by angular brackets. Throughout the paper, all algebras are over $\mathbb{C}$ and assumed to be unital, all unadorned tensor products are over $\mathbb{C}$, and all Hopf algebras are assumed to have bijective antipodes.

2.1. **Quantum Homogeneous Spaces.** We begin by briefly recalling Takeuchi's equivalence for relative Hopf modules, see [13, Appendix A] for more details. For A a Hopf algebra, we say that a left coideal subalgebra $B \subseteq A$ is a *quantum homogeneous A-space* if A is faithfully flat as a right B-module and $B^+A = AB^+$, where $B^+ := \ker(\varepsilon|_B)$. We denote by ${}^A_B\mathrm{mod}$ the category of relative Hopf modules which are finitely generated as left B-modules, and by ${}^{\pi_B}\mathrm{mod}$ the category of finite-dimensional left comodules over the Hopf algebra $\pi_B(A) := A/B^+A$, which we call the *quantum isotropy subgroup.* An equivalence of categories, known as Takeuchi's equivalence, is given by the functor $\Phi : {}^A_B\mathrm{mod} \to {}^{\pi_B}\mathrm{mod}$, where $\Phi(\mathcal{F}) = \mathcal{F}/B^+\mathcal{F}$, for any relative Hopf module $\mathcal{F}$, and the functor $\Psi : {}^{\pi_B}\mathrm{mod} \to {}^A_B\mathrm{mod}$ is defined using the cotensor product $\square_{\pi_B}$ over $\pi_B(A)$. A unit for the equivalence is given by $\mathrm{U} : \mathcal{F} \to (\Psi \circ \Phi)(\mathcal{F})$, where $\mathrm{U}(f) = f_{(1)} \otimes [f_{(0)}]$, and $[f_{(0)}]$ denotes the coset of $f_{(0)}$ in $\Phi(\mathcal{F})$. For the special case where $B = A$, Takeuchi's equivalence is known as the fundamental theorem of Hopf modules [36].

2.2. **Covariant Differential Calculi over Hopf algebras.** A *differential calculus*, or a *dc*, is a differential graded algebra (dga)

$$\left(\Omega^\bullet \simeq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \Omega^k, \mathrm{d}\right)$$

that is generated as an algebra by the elements $a, \mathrm{d}b$, for $a, b \in \Omega^0$. When no confusion arises, we denote the dc by $\Omega^\bullet$, omitting the exterior derivative d. We denote the degree of a homogeneous element $\omega \in \Omega^\bullet$ by $|\omega|$. For a given algebra B, a differential calculus *over* B is a differential calculus such that $B = \Omega^0$. We say that $\omega \in \Omega^\bullet$ is *closed* if $\mathrm{d}\omega = 0$. If B' is a subalgebra of B then the *restriction* of $\Omega^\bullet(B)$ to B' is the dga generated by the elements $\mathrm{d}b'$, for $b' \in B'$.

A *first-order differential calculus* (fodc) over an algebra B is a pair $(\Omega^1(B), \mathrm{d})$, where $\Omega^1(B)$ is a B-bimodule and $\mathrm{d} : B \to \Omega^1(B)$ is a derivation such that Ω^1 is generated as a left (or equivalently right) B-module by those elements of the form $\mathrm{d}b$, for $b \in B$. We say that a dc $(\Gamma^\bullet, \mathrm{d}_\Gamma)$ *extends* a fodc $(\Omega^1, \mathrm{d}_\Omega)$ if there exists a bimodule isomorphism $\varphi : \Omega^1 \to \Gamma^1$ such that $\mathrm{d}_\Gamma = \varphi \circ \mathrm{d}_\Omega$. It can be shown [38, §2.5] that any fodc admits an extension $\Omega^\bullet$ which is *maximal* in the sense that there exists a unique differential map from $\Omega^\bullet$ onto any other extension of Ω^1. We call this extension the *maximal prolongation* of Ω^1.

For A a Hopf algebra, and $B \subseteq A$ a quantum homogeneous space, a dc $\Omega^\bullet$ over B is said to be *left covariant* if the coaction $\Delta_L : B \to A \otimes B$ extends to a (necessarily unique) coaction $\Delta_L : \Omega^\bullet \to A \otimes \Omega^\bullet$, giving $\Omega^\bullet$ the structure of a left A-comodule algebra such that d is a left A-comodule map. We see that $\Omega^\bullet$ is naturally an object in ${}^A_B\mathrm{Mod}_B$. For any covariant dc $\Omega^\bullet$ we usually find it notationally convenient to denote $V^\bullet := \Phi(\Omega^\bullet)$.

2.3. **Some Remarks on Quantum Homogeneous Tangent Spaces.** Let A be a Hopf algebra, and $W \subseteq A^\circ$ a Hopf subalgebra of A°, such that

$$B := {}^W A = \Big\{ b \in B \,|\, b_{(1)} \langle b_{(2)}, w \rangle \Big\}$$

is a quantum homogeneous A-space, and denote by B° its dual coalgebra. A *tangent space* for B is a subspace $T \subseteq B^\circ$ such that $T \oplus \mathbb{C}1$ is a right coideal of B° and $WT \subseteq T$. For any tangent space T, a right B-ideal of B^+ is given by

$$I := \big\{ x \in B^+ \,|\, X(x) = 0, \text{ for all } X \in T \big\},$$

meaning that the quotient $V^1 := B^+/I$ is naturally an object in the category ${}^{\pi_B}\mathrm{Mod}_B$. We call V^1 the *cotangent space* of T. Consider now the object

$$\Omega^1(B) := A \square_{\pi_B} V^1.$$

If $\{X_i\}_{i=1}^n$ is a basis for T, and $\{e_i\}_{i=1}^n$ is the dual basis of V^1, then the map

$$\mathrm{d} : A \to \Omega^1(B), \qquad\qquad a \mapsto \sum_{i=1}^n (X_i^+ \triangleright a) \otimes e_i$$

is a derivation, and the pair $(\Omega^1(B), \mathrm{d})$ is a left A-covariant fodc over B. This gives a bijective correspondence between isomorphism classes of finite-dimensional tangent spaces and finitely-generated left A-covariant fodc [22].

In order to give an explicit presentation of the maximal prolongation of a left A-covariant fodc $\Omega^1(B)$, we need to recall the notion of a framing calculus: A *framing calculus* for $\Omega^1(B)$ is a left A-covariant fodc $\Omega^1(A) \simeq A \otimes \Lambda^1$ over A that restricts to $\Omega^1(B)$, such that V^1 embeds into Λ^1, and the image of V^1 in Λ^1 is a right A-submodule of Λ^1.

Let $\Omega^1(A) \simeq A \otimes \Lambda^1$ be a framing calculus for $\Omega^1(B)$, and let $I \subseteq B^+$ be the corresponding ideal of the fodc. Consider the subspace

$$I^{(2)} := \Big\{ \omega(y) := [y^+_{(1)}] \otimes [y^+_{(2)}] \mid y \in I \Big\} \subseteq V^1 \otimes V^1 \subseteq \Lambda^1 \otimes \Lambda^1.$$

Starting from the the tensor algebra $\mathcal{T}(V^1$ of V^1, we construct the $\mathbb{Z}_{\geq 0}$-graded algebra

$$V^\bullet := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} V^k := \mathcal{T}\big(V^1\big)/\langle I^{(2)} \rangle,$$

which we call the *quantum exterior algebra* of $\Omega^1(B)$, and whose multiplication we denote by $\wedge$, motivated by the classical situation. An isomorphism between $F(\Omega^k(B))$ and V^k, for $k \in \mathbb{Z}_{\geq 0}$, is determined by

$$[\mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_k] \;\mapsto\; [b_1^+] \wedge \cdots \wedge [b_k^+],$$

giving us an explicit description of the maximal prolongation.

2.4. **First-Order Differential $*$-Calculi.** A *$*$-differential calculus*, or a *$*$-dc*, over a $*$-algebra B is a differential calculus over B such that the $*$-map of B extends to a (necessarily unique) conjugate-linear involution $* : \Omega^\bullet(B) \to \Omega^\bullet(B)$ satisfying $\mathrm{d}(\omega^*) = (\mathrm{d}\omega)^*$, and

$$(\omega \wedge \nu)^* = (-1)^{kl} \nu^* \wedge \omega^*, \qquad \text{for all } \omega \in \Omega^k,\ \nu \in \Omega^l.$$

If we now assume that A is a Hopf $*$-algebra, and that $\Omega^1(A)$ is a left A-covariant fodc over A, with corresponding tangent space T, then $\Omega^1(A)$ is a $*$-fodc iff $T^* = T$, [29, Proposition 14.1.2] for details. Consider next the case where $T^* \neq T$. Since

$$\Delta(X^*) = X^*_{(1)} \otimes X^*_{(2)} \in (T^* \oplus \mathbb{C}) \otimes A^\circ,$$

T^* is a tangent space. implying in turn that $T + T^*$ is a tangent space. We call $T + T^*$ the *$*$-extension* of T.

2.5. **Preliminaries on the Full Quantum Flag Manifold of $\mathcal{O}_q(\mathrm{SU}_3)$.** In this subsection we recall the definition of the Drinfeld–Jimbo quantised enveloping algebra of the simple Lie algebra $\mathfrak{sl}_3$, the quantum coordinate algebra $\mathcal{O}_q(\mathrm{SU}_3)$. For more details, we direct the reader to [29] where the general definitions of the Drinfeld–Jimbo quantised enveloping algebras and their dual quantum coordinate algebras are given.

The algebra $U_q(\mathfrak{sl}_3)$ is generated by the elements $E_1, E_2, F_1, F_2, K_1^{\pm}$, and $K_2^{\pm}$, subject to the relations

$$K_i^{\pm 1} E_j = q^{\pm a_{ij}} E_j K_i^{\pm 1}, \qquad K_i^{\pm 1} F_j = q^{\mp a_{ij}} F_j K_i^{\pm 1}, \tag{1}$$

$$K_i^{\pm 1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{\pm 1}, \qquad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{2}$$

where (a_{ij}) is the Cartan matrix of $\mathfrak{sl}_3$, and the two *Serre relations*

$$E_1^2 E_2 - (q + q^{-1}) E_1 E_2 E_1 + E_1 E_2^2, \qquad F_1^2 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_1 F_2^2. \tag{3}$$

A Hopf algebra structure on $U_q(\mathfrak{sl}_3)$ is determined by the coproduct formulae

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \qquad \Delta(K_i) = K_i \otimes K_i,$$

and the counit formulae

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \varepsilon(K^{\pm 1}) = 1.$$

Moreover, a Hopf $*$-algebra structure is given by

$$E_i^* = K_i F_i, \qquad F_i^* = E_i K_i^{-1}, \qquad K_i^* = K_i.$$

Let u_{ij}, for $i, j = 1, 2, 3$, be the elements of $U_q(\mathfrak{sl}_3)^\circ$, the Hopf dual of $U_q(\mathfrak{sl}_3)$, defined by the coproduct formula $\Delta(u_{ij}) = \sum_{a=1}^{3} u_{ia} \otimes u_{aj}$, and the fact that their only non-zero pairings with the non-unital generators of $U_q(\mathfrak{sl}_3)$ are given by

$$\langle E_1, u_{21} \rangle = \langle E_2, u_{32} \rangle = \langle F_1, u_{12} \rangle = \langle F_2, u_{23} \rangle = 1, \qquad \langle K_i^{\pm 1}, u_{jj} \rangle = q^{\pm(\delta_{i,j-1} - \delta_{ij})}.$$

We denote by $\mathcal{O}_q(\mathrm{SU}_3)$ the subalgebra of $U_q(\mathfrak{sl}_3)^\circ$ generated by the elements u_{ij}. for $i, j = 1, 2, 3$, and call it the *quantum coordinate algebra* of SU_3. It is clear that $\mathcal{O}_q(\mathrm{SU}_3)$

is a sub-bialgebra of $U_q(\mathfrak{sl}_3)^\circ$. In fact, it is a Hopf subalgebra of $U_q(\mathfrak{sl}_3)^\circ$ whose antipode satisfies

$$S(u_{ij}) = (-q)^{i-j}(u_{km}u_{ln} - q u_{kn}u_{lm}),$$

where $\{k,l\} := \{1,2,3\}\backslash\{j\}$, and $\{m,n\} := \{1,2,3\}\backslash\{i\}$. Moreover, it is a Hopf $*$-algebra with respect to the $*$-map defined by $(u_{ij})^* = S(u_{ji})$, for all $i, = 1,2,3$.

2.6. **Preliminaries on $\mathcal{O}_q(\mathrm{F}_3)$ the Full Quantum Flag Manifold of $\mathcal{O}_q(\mathrm{SU}_3)$.** Consider now the commutative subalgebra of $U_q(\mathfrak{sl}_3)$ generated by $\{K_i^{\pm 1} \mid i = 1,2\}$ which we denote by $U_q(\mathfrak{h})$. This is a Hopf subalgebra of $U_q(\mathfrak{sl}_3)$ and we define the *full quantum flag manifold* $\mathcal{O}_q(\mathrm{F}_3)$ to be the space of invariants of $\mathcal{O}_q(\mathrm{SU}_3)$ with respect to the restriction of the action of $U_q(\mathfrak{sl}_3)$ to $U_q(\mathfrak{h})$:

$$\mathcal{O}_q(\mathrm{F}_3) := {}^{U_q(\mathfrak{h})}\mathcal{O}_q(\mathrm{SU}_3).$$

The full quantum flag manifold is a quantum homogeneous space, and a special example of the general class of quantum homogenous spaces called quantum flag manifolds, see [21, 24, 13] for more details.

