Dynamic Envy-Free Permanency in Child Welfare Systems

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Abstract

Caseworkers in foster care systems seek to place waiting children in the most suitable homes. Furthermore, social work guidelines prioritize heterogeneous attributes of children and homes when deliberating placements. We use insights from market design and dynamic matching to characterize a class of dynamically envy-free mechanisms that incentivize expedient placements when children and homes arrive to the market over time and homes may accept or decline placements. The mechanisms have robustness against justified envy and costly patience. We analyze strategic incentives and efficiency properties of dynamic envy-freeness. Finally, we conduct empirical simulations that affirm that our mechanisms drastically increase placements and reduce waiting costs while maintaining robustness to prediction error versus a naive mechanism that always sequentially runs Deferred Acceptance. Practitioners can implement our mechanisms through assigning priority to child-home matches. Journal of Economic Literature Classification: D47

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1 Introduction

The growing literature on dynamic matching theory references foster care and child adoption as a prime example of a dynamic matching market. Nevertheless, none have presented applicable mechanisms that can pair waiting children with adoptive homes. In this paper, we break the theory-application barrier. We find that existing tools from dynamic matching do not satisfy the necessary constraints for adoption from child welfare systems, and existing tools from the field fail to provide consistent, efficient outcomes. Our paper develops a fitting framework and accompanying stability notion when children and homes arrive to the system over time, the authority may unilaterally allow or disallow some matches, and homes can choose to accept or decline the authority's proposed match.

In 2022, about 109 thousand children were waiting for adoption. The median child had been waiting for over 29 months [U.S. Children's Bureau \(2024\).](#page-34-0) A systemic review of outcomes for children languishing in foster care aptly states, "outcomes of foster youth are troubling on all domains" [Gypen et al. \(2017\).](#page-33-0) Interwoven mechanisms drive longer waits for these children, but one clear flaw is a lack of consistent, institutional systems that prioritize efficient matching. At present, the process is very labor intensive, and only some counties responsible for waiting children use matching tools that could expedite the process. The matching tools that do exist are often simple spreadsheets or predictive measures that do not take into account systemic efficiency. [Hansen and Hansen \(2006\)](#page-33-1) note that in some cases, states have had oversupplies of willing adopters while simultaneously maintaining large populations of waiting children. Our current work does not have a complete analysis of the reasons behind adoption delays, but we plan to incorporate this with data partnerships with U.S. county authorities in the future.

Using these institutional drawbacks as motivation, we develop several matching mechanisms that practitioners can employ to better dynamically manage waiting children populations and create quicker, more efficient placements. In our framework, we model a dynamic matching market where children and homes arrive over time. As the matchmaker effectively has unilateral rights over placing children in homes, we primarily model the homes' strategic decisions to accept or decline proposed placements. Children and homes both have cardinal utility and waiting costs; these forces create dynamic tradeoffs between accepting a subpar match in the present versus waiting for the perfect match in the future. We focus on analyzing which classic properties our mechanisms satisfy: justified envy-freeness, non-wastefulness, individual rationality, and stability.

However, in addition, we introduce important new properties for matching mechanisms: dynamic envy-freeness and weak non-wastefulness. A matching mechanism is dynamically envy-free if it is justified envy-free in the matching market in every time period t, individually rational at every t , and if, in every period t , no home has dynamic incentives to reject any offered placement. We combine this with weak non-wastefulness which requires that when homes accept the matchmaker's placements, the mechanism is non-wasteful. These notions imply that the matchmaker can guarantee dynamic incentives for homes to accept placements by imposing penalties such as delays or less preferred placements when homes reject placements. Dynamic envy-free mechanisms are not unique, and one selection criteria that we highlight in our empirical work is selecting a dynamic envy-free mechanism that is least wasteful when homes do not accept the matchmaker's placements.

Mechanisms creating dynamic incentives for homes to accept earlier placements ap-

pear similar to policies implemented in a select few counties, showing that they may be feasible in practice. We also prove that dynamic envy-free mechanisms induce a weakly dominant strategy for homes to accept the first placement that the matchmaker offers (Proposition [1\)](#page-16-0). Unlike many other stability notions in dynamic matching literature, a dynamic envy-free mechanism always exists; the theorist and practitioner both gain from this (Corollary [1\)](#page-13-0). We design mechanisms that are dynamic envy-free and non-wasteful. They simultaneously properly incentivize homes to accept placements with sufficient data on homes' preferences. In the second half of our theoretical developments, we provide multiple results on the strategic incentives of our mechanisms. We find that not all are strategy-proof: homes might misrepresent their desire to accept certain placements. We devise a mechanism that is strategy-proof (Theorem [5\)](#page-22-0), although manipulable mechanisms may create more placements.

With data from a U.S. county, we plan to exploit knowledge about historic placements, homes' preferences, and matchmaker preferences' to simulate counterfactual placements. Currently, our simulations use synthetic data. We find that dynamic envy-free, nonwasteful mechanisms create at least 30% more placements and up to 49.48% more placements versus a mechanism that creates sequentially stable placements, even when the matchmaker has significant mean squared error, in bias and variance, in preference estimators. Our mechanisms likewise significantly decrease the costs that counties bear while caring for waiting children. Average per-month costs decrease by \$20,000 to \$50,000 for a medium-sized county with up to 240 children arriving per year. Because we compare against a stable, sequential placement mechanism, our gains are likely to be even greater compared to existing systems that have substantial inefficiencies. As our work develops, we will compare our counterfactual placements with observed placements.

Lastly, we provide insights on practically implementing our system within child welfare systems. Our simulations will include modifications to demonstrate the mechanisms' sensitivity to implementation as a match recommendation system wherein caseworkers carry out placements without caring for the externalities that imposes on the overall number of placements in the system. Match recommendation systems fail in comparison to centralization. We will discuss how an authority can implement centralization as a priority system that reserves certain homes for certain workers representing children over other workers and will prove in the Online Appendix that the system will enforce our mechanism's placements. The priority system preserves workers' autonomy and formal decision rights.

Theoretically, our model framework borrows prominently from [Baccara, Lee, and](#page-33-2) [Yariv \(2020\)](#page-33-2) and our approach to the solution concept from [Doval \(2022\).](#page-33-3) The former paper analyzes the optimal market thickness in two-sided dynamic matching problems. We add generality in agents' preferences and agents' arrival to the market to capture relevant aspects of child adoption from foster care. In practice, county authorities have little to no desire to wait for thicker market. Instead, they would prefer to expedite children's exits from the foster care system. Timely participation is exactly the focus of the latter paper. It departs from optimal dynamic matching to spotlight dynamic stability, a concept that ensures that both sides of the market follow the matchmaker's recommendation at the right times. However, dynamic stability is stringent, and, as Doval points out, whether or not an algorithm to compute dynamically stable matchings exists is an open question. Our setting allows us to use a more relaxed stability notion as one side of the market—the county authority—is a guaranteed participant and can block homes from achieving undesired placements. Nevertheless, dynamic envy-freeness comes with a wastefulness tradeoff. We measure efficiency as non-wastefulness, or utilizing all capacity for placements, and prove that stronger notions of non-wastefulness are incompatible with dynamic envy-freeness (Theorem [1\)](#page-9-0).

Institutional Details

In the United States, children that are victims of abuse or neglect enter into the local child welfare—alternatively, foster care—system. Typically, the local county manages these systems. Once a child is in the system, they enter the management of a caseworker. A worker's first priority is to reunify the child with the home of origin after rectifying the situation that warranted the child entering foster care. Failing this, many counties attempt to place the child with kin of the family of origin. Finally, if the county cannot reunify or place with kin, they must enter a legally laborious process to terminate parental rights (TPR) before placing the child with a permanent adoptive home. Federal law requires that workers pursue TPR if the child has been out-of-home for twelve consecutive months or fifteen of twenty two consecutive months.

After TPR, if a foster family has cared for the child, the worker typically attempts to place the child with that caregiver. Unfortunately, this may not always be possible. In these cases, the worker undergoes an intensive search for a new permanent home. This search process varies greatly across counties. With the proliferation of AI-assisting decision-making tools, some counties have taken up tools that automatically recommend matches. A worker might investigate several of these or simply approve the match after a cursory review.

Our work fits into this final step. Existing match tools do not use insights from matching theory. At best, they may recommend placements based on causally identified parameters. However, a child welfare system cannot maximize placements without considering that utilizing some homes for some children may necessitate not utilizing those same homes for other children. A combinatorics problem can drastically harm the number of placements made. Our work on implementing centralization when individual workers make final decisions is an attempt to address some of these issues.

Once a worker finds a suitable, willing home for a children, in most states, the home tentatively fosters the child for six months before the adoption can be finalized. The greatest threat in this stage is disruption. If a home decides the child is not the right fit, they can decline to adopt. Although formal research on disruption is lacking, workers, children, and families all anecdotally report that disruption is a traumatic, highly harmful event. For this reason, most workers' efforts are aimed toward finding matches that will not disrupt. Otherwise, at the end of the six months, the adoption is finalized and the family gains formal parental rights.

Literature Contribution

The literature on market design for child welfare systems is small. The intersection between matching market design and adoption is even smaller. The first notable work in this space is from [Baccara et al. \(2014\).](#page-33-4) They estimate preferences for private adoptions and find significant heterogeneity. They also provide empirical evidence for discrimination against boys and African American children put up for adoption. Parallels concerns exist in the foster care domain. Their policy recommendations allude to centralizing private adoptions, and we take a similar stance with respect to permanency for foster

care children. [Slaugh et al. \(2016\)](#page-34-1) take an operations approach to improve the Pennsylvania Adoption Exchange's matching process. The paper provides extensive descriptive and quantitative information about matching in foster care. They simulate counterfactual placements when utilizing a spreadsheet matching tool. Their results indicate that matching tools and combining geographic regions (i.e., inducing more centralization) substantially increase placements. Matching tools and technology that specifically guide caseworkers to the right home rather than announcing potential matchings to multiple homes also improve match quality and adoption prospects [\(Dierks, Slaugh, and](#page-33-5) Unver [2024\)](#page-33-5). This observation is key. When caseworkers announce children in need of permanency to a large pool of homes, organized and coordinated matching becomes nearly impossible. However, a caseworker-driven matching process can facilitate the centralization efforts we propose.

A few other works, including the aforementioned, pursue dynamic matching approaches to characterize foster care adoption. All of these do so from a decentralized perspective. [Mac Donald \(2023\)](#page-34-2) explicitly models a fostering process that can transition to adoption. Lower subsidies to foster parents after adopting children in their care distorts incentives away from adopting children with disabilities. Children with disabilities, in general, are much less likely to find non-institutional and permanent homes. Robinson-Cortés (2021) develops an empirical matching framework to estimate foster children outcomes under a policy that increases market thickness through delays and coarser regions. Compared to delaying, the decrease in match disruption rates is much larger from coarser regions.

A last group of papers study market design problems tangential to matching in foster care. [Dierks, Slaugh, and](#page-33-5) Unver (2024) collaborate with a match recommendation service to study families' misreporting behavior under different placement mechanisms in a static framework. They find that different mechanisms incorporating incentive compatibility come at the cost of placements for low needs and high needs children. [Altinok](#page-33-6) [and MacDonald \(2023\)](#page-33-6) design the optimal licensing structure when a regulator needs to screen families that may declare that they can care for needs beyond their capabilities. [Baron et al. \(2024\)](#page-33-7) use mechanism design and econometrics machinery to reallocate investigations among Child Protective Services investigators to reduce the number of children unnecessarily sent into foster care and appropriately place the children that are victims of abuse into foster care.

This paper creates a unique bridge between dynamic matching algorithms and child welfare market design to provide the technical foundation for centralized placements and better placement recommendations in child welfare systems. It may function as a crucial framework for future works studying matching in child welfare systems. Unsurprisingly, our main results show major benefits to centralization as papers like Robinson-Cortés [\(2021\)](#page-34-3) and [Slaugh et al. \(2016\)](#page-34-1) suggest. We offer more insights into why decentralized matching tends to lead to systemic inefficiency with theoretical and empirical analysis. Unlike any previous works, our main results show that it is possible to provide dynamic incentives for homes to accept placements earlier and sustain truth-telling as a weakly dominant strategy. Furthermore, we demonstrate our mechanisms' robustness to biased and inefficient preference estimators, offering evidence that dynamic envy-freeness is effective even when the matchmaker cannot perfectly discern homes' preferred placements. One can see our research as an argument for priority-based or centralized placements in child welfare systems and as an initial foray into developing tools that practitioners can use immediately.

Paper Organization

The rest of this paper proceeds as follows. In section two, we present our model. In section three, we present the main theoretical results on dynamic envy-free and nonwasteful mechanisms, their existence, and their theoretical performance. In section four, we explore our mechanisms' strategic incentives and offer a strategy-proof mechanism. In section five, we present our empirical simulations. Finally, we conclude in section six with applied insights and discussions on operating and implementing our mechanisms.

2 Model

We outline our model and key primitives below.

2.1 Primitives

Children arrive to the market (welfare system) over a finite time horizon T . A group arrives at every time $t \in \{1, 2, ..., T\}$ $t \in \{1, 2, ..., T\}$ $t \in \{1, 2, ..., T\}$ which we denote as $c(t)^{1}$ with an individual child as $c \in c(t)$. A group of homes also arrive at every time t which we denote as $h(t)$ with an individual home $h \in h(t)$. We think of every time period t as one month, but any arbitrary interpretation of time can fit our model.

A matchmaker would like to assign children to homes. A home h matching with child c receives utility $V_h(c)$. Conversely, a child receives utility $U_c(h)$ from the same match. In the context of child welfare systems, one should interpret a child's utility as the value of the match from the matchmaker's perspective. We do not make any assumptions about these utilities such as supermodularity or assortativity as is common in the literature. Our generality over the preferences and type space contributes to our model's validity as a useful empirical tool to set guidelines in child welfare decision-making. However, following [Baccara, Lee, and Yariv \(2020\),](#page-33-2) we include additive waiting costs in our model, where we assume that a common waiting cost $w_c > 0$ incurs for every child that remains unmatched after a time t . A child arriving at t has a time-dependent utility at time k equal to $U_c^k(h) = U_c(h) - (k - t)w_c$. Likewise, homes incur a common waiting cost w_h with time-dependent utility $V_h^k(c) = V_h(c) - (k - t)w_h$. Some homes might have negative utility for certain children (for example, a home that never wants to adopt a child older than 13). We say that a child c is **acceptable** to a home h if $V_h(c) \geq 0$, and a home h is acceptable to a child c if $U_c(h) \geq 0$.

Assumption 1. All children and homes have a strict preference ordering, that is, $\forall c$, there does not exist two h, h' with $h \neq h'$ such that $U_c(h) = U_c(h')$, and, $\forall h$, there does not exist two c, c' with $c \neq c'$ such that $V_h(c) = V_h(c')$.

Our model's formulation of stability will capture the fact that homes may accept or decline placements, but matchmakers have unilateral control over childrens' placements^{[2](#page-5-1)}. Voluntary participation only has bite for homes, and stability becomes a crucial feature

¹For the sake of tractability, we ignore sibling groups in our analysis. We make dynamic sibling group placements the goal of future work.

