

FOURIER UNCERTAINTY PRINCIPLES ON RIEMANNIAN MANIFOLDS

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ABSTRACT. The purpose of this paper is to develop a Fourier uncertainty principle on compact Riemannian manifolds and contrast the underlying ideas with those arising in the setting of locally compact abelian groups. The key obstacle is the growth of eigenfunctions, and connections to Bourgain's celebrated Λ_q theorem are discussed in this context.

1. INTRODUCTION

The classical uncertainty principle in Euclidean space says that if a suitably regular function is supported on a set $E \subset \mathbb{R}^d$ and its Fourier transform is concentrated on a set $S \subset \mathbb{R}^d$, then $|E| \cdot |S| \geq c > 0$, where $|\cdot|$ denotes the d -dimensional Lebesgue measure. See, for example, [4] and the references contained therein. The main thrust of this paper is to establish the uncertainty principle and exact recovery mechanisms on Riemannian manifolds, though in the process we shall also establish new results in the well-traveled terrain of locally compact abelian groups. See, for example, [12], [7] for classical results in that setting.

1.1. The uncertainty principle on locally compact abelian groups. While the main thrust of this paper is to study the uncertainty principle on Riemannian manifolds, we set the table by describing this problem in the setting of locally compact abelian groups. Indeed, by Pontryagin duality, if G is a locally compact abelian group, then for the Haar measure μ on G (which is unique up to a positive constant), there exists a unique Haar measure ν on \widehat{G} such that if $f \in L^1(G)$, $\widehat{f} \in L^1(\widehat{G})$, then

$$(1.1) \quad f(x) = \int_{\widehat{G}} \widehat{f}(\chi) \chi(x) d\nu(\chi),$$

where

$$\widehat{f}(\chi) = \int_G f(x) \overline{\chi(x)} d\mu(x),$$

and $\chi \in \widehat{G}$ is the character¹ on G . Note that the equality in (1.1) holds μ -almost everywhere if $f \in L^1(G)$ and point-wise if f is also continuous. The definition of the Fourier transform for functions $f \in L^2(G)$ is done in standard manner, by extending the definition from $L^1(G) \cap L^2(G)$ to $L^2(G)$.

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¹Recall that a character χ in \widehat{G} is a continuous group homomorphism function $\chi : G \rightarrow S^1$, S^1 the unit circle in \mathbb{C} .

Recall that we say $f \in L^2(G)$ is supported in a set $E \subset G$ if $f(x) = 0$ for μ -almost every $x \in G \setminus E$. In the same fashion, we say \widehat{f} is supported in $S \subset \widehat{G}$ if $\widehat{f}(\xi) = 0$ for ν -almost $\xi \in \widehat{G} \setminus S$.

Let $E \subset G$ with $0 < \mu(E) < \infty$ and $S \subset \widehat{G}$ with $0 < \nu(S) < \infty$. Let $f \in L^2(G)$ be supported on $E \subset G$ and \widehat{f} is supported on $S \subset \widehat{G}$, then for μ -almost $x \in E$, by the Cauchy-Schwarz inequality we have

$$(1.2) \quad |f(x)|^2 = \left| \int_S \widehat{f}(\chi) \chi(x) d\nu(\chi) \right|^2 \leq \nu(S) \cdot \int_S |\widehat{f}(\chi)|^2 d\nu(\chi).$$

Integrating both sides of (1.2) over E (where f is assumed to be supported on E), we see that

$$\|f\|_{L^2(E)}^2 \leq \nu(S) \cdot \mu(E) \cdot \int_S |\widehat{f}(\chi)|^2 d\nu(\chi) = \nu(S) \cdot \mu(E) \cdot \|f\|_{L^2(E)}^2$$

by the Plancherel identity. From this inequality we conclude that the following variant of uncertainty principle:

$$(1.3) \quad \mu(E) \cdot \nu(S) \geq 1.$$

The assumption that f is supported in E and \widehat{f} is supported in S can easily be relaxed and replaced by the assumption that f, \widehat{f} are concentrated in those sets in a suitable sense. The following definition will be used throughout the paper in different settings.

Definition 1.1. If X is a measure space with measure μ and $E \subset X$, we say that $f \in L^p(X, \mu)$ is L^p -concentrated on E at level $L \geq 1$ if

$$(1.4) \quad \left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} \leq L \left(\int_E |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}.$$

It is not difficult to see that (1.4) is equivalent to the condition

$$(1.5) \quad \|f - 1_E f\|_{L^p(X)} \leq \epsilon \|f\|_{L^p(X)}$$

for some $0 \leq \epsilon < 1$. Indeed (1.4) holds with $L = \frac{1}{(1-\epsilon^p)^{\frac{1}{p}}}$ if and only if (1.5) holds.