The decomposition of $\mathcal{O}_q(\mathrm{SU}_3)$ into homogeneous components with respect to the action of $U_q(\mathfrak{h})$ is equivalent to having a $\mathcal{P}^+ = \mathbb{Z}^2$ grading

$$\mathcal{O}_q(\mathrm{SU}_3) = \bigoplus_{\lambda \in \mathcal{P}^+} \mathcal{E}_\lambda, \tag{4}$$

where each $\mathcal{E}_k$ is an *equivariant line bundle* over $\mathcal{O}_q(\mathrm{F}_3)$, that is an invertible object in the category of relative Hopf bimodules over $\mathcal{O}_q(\mathrm{F}_3)$. We see that $\mathcal{E}_0 = \mathcal{O}_q(\mathrm{F}_3)$. See [13, §5] for further details. Moreover, on generators we see that

$$u_{i1} \in \mathcal{E}_{-\varpi_1}, \qquad u_{i2} \in \mathcal{E}_{\varpi_1 - \varpi_2}, \qquad u_{i3} \in \mathcal{E}_{\varpi_2}, \tag{5}$$

for all $i = 1,2,3$, which completely determines the $\mathcal{P}^+$-grading.

The subalgebra $\mathcal{O}_q(\mathbb{CP}^2)$ of $\mathcal{O}_q(\mathrm{SU}_3)$ generated by the elements $z_{ij}^{\alpha_1} := u_{i1}u_{j1}^*$, for $i,j = 1,2,3$, is called the *quantum projective plane* [15, 21, 27, 34, 37]. An isomorphic copy of the quantum projective plane is generated by the elements $z_{ij}^{\alpha_2} := u_{i3}u_{j3}^*$, for $i,j = 1,2,3$. Both algebras are contained in $\mathcal{O}_q(\mathrm{F}_3)$, and together they generate $\mathcal{O}_q(\mathrm{F}_3)$ as an algebra. Moreover, both subsalgebras are quantum homogeneous spaces, and in fact also examples of quantum flag manifolds, again see [13, §5] for further details.

Let us next introduce some notation

$$z_i^{\alpha_1} := u_{i1}, \qquad \text{for } i = 1,2,3, \qquad z_i^{\alpha_2} := u_{i3}, \qquad \text{for } i = 1,2,3. \tag{6}$$

The $*$-algebra $\mathcal{O}_q(S^5)$ generated by the elements $z_i^{\alpha_1}$ is known as the *quantum* 5*-sphere*, or the *Vaksmann–Soibelmann* 5*-sphere*. An isomorphic $*$-algebra is generated by the elements $z_i^{\alpha_2}$. From (5) above, we immediately see that

$$z_i^{\alpha_1} \in \mathcal{E}_{\varpi_1}, \qquad \overline{z}_i^{\alpha_1} := (z^{\alpha_1})^* \in \mathcal{E}_{-\varpi_1} \qquad z_i^{\alpha_2} \in \mathcal{E}_{\varpi_2}, \qquad \overline{z}_i^{\alpha_2} := (z_i^{\alpha_2})^* \in \mathcal{E}_{-\varpi_2}.$$

In fact, since these elements generate $\mathcal{O}_q(\mathrm{SU}_3)$ as a $*$-algebra, this gives an alternative complete description of the $\mathcal{P}^+$-grading.

3. The Lusztig–de Rham Complex of $\mathcal{O}_q(F_3)$

In this section we present the main result of the paper, a q-deformation of the classical de Rham complex of the full flag manifold of SU_3. We extend the construction of [40], where a quantum tangent space $T^{(0,1)}$ was constructed that q-deforms the classical anti-holomorphic tangent space of F_3. We start by considering the $*$-extension of $T^{(0,1)}$, and then look at the maximal prolongation of its associated fodc. Finally, the quantum exterior of the prolongation is considered as a Frobenius algebra.

3.1. **The $*$-Extension of $T^{(0,1)}$.** In what follows, we find it convenient to denote the positive simple generators E_1 and E_2 by E_{α_1} and E_{α_2} respectively. In addition we will also consider the non-simple root vector

$$E_{\alpha_1+\alpha_2} := [E_2, E_1]_{q^{-1}}.$$

We can understand $E_{\alpha_1+\alpha_2}$ as a Lusztig root vector. Explicitly, recall that the Weyl group of $\mathfrak{sl}_3$ is the symmetric group S_3, with standard generators s_1 and s_2. Then, with respect to the reduced decomposition $s_2s_1s_2$ of the longest element of $\mathfrak{sl}_3$, the element $E_{\alpha_1+\alpha_2}$ is the associated non-simple quantum root vector. See [29, §6.2], or [40, Appendix A], for a more detailed presentation of Lusztig's root vectors. Consider now the subspace

$$T^{(0,1)} := \mathrm{span}_{\mathbf{C}}\Big\{E_{\alpha_1},\, E_{\alpha_2}, E_{\alpha_1+\alpha_2}\Big\}. \tag{7}$$

As shown in [40, Example 3.2.], we have the identity

$$\Delta(E_{\alpha_1+\alpha_2}) = E_{\alpha_1+\alpha_2} \otimes K_1K_2 + q^{-1}\nu E_{\alpha_1} \otimes E_{\alpha_2}K_1 + 1 \otimes E_{\alpha_1+\alpha_2}. \tag{8}$$

Thus we see that $T^{(0,1)}$ is a quantum tangent space for $\mathcal{O}_q(\mathrm{SU}_3)$.

Next we consider the quantum tangent space T, the $*$-extension of $T^{(0,1)}$. We denote $T^{(1,0)} := (T^{(0,1)})^*$ and we see it is spanned by the elements

$$F_{\alpha_1} := E^*_{\alpha_1} = K_1F_1, \quad F_{\alpha_2} := E^*_{\alpha_2} = K_2F_2, \quad F_{\alpha_1+\alpha_2} := E^*_{\alpha_1+\alpha_2} = q^{-1}K_1K_2[F_1, F_2]_{q^{-1}}.$$

Moreover, we can conclude the following coproduct formula

$$\Delta(F_{\alpha_1+\alpha_2}) = F_{\alpha_1+\alpha_2} \otimes K_1K_2 + \nu F_{\alpha_1} \otimes F_{\alpha_2}K_1 + 1 \otimes F_{\alpha_1+\alpha_2}. \tag{9}$$

Denote the associated $*$-fodc by $\Omega^1_q(\mathrm{SU}_3)$, its cotangent space by Λ^1, and the basis of Λ^1 dual to the defining basis of T by

$$\Big\{e_\gamma,\, f_\gamma \,|\, \gamma \in \Delta^+\Big\}.$$

The following lemma gives explicit representatives for the the cosets of the dual basis. These representatives will be used in a number of calculations in this section.

Lemma 3.1. *It holds that*

$$e_{\alpha_1} = [u_{21}], \qquad e_{\alpha_2} = [u_{32}], \qquad e_{\alpha_1+\alpha_2} = [u_{31}],$$
$$f_{\alpha_1} = [qu_{12}], \qquad f_{\alpha_2} = [qu_{23}], \qquad f_{\alpha_1+\alpha_2} = [q^2u_{13}].$$

Proof. The representatives for the positive basis elements were established in [40, Lemma 3.6]. The negative representatives are produced similarly. For example, the calculation

$$\langle F_{\alpha_1}, u_{12}\rangle = \langle K_1F_1, u_{12}\rangle = \langle K_1, u_{11}\rangle\langle F_1, u_{12}\rangle = q^{-1},$$

implies that qu_{12} is a representative for the coset f_{α_1}. □

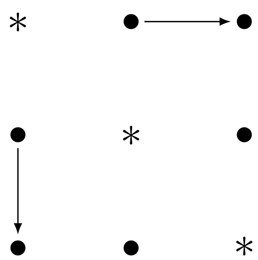
The following proposition determines the right $\mathcal{O}_q(\mathrm{SU}_3)$-module structure of Λ^1. The proof is completely analogous to that of [40, Proposition 3.7] and so we omit it.

Proposition 3.2. *The right $\mathcal{O}_q(\mathrm{SU}_3)$-module structure of Λ^1 is determined by*

$$e_\gamma u_{kk} = q^{-(\gamma,\varepsilon_k)}e_\gamma, \qquad f_\gamma u_{kk} = q^{-(\gamma,\varepsilon_k)}f_\gamma,$$
$$e_{\alpha_1}u_{32} = \nu e_{\alpha_1+\alpha_2}, \qquad f_{\alpha_1}u_{23} = q^{-1}\nu f_{\alpha_1+\alpha_2},$$

with all other actions by the generators u_{ij} being zero.

Just as in [40, §3.4], we now find it instructive to present the non-diagonal actions in the form of a graph. Arrange the basis elements in their natural upper and lower triangular form and draw an arrow from one basis element b to another b' if there exists a generator u_{ji} such that bu_{ji} is a scalar multiple of b':



3.2. **A Quantum Principal Bundle.** In this section we prove that the pair $(\mathcal{O}_q(\mathrm{SU}_3), \Omega^1_q(\mathrm{SU}_3))$ gives a quantum principal bundle according to Brzeziński and Majid [4, 9], whose definition we recall here.

Definition 3.3. Let H be a Hopf algebra. A *quantum principal H-bundle* is a pair $(P, \Omega^1(P))$, consisting of a right H-comodule algebra (P, Δ_R) and a right-H-covariant calculus $\Omega^1(P)$, such that:

1. P is a Hopf–Galois extension of $B = P^{\mathrm{co}(H)}$.
2. If $N \subseteq \Omega^1_u(P)$ is the sub-bimodule of the universal calculus corresponding to $\Omega^1(P)$, we have $\mathrm{ver}(N) = P \otimes I$, for some Ad-sub-comodule right ideal
$$I \subseteq H^+ := \ker(\varepsilon : H \to \mathbb{C}),$$
where $\mathrm{Ad} : H \to H \otimes H$ is defined by $\mathrm{Ad}(h) := h_{(2)} \otimes S(h_{(1)})h_{(3)}$.

Denoting by $\Omega^1(B)$ the restriction of $\Omega^1(P)$ to B, and $\Lambda^1(H) := H^+/I$, the quantum principal bundle definition implies that an exact sequence is given by

$$0 \longrightarrow P\Omega^1(B)P \overset{\iota}{\longrightarrow} \Omega^1(P) \overset{\mathrm{ver}}{\longrightarrow} P \otimes \Lambda^1(H) \longrightarrow 0, \tag{10}$$

where by abuse of notation ver denotes the map induced on $\Omega^1(P)$ by identifying $\Omega^1(P)$ as a quotient of $\Omega^1_u(P)$. We denote $\Omega^1(P)_{\mathrm{hor}} := P\Omega^1(B)P$ and call this subspace the *horizontal forms* of the bundle.

In this subsection we construct a quantum principal bundle from our fodc $\Omega^1_q(\mathrm{SU}_3)$.

Proposition 3.4. *For the quantum homogeneous space $\mathcal{O}_q(\mathrm{F}_3)$, a quantum principal bundle is given by the pair $(\mathcal{O}_q(\mathrm{SU}_3), \Omega^1_q(\mathrm{SU}_3))$.*

Proof. Since $\mathcal{O}_q(\mathrm{F}_3)$ is a quantum homogeneous space, it suffices to show that $\Omega^1_q(\mathrm{SU}_3)$ is a right covariant fodc. As discussed in [40, §2.2], this would follow from the identity $U_q(\mathfrak{h})T = T$, where the action of $U_q(\mathfrak{h})$ on T is given by the right adjoint action. However, since T is spanned by root vectors, this is clear. □

Corollary 3.5. *Denoting the cotangent space of $\Omega^1(B)$ by V^1, an isomorphism in the category of $U_q(\mathfrak{h})$-modules is given by*

$$V^1 \to \Lambda^1, \qquad\qquad [b] \mapsto [b].$$

Proof. The fact that this is an injective module map follows from [30, Theorem 2.1]. (See also the discussion in [40, §4.2].) Surjectivity follows from the fact that each basis element of the tangent space T pairs non-trivially with an element of $\mathcal{O}_q(\mathrm{F}_3)$. Explicitly, the pairings

$$\begin{array}{llll} \langle E_{\alpha_1}, z^{\alpha_1}_{21}\rangle, & \langle E_{\alpha_2}, z^{\alpha_2}_{32}\rangle, & \langle E_{\alpha_1+\alpha_2}, z^{\alpha_1}_{31}\rangle, & \langle E_{\alpha_1+\alpha_2}, z^{\alpha_2}_{31}\rangle, \\ \langle F_{\alpha_1}, z^{\alpha_1}_{12}\rangle, & \langle F_{\alpha_2}, z^{\alpha_2}_{23}\rangle, & \langle F_{\alpha_1+\alpha_2}, z^{\alpha_1}_{13}\rangle, & \langle F_{\alpha_1+\alpha_2}, z^{\alpha_2}_{13}\rangle. \end{array}$$

are all non-zero scalars. □

Let us now recall the definition of a framing calculus for a quantum homogeneous space $B \subseteq A$. For any first-order differential calculus $\Omega^1(B)$ over B, a framing calculus $\Omega^1(B)$ is a fodc over A such that $\Omega^1(B)$ is the restriction of $\Omega^1(A)$ to B and $\Omega^1(B)A \subseteq A\Omega^1(B)$.

