Optimal Capacity Modification for Strongly Stable Matchings

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Abstract. We consider the Hospital/Residents (HR) problem in the presence of ties in preference lists. Among the three notions of stability, viz. weak, strong, and super stability, we focus on the notion of strong stability. Strong stability has many desirable properties both theoretically and practically; however, its existence is not guaranteed.
In this paper, our objective is to optimally increase the quotas of hospitals to ensure that a strongly stable matching exists in the modified instance. First, we show that if ties are allowed in residents' preference lists, it may not be possible to augment the hospital quotas to obtain an instance that admits a strongly stable matching. When residents' preference lists are strict, we explore two natural optimization criteria: (i) minimizing the maximum capacity increase for any hospital (MinMax), and (ii) minimizing the total capacity increase across all hospitals (MinSum). We show that the MinMax problem is NP-hard in general. When hospital preference lists can have ties of length at most $\ell + 1$, we give a polynomial-time algorithm that increases each hospital's quota by at most ℓ, ensuring the resulting instance admits a strongly stable matching.
We show that the MinSum problem admits a polynomial-time algorithm. However, when each hospital incurs a cost for each capacity increase, the problem becomes NP-hard, even if the costs are 0 or 1. This also implies that the problem cannot be approximated to any multiplicative factor. We also consider a related problem under the MinSum objective. Given an HR instance and a forced pair (r^*, h^*), the goal is to decide if it is possible to increase hospital quotas (if necessary) to obtain a strongly stable matching that matches the pair (r^*, h^*). We show a polynomial-time algorithm for this problem.

Keywords: Hospital/Residents Problem · Ties · Preferences · Strongly Stable Matching · Capacity Modification.

1 Introduction

The Hospital/Residents (HR) problem [19,31] is a many-to-one generalization of the classical stable marriage problem [16]. As the name suggests, the HR problem models the assignment of junior doctors (residents) to hospitals where agents in both sets are allowed to rank acceptable agents from the other set in a preference order. The problem is extensively investigated since it has applications in a number of centralized matching schemes in many countries, including the National Resident Matching Program in the USA (NRMP), the Canadian Resident Matching Service (CaRMS), and the Scottish Foundation Allocation Scheme (SFAS), to name a few. In addition, the HR problem models several real-world applications like assigning children to schools [1] and students to undergraduate programs [5] where agents need to be matched to programs, and both sets express preferences over each other.

We consider a generalization of the HR problem where *ties* are allowed in the preference lists. That is, agents can be indifferent between multiple agents from the other set. This problem is known as the Hospital/Residents problem with ties (HRT). Ties in preference lists play an important role in real-world matching applications. For instance, hospitals with a large number of applicants often find it difficult to generate strict preference lists. Many of these hospitals, within the framework of a centralized matching scheme, have expressed the desire to include ties in their preference lists [21]. In case of college admissions, it is natural for colleges to have all the students with equal scores in a single tie in their preference lists.

The classical notion of stability defined for strict preferences has been generalized in the literature for the case of ties, in three different ways – weak stability, strong stability and super stability (see Definition 1.1 and the footnote therein). As indicated by the names, super stability is the strongest notion and weak stability is the weakest among the three. It is well-known that every instance of the HRT problem admits a weakly stable matching (and it can be obtained by breaking ties arbitrarily and computing a stable matching in the resulting instance); however, strong or super stable matchings are not guaranteed to exist [20].

The strongest notion of stability is super-stability. However, as highlighted in [22], insisting on super-stability in practical scenarios can be overly restrictive and is less likely to be attainable. Moreover, in applications like college admissions, it is natural to require students to express strict preferences over colleges, although colleges need to put students with equal scores in a tie[3]. In such scenarios, super and strong stability coincide. On the other hand, weak stability is too weak, and as justified in [29], it is susceptible to compromise through persuasion or bribery (also see [22,27] for further details). Moreover, from a social perspective, weak stability may not be an acceptable notion despite its guaranteed existence. For instance, according to the equal treatment policy used in Chile and Hungary, it is not acceptable that a student is rejected from a college preferred by her, even though other students with the same score are admitted (see [12] and the references therein). Thus, strong stability is not only appealing but also essential.

Given that strong stability is desirable but not guaranteed to exist, a natural option for applications is to adjust the quotas (of, say, colleges and hospitals) so that a strongly stable matching exists after the adjustment. We address this problem in this paper. We use the hospital-residents terminology, as is customary in many-to-one stable matchings. Thus we seek to increase or *augment* the hospital quotas to obtain a modified instance which admits a strongly stable matching.

We explore two natural optimization criteria: (i) minimize the total increase (sum) in quotas across all hospitals (MinSum), and (ii) minimize the maximum increase in quota for any hospital (MinMax). Our work falls in the broad theme of capacity planning / modification which has received significant attention [11,18,8,6,2,7] motivated by practical applications where quotas are not rigid. To the best of our knowledge, our work is the first one to explore capacity augmentation for the notion of strong stability.

1.1 Preliminaries and notation

The input to our problem is a bipartite graph $G = (\mathcal{R} \cup \mathcal{H}, E)$, where the vertex set $\mathcal{R}$ represents the set of residents, $\mathcal{H}$ represents the set of hospitals and the edge set

[3] We refer to this as HR-HT in this paper

E represents mutually acceptable resident-hospital pairs. We define $n = |\mathcal{R}| + |\mathcal{H}|$ and $m = |E|$. Every hospital $h \in \mathcal{H}$ has an associated quota $q(h)$ denoting the maximum number of residents that can be assigned to h in any assignment. Each vertex $v \in \mathcal{R} \cup \mathcal{H}$ ranks its neighbors as per its preference ordering, referred to as *the preference list of* v, denoted as $\mathsf{Pref}(v)$. We say that a vertex strictly prefers a neighbor with a smaller rank over another neighbor with a larger rank. If a vertex is allowed to be indifferent between some of its neighbors and is allowed to assign the same rank to such neighbors, it is referred to as a *tie*. The length of a tie is the number of neighbors having equal rank. If ties are not allowed (or equivalently, all ties have length 1), the preference lists are said to be *strict*. We use $u_1 \succ_v u_2$ to denote that v strictly prefers u_1 over u_2 and $u_1 \succeq_v u_2$ to denote that v either strictly prefers u_1 over u_2 or is indifferent between them.

A matching M in G is a subset of E such that $|M(r)| \leq 1$ and $|M(h)| \leq q(h)$ for each resident $r \in \mathcal{R}$ and hospital $h \in \mathcal{H}$ where $M(v)$ denotes the set of matched partners of v in M. For a resident r, if $|M(r)| = 0$, then r is unmatched in M. In this case, we denote the matched partner of r by $M(r) = \perp$. A hospital $h \in \mathcal{H}$ is said to be fully subscribed in M with respect to its quota $q(h)$, if $|M(h)| = q(h)$, under-subscribed in M if $|M(h)| < q(h)$. We abuse the term matching and say that h is over-subscribed in M if $|M(h)| > q(h)$. If left unspecified, the quota under consideration for these terms is the original quota $q(h)$. If h is under-subscribed, then we implicitly match the remaining $q(h) - |M(h)|$ many positions of h to as many copies of $\perp$. A vertex prefers any of its neighbors over $\perp$.

Definition 1.1 (Strong stability:) *For a matching M, an edge $(r, h) \in E \setminus M$ is a strong blocking pair w.r.t. M, if either (i) or (ii) holds:*

(i) $h \succ_r M(r)$ and there exists $r' \in M(h)$ such that $r \succeq_h r'$
(ii) $h \succeq_r M(r)$ and there exists $r' \in M(h)$ such that $r \succ_h r'$.

A matching M is strongly stable matching if there does not exist any strong blocking pair w.r.t. M. [4]

Throughout the paper, we refer to a strong blocking pair as a blocking pair. We give a simple example to illustrate that a strongly stable matching is not guaranteed to exist. Consider an instance with one resident r and two hospitals h_1, h_2, where r has h_1 and h_2 tied at rank-1, whereas h_1, h_2 have unit quota each and both of them rank r as a rank-1 vertex. No matching in this instance is strongly stable, since the matching $M_1 = \{(r, h_1)\}$ is blocked by (r, h_2) and $M_2 = \{(r, h_2)\}$ is blocked by (r, h_1).

$r : (h_1, h_2)$	$[1]\ h_1 : r$
	$[1]\ h_2 : r$

Moreover, the same example illustrates that increasing hospital quotas (alone) may not help in obtaining an instance which admits a strongly stable matching. This happens because there are ties in residents' preference lists whereas quota augmentation is possible for hospitals only.

[4] A pair (r, h) is a *super blocking pair* if both prefer each other strictly or equally to their matched partners. Also, (r, h) form a *weak blocking pair* if they prefer each other strictly more than their matched partners.

What if resident preference lists are strict and ties appear only in hospitals' preferences? We call such instances as HR-HT (Hospital/Residents problem with ties on hospitals' side only). There exist simple instances of HR-HT which do not admit a strongly stable matching, however, for any such instance, we can construct an *augmented instance* G' by setting the quota of each hospital h equal to its degree in G. It is easy to observe that the matching M' that assigns each resident to its rank-1 hospital, is a strongly stable matching in G'. Thus, unlike the general HRT case, an HR-HT instance can always be augmented so that the instance admits a strongly stable matching. Our objective in this paper is to optimally increase hospitals' quotas to ensure that a strongly stable matching exists in the modified instance. We are ready to formally define our problems.