Here, 1_E is the indicator of the set E and is defined by $1_E(x) = 1$ when $x \in E$ and 0 otherwise. Note that any measurable function in L^p that is supported on a measurable set is automatically L^p -concentrated on the set at level $L = 1$. This explains why concentration can be seen as a relaxed version of support.

If \widehat{f} is supported in a generic set (see Remark 1.4), the situation improves significantly. To see this, we recall the following result due to Bourgain ([2]).

Theorem 1.2. *Let G be a locally compact abelian group and let $\Psi = (\psi_1, \dots, \psi_n)$ denote a sequence of n mutually orthogonal functions, with $\|\psi_i\|_{L^\infty(G)} \leq 1$. Then for any $q > 2$, there exists an index subset S of $\{1, 2, \dots, n\}$, with size $|S| > n^{\frac{2}{q}}$, such that*

$$\left\| \sum_{i \in S} a_i \psi_i \right\|_{L^q(G)} \leq C(q) \left(\sum_{i \in S} |a_i|^2 \right)^{\frac{1}{2}}.$$

The constant $C(q)$ depends only on q and the estimate above holds for a generic index set of size $\lceil n^{\frac{2}{q}} \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than x .

Remark 1.3. Note that Theorem 1.2 holds automatically for $q = 2$. In this case, $C(2) = 1$ (by orthogonality) for any set $S \subset \{1, 2, \dots, n\}$. When $q = \infty$, it is easy to see that $C(q) \leq |S|^{\frac{1}{2}}$ by Cauchy-Schwarz and the assumption $\|\psi_i\|_{L^\infty(G)} \leq 1$. By Riesz-Thorin interpolation theorem applied to the identity operator on the Paley-Wiener subspace of $L^2(G)$ generated by $\{\psi_i\}_{i \in S}$, $C(q) \leq |S|^{\frac{1}{2} - \frac{1}{q}}$. The point of Theorem 1.2 is that $C(q)$ depends only on q if S is a random subset of $\{1, 2, \dots, n\}$ of size $\lceil n^{\frac{2}{q}} \rceil$.

Remark 1.4. The notion of generic in Theorem 1.2 means the following. Let $0 < \delta < 1$ and let $\{\xi_j\}_{1 \leq j \leq n}$ denote independent $0, 1$ random variables of mean $\int \xi_j(\omega) d\omega = \delta$, $1 \leq j \leq n$. Choosing $\delta = n^{\frac{2}{q}-1}$ generates a random subset $S_\omega = \{1 \leq j \leq n : \xi_j(\omega) = 1\}$ of $\{1, 2, \dots, n\}$ of expected size $\lceil n^{\frac{2}{q}} \rceil$. Theorem 1.2 holding for a generic set S means that the result holds for the set S_ω with probability $1 - o_N(1)$. In simpler language, if we randomly choose a subset of $\{1, 2, \dots, n\}$ by choosing each element with probability $p = n^{\frac{2}{q}-1}$, then Theorem 1.2 holds for such a set with probability close to 1.

Theorem 1.2 has a built-in uncertainty principle mechanism. Indeed, we have the following theorem, which is the first result of this paper.

Theorem 1.5. *Let G be a locally compact abelian group with a Haar measure μ , and let $\Psi = (\psi_1, \dots, \psi_n)$ denote a sequence of n mutually orthogonal functions, with $\|\psi_i\|_{L^\infty(G)} \leq 1$.*

Let $q > 2$, and let S be a generic index subset of $\{1, 2, \dots, n\}$ of size $\lceil n^{\frac{2}{q}} \rceil$. Define $f(x) = \sum_{i \in S} a_i \psi_i(x)$, for almost every $x \in G$. Suppose that f is L^2 -concentrated on $E \subset G$ at level L . Then

$$\mu(E) \geq \frac{1}{(LC(q))^{\frac{1}{\frac{1}{2} - \frac{1}{q}}}}.$$

In particular, the results stated above hold for \mathbb{T}^d , the torus. This case brings us into the realm of Riemannian manifolds. The key assumption in the arguments above is that the orthogonal functions have modulus at most one, which underlines the fundamental difficulty of the manifold setting where the results of this paper will be stated. Recall that if M is a compact Riemannian manifold without a boundary, then the eigenfunctions $\{e_j\}_{j=1}^\infty$ of $\sqrt{-\Delta_g}$ form an orthonormal basis for $L^2(M)$. As we shall see, the reason the argument given for locally compact abelian groups above does not transfer directly to the manifold setting is that the eigenfunctions of the Laplacian are not typically uniformly bounded.