Corollary 3.6. *It holds that $\Omega^1_q(\mathrm{SU}_3)$ is a framing calculus for $\Omega^1(\mathrm{F}_3)$.*

Proof. Since the embedding in Corollary 3.5 above is surjective, the image of V^1 is obviously a right $\mathcal{O}_q(\mathrm{SU}_3)$ submodule of Λ^1. It now follows from [38] that

$$\Omega^1_q(\mathrm{F}_3)\mathcal{O}_q(\mathrm{SU}_3) = (\mathcal{O}_q(\mathrm{SU}_3)\Box_{\pi_B} V^1)\mathcal{O}_q(\mathrm{SU}_3) \subseteq \mathcal{O}_q(\mathrm{SU}_3) \otimes V^1 = \mathcal{O}_q(\mathrm{SU}_3)\Omega^1_q(\mathrm{F}_3).$$

Thus $\Omega^1_q(\mathrm{SU}_3)$ is a framing calculus, as claimed. □

Finally, following the argument of [11, Proposition 5.8], we see that Corollary 3.5 implies the existence of a principal connection for the quantum bundle.

Corollary 3.7. *The zero map on $\Omega^1_q(\mathrm{SU}_3)$ is a left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant connection for the quantum principal bundle $(\mathcal{O}_q(\mathrm{SU}_3), \Omega^1_q(\mathrm{SU}_3))$.*

3.3. **The Higher Forms.** In this section we use the results of [38, §5] to calculate the degree two relations of the maximal prolongation of the fodc $\Omega^1_q(\mathrm{F}_3)$.

Theorem 3.8. *A full set of relations for the algebra $V^\bullet$ is given by the following three sets of identities: The first set is given by*

$$e_\gamma \wedge e_\beta = -q^{(\beta,\gamma)} e_\beta \wedge e_\gamma, \qquad f_\gamma \wedge f_\beta = -q^{-(\beta,\gamma)} f_\beta \wedge f_\gamma, \qquad \text{for all } \beta \leq \gamma \in \Delta^+,$$

the second set of relations is given by

$$e_\gamma \wedge f_\beta = -q^{(\beta,\gamma)} f_\beta \wedge e_\gamma, \qquad \text{for all } \beta \neq \gamma \in \Delta^+, \text{ or for } \beta = \gamma = \alpha_1 + \alpha_2,$$

and the third set is given by the two identities

$$e_{\alpha_1} \wedge f_{\alpha_1} = -q^2 f_{\alpha_1} \wedge e_{\alpha_1} - \nu f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2},$$
$$e_{\alpha_2} \wedge f_{\alpha_2} = -q^2 f_{\alpha_2} \wedge e_{\alpha_2} + \nu f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2}.$$

Proof. Using the description of the right $\mathcal{O}_q(\mathrm{SU}_3)$-module structure of Λ^1 given above, one can observe the following set of identities, analogous to those in Lemma 3.1:

$$[S(u_{21})] = -qe_{\alpha_1}, \qquad [S(u_{32})] = -qe_{\alpha_2}, \qquad [S(u_{31})] = -qe_{\alpha_1+\alpha_2},$$
$$[S(u_{12})] = -q^{-2} f_{\alpha_1}, \qquad [S(u_{23})] = -q^{-2} f_{\alpha_2}, \qquad [S(u_{13})] = -q^{-5} f_{\alpha_1+\alpha_2}.$$

Moreover, we have the second set of identities, analogous to those in Proposition 3.2:

$$e_\gamma S(u_{kk}) = q^{(\gamma,\varepsilon_k)} e_\gamma, \qquad f_\gamma S(u_{kk}) = q^{(\gamma,\varepsilon_k)} f_\gamma, \tag{11}$$

$$e_{\alpha_1} S(u_{32}) = -\nu e_{\alpha_1+\alpha_2}, \qquad f_{\alpha_1} S(u_{23}) = -q^{-3} \nu f_{\alpha_1+\alpha_2}, \tag{12}$$

with all other actions by the antipoded generators $S(u_{ij})$ being zero.
From these identities, we can now see that the following set is the dual basis of V^1:

$$e_{\alpha_1} = q^{-1}[z^{\alpha_1}_{21}], \qquad e_{\alpha_2} = -q^{-1}[z^{\alpha_2}_{32}], \qquad e_{\alpha_1+\alpha_2} = q^{-1}[z^{\alpha_1}_{31}] = -q^{-1}[z^{\alpha_2}_{31}]$$
$$f_{\alpha_1} = -q^2[z^{\alpha_1}_{12}], \qquad f_{\alpha_2} = q^2[z^{\alpha_2}_{23}], \qquad f_{\alpha_1+\alpha_2} = -q^5[z^{\alpha_1}_{13}] = q^3[z^{\alpha_2}_{13}].$$

We next introduce a set of generators for the ideal of the tangent space from the description of the right $\mathcal{O}_q(\mathrm{SU}_3)$-module structure of $\Lambda^1$is given above. We divide the set of generators according to their polynomial degree. To do so we find it convenient to introduce the subset of $\mathbb{Z}^3_{>0}$

$$B := \Big\{(1,2,1), (1,1,2), (1,3,1), (1,1,3), (2,3,2), (2,2,3), (2,3,1), (2,1,3)\Big\}.$$

Consider now the degree one polynomials

$$G_1 := \Big\{ z^{\alpha_i}_{ab} \,|\, (i,a,b) \notin B \Big\} \cup \Big\{ z^{\alpha_1}_{31} + z^{\alpha_2}_{31},\, q^2 z^{\alpha_1}_{13} + z^{\alpha_2}_{13} \Big\}$$

Next consider the quadratic polynomials

$$G_2 := \Big\{ z^{\alpha_i}_{kl} (z^{\alpha_p}_{ab})^+ \,|\, (i,k,l) \in B \backslash \{(1,2,1),\, (2,3,2)\},\, p = 1,2,\, a,b = 1,2,3 \Big\},$$
$$G_3 := \Big\{ z^{\alpha_i}_{kl} (z^{\alpha_p}_{ab})^+ \,|\, (i,k,l) \in B,\, (p,a,b) \neq (2,3,2), (2,2,3) \Big\},$$
$$G_4 := \Big\{ z^{\alpha_1}_{21} z^{\alpha_2}_{32} - \nu z^{\alpha_2}_{31},\, z^{\alpha_1}_{12} z^{\alpha_2}_{23} - \nu z^{\alpha_1}_{13} \Big\}.$$

Collecting these elements together gives us our proposed set of generators

$$G := G_1 \cup G_2 \cup G_3 \cup G_4.$$

Indeed, since it is clear that

$$\dim\left(\mathcal{O}_q(\mathrm{F}_3)^+/\langle G\rangle\right) \leq 6,$$

where $\langle G\rangle$ is the right ideal of $\mathcal{O}_q(\mathrm{F}_3)^+$ generated by the elements of G, we see that G gives a full set of generators.

Calculating the action of the map ω on these generators is now a routine calculation, as explicitly presented in [38, Proposition 5.8] for the case of quantum projective space, and in [40, §3.3] for the anti-holomoprhic complex of the A-series full quantum flag manifold. As an example, we take the generator $z^{\alpha_1}_{22}$, and note that

$$\begin{aligned} \omega(z^{\alpha_1}_{22}) &= \sum_{a,b}[u_{2a}S(u_{b2})] \otimes [z^{\alpha_1}_{ab}] \\ &= [u_{22}S(u_{12})] \otimes [z^{\alpha_1}_{21}] + [u_{23}S(u_{12})] \otimes [z^{\alpha_1}_{31}] + \\ &\quad [u_{21}S(u_{22})] \otimes [z^{\alpha_1}_{12}] + [u_{21}S(u_{32})] \otimes [z^{\alpha_1}_{13}] \end{aligned} \tag{13}$$

Using identities in 11 and 12, we computed:

$$[u_{22}S(u_{12})] = -q^{-2}f_{\alpha_1}, \qquad [u_{23}S(u_{12})] = 0$$

$$[u_{21}S(u_{22})] = q^{-1}e_{\alpha_1}, \qquad [u_{21}S(u_{32})] = -\nu e_{\alpha_1+\alpha_2}$$

Substituting these values in Eq. 13 we get:

$$w(z^{\alpha_1}_{22}) = -q^{-1}f_{\alpha_1} \otimes e_{\alpha_1} - q^{-3}e_{\alpha_1} \otimes f_{\alpha_1} + q^{-5}\nu e_{\alpha_1+\alpha_2} \otimes f_{\alpha_1+\alpha_2}.$$

Therefore, the relation $w(z^{\alpha_1}_{22}) = 0$ gives us:

$$e_{\alpha_1} \wedge f_{\alpha_1} = -q^2 f_{\alpha_1} \wedge e_{\alpha_1} + q^{-2}\nu e_{\alpha_1+\alpha_2} \wedge f_{\alpha_1+\alpha_2}.$$

Continuing as such gives us the claimed set of relations. Finally, we observe the description of the right $\mathcal{O}_q(\mathrm{SU}_3)$-module structure of Λ^1 given in Proposition 3.2 implies that the relations form a right $\mathcal{O}_q(\mathrm{F}_3)$-submodule of $V^1 \otimes V^1$. Thus they give a full set of relations. □

In the next corollary we consider $\Omega(A)$, the maximal prolongation of $\Omega^1(A)$. In view of the fundamental theorem of Hopf modules, for each $k \in \mathbb{Z}$ one hase

$$\Omega^k(A) = A \otimes \Lambda^k$$

and the following holds.

Corollary 3.9. *For $k = 1, \ldots, 6 = |\Delta|$, a basis of Λ^k is given by*

$$\left\{e_{\gamma_1} \wedge \cdots \wedge e_{\gamma_a} \wedge f_{\gamma_1} \wedge \cdots \wedge f_{\gamma_b} \,|\, \gamma_1 < \cdots < \gamma_k \in \Delta^+\right\}.$$

In particular, it holds that

$$\dim\left(V^k\right) = \binom{|\Delta|}{k}, \qquad \textit{and} \qquad \dim\left(V^\bullet\right) = 2^{|\Delta|}.$$

Proof. It is clear from the set of relations given in Theorem 3.3 that the proposed basis is a spanning set. To prove that its elements are linearly independent, let $\langle\Lambda^1\rangle$ be the free monoid generated by elements of Λ^1 and let $S_{\Delta^+} := (W_{\Delta^+}, f_{\Delta^+})$ be the reduction system in the free algebra $\mathbb{C}\langle\Lambda^1\rangle$ corresponding to the set of relations 3.3, namely

$$\begin{aligned}
&(e_\gamma \otimes e_\beta,\ -q^{(\beta,\gamma)} e_\beta \otimes e_\gamma), \quad (f_\gamma \otimes f_\beta, -q^{-(\beta,\gamma)} f_\beta \otimes f_\gamma), \qquad \text{for all } \beta \leq \gamma \in \Delta^+,\\
&(e_\gamma \otimes f_\beta,\ -q^{(\beta,\gamma)} f_\beta \otimes e_\gamma) \text{ for all } \beta \neq \gamma \in \Delta^+, \text{ or for } \beta = \gamma = \alpha_1 + \alpha_2,\\
&(e_{\alpha_1} \otimes f_{\alpha_1},\ -q^2 f_{\alpha_1} \otimes e_{\alpha_1} - \nu f_{\alpha_1+\alpha_2} \otimes e_{\alpha_1+\alpha_2},)\\
&(e_{\alpha_2} \otimes f_{\alpha_2},\ -q^2 f_{\alpha_2} \otimes e_{\alpha_2} + \nu f_{\alpha_1+\alpha_2} \otimes e_{\alpha_1+\alpha_2}).
\end{aligned}$$

Let $\ll$ denote the total ordering such that for every $\beta, \gamma \in \Delta^+$

$$f_\beta \ll e_\gamma$$

and

$$\beta \leq \gamma \in \Delta^+ \Rightarrow e_\beta \ll e_\gamma, \quad f_\gamma \ll f_\beta.$$

Then S_{Δ^+} is a reduction system compatible with the ordering $\ll$ and it is easy to verify that it has no ambiguities, hence from Bergmann's diamond lemma [6] the set of algebra relations 3.3 is linearly independent and the spanning set given above is a basis. □

Remark 3.10. The form of the anti-holomorphic relations given above imply that the dc $\Omega_q^{(0,\bullet)}(\mathrm{F}_3)$ given in [40, §5] is in fact the maximal prolongation of the fodc $\Omega_q^{(0,1)}(\mathrm{F}_3)$.

Remark 3.11. Unlike the fodc $\Omega_q^{(0,1)}(SU_{n+1})$ considered in [40, §3], the maximal prolongation of $\Omega_q^1(\mathrm{F}_3)$ does not have classical dimension, and restricts to a dc over $\mathcal{O}_q(SU_3)$ of non-classical dimension. This is our motivation for calculating the maximal prolongation of $\Omega_q^1(\mathrm{F}_3)$ directly.

