# HOMOTOPICAL RECOGNITION OF DIAGRAM CATEGORIES

BORIS CHORNY AND DAVID WHITE

Abstract. Building on work of Marta Bunge in the one-categorical case, we characterize when a given model category is Quillen equivalent to a presheaf category with the projective model structure. This involves introducing a notion of homotopy atoms, generalizing the orbits of Dwyer and Kan, [\[29\]](#page-15-0). Apart from the orbit model structures of Dwyer and Kan, our examples include the classification of stable model categories after Schwede and Shipley, [\[46\]](#page-16-0), isovariant homotopy theory after Yeakel, [\[58\]](#page-16-1), and Cat-enriched homotopy theory after Gu, [\[37\]](#page-15-1). The most surprising example is the classical category of spectra, which turns out to be equivalent to the presheaf category of pointed simplicial sets indexed by the desuspensions of the sphere spectrum. No Bousfield localization is needed.

As an application, we give a classification of polynomial functors (the sense of Goodwillie calculus, [\[36\]](#page-15-2)) from finite pointed simplicial sets to spectra, and compare it to the previous work by Arone and Ching, [\[3\]](#page-14-0).

# 1. INTRODUCTION

Sixty years ago, Marta Bunge, [\[15\]](#page-14-1), gave a criterion for when a category is equivalent to a functor category indexed by a small category. This is happens if and only if the category is equipped with a set of atoms. In this paper, we provide a homotopical version of Bunge's classification, after introducing a suitable notion of homotopy atoms.

Diagram categories were used to lay the foundations of many important constructions in homotopy theory. For example, setting up stable homotopy theory begins with sequences of spaces, and the stable model structure on spectra is a localization of the projective model structure on diagrams [\[13\]](#page-14-2). The same is true equivariantly and motivically, where even the unstable homotopy theory requires diagram categories [\[41\]](#page-15-3). Diagram categories also arise in the homotopy theory of various categories of manifolds [\[51\]](#page-16-2), and when studying morphisms in any model category [\[54\]](#page-16-3). In monoidal settings, they arise when setting up the homotopy theory of operads [\[56\]](#page-16-4), of algebras over colored operads [\[52\]](#page-16-5), of (operad-structured) ideals of ring spectra [\[57\]](#page-16-6), of polynomial monads [\[6,](#page-14-3) [7\]](#page-14-4), and in higher category theory [\[8,](#page-14-5) [49\]](#page-16-7). More generally, any combinatorial model category is Quillen equivalent to the Bousfield localization of a category of simplicial presheaves by a result by Dugger, [\[28\]](#page-15-4).

There are exceptions, however. An important example is the homotopy theory of pro-objects in a model category established by Edwards and Hastings, [\[32\]](#page-15-5), and

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later on by Isaksen, [\[43\]](#page-15-6). In order to view it as (an opposite of) a subcategory of pro-representable functors in a category of diagrams, one has to consider the category of small presheaves indexed by a large category, as considered by the first author, [\[20,](#page-14-6) [21\]](#page-15-7). These categories of small functors indexed by large categories are neither locally presentable, nor cofibrantly generated, [\[16\]](#page-14-7). They fall under a more general framework of class-combinatorial model categories developed by the first author and Rosicky, [\[23\]](#page-15-8). In this work we deal only with diagram categories indexed by small categories. We hope to extend our theory to the categories of small functors in the future.

Hence, it is natural to want to characterize when a given model category is a diagram category, up to a Quillen equivalence.

In homotopy theory, the closest analog so far to Bunge's work was the concept of an orbit introduced by Dwyer and Kan, [\[29\]](#page-15-0). Their work produces a sufficient (but not necessary) condition for a category to have a model structure Quillen equivalent to the projective model structure on diagrams of spaces, indexed by a small category. By modifying the notion of an orbit we are able to characterize when a given model category is Quillen equivalent to the projective model structure on a diagram category (see Theorem [3.3\)](#page-4-0). We work in the generality of  $\mathcal V$ -enriched model categories, rather than only simplicial model categories, and illustrate the reason for this generality with a number of examples.

As an application of Theorem [3.3](#page-4-0) we give a new and simple classification of polynomial functors from finite spaces to spectra as category of *simplicial* presheaves over some indexing category. Our classification is not the first one, hence the need to compare it with the previous work.

There are several approaches to the classification of the finitary polynomial functors from pointed spaces to spectra starting with an unpublished paper of Dwyer and Rezk, subsumed later on by the fundamental work of Arone and Ching, [\[3\]](#page-14-0). In this paper, a careful analysis of the smash powers of the identity functor allowed the authors to relate the category of the n-truncated Com-modules with the category of  $\Gamma_{\leq n}$ -indexed diagrams of spectra, where  $\Gamma_{\leq n}$  is the category of pointed sets of size at most  $n$ . This is the work we have chosen to compare our results with.

Our classification may be viewed as a variant of the above result with the exception that the diagrams we consider take values in simplicial sets instead of spectra. In order to compare the two results we convert our diagrams of simplicial sets to the diagrams of spectra using the very specific form of the indexing category.

Another paper by the same authors performs the classification in terms of an additional structure on the symmetric sequence of derivatives of a functor, [\[2\]](#page-14-8), and it is further away from our approach.

Other works on this topic include [\[1\]](#page-14-9) and [\[34\]](#page-15-9) but these works concentrate on the classification of polynomial functors from spectra to spectra in terms of the Balmer spectrum and the category of Mackey functors, respectively.

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We now describe the structure of the paper. In Section [2,](#page-2-0) we recall the notion of orbits from [\[29\]](#page-15-0), and establish notation for the rest of the paper. In Section [3,](#page-3-0) we prove our main theorem – the classification of diagram categories up to homotopy – and provide numerous examples connecting this result to equivariant homotopy theory, isovariant homotopy theory, stable homotopy theory, and Catenriched homotopy theory. In Section [4,](#page-8-0) we classify polynomial functors, our main application of Theorem [3.3.](#page-4-0) Lastly, in Section [5,](#page-11-0) we compare our classification with other known classifications.

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### 2. Preliminaries

<span id="page-2-0"></span>In this section, we introduce definitions and results that we will need to prove our main theorems, and we set down notation. We assume the reader is familiar with model categories (e.g., [\[40\]](#page-15-10)), the basics of enriched model categories (e.g., [\[46\]](#page-16-0)), and the basics of combinatorial model categories (e.g., [\[9\]](#page-14-10)).

Throughout the paper,  $S$  will denote the category of simplicial sets, with the Kan-Quillen model structure. If  $\mathscr V$  is a closed symmetric monoidal category, and M is a V-model category [\[40,](#page-15-10) Definition 4.2.1], then for  $X, Y \in \mathcal{M}$ , we write hom $(X, Y)$  for the hom object in  $\mathscr V$ . If  $K \in \mathscr V$ , we write  $X \otimes K \in \mathscr M$  for the V -tensoring.

We next review the concept of an orbit, due to Dwyer and Kan [\[29,](#page-15-0) 2.1]. A set of orbits is defined to be a set  ${O_e}_{e \in E}$  of objects of a simplicial category M, that is closed under direct limits, has good homotopical behavior of transfinite compositions of pushouts of maps of the form  $O_e \otimes K \to O_e \otimes L$  (where  $K \to L$ is the inclusion of a subcomplex of a finite simplicial set), and hom $(O_e, -)$  turns pushouts as above into homotopy pushout squares [\[29,](#page-15-0) 2.1]. A good example to keep in mind, which explains the name 'orbit', is the set of objects  $G/H$  where G is a group and H is a subgroup of G, so that  $hom(G/H, X) \simeq X^H$ .

If a simplicial category  $\mathcal M$  is equipped with a set of orbits, satisfying the axioms of [\[29,](#page-15-0) 2.1], then there exists a model structure on  $\cal{M}$  Quillen equivalent to the category of simplicial presheaves indexed by the full subcategory of orbits and equipped with the projective model structure [\[29,](#page-15-0) Theorem 3.1].