1.2. The review of basic facts regarding Riemannian manifolds. Let M be a smooth d -dimensional manifold which is compact and without boundary, and let g be a Riemannian metric tensor on M . The metric tensor g is locally represented as a real-symmetric, positive-definite $d \times d$ matrix with entries that are smooth functions of x ,

$$g(x) = \begin{bmatrix} g_{11}(x) & g_{12}(x) & \cdots & g_{1d}(x) \\ g_{21}(x) & g_{22}(x) & \cdots & g_{2d}(x) \\ \vdots & \vdots & & \vdots \\ g_{d1}(x) & g_{d2}(x) & \cdots & g_{dd}(x) \end{bmatrix}$$

where, if $\partial_i = \frac{\partial}{\partial x_i}$, we have

$$g_{ij}(x) = \langle \partial_i, \partial_j \rangle_{g(x)}.$$

The natural volume density on a Riemannian manifold (M, g) is given locally by

$$dV_g(x) = |g(x)|^{\frac{1}{2}} dx,$$

where $|g(x)|$ denotes the determinant of $g(x)$.

There is a natural generalization of the Euclidean Laplacian to (M, g) , called the Laplace-Beltrami operator, which is expressed in local coordinates by

$$\Delta_g f = |g|^{-\frac{1}{2}} \sum_{i,j=1}^n \partial_i (|g|^{\frac{1}{2}} g^{ij} \partial_j f).$$

Here g^{ij} are the matrix entries of g^{-1} . Indeed, the Laplace-Beltrami operator is defined such that

$$\int_M (\Delta_g f) h dV_g = - \int_M \langle \nabla_g f, \nabla_g h \rangle_g dV_g,$$

where

$$\nabla_g f(x) = \sum_{i,j=1}^d g^{ij}(x) \partial_i f(x) \partial_j$$

denotes the gradient of f . It follows Δ_g is both self-adjoint and negative-definite.

1.3. Laplace-Beltrami eigenfunctions and eigenvalues. We now summarize some facts about the spectrum of Δ_g . The material is standard and can be found, for example, in [9]. There exists an orthonormal basis for $L^2(dV_g)$ of eigenfunctions e_1, e_2, \dots satisfying

$$\Delta_g e_j = -\lambda_j^2 e_j, \quad j = 1, 2, \dots$$

where $\lambda_j \geq 0$. We are free to select the indices of our eigenfunctions to ensure

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \rightarrow \infty.$$

Most of these facts follow by applying the spectral theory of compact self-adjoint operators to the resolvent $(I - \Delta_g)^{-1}$.

Remark 1.6. The choice to use $-\lambda_j^2$ as the eigenvalue for e_j is a matter of convention. This allows us to think of λ_j as the “frequency” of e_j , or rather the frequency of the standing wave solution

$$u(t, x) = \cos(t\lambda_j) e_j(x)$$

to the wave equation $(\Delta_g - \partial_t^2)u = 0$.

Given a function $f \in L^2(dV_g)$, the Fourier coefficients of f are given by $\hat{f}(j) = \langle f, e_j \rangle$, the Hermitian inner product of f with e_j in $L^2(dV_g)$, given by

$$\langle f, e_j \rangle = \int_M f(x) \overline{e_j(x)} dV_g(x).$$

Indeed, if f is in $L^2(dV_g)$, then

$$f = \sum_{j=1}^{\infty} \hat{f}(j) e_j$$

where both equality and the convergence of the sum hold in $L^2(dV_g)$. If f is smooth enough, the sum converges uniformly and equality holds pointwise.

A fundamental problem in the harmonic analysis on manifolds is to estimate the count of eigenvalues (with multiplicity) under some threshold. That is, we desire nice asymptotics for the Weyl counting function

$$N(\lambda) = \#\{j \in \mathbb{N} : \lambda_j \leq \lambda\}.$$

The Weyl law gives very general asymptotics

$$(1.6) \quad N(\lambda) = (2\pi)^{-d}|M||B|\lambda^d + O(\lambda^{d-1}),$$

where $|M|$ denotes the volume of M and $|B|$ denotes the volume of the unit ball in \mathbb{R}^d . The remainder is sharp in the sense that it cannot be improved for the sphere S^d . Many manifolds enjoy an improved Weyl remainder estimate ([5]).

The flat torus $M = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ provides a key illustrative example. The Laplace-Beltrami operator is just $\sum_{\ell=1}^d \partial_{\ell}^2$. We have a complete, \mathbb{Z}^d -indexed set of eigenfunctions

$$e_m(x) = (2\pi)^{-\frac{d}{2}} e^{i\langle x, m \rangle}, \quad m \in \mathbb{Z}^d$$

with $m \in \mathbb{Z}^d$ and $\lambda_m = |m|^2$, respectively. Here, $|m|^2 = m_1^2 + \dots + m_d^2$.