3.4. **Restriction to the Heckenberger–Kolb Calculus of the Quantum Projective Plane.** As explained in §2.6, we have two copies of the quantum projective plane $\mathcal{O}_q(\mathbb{CP}^2)$, arising as subalgebras of $\mathcal{O}_q(\mathrm{F}_3)$. Each subalgebra comes endowed with a left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant fodc of classical dimension $\Omega_q^1(\mathbb{CP}^2)$, known as the *Heckenberger–Kolb fodc* [23]. For the copy of the quantum projective plane generated by the elements $z_{ij}^{\alpha_1}$, the quantum tangent space $T \subseteq \mathcal{O}_q(\mathbb{CP}^2)^\circ$ of $\Omega_q^1(\mathbb{CP}^2)$ is given by

$$T := \{F_1,\, F_2F_1,\, E_1,\, E_2E_1\}.$$

The tangent space of the second copy of the quantum projective plane is given by

$$T := \{F_2,\, F_1F_2,\, E_2,\, E_1E_2\}.$$

The following lemma is an easy extension of [40, Proposition 4.2], for the rank 2 case, to include the holomorphic parts of the Heckenberger–Kolb dc, and so, we omit it.

Lemma 3.12. *The fodc $\Omega_q^1(\mathrm{F}_3)$ restricts to the Heckenberger–Kolb fodc for both copies of the quantum projective plane.*

Let A be an algebra endowed with a dc $\Omega^\bullet(A)$, and let $B \subset A$ be a subalgebra. The *restriction* $\Omega^\bullet(B)$ of $\Omega^\bullet(A)$ to B is the subalgebra of $\Omega^\bullet(A)$ generated by the elements b

and $\mathrm{d}b'$, for $b, b' \in B$. We note that $\Omega^\bullet(B)$ is of course a dc. We note that the first-order part $\Omega^1(B)$ of the restriction dc is, of course, equal to the restriction of $\Omega^1(A)$ to B.

Denoting by $(\mathrm{Max}(\Omega^1(B)), \mathrm{d}')$ the maximal prolongation of $\Omega^1(B)$ we have a surjection

$$p : \mathrm{Max}(\Omega^1(B)) \to \Omega^\bullet(B),$$

uniquely defined by

$$p(b_0 \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_k) = b_0 \mathrm{d}'b_1 \wedge \cdots \wedge \mathrm{d}'b_k.$$

It is clear that if p is an injective map, then we have an isomorphism of dc.

For the case that A is a Hopf algebra, B a quantum homogeneous A-space, and $\Omega^\bullet(A)$ a left A-covariant dc, then it is clear that p is a morphism in the category of relative Hopf modules. Thus we can see that p is injective if and only if $\Phi(p)$ is injective.

For the case of the quantum projective plane, we can calculate the dimension of the restricted dc using the same approach as used in [40, Proposition 4.8]. This shows us that the dimension is classical, just as is well-known for the Heckenberger–Kolb dc [24, §3.3]. Thus we see that $\Phi(p)$ is injective, giving us the following proposition.

Proposition 3.13. *The dc $\Omega_q(\mathrm{F}_3)$ restricts to the Heckenberger–Kolb dc for both copies of $\mathcal{O}_q(\mathbb{CP}^2)$.*

The question of how the covariant complex structures of the Heckenberger–Kolb dc relate to the possible covariant complex structures on $\Omega^1_q(\mathrm{F}_3)$ will be considered in §5.

3.5. **A Filtration for the Quantum Exterior Algebra.** Consider the following total order on the roots of $\mathfrak{sl}_3$:

$$\alpha_1 \ \geq\ -\alpha_1 \ \geq\ \alpha_2 \ \geq\ -\alpha_2 \ \geq\ \alpha_1+\alpha_2 \ \geq\ -(\alpha_1+\alpha_2). \tag{14}$$

Following the approach of [40, Appendix B], this gives us a filtration on $V^\bullet$. We denote the associated graded algebra by $\mathrm{gr}^{\mathscr{F}}$.

Proposition 3.14. *The algebra $\mathrm{gr}^{\mathscr{F}}$ is generated by the elements e_γ, f_γ, for $\gamma \in \Delta^+$, subject to the relations, for all $\beta \leq \gamma \in \Delta^+$,*

$$e_\gamma \wedge e_\beta = -q^{(\beta,\gamma)} e_\beta \wedge e_\gamma, \qquad f_\gamma \wedge f_\beta = -q^{(\beta,\gamma)} f_\beta \wedge f_\gamma, \tag{15}$$

$$f_\gamma \wedge e_\beta = -q^{(\beta,\gamma)} e_\beta \wedge f_\gamma. \tag{16}$$

Proof. It is clear from the relation set for $V^\bullet$ given above that these relations hold in the associated graded algebra. Moreover, since these relations imply an obvious spanning of dimension 2^6, we see that they must form a complete set of relations. □

Following the argument of [40, Proposition 3.18], we can now prove the following corollary.

Corollary 3.15. *The algebra $V^\bullet$ is a Frobenius algebra.*

Remark 3.16. We note that since the space V^{2^6} is a trivial $U_q(\mathfrak{h})$-module $V^\bullet$ is a Frobenius algebra object in the category of $U_q(\mathfrak{h})$-modules. This is in contrast to $V^{(0,\bullet)}$ which is a Frobenius algebra, but not in the category of $U_q(\mathfrak{h})$-modules, nor are the anti-holomorphic subcomplexes of the Heckenberger–Kolb calculi.

For a general Frobenius algebra A, there exists an algebra automorphism $\sigma : A \to A$ of A, uniquely defined by the identity $B(x,y) = B(y,\sigma(x))$, for all $x, y \in A$. We see that the bilinear form B of a Frobenius algebra is symmetric if and only if $\sigma = \mathrm{id}$. With a view to describing the Nakayama automorphism of our quantum exterior algebra, we first describe the Nakayama automorphism of $\mathrm{gr}^{\mathscr{F}}$.

Proposition 3.17. *The Nakayama automorphism of* $\mathrm{gr}^{\mathscr{F}}$ *is determined by*

$$\sigma(e_{\alpha_1}) = -q^2 e_{\alpha_1}, \qquad \sigma(e_{\alpha_2}) = -q^2 e_{\alpha_2}, \qquad \sigma(e_{\alpha_1+\alpha_2}) = -q^4 e_{e_{\alpha_1+\alpha_2}}, \tag{17}$$

$$\sigma(f_{\alpha_1}) = -q^{-2} f_{\alpha_1}, \qquad \sigma(f_{\alpha_2}) = -q^{-2} f_{\alpha_2}, \qquad \sigma(f_{\alpha_1+\alpha_2}) = -q^{-4} f_{\alpha_1+\alpha_2}. \tag{18}$$

Proof. In the following we let $\iota : \Lambda^n \to \Lambda^0$ denote the isomorphism of $\pi_B(A)-$comodules uniquely defined by

$$\iota(_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_1+\alpha_2} \wedge f_{\alpha_1} \wedge f_{\alpha_2} \wedge f_{\alpha_1+\alpha_2}) = 1.$$

Since σ is an algebra map, it is clearly determined by its action of the algebra generators. For the generator f_{α_1} we have

$$\begin{aligned}(e_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_1+\alpha_2} \wedge f_{\alpha_2} \wedge f_{\alpha_1+\alpha_2}, f_{\alpha_1}) &= \iota[e_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_1+\alpha_2} \wedge f_{\alpha_2} \wedge f_{\alpha_1+\alpha_2} \wedge f_{\alpha_1}] \\ &= \iota[e_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_1+\alpha_2} \wedge f_{\alpha_1} \wedge f_{\alpha_2} \wedge f_{\alpha_1+\alpha_2}] \\ &= 1.\end{aligned}$$

On the other hand, it holds that

$$\begin{aligned}(f_{\alpha_1}, e_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_1+\alpha_2} \wedge f_{\alpha_2} \wedge f_{\alpha_1+\alpha_2}) &= \iota[f_{\alpha_1} \wedge e_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_1+\alpha_2} \wedge f_{\alpha_2} \wedge f_{\alpha_1+\alpha_2}] \\ &= -q^{-2}\iota[e_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_1+\alpha_2} \wedge f_{\alpha_1} \wedge f_{\alpha_2} \wedge f_{\alpha_1+\alpha_2}] \\ &= -q^{-2}.\end{aligned}$$

Thus we see that $\sigma(f_{\alpha_1}) = q^{-2} f_{\alpha_1}$. The action of σ on the other generators is calculated similarly.

□

In the following corollary we use our description of the Nakayama automorphism of $\mathrm{gr}^{\mathscr{F}}$ to produce an analogous description of the Nakayama automorphism of $\Lambda_q^{(0,\bullet)}$. The proof is completely analogous to the proof of [40, Corollary 3.21].

Corollary 3.18. *The Nakayma automorphism of* $\Lambda_q^{(0,\bullet)}$ *acts on the generators* e_γ, *and* f_γ, *for* $\gamma \in \Delta$, *just as in* (17) *and* (18).

We finish by observing that $\Lambda_q^{(0,\bullet)}$ is also a Koszul algebra. Recall that a Koszul algebra is a $\mathbb{Z}_{\geq 0}$-graded algebra admitting a linear minimal graded free resolution. We refer the reader to the standard text [41] for more details on Koszul algebras.

Proposition 3.19. *The algebra* $\Lambda_q^{(0,\bullet)}$ *is a Koszul algebra.*

Proof. The algebra $\Lambda_q^{(0,\bullet)}$ is clearly a PBW-algebra in the sense of Priddy [41, §4.1]. Thus it follows from Priddy's theorem [41, Theorem 3.1] that it is a Koszul algebra. □

4. Connections and Torsion

In this section we make some general observations about torsion for connections, and prove the existence of a covariant torsion-free connection for a dc over a quantum homogeneous space with cosemisimple quantum isotropy subgroup. Moreover, we classify covariant torsion free connections for such spaces. These general results are then applied to the dc $\Omega_q^\bullet(\mathrm{F}_3)$. One major difference with the irreducible quantum flag manifold case is that $\Omega_q^\bullet(\mathrm{F}_3)$ admits covariant connections with torsion.

Let us first recall the definition of a connection and its associated torsion operator. Let $\Omega^\bullet(B)$ be a differential calculus over an algebra B and $\mathcal{F}$ a left B-module, a *connection* on $\mathcal{F}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{F} \to \Omega^1(B) \otimes_B \mathcal{F}$ satisfying the identity

$$\nabla(bf) = \mathrm{d}b \otimes f + b\nabla f, \qquad \text{for all } b \in B,\ f \in \mathcal{F}.$$

An immediate but important consequence of the definition is that the difference of two connections $\nabla - \nabla'$ is a left B-module map.

Let $\nabla : \Omega^1(B) \to \Omega^1(B) \otimes_B \Omega^1(B)$ be a connection for $\Omega^1(B)$. The *torsion* of ∇ is the left B-module map

$$T_\nabla := \wedge \circ \nabla - \mathrm{d} : \Omega^1(B) \to \Omega^2(B).$$

We note that if $B \subseteq A$ is a quantum homogeneous space, $\Omega^1(B)$ is a left A-covariant dc over B, and ∇ is a left A-comodule map, then T_∇ is a morphism in the category ${}^A_B\mathrm{Mod}$.

We note that since the difference of two left A-covariant connections is a morphism of relative Hopf modules, the set of left A-covariant connections is an affine space for the vector space of morphisms from $\Omega^1(B)$ to $\Omega^1(B) \otimes_B \Omega^1(B)$. Moreover, for any two torsion free connections ∇ and ∇', it holds that

$$(\nabla - \nabla')(\omega) \in \ker(\wedge), \qquad \text{for } \omega \in \Omega^1(A).$$

This implies that the set of left A-covariant connections is an affine space for the vector space of morphisms from $\Omega^1(B)$ to the kernel

$$\ker\Big(\wedge : \Omega^1(B) \otimes_B \Omega^1(B) \to \Omega^1(B)\Big).$$

Moreover, since we are dealing with a quantum homogeneous space, and hence a principal comodule algebra, $\Omega^1_u(B)$ admits a left A-covariant connection [8, §3.4], which we can then quotient to a covariant connection for any covariant differential calculus [19, Corollary 3.2]. Moreover, we recall from [20, §4.5] that if the space of $\pi_B(A)$-comodule maps from $V^{(0,1)}$ to $V^{(0,2)}$ is trivial, then this connection is necessarily torsion free. The following proposition is a variation on this result for the cosemisimple case.