<span id="page-2-1"></span>**Definition 2.1.** Let M be a simplicial category with a set  $\{O_e\}_{e \in E}$  of orbits. The *orbit model structure* on  $\mathcal{M}$  [\[29,](#page-15-0) Theorem 2.2] defines a morphism f to be a weak equivalence (resp. fibration) if and only if the induced map hom $(O_e, f)$  is a weak equivalence (resp. fibration) of simplicial sets. Cofibrations are characterized by the lifting property, and generating cofibrations are of the form  $O_e \otimes i_n$  where  $i_n : \partial \Delta[n] \to \Delta[n]$  is the usual inclusion.

<span id="page-3-1"></span>**Example 2.2.** If G is a topological group, then the set of spaces  $\{G/H \mid H \leq \}$ G} is a set of orbits. The standard model structure for G-spaces, modeling the equivariant homotopy theory in sense of Bredon [\[14\]](#page-14-11), is a special case of the orbit model structure, and Elmendorf's theorem [\[33\]](#page-15-11) is a special case of the Quillen equivalence [\[29,](#page-15-0) Theorem 3.1].

Another example of orbits is a collection of representable functors in the category of simplicial presheaves indexed by a small category  $\mathscr C$ . In this case the orbits define the projective model structure on the presheaf category  $\mathcal{S}^{\mathscr{C}^{op}}$ . Generalizing the concept of homogeneous spaces to include actions of small categories, Dror Farjoun and Zabrodsky came up with the following concept of an orbit([\[27,](#page-15-12) Definition 1.1): for a small category  $\mathscr{D}$ , a diagram  $T \in \mathcal{S}$  is an orbit if colim<sub> $\mathscr{D}$ </sub>  $T = *$ . Dror Farjoun noticed later on that these new orbits are also orbits in the sense of Dwyer-Kan [\[26\]](#page-15-13). The significant difference is that there is usually a proper class of orbits [\[17,](#page-14-12) [22\]](#page-15-14). The example of representable functors interpreted as orbits led to the development of the relative homotopy theory of Balmer and Matthey [\[4\]](#page-14-13) and was further extended to classes of orbits by the first author, in [\[19\]](#page-14-14).

Together with the work of Gu [\[37\]](#page-15-1), carrying over the concept of orbits to the categories enriched in  $\mathcal{C}at$ , and work of Housden [\[39\]](#page-15-15), using orbits for equivariant stable homotopy theory, these are all the currently known examples of orbits in the sense of Dwyer and Kan. It is evident, however, that the categories of diagrams of spaces (or spectra) appear frequently in homotopy theory and its applications and are not necessarily equipped with sets (or classes) of orbits. The axioms by Dwyer and Kan provide a sufficient condition allowing for a model structure Quillen equivalent to the category of diagrams of spaces, but this condition is not necessary.

<span id="page-3-0"></span>Having reviewed these preliminary concepts and results, we turn to our main theorem.

# 3. HOMOTOPY ATOMS

In this section we define the notion of *homotopy atoms* by softening the Dwyer and Kan orbit axioms [\[29,](#page-15-0) 2.1], and we give a necessary and sufficient condition for a model category to be Quillen equivalent to a projective model structure on a category of presheaves taking values in a closed symmetric monoidal combinatorial model category.

We say that a set of functors  $\{F_i \mid i \in I\}$  jointly reflect a property if, given a morphism f, if  $F_i f$  has the property for all  $i \in I$ , then so does f.

**Definition 3.1.** Let  $\mathcal V$  be a closed symmetric monoidal combinatorial model category with  $\mathscr{I} = \{A_i \hookrightarrow B_i | i \in I\}$  a set of generating cofibrations for some set I. Suppose that  $\mathscr M$  is a  $\mathscr V$ -model category. We say that  $\mathscr M$  is equipped with a set of homotopy atoms if there exists a set of cofibrant objects  $\mathcal{H} \subset \mathcal{M}$  such that

- (1) the functors hom $(T, -)$  for all  $T \in \mathcal{H}$  jointly reflect weak equivalences between fibrant objects;
- (2) the functors hom $(T, -)$  for all  $T \in \mathcal{H}$  commute, up to weak equivalence, with  $A_i \otimes -$  and  $B_i \otimes -$  for all  $i \in I$ , with homotopy pushouts, and with sequential homotopy colimits.

<span id="page-4-1"></span>Remark 3.2. We mostly have in mind (pointed) simplicial sets or spectra,  $\mathcal{V} =$  $S, S_*,$  Sp, as the base category, but the category of chain complexes or the category of small categories are also good examples. Note that in case  $\mathscr{V} = \mathscr{S}$  or  $\mathscr{S}_{*}$ , the verification of the commutation with  $A_i \otimes -$  and  $B_i \otimes -$  for all  $i \in I$  is redundant by the inductive argument similar to [\[18,](#page-14-15) Lemma 3.1] or, more generally [\[21,](#page-15-7) Lemma 4.2]. If  $\mathcal{V} =$  Sp this verification is redundant by the generalization of [\[11,](#page-14-16) Lemma 7.2], since the set of generating cofibrations in spectra has compact domains and codomains (these are the same generating cofibrations as in the projective model structure). Thus, Spanier-Whitehead duality implies that  $A \wedge X \simeq \hom(DA, X)$  for every fibrant  $X \in Sp$ . Substitute  $X = \hom(T, \widehat{\mathcal{I}})$  to obtain

$$
A \wedge \hom(T, \widehat{-}) \simeq \hom(DA, \hom(T, \widehat{-})) \cong \hom(DA \otimes T, \widehat{-})
$$
  

$$
\cong \hom(T, \hom(DA, \widehat{-})) \simeq \hom(T, \widehat{A \otimes -}),
$$

where the last weak equivalence is an application of Spanier-Whitehead duality again (the homotopy category of compact spectra is self dual).

For  $\mathcal{M} = \mathcal{C}$  at or Ch<sub>R</sub> the full verification is required, since weighted homotopy colimits are different from ordinary homotopy colimits, see [\[47\]](#page-16-8).

With this definition in hand, we are ready to formulate our main result, which is a homotopical classification of diagram categories.

<span id="page-4-0"></span>**Theorem 3.3.** Let  $\mathcal{M}$  be a  $\mathcal{V}$ -model category for a combinatorial base category  $\mathcal{V}$ . For any small  $\nu$ -category  $\mathscr C$ , we assume that the category of  $\nu$ -valued presheaves  $V^{\mathscr{C}^{op}}$  is equipped with the projective model structure. Then there exists a small  $\mathscr V$ -category C with a Quillen equivalence  $R: \mathscr M \longrightarrow^\text{\text{Cep}}: L$  if and only if the category  $M$  is equipped with a set of homotopy atoms.

*Proof.* Suppose first that there is a Quillen equivalence of  $\mathscr M$  with the category of  $\mathscr V$ -valued presheaves. Then put  $\mathcal H = \{T_C = L(\text{hom}(-, C) | C \in \mathscr C\}$ . This is a set of cofibrant objects in  $\mathcal{M}$ , since the representable functors are cofibrant in the projective model structure.

Note that the right adjoint  $R: \mathscr{M} \to \mathscr{V}^{\mathscr{C}^{op}}$  for every  $M \in \mathscr{M}$  may be computed as  $RM(C) = \text{hom}(T_C, M)$ , since, by Yoneda's lemma,

$$
\hom_{\mathscr{M}}(L(\hom(-,C)), M) = \hom_{\mathscr{V}^{\otimes \mathrm{op}}}(\hom_{\mathscr{C}}(-,C), RM) = RM(C).
$$

The right Quillen functor  $R$  reflects weak equivalences between fibrant objects in M and, for every cofibrant  $F \in \mathscr{V}^{\otimes \text{op}}$ , the derived unit of the adjunction  $F \to R\tilde{L}\tilde{F}$  is a weak equivalence [\[40,](#page-15-10) Corollary 1.3.16(c)]. Therefore, the functors hom $(T, -)$ ,  $T \in \mathcal{H}$  jointly reflect weak equivalences between fibrant objects.