1.2 Our problems and contributions

For all our problems, unless stated explicitly, we assume that the given HR-HT instance $G = (\mathcal{R} \cup \mathcal{H}, E)$ does not admit a strongly stable matching. Deciding whether an HRT instance admits a strongly stable matching can be done in polynomial time using the algorithm by Irving *et al.* [22] (see Appendix A). Throughout this paper the *augmented* instance G' is the same as G except that $q'(h) \geq q(h)$ for each h. Our first objective is to minimize the total increase in quotas across all hospitals. Our first problem under this objective is MinSum-SS.

<u>MinSum-SS:</u> Given an HR-HT instance $G = (\mathcal{R} \cup \mathcal{H}, E)$, construct an augmented instance G' such that G' admits a strongly stable matching and the sum of the increase in quotas over all hospitals (that is, $\sum_{h \in \mathcal{H}} (q'(h) - q(h))$) is minimized.

Theorem 1.2 MinSum-SS *problem is solvable in polynomial time.*

Given the polynomial-time solution for the MinSum-SS problem, we consider the optimal total quota augmentation (if possible) for matching a pair (r^*, h^*) in G. We denote this problem as MinSum-SS-FP (forced pair) and define it formally below.

<u>MinSum-SS-FP:</u> Given an HR-HT instance $G = (\mathcal{R} \cup \mathcal{H}, E)$, which possibly admits a strongly stable matching, and an edge $(r^*, h^*) \in E$, construct an augmented instance G', if possible, such that G' admits a strongly stable matching that matches (r^*, h^*) and the sum of the increase in quotas over all hospitals (that is, $\sum_{h \in \mathcal{H}} (q'(h) - q(h))$) is minimized.

Theorem 1.3 *The* MinSum-SS-FP *problem is solvable in polynomial time.*

Next, we consider a generalization of the MinSum-SS problem, where increasing the quota of a hospital incurs a cost. This problem is denoted by MinSum-Cost.

<u>MinSum-Cost</u>**:** Given an HR-HT instance $G = (\mathcal{R} \cup \mathcal{H}, E)$ with costs associated with hospitals, construct an augmented instance G' such that G' admits a strongly stable matching, and the total cost of the increase in quotas of all hospitals is minimized.

In contrast to the polynomial-time solvability for MinSum-SS, we obtain a hardness result for the MinSum-Cost problem.

Theorem 1.4 *The* MinSum-Cost *problem is NP-hard and is inapproximable to within any multiplicative factor.*

We now turn our attention to the alternative objective: minimizing the maximum increase in quota for any hospital and define MinMax-SS problem.

MinMax-SS: Given an HR-HT instance $G = (\mathcal{R} \cup \mathcal{H}, E)$, construct an augmented instance G' such that G' admits a strongly stable matching, and the maximum increase in the quota for any hospital is minimized, that is, $\max_{h \in \mathcal{H}}\{q'(h) - q(h)\}$ is minimized.

Our result for MinMax-SS is shown in Theorem 1.5

Theorem 1.5 MinMax-SS *is NP-hard even when resident preferences are single-peaked, and hospital preferences are derived from a master list. Moreover, the same minimization objective with the goal of constructing an instance that admits a resident-perfect strongly stable matching (one that matches all residents) is also NP-hard.*

1.3 Related Work

Capacity Modification: Chen and Csáji [11] studied a problem similar to ours for the case of strict preference lists. The goal was to augment the instance by increasing hospital quotas such that the resulting instance admits a perfect stable matching. They showed that with the MinMax objective, the problem admits a polynomial-time algorithm. In contrast, somewhat surprisingly, for strongly stable matching, we get an NP-hardness result (Theorem 1.5) for MinMax. They also consider the MinSum objective, and show NP-hardness for getting an augmented instance that admits a stable and perfect matching under the MinSum objective. Note that this also implies NP-hardness for constructing an augmented instance in the HR-HT setting for achieving a strongly stable and *perfect* matching. However, without the perfectness requirement, our result in Theorem 1.2 gives a polynomial-time algorithm.

Capacity modification to achieve specific objectives has attracted significant interest in recent years. Bobbio *et al.* [8] explored the complexity of determining the optimal variation (augmentation or reduction) of hospital quotas to achieve the best outcomes for residents, subject to stability and capacity variation constraints, and showed NP-hardness results. In a follow-up work, Bobbio et al.[6] developed a mixed integer linear program to address this issue, and in [7], they provided a comprehensive set of tools for obtaining near-optimal solutions. Gokhale *et al.* [18] considered the problem of modifying hospitals' quotas to achieve two objectives – (i) to obtain a stable matching so as to match a given pair, and, (ii) to stabilize a given matching, either by only augmenting or only reducing hospital quotas. Afacan et al.[2] examined capacity design in the HR setting, to achieve a stable matching that is not Pareto-dominated by any other stable matching.

Kavitha and Nasre [24] and Kavitha *et al.* [25] addressed the capacity augmentation problem for *popular* matchings in the one-sided preference list setting (where every hospital is indifferent between its neighbours). It is known that a popular matching is not guaranteed to exist in this setting. Therefore, their objective was to optimally increase hospital quotas to create an instance that admits a popular matching. Although we focus on a different setting (two-sided preference lists) and a different optimality notion – strong stability, it is interesting to note that our results closely resemble those obtained by Kavitha and Nasre [24] and Kavitha *et al.* [25].

Strong Stability: The notion of strong stability was first studied in the one-to-one setting for balanced, complete bipartite graphs (*i.e.* $q(h) = 1$ *for all* $h \in \mathcal{H}$) by Irving [20], where they gave an $O(n^4)$ algorithm to compute a strongly stable matching if it exists. Since then, the strongly stable matching problem has received a significant attention in the literature. Manlove [28] extended Irving's results [20] to the general one-to-one setting (*i.e.* incomplete bipartite graphs) and also showed that all strongly stable matchings have the same size and match the same set of vertices. Irving *et al.* [22] further extended these results to the HRT setting and gave $O(m^2)$ algorithm for the strongly stable matching problem, which was later improved to $O(mn)$ by Kavitha *et al.* [23]. Manlove [29] studied the structure of the set of strongly stable matchings and showed that, similar to the classical stable matchings, the set of strongly stable matchings forms a distributive lattice. Kunysz *et al.* [27] showed that there exists a partial order with $O(m)$ elements representing all strongly stable matchings and also provided an $O(mn)$ algorithm to construct such a representation. In the presence of edge weights, Kunysz [26] showed that when edge weights are small, the maximum weight strongly stable matching problem can be solved in $O(mn)$ time, and in $O(mn\ \log(Wn))$ if the maximum weight of an edge is W. Strong stability w.r.t. restricted edges viz. forced, forbidden and free edges has been studied by Cseh and Heeger [12] and by Boehmer and Heeger [9].
Organization of the paper In Sections 2, 3 and 4, the objective of our problems is to minimize the total increase in quotas. In Section 5, we study the MinMax-SS problem where the objective is to minimize maximum increase in quotas.

2 MINSUM-SS problem

In this section we present an efficient algorithm for the MinSum-SS problem. Since the input instance does not admit a strongly stable matching we need to increase the quotas of certain hospitals to obtain G'. Our algorithm (pseudo-code given in Algorithm 1) involves a sequence of proposals from hospitals and is inspired by the hospital-oriented algorithm for super-stability [21].

The algorithm starts with every resident being unmatched or equivalently matching every resident to its least preferred hospital $\perp$. Call this matching M' (see line 1 of Algorithm 1). During the course of the algorithm, let h be a hospital that is under-subscribed in M', and t be the most preferred rank in $\mathsf{Pref}(h)$ at which h has not yet made a proposal. Then, h simultaneously proposes to all residents at rank t in $\mathsf{Pref}(h)$ (see Line 3). Since a hospital h proposes to all the residents at a particular rank simultaneously, it may lead to the over-subscription of that hospital. A fully subscribed or over-subscribed hospital does not propose further, and the sequence of proposals terminates when either no hospital is under-subscribed or all under-subscribed hospitals have exhausted proposing to all residents on their preference lists. When a resident r receives a proposal from h, the resident accepts or rejects the proposal based on the resident's preference between h and its current matched partner $M'(r)$. Let M' represent the set of matched edges when the proposal sequence terminates. Since G does not admit a strongly stable matching, there must exist at least one hospital h that is over-subscribed in M'. Let G' denote the instance with the modified quotas where the quota of each hospital $h \in \mathcal{H}$ is set to

$q'(h) = \max\{q(h), |M'(h)|\}$. The algorithm returns the augmented instance G' and the matching M'.

Algorithm 1: Algorithm for MinSum-SS

$M' = \{(r, \bot) \mid \text{for every resident } r \in \mathcal{R}\}$
while $\exists$ *h that is under-subscribed in* M' *w.r.t.* $q(h)$ *and h has not exhausted* $\mathsf{Pref}(h)$ **do**
  h proposes to all residents at the most preferred rank t that h has not yet proposed
  for *every resident r that receives a proposal from h* **do**
    if $h \succ_r M'(h)$ **then**
      $M' = M' \setminus \{(r, M'(r))\} \cup \{(r, h)\}$
G' is the same as G, except quotas are set as follows
For each $h \in \mathcal{H}$, set $q'(h) = \max\{q(h), |M'(h)|\}$
return G' and M'

Next, we prove the correctness and optimality of our algorithm.

