Bounding the Chromatic Number via High Dimensional Embedding

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Abstract: We present a necessary and sufficient condition for determining whether a hypergraph can be embedded in a high-dimensional space, and establish an upper bound for the chromatic number of such hypergraph, which can be viewed as the high-dimensional extension of the Heawood conjecture and Wagner's Theorem, respectively. Based on them, for a graph of order n, there exists accordingly an algorithm with a complexity of $O(n \log^2 n)$ that can determine the upper bound for the chromatic number. Specifically, we prove: (i) The graph-closure of a *d*-uniform-topological hypergraph is a closed- R^d -graph if and only if its minors include neither K^d_{d+3} nor $K^d_{3,d+1}$; (ii) Let G be a special triangulated R^d -graph, G^v be the vertex-hypergraph of G, then G^v is $3 \cdot 2^{d-1}$ -choosable.

Keywords: embedding; coloring; chromatic number; hypergraph

1 Introduction

This paper aims to bounding the chromatic number via high dimensional embedding in quasilinear time. To achieve this goal, we generalize the concept of planar graph to a higher-dimensional form. A challenge arises: Unlike the plane, higher-dimensional Euclidean spaces are not intuitive. Most definitions and methods related to planar graphs cannot be directly extend to higher dimensions. To address this challenge, it is necessary to introduce concepts and theorems related to simplex, CW complexes, homotopy, and fundamental groups. Please refer to [1, 2, 9] for common used definitions and notations that are not specified in this work.

It is NP-complete to decide if a given graph admits a k-coloring except for the cases $k \in \{0, 1, 2\}$. In particular, computing the chromatic number (denoted by χ) is NP-hard [7]. There are many algorithms for the problem of calculating the chromatic number, such as greedy algorithms [4, 19], heuristic algorithms [11], parallel and distributed algorithms [16, 17], and graph embeddings [3, 8, 12, 13]. Previous researchers developed the theory of graph embeddings on two-dimensional surfaces and established upper bounds for the chromatic number of graphs on such surfaces, and we follow in this research line by further investigating the embedding problem in high-dimensional spaces.

1.1 Our results

We consider a *d*-uniform hypergraph as a CW complex. If the fundamental group of a *d*-uniform hypergraph is trivial, we refer to it as a special R^d -graph. All related definitions are given in Section 2.

Wagner's Theorem states that a finite graph is planar if and only if its minors include neither K_5 nor $K_{3,3}$, and we derive its higher-dimensional form.

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Theorem 1.1. The graph-closure of a d-uniform-topological hypergraph is a closed- R^d -graph if and only if its minors include neither K_{d+3}^d nor $K_{3,d+1}^d$.

The Heawood conjecture (or Ringel-Youngs Theorem) states that $\chi(G) \leq \frac{1}{2}(7+\sqrt{49-24c'})$ if G is a graph which can be embedded in a surface Σ of Euler characteristic c', and the following theorem presents a kind of its higher-dimensional extension.

Theorem 1.2. Let G be a special triangulated R^d -graph, G^v be the vertex-hypergraph of G, then G^v is $3 \cdot 2^{d-1}$ -choosable.

Theorem 1.1 provides conditions for determining whether a hypergraph can be embedded in *d*-dimensional Euclidean space (denoted by R^d), and Theorem 1.2 gives an upper bound for the chromatic number of R^d . Since a hypergraph can be regarded as a CW complex and a graph can be regarded as the 1-skeleton of a CW complex, we can use the aforementioned two theorems to estimate the upper bound for the chromatic number of a graph.

Let G be a graph and G_d be a d-uniform-topological hypergraph, if G is the 1-skeleton of the graph-closure of G_d , then we can conclude from Theorem 1.1 that G_d can be embedded in \mathbb{R}^d . Furthermore, we obtain an upper bound for the chromatic number of G_d by Theorem 1.2. Since the chromatic number of G is less than that of G_d , we can obtain the following corollary. It should be noted that the method of transforming a graph into a CW complex is not unique, and there should exist an optimal embedding method that yields the better estimate, but we have not yet solved this problem, which remains one of our future work.

Corollary 1. Let G be a graph, if its minors include neither K_{d+3} nor $K_{3,d+1}$, then G is $3 \cdot 2^{d-1}$ -choosable.

Corollary 1 transforms the problem of determining the upper bound for the chromatic number into the problem of minor testing. Analogous to the embedding problem for two-dimensional closed surfaces [14, 12, 13], if there exists an algorithm that can determine whether a given minor exists in a graph, we can use binary search to obtain an upper bound for the chromatic number. Given a graph of order n, Robertson and Seymour [15] proved that for every graph M there exists a algorithm with a complexity of $O(n^3)$ that decides whether G contains M as a minor. Later, Kawarabayashi and Reed [10] improved it to $O(n \log n)$.

Since the complexity of binary search is $O(\log n)$ after noting that the chromatic number of any graph does not exceed its order n and the complexity of minor testing is $O(n \log n)$ [10], by multiplying the two complexities, we can naturally obtain an algorithm with a complexity of $O(n \log^2 n)$ based on the binary search to compute the upper bound for the chromatic number of any graph.

Remark. Several remarks related to above results follow.

• We observe that Corollary 1 establish a connection between graph minors and chromatic numbers, which is very similar to Hedwiger's Conjecture [2]. In [18], Thomason proved that if a graph G does not contain a K_t -minor, then the chromatic number of G is at most $2.68t\sqrt{\log_2 t} \cdot (1 + O(1))$. Substituting t = 7, we obtain the following conclusion: If a graph G does not contain a K_7 -minor, then the chromatic number of G is at most 26. Now, substituting d = 4 into Corollary 1, we know that if G contains neither a K_7 -minor nor a $K_{3,5}$ -minor, then the chromatic number of G is at most 24. Clearly, even without optimize the bound of Corollary 1, our conclusion is partially superior to previous results in low dimensions. Furthermore, if we can introduce the *Discharging* method into R^d -graph to improve the conclusion of Corollary 1, it may help us find another approach to solving Hedwiger's conjecture. Note that the bound in Theorem 1.2 is estimated only by the relationship between the number of vertices and edges; therefore, there remains a gap between Theorem 1.2 and the sharp upper bound for the chromatic number. By introducing mathematical tools from the field of planar graphs, such as *Discharging*, we can further refine the bound of Theorem 1.2. However, due to space limitations, we will not discuss this issue in detail here, and relevant conclusions will be presented in our future work. (We conjecture that if G is a special triangulated R^d -graph, then there exists a polynomial f(d) such that G^v is f(d)-choosable.)

- Taking planar graph G as an example, from Wagner's Theorem, we know that if its minors include neither K_5 nor $K_{3,3}$, it can be embedded in the plane, thereby implying that the upper bound for the chromatic number of G is 4. For any arbitrary two-dimensional closed surface, there is a similar conclusion. In 1979, Filotti et al. [6] presented a polynomial-time algorithm with a complexity of $O(n^{\alpha k+\beta})$, capable of determining whether a graph G can be embedded in an orientable surface. This algorithm was subsequently optimized, with the most efficient version provided by Mohar [12, 13] in 1996, achieving a complexity of O(n).
- Since there exists an algorithm with a complexity of O(n) that can determine whether a graph can be embedded in any given two-dimensional closed surface, and since the chromatic number of a graph does not exceed the number of its vertices n, we can naturally obtain an algorithm with a complexity of $O(n \log n)$ based on binary search to compute an upper bound for the chromatic number of any graph. Unfortunately, for many classes of graphs, the upper bound for the chromatic number obtained by the aforementioned algorithm is not good enough, even for special graph classes such as bipartite graphs. For example, the bipartite graph $K_{3,(5-c')}$ cannot be embedded in a two-dimensional closed surface with Euler characteristic c', which suggests that when using this algorithm to estimate the upper bound for the chromatic number for bipartite graphs, the result can be arbitrarily large. However, it is easy to construct a closed- R^3 -graph G such that $K_{3,(5-c')}$ is the 1-skeleton of it. Therefore, the upper bound for the chromatic number of G is also an upper bound for the chromatic number of $K_{3,(5-c')}$. According to Theorem 1.2, this bound is 12, which reduces the gap between the upper bound of the chromatic number and the actual chromatic number of $K_{3,(5-c')}$ from infinite to finite.
- Over the past few decades, many researchers have attempted to study the embedding of graphs in higherdimensional spaces. In 1973, Bothe defined the *linkless embedding*: A *linkless embedding* of an undirected graph is an embedding of the graph into three-dimensional Euclidean space in such a way that no two cycles of the graph are linked (the *knotless embedding* can also be defined in a similar manner). In 1995, Cohen et al. [5] proved that any finite graph can be embedded into three-dimensional Euclidean space. We focus on hypergraphs and use polytopes in higher-dimensional spaces to represent the hyperedges of hypergraphs, then extend the problem of graph embedding to higher dimensions.