²Child welfare authorities and children are well-aligned. The authorities have enormous leeway in placement decisions when children are in foster care, and children, not having a firm idea of how to evaluate a home, most often follow the authorities' decisions. If a child prefers some homes over others, the authorities take this into consideration in their own preferences.

that encourages homes to follow a matchmaker's recommendation in a timely fashion. Formally, each home h decides to either accept or reject a placement at time t where $a_h^t \in \{0, 1\}$ represents this decision.

We assume that all homes have deterministic knowledge about future arrivals and their types. Designing an algorithm that encourages timely participation relies on this assumption. A home's option value of waiting is endogenously formed by both the expectation of future arrivals and what mechanism would offer. If we assume that homes have expectations over future arrivals, any mechanism should be responsive to these expectations. However, in reality, such expectations may not be known to the matchmaker. Assuming deterministic knowledge forces frees the mechanism we design to guarantee timely participation for homes under any possible realization of arrivals. This implies that our mechanisms would do the same when homes form rational expectations based on some prior. Therefore, matchmakers can use the algorithms that we design without having access to any information about homes' expectations. Restricting homes' knowledge to distributional information about future arrivals would relax the matchmaker's problem; in this sense, our assumption improves our work's robustness.

We also assume that the matchmaker has access to children and homes' utilities and waiting costs. The former is simple as a child's welfare from a match and costs are typically evaluated by the matchmaker. The latter is more complicated to elicit in practice, but matchmakers in some child welfare systems do estimate homes' preferences and costs. We show our results' responsiveness to estimator bias and variance in our empirical section.

2.2 Dynamic Matching

Matchmakers may assign any placements they like, but homes must accept them for the match to proceed. Define $a_h(t) \equiv (a_h^k)_{k\leq t}$ and $\mathbf{a}(t) \equiv (a_h(t))_{\forall h}$. We call $A(t)$ the action space at t where $a(t) \in A(t)$. At time t, children and homes matched in prior periods do not participate in the market. The set $C(t)$ contains all active children after arrival at t, and the set $H(t)$ contains all active homes after arrival at t. The market at time t is the undirected bipartite graph $M(t) = (C(t), H(t), E(t))$ where $E(t) \equiv \{\{h, c\} : c \in$ $C(t)$, $h \in H(t)$ is the possible edges (matches) at time t.

An admissible matching $\mu \subset E(t)$ at time t selects one-to-one matchings, i.e., on μ , each vertex in $C(t)$ and $H(t)$ only have one incident edge. We write that c and h are matched if $\{c, h\} \in \mu$. Alternatively, we denote this as $\mu(c) = h$ and $\mu(h) = c$. Say that $\mu(c) = c$ if c has no incident edge on μ and likewise for h. From hereon, we refer to an admissible matching as a matching.

A dynamic mechanism is a (deterministic) rule Q creating matchings at every t on the market at t which may depend on the history. Formally, $Q = (\mu_t)_{1 \leq t \leq T}$ where $\mu_t: (M(k))_{k\leq t}\times (\mu_k)_{k\leq t}\times A(t-1)\to E(t)$. Q and homes' actions endogenize the market at $1 < t \leq T$ so that, given the sets of children and homes that receive placements and accept them

$$
A^{C}(t|\mathbf{a}(t)) \equiv \{c : \mu_{k}(c) \neq c \text{ and } a_{\mu_{k}(c)}^{k} = 1 \text{ for some } k \leq t\}
$$

$$
A^{H}(t|\mathbf{a}(t)) \equiv \{h : \mu_{k}(h) \neq h \text{ and } a_{h}^{k} = 1 \text{ for some } k \leq t\}
$$

then:

$$
C(t|\mathbf{a}(t-1)) \equiv \bigcup_{k=1}^{t} c(t) - A^{C}(t-1|\mathbf{a}(t-1))
$$

$$
H(t|\mathbf{a}(t-1)) \equiv \bigcup_{k=1}^{t} h(t) - A^{H}(t-1|\mathbf{a}(t-1))
$$

A child remains in the market if and only if no home has adopted her up till the present, and a home remains if and only if it makes no adoptions until the present. A home receives payoff $V_h(\mu_t(h))$ if $h \in A^H(t|\mathbf{a}(t))$, and the matchmaker receives payoff $U_c(\mu_t(c))$ if $c \in A^C(t|({\bf a}(t)))$. Every period that $h \in H(t|{\bf a}(t-1)), h \notin A^H(t|{\bf a}(t))$ it incurs a waiting cost w_h . The matchmaker incurs the aggregate waiting costs for children:

$$
W^C \equiv \sum_{t=1}^T \sum_{c \in C(t|\mathbf{a}(t-1))} \mathbb{1}\{c \notin A^C(t|\mathbf{a}(t))\} w_c
$$

While our primary focus is the design of the dynamic mechanism, we specify the timing of the extensive form game induced on the mechanism below:

- 1. The matchmaker announces Q.
- 2. At period t, each home h receives a placement according to Q.
- 3. Each home h simultaneously decides to accept or decline the placement.
- 4. t ends.

This continues until $t = T$, at which point the game ends at (4).

2.3 Stability Criterion and Efficiency

Determining a definition of stability in dynamic markets stirs debates among economic theorists. Early attempts posited that blocking pairs can form inter-temporally, and a satisfactory dynamic matching mechanism must preclude the formation of present and future blocking pairs. [Doval \(2022\)](#page-33-3) counters, citing that dynamic cores induce absurd conclusions and are often empty. Instead, she suggests that a participant in the market reasons over the worst possible conjectures consistent with a dynamic stability notion; if the matchmaker's recommendation is superior to any outcome that could happen if the agent delays, the agent assents. We pursue a different course for our application for three reasons. First, experimental literature^{[3](#page-7-0)} shows that agents fail to play weakly dominant strategies in fairly plain environments. Dynamic stability a la recent developments requires advanced reasoning, therefore it is not clear ex-ante that it should achieve the same benefits of stability that algorithms like Deferred Acceptance (DA) enjoy in practice. Second, when the matching mechanism is transparent to participants, participants ought not posit conjectures that are inconsistent with the actual matchings that would realize under non-participation in a given time period. Third, as [Doval \(2022\)](#page-33-3) points out, the existence of an algorithm that can compute dynamically stable matchings is a highly challenging, open question in itself.

 3 See [Chou et al. \(2008\).](#page-33-8)

We take a middle route that, admittedly, does not aim to resolve issues in the current debate. Our motivation is developing a concept of stability fitting our applied setting that also provides useful insights for subsequent theoretical work. Two key observations are that, first, if a matchmaker can commit to a contingent matching that, in a very weak sense, is sequentially inefficient, then the matchmaker can motivate timely matches. Second, requiring that participant blocking pairs are consistent with possible counterfactual assignments of the mechanism limits deviations. The theorist, then, rejoices in a guarantee of the existence of "stable" assignments.

Our solution is dynamic envy-freeness. It relies on familiar properties in matching theory which we review below.

Definition 1. A matching μ , child c, and home h have justified envy if c and h have an assignment $(\mu(c) \neq c, \mu(h) \neq h)$, but $U_c(h) > U_c(\mu(c))$ and $V_h(c) > V_h(\mu(h))$.

Definition 2. A matching μ is justified envy-free on C and H if there does not exist any $c \in C$ and $h \in H$ with justified envy.

In standard static environments, justified envy-free is almost equivalent to stability. Its only difference is that it does not require capacity filling; i.e., if some home were not matched under μ_t , it cannot be subject to a child's envy and vice versa. We use justified envy-freeness to capture a limited sense of fairness. Conditional on whatever placements occur, there should not exist children and homes that could mutually improve on their existing partners. Our definition allows for some necessary waste that will guarantee the matchmaker's ability to induce timely participation without commitment to harmful or unfair placements.

Definition 3. A matching μ is **individually rational** on C and H if, for all $c \in C, h \in$ H, $U_c(\mu(c)) \geq 0$ and $V_h(\mu(h)) \geq 0$.

In any child welfare system, a matching that is not individually rational cannot be a good outcome as either a child suffers harm or a home must be forced to accommodate a child that it does not want.

Definition 4. Fix a mechanism Q and action profile $a(t)$. A counterfactual action profile $\hat{a}(t, h)$ is the profile where $a_h^k = 0$ for all $k < t$, $a_h^t = 1$, and otherwise actions are as in $\mathbf{a}(t)$. Counterfactual homes and children are thus $(C(k|\hat{a}(k-1,h)))_{k\leq t}$ and $(H(k|\hat{a}(k-1,h)))_{k\leq t}$. $(\hat{E}(k))_{k\leq t}$ are the spaces of admissible matchings on the counterfactual homes and children. The counterfactual matching for time t is the mapping $\hat{\mu}_t^h$ from $(\hat{M}(k, h))_{k \leq t} \times (\hat{\mu}_k^h)_{k < t} \times \hat{a}(t - 1, h)$ to $\hat{E}(t)$.

Definition 5. A mechanism Q is dynamic envy-free if

- 1. (Justified Envy-Free) For all $t, \mathbf{a}(t), \mu_t$ is justified envy-free.
- 2. (Patience-Free) For all t, h , and $\mathbf{a}(T)$, if $\mu_t(h) \neq h$, then $V^t_h(\mu_t(h)) \geq V^t_h$ $\hat{h}^{t'}(\hat{\mu}^h_{t'}(h))$ for all $t' > t$.
- 3. (Individually Rational) For all $t, \mathbf{a}(t), \mu_t$ is individually rational.

Condition (1) and (3) require properties to hold on- and off-path. We assume this because matchmakers may be unwilling to enforce desirable on-path placements through off-path harmful placements. In condition (2), patience embeds the idea that when a home

contemplates rejecting a placement, it does not "go outside" of the market to attempt to form a blocking pair. Rather, the home reasons based on what the mechanism would deliver it if it instead participates at a later time given its knowledge of the market's evolution^{[4](#page-9-1)}. It accepts the present match if it is better than the future time-discounted match. In another sense, under a patience-free mechanism, homes are always ex-post satisfied with their placements.

Definition 6. A dynamic mechanism Q is (weakly) **non-wasteful** if, given that $a_h^{t'} = 1$ for all t', h, for all t, there does not exist a $c \in C(t|\mathbf{a}(t))$ and $h \in H(t|\mathbf{a}(t))$ such that $U_c(h) \geq 0$, $V_h(c) \geq 0$, $\mu_t(c) = c$, and $\mu_t(h) = h$.

Definition 7. A dynamic mechanism Q is **strictly non-wasteful** if for all t and $a(t)$, there does not exist a $c \in C(t|\mathbf{a}(t))$ and $h \in H(t|\mathbf{a}(t))$ such that $U_c(h) \geq 0$, $V_h(c) \geq 0$, $\mu_t(c) = c$, and $\mu_t(h) = h$.

When a mechanism is non-wasteful^{[5](#page-9-2)}, it will always place all possible children and homes when homes accept the matchmaker's placements. It is an attractive property for matchmakers because they can place more children in permanent homes. Strict nonwastefulness must hold on- and off-path. A mechanism that is justified envy-free and strictly non-wasteful is, by definition, always stable on- and off-path. Non-wastefulness is a key property that we exploit to allow the matchmaker to refuse placements following homes' refusal to comply. This shapes homes' dynamic incentives toward accepting the matchmaker's offers.

3 Dynamic Envy-Free Mechanisms

In this section, we first offer key existence and non-existence proofs. Then, we explore important implications of dynamic envy-freeness that guarantee expedient, voluntary participation among homes. Finally, we highlight how a multiplicity of dynamic envyfree mechanism warrants further exploration, especially for discovering the least wasteful, dynamic envy-free mechanism.

3.1 Existence

Our primary measure of efficiency for a dynamically envy-free mechanism is wastefulness. In particular, we find that there is a significant tradeoff between dynamic envy-freeness and stronger non-wastefulness requirements.

Theorem 1. There does not exist a mechanism that is strictly non-wasteful and dynamic envy-free.

Proof. We demonstrate Theorem 1 via. counterexample. Consider an environment where $T = 2$, $h(1) = \{h_1\}$, $c(1) = \{c_1\}$, $h(2) = h_2$, and $c(2) = \{c_2\}$.

⁴Child welfare systems fit this description. As aforementioned, the few outside options for a home are very costly, and the matchmaker dictates the child's placement. When a home decides not to accept a placement, it is rarely, if ever, possible for that home to foster or adopt another child that the matchmaker does not consent to.

 $^{5}\mathrm{We}$ omit "weakly" except where necessary to ease exposition.

Players' preferences are:

$$
V_{h_1}(c_2) = 2 > V_{h_1}(c_1) = 1
$$

\n
$$
V_{h_2}(c_1) = 2 > V_{h_2}(c_2) = 1
$$

\n
$$
U_{c_1}(h_2) = 3/2 > U_{c_1}(h_1) = 1
$$

\n
$$
U_{c_2}(h_1) = 3/2 > U_{c_2}(h_2) = 1
$$

with $w_c = 2$ and $w_h = 1/2$. Suppose that Q is a strictly non-wasteful mechanism. For any a_h^1 , it must specify $\mu_1(h) = c_1$. Suppose that $a_h^1 = 0$, then $C(2|a_h^1) = \{c_1, c_2\}$ and $H(2|a_h^1) = \{h_1, h_2\}$. By strict non-wastefulness, $\mu_2(h_1) \in \{c_1, c_2\}$ and $\mu_2(h_2) \in$ ${c_1, c_2}$. If Q is dynamic envy-free, μ_2 must satisfy justified envy-freeness, hence $\mu_2(h_1)$ = c_2 and $\mu_2(h_2) = c_1$. Since every agent has positive utility for any match, this is also individually rational. However, Q must also satisfy patience-freeness. Since $\mu_2 = \hat{\mu}_2^{h_1}$, patience-freeness implies that $V_{h_1}^1(\mu_1(h)) = 1 \ge V_{h_1}^2(\mu_2(h)) = 2 - 1/2 = 3/2$. This is a contradiction. Therefore, Q cannot be both strictly non-wasteful and dynamic envyfree. \Box

Furthermore, a matchmaker would prefer to place h_1 with c_1 and h_2 with c_2 . These placements would yield payoff 2 to the matchmaker, whereas waiting until $t = 2$ to match h_1 with c_2 and h_2 with c_1 yields payoff 1 because of c_1 's waiting cost. The home's patience combined with strict non-wastefulness renders the matchmaker's first-best outcome impossible.

Capacity and Dynamic Incentives

The fact that strict non-wastefulness requires the matchmaker to fully utilize capacity after any history incapacitates the matchmaker's ability to provide dynamic incentives to homes. The Pennsylvania Adoption Exchange (PAE), a permanency matchmaker, realized a similar connection in the context of utilizing matching delays to induce truthtelling. In [Slaugh et al. \(2016\),](#page-34-1) the authors note that the exchange implemented a policy to wait thirty days between successive placements for the same home when the home rejected a placement. They believed that this might discipline homes away from overstating preferences for children they would not ultimately adopt, even though the matchmakers might, in practice, make less placements as a consequence. Later, we partially affirm their reasoning with formal results. Theorem 1 suggests that a similar policy could also induce homes to accept placements *sooner* even though it can create wastefulness of f-path.

