On the stability of hyperbolicity under quantitative measure equivalence

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Abstract

A well-known result of Shalom says that lattices in $\mathrm{SO}(n,1)$ are L^p measure equivalent for all $p < n-1$. His proof actually yields the following stronger statement: the natural coupling resulting from a suitable choice of fundamental domains from a uniform lattice Λ to a uniform one Γ is (L^∞, L^p). Moreover, the fundamental domain of Γ is contained in a union of finitely many translates of the fundamental domain of Λ. The purpose of this note is to prove a converse statement. More generally, it is proved that if a ME-coupling from a non-hyperbolic group Λ to a hyperbolic group Γ is (L^∞, L^p) and the fundamental domain of Γ is contained in a union of finitely many translates of the fundamental domain of Λ, then p must be less than some p_0 only depending on Γ.

Contents

1 Introduction

Gromov introduced measure equivalence between countable groups as a measured analogue of quasi-isometry. A classical instance of a pair of measure equivalent groups is given by lattices in a common locally compact group. Another source of examples is given by orbit equivalent groups. Recall that two groups Γ and Λ are orbit-equivalent if they admit free measure-preserving actions on a same standard probability space (X, μ) which share the same orbits: for almost every $x \in X$, $\Gamma \cdot x = \Lambda \cdot x$.

The notion of measure equivalence has been extensively studied over the past 20 years, and we refer the reader to [Gab05, Sec. 2] for an overview of its main properties as well as its tight connections with invariants such as cost or ℓ^2 Betti numbers. Various rigidity phenomenons have also been uncovered. A famous example is Furman's superrigidity results for lattices in higher rank semi-simple Lie groups [Fur99], which implies for instance that any countable group which is measure-equivalent to a lattice

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in $\mathrm{PSL}_3(\mathbb{R})$ is commensurable up to finite kernel to another lattice in $\mathrm{PSL}_3(\mathbb{R})$. Another example is provided by Kida's work on mapping class groups of surfaces: he showed that most surfaces can be reconstructed from the measure equivalence class of their mapping class group [Kid08], and that every group which is measure equivalent to a mapping class group must actually be commensurable up to finite kernel to it [Kid10].

In the opposite direction of flexibility, a celebrated result of Ornstein and Weiss implies that all infinite countable amenable groups are orbit equivalent and hence measure equivalent [OW80]. So most coarse geometric invariants (such as volume growth) are not preserved under orbit equivalence. Also, it is known that the class of groups measure equivalent to lattices in $\mathrm{PSL}_2(\mathbb{R})$ is very diverse and contains groups that are not virtually isomorphic to lattices of the latter (for instance, all free products of infinite amenable groups belong to this class). But as we will now see, measure equivalence admits natural refinements which capture meaningful coarse geometric invariants.

The main motivation for study quantitative versions of measure equivalence is to distinguish groups that are measure equivalent but have different geometric properties. Beyond the case of amenable groups, another source of examples are lattices in a same locally compact group. Recall[1] that uniform lattices in rank one simple Lie groups are hyperbolic but non-uniform ones are not hyperbolic, except for $\mathrm{SL}(2,\mathbb{R})$ where all lattices are hyperbolic. Hence these are interesting instances of groups that are measure equivalent, yet with different geometric properties. With this main application in mind, we will look for general integrability conditions ensuring that hyperbolicity is preserved under measure equivalence. We will not state our main results in this introduction as these are quite technical (see Theorem 4.2 and Theorem 4.1). Instead we shall focus on two specific corollaries.

Uniform versus non-uniform lattices in rank one simple Lie groups. Shalom proved that any two lattices in $\mathrm{SO}(n,1)$ are L^p measure equivalent for all $p < n-1$. When one of the two lattices is uniform, one can strengthen this statement as follows. Let $n \geqslant 2$, and let Γ (resp. Λ) be a uniform (resp. non-uniform) lattice in $\mathrm{SO}(n,1)$. We consider the coupling associated to the action of Λ and Γ respectively by left and right-translations on the measure space $\mathrm{SO}(n,1)$ equipped with an invariant Haar measure. Shalom showed that for a suitable fundamental domain X_Λ for Λ and any relatively compact fundamental domain X_Γ for Γ, the resulting coupling is an $(\mathrm{L}^\infty,\mathrm{L}^p)$ measure equivalence coupling from Λ to Γ for all $p < n-1$. Exploiting the relative compactness of X_Γ, one can check that this coupling satisfies the following additional property: there is a finite subset $F \subset \Lambda$ such that $X_\Gamma \subset F X_\Lambda$. In what follows, we shall refer to this property that as *coboundedness* of the coupling. We summarize this as follows.

Theorem 1 (Shalom [Sha00, Thm. 3.6])**.** *Let Γ and Λ be two lattices in $SO(n,1)$ such that Γ is uniform. Then there exists a cobounded coupling from Λ to Γ that is $(\mathrm{L}^\infty,\mathrm{L}^p)$-integrable for all $p < n-1$.*

By contrast we prove the following rigidity result.

Theorem 2 (see Cor. 4.3)**.** *Let Γ be a finitely generated hyperbolic group. There exists $p > 0$ such that if there exists a cobounded $(\mathrm{L}^\infty,\mathrm{L}^p)$ measure equivalence coupling from a finitely generated group Λ to Γ, then Λ is also hyperbolic.*

Remark 1.1. The value of p for which the conclusion holds is explicit: assuming that Γ admits a Cayley graph that is δ-hyperbolic and has volume entropy at most α, one can take $p = 75\delta\alpha + 2$. For the definition of volume entropy, see the paragraph which precedes Theorem 4.1.

We immediately deduce the following converse of Shalom's result.

Corollary 3. *Assume that Γ is a uniform lattice in a center-free, real rank 1 simple Lie group G and Λ is another lattice of G. There exists p only depending on G such that if there exists a cobounded $(\mathrm{L}^\infty,\mathrm{L}^p)$ coupling from Λ to Γ, then Λ must be uniform as well.*

[1]This is classical: see for instance [DK18, Example 22.1 and Proof of Theorem 22.32].

This raises the following question.

Question 1.2. What is the infimum over all p such that the previous result holds? Does it match Shalom's value $n-1$ for lattices in $\mathrm{SO}(n,1)$?

As observed by Mikael de la Salle, Corollary 3 is in sharp contrast with what happens for lattices in higher rank simple Lie groups: indeed if Γ and Λ are lattices in a simple Lie group Γ of rank $\geqslant 2$, then if X_Λ and X_Γ are Dirichlet fundamental domains for Λ and Γ, the resulting measure equivalence coupling is exponentially integrable [de 19, Lemme 5.6]. Hence we obtain:

Theorem 4 (de la Salle)**.** *Let Γ and Λ be two lattices in a simple Lie group of rank at least 2, such that Γ is uniform. Then there exists a cobounded $(\mathrm{L}^\infty, \varphi)$ measure equivalence coupling from Λ to Γ, where $\varphi(t) = \exp(ct)$ for some $c > 0$.*

Coming back to Theorem 2, observe that the L^∞ condition from Λ to Γ is the strongest possible. It can be relaxed to an L^p-type condition, but at the cost of imposing a stretched exponential integrability condition in the other direction. More precisely, we obtain:

Theorem 5 (see Cor. 4.4)**.** *Let Γ be a finitely generated hyperbolic group. For every $p > q > 0$, if there is a cobounded (ψ, φ)-integrable measure equivalence coupling from a finitely generated group Λ to Γ, where $\varphi(t) = \exp(t^p)$ and $\psi(t) = t^{1+1/q}$, then Λ is also hyperbolic.*

Remark 1.3. This shows for instance that there does not exist $(\mathrm{L}^{2+\varepsilon}, \varphi)$-integrable measure equivalence couplings from a hyperbolic group to a non hyperbolic group, where $\varphi(t) = \exp(ct)$ for any $c > 0$ and $\varepsilon > 0$.