1.2 Overview of the paper

Section 2 mainly talks about the definitions of hypergraphs that can be embedded in \mathbb{R}^d . In Section 3, we extend the concept of triangulation to higher-dimensional spaces. In Section 4, we prove Theorem 1.2. In Sections 5 and 6, we prepare the groundwork for proving Theorem 1.1, and the proof is laid out in Section 7.

2 R^d -graph: Definition and property

By considering (d-1)-dimensional simplices as hyperedges of a *d*-uniform hypergraph, the concept of planar graphs can be extended to higher-dimensional spaces. We note that the line segments (1-dimensional simplices) in a planar graph are allowed to undergo topological deformation. Therefore, we allow hyperedges embedded in higher-dimensional space to undergo topological deformation as well. To avoid confusion, we refer to these hyperedges as simplexoid, defined as follows.

Definition 1 (Simplexoid). If A is a simplex of dimension k with vertex set $V(A) = \{v_0, v_1, ..., v_k\}$, B is a face of A, A_0 is homeomorphic to A, the images of V(A) under homeomorphism is $V(A_0) = \{u_0, u_1, ..., u_k\}$, then we say A_0 is a simplexoid of dimension k, and $V(A_0)$ is the vertex set of A_0 . Furthermore, if the image of B under homeomorphism is B_0 , then we say B_0 is a (i-1)-sub-simplexoid of A_0 (or (i-1)-face) if B_0 contains exactly i vertices.

Definition 2 (*d*-Uniform-Topological hypergraph). Let H = (X, E) be a *d*-uniform hypergraph. If each hyperedge of H is regarded as a simplexoid of dimension (d-1), we refer to such a hypergraph as a *d*-uniform-topological hypergraph.

Definition 3 (General \mathbb{R}^d -graph). Let $T_1, T_2, ..., T_m$ be simplexoids of dimension (d-1) which are embedded in \mathbb{R}^d . We say $\bigcup_{i=1}^m T_i$ is a general \mathbb{R}^d -graph if the following conditions hold: If T_i and T_j are simplexoids in $\{T_1, T_2, ..., T_m\}$ and $T' = T_i \cap T_j \neq \emptyset$, then T' is a sub-simplexoid of both T_i and T_j . (Each simplexoid T_i can be regarded as a hyperedge.)

From Definitions 2 and 3, we know that if a *d*-uniform-topological hypergraph can be embedded in \mathbb{R}^d , then it is a general \mathbb{R}^d -graph.

Definition 4 (Special R^d -graph). If the fundamental group of a general R^d -graph is trivial, then this graph is called a special R^d -graph.

Note that in the case where the fundamental group is trivial, there will be no knot or other complex structures in the manifold. According to Definition 3 and 4, it is obvious that the *special* R^d -graph is a special kind of the general R^d -graph. Since the fundamental group of a *special* R^d -graph is trivial, it can be regarded, up to isomorphism, as a union of internally disjoint (d-1)-dimensional spheres and (d-1)-dimensional balls. (An *i-sphere* is a topological space that is homeomorphic to a standard *i*-sphere. The space enclosed by a *i*-sphere is called an (i + 1)-ball.)

Lemma 2.1 (The Construction of Special \mathbb{R}^d -graph). Every special \mathbb{R}^d -graph can be constructed by the following procedure.

Procedure X: $T_1, T_2, ..., T_m$ are simplexoids of dimension (d-1) in \mathbb{R}^d . Starting from T_0 , add $T_1, T_2, ..., T_m$ in \mathbb{R}^d one by one, and this procedure satisfies the following conditions:

- 1. Let T_i and T_j be arbitrary simplexoids in $\{T_1, T_2, ..., T_m\}$ and $T' = T_i \cap T_j \neq \emptyset$, then T' is a sub-simplexoid of both T_i and T_j .
- 2. Suppose our procedure has reached the *i*-th step $(T_i \text{ has been added in } R^d)$. Let $K_i = \bigcup_{i=0}^{i} T_j$, $V(T_{i+1}) =$

 $\{v_0, v_1, ..., v_{d-1}\}, B_{v_i}$ be the sub-simplexoid with vertex set $V(T_{i+1}) \setminus \{v_i\}, \mathscr{B} = \bigcup_{i=0}^{d-1} \{B_{v_i}\}$ be the sub-simplexoid set of T_{i+1} of dimension (d-2). T_{i+1} satisfies the following condition when adding T_{i+1} to R^d : If $\mathscr{A} = T_{i+1} \cap K_i$, then $\exists \mathscr{B}_0 \subseteq \mathscr{B}$ such that $\mathscr{A} = \bigcup_{B_{v_i} \in \mathscr{B}_0} B_{v_i}$ and $\mathscr{B}_0 \neq \emptyset$.

Proof. Let K be a special \mathbb{R}^d -graph which is constructed by **Procedure X**. We prove that the fundamental group of K is trivial. Since K is obtained by adding simplexoid $T_0, T_1, T_2, ..., T_m$ be one by one, this procedure does not result in any change of the fundamental group, thus the fundamental group of K is the same as that of T_0 .

Next, let K be a special \mathbb{R}^d -graph which is constructed by Definition 4, then it is evident that there exists a simplexoid T_i in the special \mathbb{R}^d -graph K such that the fundamental group remains trivial after removing T_i from K. By repeatedly removing simplexoids from K, we eventually obtain an empty graph. Note that the fundamental group remains trivial in this process and the process is reversible, which implies that K can be constructed by **Procedure X**.

Definition 5 (Non-special \mathbb{R}^d Graph). If the fundamental group of a d-uniform-topological hypergraph $K = \bigcup_{i=1}^m T_i$ is trivial and K cannot embedded in \mathbb{R}^d , then K is called a non-special \mathbb{R}^d -graph (or non- \mathbb{R}^d -graph for short).

Note that the generalized Poincaré conjecture ensures that the R^d -graph can always be embedded on the d-sphere.

In high-dimensional spaces, some common definitions can be generalized as follows.

Definition 6 (Multiple Simplexoids). Let T_1 and T_2 be *i*-dimensional simplexoids of an \mathbb{R}^d -graph G. If $V(T_1) = V(T_2)$, and there exists an open set $W \subseteq \mathbb{R}^d \setminus G$ such that the boundary of W contains both T_1 and T_2 , then T_1 and T_2 are called *i*-dimensional multiple simplexoids.

It should be noted that the above definition requires two simplexoids to lie on the boundary of the same open set; otherwise, we do not consider they are multiple simplexoids. **Definition 7** (R^d -Loops). Let T_1 be an *i*-dimensional simplexoid of an R^d -graph. $V(T_1) = \{u_0, u_1, ..., u_k\}$. If there exists $u_i, u_j \in V(T_1)$ ($i \neq j$) such that u_i and u_j overlap, then T_1 is called an R^d -loop.

Since higher-dimensional simplexoids have more than two vertices, the definition of the loops in higher-dimensional manifolds differs slightly from that in planar graphs. As long as two vertices of a simplexoids overlap, we consider it as an R^d -loop.

Definition 8 (Simple R^d -Graph). If R^d -graph G does not contain multiple simplexoid and R^d -loop, then G is called a simple R^d -graph.

Unless otherwise specified, all R^d -graphs mentioned hereafter will be simple R^d -graphs. Analogous to the definition of incident and adjacent in graph theory, we can define the notions of incident and adjacent in R^d -graphs.