Lemma 2.1 *The matching M' returned by Algorithm 1 is a strongly stable matching in the augmented instance G'.*

Proof. By the way quotas of the hospitals are set in G', it is clear that M' is a valid matching in G'. Suppose for contradiction, M' is not strongly stable in G'. This implies that there exists a strong blocking pair, say (r, h) w.r.t. M' in G'. Therefore, $h \succ_r M'(r)$ and there exists $r' \in M'(h)$ such that $r \succeq_h r'$. Since hospitals propose in order of their preference list, h must have proposed r during the course of Algorithm 1. The fact that $h \neq M'(r)$ implies that the resident r must have rejected h. Thus, at the time when r rejected h, the resident r must have been matched to a better-preferred hospital, say h'. Since during the course of the algorithm residents improve their matched hospital, the final matched hospital $M'(r)$ of r must be such that $M'(r) \succeq_r h' \succ_r h$. This contradicts the fact that $h \succ_r M'(r)$ and completes the proof. $\square$

To prove the optimality of our capacity increase, we establish useful properties of *any* augmented instance $\widetilde{G}$ (not necessarily optimal), obtained from G, such that $\widetilde{G}$ admits a strongly stable matching. Let $\widetilde{M}$ be a strongly stable matching in $\widetilde{G}$. In Claim 2.2, we show that if a resident r is matched to h (not equal to $\bot$) in M' output by Algorithm 1, then r is matched in $\widetilde{G}$ and is matched to either h or a better-preferred hospital in $\widetilde{G}$.

Claim 2.2 *Let r be matched to $h \in \mathcal{H}$ in M' at the end of Algorithm 1. Then the resident r is matched in $\widetilde{M}$, and $\widetilde{M}(r) \succeq_r h$.*

Proof. Suppose for contradiction, that $\widetilde{M}(r) \prec_r h$. Starting at r we build a path alternately using the matched edges of M' and $\widetilde{M}$ and show that each time we are able to find a new resident along this path.

Since $\widetilde{M}$ is a strongly stable matching, the hospital h must be fully subscribed in $\widetilde{M}$ w.r.t. its quota $\tilde{q}(h)$. Furthermore, each $r' \in \widetilde{M}(h)$ is strictly preferred over r by h. The fact that h proposed to the resident r during the execution of Algorithm 1 implies that the number of residents matched to h *before* it proposed to r must have been less than $q(h)$. This further implies that, in Algorithm 1, h proposed to all residents in $\widetilde{M}(h)$ as they are strictly preferred over r. Note that $\tilde{q}(h) \geq q(h)$. Thus, there must exist a resident, say $r_1 \in \widetilde{M}(h)$, such that r_1 rejected h during the execution of Algorithm 1. The fact that $r_1 \succ_h r$, implies that $M'(r_1) \succ_{r_1} h$. Suppose $M'(r_1) = h_1$.

Now, consider the residents in $\widetilde{M}(h_1)$. Since $h_1 \succ_{r_1} h$, the hospital h_1 must be such that $|\widetilde{M}(h_1)| = \tilde{q}(h_1)$ in the instance $\widetilde{G}$ with all $r' \in \widetilde{M}(h_1)$ strictly preferred over r_1 by h_1. Since $\tilde{q}(h_1) \geq q(h_1)$ and $r_1 \in M'(h_1)$, there must exist a resident $r_2 \in \widetilde{M}(h_1)$ such that $r_2 \notin M'(h_1)$. Since h_1 proposed to r_1 during the execution of Algorithm 1, it must have proposed to r_2. The fact that r_2 rejected h_1 implies that r_2 is matched in M' to a strictly preferred hospital over h_1. Let $M'(r_2) = h_2$. Note that $h_2 \neq h_1$ but it may be the case that $h_2 = h$. However, if $h_2 = h$, then since $|\widetilde{M}(h)| = \tilde{q}(h) \geq q(h)$, there must exist r_3 different from r_1 such that $r_3 \notin M'(h)$ and $r_3 \in \widetilde{M}(h)$. Since $h \succ_{r_2} h_1$, it must be the case that $r_3 \succ_h r_2$. If h_2 is different from all the hospitals explored previously, then it is easy to see that there exists $r_3 \in \widetilde{M}(h_2)$ such that $r_3 \notin M'(h_2)$, and $r_3 \succ_{h_2} r_2$. So, in both cases, we have found a new resident who had not been seen earlier. We continue this procedure. Note that each time we find a new resident, we see that it is matched in M', and we use the M'-edge incident on that resident to extend the path. Similarly, each time we reach a hospital, we find that it is fully subscribed in $\widetilde{M}$ with strictly preferred residents, and we use an $\widetilde{M}$-edge incident to that hospital to find a new resident which we have not seen earlier. Since the process does not terminate and the graph is finite, we get the desired contradiction. □

In the next claim we show that any hospital that remains under-subscribed in M' w.r.t. $q(h)$ continues to remain under-subscribed (to the same extent or more) in a strongly stable matching $\widetilde{M}$ of any augmented instance $\widetilde{G}$.

Claim 2.3 *Let h be a hospital such that $|M'(h)| < q(h)$. Then, $|\widetilde{M}(h)| \leq |M'(h)|$.*

Proof. Since $|M'(h)| < q(h)$, the hospital h exhausted proposing all residents in $\mathsf{Pref}(h)$ during the execution of Algorithm 1. Clearly, all neighbors of h received proposals from h. If there exists any resident, say r, who rejected h during the execution of Algorithm 1, then r must have been matched in M' to $M'(r)$ where $M'(r) \succ_r h$. Using Claim 2.2, we conclude that $\widetilde{M}(r) \succeq_r M'(r) \succ h$. Thus, no resident who rejected h during the execution of Algorithm 1 can be matched to h in $\widetilde{M}$ implying that $|\widetilde{M}(h)| \leq |M'(h)|$. □

Now, we show that the total increase in quotas of all hospitals incurred by Algorithm 1 is optimal.

Lemma 2.4 *The total quota increase by Algorithm 1 is optimal.*

Proof. Let $\mathcal{R}_m \subseteq \mathcal{R}$ be the set of residents who received some proposal during the execution of Algorithm 1 and hence residents in $\mathcal{R}_m$ are matched in M'. By Claim 2.2, every

$r \in \mathcal{R}_m$ must be matched in $\widetilde{M}$. Let $\mathcal{H}_u$ be the set of hospitals such that $|M'(h)| < q(h)$, and $\mathcal{H}_f = \mathcal{H} \setminus \mathcal{H}_u$. Let the $\mathcal{R}_m^u$ denote the set of residents matched in M' to hospitals in $\mathcal{H}_u$. By Claim 2.3, the quota utilization over all hospitals $\mathcal{H}_u$ in a strongly stable matching $\widetilde{M}$ of any instance $\widetilde{G}$ must be at most $|\mathcal{R}_m^u|$. This implies that at least $|\mathcal{R}_m \setminus \mathcal{R}_m^u|$ many residents must be matched to hospitals in $\mathcal{H}_f$ in the matching $\widetilde{M}$. Let $k = |\mathcal{R}_m \setminus \mathcal{R}_m^u| - \sum_{h \in \mathcal{H}_f} q(h)$. Thus, the total quota increase in any instance $\widetilde{G}$ is at least k. Algorithm 1 increases the quotas of hospitals in $\mathcal{H}_f$ only and matches the residents in $\mathcal{R}_m \setminus \mathcal{R}_m^u$ to hospitals in $\mathcal{H}_f$. Thus, the total quota increase of hospitals in G' is exactly k which is optimal. □

Lemma 2.1 and Lemma 2.4 together imply Theorem 1.2.

It is well known that when an HR-HT instance admits a strongly stable matching, all strongly stable matchings of the instance match the same set of residents [22]. In a similar spirit, we prove that all optimal solutions of a given MinSum-SS instance match the same set of residents.

Theorem 2.5 *Let G' be the instance returned by Algorithm 1 and $\mathcal{R}_m$ denote the set of residents matched in the strongly stable matching M'. Then for any optimal augmentation G_{opt}, the set of residents matched in a strongly stable matching is exactly $\mathcal{R}_m$.*

Proof. Theorem 1.2 asserts that the instance G' returned by Algorithm 1 is an optimal augmented instance for G. Let M_{opt} be a strongly stable matching in G_{opt}. Applying Claim 2.2, we know that M_{opt} must match all residents in $\mathcal{R}_m$. If M_{opt} matches any resident $r \notin \mathcal{R}_m$, then M_{opt} must match more than $|\mathcal{R}_m|$ many residents for the instance G_{opt}. Using Claim 2.3, we observe that any hospital h that is under-subscribed in M' w.r.t. $q(h)$ is matched to at most $|M'(h)|$ many residents in M_{opt}. Thus, the matching M_{opt} must match r to a hospital h such that $|M'(h)| \geq q(h)$. Therefore, the total increase in quotas by G_{opt} is more than that of G'. This contradicts the optimality of G_{opt}. □

Using Claim 2.3 and Theorem 2.5 we have the following theorem

Theorem 2.6 *Let G' be the instance returned by Algorithm 1. Also, assume that G_{opt} be any optimal augmentation and M_{opt} be a strongly stable matching in G_{opt}. Then, $|M'(h)| \geq q(h)$ for a hospital h implies that $|M_{opt}(h)| \geq q(h)$. Moreover, if $|M'(h)| < q(h)$, then $|M_{opt}(h)| = |M'(h)|$.*

Now, let us consider a variant of MinSum-SS problem where our goal is to determine the existence of an augmented instance which admits a resident-perfect strongly stable matching. Let us denote this problem by MinSum-SS-RP. Chen and Csáji [11] studied a special case of this problem (called MinSum Cap Stable and Perfect) in the strict list setting and showed that this problem is NP-complete even for a very restricted case. Therefore, we conclude that MinSum-SS-RP problem is NP-complete.