In order to show that the functor hom $(T, \hat{\text{-}})$  commutes with the homotopy colimits for all  $T \in \mathcal{H}$ , it suffices to show that  $R^{\widehat{-}}$  commutes with homotopy colimits. For all  $M \in \mathcal{M}$  there exists a zigzag of weak equivalences  $M \rightarrow \widehat{M} \leftarrow L(R\widehat{M})_{\text{cof}}$  by [\[40,](#page-15-10) Corollary  $1.3.16(b)$ ]. Therefore,

$$
R(\text{hocolim }M_{i})_{\text{fib}} \simeq R(\text{hocolim }L(R\widehat{M}_{i})_{\text{cof}})_{\text{fib}} \simeq R(L \text{hocolim } (R\widehat{M}_{i})_{\text{cof}})_{\text{fib}}.
$$

The latter is weakly equivalent to hocolim<sub>i∈I</sub> ( $(R\widehat{M}_i)_{\text{cof}}$ )  $\simeq$  hocolim<sub>i∈I</sub> ( $R\widehat{M}_i$ ) by [\[40,](#page-15-10) Corollary 1.3.16(c)] again. In this argument, we denote by hocolim(−) the weighted homotopy colimit, i.e.,  $A \otimes -$  is a kind of homotopy colimit for a cofibrant  $A \in \mathscr{V}$  (see [\[47\]](#page-16-8)).

For the inverse direction, assume that  $\mathscr M$  is equipped with a set of homotopy atoms H. Let  $\mathscr C$  be a full  $\mathscr V$ -subcategory of  $\mathscr M$  generated by the elements of  $\mathcal H$ , and consider the adjunction  $R: \mathscr{M} \longrightarrow^{\mathscr{C}^{op}}: L$ , where  $RM(T) = \hom(T, M)$ for all  $T \in \mathcal{H}$  and  $M \in \mathcal{M}$ . Hence, the left adjoint is given by  $L(-) = (-) \otimes_{\mathcal{C}} H$ , where  $H: \mathscr{C} \to \mathscr{M}$  is the inclusion of a full subcategory generated by the homotopy atoms [\[44,](#page-16-9) 3.5].

This adjunction is a Quillen pair, since the right adjoint  $R$  readily preserves fibrations and trivial fibrations, since those are levelwise in the projective model structure. In order to show that this is a Quillen equivalence we verify the con-ditions of [\[40,](#page-15-10) Corollary 1.3.16(c)]. The right adjoint reflects weak equivalences by the first property of the homotopy atoms. It suffices to check that the map  $F \to R\widehat{LF}$  is a weak equivalence for all cellular  $F \in \mathscr{V}^{\otimes p}$ , since any cofibrant object is a retract of a cellular one. By a straightforward cellular induction, this map a weak equivalence on each stage of the cellular construction, which is a combination of cotensors with the generating cofibrations of  $\mathscr V$ , of homotopy pushouts, and of sequential homotopy colimits. They are all preserved strictly by  $L$  as a left Quillen functor and they are also preserved by  $R(-)$ , up to weak equivalence, by the second property of homotopy atoms the second property of homotopy atoms.

We now give several examples illustrating the power of Theorem [3.3.](#page-4-0) The first example is to the model categories that inspired the original definition of orbits.

**Example 3.4.** Dwyer-Kan orbits in a simplicial category  $\mathcal{M}$  are homotopy atoms with respect to the model structure they induce on  $\mathcal{M}$ , of Definition [2.1.](#page-2-1)

Another famous classification in homotopical algebra is the Schwede-Shipley classification of stable model categories. Diagrams play an essential role here, because modules over a ring with many objects are best encoded as diagrams. Thus, the connection to Theorem [3.3](#page-4-0) should not be entirely surprising.

<span id="page-6-0"></span>**Example 3.5.** Let  $\mathcal{M}$  be a stable simplicial model category equipped with a set of (weak) compact generators  $\mathcal G$  in the sense of Schwede and Shipley [\[46\]](#page-16-0). Consider the Quillen equivalence  $F_0: \mathscr{M} \longrightarrow_{\mathbb{R}} \mathrm{Sp}^{\Sigma} \mathscr{M} : \mathrm{Ev}_0$  of  $\mathscr{M}$  with a model category enriched over the category  $\mathscr{V} = \text{Sp}^{\Sigma}$  of symmetric spectra [\[41,](#page-15-3) Def. 7.3]. Since the generators are defined on the level of the homotopy category, which did not change, the set  $F_0\mathcal{G}$  forms a set of homotopy generators in the category  $Sp^{\Sigma}\mathcal{M}$ . By [\[46,](#page-16-0) Theorem 3.9.3], this spectral category is Quillen equivalent to the category of spectral presheaves indexed by the endomorphism category  $\mathcal{E}((F_0 \mathcal{G})_{fib})$ , which is a full subcategory of  $\text{Sp}^{\Sigma}\mathcal{M}$  generated by the fibrant replacements of the generators. Hence, the category  $Sp^{\Sigma}\mathscr{M}$  may be equipped with a set of homotopy atoms by Theorem [3.3.](#page-4-0)

A monoidal version of [\[46,](#page-16-0) Theorem 3.9.3] has recently been proven by [\[10\]](#page-14-17) (for the monogenic setting) and by [\[24\]](#page-15-16) (for the general setting). We note that [\[24\]](#page-15-16) also has a  $\mathscr V$ -version of [\[46,](#page-16-0) Theorem 3.9.3], allowing for a generalization of Example [3.5](#page-6-0) away from the simplicial context. It would be interesting to formulate a monoidal version of Theorem [3.3,](#page-4-0) in analogy with [\[24,](#page-15-16) Theorem A].

Our next example shows how to use Theorem [3.3](#page-4-0) to provide a new monoidal model for spectra, an important problem that was open from the 1950s until the late 1990s, when several monoidal models were found.

Notation 3.6. In the next example and further occurrences of stable model categories in this paper, we will use the desuspension notation for the derived version of the loop functor:  $\lambda$ .

$$
\Sigma^{-i}(-)=\left(\Omega^i(-)_{\rm fib}\right)_{\rm cof}.
$$

<span id="page-6-1"></span>**Example 3.7.** Let  $\mathcal{M} = Sp$ , be the Bousfield-Friedlander model category of spectra enriched over  $\mathscr{V} = \mathcal{S}_{*}$  [\[13\]](#page-14-2). Consider the set of homotopy atoms  $\mathcal{H} =$  $\{\Sigma^{-i}(\Sigma^{\infty}S^{0})\,|\,i\geq 0\}$ . We denote by  $\mathscr E$  the full subcategory of spectra generated by  $H$ . Let us show that they jointly reflect weak equivalences of Ω-spectra (the fibrant objects).

Notice first that if  $X_{\bullet}$  is an  $\Omega$ -spectrum, then  $\Sigma^{i} X_{\bullet} \simeq (X_{i}, X_{i+1}, \ldots)$ , which is readily verified by application of  $\Omega^i$  on both sides.

Let  $f_{\bullet} : X_{\bullet} \to Y_{\bullet}$  be a map of  $\Omega$ -spectra. If the maps

$$
\hom(\Sigma^{-i}(\Sigma^{\infty}S^0), f_{\bullet}) \simeq \hom(\Sigma^{\infty}S^0, \widehat{\Sigma^i(f_{\bullet})}) \simeq f_i
$$

are weak equivalences for all  $i \in \mathbb{N}$ , then  $f_{\bullet}$  is a projective weak equivalence of Ω-spectra, hence a stable weak equivalence.