Once again we deduce the following corollary for lattices in rank 1 simple Lie groups, which again contrasts with the the case of higher rank lattices.

Corollary 6. *Assume that Γ is a uniform lattice in a center-free, real rank 1 simple Lie group G and Λ is another lattice of G. For every $p > q > 0$, if there is a cobounded (ψ, φ)-integrable measure equivalence coupling from Γ to Λ where $\varphi(t) = \exp(t^p)$ and $\psi(t) = t^{1+1/q}$, then Λ is uniform as well.*

Remark 1.4. In Corollaries 3 and 6, the case of $\mathrm{SL}(2, \mathbb{R})$ was already known and actually a much stronger conclusion holds in that case: Bader, Furman and Sauer have proved that non-uniform lattices and uniform ones are not L^1 measure equivalent.

Remark 1.5. Theorems 2 and 5 should be compared with a theorem of Bowen saying that if there exists an $(\mathrm{L}^1, \mathrm{L}^0)$ orbit equivalence coupling from a finitely generated accessible group Λ to a virtually free group, then Λ is virtually free.

Stability of hyperbolicity under quantitative orbit equivalence From an orbit equivalence coupling, a measure equivalence coupling can be constructed in such a way that both groups share a common fundamental domain. In particular, such a coupling is automatically cobounded. Hence Theorem 2 and Theorem 5 have the following immediate corollaries.

Corollary 7. *Let Γ be a finitely generated hyperbolic group. There is $p > 0$ such that if there exists a $(\mathrm{L}^p, \mathrm{L}^\infty)$ orbit equivalence coupling from a finitely generated group Λ to Γ, then Λ is also hyperbolic.*

Corollary 8. *Let Γ be a finitely generated hyperbolic group. For every $p > q > 0$, if there is a (ψ, φ)-integrable orbit equivalence coupling from a finitely generated group Λ to Γ where $\varphi(t) = \exp(t^p)$ and $\psi(t) = t^{1+1/q}$, then Λ is also hyperbolic.*

Hyperbolicity and embedded cycles. The proofs of Theorem 2 and Theorem 5 are based on the following characterization of *non*-hyperbolic spaces. We denote by C_n the cyclic graph of length n.

Theorem 9. *[HM20, Proposition 5.1] Let X be a connected graph that is not hyperbolic. Then for all $n \in N$, there exists a 18-bi-Lipschitz embedded cyclic subgraph in X of length at most n.*

The strategy of proof consists in confronting this result with the following one (a very close statement is proved in [VS14] for the real hyperbolic space).

Theorem 10 (see Corollary 2.3)**.** *Let $a \geqslant 0$, $b \geqslant 1$, and $\delta \geqslant 1$. There is an integer $n_0 = n_0(a, b) \geqslant 2$ such that the following holds. For all δ-hyperbolic geodesic space X, for all $n \geqslant n_0$, if there is a map $\varphi\colon C_n \to X$ such that for all $x, y \in C_n$*

$$ad_{C_n}(x, y) \leqslant d(\varphi(x), \varphi(y)) \leqslant bd_{C_n}(x, y)$$

then we have

$$a < 6\delta \cdot \frac{\log n}{n}. \tag{1}$$

Sketch of proof. We prove Theorem 2 and Theorem 5 by contradiction. Let us briefly sketch the argument for an orbit equivalence coupling. We assume that Λ is non hyperbolic and consider the map $\varphi_n\colon C_n \to \Lambda$ provided for some large integer n by Theorem 9. Identifying the orbits of Λ and Γ, we obtain for a.e. $x \in X$, a map $\psi_{n,x}\colon C_n \to \Gamma$. By exploiting the integrability condition from Λ to Γ, we obtain a bound on the Lipschitz constant of $\psi_{n,x}$, which is satisfied on a subset of X of sufficiently large measure. Observe that this step is trivial under the hypotheses of Theorem 2, as the L^∞-condition ensures that $\psi_{n,x}$ is Lipschitz, uniformly with respect to $x \in X$. The more subtle part of the argument consists in estimating the Lipschitz constant of the inverse of $\psi_{n,x}$, in order to obtain a contradiction with Theorem 10.

In case of a measure equivalence coupling, a difficulty arises in the second step of the proof: in order to estimate the Lipschitz constant of the inverse of $\psi_{n,x}$, we need to go back to Λ and exploit the lower bound $\frac{1}{18}$ on the Lipschitz constant of the inverse of φ_n. This where the coboundedness condition comes in, allowing us to relate the fundamental domains of the two groups.

Further remarks and questions. As commented above, the coboundedness assumption plays an important (though technical) role in the proofs. It would be interesting to know whether it can be avoided. In particular, this raises the following question.

Question 1.6. Assume $n \geqslant 3$, and let Γ (resp. Λ) be a uniform (resp. non-uniform) lattice in $\mathrm{SO}(n, 1)$. For what values of $p \in [1, \infty]$ are Γ and Λ L^p measure equivalent?

Coboundedness was also considered by Sauer in his PhD thesis [Sau02]. His statement (very close to Shalom's [Sha04, Theorem 2.1.7.]) is that two amenable finitely generated groups are quasi-isometric if and only if they admit an L^∞ measure coupling which is cobounded in both directions: namely there exist finite subsets $F_\Lambda \subset \Lambda$ and $F_\Gamma \subset \Gamma$ such that $X_\Gamma \subset F_\Lambda X_\Lambda$ and $X_\Lambda \subset F_\Gamma X_\Gamma$. In general, it is unknown whether being L^∞ measure equivalent implies being quasi-isometric. This justifies the following question.

Question 1.7. Is hyperbolicity invariant under L^∞ measure equivalence?

Plan of the paper. Theorem 10 is first proved in § 2. After recalling the definitions of quantitative measure equivalence in § 3, we prove our main result, namely Theorem 4.1 from which we deduce the theorems announced in the introduction.

2 Geometric preliminaries

Let X be a graph, a **discrete path** in X of length $l \geqslant 1$ is a map $\alpha\colon \{0,\dots,l\} \to X$ such that for all $i \in \{0,\dots,l-1\}$, we have that $\alpha(i)$ and $\alpha(i+1)$ are connected by an edge. We will also say that α is a path from $\alpha(0)$ to $\alpha(l)$, and we will often identify a path to its range.