The classical Weyl Law (see e.g. [13]) says that as $\lambda \rightarrow \infty$

$$N(\lambda) = \#\{m \in \mathbb{Z}^d : |m|^2 \leq \lambda\} = |B|\lambda^d + O(\lambda^{d-1}),$$

which coincides with the trivial bound for the Gauss circle problem. We will need the *local Weyl counting function*,

$$N_x(\lambda) = \sum_{\lambda_j \leq \lambda} |e_j(x)|^2, \quad x \in M, \lambda \geq 0,$$

which enjoys similar asymptotics to the Weyl counting function, namely

$$N_x(\lambda) = (2\pi)^{-d}|B|\lambda^d + O(\lambda^{d-1}), \quad \text{as } \lambda \rightarrow \infty$$

where the constants implicit in the big- O remainder are uniform for $x \in M$. We note the pointwise asymptotics imply the Weyl law since $\|e_j\|_{L^2} = 1$ and

$$N(\lambda) = \int_M N_x(\lambda) dV_g(x).$$

1.4. An uncertainty principle for compact manifolds without boundary. We begin a general technical statement that we shall proceed to clarify in a variety of instances.

Proposition 1.7. *Let S be a finite subset of the set of eigenvalues of $\sqrt{-\Delta_g}$. Let $X_S = \{j : \lambda_j \in S\}$. Suppose that $f \in L^2(M)$ is not identically 0 and f is L^2 -concentrated in $E \subset M$ at level L with respect to the Riemannian volume density (see Definition 1.1). Suppose also that \widehat{f} is L^2 -concentrated on X_S at level L' with respect to the counting measure. Then*

$$(1.7) \quad \left(\frac{1}{\#X_S} \sum_{j \in X_S} \frac{1}{|E|} \int_E |e_j(x)|^2 dx \right)^{-1} \leq (1 - \epsilon - \epsilon')^{-2} |E| (\#X_S)$$

where

$$(1.8) \quad L = (1 - \epsilon^2)^{-\frac{1}{2}} \quad \text{and} \quad L' = (1 - \epsilon'^2)^{-\frac{1}{2}}.$$

Remark 1.8. This result is valid for general orthogonal systems in $L^2(M)$ and holds without the assumption that the underlying manifold does not have a boundary. See, for example, [11], [8], and [10] for some interesting implementations of this viewpoint.

Proposition 1.7 raises the question of when the left-hand side of (1.7) is bounded below by a non-zero constant. Suppose that for each frequency λ , the sum

$$(1.9) \quad \sum_{j:\lambda_j=\lambda} |e_j(x)|^2 \equiv c \quad \text{a.e. } x \in M$$

is constant, and the constant $c := c(\lambda, M)$ is independent of x . Since the e_j 's are all L^2 -normalized, this condition is equivalent to

$$(1.10) \quad \sum_{j:\lambda_j=\lambda} |e_j(x)|^2 \equiv \frac{\#\{j : \lambda_j = \lambda\}}{|M|}, \quad \text{a.e. } x \in M$$

where we count in (1.10) with multiplicity. Indeed, notice that by (1.9), we have

$$\sum_{j:\lambda_j=\lambda} \int_M |e_j(x)| dx = c|M|.$$

On the other hand, by the L^2 -normalization of e_j , we have

$$\sum_{j:\lambda_j=\lambda} \int_M |e_j(x)| dx = \#\{j : \lambda_j = \lambda\}.$$

Putting the last two equations together, we obtain $c = \frac{\#\{j:\lambda_j=\lambda\}}{|M|}$. Recalling the value of the constant c from (1.9), we complete the proof.

All homogeneous Riemannian manifolds - those whose group of isometries have a transitive action on the points in the manifold - satisfy this condition. In dimension $d = 2$, the property (1.10) holds if and only if the manifold M is homogeneous [15]. This question is still open in higher dimensions. We note that flat tori and spheres with the standard metrics are all homogeneous, so the property (1.10) holds in these cases.

Notice that the eigenfunctions on the sphere are not uniformly bounded; however, the relation (1.10) holds due to the homogeneity of the sphere. Furthermore, the sphere is not a locally compact abelian group, which makes it a particularly important and interesting object in this context.

Corollary 1.9. *Let (M, g) be a compact Riemannian manifold without boundary satisfying the relation (1.10). Let E be a measurable subset of M and let S be a finite subset of the spectrum of $\sqrt{-\Delta_g}$. Let $X_S = \{j : \lambda_j \in S\}$. Suppose f is L^2 -concentrated in E at level L with respect to the Riemannian volume measure, and \hat{f} is L^2 -concentrated in X_S at level L' with respect to the counting measure. Then,*

$$(1 - \epsilon - \epsilon')^2 \leq \frac{|E|}{|M|} \cdot \#X_S,$$

where ϵ and ϵ' are as in (1.8).

In the case when the manifold does not satisfy (1.10), we are still able to prove an uncertainty principle that depends on the size of the frequency concentration set S . The following corollary is an immediate consequence of Proposition 1.7.

Corollary 1.10. *Assume the hypotheses of Proposition 1.7. Then,*

$$(1.11) \quad (1 - \epsilon - \epsilon')^2 \leq |E| \left(\sup_{x \in M} \sum_{\lambda_j \in S} |e_j(x)|^2 \right).$$

Note that the supremum on M can actually be replaced by the supremum on E .