Proposition 4.1. *Let $B \subseteq A$ be a quantum homogeneous space such that $\pi_B(A)$ is a cosemisimple Hopf algebra, and let $\Omega^\bullet(B)$ be a left A-covariant dc over B. Then $\Omega^1(B)$ admits a left A-covariant torsion-free connection.*

Proof. Let ∇ be a left A-covariant connection, which as discussed above, is guaranteed to exist. If it is torsion-free, then we are done. So let us assume that ∇ has non-zero torsion $\mathrm{Tor}(\nabla)$. Since we are assuming that $\pi_B(A)$ is a cosemisimple Hopf algebra, we can choose a left $\pi_B(A)$-comodule splitting of the surjection $\wedge : V^1 \otimes V^1 \to V^2$, and hence a splitting i, in the category ${}^A_B\mathrm{Mod}$, of the surjection

$$\wedge : \Omega^1(B) \otimes_B \Omega^1(B) \to \Omega^1(B)$$

Consider next the left A-comodule map

$$\nabla' := \nabla - i \circ \mathrm{Tor}(\nabla) : \Omega^1(B) \to \Omega^1(B) \otimes_B \Omega^1(B).$$

Now, for $\omega \in \Omega^1(B)$, and $b \in B$, we have

$$\begin{aligned} \nabla'(b\omega) &= \nabla(b\omega) - i \circ \mathrm{Tor}(b\nabla)(\omega) \\ &= \mathrm{d}b \otimes \omega + b\nabla(\omega) - b(i \circ \mathrm{Tor}(\nabla)(\omega)) \\ &= \mathrm{d}b \otimes \omega + b\big(\nabla(\omega) - i \circ \mathrm{Tor}(\nabla)\big)(\omega) \\ &= \mathrm{d}b \otimes \omega + b\nabla'(\omega). \end{aligned}$$

Thus we see that ∇' is a connection. Next we note that

$$\begin{aligned} \mathrm{Tor}(\nabla') &= \wedge \circ \nabla' - \mathrm{d} \\ &= \wedge \circ (\nabla - \iota \circ \mathrm{Tor}(\nabla)) - \mathrm{d} \\ &= \wedge \circ \nabla - \mathrm{d} - \mathrm{Tor}(\nabla) \\ &= \mathrm{Tor}(\nabla) - \mathrm{Tor}(\nabla) \\ &= 0. \end{aligned}$$

Thus we see that ∇' is torsion-free, and hence that a left A-covariant torsion-free connection always exists. □

A simple but useful observation is that if $\Omega^1(B)$ admits a unique left A-covariant connection, then it must be the same as the torsion-free connection just constructed. This gives us the following corollary.

Corollary 4.2. *Let $B \subseteq A$ and $\Omega^\bullet(B)$ be as above. If $\Omega^1(B)$ admits a unique left A-covariant connection, then this connection is necessarily torsion-free.*

Example 4.3. As an application of the above corollary, consider the Heckenberger–Kolb differential calculi for the irreducible quantum flag manifolds, a special subclass of the quantum flag manifolds, itself a family of quantum homogeneous spaces, containing $\mathcal{O}_q(\mathrm{F}_3)$ and $\mathcal{O}_q(\mathbb{CP}^2)$. These covariant differential calculi, extending the Podleś calculus disscused in the introduction. In [19] their 1-forms were shown to possess a unique left $\mathcal{O}_q(G)$-covariant connection, and moreover, this connection was shown to be torsion-free using a representation theoretic argument. We now see that vanishing of the torsion follows directly from Corollary 4.2.

We now apply these general results to the special case of $\Omega^1(\mathrm{F}_3)$. First we calculate the dimension of the affine space of torsion free connections. In particular, we see that in general, torsion free connections are not unique.

Corollary 4.4. *For the full quantum flag manifold $\mathcal{O}_q(\mathrm{F}_3)$, endowed with the dc $\Omega^\bullet_q(\mathrm{F}_3)$, the affine space of connections has dimension 12. The dimension of the affine space of torsion-free connections 6.*

Proof. The dimension of the space of $U_q(\mathfrak{h})$-module maps from V^1 to $V^1 \otimes V^1$ can be calculated by noting the multiplicity of the weight spaces of $V^1 \otimes V^1$ of weight $\pm\alpha_1, \pm\alpha_2$, and $\pm(\alpha_1 + \alpha_2)$. Looking at Table 1 in Appendix A, we see that each weight has multiplicity 2. Hence the dimension of the affine space of connections is 12. A similar argument confirms that the dimension of the affine space of torsion-free connections is 6. □

5. Almost-Complex Structures for the Lusztig–de Rham Complex

In this section we examine covariant complex and almost complex structures for the dc $\Omega^1_q(\mathrm{F}_3)$. We observe that the number of almost-complex structures decreases from 8 (which is 2 to the number of positive roots of $\mathfrak{sl}_3$) to 4 (which is 2 to the number of simple roots of $\mathfrak{sl}_3$). Furthermore, we demonstrate that all of these almost-complex structures are integrable, which is to say, they are both complex structures.

5.1. **Preliminaries on Complex and Almost-Complex Structures.** In this subsection, we briefly recall some preliminaries about almost-complex and complex structures. See [4, §1] or [26, 5, 38] for a more detailed discussion of complex structures.

An *almost complex structure* $\Omega^{(\bullet,\bullet)}$, for a $*$dc $(\Omega^\bullet, \mathrm{d})$, is an $\mathbb{Z}^2_{\geq 0}$-algebra grading of $\Omega^\bullet$ such that

$$\Omega^k = \bigoplus_{a+b=k} \Omega^{(a,b)}, \qquad \left(\Omega^{(a,b)}\right)^* = \Omega^{(b,a)}, \qquad \text{for all } (a,b) \in \mathbb{Z}^2_{\geq 0}.$$

If the exterior derivative decomposes into a sum $\mathrm{d} = \partial + \overline{\partial}$, for ∂ a (necessarily unique) degree $(1,0)$-map, and $\overline{\partial}$ a (necessarily unique) degree $(0,1)$-map, then we say that $\Omega^{(\bullet,\bullet)}$ is a *complex structure.* It follows that we have a double complex. The *opposite* complex structure of a complex structure $\Omega^{(\bullet,\bullet)}$ is the $\mathbb{Z}^2_{\geq 0}$-algebra grading $\overline{\Omega}^{(\bullet,\bullet)}$, defined by $\overline{\Omega}^{(a,b)} := \Omega^{(b,a)}$, for $(a,b) \in \mathbb{Z}^2_{\geq 0}$.

Finally, we restrict to the case of a covariant dc $\Omega^\bullet$ over a quantum homogeneous space $B \subseteq A$. In this case, a complex structure $\Omega^{(\bullet,\bullet)}$ for $\Omega^\bullet$ is said to be *covariant* if the $\mathbb{Z}^2_0$-decomposition of the dc is a decomposition in the category of two-sided relative Hopf modules ${}^A_B\mathrm{Mod}_B$.

5.2. **Almost-Complex Structures for the Classical Full Flag Manifold F_3.** In this subsection we briefly recall the covariant almost complex structures for the classical flag manifold F_3. We do so to highlight the novel non-classical behavior occurring for the quantum case. We refer the interested reader to [2] or [3] for more further details.

A choice of almost-complex structure for the manifold F_3 corresponds to assigning to each positive root of the root system Δ of $\mathfrak{sl}_3$ the label of holomorphic or anti-holomorphic. We see that there exist eight such labellings, meaning that up to identification of opposite almost-complex structures, we have four.

A labelling corresponds to a complex structure if and only if it gives a choice of positive roots for Δ, or equivalently a choice of base for the root system. We see that three of our almost-complex structures are integrable and one is not. The Weyl group S_3 of $\mathfrak{sl}_3$ acts

$T^{(0,1)}$	Integrable	Metric
$\Big\{E_1,\, E_2,\, [E_1, E_2]\Big\}$	✓	Kähler–Einstein
$\Big\{F_1,\, E_2,\, [E_1, E_2]\Big\},$	✓	Kähler–Einstein
$\Big\{E_1,\, F_2,\, [E_1, E_2]\Big\}$	✓	Kähler–Einstein
$\Big\{E_1,\, E_2,\, [F_1, F_2]\Big\}$	✗	Nearly Kähler

transitively on the set of bases for Δ, and hence on the set of covariant almost-complex structures.

Each almost-complex structure comes with a distinguished homogeneous Riemannian metric. In the integrable case these metrics are permuted by the Weyl group and all three are Kähler–Einstein. For the non-integrable case there is a choice of nearly-Kähler metric, recalling the special role 6-dimensional manifolds play in the theory of nearly-Kähler geometry [25]. We collect these recollections below in the form of a table.

5.3. **Complex Structures for Full Quantum Flag Manifolds.** In this subsection we classify the covariant complex structures on the dc $\Omega^1_q(\mathrm{F}_3)$. We find that two of the classical almost structures fail to extend to the quantum setting. In particular, one of the bases of the root system of $\mathfrak{sl}_3$ fails to have a corresponding foacs in the quantum setting. This breaks the classical Weyl group symmetry of the almost-complex structures on F_3.

As usual in the theory of differential calculi, we find it convenient to initially work at the level of fodc and then discuss the extension to higher forms. This motivates the following general definition.

Definition 5.1. A *first-order complex structure*, or *foacs*, for a $*$-fodc $\Omega^1(B)$ over an algebra B is a direct sum decomposition of B-bimodules

$$\Omega^1(B) \simeq \Omega^{(1,0)} \oplus \Omega^{(0,1)} \tag{19}$$

such that $(\Omega^{(1,0)})^* = \Omega^{(0,1)}$ or equivalently $(\Omega^{(0,1)})^* = \Omega^{(1,0)}$.

Just as for a complex structure, we have the corresponding notions of *opposite foacs* and *covariant foacs* for a covariant fodc over a quantum homogeneous space. Moreover, we note that any covariant dc over a quantum homogeneous space $B \subseteq A$, a covariant foacs implies a corresponding decomposition of the cotangent space V^1, in the category ${}^{\pi_B}\mathrm{Mod}_B$ of the dc over B.

Let us now recall a formula detailing the interaction of the dc $*$-map of a dc over a Hopf algebra A with the fundamental theorem of Hopf modules. Consider the commutative diagram

$$\begin{array}{ccc} \Omega^1(A) & \xrightarrow{\quad \mathrm{U} \quad} & A \otimes \Lambda^1 \\ {\scriptstyle *}\uparrow & & \uparrow {\scriptstyle \mathrm{U} \circ * \circ \mathrm{U}^{-1}} \\ \Omega^1(A) & \xleftarrow[\quad \mathrm{U}^{-1} \quad]{} & A \otimes \Lambda^1. \end{array} \tag{20}$$

As is easily shown (see [38, §2.6]) the map $\mathrm{U} \circ * \circ \mathrm{U}^{-1}$, which by abuse of notation we denote by $*$, acts explicitly as

$$*(a \otimes v) = -a^*_{(2)} \otimes v^* a^*_{(1)}, \qquad \text{for } a \otimes v \in A \otimes \Lambda^1, \tag{21}$$

where the star map $* : \Lambda^1 \to \Lambda^1$ is defined by $[a]^* = [S(a)^*]$.

Lemma 5.2. *Let $B \subseteq A$ be a quantum homogeneous space, and $\Omega^1(A)$ a left A-covariant, right $\pi_B(A)$-covariant dc for A that frames $\Omega^1(B)$, its restriction to a fodc on B. Consider a decomposition of V^1 the cotangent space of $\Omega^1(B)$*

$$V^1 \simeq V^{(1,0)} \oplus V^{(0,1)} \in {}^{\pi_B}\mathrm{Mod}_B \tag{22}$$

which is moreover, a decomposition of right A-modules, with respect to the embedding of V^1 in Λ^1 the tangent space of $\Omega^1(A)$. Then this decomposition comes from a covariant foacs on $\Omega^1(B)$ if and only if $V^{(1,0)}$ and $V^{(0,1)}$ are interchanged by the $*$-map of Λ^1.

Proof. Note first that right $\pi_B(A)$-covariance of $\Omega^1(A)$ means that we have we have a quantum principal bundle. This allows us to invoke Majid's framing theorem [30, Theorem 2.1], from which we can conclude that the natural map from V^1 to Λ^1 is actually an embedding.

Consider the A-subbimodule of $\Omega^1(A)$ given by

$$A\Omega^1(B) \simeq A \otimes V^1 \simeq A\Omega^1(B)A,$$

where the second isomorphism follows from the fact that V^1 is a right A-module, since we have assumed that $\Omega^1(A)$ is a framing calculus for $\Omega^1(B)$. The decomposition of V^1 gives us the decomposition

$$A\Omega^1(B) \simeq (A \otimes V^{(1,0)}) \oplus (A \otimes V^{(0,1)}).$$

This is again a decomposition of A-bimodules, since by assumption the decomposition of V^1 is a decomposition of right A-modules. Since we are supposing that $*$ maps $V^{(1,0)}$ to $V^{(0,1)}$, and both subspaces are right A-modules, it follows from (21) that the $*$ map interchanges $A \otimes V^{(1,0)}$ and $A \otimes V^{(0,1)}$. This of course implies that $*$ interchanges $A \Box_\pi V^{(1,0)}$ and $A \Box_\pi V^{(0,1)}$, and hence that we have a foacs.