Every connected graph X is viewed as a metric space (X,d) equipped with the **discrete path metric**, defined by setting $d(x,y)$ as the minimum length of a path from x to y. Any discrete path which realizes the discrete path metric between two points is called a **discrete geodesic**, and it is then an isometric embedding from $\{0,\dots,d(x,y)\}$ to (X,d).

Another important metric space that we can get out of a connected graph X is given by the (continuous) **path metric** which we define as in [Gro07, 1.15_+]. We first identify each edge to the interval $[0,1]$ isometrically, thus obtaining a *length structure* on our graph. The metric associated to this length structure is denoted by d_l, and it is by definition the continuous path metric on X. It agrees with the discrete path metric on the vertices of X, and it is geodesic. Every geodesic between vertices defines a discrete geodesic, and every discrete geodesic can be lifted to a geodesic between vertices.

A (geodesic) **triangle** in a metric space (X,d) with **vertices** $a_1,a_2,a_3 \in X$ is a set $[a_1,a_2,a_3] \subseteq X$ obtained by taking the union of a choice of geodesics $[a_1,a_2]$, $[a_2,a_3]$, and $[a_3,a_1]$ between its vertices. In the same way, we define a (geodesic) n**-gon** with vertices $a_1,\dots,a_n \in X$, and denote it by $[a_1,\dots,a_n]$. Given an n-gon where $n \geqslant 3$, we will frequently call any of its defining geodesics a **side**.

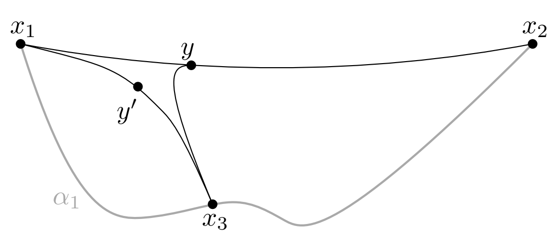
Now, recall that a geodesic space (X,d) is δ**-hyperbolic** in the sense of Rips if there exists a $\delta \geqslant 0$ such that for every geodesic triangle $[a_1,a_2,a_3]$ and for every $x \in [a_1,a_2]$, there exists an element y in either $[a_1,a_3]$ or $[a_2,a_3]$ such that $d(x,y) \leqslant \delta$; or equivalently, that the side $[a_1,a_2]$ is contained in the δ-neighborhood of $[a_1,a_3] \cup [a_2,a_3]$. Moreover, we say that a geodesic space (X,d) is **hyperbolic** if it is δ-hyperbolic for some δ, and that a finitely generated group Γ with generating set S is hyperbolic whenever its Cayley graph is hyperbolic when equipped with the continuous path metric.

We shall need the following well-known lemma.

Lemma 2.1. *Let X be a δ-hyperbolic geodesic space, let α be a path of length $\ell \geqslant 1$ between two points x_1 and x_2, and let y belong to a geodesic from x_1 to x_2. Then*

$$d(y,\alpha) \leqslant \delta \log_2(\ell) + 1.$$

Proof. Let us prove it by induction on $\lfloor \ell \rfloor$. The case $\lfloor \ell \rfloor = 1$ is clear. So assume $n \geqslant 2$ and suppose that the lemma is true for all paths of length $< n$. Let α be a path of length $\ell \in [n, n+1[$ from x_1 to x_2, represented as a gray path in the following figure. For every $x \in \alpha$, using that the geodesic triangle $[x_1,x,x_2]$ is δ-thin, then either $d(y,[x_1,x])] \leqslant \delta$ or $d(y,[x,x_2])] \leqslant \delta$. By connectedness of α, there exists x_3 such that both conditions are satisfied. By exchanging x_1 and x_2 if necessary, we can assume that the portion α_1 of α from x_1 to x_3 has length $\ell_1 \leqslant \ell/2$.



Hence there exists a point $y' \in [x_1,x_3]$ such that $d(y,y') \leqslant \delta$. Now applying the induction hypothesis to the path α_1 and the point $y' \in [x_1,x_3]$, we obtain

$$d(y',\alpha_1) \leqslant \delta \log_2(\ell_1) \leqslant \delta \log_2 \ell + 1 - \delta.$$

We deduce by the triangle inequality

$$d(y,\alpha) \leqslant d(y,y') + d(y',\alpha_1) \leqslant \delta \log_2(\ell) + 1.$$

So we are done. □

First we need an alternative definition of hyperbolicity in terms of embedded cycles. In what follows, the **cycle** C_n of length $n \geqslant 2$ is the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with respect to the generating set containing only the element 1 mod n, which we view as a *discrete* metric space denoted by (C_n, d_{C_n}).

Proposition 2.2. *Let (X,d) be a δ-hyperbolic geodesic space, let n be a positive integer. Then for every $a \geqslant 0$ and every $b \geqslant 1$, if there is a map $\varphi\colon C_{2n} \to X$ such that for every $x,y \in C_{2n}$,*

$$ad_{C_{2n}}(x,y) \leqslant d(\varphi(x),\varphi(y)) \leqslant bd_{C_{2n}}(x,y)$$

then we have

$$a \leqslant \frac{4\delta \log_2(bn) + 4 + 2b}{n}$$

Before proving the above proposition, let us note the following straightforward corollary, using the estimate $\frac{1}{\log 2} < \frac{3}{2}$.

Corollary 2.3. *Let $a \geqslant 0$, $b \geqslant 1$. There is an integer $n_0 = n_0(b) \geqslant 2$ such that the following holds. For all $n \geqslant n_0$ and all $\delta \geqslant 1$, if X is a δ-hyperbolic geodesic space and if there is a map $\varphi\colon C_n \to X$ such that for all $x,y \in C_n$*

$$ad_{C_n}(x,y) \leqslant d(\varphi(x),\varphi(y)) \leqslant bd_{C_n}(x,y)$$

then we have

$$a < 6\delta \cdot \frac{\log n}{n}. \tag{2}$$

In order to prove the proposition, we need the following additional notion. Given a discrete path β in a graph Y and a map $\varphi\colon Y \to (X,d)$ where (X,d) is geodesic, we say that a continuous path α in X is a **φ-direct image** of β if it is obtained by concatenating geodesics between $\varphi(\beta(i))$ and $\varphi(\beta(i+1))$ where i ranges from 0 to $\ell(\beta) - 1$.