Home Penalized Deferred Acceptance

Here, we introduce our first mechanism with PAE as our motivation. It is a simple modification of sequential Deferred Acceptance (DA) that incorporates previous placements and homes' decisions to reject those placements into present match decisions.

Home Penalized Deferred Acceptance

At each t, initialize each child c's and each home h's preferences as $\tilde{U}_c(\cdot) = U_c(\cdot)$ and $\tilde{V}_h(\cdot) = V_h(\cdot)$, respectively.

TRUNCATION

 $t > 1$: For each active h, if $\mu_{k'}(h) \neq h$ for some $k' < t$, then

- 1. Set k equal to the most recent k' such that $\mu_{k'}(h) \neq h$.
- 2. Set $p_h^t = 1$ if $\max_{c \in \{c': c' \in C(t | \mathbf{a}(t-1)), U_{c'}(h) \ge 0\}} V_h^t(c) > V_h^k(\mu_k(h))$ and $p_h^t = 0$ otherwise.

3. If $p_h^t = 1$, then set $\tilde{V}_h(c') = -1$ for all c' .

Proceed to MATCHING.

MATCHING

 $t \geq 1$: Using $\tilde{U}_c(\cdot)$ and $\tilde{V}_h(\cdot)$ as preferences

- 1. Each active home without a child holding its proposal proposes to its best, acceptable child that it has not yet proposed to, if it can propose to any.
- 2. Each active child holds her best, acceptable proposal and rejects all others.
- 3. Repeat 1-2 until no additional proposals are made.
- 4. Set $\mu_t(c)$ to be each child's held proposal; $\mu_t(c) = c$ otherwise. Set $\mu_t(h)$ to be the child holding a home's proposal; $\mu_t(h) = h$ otherwise.

We modify the counterexample from Theorem 1 to we illustrate how this mechanism would operate. We extend the horizon for this example to $T = 3$ and remove the arrival of h_2 , but everything else remains the same.

The black, solid line indicates that the matchmaker placed c_1 with h_1 in $t = 1$ and that $a_h^1 = 1$. In this case, the game concludes. Consider instead a case where $a_h^1 = 0$, and the black dashed line indicates that the placement was rejected.

The matchmaker does not place h with c_2 immediately. Why? h 's preferences are truncated at $t = 2$ because, as in the counterexample, $V_h^2(c_2) > V_h^1(c_1)$. Home Penalized Deferred Acceptance (HPDA) notes this and, knowing that allowing such a match would induce h to have dynamic incentives toward patience, instead penalizes h with a one-period delay. Since $V_h^3(c_2) = 1 = V_h^1(c_1)$, HPDA can place h with c_2 rather than penalizing at $t = 3$. Given that the alternative is no placement, h should accept with $a_h^3 = 1$. Thus, HPDA leaves h indifferent between accepting the first placement and waiting for a later placement. The next result shows that this insight applies to any environment.

Theorem 2. Home Penalized Deferred Acceptance is dynamic envy-free and non-wasteful.

Proof. See Appendix A.

HPDA attains dynamic envy-freeness—an ex-post property that must hold for any environment and home strategies—while only utilizing present information. The intuition behind this result is thus: regardless of what placements homes have accepted or rejected, the fact that a home h is active at time t indicates that h has received no placements or has rejected all previous placements. The matchmaker knows which case it is. In the latter, the matchmaker can delay h's placements. HPDA dynamically adjusts the relative payoff between past rejected placements and present placements.

Given that the matchmaker offers a placement at time t to a home h , that home would be in the market up to time t whether or not the home plans to accept or reject placements in the future. Therefore, given any profile $a(t)$, the market is the same up till t under a counterfactual $\hat{a}(t',h)$ for any $h, t' > t$. HPDA guarantees a home will never receive a (time-discounted) placement better than one rejected in the past. Therefore, the placement offer $\mu_t(h) = \hat{\mu}_t^h(h)$ is weakly better than all future placements under any profile $a(T)$. HPDA's non-wastefulness naturally follows. It never penalizes a home that complies with placements. If all homes are compliant, it will always use all available capacity.

Remark 1. HPDA is justified envy-free at every t, $a(t)$ even when allowing a child with no placement to have justified envy.

Proof. See Theorem [2](#page-12-0) in Appendix A.

HPDA's baseline guarantees that there is no child with justified envy toward a home that is not delayed. Any waste is necessary to dynamically incentivize homes to accept placements sooner. Aside from this, HPDA maximizes placements subject to stability constraints.

 \Box

 \Box

Authorities have multiple options for implementing HPDA. In the Online Appendix, we operationalize our framework with a computerized system that assigns priority according to placements that would occur under HPDA. If the system finds that HPDA would place a home and a child, then the worker representing the child has first claim to attempt a placement at that home over any other workers. If the worker chooses to not to pursue the placement or if the home declines the placement, then the system frees the home for a placement with a different child. Conditional on the algorithm accurately predicting workers' preferences for placements and workers utilizing the system, it always produces the same assignments as centralization. It also retains the advantage of worker autonomy and hedges against error in workers' preferences for child placements^{[6](#page-13-1)}. Many authorities already have placement recommendation systems, so another idea is to replace these systems' matching algorithms with HPDA's placements. We expect that our semi-centralization simulations in the empirical section will indicate that this operations strategy comes at a significant loss of efficiency.

Corollary 1. A dynamic envy-free and non-wasteful mechanism always exists.

Proof. This follows directly from HPDA's existence.

However, HPDA is not unique in the class of non-wasteful, dynamic envy-free mechanisms. Theorem 1 suggests that other mechanisms satisfying these properties must operate similarly to HPDA, imposing delays on non-compliant homes. Nevertheless, one can imagine very different truncation rules than specified in HPDA. We conjecture that the most efficient truncation rule should only exclude a home if it would actually receive a better match than a previous rejected match during the matching process. This rule may come at computational costs and could be overly sensitive to inaccurate preference estimates. HPDA's truncation rule uses the best hypothetical placement, meaning that it is simple and can effectively penalize even when the matchmaker has poor—even biased estimates. We provide examples of different truncation procedures in other mechanisms but leave the question of the least wasteful truncation rule open.

 \Box

3.2 Unfilled Capacity

Matchmakers in many counties face shortages of adoptive homes. HPDA excludes noncompliant homes from the market, further reducing the number of possible placements. Our example of HPDA's operation demonstrates this. At $t = 2$, the matchmaker desires a placement for c_1 or c_2 , but commitment to HPDA precludes making any placement for h. In this subsection, we address these concerns with Child Rotating Deferred Acceptance (CRDA).

Two main differences distinguish CRDA from HPDA. First, CRDA uses child-proposing DA. CRDA minimizes each home's utility while maintaining justified envy-free assignments because child-proposing DA produces the least optimal matching for homes [\(Knuth](#page-34-4) [1976\)](#page-34-4). Why? Minimizing homes' utility maximally avoids truncation, and, therefore, creates many placements even when homes deviate. Second, CRDA replaces—"rotates" children's and homes' placements when it detects any patience ex-post. Only children that could cause home justified envy or patience experience rotation. Ex-ante truncation

⁶When the algorithm does not align with workers' preferences, the assignments differ. We plan to explore estimation error in workers' preferences and worker bias in a separate paper.

Child Rotating Deferred Acceptance

MATCHING

 $t \geq 1$: Using $U_c(\cdot)$ and $V_h(\cdot)$ as preferences

- 1. Each active child without a home holding her proposal proposes to her best, acceptable home that she has not yet proposed to, if she can propose to any.
- 2. Each active home holds its best, acceptable proposal and rejects all others.
- 3. Repeat 1-2 until no additional proposals are made.
- 4. Set $\mu_t(h)$ to be each home's held proposal; $\mu_t(h) = h$ otherwise. Set $\mu_t(c)$ to be the home holding a child's proposal; $\mu_t(c) = c$ otherwise.

Proceed to TRUNCATION.

TRUNCATION

 $t > 1$: If $\mu_k(h) \neq h$ for some $h, k < t$ AND $V_h^t(\mu_t(h)) > V_h^k(\mu_k(h))$

- 1. For each active c, set $r_c^t = 1$ if $\max_{h \in \{h : h \in H(t), V_h(c) \geq 0\}} U_c(h) > U_c(\mu_t(c))$ or $V_{\mu_t(c)}^t(c) > V_{\mu_t(c)}^k(\mu_t(\mu_t(c)))$. $r_c^t = 0$ otherwise.
- 2. Set $\hat{R}_C^t \equiv \{c : r_c^t = 1 \text{ or } \mu_t(c) = c\}$ and $R_H^t \equiv \{h : \mu_t(h) \in \hat{R}_C^t \text{ or } \mu_t(h) = h\}.$
- 3. Set $R_C^t \equiv \{c : c \in \hat{R}_C^t \text{ and } V_h^t(c) \leq V_h^k(\mu_k(h)) \ \forall h \in \{h' : h' \in R_H^t, U_c(h') \geq 0\}\}\$

Proceed to ROTATION.

ROTATION

 $t > 1$: Using $U_c(\cdot)$ and $V_h(\cdot)$ as preferences, re-run MATCHING on R_C^t and R_H^t , and, for $c \in R_C^t$ and $h \in R_H^t$, update the old matching with the result. If $c \in \hat{R}_C^t$, $c \notin R_C^t$, set $\mu_t(c) = c$.

could eliminate desirable children from the market after any deviation by a home. Instead, with CRDA, desirable children—such as babies or low-needs children—can find a placement if the child avoids rotation, meaning that the child's placement is a very good fit. In general, CRDA does not outperform HPDA in wastefulness. However, CRDA satisfies a very useful property for counties that face shortages of homes. First, we state the main result for CRDA. Then, we show the aforementioned property.

Theorem 3. Child Rotating Deferred Acceptance is dynamic envy-free and non-wasteful.

Proof. See Appendix A.

The same steps in Theorem 2's proof guarantee that CRDA is patience-free and nonwasteful. CRDA also never generates justified envy. One must show this for the case when the blocking pair can arise across children and homes that CRDA matches during the MATCHING or ROTATION phases. By the usual properties of DA, there cannot be a blocking pair between a child and home that both find their placement in MATCHING

 \Box

or both find their placement in ROTATION. However, there may exist one between a child matched in MATCHING and a home matched in ROTATION.

We handle this case with rotation. If a child does not find her optimal match during MATCHING, that child must rotate. Otherwise, post-rotation, a home could find itself worse off from rotation. Then, a home might prefer a child that did not rotate and that the home rejected during MATCHING. If this child also prefers the home, then justified envy exists. The aforementioned technical step for rotation averts this. The opposite case is distinct. A home matched in MATCHING and a child matched in ROTATION might block CRDA. Our proof demonstrates that this is impossible. After TRUNCATION, there are fewer children to compete for homes. One imagines that children must improve their placements through rotation, if the child is not truncated. We discover a contradiction when any child does not weakly improve on her match through rotating. Every child must weakly improve on their previous placements. Theorem 3 follows from this logic.

Remark 2. CRDA is justified envy-free at every t , $a(t)$ even when allowing a home with no placement to have justified envy.

Proof. See Theorem [3](#page-14-0) in Appendix A.

Like HPDA, the purpose of CRDA's waste is to dynamically incentivize homes to accept earlier placements. It offers a placement to a home when possible.

 \Box

Remark 3. Child Rotating Deferred Acceptance is strictly non-wasteful if, for all t , $a(t)$, there exists an *h*-perfect stable matching on R_H^t and R_C^t .

An h-perfect matching μ on R_H^t and R_C^t must span every vertex in R_H^t , i.e., $\mu(h) \neq$ $h \forall h \in R_H^t$. The condition does not assert a tautology: that CRDA must construct this matching. Rather, the mere existence of this matching, if it is stable, is sufficient for CRDA's strict non-wastefulness. Unfortunately, finding a tighter condition that guarantees strict non-wastefulness is quite difficult. Depending on preferences and the arrival structure, even one non-compliant home can force CRDA to truncate numerous children. Still, R_C^t includes many unmatched children when there is a large supply of children relative to homes. A home shortage likely secures Remark 1's condition. Alternatively, when children have preference heterogeneity, more children win their optimal placements. Very few homes would experience rotation, so unfilled capacity would be unlikely.

In contrast, HPDA is never strictly non-wasteful if there are more children than homes at some t and any one home that could receive a placement better than a previously rejected placement is non-compliant. HPDA must penalize the home with a delay, thus it cannot receive a placement. In this way, CRDA may dynamically create more placements than HPDA and non-dynamic envy free mechanisms. Additionally, the loss in home welfare may not be consequential when homes either find a child acceptable or not acceptable, which holds for many homes adopting from foster care. In this case, child-proposing DA results in similar placements to home-proposing DA.

3.3 Participation

In our interviews with a U.S. county, workers representing children express serious dismay over managing caseloads where it seems impossible to find the right home to accept a placement for a child, especially when the child has serious needs or disabilities. Currently, our work does not present empirical evidence on the amount of time that homes may wait for a placement. Anecdotally, our interviewees report that there is significant variation in homes' waiting times and that many homes do wait in hopes of achieving an ideal placement. We are in discussions with our partner county to collect data on homes' waiting times and to survey homes on anecdotal reasons for delaying placement decisions. Nevertheless, our theoretical result below shows that when a matchmaker implements a dynamic envy-free mechanism, homes do not have an incentive to reject placements.

Proposition 1. Accepting the first placement is a weakly dominant strategy for all homes under a patience-free mechanism.

 \Box

Proof. See Appendix A.

The definition of dynamic envy-free incorporates patience-freeness and individual rationality, so this result implies that a home always wants to participate in the mechanism and always weakly prefers the first placement and its timing. This strategy's weak dominance arises from the fact that any strategy where a home does not accept a placement is equivalent to a counterfactual strategy where the home only accepts at some time t ′ . Patience-freeness asserts that any placement the mechanism offers is better than a counterfactual placement, and this yields Proposition 1.

One implicit assumption of our model is critical to participation. The matchmaker must be able to limit homes' dynamic incentives through controlling the placement offers that a home receives. This assumption is only violated when non-local authorities include the local pool of homes in their assignments. Non-local authorities might not collaborate with the local authority to impose the necessary delays on homes; homes then receive "outside options" in the future that may induce them to wait rather than accepting the local authority's first placement. Unfortunately, because of the great lack of uniformity in adoption procedures across the United States, we cannot confirm which counties this assumption may hold for.

In most child welfare systems, homes register with a provider agency responsible for representing the home to the local county. Our partner county writes contracts with the local provider agencies that specify a number of placements for children that the providers must guarantee to the county, meaning that the provider must reserve a certain number of homes for the local county. Our partner also does not pursue placements out-of-county unless the prospective home is kin to the child. These features imply that the matchmaker possesses power over the placement offers for at least some homes residing in the local county.

As we develop our empirical work, we will analyze where placement offers originate and which offers homes tend to accept. Even if homes occasionally receive outside options such as from out-of-state counties, we conjecture that dynamic envy-free mechanisms should maintain their efficacy as long as these options are infrequent or unattractive relative to the local authorities' placement offers. We offer evidence for this conjecture in our semi-centralization simulations. We split children into separate region and only allow delays for the region wherein a home rejects a placement. In the other regions, homes cannot be delayed as non-local authorities do not coordinate.