In the other direction, let us assume that the decomposition of V^1 comes from an foacs. Taking an arbitrary element $a(\partial b)a'$ in $A\Omega^{(1,0)}$ we see that

$$(a\partial b)^* = (\overline{\partial} b^*)a^* \in A\Omega^{(0,1)}.$$

Thus the $*$-map of $\Omega^1(A)$ interchanges $A\Omega^{(1,0)}$ and $A\Omega^{(0,1)}$. It now follows from (21) that $*$ maps $V^{(1,0)}$ to $V^{(1,0)}$, giving the opposite implication. □

Proposition 5.3. *The fodc $\Omega^1_q(\mathrm{F}_3)$ admits, up to identification of opposite structures, two covariant foacs. Explicitly, one decomposition of V^1 is given by*

$$V^{(1,0)} = \mathrm{span}_{\mathbb{C}}\Big\{e_{\alpha_1},\, e_{\alpha_2},\, e_{\alpha_1+\alpha_2}\Big\}, \qquad V^{(0,1)} := \mathrm{span}_{\mathbb{C}}\Big\{f_{\alpha_1},\, f_{\alpha_2},\, f_{\alpha_1+\alpha_2}\Big\},$$

and the other is given by

$$V^{(1,0)} = \mathrm{span}_{\mathbb{C}}\Big\{f_{\alpha_1},\, e_{\alpha_2},\, e_{\alpha_1+\alpha_2}\Big\}, \qquad V^{(0,1)} := \mathrm{span}_{\mathbb{C}}\Big\{e_{\alpha_1},\, f_{\alpha_2},\, f_{\alpha_1+\alpha_2}\Big\},$$

Proof. Consider a general left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant foacs on $\Omega^1_q(\mathrm{F}_3)$, and denote by

$$V^1 \simeq V^{(1,0)} \oplus V^{(0,1)}.$$

the corresponding decomposition of the cotangent space V^1 into two left $\mathcal{O}(\mathbb{T}^2)$-comodule right $\mathcal{O}_q(\mathrm{F}_3)$-modules. Since the basis elements all have mutually distinct weights, we see that each basis element is contained in either $V^{(1,0)}$ or $V^{(0,1)}$. The right $\mathcal{O}_q(\mathrm{F}_3)$-module requirement, together with Lemma 3.1, implies that if e_{α_1} is contained in $V^{(1,0)}$, then $e_{\alpha_1+\alpha_2}$ is also contained in $V^{(1,0)}$, and analogously, if f_{α_1} is contained in $V^{(0,1)}$, then $f_{\alpha+\alpha_2}$ is contained in $V^{(0,1)}$. In other words, any complex structure is determined by knowing whether the basis elements e_α, f_α, for $\alpha \in \Pi$, are contained in $V^{(1,0)}$ or $V^{(0,1)}$. We now note that any such $\mathcal{O}_q(\mathrm{F}_3)$-decomposition of V^1 will necessarily be a decomposition of right $\mathcal{O}_q(\mathrm{F}_3)$-modules. This allows us to appeal to Lemma 5.2. Considering V^1 as a subspace of Λ^1, the cotangent space of the fodc $\Omega^1_q(\mathrm{SU}_3)$, and recalling that $e^*_\gamma = f_\gamma$, for all $\gamma \in \Delta^+$, we now see that the only possible decompositions are those two decompositions given in the statement of the proposition. □

Given a foacs on a fodc, there is at most one extension to an almost complex structure on its maximal prolongation, or indeed any quotient thereof (see [38, Proposition 6.1] for details). The following proposition tells that both our foacs extend.

Corollary 5.4. *Both foacs on $\Omega^1_q(\mathrm{F}_3)$ extend to a factorisable almost complex structure on $\Omega^\bullet_q(\mathrm{F}_3)$.*

Proof. The fact that both first-order structures extend to covariant almost-complex structures, follows directly from the explicit form of the relations given in Theorem 3.3 and [38, Theorem 6.4]. Moreover, factorisability follows from the explicit form of the relations and [38, Corollarly 6.8]. □

5.4. **Integrability for the Full Quantum Flag Almost-Complex Structures.** As shown in [38, Lemma 7.2], an almost-complex structure $\Omega^{(\bullet,\bullet)}$ on a dc $\Omega^\bullet$ is integrable if and only if the maximal prolongation of the fodc $\Omega^{(0,1)}$ is isomorphic to the subalgebra $\Omega^{(0,\bullet)}$. Using this reformulation of integrability, we now observe that, just as in the classical case, both the covariant almost-complex structures on $\Omega^\bullet_q(\mathrm{F}_3)$ are integrable. Interestingly, this means that $\Omega^\bullet_q(\mathrm{F}_3)$ does not admit a non-integrable covariant almost-complex structure.

Proposition 5.5. *Both covariant almost-complex structures of the dc $\Omega^\bullet_q(\mathrm{F}_3)$ are integrable.*

Proof. We will treat the case of the almost-complex structure

$$V^{(0,1)} = \{e_{\alpha_1},\, e_{\alpha_2},\, e_{\alpha_1+\alpha_2}\},$$

the other case being entirely analogous. We need to calculate the dimension of the maximal prolongation of the associated fodc $\Omega^{(0,1)}$. We note that $\Omega_q^{(0,1)}(\mathrm{SU}_3)$ is a framing calculus for $\Omega_q^{(0,1)}(\mathrm{F}_3)$, allowing us to use the appraoch of §2.3 to calculate the degree two relations of the maximal prolongation of $\Omega_q^{(0,1)}(\mathrm{F}_3)$.

We see that the ideal $I' \subseteq \mathrm{O}_q(\mathrm{F}_3)^+$ corresponding the to the $\Omega_q^{(0,1)}(\mathrm{F}_3)$ contains the elements

$$I \cup \{z_{12}^{\alpha_1},\, z_{23}^{\alpha_2},\, z_{13}^{\alpha_1}\}.$$

Moreover, since the quotient of $\mathrm{O}_q(\mathrm{F}_3)^+$ by I' is three dimensional, we see that this is in fact the whole ideal.

Operating on the elements of I by ω we clearly reproduce the degree-$(0,2)$ elements from those given in Theorem 3.8. For the element $z_{23}^{\alpha_2}$, recalling [40, Lemma 3.8] we see that

$$\omega(z_{23}^{\alpha_2}) = \omega(u_{23}S(u_{33})) = \sum_{a=1}^{3}[(u_{23}S(u_{b3})] \otimes [S(u_{3b})^+] + \sum_{a=1}^{3}[S(u_{b3})^+] \otimes [(u_{23}^+S(u_{3b})]$$
$$+ \sum_{a=1}^{3}[(u_{2a}^+S(u_{b3})] \otimes [u_{a3}^+S(u_{3b})].$$

Since each of the elements

$$u_{23},\, u_{13},\, u_{22}^+,\, u_{23}$$

pair trivially with each element of $T^{(0,1)}$, we now see that $\omega(z_{23}^{\alpha_2}) = 0$. Analogous calculations establish that

$$\omega(z_{23}^{\alpha_2}) = \omega(z_{13}^{\alpha_1}) = 0.$$

Thus we see that the maximal prolongation of $\Omega_q^{(0,1)}(\mathrm{F}_3)$ is isomorphic to the subalgebra $\Omega_q^{(0,\bullet)}(\mathrm{F}_3)$, and so, the almost-complex structure is integrable. □

5.5. **Restriction of the Almost Complex Structures.** Throughout this subsection, P will denote a $*$-algebra and B a $*$-subalgebra. We note that, for $\Omega^\bullet(P)$ a $*$-dc over P, the restriction to a dc on B is again a $*$-dc calculus.

Proposition 5.6. *Let $\Omega^\bullet(P)$ be a $*$-dc over P, and let $\Omega^{(\bullet,\bullet)}(P)$ an almost complex structure for $\Omega^\bullet(P)$. Denote by $\Omega^\bullet(B)$ the restriction of $\Omega^\bullet(P)$ to a $*$-dc on B. Then an almost complex structure on $\Omega^\bullet(B)$ is given by $\Omega^{(\bullet,\bullet)}(B)$, where*

$$\Omega^{(a,b)}(B) := \Omega^{(a,b)}(P) \cap \Omega^{a+b}(B)$$

if and only if the following three equivalent conditions hold

1. $\partial b \in \Omega^1(B)$,
2. $\overline{\partial} b \in \Omega^1(B)$,
3. $\Omega^1(B)$ *is homogeneous with respect to the decomposition* $\Omega^1(P) \simeq \Omega^{(1,0)} \oplus \Omega^{(1,0)}$.

Proof. Since $\Omega^\bullet(B)$ is a $*$-subspace of $\Omega^\bullet(P)$, we see that

$$(\Omega^{(a,b)}(B))^* = (\Omega^{(a,b)}(P) \cap (\Omega^{a+b}(B)))^* = (\Omega^{(b,a)}(P) \cap \Omega^{a+b}(B)) \subseteq \Omega^{(b,a)}(B).$$

The fact that $*$ is an involution implies that $(\Omega^{(a,b)}(B))^* = \Omega^{(b,a)}(B)$.
Thus it remains to show that homogeneity of $\Omega^\bullet(B)$ with respect to the $\mathbb{Z}^2_{\geq 0}$-grading $\Omega^{(\bullet,\bullet)}(B)$. one direction is obvious, so let us assume homogeneity of $\Omega^1(B)$ with respect to the decomposition $\Omega^1(P) \simeq \Omega^{(1,0)}(P) \oplus \Omega^{(0,1)}(P)$. Now every form in $\Omega^k(B)$ is a linear combination of elements of the form

$$b_0 \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_k = b_0(\partial b_1 + \overline{\partial} b_1) \wedge \cdots \wedge (\partial b_k + \overline{\partial} b_k).$$

Hence each $\omega \in \Omega^k(B)$ is a linear combination of 1-forms of the form $b\partial b'$ or $c\overline{\partial} c'$, for $b, b', c, c' \in B$. Now each product is homogeneous with respect to the $\mathbb{Z}^2_{\geq 0}$-grading. Moreover, since ∂b and ∂c are in $\Omega^1(B)$ by assumption, these products are actually contained in $\Omega^1(B)$. Thus we see that, as required, $\Omega^\bullet(B)$ is a homogeneous subspace with respect to the $\mathbb{Z}^2_{\geq 0}$-grading.

Finally, we see that since $\Omega^{(\bullet,\bullet)}(P)$ is an almost-complex structure, ∂b is contained in $\Omega^1(B)$ if and only if $\overline{\partial} b^*$ is contained in $\Omega^1(B)$, which is of course equivalent to $\Omega^1(B)$ being homogeneous with respect to the decomposition of $\Omega^1(P)$. □

The proof of the following corollary, discussing the relationship of integrability and restriction, is clear, and so, we omit it.

Corollary 5.7. *If $\Omega^{(\bullet,\bullet)}(P)$ is an integrable complex structure that restricts to an almost complex structure $\Omega^{(\bullet,\bullet)}(B)$ on B, then $\Omega^{(\bullet,\bullet)}(P)$ is also integrable.*

5.6. **Restriction of Covariant Almost complex Structures.** In this subsection we deal with the restriction of covariant almost-complex structures for nested of pairs of quantum homogeneous spaces. Throughout A will denote a Hopf algebra, and $P \subseteq A$ and $B \subseteq A$ a pair of quantum homogeneous A-spaces, such that $B \subseteq P$, that is to say a *nested pair* of quantum homogeneous spaces [13]. Moreover, let $\Omega^\bullet(P)$ be a left A-covariant dc over P and $\Omega^\bullet(B)$ the restriction to a dc over B. Since we have two quantum homogeneous spaces, we have two versions of Takeuchi's equivalence. We denote the functors of the two equivalences by Φ_P and Ψ_P for P, and by Φ_P and Ψ_P for B. Moreover, we denote $V^1_P := \Phi_P(\Omega^1(P))$ and $V^1_B := \Phi_B(\Omega^1(B))$.

Proposition 5.8. *Assume that the embedding*

$$\iota : V_B \mapsto V_P, \qquad\qquad [\mathrm{d}b] \mapsto [\mathrm{d}b]$$

is injective, and identify V^1_B with its image. Then any left A-covariant almost-complex structure on $\Omega^\bullet(P)$ descends to a complex structure on $\Omega^\bullet(B)$ if and only if $V^1_B = V^{(1,0)} \oplus V^{(0,1)}$, and either, or equivalently both, of the subspaces

$$V^{(1,0)}_B := V^1_B \cap V^{(1,0)}_P, \qquad\qquad V^{(0,1)}_B := V^1_B \cap V^{(0,1)}_P,$$

are $\pi_B(A)$-subcomodules of V^1_B.

Proof. Let us denote

$$\mathrm{d}b = \sum_i a_i \otimes v_i \in A \square_{\pi_B} V^1_B.$$

Assume that V^1_B is homogeneous with respect to the decomposition $V^1_P \simeq V^{(1,0)}_P \oplus V^{(0,1)}_P$. Denoting the corresponding decomposition of each element v_i by $v_i = v_i^+ + v_i^-$, we see that

$$\mathrm{d}b = \sum_i a_i \otimes v_i^+ + \sum_i a_i \otimes v_i^1 - .$$

In particular, we see that

$$\partial b = \sum_i a_i \otimes v_i^+ \in A \otimes V^{(1,0)}.$$

Now if $V^{(1,0)}_B$ and $V^{(1,0)}_B$ are left $\pi_B(A)$-comodules, then the fact that $\mathrm{d}b$ is an element of $A\square_{\pi_B} V^1_B$ implies that ∂b is an element of $A\square_{\pi_B} V^1_B$, which is to say, ∂b is an element of $\Omega^1(B)$, which is to say the complex structure is an almost complex structure.