3 MINSUM-SS for a Forced Pair

In this section, we consider the MinSum-SS-FP problem. We first note that it may not always be possible to get an augmented instance where a strongly stable matching matches the given pair. For example, consider an instance with one resident r and two hospitals h_1, h_2 where r prefers h_1 over h_2. Then there is no way to augment the quotas to make the pair (r, h_2) a part of a strongly stable matching.

Here, we show that given an HR-HT instance G, that possibly admits a strongly stable matching, and an edge (r^*, h^*) in E, it is possible to decide in polynomial time whether there exists an augmented instance G' that admits an strongly stable matching which matches (r^*, h^*). Whenever possible, we output the optimally augmented instance.

Our algorithm is similar to the algorithm in [18] for an analogous problem in the strict list setting. We now describe our algorithm. In order to obtain a possible augmented instance where a strongly stable matching matches the given pair (r^*, h^*), we obtain a pruned graph G_p to avoid the potential blocking pairs.

1. Let $\mathcal{R}_d$ denote the set of distracting residents defined as $\mathcal{R}_d = \{r \mid r \succeq_{h^*} r^*\}$. For each $r \in \mathcal{R}_d$, we delete all h from $\mathsf{Pref}(r)$ such that $h^* \succ_r h$.
2. Let $\mathcal{H}_d$ denote the set of distracting hospitals defined as $\mathcal{H}_d = \{h \mid h \succ_{r^*} h^*\}$. For each $h \in \mathcal{H}_d$, we delete all r from $\mathsf{Pref}(h)$ such that $r^* \succeq_h r$.
3. We remove the resident r^* from the instance and also reduce the quota of h^* by 1 (to potentially match r^* to h^*).

Note that the pruned graph G_p has one resident less, the hospital quota of h^* is decremented by 1 and the preference lists of $\mathcal{R}_d$ and $\mathcal{H}_d$ are modified. For an hospital h, the quota of h in G_p is denoted by $q_p(h)$.

Given the instance G_p, we run Algorithm 1 for the MinSum-SS problem to obtain an instance G'_p. Note that by construction G'_p admits a strongly stable matching say M'_p. To finally decide whether there exists an augmented instance which matches the pair (r^*, h^*), we perform the following check.

4. If there exists a hospital $h \in \mathcal{H}_d$ which is under-subscribed in M'_p with respect to its (original) quota $q(h)$, then we declare that "No augmentation possible to match (r^*, h^*)".
5. Otherwise, we obtain a further augmented instance G^*_p and a matching M^*_p as follows. Let $\mathcal{R}^u_d$ denote the residents in $\mathcal{R}_d$ that are left unmatched in the matching M'_p. For every $r \in \mathcal{R}^u_d$, match r to h^* and increase the quota of h^* by 1. Note that this step matches r^* to h^*. We add the edges missing from E to the edge set of G^*_p and then return the instance G^*_p and the matching M^*_p.

Next, we prove the correctness and optimality of our algorithm.

We first observe that pruning the preferences of vertices $\mathcal{R}_d$ and $\mathcal{H}_d$ in Step 1 and Step 2 (and hence deleting the corresponding edges from the graph) is necessary. This is because any strongly stable matching containing the pair (r^*, h^*) cannot match a distracting resident r to a hospital h such that $h^* \succ_r h$; otherwise, (r, h^*) blocks that matching. Similarly, any strongly stable matching containing the pair (r^*, h^*) cannot match a distracting hospital h to a resident r such that $r^* \succ_h r$; otherwise, (r^*, h) blocks that matching.

We first show that if our algorithm says "No augmentation possible", there is no way to obtain an augmented instance which admits a strongly stable matching containing the pair (r^*, h^*).

Lemma 3.1 *If there exists a hospital $h \in \mathcal{H}_d$ such that $|M'_p(h)| < q(h)$, then the given instance of the* MinSum-SS-FP *problem has no solution.*

Proof. We note that for any hospital $h \in \mathcal{H}_d$ that remains under-subscribed in M'_p with respect to its quota $q(h)$, Algorithm 1 did not augment its quota. Let $\widetilde{G}_p$ be any augmented instance obtained from G_p such that $\widetilde{G}_p$ admits a strongly stable matching say $\widetilde{M}_p$. By Claim 2.3, we know that $|\widetilde{M}_p(h)| \leq q(h)$. Furthermore, by definition r^* strictly prefers h over h^* and h is under-subscribed in $\widetilde{M}_p$. Thus, (r^*, h) blocks $\widetilde{M}_p$. Since this is true for *any* augmented instance obtained from G_p, we conclude that there is no augmentation that matches (r^*, h^*). □

Lemma 3.2 *If our algorithm returns an augmented instance G^*_p, then G^*_p is optimal.*

Proof. As mentioned above, it suffices to start with the graph G_p. By Theorem 1.2, we know that G'_p is an optimal augmented instance, obtained from G_p, such that G'_p admits a strongly stable matching. We also note that if the given instance of MinSum-SS-FP admits a solution, then each $r \in \mathcal{R}^u_d$ (see step 5 for the definition of $\mathcal{R}^u_d$) must be matched to one of its neighbors h such that $h \succeq_r h^*$. By definition, all such neighbors of r are in $\mathcal{H}_d \cup \{h^*\}$.

Lemma 3.1 states that if the given instance of the MinSum-SS-FP problem admits a solution, then for each $h \in \mathcal{H}_d$, $|M'_p(h)| \geq q(h)$. Applying Theorem 2.6, we note that each $h \in \mathcal{H}_d$ is fully subscribed or over-subscribed in every strongly stable matching of any optimal augmented instance which admits a strongly stable matching, and is obtained from G_p.

Clearly, any solution for the given instance of MinSum-SS-FP must use at least $|\mathcal{R}^u_d|$ many additional capacities above the extra capacities used by G'_p. We increase the capacity $q_p(h^*)$ of the hospital h^* by precisely $|\mathcal{R}^u_d|$. Since G'_p is an optimal augmented instance of G_p, which admits a strongly stable matching, we conclude that M^*_p and G^*_p are optimal solutions for the given instance. □

Using Lemma 3.1 and Lemma 3.2 we conclude Theorem 1.3.

4 MINSUM-COST Problem

In this section, we consider the MinSum-Cost problem and show that this problem is NP-hard even when the costs of the hospitals are 0 or 1. We prove our hardness by reducing from an instance of the Monotone 1-in-3 SAT problem. The Monotone 1-in-3 SAT problem is a variant of the boolean satisfiability problem where the input is a conjunction of clauses. Each clause is a disjunction of exactly three variables, and no variable appears in negated form. The goal is to determine whether there exists a truth assignment to the variables such that for each clause exactly one variable is set to true. This problem is known to be NP-complete [32,17] even when each variable occurs in at most 3 clauses [14].

Gadget reduction. Let $\mathcal{I}$ be an instance of MONOTONE 1-IN-3 SAT problem, where each variable occurs in at most 3 clauses. Let $\{X_1, X_2, \ldots, X_\beta\}$ be the set of variables in $\mathcal{I}$ and $\{C_1, C_2, \ldots, C_\alpha\}$ be the set of clauses in $\mathcal{I}$.

Given $\mathcal{I}$, we construct an instance $G = (\mathcal{R} \cup \mathcal{H}, E)$ of MINSUM-COST problem with quota $q(h) = 1$ for each $h \in \mathcal{H}$ such that G does not admit a strongly stable matching. We also associate a cost $c(h) \in \{0, 1\}$ with each $h \in \mathcal{H}$. We show that there exists an augmented instance $G' = (\mathcal{R} \cup \mathcal{H}, E)$ with a total augmentation cost 0 that admits a strongly stable matching if and only if there exists an assignment of variables in $\mathcal{I}$ with exactly one variable in each clause set to true.

Let $C_s = (X_i \vee X_j \vee X_k)$ be a clause in the instance $\mathcal{I}$. Corresponding to the clause C_s, there exists a gadget G_s in our reduced instance G. The gadget G_s consists of the resident set $\mathcal{R}_s = \{a_i^s, a_j^s, a_k^s, b_i^s, b_j^s, b_k^s, d_1^s, d_2^s, d_3^s\}$, and the hospital set $\mathcal{H}_s = \{v_i^s, v_j^s, v_k^s, w^s\}$. The augmentation cost of each hospital in the gadget G_s is given as: $c(v_i^s) = c(v_j^s) = c(v_k^s) = 0$ and $c(w^s) = 1$.

The preference lists of residents and hospitals in G_s are given in Figure 1. The preference list of a resident b_p^s corresponding to the variable X_p for $p \in \{i, j, k\}$ consists of four hospitals – two within the gadget G_s, and two outside the gadget G_s. Assume that in $\mathcal{I}$, the variable X_i appears in three clauses, namely C_s, C_{i_1} and C_{i_2}. Then the preference list of b_i^s consists of hospitals $v_i^s, v_i^{i_1}, v_i^{i_2}$ and w^s in this order. Analogously, the hospital v_i^s corresponding to X_i ranks the resident a_i^s as its top choice, followed by a tie of length three consisting of the three b-residents, namely $b_i^s, b_i^{i_1}, b_i^{i_2}$, from three different gadgets. This completes the description of our reduction.

$$
\begin{aligned}
a_i^s &: v_i^s \\
a_j^s &: v_j^s \\
a_k^s &: v_k^s \\
b_i^s &: v_i^s, v_i^{i_1}, v_i^{i_2}, w^s \\
b_j^s &: v_j^s, v_j^{j_1}, v_j^{j_2}, w^s \\
b_k^s &: v_k^s, v_k^{k_1}, v_k^{k_2}, w^s \\
d_1^s &: w^s \\
d_2^s &: w^s \\
d_3^s &: w^s
\end{aligned}
$$

(i)

$$
\begin{aligned}
v_i^s &: a_i^s, (b_i^s, b_i^{i_1}, b_i^{i_2}) \\
v_j^s &: a_j^s, (b_j^s, b_j^{j_1}, b_j^{j_1}) \\
v_k^s &: a_k^s, (b_k^s, b_k^{k_2}, b_k^{k_2}) \\
w^s &: (b_i^s, b_j^s, b_k^s), (d_1^s, d_2^s, d_3^s)
\end{aligned}
$$

(ii)

Fig. 1: (i) Preference lists of residents in the gadget G_s. (ii) Preference lists of hospitals in the gadget G_s.