Definition 9 (Neighbour). Let G be an \mathbb{R}^d -graph, and an i-dimensional simplexoid of G is denoted by a_i , and the set of *i*-dimensional simplexoid of G is denoted as $A_i(G) = \{a_i | a_i \text{ is a simplexoid of } G \text{ of dimensional } i\}$. For convenience, we use V(G) to denote the vertex set of G, use $V(a_i)$ to denote the vertex set of a_i . Let $u, v \in V(G)$, we say u is adjacent to v if $\exists a_i \in A_i(G)$ such that $u, v \in V(a_i)$. The set of all vertices that adjacent to point u is denoted by $N_G(u)$, the degree of u is denoted by $d_G(u) = |N_G(u)|$.

Definition 10 (Incident and Adjacent). Let a_i be an *i*-dimensional simplexoid, and a_j be a *j*-dimensional simplexoid $(i \leq j)$. We say a_i is incident to a_j (or a_j is incident to a_i) if $a_i \cap a_j = a_i, i \neq j$; we say a_i is adjacent to a_j if i = j and $a_i \cap a_j$ is an (i-1)-dimensional simplexoid. The set of all *j*-dimensional simplexoid incident (adjacent) to a_i is denoted by $N_{Gj}(a_i)$. We say $d_{Gj}(a_i) = |N_{Gj}(a_i)|$ is the *j*-dimensional degree of a_i .

Definition 11 (Merging of Multiple Simplexoids). Given two multiple simplexoids x and y, merging of x and y refers to combining x and y into a new simplexoid z, and all simplexoids incident with x or y are incident with z.

Definition 12 (Simplexoid Deletion). Given an \mathbb{R}^d -graph G, there are two natural ways of deriving smaller graphs from G. If e is a (d-1)-dimensional simplexoid of G, we may obtain a graph with m-1 (d-1)-dimensional simplexoids by deleting e from G but leaving the vertices and the remaining simplexoids intact. The resulting graph is denoted by $G \setminus e$. Similarly, if v is a vertex or an i-dimensional simplexoid (i < d-1) of G, we may obtain a graph by deleting from G the vertex (or simplexoid) v together with all the (d-1)-dimensional simplexoids incident with v. The resulting graph is denoted by G - v or $G \setminus v$.

Definition 13 (Simplexoid Contraction). To contract a simplexoid e of an \mathbb{R}^d -graph G is to delete the simplexoid and then identify its incident vertices. The resulting graph is denoted by G/e. It is important to note that, during the process of simplexoid contraction, if multiple simplexoids emerge, we need to merge them to ensure that the resulting graph is a simple graph.

Definition 14 (R^d -Embedding). Let G be a d-uniform-topological hypergraph, an R^d -embedding G' of G can be regarded as a graph isomorphic to G and is embedded in R^d .

Finally, let us provide a brief summary. The R^d -graph graph can be considered as an extension of the definition of the planar graph into higher-dimensional space or as a special type of CW complex. Whether it is a general R^d -graph, a special R^d -graph, or a non- R^d -graph, they are essentially special cases of CW complex.

A general R^d -graph requires that this CW complex can be embedded in R^d . A special R^d -graph requires that this CW complex can be embedded in R^d and has a trivial fundamental group. A non- R^d -graph requires that this CW complex cannot be embedded in R^d and has a trivial fundamental group. A thorough understanding of these definitions lays a solid foundation for subsequent proofs.

3 Triangulation of special R^d -graph $(d \ge 3)$

A simple connected plane graph in which all faces have degree three is called a plane triangulation or, for short, a triangulation. This section extends the concept of triangulation to higher-dimensional spaces. We will divide the triangulation process into two steps: Graph-closure and triangulation. In the following part, the term R^d -graph refers to special R^d -graph unless otherwise specified.

3.1 Step I: Graph-closure

First, it is necessary to have a clear understanding of the structure of R^d -graphs. From Definition 4 and Lemma 3.1, an R^d -graph is homeomorphic to the union of a finite collection of (d-1)-dimensional balls and (d-1)-spheres with trivial fundamental group.

Lemma 3.1. [1] A path connected space whose fundamental group is trivial is simply connected.

The simplexoids in R^d -graphs can be divided into two categories which are similar to the hanging edge and nonhanging edge in planar graphs. Next, we will extend the concept of hanging edges to higher-dimensional spaces.

Definition 15 (Hanging Simplexoid). Let T be a (d-1)-dimensional simplexoid of an \mathbb{R}^d -graph K, $N_{K(d-2)}(T)$ represent the set of all (d-2)-dimensional simplexoids that is incident to T. If $\forall J \in N_{K(d-2)}(T)$, there exists a (d-1)-dimensional simplexoid $T' \subseteq K$ such that $J = T \cap T'$, then T is called non-hanging simplexoid, if not T is called hanging simplexoid.

Since hanging simplexoids can cause some difficulties in our subsequent research, we need to find a way to eliminate hanging simplexoids in R^d -graphs.

Before defining the graph-closure, we first need to introduce the definition of hyper-polytopes. Similar to planar graphs, the following lemma implies that an R^d -graph K can be regarded as the boundary of the maximal connected open sets in set $R^d \setminus K$.

Lemma 3.2. Let W_0 be a maximal connected open set of $\mathbb{R}^d \setminus K$, $W = \overline{W_0}$ be the closure of W_0 , $\partial W = W \setminus W_0$ be the boundary of W, then $\partial W \subseteq K$.

Proof. Suppose to the contrary that $\partial W \nsubseteq K$. Let $x \in \partial W \setminus K$, d(x, K) = r be the minimum distance from x to K.

Let $B(x, \frac{r}{2})$ be a *d*-dimensional open ball with center x and radius $\frac{r}{2}$, then $B(x, \frac{r}{2}) \cap K = \emptyset$.

Let $W' = W_0 \cup B(x, \frac{r}{2})$, then W' is a connected open set with $W' \cap K = \emptyset$ and $W_0 \subsetneq W'$, contradict to the fact that W_0 is a maximal connected open set of $\mathbb{R}^d \setminus K$.

If a planar graph does not contain hanging edges, then each face of the planar graph can be regarded as a polygon. Similarly, if an R^d -graph K does not contain hanging simplexoids, then each maximal connected open set in the R^d -graph can be regarded as a high-dimensional polytope. Therefore, eliminating all hanging edges in R^d -graphs can greatly facilitate subsequent research.

Let $W = \overline{W_0}$ be the closure of a maximal connected open set of $\mathbb{R}^d \setminus K$, then we say W is a *polytope* of K. If there is no *hanging simplexoids* in K and W is a *polytope* of an \mathbb{R}^d -graph K, then the boundary of W is a (d-1)-sphere in K by Definition 4 and Lemma 3.2.

The (d-1)-dimensional degree of W is denoted by $d_{G(d-1)}(W)$ by Definition 10. We say W is a unit polytope of dimension d if $d_{G(d-1)}(W) = d+1$. Note that the definition of unit polytope of dimension d is equivalent to d-dimensional simplexoid. We will prove that each polytope of R^d -graph K can be triangulated into a finite collection of unit polytope of dimension d in Section 3.2.



Figure 1: graph-closure

Definition 16 (Graph-Closure). Let K be an \mathbb{R}^d -graph and \mathscr{K} be a set of simplexoids such that

 $\mathscr{K} = \{K_i: V(K_i) = V(K), A_{d-1}(K) \subseteq A_{d-1}(K_i), \text{ and there is no hanging simplexoid in } K_i\}.$

The minimal R^d -graph K' in set \mathscr{K} is called the graph-closure of K, denoted as $K' = \overline{K}$. A minimal R^d -graph means that for any (d-1)-dimensional simplexoid s in K', $K' \setminus s \notin \mathscr{K}$.

Definition 17 (*Closed-R^d-Graph*). Let \overline{K} be the graph-closure of a special R^d -graph K, then K is called the closed- R^d -graph if $K = \overline{K}$.

Next, we prove that for any R^d -graph K, simplexoids can be added sequentially to obtain the graph-closure of K.