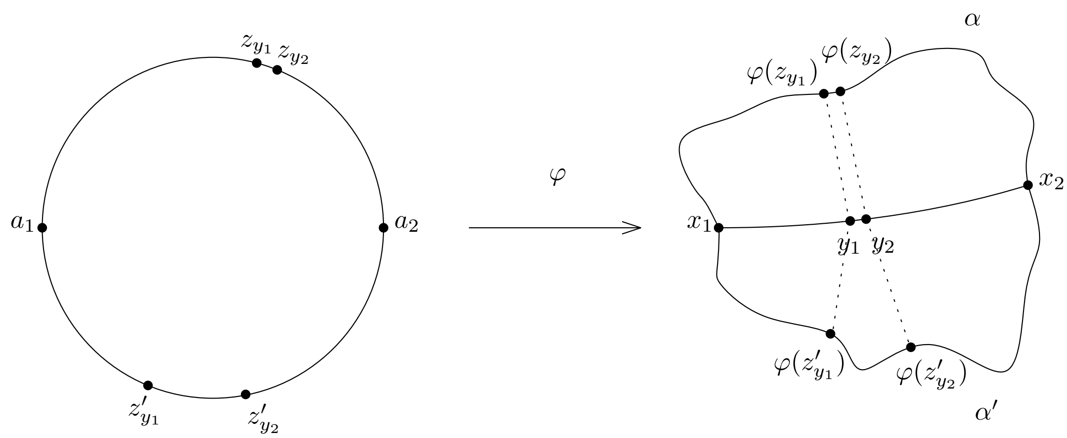
Proof of Proposition 2.2. Let $a_1, a_2 \in C_{2n}$ be such that $d_{C_{2n}}(a_1,a_2) = n$ and define $x_1 = \varphi(a_1)$, $x_2 = \varphi(a_2)$. Consider a geodesic $[x_1,x_2]$ from x_1 to x_2. In C_{2n} there are two discrete geodesic paths from a_1 to a_2, both with length n. Denote by α and α' some respective φ-direct images of those paths in X, which by assumption have length at most bn. Let $y \in [x_1,x_2]$, we deduce from Lemma 2.1 that

$$d(y,\alpha) \leqslant \delta \log_2(\ell(\alpha)) + 1 \leqslant \delta \log_2(bn) + 1,$$

where $\ell(\alpha)$ is the length of α, and by the same argument $d(y,\alpha') \leqslant \delta \log_2(bn) + 1$. If we then pick z_y and z'_y points in C_{2n} such that their φ-images are in α and α' respectively and minimize the distance to y, we have

$$\max\left(d(y,\varphi(z_y)), d(y,\varphi(z'_y))\right) \leqslant \delta \log_2(bn) + 1 + \frac{b}{2}. \tag{3}$$

Note that for any $y \in [x_1,x_2]$, any geodesic from z_y to z'_y must pass through x_1 or through x_2. Moreover there are some $y \in [x_1,x_2]$ for which the first case occurs, and some for which the second case occurs. For all $\varepsilon > 0$, we may thus find $y_1,y_2 \in [x_1,x_2]$ such that $d(y_1,y_2) \leqslant \varepsilon$, the geodesic from z_{y_1} to z'_{y_1} passes through x_1 and the geodesic from z_{y_2} to z'_{y_2} passes through x_2.



Then we have that

$$d_{C_n}(z_{y_1}, z_{y_2}) + d_{C_n}(z_{y_2}, z'_{y_2}) + d_{C_n}(z'_{y_2}, z'_{y_1}) + d_{C_n}(z'_{y_1}, z_{y_1}) = 2n.$$

Hence one of these four distances is at least $\frac{n}{2}$. On the other hand, combining (3) and the fact that $d(y_1, y_2) \leqslant \varepsilon$, we obtain the following inequality

$$d(\varphi(z_{y_1}), \varphi(z_{y_2})) + d(\varphi(z_{y_2}), \varphi(z'_{y_2})) + d(\varphi(z'_{y_2}), \varphi(z'_{y_1})) + d(\varphi(z'_{y_1}), \varphi(z_{y_1})) \leqslant 2(\delta \log_2 (bn) + 1 + \frac{b}{2}) + \varepsilon.$$

Using our assumption on φ, we thus have the following inequality: for all $\varepsilon > 0$,

$$\frac{an}{2} \leqslant 2\delta \log_2 (bn) + 2 + b + \varepsilon.$$

So the proposition follows. $\square$

3 Preliminaries on quantitative measure equivalence

We now make the statements from the introduction more precise by enriching a bit the terminology from [DKLMT22, Sec. 2] and introducing in details quantitative measure equivalence. We denote systematically smooth actions by $*$ (recall that by definitions, smooth actions are those which admit a Borel fundamental domain, i.e. a Borel subset intersecting each orbit exactly once).

3.1 Relations between fundamental domains

In this section, we fix a smooth measure-preserving action $\Gamma \curvearrowright (\Omega, \mu)$.

Let X_1, X_2 be two fundamental domains, we denote by $\pi_{X_1,X_2} : X_1 \to X_2$ the map which takes every $x \in X_1$ to the unique $x' \in X_2 \cap \Gamma * x$. The map π_{X_1,X_2} belongs to the pseudo full group of the action, in particular it is measure-preserving, and its inverse is π_{X_2,X_1}.

Say that two fundamental domains X_1 and X_2 are L^∞-equivalent if there is a finite subset $F \Subset \Gamma$ such that for all $x \in X_1$, there is $\gamma \in F$ such that such that $\pi_{X_1,X_2}(x) = \gamma * x$. Observe that L^∞-equivalence is an equivalence relation, and that if some measure-preserving $T \in \mathrm{Aut}(\Omega, \mu)$ commutes with the Γ-action, then X_1 is L^∞-equivalent to X_2 iff $T(X_1)$ is L^∞-equivalent to $T(X_2)$.

Now let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function and assume that Γ is generated by a finite set S_Γ, allowing us to endow Ω with the Schreier metric d_{S_Γ} whose definition is recalled in [DKLMT22, Def. 2.14]. For the purpose of this paper, we introduce some further terminology and say that two fundamental domains X_1, X_2 are **φ-similar** if

$$\int_{X_1} \varphi(d_{S_\Gamma}(x, \pi_{X_1,X_2}(x)))d\mu(x) < +\infty.$$

Observe that L^∞-equivalence can be recast as the fact that the map $x \in X_1 \mapsto d_{S_\Gamma}(x, \pi_{X_1,X_2}(x))$ takes only finitely many values, in particular it implies φ-similarity. However, as the name suggests, φ-similarity is not an equivalence relation in general, for instance when $\varphi(t) = e^t$ transitivity might fail (see also [DKLMT22, Rem. 2.16]). Even worse, it is a priori dependent on the choice of the finite generating set S_Γ we made.

In order to correct this, we introduce as in [DKLMT22] a coarser relation that we called φ-equivalence: two fundamental domains X_1 and X_2 are **φ-equivalent** if there is some $\varepsilon > 0$ such that they are φ_ε-similar, where $\varphi_\varepsilon(t) = \varphi(\varepsilon t)$. We checked in [DKLMT22, Cor. 2.19] that φ-equivalence is an equivalence relation.

It is important to note that φ-equivalence is equivalent to φ-similarity when φ satisfies that for all $c > 0$, there is $C > 0$ such that for all $t \geqslant 0$ we have $\varphi(ct) \leqslant C\varphi(t)$. This is notably the case when $\varphi(t) = t^p$ for some $p > 0$, and we will make use of this fact without explicit mention.

3.2 (φ, ψ)-integrable measure equivalence

In this section we fix two finitely generated groups $\Gamma = \langle S_\Gamma \rangle$ and $\Lambda = \langle S_\Lambda \rangle$.

A **measure equivalence coupling** from Γ to Λ is a measured space (Ω, μ) endowed with commuting smooth free measure-preserving actions of Γ and Λ and Borel fundamental domains X_Γ for the Γ-action, X_Λ for the Λ-action, which both have finite measure.