Moreover, homotopy pushouts of spectra are homotopy pullbacks. Thus, the functors hom $(\Sigma^{\infty}S^i, \widehat{-})$  commute with homotopy pushouts and also with sequential colimits, since the desuspensions of the sphere spectrum are compact spectra. They also commute, up to a weak equivalence, with cotensoring with a finite simplicial set,  $A \wedge -$ , by Remark [3.2.](#page-4-1)

Note that the full simplicial subcategory  $\mathscr E$  of Sp may be chosen to be monoidal. Even though  $\Sigma^{-i}(\Sigma^{\infty}S^{0}) \wedge \Sigma^{-j}(\Sigma^{\infty}S^{0}) \simeq \Sigma^{-(i+j)}\Sigma^{\infty}S^{0}$ , we can find a Dwyer-Kan equivalent model with a strict monoidal structure induced by the smash product. One possibility is to take the opposite category of the positive-dimension spheres. By Spanier-Whitehead duality it is equivalent to  $\mathscr{E}$  [\[45,](#page-16-10) Proposition 17.16]. The monoidal structure on the positive-dimension spheres is evidently defined, since all objects are suspension spectra:  $\Sigma^{\infty} S^i \wedge \Sigma^{\infty} S^j = \Sigma^{\infty} S^{i+j}$ . Another option is to take the subcategory  $\mathscr E$  in some closed symmetric monoidal model of spectra. Given a monoidal structure on  $\mathscr{E}$ , the Day convolution product defines a closed symmetric monoidal model structure on the category of simplicial presheaves  $\mathcal{S}^{\mathscr{E}^{\text{op}}}$ equipped with the projective model structure. This is a new model for spectra with a highly structured smash product.

<span id="page-7-0"></span>Our next example is related to a new use of orbits, to isovariant homotopy theory, which we will describe.

Example 3.8. Yeakel's isovariant homotopy theory [\[58\]](#page-16-1) and her isovariant Elmendorf's theorem can be viewed as a special case of Theorem [3.3,](#page-4-0) in analogy with Example [2.2.](#page-3-1) Let G be a finite group. Let  $\mathscr V$  be S, the category of simplicial sets. Let  $\mathcal{M}$  be Yeakel's isvt-Top, the category of compactly generated  $G$ -spaces with isovariant maps (and an added formal terminal object), i.e., equivariant maps  $f: X \to Y$  with an equality of stabilizers  $G_x = G_{f(x)}$  for all  $x \in X$ . Let O be Yeakel's link orbit category  $\mathcal{L}_G$  [\[58,](#page-16-1) Definition 2.1]. This category contains all orbits  $G/H$  but not all maps between them, because the goal is isovariant rather than equivariant homotopy theory. Yeakel's model structure on  $\mathscr{M}$  [\[58,](#page-16-1) Theorem 3.2] is a special case of the orbit model structure of Definition [2.1,](#page-2-1) and the Quillen equivalence of her isovariant Elmendorf's theorem [\[58,](#page-16-1) Theorem 4.1] is a special case of Theorem [3.3.](#page-4-0)

Remark 3.9. There are many interesting questions that can be formulated for isovariant homotopy theory, with Example [3.8](#page-7-0) in place. For instance, one could create spectra on isovariant spaces, following the model of [\[42\]](#page-15-17), and obtain a stable version of the isovariant Elmendorf's theorem as a special case of Theorem [3.3.](#page-4-0) Or, one could work out a global homotopy theory, e.g., following [\[5\]](#page-14-18). It is also possible to investigate the monoidal properties of the category of isovariant spaces, and work out the homotopy theory of operads and algebras in the isovariant context (which should also have versions of Theorem [3.3\)](#page-4-0), following [\[48,](#page-16-11) [38,](#page-15-18) [50\]](#page-16-12). Another option would be to work out spectral Mackey functors in the isovariant context.

Our next example is for model categories enriched in the category of small categories (or acyclic categories, or posets), and illustrates the value of working with  $\nu$ -model categories in Theorem [3.3](#page-4-0) rather than only with simplicial model categories as we did in [\[25\]](#page-15-19).

Example 3.10. In 2016, Gu studied orbit model structures in the case where  $\mathcal M$  is the category Cat of small categories, or Ac of acyclic categories, or Pos of posets. Gu's main result [\[37,](#page-15-1) Theorem 1.1] proves the existence of the orbit model structure of Definition [2.1](#page-2-1) on  $Cat^I$ , where I is a small category and O is a set of orbits (or, more generally, a locally small class), and further proves that the Thomason Quillen equivalence  $S \leftrightarrows Cat$  induces a Quillen equivalence  $S^I \leftrightarrows Cat^I$ with the orbit model structures. Since  $Cat$  is a simplicial category (in more ways than one), the existence of Gu's model structure follows from [\[22\]](#page-15-14), since one can replace the internal hom of Cat by its nerve, in Gu's proof. Furthermore, because [\[37\]](#page-15-1) was never published, we mention that the Quillen equivalence of [\[37,](#page-15-1) Theorem 1.1] follows from Theorem [3.3](#page-4-0) (to reduce the question to a Quillen equivalence of presheaf categories), together with [\[25,](#page-15-19) Proposition 1.3] (in place of [\[37,](#page-15-1) Lemma 4.2]) and the argument of [\[25,](#page-15-19) Proposition 1.4] (in place of [\[37,](#page-15-1) Lemma 4.3]) to compare the model structures on presheaf categories. The same holds for  $Ac$ , Pos [\[37,](#page-15-1) Theorem 6.1], and for the G-projective model structure on any of the three choices for  $\mathcal M$  for a discrete group  $G$  ([\[37,](#page-15-1) Proposition 7.1] for the model structure and [\[37,](#page-15-1) Corollary 7.3] for the Quillen equivalence), since these are again orbit model categories.

Remark 3.11. While we are on the topic of Gu's unpublished paper, we wish to point out that [\[37,](#page-15-1) Remark 5.11] is wrong, because the orbit model structure on  $Cat<sup>I</sup>$  will be a simplicial model structure (i.e., satisfy axiom SM7, by [\[29,](#page-15-0) Theorem 2.2]), but that's not true for the Thomason model structure with Gu's structure. The issue is that the  $(sd, Ex)$  enrichment (where sd is for 'subdivision') is not a simplicially enriched adjunction, since sd does not preserve simplicially enriched colimits, since finite products in  $\mathcal S$  are tensors. To see that the SM7 axiom fails for the Thomason model structure, let  $A$  be the one-point category, note that it's Thomason cofibrant, and note that the internal hom satisfies  $Fun(A, B) = B$ for any B. Now, if B were Thomason fibrant, and if the SM7 axiom held, it would imply that the simplicial mapping space  $hom(A, B) = NFun(A, B) = NB$ is a Kan complex. But that only happens if  $B$  is a groupoid, and not every Thomason fibrant B must be a groupoid, e.g., any category with pushouts is Thomason fibrant. The same issue arises if one uses the simplicial mapping space  $hom(A, B) = Ex<sup>2</sup> N Fun(A, B)$ . The error is in the last line of [\[37,](#page-15-1) Remark 5.11], where Gu states that the orbit model structure on  $Cat^I$ , where I is the one-point category and  $O = \{*\}$ , coincides with the Thomason model structure. In fact, it coincides with the discrete model structure which, like the folk model structure, does satisfy the SM7 axiom. No choice of orbits can yield the Thomason model structure as an orbit model category.

### 4. A classification of (finitary) polynomial functors

<span id="page-8-0"></span>In this section we apply Theorem [3.3](#page-4-0) to another stable model category, the category of simplicial functors (enriched over the category of pointed simplicial sets,  $\mathscr{V} = \mathcal{S}_{*}$  from finite pointed simplicial sets to the category of symmetric

spectra,  $Sp^{\mathcal{S}_{*}^{\text{fin}}}$ . The goal is to classify the polynomial functors in the sense of Goodwillie [\[35\]](#page-15-20) as diagrams of spaces. In the next section we will show how to convert this representation to be in terms of diagrams of spectra.