Correctness: We claim that the reduced instance G does not admit a strongly stable matching. Recall that the quota of each hospital is one. Any strongly stable matching M in the reduced instance G, must match a_p^s to v_p^s for the gadget G_s, as otherwise, (a_p^s, v_p^s) blocks M. The matching M cannot leave w^s unmatched, otherwise (d_t^s, w^s) for some

$t \in \{1, 2, 3\}$ is a strong blocking pair. Since w^s has a unit quota, it cannot be matched with any of the d-vertices. For the same reason, w^s cannot accommodate all of the three b-vertices in the gadget G_s. This implies that there exists a b-vertex, say b_j^s, which is not matched to w^s in M. Thus, the pair (b_j^s, w^s) blocks M. Hence, the reduced instance G does not admit a strongly stable matching.

Lemma 4.1 *If $\mathcal{I}$ admits a satisfying assignment, then there exists an instance G' obtained from G with an augmentation cost 0 such that G' admits a strongly stable matching.*

Proof. Given a satisfying assignment for the instance $\mathcal{I}$ of the Monotone 1-in-3 SAT problem, we know that each clause has exactly one variable that is set to true. We construct G' and M' as follows. Without loss of generality, assume that for the clause C_s= $(X_i \vee X_j \vee X_k)$, the variable X_k is set to true by the satisfying assignment. We set the quota $q'(h) = 2$ for each $h \in \{v_i^s, v_j^s\}$ and $q'(h) = 1$ for $h \in \{w^s, v_k^s\}$. Note that the total augmentation cost for each clause is $c(v_i^s) + c(v_j^s) = 0$. Hence, the total augmenting cost is 0. Let $M' = \bigcup_{s=1}^{\alpha} \{(a_i^s, v_i^s), (b_i^s, v_i^s), (a_j^s, v_j^s), (b_j^s, v_j^s), (a_k^s, v_k^s), (b_k^s, w^s)\}$. Recall that α denotes the number of clauses in the instance $\mathcal{I}$.

To show the strong stability of M', we prove that for any s, no resident in the gadget G_s, participates in a strong blocking pair w.r.t. M. Each of $a_i^s, a_j^s, a_k^s, b_i^s$ and b_j^s are matched with their rank-1 hospitals. The resident b_k^s is matched with its rank-4 hospital w^s. However, the three hospitals which b_k^s prefers over w^s have unit quotas, and they are matched to their respective rank-1 residents. The d-vertices in gadget G_s cannot block M' because the unit quota of w^s is occupied by a better-preferred resident. Thus, M' is a strongly stable matching. □

To prove the other direction we require the following two claims.

Claim 4.2 *Let G' be an instance obtained from G with zero total augmentation cost. Assume that G' admits a strongly stable matching M'. Then, for any gadget G_s, we have $|M'(w^s)| = 1$ and $M'(w^s) \in \{b_i^s, b_j^s, b_k^s\}$.*

Proof. The fact that the augmentation cost for w_s is one, and we have a zero total augmentation cost instance G' implies that the quota of w^s is not augmented in G'. Since the original quota of w^s is one, it implies that $|M'(w_s)| \leq 1$. We note that M' cannot leave w^s unmatched, neither can M' match w^s to one of the d-vertices; otherwise, an unmatched d-vertex along with w^s forms a strong blocking pair w.r.t. M'. Therefore, $|M'(w^s)| = 1$ and $M'(w^s) \in \{b_i^s, b_j^s, b_k^s\}$. □

Claim 4.3 *Let G' be an instance obtained from G with zero total augmentation cost. Assume that G' admits a strongly stable matching M'. For any gadget G_s, if $(b_i^s, w^s) \in M'$, then $(b_i^{i_1}, w^{i_1}) \in M'$ and $(b_i^{i_2}, w^{i_2}) \in M'$.*

Proof. We show that $(b_i^{i_1}, w^{i_1}) \in M'$. The proof for $(b_i^{i_2}, w^{i_2}) \in M'$ is analogous. Suppose for contradiction that $M'(b_i^{i_1}) \neq w^{i_1}$. Note that $b_i^{i_1}$ cannot remain unmatched in M'; otherwise, $(b_i^{i_1}, w^{i_1})$ is a strong blocking pair w.r.t. M'. This implies that $M'(b_i^{i_1}) \in \{v_i^s, v_i^{i_1}, v_i^{i_2}\}$. Let us first assume that $M'(b_i^{i_1}) = v_i^{i_2}$. Since $v_i^{i_2}$ ranks b_i^s at rank 2, the hospital $v_i^{i_2}$ is indifferent between residents b_i^s and $M'(v_i^{i_2}) = b_i^{i_1}$. The fact that $v_i^{i_2} \succ_{b_i^s} w^s$ implies that the edge $(b_i^s, v_i^{i_2})$ is a strong blocking pair w.r.t. M'. A similar argument works when $M'(b_i^{i_1}) = v_p^s$ or $M'(b_i^{i_1}) = v_i^{i_1}$. □

Lemma 4.4 *If there exists an instance G' obtained from G with zero total augmentation cost such that G' admits a strongly stable matching, say M', then the instance $\mathcal{I}$ admits a satisfying assignment.*

Proof. By Claim 4.2, we know that $|M'(w^s)| = 1$ and $M'(w^s) \in \{b_i^s, b_j^s, b_k^s\}$ for each $s \in \{1, 2, \ldots, \alpha\}$. We obtain the assignment of variables as follows: For the gadget G_s, if $M'(b_k^s) = w^s$, then we set X_i and X_j to false and X_k to true in clause C_s. Clearly, clause C_s is satisfied, and exactly one variable is set to true in C_s. Using Claim 4.3, we note that this assignment is consistent. Thus, every clause has exactly one variable in that clause set to true, and we have a satisfying assignment for $\mathcal{I}$. □

This completes the hardness reduction which shows that the decision version of MinSum-Cost problem is NP-hard for the budget $\ell = 0$. This completes the proof of Theorem 1.4.

5 MINMAX-SS Problem

In this Section, we consider the MinMax-SS problem. We show that the MinMax-SS problem is NP-hard even for a very restricted setting. For this, we consider a special case of this problem, where $q(h) = 1$ for all $h \in \mathcal{H}$ and budget $\ell = 1$. We call this special case 1-or-2 Capacity-SS problem as the quota of each hospital in the augmented instance is restricted to 1 or 2. Next, we show the hardness of 1-or-2 Capacity-SS problem.

5.1 NP-hardness of 1-or-2 Capacity-SS problem

We prove the hardness of 1-or-2 Capacity-SS problem by reducing from an instance of the Monotone Not-All-Equal 3-SAT problem. Let α and β be two non-negative integers. The Monotone Not-All-Equal 3-SAT problem is a variant of the boolean satisfiability problem where the input is a conjunction clauses. Each clause is a disjunction of exactly three variables, and no variable appears in negated form. The goal is to determine whether there exists a truth assignment to the variables such that for each clause, at least one variable is set to true and at least one to false. This problem is known to be NP-complete [30] even when (i) the formula is linear (where each pair of distinct clauses shares at most one variable), and (ii) each variable appears in exactly four clauses[13].
Gadget reduction. Let $\mathcal{I}$ be an instance of the Monotone Not-All-Equal 3-SAT problem, where each variable appears in exactly four clauses. Let $\{X_1, X_2, \ldots, X_\beta\}$ be the the set of variables in $\mathcal{I}$ and $\{C_1, C_2, \ldots, C_\alpha\}$ be the set of clauses in $\mathcal{I}$.

It will be convenient to order all the clauses of $\mathcal{I}$ in a cyclical order $\mathcal{C}$. Let $C_s = (X_i \vee X_j \vee X_k)$ be a clause in $\mathcal{C}$. Suppose C_{i_1} denotes the next clause where X_i appears after C_s in the cyclical order $\mathcal{C}$. We also assume that C_{i_3} denotes the clause in the cyclical order $\mathcal{C}$ where X_i appears for the fourth time starting at C_s. Similarly, we define clauses C_{j_1}, C_{k_1}, C_{j_3} and C_{k_3}.

Given $\mathcal{I}$, we construct an instance $G = (\mathcal{R} \cup \mathcal{H}, E)$ of 1-or-2 Capacity-SS problem with quota $q(h) = 1$ for each $h \in \mathcal{H}$ such that G does not admit a strongly stable matching. We show that there exists an augmented instance $G' = (\mathcal{R} \cup \mathcal{H}, E)$ where the maximum increase in quota for any hospital is at most 1 such that G' admits a strongly stable matching if and only if there exists an assignment of variables in $\mathcal{I}$ with at least one and at most two variables in each clause set to true.