Definition 3.1. Let $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be two non-decreasing functions. We say that a measure-equivalence coupling $(\Omega, \mu, X_\Gamma, X_\Lambda)$ from Γ to Λ is **(φ, ψ)-integrable** if for every $\gamma \in S_\Gamma$ the Λ-fundamental domain $\gamma * X_\Lambda$ is φ-equivalent to X_Λ , and for every $\lambda \in S_\Lambda$ the Γ-fundamental domain $\lambda * X_\Gamma$ is ψ-equivalent to X_Γ.

As explained in [DKLMT22, Prop. 2.22], since φ-equivalence and ψ-equivalence are equivalence relations, the definition is unchanged if we quantify over all $\gamma \in \Gamma$ or all $\lambda \in \Lambda$ rather than over the finite generating sets S_Γ and S_Λ.

We now give the cocycle versions of these definitions: given a measure equivalence coupling $(\Omega, \mu, X_\Gamma, X_\Lambda)$, we have the associated cocycles

$$\alpha : X_\Lambda \times \Gamma \to \Lambda \text{ and } X_\Gamma \times \Lambda \to \Gamma$$

uniquely defined by the following statements: for all $x \in X_\Lambda$ and $\gamma \in \Gamma$, $\alpha(\gamma, x) * (\gamma * x) \in X_\Lambda$, and similary for all $x \in X_\Gamma$ and $\lambda \in \Lambda$, $\beta(\lambda, x) * (\lambda * x) \in X_\Gamma$.

When we endow Γ and Λ with the natural norms $|\cdot|_{S_\Gamma}$ and $|\cdot|_{S_\Lambda}$ associated to their respective generating sets S_Γ and S_Λ, we can now state that a measure equivalence coupling $(\Omega, \mu, X_\Gamma, X_\Lambda)$ from Γ to Λ is **(φ, ψ)-integrable** iff the associated cocycles satisfy: for all $\gamma \in S_\Gamma$ there is $\varepsilon_\gamma > 0$ such that

$$\int_{X_\Lambda} \varphi(\varepsilon_\gamma \left|\alpha(x, \gamma)\right|_{S_\Lambda}) d\mu(x) < +\infty$$

and similarly for all $\lambda \in S_\Lambda$ there is $\varepsilon_\lambda > 0$ such that

$$\int_{X_\Lambda} \psi(\varepsilon_\lambda \left|\beta(x, \lambda)\right|_{S_\Gamma}) d\mu(x) < +\infty.$$

Remark 3.2. As in the end of the previous section, note that if φ satisfies that for all $c > 0$, there is $C > 0$ such that for all $t \geqslant 0$ we have $\varphi(ct) \leqslant C\varphi(t)$, then we can get rid of the factor ε_γ in the first inequality, and the same applies to the second inequality mutatis mutandis. This applies in particular for $\varphi(t) = t^p$, and thus for L^p conditions, but not for exponential integrability.

Let us denote by $\cdot$ the natural Γ (resp. Λ) action on X_Λ (resp. X_Γ) given by : for all $\gamma \in \Gamma$ and $x \in X_\Lambda$, $\gamma \cdot x$ is the unique element of $\Lambda * (\gamma * x) \cap X_\Lambda$ (and symmetrically for all $\lambda \in \Lambda$ and all $x \in X_\Gamma$,

$\lambda \cdot x$ is the unique element of $\Gamma * (\lambda * x) \cap X_\Gamma$). Observe that for all $x \in X_\Lambda$ and $\gamma \in \Gamma$, the cocycle $\alpha(\gamma, x)$ is uniquely defined by the equation $\alpha(\gamma, x) * (\gamma * x) = \gamma \cdot x$, and a similar statement holds for β.

The equivalence between the above definition and Definition 3.1 is then clear once one notes that α is connected to the the cocycle of $\pi_{X_\Lambda, \gamma * X_\Lambda}$: it is uniquely defined by the equation

$$\pi_{X_\Lambda, \gamma * X_\Lambda}(x) = \alpha(\gamma, \gamma^{-1} \cdot x) * x.$$

Indeed it follows that $d_{S_\Gamma}(x, \pi_{X_\Lambda, \gamma * X_\Lambda}(x)) = \left|\alpha(\gamma, \gamma^{-1} \cdot x)\right|_{S_\Gamma}$, so since the action of γ^{-1} on X_Λ is measure-preserving, the φ-integrability of $x \mapsto d_{S_\Gamma}(x, \pi_{X_\Lambda, \gamma * X_\Lambda}(x))$ is equivalent to that of $x \mapsto |\alpha(\gamma, x)|_{S_\Gamma}$ (and the symmetric phenomenon holds for β).

Finally, let us recall the cocycle relations satisfied by α (and β mutatis mutandis): for all $x \in X_\Lambda$ and all $\gamma, \gamma' \in \Gamma$ we have

$$\alpha(\gamma'\gamma) = \alpha(\gamma', \gamma \cdot x)\alpha(\gamma, x).$$

4 Rigidity of hyperbolicity

We now prove rigidity results, saying that hyperbolicity is preserved under cobounded measure equivalence couplings satisfying certain integrability conditions. To simplify the exposition, we choose to state two results, but the second one should be seen as a degenerate version of the first one (where one of the conditions becomes an L^∞-condition). Anyway the two results share basically the same proof, which consists in confronting Proposition 2.2 with Theorem 9. Since these results are quite technical, we shall first deduce two more appealing corollaries (namely Theorem 2 and Theorem 5 from the introduction).

Let us start with some notation: given a group Γ equipped with a finitely generated subset S_Γ, we denote the growth function $\mathrm{Vol}_{S_\Gamma}(n) = |S_\Gamma^n|$ and define the associated **volume entropy** as $\mathrm{Ent}(S_\Gamma) = \limsup_{r\to\infty} \frac{\log(\mathrm{Vol}_{S_\Gamma}(r))}{r}$.

Theorem 4.1. *Let Γ and Λ be two finitely generated groups such that Γ is δ-hyperbolic. We let $L \geqslant 1$ and φ, ψ and r be non-decreasing unbounded functions. Assume that the following conditions are satisfied:*

$$\lim_{n\to\infty} \frac{n^2 r(n)\,\mathrm{Vol}_{S_\Gamma}(r(n))}{\varphi(n/r(n))} = 0; \tag{4}$$

and[2] *for all large enough n,*

$$\frac{r(n)}{18} \geqslant 4(\delta+1)\log_2 n + 3\psi^{-1}(3Ln). \tag{5}$$

Assume that (Ω, μ) is a cobounded measure equivalence coupling from Λ to Γ, normalized so that $\mu(X_\Gamma) = 1$, and such that the associated cocycles $\alpha\colon \Gamma \times X_\Lambda \to \Lambda$ and $\beta\colon \Lambda \times X_\Gamma \to \Gamma$ satisfy the following properties.

(i) for all $s \in S_\Gamma$,

$$\int_{X_\Lambda} \varphi(|\alpha(s,x)|_{S_\Lambda})d\mu(x) < \infty;$$

(ii) for all $t \in S_\Lambda$

$$\int_{X_\Gamma} \psi(|\beta(t,x)|_{S_\Gamma})d\mu(x) \leqslant L.$$

Then Λ is hyperbolic.