4 Strategic Incentives

In this section, we explore homes' strategic incentives under several mechanisms. We first propose an augmented version of strategy-proofness that considers limited misreports and give rationales. Next, we construct benchmark strategic properties using sequential home-proposing DA. Last, we give strategic properties for HPDA, CRDA, and a third, new mechanism: Home Endowed Deferred Acceptance (HEDA).

4.1 Manipulability

What does it mean for a home to *manipulate* a placement mechanism? Practitioners report at least one major concern: some homes report a desire to adopt a child that they cannot care for or do not actually want to adopt [\(Dierks, Slaugh, and](#page-33-5) Unver 2024). Homes may do so to increase their visibility as workers search for suitable homes or because they naively overestimate their own ability when there are no incentives for truth-telling [\(Slaugh et al. 2016\)](#page-34-1). On the other hand, homes that report unwillingness to adopt any child that does not meet stringent characteristics burden many counties. In practice, these homes wait to accept placements for children that do not ultimately meet their original preference reports. In our talks with our partner county, workers say that homes often broaden their preferences after experience with the system, suggesting that homes could be attempting to manipulate matchmaking through truncating their preferences. Another plausible explanation is that homes simply have imperfect information and learn over time. We believe that both mechanisms may be at work, and eliminating manipulability has promising benefits.

Keeping in trend with the overall variation in child welfare systems, the process to report and use preferences in matchmaking between children and homes is non-standardized across the United States. In the state where our partner county resides, the statewide adoption match recommendation service requests that homes seeking a match fill a form that indicates preferences over childrens' characteristics. For example, a home can indicate that a child with severe allergies is either possibly acceptable or not acceptable. If a home indicates a characteristic is unacceptable, the county will not offer the home a placement for a child with that characteristic. Otherwise, the county might estimate measures correlated with preferences for a specific child that is acceptable, such as regressing ex-post adoption satisfaction on home and child characteristics. Then, the county uses this data as a proxy for home utility. We show that our mechanisms operate very well under this strategy in the next section.

We assume that a home reports acceptability for each child available. If strategyproofness holds in this case, it will hold when homes can only make coarser reports over acceptable or unacceptable characteristics. Formally, a home's preference report is a strategy $\forall h, c$:

$$
\sigma_h(c) = \begin{cases} 1 & \text{if } c \text{ is acceptable to } h \\ 0 & \text{otherwise} \end{cases}
$$

Denote $\sigma_h \equiv (\sigma_h(c))_{\forall c}, \sigma \equiv (\sigma_h)_{\forall h}, \sigma_{-h} \equiv (\sigma_{h'})_{\forall h' \neq h}$, and $a_{-h}(t) \equiv (a_{h'}(t))_{\forall h' \neq h}$. The matchmaker observes h's true utility for c if h reports that c is acceptable. Otherwise, the matchmaker can only observe that c is unacceptable to h . Preserving continuity in notation, we denote $V_h(c)$ as the utility that the matchmaker observes and $\hat{V}_h(c)$ as h's true utility for c. Then, we have:

EDUCATION				
Characteristic:	Acceptable	Will Consider	Unacceptable	
12. High achiever				
Achieves on grade level in regular classes 13.				
Achieves below grade level in regular classes 14.				
Needs special education classes 15.				
16. Needs learning disability classes (LD)				
Needs classes for the emotionally or behaviorally handicapped 17.				
18. Needs tutoring in one or more subjects				
19. Has serious behavior problems at school				
CHARACTERISTICS AND BEHAVIORS				
Characteristic:	Acceptable	Will Consider	Unacceptable	
20. Generally quiet and shy				
21. Generally outgoing and noisy				
Emotional issues require ongoing therapy 22.				
Tends to reject father figures 23.				
24. Tends to reject mother figures				
Difficulty making friends and relating to other children 25.				
Frequently wets the bed 26.				
Frequently wets during the day 27.				
28. Frequently soils him/herself				
29. Masturbates frequently or openly				
30. Poor social skills				
31. Problem with lying				
32. Problem with stealing				
Frequently starts physical fights with other children 33.				
Tends to abuse animals 34.				
Tends to be destructive of clothing, toys, etc. 35.				
Frequently uses foul or bad language 36.				
Frequent temper tantrums 37.				
38. Difficulty accepting and obeying rules				
39. History of inappropriate sexual behavior				
40. History of running away				
41. History of playing with matches, setting fires				

Figure 1: An example of a preference form. We treat the "Will Consider" category as "possibly acceptable".

$$
V_h(c) = \begin{cases} \hat{V}_h(c) & \text{if } \sigma_h(c) = 1\\ -1 & \text{otherwise} \end{cases}
$$

Assumption 2. No home ever accepts a placement for an unacceptable child.

We did not need to assume this before as a matchmaker would never offer an unacceptable placement. However, Assumption [2](#page-18-0) is necessary for strategic analysis. Since homes have additive waiting costs, they might calculate that accepting an unacceptable child today is better than waiting for an acceptable child tomorrow. This would never happen. Rather than incur further waiting costs, a home would leave the child welfare system. Assumption [2](#page-18-0) accommodates this outside option.

Definition 8. A report σ_h is truthful if and only if $\sigma_h(c) = 1$ when $\hat{V}_h(c) \geq 0$ and $\sigma_h(c) = 0$ when $\hat{V}_h(c) < 0$.

Definition 9. Fix a mechanism Q, action profile $a(T)$, and reports σ . Denote t_h as the period where $h \in A^H(t_h|\mathbf{a}(t_h), \sigma)$ and $t_h \equiv T$ if no such period exists. For a home h arriving at k, h's realized utility is $\hat{V}_h^r(\mathbf{a}(T), \sigma) = \hat{V}_h(\mu_{t_h}(h)) - (t_h - k)w_h$.

Definition 10. An action profile and report pair $(a_h(T), \sigma_h)$ is weakly dominant for h under a mechanism Q if $\hat{V}_h^r(a_h(T), a_{-h}(T), \sigma_h, \sigma_{-h}) \geq \hat{V}_h^r(a_h'(T), a_{-h}(T), \sigma_h', \sigma_{-h})$ for all $a'_h(T), a_{-h}(T), \sigma'_h, \sigma_{-h}$

Definition 11. A mechanism Q is strategy-proof if, for all h, $(a_h(T), \sigma_h)$ is weakly dominant, where σ_h is truthful and $a_h^t = 1 \ \forall t$.

Strategy-proofness in child welfare systems is multidimensional. Matchmakers want all homes want to report that acceptable children are acceptable and unacceptable children are unacceptable. Avoiding placing a child with a home that would not accept the child saves valuable time and resources, and correctly discerning what children are acceptable to homes assists in matching, recruitment, and planning. Simultaneously, the matchmaker would like homes to comply with acceptable placements. Strategy-proofness reframes and encapsulates patience-freeness for strategic analysis. Our modeling decisions follow common practice in multidimensional mechanism design to require incentivecompatibility for compliance in all dimensions. Although, because a home's action profile does not reflect private information, many results are likely similar if one reduces to strategy-proofness only in reports.

We change the timing of the game as follows:

- 1. The matchmaker announces Q.
- 2. Each home h reports $\sigma_h(c)$ for all c.
- 3. At period t, each home h receives a placement according to Q and reported preferences.
- 4. Each home h simultaneously decides to accept or decline the placement.
- 5. t ends.

This continues until $t = T$, at which point the game ends at (5).

4.2 Benchmark

In our empirical section, we plan to perform two comparisons. The first is dynamic envy-free placements contra observed placements. We do not yet have data on observed placements from our partner county. The second is dynamic envy-free placements contra sequentially stable placements. We currently focus on this comparison. Here, we introduce sequential Deferred Acceptance (DA) which is a spot placement mechanism that indiscriminately runs home-proposing or child-proposing DA. In our main specifications, we use home-proposing DA.

The proposition below offers insight on theoretical differences between sequential DA's properties and our dynamic envy-free mechanisms.

Proposition 2. Sequential Deferred Acceptance is justified envy-free, individually rational, and strictly non-wasteful, but it is not patience-free nor strategy-proof.

Proof. Sequential DA is invariant to $a(T)$ and the histories, and it is stable at every t, therefore it is justified envy-free, individually rational, and strictly non-wasteful at every $t, \mathbf{a}(T)$. Since it is strictly non-wasteful, by Theorem 1, it cannot be dynamic envy-free.

Sequential Deferred Acceptance

- $t \geq 1$: Using $U_c(\cdot)$ and $V_h(\cdot)$ as preferences
	- 1. Each active home without a child holding its proposal proposes to its best, acceptable child that it has not yet proposed to, if it can propose to any.
	- 2. Each active child holds her best, acceptable proposal and rejects all others.
	- 3. Repeat 1-2 until no additional proposals are made.
	- 4. Set $\mu_t(c)$ to be each child's held proposal; $\mu_t(c) = c$ otherwise. Set $\mu_t(h)$ to be the child holding a home's proposal; $\mu_t(h) = h$ otherwise.

Yet, it satisfies justified envy-freeness and individual rationality, so it must be that it is not patience-free. Strict non-wastefulness implies that sequential DA is non-wasteful. Then, by Theorem 4 below, it cannot be strategy-proof. \Box

Our simulations in Section 5 show that dynamic envy-free violations trigger severe delays for waiting children in child welfare systems. Its strict non-wastefulness does not appear to outweigh the additional placements and reductions in waiting costs that patience-freeness achieves. Sequential DA is not strategy-proof under our definition. Our proof for Theorem [4](#page-20-0) shows that a home that plans to accept the matchmaker's placements can, in some circumstances, increase its utility from a misreport. A home that expects very good placements in the future but does not want to reject prospective adoptive children can declare unacceptability for large swathes of children to avoid less desirable placements. Moreover, this alters the matchmaker's consideration set for some placements and may even, from the home's perspective, save the matchmaker from unnecessary effort costs. On the contrary, if the home truthfully reports, then the matchmaker might offer placements that the home would find optimal to reject. This blocks the rejected child from other placements that might be accepted in that same time period. Therefore, preference misreports and rejections have meaningful outcome differences under sequential DA, and it is not clear that there is a strong incentive for homes to report truthfully under sequential DA.

4.3 Dynamic Envy-Free and Strategy-Proof Mechanisms

We turn our attention toward examining the strategic properties of HPDA, CRDA, and a new mechanism. Our first result dispenses immediate conclusions for HPDA and CRDA.

Theorem 4. There does not exist a justified envy-free, non-wasteful and strategy-proof mechanism.

Proof. We show this via. counterexample using the environment for HPDA's operation where $T = 2$, $h(1) = \{h\}$, $c(1) = \{c_1\}$, $h(2) = \emptyset$, and $c(2) = \{c_2\}$.

Players' preferences are:

$$
\hat{V}_h(c_2) = 2 > \hat{V}_h(c_1) = 1
$$

 $U_{c_1}(h) = 3/2$
 $U_{c_2}(h) = 2$

with $w_c = 1$ and $w_h = 1/2$. Suppose that Q is a non-wasteful mechanism. Consider the action profile $a_h = (1, 1)$ and strategy profile $\sigma'_h(c_1) = 0, \sigma'_h(c_2) = 1$. Q must specify $\mu_1(h) = \emptyset$ by individual rationality. Thus, $C(2|a_h^1) = \{c_1, c_2\}$ and $H(2|a_h^1) = \{h\}$. By non-wastefulness, $\mu_2(h) \in \{c_1, c_2\}$. By justified envy-freeness, $\mu_2(h) = c_2$. h exits with payoff $2 - 1/2 = 3/2$. Now, instead, suppose that h reports truthfully under the same action profile, i.e., some report $\sigma_h(c_1) = \sigma_h(c_2) = 1$. Non-wastefulness implies that $\mu_1(h) = c_1$ and h exits with payoff 1. Thus, we have that for $a_h, \sigma_h, \sigma'_h, \hat{V}_h^r(a_h, \sigma'_h)$ $3/2 > \hat{V}_h^r(a_h, \sigma_h) = 1.$ \Box

Strategy-proofness demands that truthful reporting is weakly optimal for homes when homes plan to accept placements. In the above example, a home that plans to accept the matchmaker's placements benefits from misreporting and violates strategy-proofness. The home could achieve the higher payoff with a truthful report and rejection at $t = 1$. Nevertheless, this equivalence does not eliminate rationale for misreporting. Theorem [4](#page-20-0) uses a simple environment where the economic theorist can find the equivalence. In more complex environments with more agents and a longer horizon, as we noted in our discussion about sequential DA, misreports and rejections can produce different outcomes. Not even the theorist might find the equivalence, suggesting scant hopes of homes finding the equivalence. Additionally, some homes might innocuously or even unconsciously err toward reporting that acceptable children are unacceptable given sufficient patience. We suggest that preference truncations could be a potential mechanism explaining child welfare systems' perpetual lack of homes to care for older children and children with higher needs, and strategy-proof mechanisms—under our strong definition—could be a solution.

Corollary 2. Home Penalized Deferred Acceptance and Child Rotating Deferred Acceptance are not strategy-proof.

Proof. Both are justified envy-free and non-wasteful. By Theorem [4,](#page-20-0) neither are strategyproof. \Box

The key manipulation that both exploit is the same as in the above counterexample, but homes have even stronger incentives to misreport. Since HPDA and CRDA are dynamically envy-free, they cannot allow h to match with c_2 if h rejects c_1 at $t = 1$. Hence, h can only match with c_2 if it reports that c_1 is unacceptable.

Home Endowed Deferred Acceptance

ENDOWMENT

INITIALIZE: Fix some arbitrary maximum and minimum utilities \overline{V} and \underline{V} . Fix an arbitrary set of N disjoint, compact intervals $(E_i)_{0 \leq i \leq N}$ spanning $[V, \overline{V}]$ where the minimal element in E_i is greater than the maximal element in E_{i+1} . Home h arriving in period k has a time-dependent endowment $B_t(h) \equiv E_{t-k}$ if $t - k \leq N$ and $B_t(h) \equiv \{V\}$ } otherwise.

 $t \geq 1$: Initialize $\tilde{V}_h(\cdot) = V_h(\cdot)$, then

- 1. Set $e_h^t(c) = 1$ if $V_h^t(c) \in B_t(h)$ and $V_h(c) \geq 0$. $e_h^t(c) = 0$ otherwise.
- 2. If $e_h^t(c) = 0$, then set $\tilde{V}_h(c) = -1$.

MATCHING

 $t \geq 1$: Using $U_c(\cdot)$ and $\tilde{V}_h(\cdot)$ as preferences

- 1. Each active home without a child holding its proposal proposes to its best, acceptable child that it has not yet proposed to, if it can propose to any.
- 2. Each active child holds her best, acceptable proposal and rejects all others.
- 3. Repeat 1-2 until no additional proposals are made.
- 4. Set $\mu_t(c)$ to be each child's held proposal; $\mu_t(c) = c$ otherwise. Set $\mu_t(h)$ to be the child holding a home's proposal; $\mu_t(h) = h$ otherwise.