Remark 4.1. We would like to stress that because our functors are enriched over the category of *pointed* simplicial sets  $S_*$ , it follows that all functors  $F \in \text{Sp}^{\mathcal{S}_*^{\text{fin}}}$  are reduced, i.e.,  $F(*) = 0$ . This follows from the representation of F as a weighted colimit of representable functors:

$$
F(X) = \int^{Y \in \mathcal{S}_*^{\text{fin}}} R^X(Y) \wedge F(X).
$$

There are other classifications of polynomial functors from pointed spaces to spectra. Dwyer and Rezk showed that polynomial functors are equivalent to the functors indexed by the category of finite sets and surjections (unpublished); a different set of classification results is due to Arone and Ching [\[2,](#page-14-8) [3\]](#page-14-0). They show that the homotopy category of polynomial functors is equivalent to the category of symmetric sequences of spectra equipped with an additional structure of a divided power right module over the operad formed by the derivatives of the identity on based spaces, as in [\[2\]](#page-14-8), or, alternatively, they consider the category of coalgebras in symmetric sequences of spectra over the comonad  $C_{KE_{\bullet}}$ , where  $KE_n$  is the Koszul dual of the little *n*-disc operad and  $KE_{\bullet}$  is an inverse sequence of operads, or a pro-operad, as in [\[3\]](#page-14-0). For  $\nu$ -enriched contexts, the homotopy theory of algebras over operads is described in [\[52,](#page-16-5) [55\]](#page-16-13), and for coalgebras over cooperads, with connections to Koszul duality, in [\[53\]](#page-16-14).

<span id="page-9-0"></span>**Lemma 4.2.** Let  $\mathcal{H} = \left\{ \sum_{k} P_{k} \left( \sum_{i=1}^{k} R_{i}^{S_{i}} \right)_{c \in \mathbb{R}} \right\} \mid 1 \leq k \leq n, p \geq 0 \right\} \subset \text{Sp}^{\mathcal{S}_{*}^{\text{fin}}}$  be a set of objects, and let  $f: F \to G$  be a map of fibrant functors in  $\text{Sp}^{\mathcal{S}_*^{\text{fin}}}$  equipped with the *n*-excisive model structure, [\[12,](#page-14-19) Theorem 4.6]. If the induced map

 $hom(H, f): hom(H, F) \to hom(H, G)$ 

is a weak equivalence of (pointed) simplicial sets for all  $H \in \mathcal{H}$ , then the map f is a projective weak equivalence. In other words, the objects of the set  $H$  jointly reflect weak equivalences of fibrant objects.

*Proof.* Put  $H_{k,p} = \Sigma^{-p} \left( \Sigma^{\infty} (\bigwedge_{i=1}^{k} R^{S^0})_{\text{cof}} \right)$ . Recall from [\[12,](#page-14-19) Lemma 8.2(ii)] that for any projectively fibrant  $F \in \text{Sp}^{\mathcal{S}_{*}^{\text{fin}}},$  the following holds:

$$
\begin{split} \operatorname{hom}_{\operatorname{Sp}^{S_*^{\text{fin}}}}(H_{k,p}, F) &= \operatorname{hom}_{\operatorname{Sp}^{S_*^{\text{fin}}}}\left(\Sigma^{-p}\Sigma^\infty(\bigwedge_{i=1}^k R^{S^0})_{\operatorname{cof}}, F\right) \\ &\simeq \operatorname{hom}_{\mathcal{S}_*^{S_*^{\text{fin}}}}\left((\bigwedge_{i=1}^k R^{S^0})_{\operatorname{cof}}, \Omega^\infty \widehat{\Sigma^p F}\right) \\ &= cr_k(\Omega^\infty \widehat{\Sigma^p F})(S^0, \dots S^0) \\ &\simeq \Omega^\infty \left(\Sigma^p (cr_k F(S^0, \dots, S^0))\right)_{\operatorname{fib}}. \end{split}
$$

Similarly to Example [3.7,](#page-6-1)  $\Omega^{\infty} (\Sigma^p (cr_k F(S^0, \ldots, S^0)))_{\text{fib}}$  is equivalent to the *p*-th layer of the spectrum  $cr_kF(S^0,\ldots,S^0)_{\text{fib}}$ . Let  $f: F \to G$  be a natural transformation of projectively fibrant functors from finite spaces to spectra such that hom $(H_{k,p}, f)$  is a weak equivalence for all  $1 \leq k \leq n, p \geq 0$ . Fix k and q and let p run from 0 to  $\infty$ . Then we obtain a weak equivalence of spectra

<span id="page-10-0"></span>
$$
(1) \t cr_k f(S^0, \dots S^0) \colon cr_k F(S^0, \dots S^0) \to cr_k G(S^0, \dots S^0)
$$

for all  $1 \leq k \leq n$ , as these spectra are levelwise weakly equivalent after fibrant replacement.

Since the model category of polynomial functors  $Sp^{\mathcal{S}_{*}^{\text{fin}}}$  is stable (the suspension commutes with polynomial approximation), the fibre sequence  $D_k F \to P_k F \to$  $P_{k-1}F$  is part of the exact triangle  $D_kF \to P_kF \to P_{k-1}F \to \Sigma^{-1}D_kF$ . We will use this to prove that  $f : F \to G$  is a weak equivalence. Put  $\Sigma^{-1}D_kF = R_kF$ , cf [\[36,](#page-15-2) Lemma 2.2], and consider the fibre sequence  $P_k F \to P_{k-1} F \to R_k F$ , where  $R_k F$ is a k-homogeneous functor. The same construction applies to G, and  $f: F \to G$ induces a morphism of exact triangles. Assume for the sake of induction that  $P_{k-1}f: P_{k-1}F \to P_{k-1}G$  is a weak equivalence. The base case is satisfied since both functors F and G are reduced.

Notice that the weak equivalence [\(1\)](#page-10-0) implies, in particular, that

$$
cr_kR_kf: cr_kR_kF(S^0,\ldots,S^0)\to cr_kR_kG(S^0,\ldots,S^0)
$$

is a weak equivalence, hence, by [\[36,](#page-15-2) Proposition 5.8], the induced map of the multilinear functors

$$
cr_kR_kf: cr_kR_kF(X_1,\ldots,X_k)\to cr_kR_kG(X_1,\ldots,X_k)
$$

is a weak equivalence for all  $X_1 \ldots, X_k \in \mathcal{S}_{*}^{\text{fin}}$ . Therefore, [\[36,](#page-15-2) Proposition 3.4] implies that the induced map of the k-homogeneous functors  $R_k f: R_k F \to R_k G$ is a weak equivalence. Hence, the induced map of the homotopy fibers is a weak equivalence.

We conclude by induction that, for any n, the map  $f: F \to G$  of n-excisive functors is a weak equivalence.

<span id="page-11-1"></span>**Theorem 4.3.** Let  $\mathcal{M} = \text{Sp}^{\mathcal{S}_{*}^{\text{fin}}}$  be the model structure for n-polynomial functors enriched over  $\mathscr{V} = \mathcal{S}_{*}$ , obtained from the projective model structure by left Bousfield localization [\[12,](#page-14-19) Theorem 4.6]. Then the set

$$
\mathcal{H} = \left\{ \left. \Sigma^{-p} \left( \sum_{i=1}^{k} R^{S^0} \right)_{\text{cof}} \right) \middle| 1 \leq k \leq n, p \geq 0 \right\}
$$

is a set of homotopy atoms for  $\mathcal M$ . In other words, the category  $\mathcal M$  is Quillen equivalent to the category of simplicial presheaves indexed by the full subcategory  $\mathscr C$  of  $\mathscr M$  generated by  $\mathcal H$ .