Assuming that β is bounded, we have the following variant.

[2]The constant 18 in (5) comes from the use of Theorem 9, while the right-hand term comes from the use of Proposition 2.2.

Theorem 4.2. *Let Γ and Λ be two finitely generated groups such that Γ is δ-hyperbolic. We let φ and r be increasing unbounded functions. Assume that (4) holds, and that for all large enough n,*

$$\frac{r(n)}{18} \geqslant 13(\delta+1)\log n. \tag{6}$$

Assume that (Ω, μ) is a cobounded measure equivalence coupling from Λ to Γ, normalized so that $\mu(X_\Gamma) = 1$, and such that the associated cocycles $\alpha\colon \Gamma \times X_\Lambda \to \Lambda$ and $\beta\colon \Lambda \times X_\Gamma \to \Gamma$ satisfy the following properties.

(i) for all $s \in S_\Gamma$,

$$\int_{X_\Lambda} \varphi(|\alpha(s,x)|_{S_\Lambda}) d\mu(x) < \infty;$$

(ii) for all $t \in S_\Lambda$

$$|\beta(t,\cdot)|_{S_\Gamma} \in \mathrm{L}^\infty(X_\Gamma).$$

Then Λ is hyperbolic.

Theorem 2 is an immediate consequence of the following corollary of Theorem 4.2.

Corollary 4.3. *Let Γ be a finitely generated δ-hyperbolic group and let $p > 75\delta\,\mathrm{Ent}(S_\Gamma) + 2$. Assume that there exists a cobounded $(\mathrm{L}^\infty, \mathrm{L}^p)$-integrable measure equivalence coupling from a finitely generated group Λ to Γ. Then Λ is hyperbolic.*

Proof. We apply Theorem 4.2 with $p = 75\delta(\mathrm{Ent}(S_\Gamma) + \varepsilon) + 2$ for some $\varepsilon > 0$. We let $r(n) = 75\delta \log n$. By definition of $\mathrm{Ent}(S_\Gamma)$, we have $\mathrm{Vol}_{S_\Gamma}(r(n)) = o(e^{(\mathrm{Ent}(S_\Gamma)+\varepsilon/2) r(n)})$. Hence we deduce that

$$\mathrm{Vol}_{S_\Gamma}(r(n)) = o(n^{75\delta(\mathrm{Ent}(S_\Gamma)+\varepsilon/2)}),$$

which combined with the fact that $\varphi(t) = t^{75\delta(\mathrm{Ent}(S_\Gamma)+\varepsilon)+2}$ implies that (4) is satisfied. Thus, Λ is hyperbolic. □

Corollary 4.4. *Let Γ be a finitely generated hyperbolic group. For every $p > q > 0$ such that if there is a cobounded (ψ, φ)-integrable measure equivalence coupling from a finitely generated group Λ to Γ where $\varphi(t) = \exp(t^p)$ and $\psi(t) = t^{1+1/q}$, then Λ is also hyperbolic.*

Proof. We consider our cobounded (φ, ψ)-integrable measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ and normalize μ such that $\mu(X_\Gamma) = 1$. We pick η strictly between q and p, we let $r(n) = n^{\frac{\eta}{1+\eta}}$, and we define

$$L = \max_{s \in S_\Lambda} \int_{X_\Gamma} \psi(|\beta(s,x)|_{S_\Gamma}) d\mu(x).$$

Note that $\psi^{-1}(t) = t^{\frac{q}{1+q}}$. So (5) follows from the fact that $\eta > q$. Finally, take $\varepsilon > 0$ such that

$$\max_{s \in S_\Gamma} \int_{X_\Lambda} \varphi_\varepsilon(|\alpha(s,x)|_{S_\Lambda}) d\mu(x) < \infty,$$

where $\varphi_\varepsilon(t) = \varphi(\varepsilon t)$. Note that $n/r(n) = n^{\frac{1}{1+\eta}}$. Hence $\varphi_\varepsilon(n/r(n)) = \exp(\varepsilon^p n^{\frac{p}{1+\eta}})$, while

$$\mathrm{Vol}_{S_\Gamma}(r(n)) \leqslant |S_\Gamma|^{n^{\frac{\eta}{1+\eta}}}$$

Hence since $\eta < p$, we have

$$\mathrm{Vol}_{S_\Gamma}(r(n))/\varphi_\varepsilon(n/r(n)) = O(n^{-k})$$

for any $k > 0$. So (4) is satisfied and we conclude by Theorem 4.1 that Λ is hyperbolic. □

Proof of Theorems 4.1 and 4.2. Recall that we use the notation $*$ for smooth actions and $\cdot$ for the induced actions on the respective fundamental domains. Let us start by strengthening the coboundedness condition.

Claim 4.5. Replacing Condition (5) and Condition (6) respectively by the slightly weaker conditions

$$\frac{r(n)}{18} \geqslant 4\delta \log_2 n + 3\psi^{-1}(3Ln) - 3, \tag{7}$$

and

$$\frac{r(n)}{18} \geqslant 13\delta \log n, \tag{8}$$

it is enough to prove Theorem 4.1 and Theorem 4.2 under the assumption that $X_\Gamma \subseteq X_\Lambda$.

Proof of the claim. Assuming we have a coupling satisfying the conditions of Theorem 4.1 (resp. Theorem 4.2), we build a new coupling satisfying Condition (7) (resp. Condition (8)) with $X_\Gamma \subseteq X_\Lambda$.

Since our initial coupling is cobounded, there exists a finite subset F of Λ such that $X_\Gamma \subseteq F * X_\Lambda$. Consider the new coupling space $\tilde{\Omega} \colon = \Omega \times F$, let K be a finite group which acts simply transitively on F, and let $\tilde{\Gamma} = \Gamma \times K$ act on $\tilde{\Omega}$ by $(\gamma, k) * (\omega, f) = (\gamma * \omega, kf)$. This action is smooth, and we take as a fundamental domain the set

$$\tilde{X}_{\tilde{\Gamma}} \colon = \bigsqcup_{f \in F} (X_\Gamma \cap f * X_\Lambda) \times \{f\}$$

The Λ-action on $\tilde{\Omega}$ is the action on the first coordinate; a fundamental domain is provided by

$$\tilde{X}_\Lambda = \bigsqcup_{f \in F} (f * X_\Lambda) \times \{f\}.$$

Viewing both Γ and K as subgroups of $\tilde{\Gamma}$, the latter has $S_{\tilde{\Gamma}} = S_\Gamma \cup K$ as a finite generating set. In fact, with this generating set $\tilde{\Gamma}$ is $\tilde{\delta}$-hyperbolic, with $\tilde{\delta} = \delta + 1$. It follows that Condition (8) holds in this new setup. Also observe that the volume growth of $\tilde{\Gamma}$ is at most $|K|$ times that of Γ, so Condition (4) is preserved.

In what follows, we implicitely use the fact that our quantitative conditions (i) and (ii) can be recast using the notion of φ-similarity between fundamental domains of a smooth action as explained in Section 3. We also use the straightforward fact that L^∞-equivalence refines φ-similarity.