These results temper overly optimistic expectations of any non-wasteful mechanism's performance and motivate a search for a mechanism that is dynamically envy-free, strategyproof, and, at least in some sense, limited in wastefulness. We develop this, partially, in our new mechanism. Home Endowed Deferred Acceptance (HEDA) fixes a range of utility that a home can receive at any time period relative to its entry into the child welfare system; i.e., each home receives an endowment at each time period. Following this, homes receive their best justified envy-free placement within each endowment. We show that HEDA is strategy-proof, but it is not quite dynamically envy-free. We hypothesize that it is possible to derive a dynamically envy-free and strategy-proof mechanism, but it would exhibit serious deficiencies in wastefulness that defeat the purpose of fairness. We test the performance of HEDA in Section 5.

Theorem 5. Home Endowed Deferred Acceptance is patience-free, individually rational, and strategy-proof.

Proof. See Appendix A.

The algorithm's success rests on the endowment's invariance to homes' strategies. Because a home always receives its best placement at the earliest period relative to its arrival, a home's optimal strategy is to accept the earliest placement. HEDA's strategyproofness is non-trivial. In the first period where a home's misreport would cause any

 \Box

match for any home to differ compared to when the home is truthful, we prove that it must be that the misreporting home receives the same match under misreporting/truth-telling or else the misreporting home's match differs. In the former case, the home should accept the placement by the same logic that we use to show HEDA is patience-free; waiting until later yields a worse placement. Misreporting cannot increase the home's utility since the first placement is the same as under truth-telling. The latter case reduces to demonstrating that HEDA is strategy-proof in a static sense; we show this using a novel, amended proof strategy based on [Roth \(2017\).](#page-34-5) Misreporting cannot improve the home's utility even when the home's match changes. Tying the two cases together yields the theorem.

HEDA is not justified envy-free because it partitions homes' potential placements using endowments that are ex-ante fixed. Therefore, it is possible for a home and child to mutually benefit from an endowment violation. Moreover, HEDA is not non-wasteful for the same reason. Even a compliant home might not receive a placement at t if no children exist within its endowment, and some other acceptable children outside of that home's endowment might not be placed elsewhere. These two features are the primary downsides of HEDA. Still, HEDA has many attractive properties. Homes that do not receive placements eventually disperse into separate endowments and, when children outnumber homes, it is unlikely that a home would not find at least one acceptable match. Since HEDA is patience-free, whenever a home receives a placement, it will accept that placement, ameliorating some wastefulness concerns. Finally, strategy-proofness incentivizes homes to carefully and truthfully report preferences. It eliminates *false negatives*: homes that would adopt a child yet report the child as unacceptable. If false negatives are prevalent in child welfare systems, as we conjecture, then HEDA advances as a forefront solution that increases the number of home willing to adopt children that might otherwise experience long waits.

Theorem [4](#page-20-0) cautions against pursuing a dynamic envy-free and strategy-proof mechanism. Dynamic envy-freeness and non-wastefulness imply that any unfilled capacity exists to penalize noncompliant homes. In contrast, a dynamic envy-free mechanism without the binding restrictions of non-wastefulness—as a dynamic envy-free, strategyproof mechanism must be—may be arbitrarily unfair. Consider this possible direction to extend HEDA to dynamic envy-freeness: at every period after running DA, dissolve placements that cause justified envy for any home. One cannot rearrange the placements to match the home and child with justified envy if the home would require a higher endowment; this would violate patience-freeness. Therefore, eliminating the placement is necessary, but surely this is less fair than allowing the placement to persist.

Yet, HEDA's structure is amenable to one useful alteration. We have assumed that the matchmaker has access to a reliable estimate for w_h —and we will explore unreliable estimates in the next section—but this may not hold for every county. We propose HEDA^{*} when w_h is unknown: it sets a home's endowment $e_h^t(c) = 1$ if $V_h(c) \in B_t(h)$ rather than relying on the home's time-discounted utility. HEDA* does not differ theo-retically from HEDA so Theorem [5](#page-22-0) still applies^{[7](#page-23-0)}. HEDA is generally better than HEDA^{*} when waiting costs are known. HEDA^{*} has a limited number of endowments before a home must receive a zero utility placement which harms the number of placements that homes in the bottom endowment can receive. HEDA can have an arbitrarily large number of endowments so that a home's utility for a placement can always be in any arbitrary

⁷HEDA is equivalent to running HEDA* with the same endowment interval bounds shifted upward by $(t - k)w_h$.

interval, and a home's chances of receiving a placement are much greater.

5 Empirical Evidence

In this section, we describe our current and future empirical simulations. Our theoretic work confirms properties of the mechanisms we examine, but it does not speak to these properties' empirical performance. At present, we generate synthetic data representing a child welfare system, its participants, and their preferences. We then run simulations to benchmark each mechanisms' placement count, waiting costs for children, and number of homes with justified envy against sequential DA. However, our strong assumptions about the matchmaker's knowledge of homes' preferences and waiting costs spur a simple concern: what if the matchmaker cannot estimate these parameters? We run additional simulations to evaluate our mechanisms when the matchmaker's preference estimator has substantial root mean squared error (RMSE) from bias and variance.

Our collaborating U.S. county is processing our requests for data. Once data is granted, we will use it to construct the true, observed market and compute child preferences. Still, data for home preferences is lacking. We will complement the county data with surveys of adoptive homes to understand how the match-specific interaction between child and home behavioral characteristics affect homes' preferences. We will combine observable child arrivals with simulated home arrivals based on known distributions of preferences from the survey data geographically restricted to the county. We fix the number of home arrivals from the distribution observed in county data. Then, we plan to benchmark our mechanisms against sequential DA *and* observed placements, observed waiting costs for children, and observed homes with justified envy. Finally, we will devise two key extensions. One, we will test HPDA, CRDA, and HEDA's sensitivity to heterogeneous home waiting costs; two, we will test the effects of semi-centralization when workers may imperfectly coordinate on the ideal centralized assignment.

In what follows, we describe our synthetic data and preferences, then we present our preliminary main results.

5.1 Data

We simulate two years $(T = 24)$ of children and homes arriving to a child welfare systems. Our results focus on the first year because incentives for patience mechanically decrease as t approaches T . This is a technical limitation which we plan to overcome with more computing power and simulating a longer horizon. The number of children arriving per month is N_t^H and the number of homes arriving per month is N_t^H where $N_c^C \sim$ $U[15, 20]$ $\forall t$ and $N_t^H \sim U[12, 15]$ $\forall t$, reflecting common shortages of homes across child welfare systems. Each child c has a bundle of characteristics $\theta(c) = \{age(c), needs(c)\}\,$ where $age(c) \in [0, 18]$ and $needs(c) \in \{0, 1\}$. Each home h has a preference survey $P(h) = {need s(h)}, need s(h) \in {0, 1}$ indicating whether or not h reports a willingness to care for a child with high needs. Following trends in our partner county, we assume that $age(c) \sim N(8, 4)$ and $Pr(needs(c) = 1) = \frac{1}{3}$ $\forall c$. However, there is also a shortage of homes willing to care for high-needs children, and we model this through assuming $Pr(needs(h) = 1) = \frac{1}{5} \forall h.$

5.2 Preferences

The main goal for most child welfare systems is to guarantee a safe, healthy permanent placement for all children. One threat to permanency is disruption. After the matchmaker places a child in a home, the home becomes the child's pre-adoptive home. The home must foster the child for six months before the adoption is finalized. If the home decides not to adopt the child, it returns the child to the welfare system, i.e., the adoption disrupts. In our model, a safe and healthy placement is one where the child is acceptable to the home and the home is acceptable to the child. The matchmaker can define the acceptability threshold to include only homes that would adequately care for the child. Conditional on a placement being acceptable, the matchmaker's utility for a child c matching with a home h, $U_c(h)$, is the predicted non-disruption probability. Our partner county's data will allow us to predict non-disruption on characteristics observable to caseworkers. Our interviews with caseworkers indicate that they believe disruption is more or less likely depending on the fitness of the child-home match. We model this in our synthetic data using a match-specific parameter:

$$
U_c(h) = 1 - \epsilon_{c,h}
$$

where $\epsilon_{c,h} \sim N(-\frac{3}{10}, \frac{1}{10})$. Note that child-specific characteristics that affect disruption are irrelevant for matching as they cannot affect the preference ordering over homes. Home-specific effects are important, but we are not currently aware of the most salient home traits that affect disruption.

Next, we turn to home preferences. In our baseline, we assume that the matchmaker perfectly knows all homes' preferences. Across the United States, younger children almost always find placements much sooner than older children, and homes almost universally report desires to adopt younger children. We model this as vertical differentiation among children: younger children are more desirable to homes. Simultaneously, our county reports that homes may have very heterogeneous willingness to adopt a child depending on behavioral characteristics. Therefore, we also add horizontal differentiation to home preferences.

$$
\hat{V}_h(c) = \begin{cases} \bar{V} - \bar{V}(\frac{age(c)}{18})^2 + \delta_{h,c} & \text{if } need(c) = 0 \text{ or } need(c) = 1, need(h) = 1\\ -1 & \text{otherwise} \end{cases}
$$

where we normalize $\bar{V} = 100$, and $\delta_{h,c} \sim N(0, \frac{\bar{V}}{10})$. The cost for a child's age is quadratic as placement outcomes for children tend to worsen rapidly as a child approaches eighteen. We calibrate $w_c = \frac{14000}{12}$ as the median Title-IV payment per out-of-home foster youth per year to a county is approximately \$14,000. Finally, we set $w_h = 4$ so that a home's tradeoff for achieving the "ideal" match \overline{V} is about two years (25 months).

Our approach to estimating home preferences using the county's data will inevitably not be able to identify the preferences. It might be possible to identify home's preferences in this setting through observing the placements that a home receives and the placement that a home eventually accepts. Nevertheless, our objective is to provide a minimally operational tool that counties across the United States can utilize. Identifying preferences is a notoriously difficult exercise that highly skilled economists can sometimes accomplish; it is likely not something that a resource-strapped county could do. Therefore, we aim to understand how effective our mechanisms can be when the matchmaker uses a preference estimator that is, at best, correlated with a home's true utility. Hence, we model the

matchmaker's preference estimator as:

$$
V_h(c) = \hat{V}_h(c) + \gamma_{h,c}
$$

with the following specifications for $\gamma_{h,c}$:

K-bias: The matchmaker's estimator systemically produces biased estimates with $E[\gamma_{h,c}] = K \ \forall h, c$ and contains a small amount of variance. The empirical root mean squared error (RMSE) converges to K over a large number of simulations.

K-variance: The matchmaker's estimator is unbiased, but it has a large amount of variance such that the empirical root mean squared error (RMSE) converges to K over a large number of simulations, where:

RMSE =
$$
\sqrt{\frac{\sum_{h \in H(t) \forall t} \sum_{c \in C(t) \forall t} (\hat{V}_h(c) - V_h(c))^2}{\sum_{t=1}^{T} |C(t)| * \sum_{t=1}^{t} |H(t)|}}
$$

Under the K-bias specification, we set the bias so that the matchmaker underestimates the true utility. Overestimating homes' utility does not pose a problem for most of our mechanisms. Patience-freeness enforces stronger penalties on homes when utility is overestimated because a patience-free mechanism must guarantee that a home's placements after noncompliance are worse than the denied placement. Overestimation means that future placements will be even less preferable than when the matchmaker knows a home's true utility. Homes have an even stronger incentive to accept placements earlier. In contrast, underestimation creates a problem because the matchmaker may not sufficiently punish a home for waiting.

We use four different values for K: true utility $(K = 0$ and $V_h(c) = \hat{V}_h(c)$, ten percent error $(K = \frac{\bar{V}}{10})$, twenty-five percent error $(K = \frac{\bar{V}}{4})$ $\left(\frac{\tilde{V}}{4}\right)$, and fifty-percent error $\left(K=\frac{\tilde{V}}{2}\right)$ $\frac{V}{2}$.

Finally, how does a home decide to accept or decline a placement? If we compute the home's equilibrium strategy, the computation requires fixing an arbitrary action profile $a(T)$ then checking which homes maximize their utility. A home then updates its strategy if the change would yield higher immediate utility or higher utility in the future given the market's evolution and future placements. Yet, this changes the placements that might occur for other homes in the future. We would have to repeat this computation until $a(T)$ no longer updates. Unfortunately, this calculation is too expensive.

Instead, we borrow an insight from dynamic envy-freeness. In the $K = 0$ case, accepting the first placement is a weakly dominant strategy for all homes. It is arguably without loss to assume that a home h takes it as given that all other homes will accept their placements, and then h's deviation reduces to computing its utility under its counterfactual action profile. This deviation is much more tractible and still allows us to simulate plausible scenarios where homes might desire to wait for a future placement. A home h accepts its placement if, when all other homes accept their placements, it maximizes its utility through accepting the placement rather than waiting to accept a future placement.

5.3 Main Results

We compare HPDA, CRDA, and HEDA against sequential DA. In the future, we will contrast each mechanism against observed outcomes and note their individual strengths and weaknesses. Our outcomes focus on three measures: cumulative placements, cumulative child waiting costs, and the percentage of placed homes with justified envy at the time of placement. We count a home h that accepts its placement at time t as having justified envy, saying that $h \in J(t)$, if $\hat{V}_h(c) > \hat{V}_h(\mu_t(h))$ and $U_c(h) > U_c(\mu_t(c))$ for any $c \in C(t)$ such that $\mu_t(c) \neq c$. The number of placed home with justified envy at t is $|J(t)|$. Fixing some $\mathbf{a}(T)$, at some time t', the last measure is $\left(\sum_{t=1}^{t'} |J(t)|\right)/A^H(t')$.

For each simulation, we run HPDA, CRDA, HEDA, and sequential DA on the same market. We set HEDA's endowments at a width of $\bar{V}/4$ for the first four months, then the width reduces to w_h for all future months. We run 5 simulations for every K-bias and for K-variance type, averaging over all simulations^{[8](#page-27-0)}. The results for cumulative placements are in Figures [2](#page-27-1) and [3.](#page-28-0)

Figure 2: Placements under K-Bias

Placements—Our placement results are all very promising. As theoretically predicted, HPDA and CRDA are equivalent when the matchmaker knows the true utility. No home will ever deviate, so they both run DA at every t without any truncations. Both mechanisms also produce far more placements than sequential DA: nearly an additional 50%. HEDA is quantitatively similar, although it differs by an insignificant magnitude. These results suggest that patience-freeness is an important property for guaranteeing not only *expedient* placements but also *more* placements. Without patience-freeness, homes effectively always have an incentive to delay before encountering a satisficing placement. As T approaches infinity, the constant expectation that a better placement might arrive tomorrow induces a consistent downward pressure on the amount of potential placements. Patience-freeness eliminates such incentives.

⁸We plan to run 100 simulations per type after fine-tuning the paper.

Turning to the K-bias results, overall, placements decrease for CRDA. As our previous discussion highlighted, the underestimation of preferences causes violations in patience constraints. Interestingly, HPDA and HEDA are mostly unaffected. Sequential DA does not change greatly because the misestimation mostly cannot affect proposals aside from the small amount of variance we introduce. Nevertheless, all patience-free mechanisms continue to dominate sequential DA, showing that they are robust even up to fifty percent error.