In the other direction, assume that the almost complex structure on $\Omega^1(P)$ restricts to a complex structure on $\Omega^1(B)$. In particular, assume that $\overline{\partial} b$ is an element of $\Omega^1(B)$. Then since $[\mathrm{d}b] = [\partial b] + [\overline{\partial} b]$, and $[\partial b], [\overline{\partial} b] \in \iota(V^1_B)$, we see that $V^1_B = V^{(1,0)} \oplus V^{(0,1)}$. Looking next at the element $[\overline{\partial}] \in V^{(1,0)}$, we see that

$$\Delta_L([\partial b]) = \pi_B(b_{(1)}) \otimes [\partial b_{(2)}],$$

which is to say, $V^{(1,0)}$ is a left $\pi_B(A)$-comodule. The proof that $V^{(0,1)}$ is a left $\pi_B(A)$-comodule is analogous. Thus we have established the opposite implication. □

The following corollary is a simple dualisation of this result to the tangent space setting, under the assumption that B is a quantum homogeneous space of the form ${}^W A$, for $W \subseteq A^\circ$, as discussed in §2.3. Note that a covariant almost complex structure $\Omega^{(\bullet,\bullet)}(P)$ on $\Omega^\bullet(P)$ induces a direct sum decomposition of its corresponding the tangent space $T \simeq T^{(1,0)} \oplus T^{(0,1)}$, where $T^{(1,0)}$ is the subspace of elements of T that vanish on $V^{(1,0)}$, and $T^{(0,1)}$ is the subspace of elements of T that vanish on $V^{(0,1)}$.

Corollary 5.9. *The almost complex structure $\Omega^{(\bullet,\bullet)}$ restricts to an almost complex structure on $\Omega^\bullet(B)$ if*

$$WT^{(1,0)}|_B = T^{(1,0)}|_B, \qquad \text{and} \qquad WT^{(0,1)}|_B = T^{(0,1)}|_B. \tag{23}$$

Proof. Note first that if (23) holds then $WT|_B = T|_B$, and hence the map ι is an injection (see [40, §4.2] for a discussion of this). The equivalence of the requirements of (23) and those given in Proposition 5.8 now follows from a routine dualisation argument. □

5.7. **Restriction of the Complex Structures to $\mathcal{O}_q(\mathbb{CP}^2)$.** In this subsection we address the question of the restriction of the complex structures on $\Omega^\bullet_q(\mathrm{F}_3)$ to the dc $\Omega^\bullet_q(\mathbb{CP}^2)$. We see that just as in the classical case, sometimes a complex structure restricts, while other times it does not. In particular, we see that the unique left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant complex structures on the Heckenberger–Kolb conplex dc can be realised as the restriction of a complex structure on $\Omega^\bullet_q(\mathrm{F}_3)$.

Proposition 5.10. *It holds that*

1. *the complex structures* $V_I^{(\bullet,\bullet)}$ *and* $V_{II}^{(\bullet,\bullet)}$ *restrict to a left* $\mathcal{O}_q(\mathrm{SU}_3)$*-covariant complex structure on* $\Omega_q^\bullet(\mathbb{CP}^2_{\alpha_1})$*,*
2. *the complex structure* $V_I^{(\bullet,\bullet)}$ *restricts to a a left* $\mathcal{O}_q(\mathrm{SU}_3)$*-covariant complex structure on* $\Omega_q^\bullet(\mathbb{CP}^2_{\alpha_2})$*, while* $V_{II}^{(\bullet,\bullet)}$ *does not restrict.*

Proof. By Corollary 5.9 we simply need to check that $T^{(1,0)}$ and $T^{(0,1)}$ are $U_q(\mathfrak{l}_S)$-modules. For example, for the complex structure $V_{II}^{(\bullet,\bullet)}$,

$$F_1 \triangleright F_2 = F_1F_2 \notin T^{(0,1)},$$

and so, the complex structure does not restrict to a complex structure on $\Omega_q^\bullet(\mathbb{CP}^2_{\alpha_2})$. □

Remark 5.11. The asymmetry between the case of $\Omega_q^\bullet(\mathbb{CP}^2_{\alpha_1})$ and $\Omega_q^\bullet(\mathbb{CP}^2_{\alpha_2})$ can be understood as follows: In the classical case, each copy of the complex projective plane has two complex structure on F_3) that restrict to its covariant complex structure. However, in the noncommutative setting, we have fewer complex structures, meaning we have only one *lift* of the complex structure on $\Omega_q^\bullet(\mathbb{CP}^2_{\alpha_2})$ to a complex structure on $\Omega_q^\bullet(\mathrm{F}_3)$. However, we we instead look at the case of the Lusztig dc on $\mathcal{O}_q(\mathrm{SU}_3)$ associated to the reduced decomposition of the longest element of the Weyl group $w_0 = s_1s_1s_1$, then this situation is reversed, with $\Omega_q^{(\bullet,\bullet)}(\mathbb{CP}^2_{\alpha_2})$ having two lifts and $\Omega_q^{(\bullet,\bullet)}(\mathbb{CP}^2_{\alpha_1})$ having only one. Thus the symmetry is preserved by considering the alternative reduced decomposition.

5.8. **Some Remarks about the Higher Rank Full Quantum Flag Manifolds.** In this subsection, which is in effect an extended remark, we discuss the extension of our results for $\mathcal{O}_q(\mathrm{F}_3)$ to the higher rank quantum flag manifolds. The definition of the quantum flag manifolds directly extends from the Podleś sphere, and $\mathcal{O}_q(\mathrm{F}_3)$, to a general definition of full quantum manifold. Following the conventions of [29, §7.1], we denote by $U_q(\mathfrak{sl}_{n+1})$ the Drinfeld–Jimbo quantisation of the universal enveloping algebra of $\mathfrak{sl}_{n+1}$, and by $\mathcal{O}_q(\mathrm{SU}_{n+1})$ the dual quantised coordinate algebra. We then define the *full quantum flag manifold* of $\mathcal{O}_q(\mathrm{SU}_{n+1})$ to be the coideal subalgebra

$$\mathcal{O}_q(\mathrm{F}_{n+1}) := \{b \in \mathcal{O}_q(\mathrm{SU}_{n+1}) \,|\, K_i^{\pm 1} \triangleright b = b\}$$

Just as for $\mathcal{O}_q(\mathrm{F}_3)$, this is a quantum homogeneous space. Recall next that the Weyl group of $\mathfrak{sl}_{n+1}$ is the symmetric group S_{n+1}, and that for any reduced decomposition of ω_0, the longest element of S_{n+1}, we have an associated set of root vectors

$$\{X_\gamma \,|\, \gamma \in \Delta\} \subseteq U_q(\mathfrak{sl}_{n+1}),$$

labeled by Δ, the set of roots of $\mathfrak{sl}_{n+1}$. (See [29, §6.2], or [40, Appendix A], for a more detailed presentation of Lusztig's root vectors.)

As shown in [40], for either of the reduced decompositions

$$\begin{aligned} w_0 &= (s_ns_{n-1}\cdots s_1)(s_ns_{n-1}\cdots s_2)\cdots(s_ns_{n-1})s_n \\ &= (s_1s_2\cdots s_n)(s_1s_2\cdots s_{n-1})\cdots(s_1s_2)s_1 \end{aligned}$$

the associated space of positive Lusztig root vectors, that is the space spanned by the elements X_γ, for $\gamma \in \Delta^+$, is a quantum tangent space $T^{(0,1)}$ for $\mathcal{O}_q(\mathrm{F}_{n+1})$. For the special case of $\mathfrak{sl}_3$, this reduces to the $\mathcal{O}_q(\mathrm{F}_3)$ tangent space presented in §3.1, and for the special case of the Podleś sphere it reduces to anti-holomorphic part of the tangent space of the Podleś calculus.

This quantum tangent space is a direct q-deformation of the holomorphic tangent space of the classical full flag manifold, and we denote the associated covariant dc by $\Omega^{(0,1)}$. We denote the basis elements of the cotangent space $V^{(0,1)}$ by e_γ, for $\gamma \in \Delta^+$.

The space of Lusztig root vectors in fact forms a tangent space for $\mathcal{O}_q(\mathrm{SU}_{n+1})$. Just as for the rank 2 case, we can consider $(T^{(0,1)})^*$ the $*$-extension of $T^{(0,1)}$ and then restrict to the full quntum flag manifold. We denote the associated fodc by $\Omega^1_q(\mathrm{F}_{n+1})$ and observe that by construction it admits a direct sum decomposition

$$\Omega^1_q(\mathrm{F}_{n+1}) \simeq \Omega^{(1,0)} \oplus \Omega^{(0,1)},$$

where $\Omega^{(1,0)}$. We denote the basis of the associated cotangent space by f_γ, for $\gamma \in \Delta^+$. We now establish a direct generalisation of Proposition 5.3 to this higher rank setting, with a sketched proof.

Proposition 5.12. *For the dc $\Omega^1_q(\mathrm{F}_{n+1})$, there exist, up to identification of opposite structures, $2^{|\Pi|}$ left $\mathcal{O}_q(\mathrm{SU}_{n+1})$-covariant foacs.*

Proof. (Sketch) The proof is a direct extension of the proof for $\mathcal{O}_q(\mathrm{F}_3)$. The right $\mathcal{O}_q(\mathrm{SU}_{n+1})$-module structure of $V^{(0,1)}$ is given explicitly in [40, Proposition 3.7]. The right $\mathcal{O}_q(\mathrm{SU}_{n+1})$-module structure of $V^{(1,0)}$ can then be concluded from

$$[S(\omega^*)]b = [S(\omega S^{-1}(b^*))].$$

In short, this implies that

$$V^{(1,0)} \simeq \bigoplus_{\gamma \in \Pi} f_\gamma \mathcal{O}_q(\mathrm{SU}_{n+1}), \qquad V^{(0,1)} \simeq \bigoplus_{\gamma \in \Pi} e_\gamma \mathcal{O}_q(\mathrm{SU}_{n+1}).$$

It now follows that, just as for $\mathcal{O}_q(\mathrm{F}_3)$, a covariant foacs on the dc is determined by assigning to the basis elements e_γ and f_γ, for γ a simple root, the label of holomorphic or anti-holomorphic. Moreover, any such assignment necessarily gives a decomposition of right $\mathcal{O}_q(\mathrm{F}_{n+1})$-modules

$$V^1 \simeq V^{(1,0)} \oplus V^{(0,1)}.$$

This allows us to appeal to Lemma 5.2. Considering V^1 as a subspace of Λ^1, the cotangent space of the fodc on $\mathcal{O}_q(\mathrm{SU}_{n+1})$.

Noting that by construction $e^*_\gamma = f_\gamma$, for all $\gamma \in \Delta^+$, we now see that the only decompositions that give foacs are those two decompositions where e_γ and f_γ, for $\gamma \in \Pi$, are contained in complementary summands of the decomposition. Thus we see that we have $2^{|\Pi|}$ covariant foacs for the dc. □

The relations of the maximal prolongation of $\Omega^1_q(\mathrm{F}_{n+1})$ have not, as of now, been calculated. However, we expect that the results for the $\mathcal{O}_q(\mathrm{F}_3)$-case extend directly. This is formally presented in the following conjecture.

Conjecture 5.13. *For each full quantum flag manifold $\mathcal{O}_q(\mathrm{F}_{n+1})$, endowed with the fodc $\Omega^1_q(\mathrm{F}_{n+1})$, each of its $2^{|\Pi|}$ left $\mathcal{O}_q(\mathrm{SU}_{n+1})$-covariant structures extends to a factorisable, integrable, left $\mathcal{O}_q(\mathrm{SU}_n)$-covariant almost complex structure on $\Omega^\bullet_q(\mathrm{F}_{n+1})$, the maximal prolongation of $\Omega^1_q(\mathrm{F}_{n+1})$.*

Whether this conjecture is true or not, we note that we still have a much smaller number of covariant complex structures than in the classical case, with an upper bound being expressed in terms of the number of simple roots, as opposed to the classical case where it is the number positive roots. Moreover, this conjecture claims that non-integrable almost-complex structures for the full flags are a classical phenomenon.

5.9. **The Non-existence of a Covariant Kähler Structure.** Classically, the flag manifolds possess not only a complex structure, but a Kähler structure. Indeed, much of the classical Kähler geometry of the irreducible flag manifolds carries over to the quantum setting. The notion of a noncommutative Kähler structure was introduced in [39] to provide a framework in which to describe this q-deformed geometry. Moreover, the existence of a Kähler structure in general, was shown to imply direct noncommutative generalisations of many classical results of Kähler geometry, such as, Lefschetz decomposition and the Kähler identities.

It is thus natural to ask if the dc $\Omega^\bullet_q(\mathrm{F}_3)$ admits a noncommutative Kähler structure, and in particular, if it admits a left $\mathcal{O}_q(\mathrm{SU}_3)$-covariant Kähler structure. The definition [39, Definition 7.1] of a noncommutative Kähler structure requires a central element of the algebra $\Omega^\bullet_q(\mathrm{F}_3)$ that is *non-degenerate*, that is to say, a form satisfying $\kappa^3 \neq 0$. Moreover, if the Kähler structure is covariant, then κ must be a left $\mathcal{O}_q(\mathrm{F}_3)$-coinvariant element. We will prove the non-existence of a covariant Kähler structure by showing that no such form κ exists.

We begin with two simple general lemmas, which are undoubtly well-known to the experts, but which we include for the reader's convenience.