*Proof.* We have to verify two properties of  $H$ . The first property is that the functors  $\{\text{hom}(T, -)| T \in \mathcal{H}\}\$  jointly reflect weak equivalences of fibrant objects. This is proven in Lemma [4.2.](#page-9-0)

The second property is the commutation of  $\hom(\Sigma^{-p} \Sigma^{\infty} (\bigwedge_{i=1}^k R^{S^0})_{\text{cof}}, \widehat{-})$  with homotopy pushouts and sequential homotopy colimits. The commutation with the functors  $A \wedge -$  for finite  $A \in \mathcal{S}_*$  is taken care of by Remark [3.2.](#page-4-1) Since  $\mathcal{M} =$  $Sp^{\mathcal{S}_{*}^{\text{fin}}}$  is a stable model category, homotopy pushout squares are also homotopy pullbacks and vice versa, hence the required commutation is obvious. To verify that these mapping spaces commute with the sequential homotopy colimits recall that applying these functors we just compute the cross-effects levelwise with respect to the levels of the spectrum [\[12,](#page-14-19) Lemma  $8.2(ii)$ ]. But cross-effects are the total homotopy fibers of certain cubical diagrams, i.e., a sequence of finite homotopy limits, which commute with the sequential homotopy colimits.  $\Box$ 

# <span id="page-11-0"></span>5. Comparison to other classifications of polynomial functors

This section is devoted to the comparison of our classification of polynomial functors to other results in this field.

<span id="page-11-2"></span>**Proposition 5.1.** The full subcategory  $\mathscr C$  of the category of simplicial functors  $\text{Sp}^{\mathcal{S}^{\text{fin}}_{*}}$ , defined in Theorem [4.3,](#page-11-1) may be decomposed, up to a Dwyer-Kan equivalence, into a Kelly product of two categories the  $\mathscr E$  from Example [3.7](#page-6-1) and the full subcategory  $\mathscr{F} \subset \text{Sp}^{\mathcal{S}_*^{\text{fin}}}$  generated by the set of object  $\left\{ \sum^{\infty} (\bigwedge_{i=1}^k R^{S^0})_{\text{cof}} \middle| 1 \leq k \leq n \right\}$ .

Reminder 5.2. Let  $\mathscr V$  be a closed symmetric monoidal category. Kelly product of two *V*-categories  $\mathscr A$  and  $\mathscr B$  is a *V*-category  $\mathscr A \otimes \mathscr B$  with  $obj(\mathscr A \otimes \mathscr B)$  =  $obj(\mathscr{A}) \times obj(\mathscr{B})$  and  $hom_{\mathscr{A} \otimes \mathscr{B}}((A, B), (A', B')) = hom_{\mathscr{A}}(A, A') \otimes hom_{\mathscr{B}}(B, B'),$ [\[44,](#page-16-9) Section 1.4]. Moreover, if  $\mathscr{A}, \mathscr{B}, \mathscr{C}$  are  $\mathscr{V}\text{-categories, then the exponential}$ rule is satisfied, [\[44,](#page-16-9) Section 6.5].

$$
\mathscr{C}^{\mathscr{A}\otimes\mathscr{B}}\cong\left(\mathscr{C}^{\mathscr{A}}\right)^{\mathscr{B}}.
$$

 $\Box$ 

Proof of [5.1.](#page-11-2) Let us put

$$
C_{p,k} = \Sigma^{-p} \left( \Sigma^{\infty} (\bigwedge_{i=1}^k R^{S^0})_{\text{cof}} \right) \in \mathscr{C}, E_p = \Sigma^{-p} (\Sigma^{\infty} S^0), \text{ and } F_k = \Sigma^{\infty} (\bigwedge_{i=1}^k R^{S^0})_{\text{cof}} \in \mathscr{F},
$$

for  $1 \leq k \leq n$  and  $p \geq 0$ .

Now we are going to establish a Dwyer-Kan equivalence of simplicial categories  $T: \mathscr{C} \to \mathscr{E} \wedge \mathscr{F}$ , assigning  $T(C_{p,k}) = (E_p, F_k)$  and for each pair of objects

$$
T_{C_{p_1,k_1},C_{p_2,k_2}}: \text{hom}_{\mathscr{C}}(C_{p_1,k_1}, C_{p_2,k_2}) \longrightarrow
$$

$$
\text{hom}_{\mathscr{E}\otimes\mathscr{F}}((E_{p_1}, F_{k_1}), (E_{p_2}, F_{k_2}))
$$

$$
= \text{hom}_{Sp}(\Sigma^{-p_1}(\Sigma^{\infty}S^0), \Sigma^{-p_2}(\Sigma^{\infty}S^0)) \wedge \text{hom}_{Sp}^{f_{\text{fin}}}(F_{k_1}, F_{k_2})
$$

$$
= \begin{cases} S^{p_1-p_2} \wedge \text{hom}_{Sp}^{f_{\text{fin}}}(F_{k_1}, F_{k_2}), & \text{if } p_1 \ge p_2; \\*, & \text{if } p_1 < p_2 \end{cases}
$$

is assigned to be a natural weak equivalence of pointed simplicial sets, since hom<sub> $\mathscr{C}(C_{p_1,k_1}, C_{p_2,k_2}) = \text{hom}_{\mathscr{C}}((\bigwedge_{i=1}^{k_1} R^{S^0})_{\text{cof}}, \Omega^{\infty} \Sigma^{p_1-p_2}(\Sigma^{\infty}(\bigwedge_{i=1}^{k_2} R^{S^0})_{\text{cof}})).$  By [\[12,](#page-14-19)</sub> Lemma 8.2(ii)], this is  $cr_{k_1}(\Omega^{\infty} \Sigma^{p_1-p_2}(\Sigma^{\infty}(\bigwedge_{i=1}^{k_2} R^{S^0})_{\text{cof}}))(S^0, \ldots, S^0)$ . In case  $p_1 <$  $p_2$ , this cross-effect is contractible, since any suspension spectrum is connective and its desuspensions can only produce contractible entries at the 0-th level, therefore the cross-effect of a contractible diagram is contractible. In case  $p_1 \geq p_2$ , we notice that  $(p_1 - p_2)$ -fold suspension in spectra may be modeled as a smash product with a simplicial sphere, which may be viewed as a homotopy colimit, i.e., it commutes with cross-effects for functors taking values in spectra, since it may be equivalently expressed as a co-cross-effect (a finite sequence of homotopy colimits), [\[3,](#page-14-0) Definition 1.5]. Therefore, if  $p_1 \geq p_2$ , then

$$
\begin{split} \hom_{\mathscr{C}}(C_{p_1,k_1}, C_{p_2,k_2}) &\simeq \Omega^{\infty}(S^{p_1-p_2} \wedge cr_{k_1}(\Sigma^{\infty}(\bigwedge_{i=1}^{k_2} R^{S^0})_{\text{cof}})(S^0, \dots, S^0)) \\ &= S^{p_1-p_2} \wedge \Omega^{\infty} cr_{k_1}(\Sigma^{\infty}(\bigwedge_{i=1}^{k_2} R^{S^0})_{\text{cof}})(S^0, \dots, S^0) \\ &= S^{p_1-p_2} \wedge \hom_{\text{Sp}} \mathfrak{sl}_n^{\text{fin}}(F_{k_1}, F_{k_2}). \end{split}
$$

In other words, there is a natural weak equivalence of simplicial sets

$$
T_{C_{p_1,k_1},C_{p_2,k_2}}: \hom_{\mathscr{C}}(C_{p_1,k_1},C_{p_2,k_2}) \tilde{\to} \hom_{\mathscr{E} \otimes \mathscr{F}}((E_{p_1},F_{k_1}),(E_{p_2},F_{k_2}))
$$

for every pair of objects of  $\mathscr{C}$ , or T is a Dwyer-Kan equivalence.

**Corollary 5.3.** The category of functors  $Sp^{S^{\text{fin}}}_{*}$  equipped with the n-excisive model structure is Quillen equivalent to the projective model structure on the category  $Sp^{\mathscr{F}^\mathrm{op}}$  .