To be useful, this corollary requires as input some estimates on the sup-norms of eigenfunctions. For large $\lambda > 0$, the general bounds are

$$(1.12) \quad \sup_{x \in M} \sum_{\lambda_j \in [\lambda, \lambda+1]} |e_j(x)|^2 = O(\lambda^{d-1}) \quad \text{as } \lambda \rightarrow \infty.$$

(see [13] and the references therein). These bounds are sharp, but can be improved if one both makes some assumptions on the geodesic flow on M and replaces the unit interval $[\lambda, \lambda + 1]$ with an interval $[\lambda, \lambda + o(1)]$ that slowly shrinks in length as $\lambda \rightarrow \infty$ (see e.g. [1, 14]).

Corollary 1.10 and the bound (1.12) lead us to the following result.

Corollary 1.11. *Assume the hypotheses of Proposition 1.7 and suppose that S can be covered by unit intervals $\cup_{k=1}^n [\mu_k, \mu_k + 1]$, i.e., $S \subset \cup_{k=1}^n [\mu_k, \mu_k + 1]$. Then*

$$(1.13) \quad (1 - \epsilon - \epsilon')^2 \leq |E| \cdot C_M \sum_{k=1}^n \mu_k^{d-1},$$

where C_M is the constant implicit on the right-hand side of the equation (1.12).

Remark 1.12. The quantity on the right-hand side of (1.13) carries a lot of information. Assume that $\lambda_m = \max\{\lambda \in S\}$. We always have

$$(1.14) \quad \sum_{k=1}^n \mu_k^{d-1} \leq \#S \cdot (\lambda_m + 1)^{d-1},$$

since $\#S \geq n$. On the other hand, suppose that $\mu_k \approx 2^k$. Then $\lambda_m \leq 2^N$, and we obtain

$$\sum_{k=1}^n \mu_k^{d-1} \approx \sum_{k=1}^N 2^{k(d-1)} \leq 2^{N(d-1)+1},$$

which is much better than (1.14).

1.5. Functions on Riemannian manifolds with random spectra. The following is a manifold analog of Theorem 1.5.

Theorem 1.13. *Let M be a compact Riemannian manifold without a boundary, and let $f \in L^2(M)$. Let $\{e_1, e_2, \dots, e_n\}$ denote a subset of the eigenbasis of Δ_g such that*

$$\|e_i\|_\infty \leq B.$$

Then there exists $C, c > 0$ such that for a generic index subset $I \subset \{1, 2, \dots, n\}$ with

$$\left| \#I - \frac{n}{2} \right| \leq c\sqrt{n}$$

such that if

$$f(x) = \sum_{i \in I} a_i e_i \quad \text{is } L^1\text{-concentrated in } E \subset M \text{ at level } A,$$

then

$$|E| \geq \frac{1}{C^2 \cdot B^2 \cdot A^2 \cdot \log(n)^2 \cdot (\log(\log(n)))^5}.$$

We now compare Theorem 1.13 to the results obtained above, specifically in the case where S is a generic subset of the first n eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

repeated with multiplicity. The Weyl law (1.6) guarantees positive constants c and C for which

$$c\lambda_n^d \leq n \leq C\lambda_n^d \quad \text{for large } n,$$

and hence $\lambda_n \approx n^{\frac{1}{d}}$. The corresponding eigenfunctions e_1, \dots, e_n have L^∞ norms uniformly bounded by $O(\lambda_n^{\frac{d-1}{2}})$, which can often be improved (see e.g. [1, 14, 3]). The lower bound in the theorem then reads as

$$(1.15) \quad |E| \geq \frac{1}{C^2 \cdot A^2 \cdot \lambda_n^{d-1} \cdot \log(\lambda_n)^2 \cdot (\log(\log(\lambda_n)))^5},$$

after perhaps adjusting the constant C . On the other hand, Corollary 1.11 yields the bound

$$|E| \geq \frac{1}{C^2 \cdot A^2 \cdot \lambda_n^d}$$

in this situation, which is worse than (1.15) by nearly a full power of λ_n .

1.6. Jointly elliptic operators. We now bridge the conceptual gap between the Fourier uncertainty principal on manifolds and on locally compact abelian groups. The torus, being a model member of both settings, will be our focal point.

Let M be a compact smooth manifold without boundary which comes equipped with a volume density which we will simply denote as dx . Suppose we have some differential operators L_1, \dots, L_m , which together have the following properties.

- (i) L_i is self-adjoint in $L^2(dx)$ for each i .
- (ii) L_i and L_j commute for each i, j .
- (iii) L_1, \dots, L_m are jointly elliptic in the sense that

$$T := I + \sum_{i=1}^m L_i^2$$

is an elliptic differential operator on M .