Figure 3: Placements under K-Variance

Under K-variance, a different pattern emerges. Again, the $K = 0$ case is unaffected. However, under higher RMSE from variance, sequential DA suffers far more. High variance leads to different proposal orders under DA, and homes gain more incentives to wait for a better, future placement. Patience-free mechanisms are still robust, because even under the error, they provide some dynamic incentives that will always push the home toward accepting earlier placements. The lone exception is CRDA which has similar trends to sequential DA. HEDA creates the most cumulative one-year placements under 25% and 50% error. We conjecture that the endowments are more effective at enforcing dynamic incentives. They do not rely on selective punishment of homes when the matchmaker detects a possible patience violation. Whereas, high variance may frequently trick HPDA and CRDA into allowing a placement that should be truncated.

In Table [1,](#page-29-0) we show average-over-months placement outcomes for K-bias. HPDA maximizes benefits at a 49.48% additional (over sequential DA) placements per month, and the minimum benefit is from CRDA at 35.17% additional placements per month. We also show outcomes for traditionally hard-to-place children: teenagers and highneeds. The percent of teenagers placed at t is the number of teenagers with an accepted placement over the number of teenagers in the market at t . The average percent of

K -Bias Type	$\overline{0\%}$	10%	25%	50%
Average placements per month				
Sequential DA	9.3	9.17	9.07	8.73
HPDA	13.15	13.63	13.42	13.05
CRDA	13.15	13.17	12.8	11.8
HEDA	13	13.27	13.35	13.02
Average percent of teenagers placed				
Sequential DA	0%	0%	0%	0%
HPDA	11\%	18%	18%	12\%
CRDA	11\%	21\%	21\%	16\%
HEDA	7%	11\%	8\%	8\%
Average percent of high-needs placed				
Sequential DA	10%	11\%	9%	10%
HPDA	14\%	18\%	12\%	13\%
CRDA	14\%	15\%	11\%	13\%
HEDA	11\%	12\%	12\%	12\%
Average waste per month				
Sequential DA	0.45	0.11	0.09	1.63
HPDA	0.53	2.45	3.02	2.92
CRDA	0.53	5.17	7.9	9.43
HEDA	3.38	5.53	3.02	4.32

Table 1: Placements for Sequential DA, HPDA, CRDA, and HEDA under K-Bias

Note: waste is the number of homes in $H(t)$ that did not receive a placement offer. Some homes are unplaced in every stable match which explains DA's slightly positive waste.

teenagers placed is the average over all t. The same holds for high-needs children. One stark result is that sequential DA never (rounded down) successfully places teenagers into permanent homes. The homes with incentives to decline the placements when the mechanism is not patience-free are the same homes that are dynamically incentivized to accept teenagers under our mechanisms. All of our mechanisms improve outcomes for teenagers—CRDA does so by 21 percentage points under 10% and 25% error. Teen placements increase under CRDA with $K > 0$ because younger children are frequently envied with vertically differentiated preferences, and they tend not to enter rotation. CRDA offers a sense of priority to less envied children, in this case, teenagers. The same observations hold for high-needs children as CRDA places at least 12% and at most 18%. When $K = 0$, CRDA never rotates any children, so benefits for the less-envied cannot materialize.

Outcomes are similar for K-variance in Table [2.](#page-30-0) The biggest distinction is that CRDA's performance is much worse at 25% and 50% error. Its waste and HPDA's waste drastically increase. Unfortunately, it appears that CRDA truncates too many children for Remark [3](#page-15-0) to hold. These empirical results suggest that homes fail to comply moreso under K-variance than K-bias under dynamic envy-free and non-wasteful mechanisms.

CRDA's saving grace is that it still maintains its advantage in placements for teenagers. HEDA produces the largest increase in placements relative to sequential DA—49.08% at 25% error. HPDA's minimum increase over sequential DA is 30.23% at 50% error. Sequential DA is strictly non-wasteful, and HEDA's waste is fairly constant across all specifications.

K -Variance Type	0%	10%	25%	50%
Average placements per month				
Sequential DA	9.3	8.97	8.72	8.27
HPDA	13.15	13.02	12.27	10.77
CRDA	13.15	12.5	10.65	7.65
HEDA	13	13.08	13	12.3
Average percent of teenagers placed				
Sequential DA	0%	0%	0%	1%
HPDA	11\%	11\%	7%	3%
CRDA	11\%	17\%	17\%	12\%
HEDA	7%	9%	9%	12\%
Average percent of high-needs placed				
Sequential DA	10%	9%	8\%	7%
HPDA	14%	12\%	9%	8%
CRDA	14\%	13%	11\%	7%
HEDA	11\%	12%	10\%	10\%
Average waste per month				
Sequential DA	0.45	0.28	$\mathbf{1}$	0.18
HPDA	0.53	2.27	7.6	13.45
CRDA	0.53	3.85	18.53	32.42
HEDA	3.38	3.63	3.57	3.12

Table 2: Placements for Sequential DA, HPDA, CRDA, and HEDA under K-Variance

Waiting Costs—Our results for child waiting costs (Figures [6](#page-40-0) and [7](#page-41-0) in Appendix D) imposed on the child welfare system mirror the placement results as more placements mechanically implies less waiting. The gap tends to widen between sequential DA and our patience-free mechanisms over time because the stock of unplaced children increases more quickly under sequential DA. There is a natural limit to the increase in the gap as children eventually age out of the child welfare system which we do not capture in our model and figures.

Overall, per-month waiting costs are orders of magnitude smaller by at least \$20,000 in all specifications (excluding CRDA at 50% error) and nearly \$50,000 when the matchmaker knows the homes' true preferences. These estimates only grow as the county manages welfare services for more children; in our simulations, the county is a small to medium-sized system with, at most, 240 children entering in a year.

Justified Envy—Sequential DA, HPDA, and CRDA are all justified envy-free if $K = 0$ as our theoretical results proved. Even in this case, HEDA has a relatively large amount

Figure 4: Justified Envy under K-Bias

of justified envy with approximately 50% of homes having justified envy over the year. Unfortunately, this implies that some children might be able to receive placements that are less likely to disrupt, and there are homes that would adopt those children. We investigate how the non-disruption probabilities change shortly.

At any $K > 0$, justified envy appears for every mechanism, although sequential DA, HPDA, and CRDA still have significantly less justified envy than HEDA. HPDA and CRDA generally converge to the same amount of justified envy, but CRDA always maintains the least envy. Bias does not appear to affect HEDA's justified envy as it remains constant between 50 to 60% of homes.

Under K-variance, justified envy shoots up as K increases for sequential DA, HPDA, and HEDA (Figure [5\)](#page-32-0). CRDA is very robust to justified envy even up to 50% error with justified envy converging to about 10% of homes by $t = 12$. We attribute this to the fact that CRDA uses child-proposing DA, and the matchmaker knows every child's true preferences. Therefore, children always propose to their favorite homes, and it is less likely that a child could find a justified envy-free improvement than when homes propose.

Sequential DA and HPDA emerge in the middle. Both function with low justified envy under 10% and 25% error but justified envy is high at about 30% of homes under HPDA and 40% of homes under sequential DA at 50% error. Finally, HEDA is extremely erratic with nearly 80% of homes having justified envy. Unfortunately, HEDA's major pitfall—lacking dynamic envy-freeness—heightens more than other mechanisms at K increases. This may be a fundamental tradeoff for strategy-proofness. As we do not consider strategic reports in our simulations, it is difficult to tell if the gains from an increased supply of homes willing to care for high-needs children is worth increasing disruption rates. We provide two tables in Appendix D to examine this concern. Our

Figure 5: Justified Envy under K-Variance

results show that non-disruption rates stand unchanged for HPDA and HEDA, implying that while justified envy exists for children, it is small in magnitude. Sequential DA generally maintains the highest non-disruption rates with a 8% difference at most. Homes effectively select into placements that are less likely to disrupt under sequential DA because these are precisely the placements they are more likely to attain if they wait (they will match with children in the future that prefer the homes over contemporaneous options), i.e., there is a market thickness effect. CRDA partially compensates as childproposing DA elevates non-disruption's importance; when K-variance is high, CRDA beats sequential DA by about 4%.

6 Conclusion

Child welfare systems lack organized attempts to centralize placement assignments for children. Previous efforts to construct match recommendation systems have not gone as hoped, yet the area still seems to hold significant promise to be an arena for market design, and many economists have studied adjacent problems to improve outcomes for children. We show that the 109 thousand children waiting for adoption in the United States, and certainly many more abroad, could have permanent homes—now. Nevertheless, even with centralization, matchmakers must choose carefully between mechanisms that prioritize different goals. Dynamically envy-free and patience-free mechanisms almost always increase the number of placements and decrease waiting costs, but some come at the price of slightly lower non-disruption rates for adoptions and a large number of homes that never receive placement offers. We offer at least one mechanism—HEDA—that consistently has low waste, more or as many placements as other mechanisms, and good non-disruption outcomes. It is also strategy-proof, meaning that it prevents incentives for homes that can care for high-needs or older children to report that they cannot.

As this paper develops, we intend to sketch an implementation strategy for our mechanisms that allows workers to have the final say over placements while simultaneously ensuring that the best, dynamically envy-free child-home placements are prioritized. Several obstacles remain between current practice and smooth centralization—caseworkers say that previous designs failed because of poor coding, match recommendations based on irrelevant characteristics, and many other operational concerns. Going forward, we desire that child welfare systems will take seriously the challenge to design better matching for the many waiting children everywhere.

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Appendix A: Proofs

Below, we prove the theorems and propositions that we did not prove in the main text.

Theorem [2.](#page-12-0) Home Penalized Deferred Acceptance is dynamic envy-free and nonwasteful.

Proof. First, we show that HPDA is justified envy-free and individually rational for any action profiles at any time. Then, we prove that HPDA is patience-free. Last, we show that HPDA is non-wasteful. The following proofs do not rely on any features of the environment's horizon (aside from finiteness) nor arrival structure nor type space, hence, they always apply.

(i) HPDA is justified envy-free for any t and $a(t)$. Suppose for a contradiction that for some $t, \mathbf{a}(t)$, we have that μ_t is not justified envy-free. This implies that there exists some $c \in C(t|\mathbf{a}(t-1))$ and $h \in H(t|\mathbf{a}(t-1))$ with $\mu_t(h) \neq h$ such that $U_c(h) > U_c(\mu_t(c))$ and $V_h(c) > V_h(\mu_t(h))$. However, this implies that h did not have its preferences truncated at t, otherwise $\mu_t(h) = h$. Therefore, during the operation of HPDA, h proposed to c. Since $\mu_t(c) \neq h$, this implies that c rejected h. This contradicts $U_c(h) > U_c(\mu_t(c))$.

(ii) A non-individually rational match is impossible because no home would propose to an unacceptable child, and no child would hold a proposal from an unacceptable home.

(iii) HPDA is patience-free. Take any arbitrary t,h , $a(T)$ such that $\mu_t(h) \neq h$. For any arbitrary $t' > t$, take the counterfactual $\hat{a}(t',h)$. If no matching occurs at t' under the counterfactual, then trivially $V_h^t(\mu_t(h)) \ge V_h^{t'}$ $\mathcal{L}_{h}^{t'}(\hat{\mu}_{t'}^h(h))$. Otherwise, the proof proceeds in three steps. One, we prove by strong induction that $\mu_k = \hat{\mu}_k^h \forall k \leq t$. Two, we prove that the most recent match always acts as the maximum time-discounted utility bound for any subsequent matching. Three, we combine these statements to conclude (ii).

Note that for this proof we write $C(t) \equiv C(t|\mathbf{a}(t-1))$, $H(t) \equiv H(t|\mathbf{a}(t-1))$, $A^{C}(t|\mathbf{a}(t)) \equiv A^{C}(t)$, and $A^{H}(t|\mathbf{a}(t)) \equiv A^{H}(t)$. Define:

$$
\hat{A}^C(t, h) \equiv \{c : \hat{\mu}_k^h(c) \neq c \text{ and } \hat{a}_{\hat{\mu}_k^h(c)}^k = 1 \text{ for some } k \leq t\}
$$

$$
\hat{A}^H(t, h) \equiv \{h : \hat{\mu}_k^h(h) \neq h \text{ and } \hat{a}_h^k = 1 \text{ for some } k \leq t\}
$$

Induction Statement: For every natural number $k \in [1, t]$, $\mu_k = \hat{\mu}_k^h$, $\hat{A}^C(k-1, h) =$ $A^{C}(k-1)$, and $\hat{A}^{H}(k-1,h) = A^{\hat{H}}(k-1)$.

The base case is $t = 1$. By definition, $M(1) = (c(1), h(1), E(1)) = \hat{M}(1, h), \hat{A}^C(0, h) =$ $A^{C}(0) = \emptyset$, and $\hat{A}^{H}(0, h) = A^{H}(0) = \emptyset$. The matchings μ_1 and $\hat{\mu}_1$ can only take the market at $t = 1$ as input, therefore it must be that $\mu_1 = \hat{\mu}_1^h$.

Next, the induction step is for $k+1 \leq t$ if the statement holds for all natural numbers less than $k + 1$. Since the statement holds for such k (which must be $k < t$), we have that $C(k) = \hat{C}(k, h)$, $H(k) = \hat{H}(k, h)$, and $\mu_k = \hat{\mu}_k^h$. By assumption that $\mu_t(h) \neq h$, either $\mu_k(h) = \hat{\mu}_k^h(h) = h$ or $a_k^h = \hat{a}_k^h = 0$. In either case, $h \in H(k+1), \hat{H}(k+1,h)$. In the case that some c's match is $h, c \in C(k+1), \hat{C}(k+1, h)$ by the above argument. Further, by definition, $a_{h'}^k = \hat{a}_{h'}^k$ for all $h' \neq h$. Thus, if $\mu_k(c) \neq h$ and $c \in C(k)$, $\hat{C}(k, h)$, $c \in A^C(k) \iff \mu_k(c) \neq c \text{ and } a_{\mu_k(c)}^k = 1 \iff \hat{\mu}_k^h(c) \neq c \text{ and } \hat{a}_{\hat{\mu}_k}^k(c)$ $\frac{k}{\hat{\mu}_k^h(c)} = 1 \iff$ $c \in \hat{A}^C(k, h)$. The same logic holds for $h' \neq h \in A^H(k)$. Therefore, $C(k + 1) =$ $\hat{C}(k+1, h)$, and $H(k+1) = \hat{H}(k+1, h)$. By the induction assumption, the match histories $(\mu_n)_{n \leq k+1} = (\hat{\mu}_n^h)_{n \leq k+1}$. At every time, HPDA is a deterministic function of the present market and match history, which are equivalent under the actual and counterfactual; therefore $\mu_{k+1} = \hat{\mu}_{k+1}^h$. This concludes the proof for the induction statement.