Theorem 3.1. Let K be an \mathbb{R}^d -graph, we add simplexoids $T_0, T_1, T_2, T_3, ...$ into K one by one according to the following procedure. We prove that this procedure will inevitably terminate in a finite number of steps, and the resulting graph will be a graph-closure of K.

Procedure: Let $K = K_0$, we assume that K has became into K_i when adding the *i*-th simplexoid T_i .

- (1). Suppose that our procedure has reached the *i*-th step, now we need to add T_{i+1} into K_i .
 - (1.1). If K_i has no hanging-simplexoid, then we terminate the procedure, it is easy to verify that K_i is a graph-closure of K.
 - (1.2). Otherwise, execute (2).
- (2). Let W_i be a polytope of a \mathbb{R}^d -graph K_i .
 - (2.1). If there exists two (d-1)-dimensional hanging-simplexoids $T \subseteq \partial W_i$ and $T' \subseteq \partial W_i$ such that each of them contains a (d-2)-sub-simplexoid (denoted by T_h and T'_h) which is incident with no (d-1)dimensional simplexoid in K_i and $|V(T_h) \cap V(T'_h)| = d-2$. Assume that $V(T_h) = \{v_1, v_2, ..., v_{d-2}, v^*\};$ $V(T'_h) = \{v_1, v_2, ..., v_{d-2}, v'\}$. Let T_{i+1} be a (d-1)-dimensional simplexoid with $V(T_{i+1}) = V(T_h) \cup V(T'_h) =$ $\{v_1, v_2, ..., v_{d-2}, v^*, v'\}$, we add T_{i+1} into K_i and get K_{i+1} , return to (1).

(We provide a figure of the graph-closure of an \mathbb{R}^3 -graph to explain Srep(2.1)). As shown in Figure 1, K_i is an \mathbb{R}^3 -graph, both T and T' are 2-dimensional simplexoids, $V(T_{i+1}) = \{v_1, v^*, v'\}$. Note that after adding the simplexoids T_{i+1} , the number of hanging-simplexoids in K_i decreases by one.)

- (2.2). Otherwise, choose a (d-1)-dimensional hanging-simplexoid $T \subseteq \partial W_i$ and a (d-1)-dimensional simplexoid $T' \subseteq \partial W_i$ such that $|V(T) \cap V(T')| \ge d-2$. Let T_h be the (d-2)-sub-simplexoid of T that



Figure 2: hanging simplexoids

is incident with no (d-1)-dimensional simplexoid in K_i ; T'_h be the (d-2)-sub-simplexoid of T' such that $|V(T_h) \cup V(T'_h)| = d-2$. Assume that $V(T_h) = \{v_1, v_2, ..., v_{d-2}, v^*\}$; $V(T'_h) = \{v_1, v_2, ..., v_{d-2}, v'\}$. let T_{i+1} be a (d-1)-dimensional simplexoid with $V(T_{i+1}) = V(T_h) \cup V(T'_h) = \{v_1, v_2, ..., v_{d-2}, v^*, v'\}$, we add T_{i+1} into K_i and get K_{i+1} , return to (1).

(We provide a figure of the graph-closure of an \mathbb{R}^3 -graph to explain Srep(2.2)). As shown in Figure 2, K_i is an \mathbb{R}^3 -graph and e is a hanging-simplexoid. Note that after adding the simplexoids T_{i+1} and T_{i+2} , there is no hanging-simplexoid in K_i .)

Proof. First, we prove that the above process will certainly terminate in a finite number of steps. It is evident that through step (2), the number of hanging-simplexoids in K will decrease. Since K is a finite R^d -graph with no multiple simplexoids, the process will inevitably terminate in a finite number of steps.

Since the number of hanging-simplexoids in K decreases with each iteration of step (2), let K_{x-1} denote the R^d -graph after (x-1) steps, which still contains hanging-simplexoids; let K_x be the R^d -graph which no longer contains any hanging-simplexoids after x steps. By Definition 16, it is evident that K_x is the closure of K.

Corollary 2. Let K be an \mathbb{R}^d -graph and \overline{K} be the graph-colsure of K, then \overline{K} is homeomorphic to the union of a finite collection of (d-1)-spheres.

3.2 Step II: Triangulation

In this section, we will further triangulate R^d -graphs based on their closures, ensuring that each *polytope* in the R^d -graph is a *unit polytope*.

Definition 18. An R^d -graph in which all polytopes are unit polytopes of dimension d is called a triangulated R^d -graph.

Definition 19. Let K and K^T be R^d -graphs. If $V(K) = V(K^T)$, $A_{d-1}(K) \subseteq A_{d-1}(K^T)$, and K^T is a triangulated R^d -graph, then K^T is a called a triangulated R^d -graph of K.

Definition 20. An R^d -graph G is called a special triangulated R^d -graph if G is a special R^d -graph and all polytopes of G are unit polytopes of dimension d.

Next, we prove that for any R^d -graph, simplexoids can be added sequentially to obtain a triangulated R^d -graph.

Theorem 3.2. Let K be a closed- \mathbb{R}^d -graph, we add simplexoids T_1, T_2, T_3, \dots into \overline{K} one by one according to the following procedure. We prove that this procedure will inevitably terminate in a finite number of steps, and the resulting graph will be a triangulated \mathbb{R}^d -graph of K.

Procedure: Let $K = K_0$; we assume that K has become into K_i when adding the *i*-th simplexoid T_i .

• (1). Suppose that our procedure has reached the *i*-th step, now we need to add T_{i+1} into K_i .

- (1.1). If all polytopes of K_i are unit polytopes of dimension d, it is easy to verify that K_i is a graph-closure of K.
- (1.2). Otherwise, execute (2).
- (2). Let W_i be a polytope of a special \mathbb{R}^d -graph K_i . Choose two (d-2)-dimensional simplexoids $T_h \subseteq \partial W_i$ and $T'_h \subseteq \partial W_i$ such that $|V(T_h) \cap V(T'_h)| = d-2$. Assume that $V(T_h) = \{v_1, v_2, ..., v_{d-2}, v^*\}$; $V(T'_h) = \{v_1, v_2, ..., v_{d-2}, v'\}$.
 - (2.1). If $V(T'') \neq V(T_h) \cup V(T'_h)$ for any (d-1)-dimensional simplexoid $T'' \in A_{d-1}(K_i)$, then let T_{i+1} be a (d-1)-dimensional simplexoid with $V(T_{i+1}) = V(T_h) \cup V(T'_h)$. Now we add T_{i+1} into K_i and get K_{i+1} , return to (1). (Figure 3 is the procedure of (2.1) when K is a special R^2 -graph (planar graph).)
 - (2.2). If there exists a (d-1)-dimensional simplexoid $T'' \in A_{d-1}(K_i)$ such that $V(T'') = V(T_h) \cup V(T'_h) = \{v_1, v_2, ..., v_{d-2}, v^*, v'\}$, then we return to (2), reselect two different (d-2)-dimensional simplexoids $T_h \subseteq \partial W_i$ and $T'_h \subseteq \partial W_i$, and then re-execute (2.1) and (2.2).

Proof. Since K contains no multiple simplexoid, the above process always terminates in a finite number of steps. Clearly, the necessary and sufficient condition for the above process to terminate at step i is that all *polytopes* of K_i are *unit polytopes* of dimension d, which implies that K_i is called a *triangulated* R^d -graph of K.





Figure 3: How to add T_{i+1} into W_i in \mathbb{R}^d -graph

Figure 4: An example of A when d = 3

In practical applications, it is necessary not only to perform a triangulation of the R^d -graph but also to determine the number of *unit polytopes* in the triangulated R^d graph. Therefore, it is required to prove the following theorem.

Theorem 3.3. Let W be a polytope of a special \mathbb{R}^d -graph K. Each simplexoid in W is the non-hang simplexoid. We use $d_{\partial W(d-1)}(v)$ to denote the number of (d-1)-dimensional simplexoids which are incident to v in ∂W . We use $d_{G(d-1)}(W)$ to denote the number of (d-1)-dimensional simplexoids that are in ∂W . Then W can be triangulated into $d_{G(d-1)}(W) - k$ d-dimensional simplexoids if there exists $v \in \partial W$ with $d_{\partial W(d-1)}(v) = k$. Such triangulation is called the v-triangulation.