Then, the tuple of operators $L = (L_1, \dots, L_m)$ admits a discrete orthonormal basis $\{e_j : j \in \mathbb{N}\}$ of joint eigenfunctions satisfying

$$L_i e_j = \lambda_{i,j} e_j \quad \text{for each } i \in \{1, \dots, m\}, j \in \mathbb{N}.$$

The joint spectrum of this family of operators is

$$\{(\lambda_{1,j}, \dots, \lambda_{m,j}) \in \mathbb{R}^m : j \in \mathbb{N}\}.$$

We will feel free to replace \mathbb{N} with any other countably infinite indexing set.

Example 1.14 (The torus). Consider the flat torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ with operators

$$L_i = \frac{1}{i} \frac{\partial}{\partial x_i} \quad \text{for } i \in \{1, \dots, d\}.$$

Note $\{L_1, \dots, L_d\}$ is a jointly elliptic, commuting family of self-adjoint operators as described above, which also admits a joint eigenbasis of Fourier exponentials

$$e_n(x) = (2\pi)^{-\frac{d}{2}} e^{i\langle x, n \rangle} \quad n \in \mathbb{Z}^d.$$

As expected, the joint spectrum is \mathbb{Z}^d .

Example 1.15 (Cartesian products of manifolds). Let M_1 and M_2 denote compact Riemannian manifolds without boundary with respective Laplace-Beltrami operators Δ_1 and Δ_2 . By an abuse of notation, we consider Δ_1 and Δ_2 as operators acting on functions on $M_1 \times M_2$ in their respective variables. Note, $\{\Delta_1, \Delta_2\}$ is a family of commuting, jointly elliptic, self-adjoint operators. The joint eigenbasis consists of elements, each a tensor product of an eigenfunction on M_1 with one on M_2 . The joint spectrum is $\Lambda_1 \times \Lambda_2$, where Λ_1 and Λ_2 are the spectra of Δ_1 and Δ_2 , respectively.

Example 1.16 (The two-dimensional sphere). The classic spherical harmonics Y_ℓ^m on S^2 are joint eigenbasis elements for family

$$\left\{ \frac{1}{i} \frac{\partial}{\partial \theta}, -\Delta_{S^2} \right\},$$

where θ here denotes the azimuthal angle. The joint spectrum is

$$\{(m, \ell(\ell + 1)) : |m| \leq \ell\}.$$

The statement for Proposition 1.7 can be phrased and proved in nearly exactly the same way, with S being a subset of the joint spectrum of a family of operators.

Proposition 1.17. *Let M be a compact Riemannian manifold with a system of self-adjoint, commuting, jointly-elliptic differential operators L_1, \dots, L_m with joint eigenbasis e_j for $j \in \mathbb{N}$ and joint spectrum $\Lambda \subset \mathbb{R}^m$. Let $E \subset M$ be measurable, let S be a finite subset of the joint spectrum Λ . Let $X_S = \{j : \lambda_j \in S\}$, where λ_j is the vector of eigenvalues of L_1, \dots, L_m associated with the eigenfunction e_j . Suppose there is a nontrivial $f \in L^2(M)$ that is L^2 -concentrated in E at level L and whose Fourier coefficients \hat{f} are L^2 -concentrated in X_S at level L' . Then*

$$\sum_{j \in X_S} \int_E |e_j(x)|^2 dx \geq (1 - \epsilon - \epsilon')^2,$$

where

$$L = (1 - \epsilon^2)^{-\frac{1}{2}} \quad \text{and} \quad L' = (1 - \epsilon'^2)^{-\frac{1}{2}}.$$

The left-hand side can be simplified as in Corollary 1.9 provided

$$(1.16) \quad \sum_{j: \lambda_j = \lambda} |e_j(x)|^2 \text{ is constant in } x$$

for each λ in the joint spectrum. As we saw earlier in (1.10), in this case, we have

$$\sum_{j:\lambda_j=\lambda} |e_j(x)|^2 = \frac{\#\{j : \lambda_j = \lambda\}}{|M|}.$$

Corollary 1.18. *Assume the hypotheses of Proposition 1.17, and further assume (1.16). Then,*

$$\#X_S \frac{|E|}{|M|} \geq (1 - \epsilon - \epsilon')^2,$$

where X_S is as described in Proposition 1.17.

2. PROOFS OF MAIN RESULTS

In this section we prove the main results of the paper.