We can now show that condition (i) is still met by the new generating set $S_{\tilde{\Gamma}} = S_\Gamma \cup K$. Indeed $\tilde{X}_\Lambda$ is L^∞-equivalent to the fundamental domain $X_\Lambda \times F$, and for all $\gamma \in \Gamma$, $x \in X_\Lambda$ and $f \in F$ we have $d_{S_\Lambda}(\gamma * (x, f), \gamma \cdot (x, f)) = d_{S_\Lambda}(\gamma * x, \gamma \cdot x)$, so for all $\tilde{\gamma} \in S_{\tilde{\Gamma}}$ we have that $\tilde{\gamma} * X_\Lambda \times F$ is φ-similar to $X_\Lambda \times F$, so $\tilde{\gamma} * \tilde{X}_\Lambda$ is φ-similar to $\tilde{X}_\Lambda$.

For condition (ii), we have, by construction, for all $x \in X_\Gamma$ and all $f, f' \in F$ the inequality $d_{S_{\tilde{\Gamma}}}((x, f), (x, f')) \leqslant 1$, so for every $\lambda \in \Lambda$ we have

$$d_{S_{\tilde{\Gamma}}}(\lambda \cdot (x, f), \lambda * (x, f)) \leqslant 1 + d_{S_\Gamma}(\lambda \cdot x, \lambda * x) = 1 + |\beta(\lambda, x)|_{S_\Gamma},$$

hence the new coupling satisfies the same conditions replacing $\psi(t)$ by $\tilde{\psi}(t) = \psi(\max\{t-1, 0\})$ in Theorem 4.1. Note that for n large enough, $\psi^{-1}(3Ln) \geqslant 1$. Since $\tilde{\psi}(t) = \psi(t-1)$ for all $t \geqslant 1$, we deduce that for large enough n,

$$\frac{r(n)}{18} \geqslant 4\tilde{\delta} \log_2 n + 3\tilde{\psi}^{-1}(3Ln) - 3.$$

So the claim is proved. $\square_{\text{claim}}$

From now on, we assume that $X_\Gamma \subseteq X_\Lambda$ and we normalize the measure so that $\mu(X_\Gamma) = 1$. Suppose by contradiction that Λ is not hyperbolic. Theorem 9 provides us with a cycle C_n of arbitrary large length n, and a map $C_n \to \Lambda$ which is 1-Lipschitz and contracts distances at most by a factor 18. In what follows we consider C_n as a subset of Λ. Let K be such that $\int_{X_\Lambda} \varphi(|\alpha(s,x)|_{S_\Lambda}) d\mu(x) \leqslant K$ for all $s \in S_\Gamma$.

For every $x \in X_\Gamma$ we denote by $b_x \colon \Lambda \to \Gamma$ the map defined by $b_x(\lambda) = \beta(\lambda^{-1}, x)^{-1}$ for every $\lambda \in \Lambda$. We will use throughout the following straightforward consequence of the cocycle relation: for all $u, v \in \Lambda$, we have

$$b_x(u)^{-1} b_x(v) = \beta(v^{-1}u, u^{-1} \cdot x)^{-1}. \tag{9}$$

We endow Γ and Λ with their usual left-invariant Cayley metrics, denoted by d_{S_Γ} and d_{S_Λ} respectively (so the map $\gamma \mapsto \gamma^{-1} * x$ is an isometry if we endow $\Gamma * x$ with the Schreier metric that we previously used and denoted by d_{S_Γ} as well).

Upper estimates for the restriction of b_x to C_n. In the case of Theorem 4.2, we trivially have that b_x is a.e. L-Lipschitz for some constant L.

Under the assumption of Theorem 4.1, we claim that with probability at least 2/3, the restriction of b_x to C_n is $\psi^{-1}(3Ln)$-Lipschitz. Here we use the integrability condition for β. For every u and v adjacent in C_n there exists an $s_{u,v} \in S_\Lambda$ such that $u = v s_{u,v}$. By (9), we have:

$$d_{S_\Gamma}(b_x(v), b_x(u)) = |b_x(u)^{-1} b_x(v)|_{S_\Gamma} = |\beta(s_{u,v}, u^{-1} \cdot x)|_{S_\Gamma}.$$

Next consider for any $u \in C_n$ the set of all $x \in X_\Gamma$ such that $\psi(|\beta(s_{u,v}, u^{-1} \cdot x)|_{S_\Gamma}) \geqslant 3Ln$. By Markov's inequality, these sets have measure at most $\frac{1}{3n}$ and therefore the set of all $x \in X_\Gamma$ such that b_x is $\psi^{-1}(3Ln)$-Lipschitz in restriction to C_n has measure at least $1 - n \cdot \frac{1}{3n} = \frac{2}{3}$. So our claim follows.

Lower estimates for the restriction of b_x to C_n. Providing lower estimates on the quasi-isometric embedding constants is more involved as this requires to apply the cocycle α to C_n. We shall use the inverse relation between α and β and the inclusion $X_\Gamma \subseteq X_\Lambda$: for all $x \in X_\Gamma$ and $\lambda \in \Lambda$

$$\alpha(\beta(\lambda, x), x) = \lambda \tag{10}$$

We claim that we have the following key inequality.

Claim 4.6. For every $R > 0$, and u and v in Λ , we have

$$\mu\left(\left\{x \in X_\Gamma \colon d_{S_\Gamma}\big(b_x(v), b_x(u)\big) \leqslant R\right\}\right) \leqslant KR \frac{\mathrm{Vol}_{S_\Gamma}(R)}{\varphi\left(\frac{d_{S_\Lambda}(u,v)}{R}\right)}.$$

Proof of the claim. For any $\gamma \in \Gamma$, we define the set

$$A_\gamma = \{x \in X_\Gamma \colon b_x(u)^{-1} b_x(v) = \gamma^{-1}\}.$$

By (10) and (9), we have that for every $x \in A_\gamma$,

$$\alpha\left(\gamma, u^{-1} \cdot x)\right) = \alpha\left(\beta(v^{-1}u, u^{-1} \cdot x), u^{-1} \cdot x\right) = v^{-1}u,$$

from which we deduce that $\left|\alpha\left(\gamma, u^{-1} \cdot x)\right)\right|_{S_\Lambda} = d_{S_\Lambda}(u, v)$.