Proof. Let B_d be a *d*-dimensional closed ball, $\partial B_d = S_{d-1}$, then *W* is homeomorphic to the B_d with homeomorphism $\phi: W \to B_d$. Furthermore, ϕ takes the interior of *W* to the interior of B_d , and the ∂W to the boundary of S_d by Lemma 3.3.

Let $T \subseteq \partial W$ be a (d-1)-dimensional simplexoid which satisfy $v \notin V(T)$, then $\phi(T) \subseteq S_d$ is also a (d-1)-dimensional simplexoid. The number of such T is $d_{G(d-1)}(W) - k$.

 $\forall x_i \in \phi(T)$, join $\phi(v)$ and x_i . All such line segments form a *d*-dimensional manifold *A*, and it is obvious that $\phi^{-1}(A)$ is a *d*-dimensional simplexoid in *W* (Figure 4 is an example of *A* when d = 3). For every such *T* that satisfies the above conditions, we repeat this process and get a triangulation of B_d , such triangulation is also a triangulation of *W* since *W* is homeomorphic to the B_d .

Note that the number of such d-dimensional simplexoids in W is equal to the number of (d-1)-dimensional simplexoids (denoted by T) which satisfy $T \subseteq \partial W$ and $v \notin T$, thus we can triangulate W into $d_{G(d-1)}(W) - k$ d-dimensional simplexoids.

Lemma 3.3. [9] Let $h : S_1 \to S_2$ be a homeomorphism between two manifolds, then h takes the interior of S_1 to the interior of S_2 , and the boundary of S_1 to the boundary of S_2 .

4 Coloring of special *R*^{*d*}-graph

In order to estimate the chromatic number of special R^d -graph, we need a new definition that connects the special R^d -graph with coloring. In fact, the *vertex-hypergraph* can be considered as the skeleton of the special R^d -graph.

4.1 Relationship between the number of vertices and edges

Definition 21 (Vertex-Hypergraph). Let G be an \mathbb{R}^d -graph. If we regard each T as a hyperedge with vertex set V(T), then we get a d-uniform hypergraph G^v which is called the vertex-hypergraph of G. A proper coloring of the hypergraph G^v is a coloring such that $c(u) \neq c(v)$ for any adjacent vertices u and v (id est, there exists a hyperedge T such that $\{u, v\} \subseteq V(T)$), the chromatic number $\chi(G^v)$ of G^v is the smallest integer k such that G^v is k-colorable. Let L be a list assignment of G^v with L(v) = k for any $v \in V(G^v)$, an L-coloring c of G^v is a coloring such that $c(v) \in L(v)$ for any $v \in V(G^v)$. G^v is k-choosable if G^v has an L-coloring for any list L with |L(v)| = k. The choice number of G^v , denoted by $\chi^l(G^v)$, is the least integer k such that G^v is k-choosable.

We observe that performing graph-closure or triangulation on an R^d -graph K does not decrease the chromatic number of the vertex-hypergraph.

Theorem 4.1. Let G be a special triangulated \mathbb{R}^d -graph, E(G) be the set of 1-dimensional simplexoids of G, then $|E(G)| \leq (3 \cdot 2^{d-2})|V(G)| - 3 \cdot 2^{d-2} \cdot (d+1) + \frac{d(d+1)}{2}$ if $|V(G)| \geq d+1$.

Proof. By induction on d (denoted by **Induction** α). The theorem holds trivially for d = 2. Assume that it holds for all integers less tha d, and let G be a triangulated special R^{d-1} -graph with

$$|E(G)| \le (3 \cdot 2^{d-3})|V(G)| - 3 \cdot 2^{d-3} \cdot d + \frac{d(d-1)}{2},$$

then

$$\frac{2|E(G)|}{|V(G)|} \leq 3 \cdot 2^{d-2} - \frac{3 \cdot 2^{d-3} \cdot d - \frac{d(d-1)}{2}}{|V(G)|} < 3 \cdot 2^{d-2},$$

which implies that there exists a vertex $v \in V(G)$ such that $d_G(v) \leq 3 \cdot 2^{d-2} - 1$.

Now we prove that the theorem holds for d.

Claim 4.1. Theorem 4.1 holds for d.

Proof. By induction on |V(G)| (denoted by **Induction** β). The theorem holds trivially for |V(G)| = d + 1. Let m be an integer with m > d + 1. Assume that the theorem holds for |V(G)| < m.

Now we prove the theorem holds for |V(G)| = m.

 $\forall u \in V(G)$, let $d_{G0}(u)$ be the number of vertices which are adjacent to $u, G_1 = G - u$ be a special \mathbb{R}^d -graph with $V(G_1) = V(G) \setminus \{u\}$ and $A_{d-1}(G_1) = \{T \mid T \in A_{d-1}(G), u \notin V(T)\}$. It is obvious that there is only one polytope W which is not the unit polytope of G_1 , and ∂W is a special \mathbb{R}^{d-1} -graph (denoted by G_0).

Observe that G_0 is homeomorphic to S_{d-1} , and $|V(G_1)| \ge d+1$ since |V(G)| = m > d+1, then there exists a vertex $v \in V(G_0)$ such that $d_{G_0}(v) \le 3 \cdot 2^{d-2} - 1$ by the induction hypothesis (**Induction** α).

We triangulate G_1 into a triangulated R^d -graph G'_1 , and such a triangulation is a v-triangulation in Theorem 4.2. Observe that G'_1 is also a special R^d -graph, then $|E(G'_1)| \leq 3 \cdot 2^{d-2} |V(G'_1)| - 3 \cdot 2^{d-2} \cdot (d+1) + \frac{d(d+1)}{2}$ by the induction hypothesis (**Induction** β). Note that the triangulation is a v-triangulation, then

$$|E(G)| - d_{G_0}(u) = |E(G_1)| = |E(G_1')| - (d_G(u) - d_{G_0}(v) - 1)$$
$$|E(G_1')| = |E(G)| - d_{G_0}(v) - 1.$$

On the other hand, $|E(G'_1)| = |E(G)| - d_{G_0}(v) - 1 \ge |E(G)| - 3 \cdot 2^{d-2}$ since $d_{G_0}(v) \le 3 \cdot 2^{d-2} - 1$, and we also know that $|V(G_1)| = |V(G)| - 1$. Therefore,

$$|E(G)| - 3 \cdot 2^{d-2} \le |E(G'_1)| \le 3 \cdot 2^{d-2} (|V(G)| - 1) - 3 \cdot 2^{d-2} \cdot (d+1) + \frac{d(d+1)}{2},$$
$$|E(G)| \le 3 \cdot 2^{d-2} |V(G)| - 3 \cdot 2^{d-2} \cdot (d+1) + \frac{d(d+1)}{2},$$

and the claim follows by induction.

We infer that Theorem 4.1 holds for d, and the theorem follows by induction.

Corollary 3. Let G be a special triangulated R^d -graph, G^v be the vertex-hypergraph of G, then there exists a $(3 \cdot 2^{d-1} - 1)^-$ -vertex in G^v . (An i^- -vertex v represent that d(v) < i.)

Proof. By Lemma 4.1,

$$\frac{2|E(G)|}{|V(G)|} \le 3 \cdot 2^{d-1} - \frac{3 \cdot 2^{d-2} \cdot (d+1) - \frac{d(d+1)}{2}}{|V(G)|} < 3 \cdot 2^{d-1},$$

which implies that there exists a $(3 \cdot 2^{d-1} - 1)^{-1}$ -vertex in G^{v} .

4.2 Proof of Theorem 1.2

Proof. Suppose to the contrary that there exists a minimum counterexample G of Theorem 1.2 with fewest vertices. Note that there exists a $(3 \cdot 2^{d-1} - 1)^-$ -vertex v in G^v by Corollary 3.

By the minimality of G^v , $G' = G^v - v$ is $3 \cdot 2^{d-1}$ -choosable. Let $N_G(v) = \{v_i | \exists e \in E \text{ such that } v_i \in e \text{ and } v \in e\}$, L be a list assignment of G with $L(v) = 3 \cdot 2^{d-1}$. It is obvious that G' has an L-coloring c. Observe that $d_G(v) \leq 3 \cdot 2^{d-1} - 1$, we color v by $c(v) \in L(v) \setminus \{c(v_i) | v_i \in N_G(v)\}$ and get a L-coloring of G, a contradiction.