2.1. Proof of Theorem 1.5. In this subsection, we shall use the notation $|E| = \mu(E)$, where μ is the Haar measure on G . By Theorem 1.2,

$$(2.1) \quad \|f\|_{L^q(G)} \leq C(q) \|f\|_{L^2(G)}.$$

The left-hand side of (2.1) is bounded from below by

$$(2.2) \quad \left(\int_E |f(x)|^q d\mu(x) \right)^{\frac{1}{q}} = |E|^{\frac{1}{q}} \cdot \left(\frac{1}{|E|} \int_E |f(x)|^q d\mu(x) \right)^{\frac{1}{q}}.$$

Since f is assumed to be L^2 concentrated on E at level L , the right-hand side of (2.1) is bounded from above by

$$(2.3) \quad C(q) \cdot L \cdot \left(\int_E |f(x)|^2 d\mu(x) \right)^{\frac{1}{2}} = L \cdot C(q) \cdot |E|^{\frac{1}{2}} \cdot \left(\frac{1}{|E|} \int_E |f(x)|^2 d\mu(x) \right)^{\frac{1}{2}}.$$

Plugging (2.2) and (2.3) into (2.1), we obtain

$$|E|^{\frac{1}{q}} \cdot \left(\frac{1}{|E|} \int_E |f(x)|^q d\mu(x) \right)^{\frac{1}{q}} \leq C(q)L \cdot |E|^{\frac{1}{2}} \cdot \left(\frac{1}{|E|} \int_E |f(x)|^2 d\mu(x) \right)^{\frac{1}{2}}.$$

It follows that

$$|E|^{\frac{1}{2} - \frac{1}{q}} \geq \frac{1}{LC(q)} \cdot \frac{\left(\frac{1}{|E|} \int_E |f(x)|^q d\mu(x) \right)^{\frac{1}{q}}}{\left(\frac{1}{|E|} \int_E |f(x)|^2 d\mu(x) \right)^{\frac{1}{2}}} \geq \frac{1}{LC(q)}.$$

We conclude that

$$|E| \geq \frac{1}{(LC_q)^{\frac{1}{\frac{1}{2} - \frac{1}{q}}}},$$

as desired.

2.2. Proof of Proposition 1.7. To prove the proposition, we first need to introduce some notation:

We denote by P_E the cut-off map $f \rightarrow 1_E f$, and by B_S the band-limiting projection

$$B_S(f) = \sum_{\lambda_j \in S} \widehat{f}(j) e_j = \sum_{\lambda_j \in S} \left(\int_M f(x) \overline{e_j(x)} d\mu(x) \right) e_j.$$

Proposition 1.7 follows immediately from the next two lemmas, and the assumption that $P_E B_S f \neq 0$. As per Remark 1.8, the statements and proofs only require that $\{e_j : j \in \mathbb{N}\}$ be a set of orthonormal functions in $L^2(M)$.

Lemma 2.1. *Suppose f is a L^2 -concentrated on E at level L and that \widehat{f} is ℓ^2 -concentrated on S at level L' . Then,*

$$(1 - \epsilon - \epsilon') \|f\|_{L^2} \leq \|P_E B_S f\|_{L^2}.$$

where $L = (1 - \epsilon^2)^{-\frac{1}{2}}$ and $L' = (1 - \epsilon'^2)^{-\frac{1}{2}}$ as per Definition 1.1.

Lemma 2.2. *With the hypothesis of the previous lemma, we have*

$$\|P_E B_S f\|_{L^2} \leq \left(\int_E \sum_{\lambda_j \in S} |e_j(x)|^2 dx \right)^{\frac{1}{2}} \|f\|_{L^2}$$

Proof of 2.1. Note

$$\|P_E f\|_{L^2} \leq \|f\|_{L^2} \quad \text{and} \quad \|B_S f\|_{L^2} \leq \|f\|_{L^2} \quad \text{for all } f \in L^2.$$

Hence,

$$\begin{aligned} 0 &\leq \|f\|_{L^2} - \|P_E B_S f\|_{L^2} \\ &\leq \|f - P_E B_S f\|_{L^2} \\ &\leq \|f - P_E f\|_{L^2} + \|P_E f - P_E B_S f\|_{L^2} \\ &\leq \|f - P_E f\|_{L^2} + \|f - B_S f\|_{L^2}. \end{aligned}$$

By hypothesis, f satisfies

$$\|f - P_E f\|_{L^2} \leq \epsilon \|f\|_{L^2} \quad \text{and} \quad \|f - B_S f\|_{L^2} \leq \epsilon' \|f\|_{L^2},$$

from which the lemma follows. □

Proof of Lemma 2.2. We have

$$\begin{aligned} \|P_E B_S f\|_{L^2}^2 &= \int \left| 1_E(x) \sum_{\lambda_j \in S} \widehat{f}(j) e_j(x) \right|^2 dx \\ &\leq \int_E \left(\sum_{\lambda_j \in S} |e_j(x)|^2 \right) \left(\sum_{\lambda_j \in S} |\widehat{f}(j)|^2 \right) dx \\ &\leq \left(\int_E \sum_{\lambda_j \in S} |e_j(x)|^2 dx \right) \|f\|_{L^2}^2. \end{aligned}$$

□

2.3. Proof of Corollary 1.9.

Proof. From (1.10), we have

$$\frac{1}{\#\{j : \lambda_j \in S\}} \sum_{\lambda_j \in S} \frac{1}{|E|} \int_E |e_j(x)|^2 = \frac{1}{|M|}.$$

The corollary follows immediately from Proposition 1.7. \square

2.4. Proof of Theorem 1.13. We shall need the following result due to Guedon, Mendelson, Pajor, and Tomczak-Jaegermann ([6]).