In our reduced instance G, there exists a gadget G_s corresponding to the clause C_s in $\mathcal{I}$. The gadget G_s is the same as the one constructed in Section 4. Hence, the intra-gadget edges are exactly the same. However, the set of inter-gadget edges is different. The preference lists of residents and hospitals are given in Figure 2. The preference list of a resident b_p^s corresponding to the variable X_p for $p \in \{i, j, k\}$ consists of three hospitals – two within the gadget G_s, and one outside the gadget G_s. Specifically, the preference list of b_i^s consists of hospitals $v_i^s, v_i^{i_1}$ and w^s in this order. Analogously, the hospital v_i^s corresponding to X_i ranks the resident a_i^s as its top choice, followed by a tie of length two consisting of the two b-residents, namely $b_i^s, b_i^{i_3}$, from two different gadgets. This completes the description of our reduction.

$$
\begin{array}{ll}
a_i^s : v_i^s & \\
a_j^s : v_j^s & \\
a_k^s : v_k^s & \\
b_i^s : v_i^s, v_i^{i_1}, w^s & \\
b_j^s : v_j^s, v_j^{j_1}, w^s & v_i^s : a_i^s, (b_i^s, b_i^{i_3}) \\
b_k^s : v_k^s, v_k^{k_1}, w^s & v_j^s : a_j^s, (b_j^s, b_j^{j_3}) \\
d_1^s : w^s & v_k^s : a_k^s, (b_k^s, b_k^{k_3}) \\
d_2^s : w^s & w^s : (b_i^s, b_j^s, b_k^s), (d_1^s, d_2^s, d_3^s) \\
d_3^s : w^s & \\
\text{(i)} & \text{(ii)}
\end{array}
$$

Fig. 2: (i) Preference lists of residents in the gadget G_s. (ii) Preference lists of hospitals in the gadget G_s.

Correctness: Using exactly the same argument as in Section 4, we claim that the reduced instance G does not admit a strongly stable matching.

Lemma 5.1 *If $\mathcal{I}$ admits a satisfying assignment, then there exists an instance G' obtained from G by setting each of hospital quotas either 1 or 2 such that G' admits a strongly stable matching.*

Proof. We obtain an instance G' from G by modifying the hospital quotas as follows. Set $q'(v_r^s) = 2$ if and only if X_r in clause C_s is set to true. Since the satisfying assignment has the property that each clause C_s in $\mathcal{I}$ has at least one and at most two variables set to true, the gadget G_s must have the property that at least one and at most two of

$q'(v_i^s), q'(v_j^s), q'(v_k^s)$ are set to 2. Additionally, we set $q'(w^s) = 2$ if and only if clause C_s is satisfied by setting exactly one of X_i, X_j, and X_k to true in the given assignment.

Next, we construct a strongly stable matching in G'. We consider two cases based on the number of variables in clause C_s that are set to true in the satisfying assignment.
Case 1: Exactly one variable in C_s is set to true. Without loss of generality, assume that X_i is true, and X_j and X_k are false in C_s. Clearly, $q'(v_i^s) = q'(w^s) = 2$ and $q'(v_j^s) = q'(v_k^s) = 1$. We construct a matching M_s in G_s as follows:
$M_s = \{(a_i^s, v_i^s), (b_i^s, v_i^s), (a_j^s, v_j^s), (b_j^s, w^s), (a_k^s, v_k^s), (b_k^s, w^s)\}$.
Case 2: Exactly two variables in C_s are set to true. Without loss of generality, assume that X_i and X_j are true and X_k is false in C_s. Clearly, $q'(v_i^s) = q'(v_j^s) = 2$ and $q'(v_k^s) = q'(w^s) = 1$. We construct a matching M_s in G_s as follows:
$M_s = \{(a_i^s, v_i^s), (b_i^s, v_i^s), (a_j^s, v_j^s), (b_j^s, v_j^s), (a_k^s, v_k^s), (b_k^s, w^s)\}$.

Now, let $M' = \bigcup_{s=1}^{\alpha} M_s$. To show the strong stability of M', we prove that for any s, no resident in the gadget G_s participates in a strong blocking pair w.r.t. M'. We consider two cases based on how M_s was constructed.
Case (a): M_s was constructed as in Case 1. Each of Each of a_i^s, a_j^s, a_k^s and b_i^s are matched with their rank-1 hospitals. The residents b_j^s and b_k^s are matched with their rank-3 hospital w^s. However, the two hospitals which they prefer over w^s have unit quotas (because the satisfying assignment is consistent), and they are matched to their respective rank-1 residents. The d-vertices in gadget G_s cannot block M' because the quota of w^s is two, and both of it is occupied by better-preferred residents.
Case (b): M_s was constructed as in Case 2. Each of $a_i^s, a_j^s, a_k^s, b_i^s$ and b_j^s is matched with its rank-1 hospital. The resident b_k^s is matched with its rank-3 hospital w^s. However, the two hospitals v_k^s and $v_k^{k_1}$ which b_k^s prefers over w^s have unit quotas (because the satisfying assignment is consistent), and they are matched to their respective rank-1 residents. The d-vertices in gadget G_s cannot block M' because the unit quota of w^s is occupied by a better-preferred resident. Thus, M' is a strongly stable matching. □

We require the following claims to prove the other direction.

Claim 5.2 *Let G' be an instance obtained from G with $q'(h) \in \{1, 2\}$ for all $h \in \mathcal{H}$. Assume that G' admits a strongly stable matching M'. Then, for any gadget G_s, $q'(h) = 2$ for some $h \in \{v_i^s, v_j^s, v_k^s\}$.*

Proof. Suppose for contradiction that $q'(h) = 1$ for all $h \in \{v_i^s, v_j^s, v_k^s\}$. Note that $(a_p^s, v_p^s) \in M$ for all $p \in \{i, j, k\}$, otherwise, (a_p^s, v_p^s) blocks M'. Also, note that all three b_p^s for $p \in \{i, j, k\}$ are matched in M', as otherwise (b_p^s, w^s) blocks M'. Clearly, none of the b-vertices in G_s can be matched to their rank-1 hospitals in M'. We claim that b_p^s for $p \in \{i, j, k\}$ cannot be matched to its rank-2 hospital in M'. To see this, wlog, let us assume that b_i^s is matched with its rank-2 hospital $v_i^{i_1}$, that is, $(b_i^s, v_i^{i_1}) \in M'$. The fact that $(a_i^{i_1}, v_i^{i_1}) \in M'$ and $q'(v_i^{i_1}) \leq 2$ implies that the resident $b_i^{i_1}$ is not matched with $v_i^{i_1}$. This implies $(b_i^{i_1}, v_i^{i_1})$ blocks the matching M. Therefore, it must be the case that all three of b_i^s, b_j^s and b_k^s are matched with their rank-3 neighbor w^s. But, $q'(w^s) \leq 2$, the hospital w^s can accommodate at most two of these b-vertices. Thus, the unmatched resident, say b_i^s, along with w^s blocks M'. □

Claim 5.3 *Let G' be an instance obtained from G with $q'(h) \in \{1, 2\}$ for all $h \in \mathcal{H}$. Assume that G' admits a strongly stable matching M'. Then, $q'(h) = 1$ for some $h \in \{v_i^s, v_j^s, v_k^s\}$.*

Proof. Suppose for contradiction that $q'(h) = 2$ for all $h \in \{v_i^s, v_j^s, v_k^s\}$. This implies that $(b_p^s, v_p^s) \in M$ for all $p \in \{i, j, k\}$, otherwise, (b_p^s, v_p^s) blocks the matching M. This implies that w^s is not matched to any of its rank-1 residents in M'. Since $q'(w^s) \leq 2$, an unmatched d-vertex in G_s, say d_j^s, along with w^s blocks M'. □

Claim 5.4 *Let G' be an instance obtained from G with $q'(h) \in \{1, 2\}$ for all $h \in \mathcal{H}$. Assume that G' admits a strongly stable matching M'. For any gadget G_s, if $q'(v_i^s) = 2$, then $q'(v_i^{i_3}) = 2$.*

Proof. Clearly, $(a_i^s, v_i^s) \in M'$. Since $q'(v_i^s) = 2$, it must be the case that $(b_i^s, v_i^s) \in M'$, as otherwise, (b_i^s, v_i^s) blocks M'. This implies $|M'(v_i^s)| = 2 = q'(v_i^s)$, and therefore, both positions of v_i^s are occupied by the residents in G_s. Since v_i^s is matched to a rank-2 resident, the other rank-2 resident $b_i^{i_3}$ of v_i^s which belongs to a different gadget G_{i_1} must be such that $M'(b_i^{i_3})$ is within rank-2 in $\mathsf{Pref}(b_i^{i_3})$. The fact that both positions of v_i^s are occupied by the residents in G_s implies that $M'(b_i^{i_3}) = v_i^{i_3}$. Also, $M'(a_i^{i_3}) = v_i^{i_3}$, as otherwise, $(a_i^{i_3}, v_i^{i_3})$ blocks M'. This implies $q'(v_i^{i_3}) = 2$. □

Lemma 5.5 *If there exists an instance G' obtained from G with $q'(h) \in \{1, 2\}$ for all $h \in \mathcal{H}$ such that G' admits a strongly stable matching, say M', then the instance $\mathcal{I}$ admits a satisfying assignment.*

Proof. We obtain a truth assignment for variables in $\mathcal{I}$ using the quotas of hospitals in G'. Set X_i in clause C_s to true if $q'(v_i^s) = 2$. Otherwise, set X_i in clause C_s to false. By using Claim 5.2 and Claim 5.3, we know that each clause has at least one and at most two variables set to true. This implies that every clause is satisfied. Using Claim 5.4, we conclude that this assignment is consistent. Therefore, every clause has at least one variable set to true and at least one to false. Thus, we have a satisfying assignment for $\mathcal{I}$. □

Next, we show that the preference lists of residents admit single-peakedness[5] property. We note that the preference list of a degree-2 vertex is always single-peaked, and the preference list of a degree-3 vertex is single-peaked w.r.t. the strict linear ordering if its least-preferred neighbor does not lie between its two other neighbors in that ordering. Now, we give the strict linear ordering of hospitals in G as follows. Let V^s denote the strict order $\langle v_i^s, v_j^s, v_k^s \rangle$ and W to denote the strict order $\langle w^1, w^2, \ldots, w^\alpha \rangle$. Then the strict linear ordering of hospitals is given as $\langle V^1, V^2, \ldots, V^\alpha, W \rangle$. It is trivial to observe that for

[5] Single-peaked preferences originate from the problems in the context of elections where preferences are strict and complete. Preferences are called single-peaked if there is an arrangement of alternatives such that each voter's preference graph has only one local maximum. In this graph, alternatives are on the x-axis, and scores assigned by voters are on the y-axis. Single-peaked preferences have also been considered for stable matchings [4,3,10]. Social choice literature has adapted these notions to incomplete preferences as well [10,15]. For the bipartite graph $G = (\mathcal{R} \cup \mathcal{H}, E)$, providing a strict linear ordering over $\mathcal{R}$ and $\mathcal{H}$ separately, suffices.

any resident r, the preference list of r has a single peak over the above linear ordering. Thus, resident preferences are single-peaked.