Since an R^d -graph is a special type of (d-1)-dimensional CW complex, and a graph can be regarded as the 1-skeleton of an R^d -graph, the upper bound for the chromatic number of R^d -graphs can be easily extended to graphs.

5 Bridges

In this section, we aim to establish some lemmas of *bridge* in higher-dimensional spaces. Let H be a proper subgraph of a connected R^d -graph G. The set $A_{d-1}(G) \setminus A_{d-1}(H)$ may be partitioned into classes as follows. For each component F of G[V(G) - V(H)], there is a class consisting of the d-dimensional simplexoids of F together with the d-dimensional simplexoids linking F to H. Each remaining d-dimensional simplexoid e defines a singleton class $\{e\}$. The subgraphs of G induced by these classes are the bridges of H in G. It follows immediately from this definition that bridges of H can intersect only in i-dimensional simplexoids of H with $i \leq d-1$, and that any two vertices of a bridge of H are connected by a path in the bridge that is internally disjoint from H.

For a bridge B of H, the elements of $V(B) \cap V(H)$ are called its vertices of attachment to H; the remaining vertices of B are its internal vertices. A bridge is trivial if it has no internal vertices (that is, if it is of the second type). A bridge

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Figure 5: B_1 and B_2 are skew of S_2

with k vertices of attachment is called a k-bridge. Two bridges with the same vertices of attachment are equivalent bridges.

We are concerned here with bridges of (d-1)-sphere, and all bridges are understood to be bridges of a given (d-1)-sphere S_{d-1} . Thus, to avoid repetition, we abbreviate 'bridge of S_{d-1} ' to 'bridge' in the coming discussion.

Lemma 5.1. Let G be a special \mathbb{R}^d -graph, B be a bridge of (d-1)-sphere S_{d-1} , the projection of B which is denoted by $p(B) = B \cap S_{d-1}$ is a connected special \mathbb{R}^{d-1} -graph.

Proof. Let p_1 and p_2 be two distinct connected components of p(B), $v_1 \in p_1$ and $v_2 \in p_2$, $P_B(v_1, v_2) \subseteq B$ denotes a path between vertices v_1 and v_2 , $P_S(v_1, v_2) \subseteq S_{d-1}$ denotes another path between vertices v_1 and v_2 , then $C = P_B(v_1, v_2) \cup P_S(v_1, v_2)$ is evidently a cycle, and in graph G, cycle C cannot be continuously contracted to a point. Thus, it follows that the fundamental group of graph G is nontrivial, leading to a contradiction.

The projection of a k-bridge B with $k \ge d-1$ effects a partition of S_{d-1} into r disjoint segments, called the segments of B. Two bridges avoid each other if all the vertices of attachment of one bridge lie in a single segment of the other bridge; otherwise, they overlap.

Two bridges B and B' are skew if p(B) contains a (d-2)-sphere C(B) as a subgraph which effects a partition of S_{d-1} into two disjoint segments $\{R_1, R_2\}$, and there are distinct vertices of attachment u, v of B' such that u and v are in different segment of $\{R_1, R_2\}$, note that there is a uv-path P(u, v) in S_{d-1} such that $P(u, v) \cap C(B) \neq \phi$ by Lemma 5.1 and Lemma 7.6.

We give an example of skew for S_2 . As shown in Figure 5, both u and v are in the inner region of S_2 , the bridge induced by $\{u, u_1, u_2, ..., u_5\}$ is denoted by B_1 , the bridge induced by $\{v, v_1, v_2\}$ is denoted by B_2 . It is obvious that B_1 effects a partition of S_2 into two disjoint segments (two hemispheres), and v_1 and v_2 lie in different segments.

Lemma 5.2. Overlapping d-hyper-bridges of a closed- R^d -graph are either skew or else equivalent (d + 1)-bridges.

Proof. Suppose that bridges B and B' overlap. Clearly, each of them must have at least d vertices of attachment. If either B or B' is a d-bridge, it is easily verified that they must be skew. We may therefore assume that both B and B' have at least (d + 1) vertices of attachment.

If B or B' are not equivalent bridges, then all the vertices of attachment of one bridge cannot lie in a single segment of the other bridge. Without loss of generality, let $C(B) \subseteq p(B)$ be a (d-2)-sphere which effects a partition of S_{d-1} into 2 disjoint segments $\{R_1, R_2\}$, then there exist distinct vertices of attachment u', v' of B' such that u' and v' are in different segment of $\{R_1, R_2\}$. It follows that B and B' are skew.

If B and B' are equivalent k-bridges, then $k \ge d+1$. If $k \ge d+2$, B and B' are skew by Lemma 5.3; if k = d+1, they are equivalent (d+1)-bridges.

Lemma 5.3. Let G be an special \mathbb{R}^d -graph which is isomorphic to S_{d-1} with $|V(G)| \ge d+2$, then there is a subgraph C of G which is isomorphic to (d-2)-sphere that effects a partition of S_{d-1} into 2 disjoint segments $\{R_1, R_2\}$, and there are distinct vertices of attachment u', v' of G such that u' and v' are in different segment of $\{R_1, R_2\}$.

Proof. Firstly, we prove that there exists a vertex $v' \in V(G)$ such that $d(v') \leq |V(G)| - 2$, if not, we know that d(v) = |V(G)| - 1 for every vertex $v \in V(G)$, which implies that G is a complete graph. It is impossible since G is isomorphic to S_{d-1} .

Let $N_G(v')$ be the neighbor of v', then $G[N_G(v')]$ is isomorphic to a (d-2)-sphere, note that $G[N_G(v')]$ effects a partition of S_{d-1} into 2 disjoint segments $\{R_1, R_2\}$, without loss of generality, let $v' \in R_1$.

It is easy to verify that $V(G) \setminus [N_G(v') \cup \{v'\}] \neq \phi$ since $|V(G)| \geq d+2$, let $u' \in V(G) \setminus [N_G(v') \cup \{v'\}]$, it follows that $u' \in R_2$.

6 Hyper ear decomposition

In this section, we aim to establish some lemmas of *ear decomposition* in higher-dimensional spaces. Let K be a d-connected R^d -graph. Apart from complete graph K_i $(i \leq d+1)$, the d-connected closed- R^d -graph contains a subgraph G_0 which is isomorphic to S_{d-1} . We describe here a simple recursive procedure for generating any such graph starting with an arbitrary (d-1)-sphere of the R^d -graph.

Definition 22 (Hyper Ear). Let F be a subgraph of an \mathbb{R}^d -graph G. A hyper ear of F in G is a nontrivial (d-1)-ball in G whose boundary lies in F but whose internal vertices do not.

Definition 23 (Hyper Ear Decomposition). A nested sequence of a closed- \mathbb{R}^d -graph is a sequence $(G_0, G_1, ..., G_k)$ of \mathbb{R}^d -graphs such that $G_i \subseteq G_{i+1}, 0 \leq i \leq k$. A hyper ear decomposition of a d-connected closed- \mathbb{R}^d -graph G is a nested sequence $(G_0, G_1, ..., G_k)$ of d-connected subgraphs of G such that:

- G_0 is isomorphic to S_{d-1} .
- $G_{i+1} = G_i \cup P_i$ where P_i is a hyper ear of G_i in $G, 0 \le i \le k$.
- $G_k = G$.

Lemma 6.1. The d-connected closed- R^d -graph G with |V(G)| > d + 1 has a hyper ear decomposition.

Proof. Note that G is homeomorphic to the union of a finite collection of (d-1)-spheres by Corollary 2. It is obvious that G has a hyper ear decomposition by Definition 23.

Lemma 6.2. In a d-connected closed- R^d -graph G with |V(G)| > d + 1, each maximal connected region or $R_d \setminus G$ is bounded by a (d-1)-sphere.