*Proof.* By Theorem [4.3,](#page-11-1) the *n*-excisive model category on  $Sp^{\mathcal{S}_{*}^{\text{fin}}}$  is Quillen equivalent to the projective model category on  $\mathcal{S}_{*}^{\mathscr{C}^{op}}$ ∗ . The latter is Quillen equivalent  $\mathrm{to} \,\, \mathcal{S}_*^{(\mathscr{E} \wedge \mathscr{F})^{\mathrm{op}}} = \, \mathcal{S}_*^{\mathscr{E}^{\mathrm{op}} \wedge \mathscr{F}^{\mathrm{op}}} = \, (\mathcal{S}_*^{\mathscr{E}^{\mathrm{op}}}$ ∗ ) <sup>F</sup>op by Dwyer-Kan theorem, [\[30\]](#page-15-21), and Proposition [5.1.](#page-11-2)

The last two categories are equivalent as simplicial categories, but we need to compare their model structures as well, so we claim that this equivalence of categories is a Quillen equivalence if we impose the projective model structure on both sides:  $(S_{*}^{\mathscr{E}^{op}\wedge \mathscr{F}^{op}})_{proj} \leftrightarrows ((S_{*}^{\mathscr{E}^{op}})_{op})$  $(\ell^{\otimes_{\text{op}}})_{\text{proj}}$ ,  $(\ell^{\otimes_{\text{op}}})_{\text{proj}}$ . The verification is left to the reader.

Example [3.7](#page-6-1) implies that the last model category is actually the projective model category of contravariant functors from  $\mathscr F$  to the category of spectra  $\text{Sp}^{\mathscr F^{op}}$ . . □

Now we are able to compare our classification result to the Dwyer-Rezk classification of polynomial functors using the results of Arone and Ching, [\[3\]](#page-14-0).

Let  $\Omega_{\leq n}$  denote the category of non-empty finite sets with at most n points and surjections as morphisms. And let  $\Omega_{\leq n}^+$  denote the category with the same objects as  $\Omega_{\leq n}$ , and morphisms  $\hom_{\Omega_{\leq n}^+}(m,k) = \hom_{\Omega_{\leq n}^+}(m,k)_+ \in \mathcal{S}_*$ . The reason for adding the base point is to have an enrichment over  $S_*$  on  $\Omega^+_{\leq n}$ .

Recall that the categories Sp and  $\mathcal{S}_*$  may be enriched over the category of simplicial sets  $S$ , as well as over the category of pointed simplicial sets, using the half-smash product instead of the smash product.

**Corollary 5.4.** The  $S_*$ -category of functors  $Sp^{S_*^{\text{fin}}}$  equipped with the n-excisive model structure is Quillen equivalent to the projective model structure on the Scategory  $\text{Sp}^{\Omega_{\leq n}}$ .

Proof. Let S denote the sphere spectrum. Consider the weak equivalence

$$
\phi \colon \bigvee_{m \to k} \mathbb{S} \xrightarrow{\sim} \text{spt}(F_k, F_m), \, m, k \le n
$$

which appeared in [\[3,](#page-14-0) 3.16]. The notation spt for the spectrum-valued mapping space is taken from [\[12,](#page-14-19) 8.3], since the correspondent notation Nat in [\[3\]](#page-14-0) is not detailed. Looking at the 0-th level of  $\phi$  we obtain the weak equivalence of simplicial sets

$$
\mathrm{Ev}_0 \circ \phi \colon \mathrm{hom}_{\Omega_{\leq n}^+}(m,k) \longrightarrow \mathrm{hom}_{\mathrm{Sp}}\mathfrak{S}_*^{\mathrm{fin}}(F_k,F_m)
$$

In other words, there is a Dwyer-Kan equivalence of  $S_*$ -categories  $\Omega^+_{\leq n}$  and  $\mathscr{F}^{\text{op}}$ . That implies a Dwyer-Kan equivalence between  $\mathscr{E}^{\text{op}} \wedge \Omega^+_{\leq n}$  and  $\mathscr{C}^{\text{op}} = \mathscr{E}^{\text{op}} \wedge$  $\mathscr{F}^{\mathrm{op}}$ . By a pointed version of [\[31,](#page-15-22) Theorem 2.1], proved by Lukáš Vokřínek, [\[47,](#page-16-8) Theorem 10, there is a Quillen equivalence between  $S_*^{\mathscr{E}^{\text{op}}\wedge\Omega_{\leq n}^+}$  and  $S_*^{\mathscr{E}^{\text{op}}}$  with the projective model structure. The latter is Quillen equivalent to the n-excisive model structure on the category of  $S_*$ -functors  $Sp^{S_*^{\text{fin}}}$  by Theorem [4.3,](#page-11-1) while the former may be interpreted as a category of  $S_*$ -functors  $S_*^{\mathscr{E}^{\text{op}} \wedge \Omega_{\leq n}^{\neq}} = \left( \mathscr{S}_*^{\mathscr{E}^{\text{op}}} \right)$  $(\mathcal{C}^{\text{op}})^{\Omega^+_{\leq n}} = \text{Sp}^{\Omega^+_{\leq n}}.$ 

The last reduction to the original Dwyer-Rezk classification is to note that the underlying category of the  $S_*$ -category  $Sp^{\Omega_{\leq n}^+}$  is naturally equivalent to the Scategory  $Sp^{\Omega_{\leq n}}$  and the projective model structures on both underlying categories coincide. □