Let us start giving an upper bound of $\mu(A_\gamma)$ as a function of $|\gamma|$. Write $\gamma = s_1 \dots s_{|\gamma|_{S_\Gamma}}$ with $s_i \in S_\Gamma$. By the cocycle relation and the triangular inequality, there exists an i such that the set

$$\left\{x \in A_\gamma \colon \left|\alpha\left(s_i, s_{i+1} \dots s_{|\gamma|_{S_\Gamma}} \cdot (u^{-1} \cdot x)\right)\right|_{S_\Lambda} \geqslant \frac{d_{S_\Lambda}(u, v)}{|\gamma|_{S_\Gamma}}\right\}$$

has measure at least $\frac{\mu(A_\gamma)}{|\gamma|_{S_\Gamma}}$. Letting $s = s_i$, we have that

$$\mu\left(\left\{y \in X_\Lambda \colon |\alpha(s,y)|_{S_\Lambda} \geqslant \frac{d_{S_\Lambda}(u,v)}{|\gamma|_{S_\Gamma}}\right\}\right) \geqslant \frac{\mu(A_\gamma)}{|\gamma|_{S_\Gamma}},$$

from which we deduce the following upper bound on the measure of A_γ:

$$\mu(A_\gamma) \leqslant |\gamma|_{S_\Gamma}\, \mu\left(\left\{y \in X_\Lambda \colon |\alpha(s,y)|_{S_\Lambda} \geqslant \frac{d_{S_\Lambda}(u,v)}{|\gamma|_{S_\Gamma}}\right\}\right)$$

By Markov's inequality, we deduce

$$\mu(A_\gamma) \leqslant \frac{K|\gamma|_{S_\Gamma}}{\varphi\left(\frac{d_{S_\Lambda}(u,v)}{|\gamma|_{S_\Gamma}}\right)}.$$

Using that φ is non-decreasing, we get that for all $R > 0$,

$$\begin{aligned}
\mu\left(\{x \in X_\Gamma \colon |b_x(v)^{-1}b_x(u)|_{S_\Gamma} \leqslant R\}\right) &= \sum_{\gamma \in B_\Gamma(e_\Gamma, R)} \mu(A_\gamma) \\
&\leqslant \sum_{\gamma \in B_\Gamma(e_\Gamma, R)} \frac{K|\gamma|_{S_\Gamma}}{\varphi\left(\frac{d_{S_\Lambda}(u,v)}{R}\right)} \\
&\leqslant KR\frac{\operatorname{Vol}_{S_\Gamma}(R)}{\varphi\left(\frac{d_{S_\Lambda}(u,v)}{R}\right)}.
\end{aligned}$$

So the claim is proved. $\square_{\text{claim}}$

Applying Claim 4.6 with $R = \frac{r(n)}{n} d_{S_\Lambda}(u,v)$, $u, v \in C_n$, observing that $d_{S_\Lambda}(u,v) \leqslant n$, we obtain

$$\mu\left(\left\{x \in X_\Gamma \colon d_{S_\Gamma}(b_x(v), b_x(u)) \leqslant \frac{r(n)}{n} d_{S_\Lambda}(u,v)\right\}\right) \leqslant \frac{Kr(n)\operatorname{Vol}_{S_\Gamma}(r(n))}{\varphi(n/r(n))}.$$

As there are at most n^2 pairs (u,v) in C_n, we deduce that

$$\mu\left(\left\{x \in X_\Gamma \colon \exists u, v \in C_n \colon d_{S_\Gamma}(b_x(v), b_x(u)) \leqslant \frac{r(n)}{n} d_{S_\Lambda}(u,v)\right\}\right) \leqslant \frac{Kn^2 r(n)\operatorname{Vol}_{S_\Gamma}(r(n))}{\varphi(n/r(n))}.$$

By (4), there exists n_0 such that for $n \geqslant n_0$, there exists a subset B of X_Γ of measure at least $2/3$ on which for all $u, v \in C_n$,

$$d_{S_\Gamma}(b_x(v), b_x(u)) \geqslant \frac{r(n)}{n} d_{S_\Lambda}(u,v).$$

Finally, for all x in the subset $A \cap B$ which has positive measure, we deduce for every $u, v \in C_n$ that

$$a_n d_{C_n}(u,v) \leqslant d_{S_\Gamma}(b_x(v), b_x(u)) \leqslant b_n d_{C_n}(u,v),$$

where in the case of Theorem 4.1, $a_n = \frac{r(n)}{18n}$ and $b_n = \psi^{-1}(3Ln)$; and in the case of 4.2, $a_n = \frac{r(n)}{18n}$ and $b_n = L$. In the first case, assuming (7), we have

$$\frac{r(n)}{18} \geqslant 4\delta \log_2 n + 3\psi^{-1}(3Ln),$$

from which we deduce that for n large enough (as $b_n \to \infty$),

$$na_n \geqslant 4\delta \log_2 n + 3b_n > 4\delta \log_2(b_n n) + 4 + 2b_n,$$

which contradicts Proposition 2.2. Similarly, in the second case, we check that (8) contradicts Corollary 2.3. $\square$

References

[de 19] Mikael de la Salle. Strong property (T) for higher-rank lattices. *Acta Math.*, 223(1):151–193, 2019. doi:10.4310/ACTA.2019.v223.n1.a3.

[DK18] Cornelia Druţu and Michael Kapovich. *Geometric Group Theory. With an Appendix by Bogdan Nica*, volume 63 of *Colloq. Publ., Am. Math. Soc.* American Mathematical Society (AMS), Providence, RI, 2018. doi:10.1090/coll/063.

[DKLMT22] Thiebout Delabie, Juhani Koivisto, François Le Maître, and Romain Tessera. Quantitative measure equivalence between amenable groups. *Annales Henri Lebesgue*, 5:1417–1487, 2022. doi:10.5802/ahl.155.

[Fur99] Alex Furman. Gromov's Measure Equivalence and Rigidity of Higher Rank Lattices. *Annals of Mathematics*, 150(3):1059–1081, 1999. doi:10.2307/121062.

[Gab05] D. Gaboriau. Examples of groups that are measure equivalent to the free group. *Ergodic Theory and Dynamical Systems*, 25(6):1809–1827, 2005. doi:10.1017/S0143385705000258.

[Gro07] Mikhael Gromov. *Metric Structures for Riemannian and Non-Riemannian Spaces.* Modern Birkhäuser Classics. Birkhäuser, Boston, 2007.

[HM20] David Hume and John Mackay. Poorly connected groups. *Proceedings of the American Mathematical Society*, 148(11):4653–4664, 2020. doi:10.1090/proc/15128.

[Kid08] Yoshikata Kida. The mapping class group from the viewpoint of measure equivalence theory. *Mem. Amer. Math. Soc.*, 196(916):viii+190, 2008. doi:10.1090/memo/0916.

[Kid10] Yoshikata Kida. Measure equivalence rigidity of the mapping class group. *Ann. of Math. (2)*, 171(3):1851–1901, 2010. doi:10.4007/annals.2010.171.1851.

[OW80] Donald S. Ornstein and Benjamin Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc. (N.S.)*, 2(1):161–164, 1980. doi:10.1090/S0273-0979-1980-14702-3.

[Sau02] Roman Sauer. L^2-*Invariants of Groups and Discrete Measured Groupoids.* PhD thesis, Universität Münster, 2002.

[Sha00] Yehuda Shalom. Rigidity, Unitary Representations of Semisimple Groups, and Fundamental Groups of Manifolds with Rank One Transformation Group. *Annals of Mathematics*, 152(1):113–182, 2000. doi:10.2307/2661380.

[Sha04] Yehuda Shalom. Harmonic analysis, cohomology, and the large-scale geometry of amenable groups. *Acta Math.*, 192(2):119–185, 2004. doi:10.1007/BF02392739.

[VS14] Kevin Verbeek and Subhash Suri. Metric Embedding, Hyperbolic Space, and Social Networks. In *Proceedings of the Thirtieth Annual Symposium on Computational Geometry*, SOCG'14, pages 501:501–501:510, New York, NY, USA, 2014. ACM. doi:10.1145/2582112.2582139.