Next, we show by induction that any subsequent matching under HPDA yields smaller time-discounted utility than the matching at t. Formally, the induction statement is: for any $k \geq t$ where $\mu_t(h) \neq h$, $V_h^k(\mu_k(h)) \leq V_h^t(\mu_t(h))$. The base case is $k = t$. Clearly, the statement holds as the utilities are equivalent. The induction step is for $k + 1$ given that the statement holds for all natural numbers less than $k + 1$. If $\mu_{k+1}(h) = h$, then the statement holds trivially. Otherwise, let n be the most recent natural number such that $\mu_n(h) \neq h$ and $n < k+1$. By the induction hypothesis, $V_h^n(\mu_n(h)) \leq V_h^t(\mu_t(h))$. Now $\mu_{k+1}(h) \neq h \iff p_h^{k+1} = 0 \iff V_h^{k+1}$ $V_h^{k+1}(\mu_{k+1}(h)) \leq V_h^n(\mu_n(h)) \leq V_h^t(\mu_t(h))$. This proves the induction statement. This proof holds equivalently for the counterfactual matching. Thus we have that for any $t' > t$, $V_h^t(\mu_t(h)) = V_h^t(\hat{\mu}_t^h(h)) \geq V_h^{t'}$ $\hat{h}^{t'}(\hat{\mu}^h_{t'}(h)).$

(iv) Last, HPDA is non-wasteful. Suppose, for a contradiction, the contrary. Since $a_h^t = 1$ for all h, t , if $\mu_t(h) \neq h$ then $h \notin H(t+1)$. Therefore, it cannot be that any home has its preferences truncated. But by our supposition, there exists some $c \in C(t)$ and $h \in H(t)$ for some t such that $U_c(h) \geq 0$, $V_h(c) \geq 0$, $\mu_t(c) = c$, and $\mu_t(h) = h$. The child and homes' non-negative utilities for the match and $\mu_t(h) = h$ imply that, at some point, h proposed to c and c had no proposals. However, $\mu_t(c) = c$ implies that c rejected h, which contradicts $U_c(h) \geq 0$. This completes the theorem. \Box

Proposition [1.](#page-16-0) Accepting the first placement is a weakly dominant strategy for all homes under a patience-free mechanism.

Proof. Fix a patience-free mechanism Q. For any action profile $a(T)$ and home h, let t be the first time that $\mu_t(h) \neq h$, if this ever occurs. By patience-freeness, $V_h^t(\mu_t(h)) \ge$ $V_h^{t'}$ $\mathcal{L}_h^{t'}(\mu_{t'}^h(h))$ for any $t' > t$. If $a_h^t = 1$, then h receives payoff $V_h^t(\mu_t(h))$. Let \bar{a}_h be any other strategy for h. Define \bar{a} where \bar{a}_h is h's strategy and $\bar{a}_{h'} = a_{h'}$ for all $h' \neq h$. We

proved in Theorem 2 that up until and including the first period t' where $\bar{a}_h^{t'} = 1$ and h is offered a match, the matchings under this profile and the counterfactual $\hat{a}(t',h)$ are equivalent (if $\bar{a}_h^k = 0$ for all k or h never accepts another match, then h receives payoff 0.). Therefore, the payoffs are equivalent as the same placement will be accepted under these profiles, and so h is weakly better off with $a_h^t = 1$. \Box

Theorem [3.](#page-14-0) Child Rotating Deferred Acceptance is dynamic envy-free and nonwasteful.

Proof. We show that CRDA is justified envy-free for any action profiles at any time. The proof for patience-freeness, individual rationality, and non-wastefulness are the same as in Theorem [2.](#page-12-0) For any $t, a(t)$, if no home triggers truncation, then only MATCHING occurs. This is equivalent to DA which always produces a stable matching. Stability implies justified envy-freeness. Otherwise, the proof proceeds by cases. In each case, suppose that for some $t, \mathbf{a}(t)$ there exists some $c \in C(t)$, $h \in H(t)$ such that $\mu_t(c) \neq$ $c, U_c(h) > U_c(\mu_t(c)),$ and $V_h(c) > V_h(\mu_t(h)).$

Case 1: $c \notin \hat{R}_{C}^{t}$, $h \notin R_{H}^{t}$. This implies that neither c nor h had matchings replaced during ROTATION. Then, at some point, c proposed to h, but, since $\mu_t(h) \neq c$, h rejected c. This contradicts $V_h(c) > V_h(\mu_t(h))$.

Case 2: $c \in \hat{R}_C^t, h \in R_H^t$. By $\mu_t(c) \neq c$, we have that $c \in R_C^t$. As above, this now implies that c proposed to h during ROTATION, but h rejected c, which generates the same contradiction.

Case 3: $c \notin \hat{R}_C^t, h \in R_H^t$. By $c \notin \hat{R}_C^t$, we have that $r_c^t = 0 \implies U_c(\mu_t(c)) > U_c(h)$. This is a contradiction.

Case 4: $c \in \hat{R}_C^t, h \notin R_H^t$. We disprove this case via. contradiction. Again, by $\mu_t(c) \neq c$, we have that $c \in R_C^t$. Define $h' \equiv \mu_t^M(c)$ as c's match during MATCHING (if c has none, then trivially $U_c(\mu_t(c)) \ge U_c(h') = 0$.

We claim that (A) if (*) $U_c(\mu_t(c)) < U_c(h')$ for any $c \in R_C^t$, then there exists some $c' \in R_C^t$ such that $U_{c'}(\mu_t^M(c')) > U_{c'}(h'), V_{h'}(c') > V_{h'}(c),$ and $c \neq c'$. Notice that $c \in R_C^t \implies h' \in R_H^t$. If $U_c(\mu_t(c)) < U_c(h')$, then h' must, at some point, reject c during ROTATION. Hence, there must exist some more preferable $c' = \mu_t(h)$ proposing to h'. There exist two subcases. Either c' proposed to h' during MATCHING or c' did not. If c' did, then $\mu_t^M(h') = c \implies V_{h'}(c) > V_{h'}(c')$ which contradicts h' rejecting c during rotation. Therefore, it must be that c' did not propose to h' during MATCHING even though c' was active during MATCHING. For $V_{h'}(c') > V_{h'}(c)$, if this were not true, then it must be that h' does not reject c during ROTATION. Thus, $U_c(\mu_t(c)) \ge U_c(h')$, a contradiction. This proves (A).

Claim (B): if $U_c(\mu_t(c)) < U_c(h')$, then there must exist some $c' \in R_C^t$ such that $U_{c'}(\mu_t(c')) < U_{c'}(\mu_t^M(c'))$. By (A), we have that there exists some c' with $\mu_t(c') = h'$ after ROTATION that did not propose to h' during MATCHING. Thus, $U_{c'}(\mu_t(c'))$ $U_{c'}(h') < U_{c'}(\mu_t^M(c'))$. This proves (B).

Let $S \equiv \{c : c \in R_C^t, U_c(\mu_t(c)) \prec U_c(\mu_t^M(c))\}.$ Consider the first round during ROTATION where any $\mu_t^M(c)$ rejects any $c \in S$. This can only happen if $\mu_t^M(c)$ has a preferable offer from some $c' \neq c$. If c' has not proposed to $\mu_t^M(c')$ or c' has no match, then $U_{c'}(\mu_t^M(c)) > U_{c'}(\mu_t^M(c'))$. Then, however, c' and $\mu_t^M(c)$ form a blocking pair during MATCHING which contradicts DA's stability. Hence, c' has proposed to $\mu_t^M(c')$, but then $U_{c'}(\mu_t(c')) < U_{c'}(\mu_t^M(c')) \implies c' \in S$. However, by assumption that this is the first round where any child in S is rejected by her MATCHING placement, it must be that c'

has not yet proposed to $\mu_t^M(c')$. This is a contradiction. Therefore, it must be that S is an empty set.

Now we have that for all $c \in R_C^t$, $U_c(\mu_t(c)) \geq U_c(\mu_t^M(c))$. If a blocking pair $c \in \hat{R}_C^t$, $h \notin$ R_H^t exists, then this implies $U_c(h) > U_c(\mu_t(c)) \geq U_c(\mu_t^M(c))$. Then, c must have proposed to h during MATCHING. Furthermore, $h \notin R_H^t \implies \mu_t^M(h) \neq c$. Therefore, h must have rejected c during MATCHING. This contradicts $V_h(c) > V_h(\mu_t(h)) = V_h(\mu_t^M(h))$. \Box

Remark [3.](#page-15-0) Child Rotating Deferred Acceptance is strictly non-wasteful if, for all $t, \mathbf{a}(t)$, there exists an h-perfect stable matching on R_H^t and R_C^t .

Proof. Suppose, for a contradiction, that there exists an h-perfect stable matching on R_H^t and R_C^t for all $t, \mathbf{a}(t)$ and, at some $t, \mathbf{a}(t)$, there exists $c \in C(t|\mathbf{a}(t)), h \in H(t|\mathbf{a}(t))$ such that $U_c(h) \geq 0, V_h(c) \geq 0, \mu_t(c) = c$, and $\mu_t(h) = h$. $\mu_t(h) = h \implies h \in R_H^t$. The Rural Hospitals Theorem states that under Assumption 1, the set of unmatched agents is the same in any stable matching [\(Roth 1986\)](#page-34-6). If there exists an h-perfect stable matching, then every stable matching is an h-perfect matching. CRDA always constructs a stable matching with respect to R_C^t and R_H^t . Hence, $\mu_t(h) = h$ contradicts the existence of an h-perfect stable matching. \Box

Theorem [5](#page-22-0) Home Endowed Deferred Acceptance is patience-free, individually rational, and strategy-proof.

Proof. First, we show that HEDA is individually rational for any action profiles at any time. Then, we prove that HEDA is patience-free. Last, we show that HEDA is strategyproof. For individual rationality and patience-freeness, we take as given the input preferences and prove the properties with respect to the input.

(i) An non-individually rational match is impossible because no home would propose to an unacceptable child, and no child would hold a proposal from an unacceptable home.

(ii) HEDA is patience-free. Take any arbitrary $t, h, \mathbf{a}(T)$ such that $\mu_t(h) \neq h$. For any arbitrary $t' > t$, take the counterfactual $\hat{a}(t',h)$. If no matching occurs at t' under the counterfactual, then trivially $V_h^t(\mu_t(h)) \ge V_h^{t'}$ $\hat{h}_h^{lt'}(\hat{\mu}_{t'}^h(h))$. Otherwise, suppose that $\hat{\mu}_{t'}^h(h) \neq$ h. Since $\mu_t(h) \neq h$, we have that $e_h^t(\mu_t(h)) = 1 \iff V_h^t(\mu_t(h)) \in B_t(h)$. Likewise, $\hat{\mu}_{t'}^h(h) \neq h \iff e_h^{t'}$ $h'_h(\hat{\mu}_{t'}^h(h)) = 1 \iff V_h^{t'}$ $\mathcal{L}_h^{t'}(\hat{\mu}_{t'}^h(h)) \in B_{t'}(h)$. Then by definition of HEDA and $t' > t$, $V_h^t(\mu_t(h)) \geq V_h^{t'}$ $\hat{h}^{t'}(\hat{\mu}^h_{t'}(h))$

(iii) HEDA is strategy-proof. Take any arbitrary $a_{-h}(T), \sigma_{-h}$. Let σ_h be truthful and $a_h(T)$ be compliant, i.e., $a_h^t = 1 \forall t$. Denote h's realized utility under this plan as $\hat{V}_h^r(a(T), \sigma)$.

Consider any alternative pair for h: $(\hat{a}_h(T), \hat{\sigma}_h)$. Then $\hat{a}(T)$ is the profile with the specified action for h and $\hat{a}_{h'}(T) = a_{h'}(T) \ \forall h' \neq h$. Likewise for $\hat{\sigma}$. Denote h's realized utility under the alternative pair as $\hat{V}_h^r(\hat{a}(T), \hat{\sigma})$. We prove the theorem by showing that in the first period where matchings differ under the two strategy pairs, it must be that h 's match cannot improve. Under HEDA, h maximizes its utility through accepting the earliest possible match, so any subsequent match under the alternative pair is also worse. The theorem follows. We use a few last pieces of notation for the proof. As in Theorem [2,](#page-12-0) we write $C(t) \equiv C(t|a(t-1), \sigma)$, $H(t) \equiv H(t|a(t-1), \sigma)$, $A^{C}(t|a(t), \sigma) \equiv A^{C}(t)$, and

 $A^H(t|a(t),\sigma) \equiv A^H(t)$. Define:

$$
\hat{C}(t) \equiv C(t|\hat{a}(t-1), \hat{\sigma})
$$

\n
$$
\hat{H}(t) \equiv H(t|\hat{a}(t-1), \hat{\sigma})
$$

\n
$$
\hat{A}^C(t) \equiv \{c : \hat{\mu}_k(c) \neq c \text{ and } \hat{a}^k_{\hat{\mu}_k(c)} = 1 \text{ for some } k \leq t\}
$$

\n
$$
\hat{A}^H(t) \equiv \{h : \hat{\mu}'_k(h) \neq h \text{ and } \hat{a}^k_h = 1 \text{ for some } k \leq t\}
$$

Take the first period k where $\mu_k \neq \hat{\mu}_k$, if one exists. If one does not exist, the theorem follows trivially. Otherwise, we first prove that $A^{C}(k') = \hat{A}^{C}(k')$ and $A^{H}(k') =$ $\hat{A}^H(k')$ $\forall k' < k$. Suppose, for a contradiction, that this is not true. If it is not true for the latter, then take the earliest k' where there exists a $h' \in A^H(k'), h' \notin \hat{A}^H(k')$ or $h' \notin A^H(k'), h' \in \hat{A}^H(k')$. If $h' = h$, this would imply that $h \in A^H(k'), h \notin \hat{A}^H(k')$ (since h must accept the placement under the truthful, compliant pair), but then h 's realized utility is greater under the truthful, compliant pair by (ii). Thus, we can assume WLOG that $h' \neq h$. However, $\mu_{k'}(h') = \hat{\mu}_{k'}(h')$ by assumption, and $\hat{a}_{h'}^k = a_{h'}^k$. This is a contradiction, so, WLOG, $A^H(k') = \hat{A}^H(k')$. Next, suppose this is not true for the former, i.e., take the earliest k' where there exists a $c \in A^{C}(k)$, $c \notin \hat{A}^{C}(k')$ or $c \notin A^{C}(k'), c \in \hat{A}^{C}(k')$. If $c = \mu_{k'}(h)$, then, again, h's realized utility is greater under the truthful, compliant pair so we can assume, WLOG, that $c \neq \mu_{k'}(h)$. Yet, $\mu_{k'}(c) = \hat{\mu}_{k'}(c)$ by assumption, and $\hat{a}^k_{\hat{\mu}_{k'}(c)} = a^k_{\mu_{k'}(c)}$. This contradicts $A^C(k') \neq \hat{A}^C(k')$, so, WLOG, $A^{C}(k') = \hat{A}^{C}(k')$. These two facts imply that $C(k) = \hat{C}(k)$ and $H(k) = \hat{H}(k)$.