Proof. Note that G has a hyper ear decomposition by Lemma 6.1. Consider an ear decomposition $(G_0, G_1, ..., G_k)$ of G, where G_0 is isomorphic to S_{d-1} , $G_k = G$, and, for $0 \le i \le k-2$, $G_{i+1} = G_i \cup P_i$ is a d-connected subgraph of G, where P_i is an ear of G_i in G. Since G_0 is isomorphic to S_{d-1} , the two maximal connected regions of G_0 are clearly bounded by a (d-1)-sphere. Assume, inductively, that all maximal connected regions of G_i are bounded by (d-1)-spheres, where $i \ge 0$. Because G_{i+1} is a d-connected R^d -graph, the ear P_i of G_i is contained in some maximal connected region f of G_i . Each region of G_i other than f is a region of G_{i+1} as well, and so, by the induction hypothesis, is bounded by (d-1)-sphere. On the other hand, the region f of G_i is divided by P_i into two regions of G_{i+1} , and it is easy to see that these regions are also bounded by (d-1)-spheres.



Figure 6: An example of S-decomposition and marked S-decomposition of \mathbb{R}^3 -graph

Lemma 6.3. In a loopless (d + 1)-connected closed- R^d -graph G, the neighbours of any vertex lie on a common (d - 1)-sphere.

Proof. Let G be a loopless (d+1)-connected \mathbb{R}^d -graph and v be a vertex of G, then G-v is d-connected, so each maximal connected region of G-v is bounded by a sphere by Lemma 6.2. If f is the region of G-v in which the vertex v was situated, the neighbours of v lie on its bounding sphere $\partial(f)$.

7 Recognizing R^d -Graph

We extend the definition of S-component for graphs to higher-dimensional spaces before proving our main result.

7.1 S-component

Definition 24 (S-component). Let G be a connected graph which is not complete, let S be a vertex cut of G, and let X be the vertex set of a component of G - S. The subgraph H of G induced by $S \cup X$ is called an S-component of G. In the case where G is a d-connected \mathbb{R}^d -graph and $S := \{x_1, x_2, ..., x_d\}$ is a d-vertex cut of G, we find it convenient to modify each S-component by adding a new (d-1)-dimensional simplexoid with vertex set $\{x_1, x_2, ..., x_d\}$. We refer to this simplexoid as a marker simplexoid and the modified S-components as marked S-components. The set of marked S-components constitutes the marked S-decomposition of G. The graph G can be recovered from its marked S-decomposition by taking the union of its marked S-components and deleting the marker edge.

As shown in Figure 6, $S := \{x_1, x_2, x_3\}$ be a 3-cut of an R^3 -graph G, we provide an example of the S-decomposition and marked S-decomposition of an R^3 -graph. The only difference between S-decomposition and marked S-decomposition is that marked S-decomposition must contain a simplexoid with vertex set $S := \{x_1, x_2, x_3\}$. If this simplex does not exist in the original graph, it needs to be added during the construction.

We need to establish some lemmas before proving our main results.

Lemma 7.1. Let G be a closed- R^d -graph with a d-vertex cut $\{x_1, x_2, ..., x_d\}$, then each marked $\{x_1, x_2, ..., x_d\}$ -component of G is isomorphic to a minor of G.

Proof. Let H be an $\{x_1, x_2, ..., x_d\}$ -component of G, with marker simplexoid e. Let H' be another $\{x_1, x_2, ..., x_d\}$ -component of G, with marker simplexoid e, then there is a (d-1)-ball B such that $B \subseteq H'$ and $B \cup e$ is a (d-1)-sphere by Lemma 6.1. It is easy to verify that H is isomorphic to a minor of G by contract H' into a single simplexoid e.

Lemma 7.2. Let G_1 and G_2 be closed R^d -graphs whose intersection is isomorphic to K_d with $V(K_d) = \{x_1, x_2, ..., x_d\}$, then $G_1 \cup G_2$ is a closed R^d -graph.

Proof. Let H be a hyperplane, $V(K_d) \subseteq H$. At this point, the hyperplane H divides \mathbb{R}^d into two disconnected regions, denoted by R_1 and R_2 , respectively. We embed G_1 into R_1 and G_2 into R_2 in such a way that G_1 and G_2 intersect only at $V(K_d)$.

By contradiction, suppose the fundamental group of $G_1 \cup G_2$ is nontrivial, then there must exist a curve L that cannot be continuously contracted to the base point. If curve L belongs to either G_1 or G_2 , then it can be contracted continuously to base point, which leads to a contradiction. Therefore, L must intersect both G_1 and G_2 . Let $L \cap G_1 = L_1$ and $L \cap G_2 = L_2$, respectively. We first transform L_1 into L_3 by homotopy, such that L_3 belongs to H. It is easy to verify that L_2 and L_3 belong to G_2 , thus they can be continuously contracted to the base point. By combining the two homotopy transformations, we obtain that L can be continuously contracted to the base point, a contradiction. In conclusion, the assumption is invalid, and the theorem is proven.

Lemma 7.3. Let G be a closed- R^d -graph with a d-vertex cut $\{x_1, x_2, ..., x_d\}$, then G is a closed- R^d -graph if and only if each of its marked $\{x_1, x_2, ..., x_d\}$ -components is a closed- R^d -graph.

Proof. Suppose, first, that G is a closed R^d -graph. By Lemma 7.1, each marked $\{x_1, x_2, ..., x_d\}$ -component of G is isomorphic to a minor of G, hence is closed- R^d -graph.

Conversely, suppose that G has k marked $\{x_1, x_2, ..., x_d\}$ -components each of which is a closed- R^d -graph. Let e denote their common marker simplexoid. Applying Lemma 7.2 and induction on k, it follows that G + e is a closed R^d -graph, hence so is G.

By Lemma 7.3, we know that to prove a closed R^d -graph can be embedded in R^d , it is sufficient to show that all of its marked $\{x_1, x_2, ..., x_d\}$ -components can be embedded in R^d .

7.2 Connectivity

Before proving Theorem 1.1, we need a lemma regarding connectivity.

Lemma 7.4. Let G be a (d + 1)-connected graph on at least (d + 2) vertices, then G contains an edge e such that G/e is (d + 1)-connected.

Proof. Suppose that the theorem is false. Then, for any edge e = xy of G, the contraction G/e is not (d + 1)-connected. By Lemma 7.5, there exists vertex set $\{z_1, z_2, ..., z_{d-1}, w\}$ such that $\{z_1, z_2, ..., z_{d-1}, w\}$ is a d-vertex cut of G.

Choose the edge e and the vertex set $\{z_1, z_2, ..., z_{d-1}, w\}$ in such a way that $G - \{x, y, z_1, z_2, ..., z_{d-1}\}$ has a component F with as many vertices as possible. Consider the graph $G - \{z_1\}$. Because G is (d + 1)-connected, $G - \{z_1\}$ is d-connected. Moreover $G - \{z_1\}$ has the d-vertex cut $\{x, y, z_2, ..., z_{d-1}\}$. It follows that the $\{x, y, z_2, ..., z_{d-1}\}$ -component $H = G[V(F) \cup \{x, y, z_2, ..., z_{d-1}\}]$ is d-connected.

Let u be a neighbour of z_1 in a component of $G - \{x, y, z_1, z_2, ..., z_{d-1}\}$ different from F. Since f = zu is an edge of G, and G is a counterexample to Lemma 7.4, there is a vertex set $\{v_1, v_2, ..., v_{d-1}\}$ such that $\{z, u, v_1, v_2, ..., v_{d-1}\}$ is a (d + 1)-vertex cut of G, too. (The vertices $\{v_1, v_2, ..., v_{d-1}\}$ might or might not lie in H.) Moreover, because H is d-connected, $H - \{v_1, v_2, ..., v_{d-1}\}$ is connected (where, if $\exists v_i \in \{v_1, v_2, ..., v_{d-1}\}$ such that $v_i \in V(H)$, we set $H - v_i = H$),

and thus is contained in a component of $G - \{z, u, v_1, v_2, ..., v_{d-1}\}$. But this component has more vertices than F (because H has d more vertices than F), contradicting the choice of the edge e and the vertex v.