### **REFERENCES**

- <span id="page-14-9"></span><span id="page-14-8"></span>[1] G. Arone, T. Barthel, D. Heard, and B. Sanders. The spectrum of excisive functors. Preprint, arxiv::2402.04244, 2024.
- <span id="page-14-0"></span>[2] G. Arone and M. Ching. A classification of Taylor towers of functors of spaces and spectra. Adv. Math., 272:471–552, 2015.
- [3] G. Arone and M. Ching. Cross-effects and the classification of Taylor towers. Geom. Topol., 20(3):1445–1537, 2016.
- <span id="page-14-18"></span><span id="page-14-13"></span>[4] P. Balmer and M. Matthey. Codescent theory. I. Foundations. *Topology Appl.*, 145(1-3):11– 59, 2004.
- <span id="page-14-3"></span>[5] M. Batanin, F. D. Leger, and D. White. Model structures on operads and algebras from a global perspective. Preprint, 2023.
- [6] M. Batanin, F. D. Leger, and D. White. Quasi-tame substitudes and the grothendieck construction. Preprint, 2023.
- <span id="page-14-4"></span>[7] M. Batanin and D. White. Left Bousfield localization and Eilenberg-Moore categories. Homology Homotopy Appl., 23(2):299–323, 2021.
- <span id="page-14-5"></span>[8] M. Batanin and D. White. Homotopy theory of algebras of substitudes and their localisation. Trans. Amer. Math. Soc., 375(5):3569–3640, 2022.
- <span id="page-14-10"></span>[9] M. Batanin and D. White. Left Bousfield localization without left properness. J. Pure Appl. Algebra, 228(6):Paper No. 107570, 23, 2024.
- <span id="page-14-17"></span>[10] H. O. Bayındır and B. Chorny. Admissible replacements for simplicial monoidal model categories. Algebr. Geom. Topol., 23(1):43–73, 2023.
- <span id="page-14-16"></span>[11] G. Biedermann and B. Chorny. Duality and small functors. Algebr. Geom. Topol., 15(5):2609–2657, 2015.
- <span id="page-14-19"></span>[12] G. Biedermann, B. Chorny, and O. Röndigs. Calculus of functors and model categories. Adv. in Math.,  $214(1):92-115$ ,  $2007$ .
- <span id="page-14-2"></span>[13] A. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In Geometric Applications of Homotopy Theory II, number 658 in Lecture Notes in Mathematics. Springer, 1978.
- <span id="page-14-11"></span>[14] G. E. Bredon. Equivariant Cohomology Theories. Lecture Notes in Math. 34. Springer-Verlag, Berlin, 1967.
- <span id="page-14-1"></span>[15] M. Bunge. Categories of set valued functors. ProQuest LLC, Ann Arbor, MI, 1966. Thesis (Ph.D.)–University of Pennsylvania.
- <span id="page-14-7"></span>[16] B. Chorny. The model category of maps of spaces is not cofibrantly generated. Proc. Amer. Math. Soc. 131, pages 2255–2259, 2003.
- <span id="page-14-12"></span>[17] B. Chorny. Localization with respect to a class of maps. I. Equivariant localization of diagrams of spaces. Israel J. Math., 147:93–139, 2005.
- <span id="page-14-15"></span>[18] B. Chorny. Brown representability for space-valued functors. Israel J. Math., 194(2):767– 791, 2013.
- <span id="page-14-14"></span>[19] B. Chorny. Homotopy theory of relative simplicial presheaves. Israel J. Math., 205(1):471– 484, 2015.
- <span id="page-14-6"></span>[20] B. Chorny. A classification of small homotopy functors from spectra to spectra. Fund. Math., 234(2):101–125, 2016.
- <span id="page-15-7"></span>[21] B. Chorny. A classification of small linear functors. Int. Math. Res. Not. IMRN, 2016(18):5493–5517, 2016.
- <span id="page-15-14"></span><span id="page-15-8"></span>[22] B. Chorny and W. G. Dwyer. Homotopy theory of small diagrams over large categories. Forum Math., 21(2):167–179, 2009.
- <span id="page-15-16"></span>[23] B. Chorny and J. Rosický. Class-combinatorial model categories. Homology Homotopy Appl., 14(1):263–280, 2012.
- <span id="page-15-19"></span>[24] B. Chorny and D. White. Stable monoidal model categories are categories of modules over commutative monoids. Preprint, 2024.
- <span id="page-15-13"></span>[25] B. Chorny and D. White. A variant of a Dwyer-Kan theorem for model categories. Algebr. Geom. Topol., 24(4):2185–2208, 2024.
- <span id="page-15-12"></span>[26] E. Dror Farjoun. Homotopy theories for diagrams of spaces. Proc. Amer. Math. Soc. 101, pages 181–189, 1987.
- [27] E. Dror Farjoun and A. Zabrodsky. Homotopy equivalence between diagrams of spaces. J. Pure Appl. Algebra 41(2-3), pages 169–182, 1986.
- <span id="page-15-4"></span>[28] D. Dugger. Combinatorial model categories have presentation. Adv. in Math., 164(1):177– 201, December 2001.
- <span id="page-15-0"></span>[29] W. G. Dwyer and D. M. Kan. Singular functors and realization functors. Proc. Kon. Nederl. Acad. 87(2); Indag. Math. 46(2), pages 147–153, 1984.
- <span id="page-15-21"></span>[30] W. G. Dwyer and D. M. Kan. Equivalences between homotopy theories of diagrams. In Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), volume 113 of Ann. of Math. Stud., pages 180–205. Princeton Univ. Press, Princeton, NJ, 1987.
- <span id="page-15-22"></span>[31] W. G. Dwyer and D. M. Kan. Equivalences between homotopy theories of diagrams. In Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), volume 113 of Ann. of Math. Stud., pages 180–205. Princeton Univ. Press, Princeton, NJ, 1987.
- <span id="page-15-5"></span>[32] D. A. Edwards and H. M. Hastings. Cech and Steenrod homotopy theories with applications to geometric topology. Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 542.
- <span id="page-15-11"></span>[33] A. Elmendorf. Systems of fixed point sets. Trans. Amer. Math. Soc., 1983. 277 (1983), no. 1, 275–284.
- <span id="page-15-20"></span><span id="page-15-9"></span>[34] S. Glasman. Goodwillie calculus and mackey functors, 2018.
- [35] T. G. Goodwillie. Calculus. I. The first derivative of pseudoisotopy theory. K-Theory, 4(1):1– 27, 1990.
- <span id="page-15-2"></span><span id="page-15-1"></span>[36] T. G. Goodwillie. Calculus. III. Taylor series. Geom. Topol., 7:645–711 (electronic), 2003.
- <span id="page-15-18"></span>[37] Y. Gu. Generalized equivariant model structures on  $\mathbf{Cat}^i$ , 2016.
- [38] J. J. Gutiérrez and D. White. Encoding equivariant commutativity via operads. Algebr. Geom. Topol., 18(5):2919–2962, 2018.
- <span id="page-15-15"></span>[39] H. Housden. Equivariant Stable Homotopy Theory for Diagrams. ProQuest LLC, Ann Arbor, MI, 2022. Thesis (Ph.D.)–University of California, Los Angeles.
- <span id="page-15-10"></span>[40] M. Hovey. Model categories. Mathematical Surveys and Monographs 63. American Mathematical Society, Providence, RI, 1999.
- <span id="page-15-3"></span>[41] M. Hovey. Spectra and symmetric spectra in general model categories. J. Pure Appl. Algebra, 165(1):63–127, 2001.
- <span id="page-15-17"></span>[42] M. A. Hovey and D. White. An alternative approach to equivariant stable homotopy theory. Tbilisi Mathematics Journal, Special Issue on Homotopy Theory, Spectra, and Structured Ring Spectra:51–69, 2020.
- <span id="page-15-6"></span>[43] D. C. Isaksen. Strict model structures for pro-categories. In Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), volume 215 of Progr. Math., pages 179–198. Birkhäuser, Basel, 2004.
- <span id="page-16-10"></span><span id="page-16-9"></span>[44] G. M. Kelly. Basic concepts of enriched category theory, volume 64 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1982.
- [45] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. Proc. London Math. Soc. (3), 82(2):441–512, 2001.
- <span id="page-16-0"></span>[46] S. Schwede and B. Shipley. Stable model categories are categories of modules. Topology, 42(1):103–153, 2003.
- <span id="page-16-11"></span><span id="page-16-8"></span>[47] L. Vokřínek. Homotopy weighted colimits. Preprint, arXiv:1201.2970 [math.CT], 2012.
- [48] D. White. Model structures on commutative monoids in general model categories. J. Pure Appl. Algebra, 221(12):3124–3168, 2017.
- <span id="page-16-7"></span>[49] D. White. Substitudes, bousfield localization, higher braided operads, and baez-dolan stabilization. In Homotopical Algebra and Higher Structures, volume 46, pages 56–60. Mathematisches Forschungsinstitut Oberwolfach, 2021.
- <span id="page-16-12"></span>[50] D. White. Monoidal Bousfield localizations and algebras over operads. In Equivariant topology and derived algebra, volume 474 of London Math. Soc. Lecture Note Ser., pages 180–240. Cambridge Univ. Press, Cambridge, 2022.
- <span id="page-16-2"></span>[51] D. White. On colimits in various categories of manifolds. *Graduate Journal of Mathematics*, 9:1–16, 2024.
- <span id="page-16-5"></span>[52] D. White and D. Yau. Bousfield localization and algebras over colored operads. Appl. Categ. Structures, 26(1):153–203, 2018.
- <span id="page-16-14"></span><span id="page-16-3"></span>[53] D. White and D. Yau. Comonadic coalgebras and bousfield localization. Preprint, 2018.
- [54] D. White and D. Yau. Arrow categories of monoidal model categories. Math. Scand., 125(2):185–198, 2019.
- <span id="page-16-13"></span>[55] D. White and D. Yau. Homotopical adjoint lifting theorem. Appl. Categ. Structures, 27(4):385–426, 2019.
- <span id="page-16-4"></span>[56] D. White and D. Yau. Right bousfield localization and operadic algebras. Tbilisi Mathematics Journal, Special Issue on Homotopy Theory, Spectra, and Structured Ring Spectra:71– 118, 2020.
- <span id="page-16-6"></span>[57] D. White and D. Yau. Smith ideals of operadic algebras in monoidal model categories. Algebr. Geom. Topol., 24(1):341–392, 2024.
- <span id="page-16-1"></span>[58] S. Yeakel. An isovariant Elmendorf's theorem. Doc. Math., 2022. 27 (2022), 613–628.