Now, we show that the hospital preference lists are derived from a master list. We give a master list of residents as follows. Let A^s denote the strict order $\langle a_i^s, a_j^s, a_k^s \rangle$, B denote the tie containing all the b-residents and D^s denote the tie (d_1^s, d_2^s, d_3^s). Then, the master list of residents is given as $\langle A^1, A^2, \ldots, A^\alpha, B, D^1, D^2, \ldots, D^\alpha \rangle$. It can be easily verified that the preference list of each hospital is derived from this master list.

Lemma 5.1 and Lemma 5.5, together with the above discussion about single-peakedness and master lists, complete the proof of the first part of Theorem 1.5.

Resident-perfect MinMax-SS **problem:** This is a variant of the MinMax-SS problem, which asks for resident-perfect strongly stable matching. We now give a simple modification in the above reduction to show that the resident-perfect MinMax-SS problem is also NP-hard.

In the above reduction, we introduce a unique hospital corresponding to each of d_1^s, d_2^s and d_3^s for $1 \leq s \leq \alpha$, say f_1^s, f_2^s and f_3^s, respectively, such that d_p^s prefers f_p^s for $1 \leq p \leq 3$ at rank 2. These f-hospitals act as last resorts for each of d-residents. Now, it is easy to see that there exists a valid satisfying assignment for $\mathcal{I}$ if and only if there exists an instance $G' = (\mathcal{R} \cup \mathcal{H}, E)$ with $q'(h) \in \{1, 2\}$ for all $h \in \mathcal{H}$ obtained from G such that G' admits an $\mathcal{R}$-perfect strongly stable matching.

This completes the proof of Theorem 1.5.

5.2 MINMAX-SS problem with bounded ties

In this section, we consider a special case of the MinMax-SS problem, where tie lengths are bounded. We assume that the length of the ties in the preference lists of hospitals is bounded by $\ell + 1$. We denote this special problem by MinMax-SS-BT. We show that for a given MinMax-SS-BT instance G, the existence of an augmented instance G', within the budget ℓ, is guaranteed, and can be computed efficiently.

Next, we describe the algorithm for the MinMax-SS-BT problem. Our algorithm implicitly uses the algorithm by Irving *et al.* [22] (see Appendix A for a simplified version). The algorithm (pseudo-code given in Algorithm 2) starts by setting a temporary quota $q'(h) = q(h) + \ell$ for each hospital $h \in \mathcal{H}$, and by initializing a matching M, where each resident is matched to $\perp$. Line 3 to Line 11 represents the algorithm by Irving *et al.* [22] (restricted to $\mathcal{H}$-side ties). The matching M at the end of the while loop is considered. Note that in the matching M, it is possible that a hospital h is over-subscribed w.r.t. $q(h)$. Finally, our algorithm fixes the quota $q''(h)$ for each hospital h based on the matching M (see Line 12).

Next, we prove the correctness of our algorithm. In the following lemma, we show that the $\max_{h \in \mathcal{H}} \{q''(h) - q(h)\} \leq \ell$.

Lemma 5.6 *The instance $G' = (\mathcal{R} \cup \mathcal{H}, E)$ output by the Algorithm 2 has a property that $q''(h) \leq q(h) + \ell$ for all $h \in \mathcal{H}$.*

Proof. Line 5 to Line 11 of Algorithm 2 ensure that $|M(h)| \leq q(h) + \ell$ for all $h \in \mathcal{H}$. By definition of $q''(h)$ at Line 12, the lemma holds. □

Algorithm 2: Algorithm for MinMax-SS-BT

set $q'(h) = q(h) + \ell$ for all $h \in \mathcal{H}$
$M = \{(r, \perp) \quad | \quad \text{for every resident } r \in \mathcal{R}\ \}$
while $\exists$ *unmatched* $r \in \mathcal{R}$ *such that* Pref(r) is not empty **do**
  r proposes to the top-ranked hospital h in Pref(r) $M = M \cup \{(r, h)\}$
  if $|M(h)| > q'(h)$ **then**
    let r' be a least-preferred resident in $M(h)$
    suppose r' is at rank p in Pref(h)
    for *each* r'' *at rank* p *in* Pref(h) **do**
      if$(r'', h) \in M$ **then** $M = M \setminus (r'', h)$
    for *each* r'' *at rank* $\geq p$ *in* Pref(h) **do**
      delete h from Pref(r'') and r'' from Pref(h)
set the augmented capacity of each $h \in \mathcal{H}$ as $q''(h) = \max\{q(h), |M(h)|\}$
return M and $G' = (\mathcal{R} \cup \mathcal{H}, E)$ with $q''(h)$ for each $h \in \mathcal{H}$

Claim 5.7 *If a hospital h received at least $q(h)$ many proposals during the execution of Algorithm 2, then $|M(h)| \geq q(h)$.*

Proof. Suppose for contradiction that h received at least $q(h)$ many proposals during the execution of Algorithm 2, but $|M(h)| < q(h)$. This implies that h must have rejected some of the proposals it received. A hospital rejects a resident only when it becomes over-subscribed. Note that the capacity of h during the execution of Algorithm 2 was $q'(h) = q(h) + \ell$. Thus, h must have become over-subscribed with respect to the quota $q'(h)$ at some point. This further implies that h received at least $q(h) + \ell + 1$ proposals. By the design of the algorithm, h only rejects proposals at a particular rank (the least-preferred one) at any given time. Once all the proposals at the least-preferred rank are rejected, h is no longer over-subscribed. This is because, as soon as h becomes over-subscribed, it rejects these least-preferred proposals. Since $|M(h)| < q(h)$, h must have rejected at least $\ell + 2$ proposals at once, all from the same rank. This implies that there exists a tie of length at least $\ell + 2$, contradicting the assumption that the maximum length of ties in the preference list of h is at most $\ell + 1$. □

We use the following claim to prove Lemma 5.9.

Claim 5.8 *If $q''(h) \geq q(h)$, then h must be fully subscribed in M w.r.t. its augmented quota $q''(h)$.*

Proof. By definition of $q''(h)$, if $|M(h)| \geq q(h)$, then $q''(h) = |M(h)|$. □

Lemma 5.9 *The matching M is a strongly stable matching in the instance $G' = (\mathcal{R} \cup \mathcal{H}, E)$ returned by the Algorithm 2.*

Proof. Suppose for contradiction that M is not a strongly stable matching in G'. Let the pair (r, h) block the matching M in G'. Therefore, r is either unmatched in M or

$h \prec_r M(r) = h'$. Clearly, r proposed to h during the execution of Algorithm 2. Then it must be the case that either r got rejected by h or h deleted r from $\mathsf{Pref}(h)$. This can happen only if h received at least $q(h) + \ell + 1 > q(h)$ proposals. By Claim 5.7, $|M(h)| \geq q(h)$. Thus, in G', the quota $q''(h)$ of h must be such that $q''(h) \geq q(h)$. By Claim 5.8, h is fully subscribed in M w.r.t. $q''(h)$ in the instance G'. Also, note that h rejects a resident only when r is the least preferred matched resident to h. At the time of this rejection, h also deletes all the residents which are at the same or greater rank compared to r in $\mathsf{Pref}(h)$. Thus, for all $r'' \in M(h)$, it must be the case that $r'' \succ_h r$.

Therefore, h is fully subscribed in M with all residents in $M(h)$ strictly better-preferred than r in the instance G', and hence the edge (r, h) cannot block M, contradicting the assumption that it is a blocking pair. □

Lemma 5.6 and Lemma 5.9 together give us the following theorem.

Theorem 1. *The* MinMax-SS-BT *problem admits a polynomial-time algorithm.*

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A Background: An Algorithm for Strong Stability

In this section, we look at the algorithm for determining the existence of a strongly stable matching for a given instance and computing one if it exists. Irving *et al.* [22] explored strong stability in the context of the HRT problem and presented an algorithm for finding a strongly stable matching when one exists. Our work focuses on the HR-HT problem, where ties are restricted only to the hospitals' side. Therefore, we describe a simplified version of the algorithm by Irving *et al.* [22].