We now have that the set of children and homes available to match at k are the same under either strategy pair. Returning to the period k , we split into three cases. Either $\mu_k(h) = \hat{\mu}_k(h) = h, \, \mu_k(h) = \hat{\mu}_k(h) \neq h \text{ or } \mu_k(h) \neq \hat{\mu}_k(h).$

Case 1: We show the first case is impossible by retooling the proof from Theorem [3'](#page-14-0)s case 4. Suppose, for a contradiction, that it is true. Then for some $h' \neq h$, $\mu_k(h') \neq$ $\hat{\mu}_k(h')$. If $\tilde{V}_{h'}(\hat{\mu}_k(h')) > \tilde{V}_{h'}(\mu_k(h'))$, this implies the existence of some $h'' \neq h', h$ with $\mu_k(h'') \equiv c'' \neq h''$ and $\hat{\mu}_k(h') = c''$. If not, then $\mu_k(c'') = c''$, implying that h' proposed to this child, but the child rejected h' under $a(T)$, σ . Then, h' is unacceptable to this child, contradicting $\hat{\mu}_k(c'') = h'$. Furthermore, it must be that $\tilde{V}_{h''}(\hat{\mu}_k(h'')) > \tilde{V}_{h''}(\mu_k(h''))$. If not, this implies h'' proposed to c'' under $\hat{a}(T)$, $\hat{\sigma}$ but was rejected in favor of h'. But then h' and c'' form a blocking pair under $a(T)$, σ , meaning that μ_k is not stable w.r.t the constructed preferences, which is a contradiction.

Define $S = \{\hat{h} : \tilde{V}_{\hat{h}}(\hat{\mu}_k(\hat{h})) > \tilde{V}_{\hat{h}}(\mu_k(\hat{h}))\}$ and suppose it is non-empty. Consider the first round of proposals at time k under $a(T)$, σ where some $\hat{h} \in S$ is rejected by $\hat{c} \equiv \hat{\mu}_k(\hat{h})$ in favor of some \tilde{h} . If \tilde{h} not has proposed to $\hat{\mu}_k(\tilde{h})$ or is unmatched, this implies $\tilde{V}_{\tilde{h}}(\hat{c}) > \tilde{V}_{\tilde{h}}(\hat{\mu}_k(\tilde{h}))$ and $U_{\hat{c}}(\tilde{h}) > U_{\hat{c}}(\hat{h})$. Then, \hat{c} and \tilde{h} form a blocking pair on $\hat{\mu}_k$, which contradicts DA's stability on the constructed preferences. Therefore, it must be that \hat{h} has proposed to and been rejected by $\hat{\mu}_k(\tilde{h}) \implies \tilde{V}_{\tilde{h}}(\hat{\mu}_k(\tilde{h})) > \tilde{V}_{\tilde{h}}(\hat{c}) > \tilde{V}_{\tilde{h}}(\mu_k(\tilde{h})) \implies \tilde{h} \in S$. However, by assumption that this is the first round that any home in S was rejected, it cannot be that h has been rejected yet. This is a contradiction; S must be an empty set.

Instead, if $\tilde{V}_{h'}(\hat{\mu}_k(h')) < \tilde{V}_{h'}(\mu_k(h'))$, there must exist some $h'' \neq h', h$ with $\hat{\mu}_k(h'') \equiv$ $c'' \neq h''$ and $\mu_k(h') = c''$. We also we have that $\tilde{V}_{h''}(\hat{\mu}_k(h'')) < \tilde{V}_{h''}(\mu_k(h''))$. All of the above follows from the same logic as before. We can define $S' \equiv \{\hat{h} : \tilde{V}_{\hat{h}}(\hat{\mu}_k(\hat{h}))$ $\tilde{V}_{\hat{h}}(\mu_k(\hat{h}))\}$ and suppose that it is non-empty. Consider the first round of proposals at time k under $\hat{a}(T)$, $\hat{\sigma}$ where some $\hat{h} \in S'$ is rejected by $\hat{c} \equiv \mu_k(\hat{h})$ in favor of some \tilde{h} . If \tilde{h} not has proposed to $\mu_k(\tilde{h})$ or is unmatched, this implies $\tilde{V}_{\tilde{h}}(\hat{c}) > \tilde{V}_{\tilde{h}}(\mu_k(\tilde{h}))$ and $U_{\hat{c}}(\hat{h}) > U_{\hat{c}}(\hat{h})$. Then, \hat{c} and \hat{h} form a blocking pair on μ_k , which contradicts DA's stability on the constructed preferences. Therefore, it must be that \tilde{h} has proposed to and been rejected by $\mu_k(\tilde{h}) \implies \tilde{V}_{\tilde{h}}(\mu_k(\tilde{h})) > \tilde{V}_{\tilde{h}}(\hat{c}) > \tilde{V}_{\tilde{h}}(\hat{\mu}_k(\tilde{h})) \implies \tilde{h} \in S'.$ However, by assumption that this is the first round that any home in S' was rejected, it cannot be that \tilde{h} has been rejected yet. This is a contradiction; S' must be an empty set.

Since S and S' are empty, for every $h' \neq h$, it must be that $\tilde{V}_{h'}(\mu_k(h')) = \tilde{V}_{h'}(\hat{\mu}_k(h'))$. Since preferences are strict, it cannot be that $\mu_k(h') \neq \hat{\mu}_k(h')$ for any h'. Hence, case 1 is impossible.

Case 2: In this case, since h always accepts the first match under $a(T)$, and $\mu_k(h) =$ $\hat{\mu}_k(h) \neq h$, it must be that this is the first period that h receives a match under either $\hat{a}(T), \sigma$ or $\hat{a}(T), \hat{\sigma}$. Then, $\hat{V}_h^r(a(T), \sigma) = \hat{V}_h^k(\mu_k(h))$. If h accepts the match under $\hat{a}(T)$, h receives equivalent utility. If not, h receives some utility $\hat{V}_h^r(\hat{a}(T),\hat{\sigma}) = \hat{V}_h^n(\hat{\mu}_n(h))$ $\hat{V}_h^r(a(T), \sigma) = \hat{V}_h^k(\mu_k(h))$ in a period $n > k$. If h is not matched at n, the inequality holds trivially. If h is matched at n, we have that $\hat{\mu}_n(h) \neq h \iff e_h^n(\hat{\mu}_n(h)) = 1 \iff$ $\tilde{V}_h^n(\hat{\mu}_n(h)) \in B_n(h) \implies \hat{V}_h^n(\hat{\mu}_n(h)) \in B_n(h)$. The last implication holds because $e_h^n(\hat{\mu}_n(h)) = 1 \implies \tilde{V}_h(\cdot) > 0 \implies \tilde{V}_h(\cdot) = \hat{V}_h(\cdot) \implies \tilde{V}_h^n(\cdot) = \hat{V}_h^n(\cdot).$ Similarly, $\hat{V}_h^k(\mu_k(h)) \in B_k(h)$. By definition of HEDA, h must have higher realized utility from accepting the match in period k , and the inequality holds.

Case 3: Last, we adapt [Roth \(2017\)'](#page-34-5)s proof to show that h's utility cannot decrease. To ease notation, we write $v_h(\cdot) \equiv \tilde{V}_h(\cdot | \sigma_h)$ and $\hat{v}_h(\cdot) \equiv \tilde{V}_h(\cdot | \hat{\sigma}_h)$ (note for all $h' \neq h$, constructed preferences are the same under σ and $\hat{\sigma}$). Define $S = \{\hat{h} : \hat{v}_{\hat{h}}(\hat{\mu}_k(\hat{h})) > \sigma\}$ $v_{\hat{h}}(\mu_k(h))\}\$ and $R \equiv \{c : \hat{\mu}_k(c) \in S\}.$

For a contradiction, suppose that $h \in S$. (A), we show that $c \in R \iff \mu_k(c) \in S$. (B) We show that a contradiction arises from our supposition that $h \in S$.

Consider $(A \implies)$. Suppose that $h' \in S \iff c' \equiv \hat{\mu}_k(h') \in R$. Let $h'' \equiv \mu_k(c')$. If $h'' = h$, the statement follows trivially. If not, we know that $v_{h''}(\cdot) = \hat{v}_{h''}(\cdot)$. Furthermore, $h' \in S \iff \hat{v}_{h'}(c') = v_{h'}(c') > v_{h'}(\mu_k(h'))$ where the equality follows because if $h' = h$, $\hat{v}_{h'}(c') > v_{h'}(\mu_k(h')) \geq 0 \implies \hat{v}_{h'}(c') = \hat{V}_{h'}(c') = v_{h'}(c')$, and it follows trivially if $h' \neq h$. Then, because DA is stable on the constructed preferences and $\mu_k(h') \neq c'$, it must be that $U_{c'}(h'') > U_{c'}(h')$. However, again by DA's stability on $\hat{a}(T), \hat{\sigma}$, it must be that $V_{h''}(\hat{\mu}_k(h'')) > V_{h''}(c') \iff \mu_k(c') = h'' \in S$

 $(A \Leftrightarrow) \mu_k(c) \in S \implies c \in R$ if and only if the contrapositive is true, that is, $c \notin R \implies \mu_k(c) \notin S$. By the above proof, for every $h' \in S$, there exists exactly one $h'' \in S$ with $\mu_k(h'') \equiv c''$ such that $\hat{\mu}_k(h') = c''$. Define the h'' satisfying this for h' as $s(h')$. It must be that $\mu_k(s(h')) = c'' \in R$ because $\hat{v}_{h'}(c'') > v_{h'}(\mu_k(h'))$. Furthermore, for two $h_1 \neq h_2$ it cannot be that $s(h_1) = s(h_2)$ because this would imply $\hat{\mu}_k(h_1) = \hat{\mu}_k(h_2)$.

Suppose, for a contradiction, that some $c \notin R$ and $h_c \equiv \mu_k(c) \in S$. It cannot be that $s(h_p) = h_c$ for any $h_p \in S$ as $s(h_p) = h_c \implies \mu_k(h_c) \in R$. Notice, then, S contains $|S|$ elements that have some $s(.)$. But then there are $|S| - 1$ homes that can satisfy $s(.)$ at most. By the pigeonhole principle, at least one home must satisfy $s(\cdot)$ for at least two homes. As we proved, this is impossible. Hence, the contrapositive must be true, and (A \Leftarrow) is true.

(B) Consider the last round that some arbitrary $h_l \in S$ proposes under $a(T)$, σ . By definition, h_l must propose to $c_l \equiv \mu_k(\hat{h})$. For all $h' \in S$, $\hat{\mu}_k(h')$ rejects h'. Furthermore, $h' \in S \implies \mu_k(h') \in R \implies \mu_k(h') \neq h'$, so it must be that every h' has a proposal after its rejection by $\hat{\mu}_k(h')$. $h_l \in S \implies c_l \in R$ which implies that c_l receives a proposal from $h \equiv \hat{\mu}_k(c_l)$. Since this is the last round of proposals for all homes in S, c_l rejected h in favor of some $h_s \notin S$ in a previous round of proposals (if $h_s \in S$, this means h_s is rejected in favor of h_l and proposes to another child in a future round, which contradicts this being the last round of proposals for homes in S). Then, $U_{c_l}(h_s) > U_{c_l}(\hat{h})$. Moreover, h_s proposed to c_l before $\mu_k(h_s)$, so $v_{h_s}(c_l) > v_{h_s}(\mu_k(h_s)) > v_{h_s}(\hat{\mu}_k(h_s))$. Since $h_s \notin S \implies h_s \neq h$, $v_{h_s}(\cdot) = \hat{v}_{h_s}(\cdot)$, and h_s must propose to c_l under $a(T)$, σ as well as $\hat{a}(T)$, $\hat{\sigma}$. This implies that h_s and c_l form a blocking pair on $\hat{\mu}_k$, a contradiction. Therefore, it cannot be that $h \in S$.

Finally, $h \notin S \implies \hat{V}_h(\mu_k(h)) = v_h(\mu_k(h)) > \hat{v}_h(\hat{\mu}_k(h)) = \hat{V}_h(\hat{\mu}_k(h))$ where the last equality again holds because $\hat{v}_h(\hat{\mu}_k(h)) \ge 0$ by IR. Since $v_h(\mu_k(h)) > 0$, h must receive a match. Since $a_h(T)$ is compliant and this is the first period where the matchings differ under $a(T)$, σ and $\hat{a}(T)$, $\hat{\sigma}$, this implies this is h's first match under either pair. h will accept this match and receive utility $\hat{V}_h^r(a(T), \sigma) = \hat{V}_h^k(\mu_k(h)) > \hat{V}_h^k(\hat{\mu}_k(h))$. Hence, even if h accepts the match under $\hat{a}(T), \hat{\sigma}$, h receives strictly lower utility. If h waits until a future period to accept a match under $\hat{a}(T), \hat{\sigma}$, h receives lower utility by Case 2. This completes the theorem.

 \Box

Waiting Costs for Children per Period CRDA 10.0% BIASED
DA 10.0% BIASED
HEDA 10.0% BIASED
HPDA 10.0% BIASED CRDA 0% ONDA 0%
DA 0%
HEDA 0% 10 10 HPDA 0% \$ Cost (10k) 8 \$ Cost (10k) 8 $\overline{6}$ 6 $\overline{4}$ $\overline{4}$ $\overline{2}$ $\overline{\mathbf{c}}$ 12 12 $\overline{2}$ $\overline{4}$ 6 8 10 $\overline{4}$ 6 8 10 Time Time CRDA 25.0% BIASED
DA 25.0% BIASED
HEDA 25.0% BIASED
HPDA 25.0% BIASED CRDA 50.0% BIASED
DA 50.0% BIASED
HEDA 50.0% BIASED
HPDA 50.0% BIASED 10.0 10.0 \$ Cost (10k) **2 COST (LUK)** 7.5 7.5 5.0 5.0 2.5 2.5 $\overline{4}$ 6 8 $10\,$ 12 $\overline{4}$ 6 8 10 12 Time Time

Appendix D: Additional Figures

Figure 6: Waiting Costs under K-Bias

Figure 7: Waiting Costs under K-Variance

K -Bias Type	0%	10%	25\%	50%
<i>Average envy</i>				
Sequential DA	0%	9.18\%	8.9%	6.69%
HPDA	0%	8.09\%	5.52%	3.54%
CRDA	0%	5.51%	4.58%	2.72\%
HEDA	59.1%	52.74\%	53.68\%	45.55%
<i>Average non-disruption rate</i>				
Sequential DA	85\%	85\%	86\%	86\%
HPDA	79%	79%	80\%	79%
CRDA	79%	82\%	83\%	83\%
HEDA	77\%	79%	78%	78%

Table 3: Envy for Sequential DA, HPDA, CRDA, and HEDA under K-Bias

Table 4: Envy for Sequential DA, HPDA, CRDA, and HEDA under K-Variance

K-Variance Type	0%	10%	25%	50%
<i>Average envy</i>				
Sequential DA	0%	14.93\%	37.38\%	46.59%
HPDA	0%	9.21%	25.77\%	42.68\%
CRDA	0%	8.31\%	11.67\%	16.07%
HEDA	59.1\%	56.64\%	64.94\%	79.68\%
Average non-disruption rate				
Sequential DA	85\%	85\%	85\%	85\%
HPDA	79%	79\%	79\%	78%
CRDA	79%	82\%	85%	89\%
HEDA	77\%	78\%	78%	77\%