Theorem 2.3. *There exist two positive constants c and C such that for any even integer n and any uniformly bounded orthonormal system $(\phi_j)_{j=1}^n$ in L^2 with*

$$\|\phi_j\|_\infty \leq B \quad \text{for } 1 \leq j \leq n,$$

we can find an index subset $I \subset \{1, \dots, n\}$ with

$$\left| \#I - \frac{n}{2} \right| \leq c\sqrt{n}$$

such that for every $a = (a_i) \in \mathbb{C}^n$,

$$(2.4) \quad \left\| \sum_{i \in I} a_i \phi_i \right\|_{L^2} \leq CB \log n (\log \log n)^{5/2} \left\| \sum_{i \in I} a_i \phi_i \right\|_{L^1}$$

and

$$\left\| \sum_{i \notin I} a_i \phi_i \right\|_{L^2} \leq CB \log n (\log \log n)^{5/2} \left\| \sum_{i \notin I} a_i \phi_i \right\|_{L^1}.$$

Moreover, the proof of this result shows that it holds for a generic set I with the stated property.

To apply this result, consider the collection of eigenfunctions $\{e_1, \dots, e_n\}$, and apply Theorem 2.3 to this collection. Let

$$f(x) = \sum_{i \in I} a_i e_i.$$

Suppose that $f(x)$ is L^1 -concentrated in $E \subset M$ at level A . Then, using (2.4), we have

$$\begin{aligned} \|f\|_{L^2(E)} &\leq \|f\|_{L^2(M)} \\ &\leq CB \log n (\log \log n)^{5/2} \|f\|_{L^1(M)} \\ &\leq CBA \log n \cdot (\log \log n)^{5/2} \cdot \|f\|_{L^1(E)}. \end{aligned}$$

It follows that

$$|E|^{\frac{1}{2}} \left(\frac{1}{|E|} \int_E |f(x)|^2 dx \right)^{\frac{1}{2}} \leq CBA \cdot \log n \cdot (\log \log n)^{5/2} \cdot |E| \cdot \left(\frac{1}{|E|} \int_E |f(x)| dx \right).$$

The conclusion of Theorem 1.13 follows by collecting terms, since

$$\frac{1}{|E|} \int_E |f(x)| dx \leq \left(\frac{1}{|E|} \int_E |f(x)|^2 dx \right)^{\frac{1}{2}}$$

by Cauchy-Schwarz.

3. OPEN PROBLEMS

We conclude the paper with the following open problems:

Problem 3.1. One of the key questions left open in this paper is whether the eigenvalue size fundamentally affects the Fourier uncertainty principle on compact Riemannian manifolds. We have shown (Corollary 1.9) that if (M, g) be a compact, boundary-less Riemannian manifold satisfying the homogeneity condition (1.10), then if E is a measurable subset of M and S is a finite subset of the spectrum of $\sqrt{-\Delta_g}$, f is L^2 -concentrated in E at level L and \widehat{f} is L^2 -concentrated on X_S at level L' , then

$$(3.1) \quad (1 - \epsilon - \epsilon')^2 \leq \frac{|E|}{|M|} \cdot \#X_S,$$

where $L = (1 - \epsilon^2)^{-\frac{1}{2}}$ and $L' = (1 - \epsilon'^2)^{-\frac{1}{2}}$.

The question we ask is, does (3.1) hold for general compact Riemannian manifolds, with or without a boundary?

Problem 3.2. Our second question, highly related to the first, is whether there is a meaningful variant of Theorem 1.2 in the setting of compact Riemannian manifolds without a boundary. More precisely, let $X = \{e_1, e_2, \dots, e_n\}$ denote a set of n eigenfunctions of the Laplace-Beltrami operator Δ_g on a compact manifold M without a boundary.

A somewhat modest formulation of the question we have in mind is the following. Let S be a random subset of $\{1, 2, \dots, n\}$ of size $\approx \sqrt{n}$. Is it true that there exists a $q > 2$ and a constant K_p independent of n (and possibly depending on the size of S) such that for any function $f \in L^2(M)$ of the form $f = \sum_{i \in S} a_i e_i$,

$$(3.2) \quad \left\| \sum_{i \in S} a_i e_i \right\|_{L^q(M)} \leq K_p \left(\sum_{i \in S} |a_i|^2 \right)^{\frac{1}{2}},$$

with K_p independent of n ? Notice that the answer to this question is affirmative when $q = 2$. However, our question asks for the existence of $p > 2$ such that the “ q -orthogonality” relation (3.2) holds. A variant of this result is described in Theorem 1.13 above.

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