The pseudo-code for the simplified version of their algorithm is presented in Algorithm 3. The algorithm begins with every resident unmatched, that is, each resident is initially matched to $\perp$ (see Line 1). For each hospital h, the algorithm maintains a variable $full(h)$, to track whether h becomes fully subscribed during execution. Initially, $full(h)$ = false for every $h \in \mathcal{H}$ (see Line 2). During the algorithm, each unmatched resident r proposes to its top-ranked hospital, say h. When h receives a proposal from r, it provisionally accepts it (see Line 5). If h becomes fully subscribed, $full(h)$ is set to true. In this process, some hospitals may become over-subscribed. If a hospital h is over-subscribed, then it rejects all the residents matched to it in M at the least-preferred rank (see Line 9-Line 12). This is because none of the residents at this rank or lower-preferred rank can be matched to h in any strongly stable matching. Therefore, all such edges are deleted from the given instance (see Line 14). Subsequently, for each resident, $r \in \mathcal{R}$, either r is matched to some hospital or $\mathsf{Pref}(r)$ becomes empty. The proposal sequence terminates when either all residents are matched or $\mathsf{Pref}(r)$ for each unmatched resident becomes empty.

The algorithm then uses the tracking variable $full(h)$ to determine the existence of a strongly stable matching. If there is a hospital h such that $full(h)$ = true and h is under-subscribed in the final matching M. Then, M is blocked by (r', h) where r' is a resident who was rejected by h during the course of algorithm. In this case, Algorithm 3 concludes that no strongly stable matching exists for the given instance. On the other hand, if for all h such that $full(h)$ = true, it satisfies that h is not under-subscribed, then Algorithm 3 declares that M is a strongly stable matching for the given instance.

Now, we prove the correctness of this algorithm. First, we show that if Algorithm 3 returns a matching M, then M is indeed a strongly stable matching. Note that by the design of the algorithm, no hospital can be over-subscribed in M. This is because of the following reason. Since preference lists of residents are strict and at a time only one resident can make a proposal, a hospital h can become oversubscribed by at most one slot. As soon as a hospital becomes over-subscribed, Line 8 – Line 14 are executed, and h rejects at least one resident. Consequently, no hospital remains over-subscribed, ensuring that M is a valid matching. Now, we show the strong stability of M.

Lemma A.1 *The matching M returned by Algorithm 3 is a strongly stable matching.*

Proof. Suppose for contradiction that M is not a strongly stable matching. Then, there must exists a pair (r, h), that blocks M. We first show that (r, h) was not deleted during the execution of Algorithm 3. Suppose that the pair (r, h) was deleted by Algorithm 3. We consider two cases: (i) h is fully subscribed in M: In this case, by the design of

Algorithm 3: Algorithm for strongly stable matching in HR-HT Problem

1 $M = \{(r, \perp) \;\; | \;\; \text{for every resident } r \in \mathcal{R}\ \}$
2 set $full(h) =$ false for each hospital $h \in \mathcal{H}$
3 **while** $\exists$ *unmatched* $r \in \mathcal{R}$ *such that* Pref(r) is not empty **do**
4   r proposes to the top-ranked hospital h in Pref(r)
5     $M = M \cup \{(r, h)\}$
6   **if** $|M(h)| \geq q(h)$ **then**
7         $full(h) =$ true
8   **if** $|M(h)| > q(h)$ **then**
9   let r' be a least-preferred resident in $M(h)$
10   suppose r' is at rank p in Pref(h)
11   **for** *each* r'' *at rank* p *in* Pref(h) **do**
12   **if**$(r'', h) \in M$ **then** $M = M \setminus (r'', h)$
13   **for** *each* r'' *at rank* $\geq p$ *in* Pref(h) **do**
14   delete h from Pref(r'') and r'' from Pref(h)
15 **if** $\exists\ h \in \mathcal{H}$ *such that* $full(h) =$ *true and* $|M(h)| < q(h)$ **then**
16   G does not admit a strongly stable matching
17 **else**
18   **return** M as a strongly stable matching in G

the algorithm, all residents matched to h in M are strictly better-preferred over r – contradicting the assumption that (r, h) blocks M. (ii) h is under-subscribed in M: Since h rejected r, it must be the case that $full(h) =$ true. However, if $full(h) =$ true and h is under-subscribed in M, then Algorithm 3 would have concluded that G does not admit a strongly stable matching – a contradiction. Therefore, the algorithm could not have deleted the pair (r, h).

The fact that (r, h) was not deleted implies that Pref(r) did not become empty, and hence, r is matched in M. Let $M(r) = h'$. Since (r, h) blocks M, and resident preference lists are strict, it must be the case that $h \succ_r h'$. The fact that r proposed to h' implies that h rejected r and hence the pair (r, h) was deleted by the Algorithm 3 – a contradiction. □

For the rest of the proof, we assume that M denote the matching obtained when the proposal sequence terminates, that is, just before reaching Line 15. We prove the following claims which will be useful in proving the correctness of Line 16 of the Algorithm 3.

Claim A.2 *Suppose Algorithm 3 deleted a pair* (r, h) *during the execution. Then the pair* (r, h) *cannot belong to any strongly stable matching.*

Proof. Suppose for contradiction that (r, h) was deleted by Algorithm 3, but $(r, h) \in \widetilde{M}$ for some strongly stable matching $\widetilde{M}$. Without loss of generality, assume that (r, h) was the first strongly stable pair that was deleted during the execution of Algorithm 3. Note that h is not over-subscribed in $\widetilde{M}$. Suppose the pair (r, h) was deleted at time t. This implies that h was over-subscribed with residents, say $M^t(h)$, at time t. This further implies that for each $r' \in M^t(h)$ we have that $r' \succeq_h r$. Since $(r, h) \in \widetilde{M}$, there must exist $r'' \in M^t(h)$ such that $(r'', h) \notin \widetilde{M}$. We claim that there does not exist any strongly stable matching, say M'', such that $(r'', h'') \in M''$ for some $h'' \succ_{r''} h$. If this is the case, then the pair (r'', h'') would have been deleted before (r, h) – a contradiction. This implies that $h \succ_{r''} \widetilde{M}(r'')$. Since $r'' \in M^t(h)$, $r'' \succeq_h r$. The fact that $(r'', h) \notin \widetilde{M}$ implies that (r'', h) blocks the supposed strongly stable matching $\widetilde{M}$ – a contradiction. □

Claim A.3 *If a resident r is unmatched in M, then r cannot be matched in any strongly stable matching.*

Proof. Suppose a resident r is unmatched in M. This implies during the execution of Algorithm 3, $\mathsf{Pref}(r)$ becomes empty. That is, each pair $(r, h) \in E$ is deleted by the Algorithm 3. Applying Claim A.2, we conclude that r remains unmatched in each strongly stable matching. □

Claim A.4 *Suppose G admits a strongly stable matching $\widetilde{M}$, and there exists a hospital h such that $|\widetilde{M}(h)| < q(h)$. If a resident r proposed to h during the course of Algorithm 3, then $r \in \widetilde{M}(h)$.*

Proof. The fact that r proposed to h during the execution of Algorithm 3 implies that each pair (r, h') such that $h' \succ_r h$ are deleted. Thus, by applying Claim A.2, we know that $\widetilde{M}(r) \not\succ_r h$. If $h \succ_r \widetilde{M}(r)$, then (r, h) blocks $\widetilde{M}$ as $|\widetilde{M}(h)| < q(h)$. Therefore, $r \in \widetilde{M}(h)$. □

Lemma A.5 *If there exists a hospital h such that $full(h) =$ true, but $|M(h)| < q(h)$, then G does not admit a strongly stable matching.*

Proof. Suppose for contradiction that G admits a strongly stable matching, say $\widetilde{M}$. The fact (i) $full(h) =$ true, and (ii) $|M(h)| < q(h)$ together imply that there exists a resident r^* such that the proposal by r^* was provisionally accepted by h but was later rejected by h. This further implies that the pair (r^*, h) was deleted by Algorithm 3. Applying Claim A.2, we know that $(r^*, h) \notin \widetilde{M}$. Also, note that $(r^*, h') \notin \widetilde{M}$ for any h' such that $h' \succ_r h$. This is because all such pairs are deleted (as r^* proposed to the hospital h). If we show that $|\widetilde{M}(h)| < q(h)$, then it immediately implies that (r^*, h) blocks $\widetilde{M}$. Next, we show that $|\widetilde{M}(h)| < q(h)$.

Let us assume that the set of residents matched in M is denoted by $\mathcal{R}^M$, and the set of residents matched in $\widetilde{M}$ is denoted by $\mathcal{R}^{\widetilde{M}}$. Applying Claim A.4, we observe that if $|M(h)| = q(h)$, then $|\widetilde{M}(h)| = q(h)$, and if $|M(h)| < q(h)$, then $|\widetilde{M}(h)| \geq |M(h)|$. Therefore, $|\widetilde{M}(h)| \geq |M(h)|$ for each $h \in \mathcal{H}$. Thus, $|\mathcal{R}^{\widetilde{M}}| \geq |\mathcal{R}^M$—. Now, using Claim A.3, we observe that $|\mathcal{R}^{\widetilde{M}}| \leq |\mathcal{R}^M|$. Thus, $|\mathcal{R}^{\widetilde{M}}| = |\mathcal{R}^M|$. Since $|\widetilde{M}(h)| \geq |M(h)|$, we conclude that $|\widetilde{M}(h)| = |M(h)|$ for each $h \in \mathcal{H}$. Thus, $|\widetilde{M}(h)| = |M(h)| < q(h)$.

Therefore, (r^*, h) blocks $\widetilde{M}$. □

Using Lemma A.1 and Lemma A.5 we have the following theorem.

Theorem 2. *For a given* HR-HT *instance, the existence of strongly stable matching is decidable in polynomial time. Moreover, if the instance admits a strongly stable matching, one such matching can be computed in polynomial time.*