Lemma 5.14. *Let $B \subseteq A$ be a quantum homogeneous space and $\mathcal{F} \in {}^A_B\mathrm{Mod}$ a relative Hopf module. Then it holds that*

$$^{\mathrm{co}(A)}\Big(1\Box_\pi\Phi(\mathcal{F})\Big) = 1 \otimes \big({}^{\pi_B(A)}\Phi(\mathcal{F})\big).$$

Proof. Note first that any element of $1 \otimes ({}^{\pi_B(A)}\Phi(\mathcal{F}))$ is contained in the cotensor product $A\Box_{\pi_B}\Phi(\mathcal{F})$. Since any such element is clearly left A-coinvariant, we see that $1\otimes({}^{\pi_B(A)}\Phi(\mathcal{F}))$ is contained in ${}^{\mathrm{co}(A)}\mathcal{F}$. In the other direction, for any coinvariant element f in $\mathcal{F}$, we see that

$$\mathrm{U}(f) = f_{(-1)} \otimes [f_{(0)}] = 1 \otimes [f].$$

where of course $[f] \in {}^{\pi_B(A)}\Phi(\mathcal{F}))$. Thus the unit U maps the coinvariant elements of $\mathcal{F}$ into $1 \otimes ({}^{\pi_B(A)}\Phi(\mathcal{F}))$. Since U is a left A-comodule map, we see that the opposite inclusion holds, and hence we have equality. □

Lemma 5.15. *Let $f \in \mathcal{F}$ be a left A-coinvariant element. Then $fb = bf$, for all $b \in B$, if and only if $[f]b = \varepsilon(b)[f]$, for all $b \in B$.*

Proof. Since f is left A-coinvariant by assumption, it holds that $\mathrm{U}(f) = 1 \otimes [f]$. Now if $[f]b = \varepsilon(b)[f]$, for all $b \in B$, then

$$(1 \otimes [f])b = b_{(1)} \otimes [f]b_{(2)} = b_{(1)} \otimes \varepsilon(b_{(2)})[f] = b \otimes [f] = b(1 \otimes [f]).$$

Thus since U is a B-bimodule map, we see that $fb = bf$.

In the other direction, if $fb = bf$ then we see that

$$[f]b = [fb] = [bf] = \varepsilon(b)[f], \qquad \text{for all } b \in B,$$

which establishes the claimed equivalence. □

We now apply these results to the dc $\Omega^\bullet_q(\mathrm{F})$, describing first that the space of degree-2 coinvariant forms.

Lemma 5.16. *The space of left $\mathcal{O}_q(\mathrm{SU}_3)$-coinvariant 2-forms is a three-dimensional space. The corresponding space of $\mathcal{O}_q(\mathbb{T}^2)$-coinvariant degree-2 elements of $V^\bullet$ is spanned by*

$$f_{\alpha_1} \wedge e_{\alpha_1}, \qquad f_{\alpha_2} \wedge e_{\alpha_2}, \qquad f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2}.$$

Proof. The left $\mathcal{O}(\mathbb{T}^2)$-coinvariant elements of $V^\bullet$ are simply the elements of weight zero. Looking at the degree two elements of $V^\bullet$, and consulting Table 1 we see that the only weight zero basis elements are $f_{\alpha_1} \wedge e_{\alpha_1}$, $f_{\alpha_2} \wedge e_{\alpha_2}$, and $f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2}$. Thus we see that the space of coinvariants is three-dimensional as claimed. □

Lemma 5.17. *The space of left $\mathcal{O}_q(\mathbb{T}^2)$-coinvariant elements $v \in V^2$, satisfying $vb = \varepsilon(b)v$, for all $b \in \mathcal{O}_q(\mathrm{F}_2)$, is spanned by the elements*

$$f_{\alpha_2} \wedge e_{\alpha_2}, \qquad f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2}.$$

Proof. Note first that

$$(f_{\alpha_2} \wedge e_{\alpha_2})b = f_{\alpha_2} b_{(1)} \wedge e_{\alpha_2} b_{(1)} = f_{\alpha_2} b \wedge e_{\alpha_2} = \varepsilon(b) f_{\alpha_2} \wedge e_{\alpha_2},$$

with the analogous result for $f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2}$. Let us now look at the remaining $\mathcal{O}_q(\mathbb{T}^2)$-coinvariant basis element $f_{\alpha_2} \wedge e_{\alpha_2}$. It follows from (5) that the element $u_{11}u_{32}u_{33}$ is contained in $\mathcal{O}_q(\mathrm{F}_3)$. We see that

$$\begin{aligned}
(f_{\alpha_1} \wedge e_{\alpha_1})u_{11}u_{32}u_{23} &= \sum_{a,b,c=1}^{3} (f_{\alpha_1} u_{1a}u_{3b}u_{3c}) \wedge (e_{\alpha_1} u_{a1}u_{b2}u_{c3}) \\
&= \sum_{a,b,c=1}^{3} (f_{\alpha_1} u_{11}u_{23}u_{33}) \wedge (e_{\alpha_1} u_{11}u_{32}u_{33}) \\
&= -\, q^{-3}\nu^2 f_{\alpha_1} \wedge e_{\alpha_1} \\
&\neq \; 0.
\end{aligned}$$

Thus $(f_{\alpha_1} \wedge e_{\alpha_1})u_{11}u_{23}u_{33} \neq \varepsilon(b) f_{\alpha_1} \wedge e_{\alpha_1}$, for all b, meaning that the space of coinvariant forms, whose right $\mathcal{O}_q(\mathrm{F}_3)$-action is trivial, is two-dimensional as claimed. [39] □

Lemma 5.18. *For an arbitrary left $\mathcal{O}_q(\mathrm{SU}_3)$-coinvariant form*

$$\omega := c_1 f_{\alpha_1} \wedge e_{\alpha_1} + c_2 f_{\alpha_2} \wedge e_{\alpha_2} + c_3 f_{\alpha_1+\alpha_2} \wedge e_{\alpha_1+\alpha_2}$$

it holds that $\omega^3 \neq 0$ only if $c_1 \neq 0$.

Proof. Let us consider the case of $c_1 = 0$ and consider the third power of the form. This will be a linear combination of elements of the form

$$c_i c_j c_k f_{\alpha_i} \wedge e_{\alpha_i} \wedge f_{\alpha_j} \wedge e_{\alpha_j} \wedge f_{\alpha_k} \wedge e_{\alpha_k},$$

for $i, j, k = 2, 3$, and for convenience we have denoted $\alpha_3 = \alpha_1 + \alpha_2$. It follows directly from the commutation relations given in §3.8 that all such products are zero. For example, we see that the product

$$f_{\alpha_2} \wedge e_{\alpha_2} \wedge f_{\alpha_2} \wedge e_{\alpha_2} \wedge f_{\alpha_3} \wedge e_{\alpha_3}$$

is equal to

$$-q^{-2} f_{\alpha_2} \wedge f_{\alpha_2} \wedge e_{\alpha_2} \wedge e_{\alpha_2} \wedge f_{\alpha_3} \wedge e_{\alpha_3} + \nu f_{\alpha_2} \wedge f_{\alpha_3} \wedge e_{\alpha_3} \wedge e_{\alpha_3} \wedge f_{\alpha_3} \wedge e_{\alpha_3}$$

which is in turn equal to

$$q^{(\alpha_1+\alpha_2,\alpha_1+\alpha_2)} \nu f_{\alpha_2} \wedge f_{\alpha_3} \wedge f_{\alpha_3} \wedge e_{\alpha_3} \wedge e_{\alpha_3} \wedge e_{\alpha_3} = 0.$$

Thus the form ω^3 is equal to zero as claimed. □

Combining the statements of Lemma 5.17 and Lemma 5.18 we arrive at the following theorem, which means that the dc $\Omega^\bullet)q(\mathrm{F}_3)$ does not admit a covariant Kähler structure.

Theorem 5.19. *There does not exist a left $\mathcal{O}_q(\mathrm{SU}_3)$- coinvariant non-degenerate form $\sigma \in \Omega^2_q(\mathrm{F}_3)$ that commutes with the elements of $\mathcal{O}_q(\mathrm{F}_3)$.*

Remark 5.20. Despite the fact that any nondegenerate coinvariant 2-form does not commutate with the elements of $\mathcal{O}_q(\mathrm{F}_3)$, we can still define left and right Lefschetz maps. As is readily checked, each map is (either a left or right) $\mathcal{O}_q(\mathrm{F}_3)$-module isomorphism. Each has an associated Lefschetz decomposition with a corresponding Hodge map, metric, and inner product.

Remark 5.21. The nonexistence of a coinvariant non-degenerate form also implies that $\Omega^2_q(\mathrm{F}_3)$ does not admit a metric in the sense of Beggs and Majid [4]. This implies that $\Omega^1_q(\mathrm{F}_3)$ is not self-dual as an object in the category of relative Hopf modules ${}^A_B\mathrm{Mod}_B$, as explained for example in [7].

Remark 5.22. The definition of a Kähler structure also requires that the Kähler form κ is *real*, that is to say $\kappa^* = \kappa$, and *closed*, that is to say $\mathrm{d}\kappa = 0$. A family of real, closed, left $\mathcal{O}_q(\mathrm{SU}_3)$-coinvariant 2-forms can be constructed from the Kähler structures of the two copies of $\mathcal{O}_q(\mathbb{CP}^2)$ in $\mathcal{O}_q(\mathrm{F}_3)$.

As shown in §3.4, the Heckenberger–Kolb double complex of each copy of $\mathcal{O}_q(\mathbb{CP}^2)$ is realised as the restriction of the $*$-dc $\Omega^\bullet_q(\mathrm{F}_3)$. Thus the Kähler forms κ_1 and κ_2 of these dc, as introduced in [39], will be real closed left $\mathcal{O}_q(\mathrm{SU}_3)$-coinvariant elements of $\Omega^2_q(\mathrm{F}_3)$. Thus we see that the 2-form

$$\kappa_1 + \lambda\kappa_2, \qquad\qquad \text{for } \lambda \in \mathbb{C}^\times$$

is a real closed left $\mathcal{O}_q(\mathrm{SU}_3)$-coinvariant 2-form. Classically this form is the fundamental form of a Kähler metric for F_3.

TABLE 1. Sums of roots of $\mathfrak{sl}_3$

	α_1	α_2	$\alpha_1+\alpha_2$	$-\alpha_1$	$-\alpha_2$	$-(\alpha_1+\alpha_2)$
α_1	$2\alpha_1$	$\alpha_1+\alpha_2$	$2\alpha_1+\alpha_2$	0	$\alpha_1-\alpha_2$	$-\alpha_2$
α_2	$\alpha_1+\alpha_2$	$2\alpha_2$	$\alpha_1+2\alpha_2$	$\alpha_2-\alpha_1$	0	$-\alpha_1$
$\alpha_1+\alpha_2$	$2\alpha_1+\alpha_2$	$\alpha_1+2\alpha_2$	$2(\alpha_1+\alpha_2)$	α_2	α_1	0
$-\alpha_1$	0	$\alpha_2-\alpha_1$	α_2	$-2\alpha_1$	$-(\alpha_1+\alpha_2)$	$-(2\alpha_1+\alpha_2)$
$-\alpha_2$	$\alpha_2-\alpha_1$	0	α_1	$-(\alpha_1+\alpha_2)$	$-2\alpha_2$	$-(\alpha_1+2\alpha_2)$
$-(\alpha_1+\alpha_2)$	$-\alpha_2$	$-\alpha_1$	0	$-(2\alpha_1+\alpha_2)$	$-(\alpha_1+2\alpha_2)$	$-2(\alpha_1+\alpha_2)$

APPENDIX A. SOME DETAILS ON THE LIE ALGEBRA $\mathfrak{sl}_3$

In this subsection, so as to set notation, we recall some elementary definitions and results about the A_2-root system associated to the special linear Lie algebra $\mathfrak{sl}_2$. Let $\{\varepsilon_i\}_{i=1}^{n+1}$ be the standard basis of $\mathbb{R}^3$, and endow it with its canonical Euclidean structure. The root system A_3 is the pair (V,Δ), where V is the subspace of $\mathbb{R}^3$ spanned by the roots

$$\Delta := \Big\{ \pm\alpha_1 := \pm(\varepsilon_1-\varepsilon_2),\, \pm\alpha_2 := \pm(\varepsilon_2-\varepsilon_3),\, \pm(\alpha_1+\alpha_2) = \pm(\varepsilon_1-\varepsilon_3)\Big\}.$$

We take the standard subset of positive roots, and its associated set of simple roots,

$$\Delta^+ := \Big\{\alpha_1,\,\alpha_2,\,\alpha_1+\alpha_2\Big\}, \qquad\qquad \Pi := \Big\{\alpha_1,\,\alpha_2\Big\}.$$

This gives us the Cartan matrix

$$(a_{ij})_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

We also recall that the Weyl group of the root system is the symmetric group S_3 of order 6.

We finish with a table presenting all possible sums $\alpha+\beta$, where $\alpha,\beta\in\Delta^+$. The sums highlighted in blue are those that again roots of $\mathfrak{sl}_3$

We appeal to this table a number of times in the paper. For example, we refer to it when classifying the left coinvariant forms in Lemma 5.16.

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