Lemma 7.5. Let G be a (d+1)-connected graph on at least (d+2) vertices, and let e = xy be an edge of G such that G/e is not (d+1)-connected. Then there exists a vertex z such that $\{x, y, z_1, z_2, ..., z_{d-1}\}$ is a (d+1)-vertex cut of G.

Proof. Let $\{z_1, z_2, ..., z_{d-1}, w\}$ be a *d*-vertex cut of G/e. At least d-1 of these *d* vertices, say $\{z_1, z_2, ..., z_{d-1}\}$, is not the vertex resulting from the contraction of *e*. Set $F = G - \{z_1, z_2, ..., z_{d-1}\}$. Because *G* is (d+1)-connected, *F* is certainly 2-connected. However $F/e = (G - \{z_1, z_2, ..., z_{d-1}\})/e = (G/e) - \{z_1, z_2, ..., z_{d-1}\}$ has a cut vertex, namely *w*.

If w is not the vertex resulting from the contraction of e, then $\{z_1, z_2, ..., z_{d-1}, w\}$ must be a d-vertex cut of G, a contradiction. Hence w must be the vertex resulting from the contraction of e. Therefore $G - \{x, y, z_1, z_2, ..., z_{d-1}\} = (G/e) - \{z_1, z_2, ..., z_{d-1}, w\}$ is disconnected, in other words, $\{x, y, z_1, z_2, ..., z_{d-1}\}$ is a (d+1)-vertex cut of G.

7.3 Anti-d-dimension minor

The following proof demonstrates that there is a conclusion that holds in \mathbb{R}^d that similar to Wagner's Theorem. It is necessary to generalize the concepts of complete graphs and bipartite graphs to higher-dimensional spaces.

Definition 25 (Complete *i*-Uniform-Topological Hypergraph K_n^i). Let G be an *i*-uniform-topological hypergraph, V be the vertex set of G with order n, $\mathscr{V}(i)$ be the collection of all subsets of V containing *i* elements. If for any $V_j \in \mathscr{V}(i)$, there exists a simplexoids T in G such that $V(T) = V_j$, then we call G a complete *i*-uniform-topological hypergraph, which is denoted by K_n^i . (Figure 7 is an example of K_4^3 .)



Definition 26 (Complete Bipartite *i*-Uniform-Topological Hypergraph $K_{p,q}^i$). Let G be an *i*-uniform-topological hypergraph, V be the vertex set of G, V(A : B) be a partition of V in which $A = \{a_1, a_2, ..., a_p\}$ and $B = \{b_1, b_2, ..., b_q\}$. If G satisfies the following properties, we call G a complete bipartite *i*-uniform-topological hypergraph, which is denoted by $K_{p,q}^i$. (Figure 8 is an example of $K_{2,4}^3$.)

- G[B] is a complete (i-1)-uniform-topological hypergraph K_q^{i-1} .
- For any $T_j \in A_{i-2}(G[B])$ and $a_k \in A$, there exists a simplexoid T in G such that $V(T) = V(T_j) \cup \{a_k\}$ and $|A_{i-1}(G)|$ is minimal.

Next, we use the Jordan-Brouwer Separation Theorem to prove Lemma 7.7 and 7.8.

Lemma 7.6. (Jordan-Brouwer Separation Theorem) Let X be a d-dimensional topological sphere in the (d + 1)dimensional Euclidean space R_{d+1} (d > 0), i.e. the image of an injective continuous mapping of the d-sphere S_d into R_{d+1} , then the complement Y of X in R_{d+1} consists of exactly two connected components. One of these components is bounded (the interior) and the other is unbounded (the exterior). The set X is their common boundary.

Lemma 7.7. K_{d+3}^d is a non- R^d -graph.

Proof. Suppose to the contrary that K_{d+3}^d is an R^d -graph.

Let $V(K_{d+3}^d) = \{v_1, v_2, \dots, v_{d+2}, v_{d+3}\}$. Let \mathscr{V}_{d+2}^{d+1} be the collection of all subsets of $V(K_{d+3}^d) \setminus \{v_{d+3}\}$ containing (d+1)elements.

Note that for any $V_i \in \mathscr{V}_{d+2}^{d+1}$, $K_{d+3}^d[V_i]$ is isomorphic to S_{d-1} .



Figure 10: K_{3d+1}^d in \mathbb{R}^d

Let $V_i = V(K_{d+3}^d) \setminus \{v_{d+3}, v_i\}$. We assume that $K_{d+3}^d[V(K_{d+3}^d) \setminus \{v_{d+3}\}]$ has already been embedded in \mathbb{R}^d . By Lemma 7.6, each $K_{d+3}^d[V_i]$ will divide \mathbb{R}^d into two disconnected regions, with one of the regions being empty. We designate the non-empty region as the external region of $K_{d+3}^{d}[V_i]$ and the empty region as the internal region. Let In(i)be the internal region corresponding to $K_{d+3}^d[V_i]$ and Ex(i) be the external region corresponding to $K_{d+3}^d[V_i]$. (As shown in the Figure 9, when embedded K_{d+3}^d in \mathbb{R}^d , it is obvious that line v_1v_6 intersects with $K_{d+3}^d[V_i]$ at least at one point.)

Without loss of generality, we assume that v_{d+3} is in In(1), and it follows that v_1 is in Ex(1). Since K_{d+3}^d is a complete d-uniform-topological hypergraph, there must exist a simplexoid whose vertex set includes both v_1 and v_{d+3} . By Lemma 7.6, this simplexoid must intersect with $K_{d+3}^d[V_1]$. Therefore K_{d+3}^d cannot be embedded in \mathbb{R}^d .

Lemma 7.8. $K_{3,d+1}^d$ is a non- R^d -graph.

Proof. The proof of Lemma 7.8 is similar to that of Lemma 7.7. (Figure 10 is an example of $K_{3,4}^3$ in \mathbb{R}^3 .)

Definition 27 (Anti-d-dimension Minor). If a d-uniform-topological hypergraph G has a K^d_{d+3} -minor or $K^d_{3,d+1}$ -minor, then we call G an anti-d-dimension minor.

Proof of Theorem 1.1 7.4

Proof. Let G be the graph-closure of a d-uniform-topological hypergraph. If its minors include either $K_{3,d+1}^d$ or K_{d+3}^d it is obvious that G is a non- R^d -graph by Lemma 7.7-7.8.

Now we assume that G is a (d+1)-connected non- R^d -graph and G is simple. Because all graphs on d+2 or fewer vertices can be embedded in \mathbb{R}^d , we have $|V(G)| \ge d+3$. We proceed by induction on |V(G)|. By Lemma 7.4, G contains an edge e = xy such that H = G/e is (d+1)-connected. If H is non- R^d , it has an anti-d-dimension minor, by induction. Since every minor of H is also a minor of G, we deduce that G too has an anti-d-dimension minor. So we may assume that H is an R^d -graph.

Consider an R^d -embedding H' of H. Denote by z the vertex of H formed by contracting e. Because H is (d+1)connected, by Lemma 6.3 the neighbours of z lie on a (d-1)-sphere S_{d-1} , the boundary of some polytope W of H'-z. Denote by B_x and B_y , respectively, the hyper-bridges of W in $G \setminus e$ that contain the vertices x and y.



Figure 11: Minimum Skew in closed- R^3 -graph ($K^3_{3,4}$ -minor)

Suppose, first, that B_x and B_y avoid each other. In this case, B_x and B_y can be embedded in the polytope W of H'-z in such a way that the vertices x and y belong to the same polytope of the resulting R^d -graph $(H'-z)\cup B_x\cup B_y$. The edge xy and the simplexoids that incident with xy can now be drawn in that polytope so as to obtain an R^d -embedding of G itself, contradicting the hypothesis that G is non- R^d .

It follows that B_x and B_y do not avoid each other, that is, they overlap. By Lemma 5.2, they are therefore either skew or else equivalent (d+1)-bridges. In the latter case, G has a K^d_{d+3} -minor; In the former case, G has a $K^d_{3,d+1}$ -minor.

- In higher dimensions, the former case is not intuitive. Here, we provide an example of minimum skew is equivalent to $K_{3,4}^3$ in \mathbb{R}^3 , which can be analogized to higher-dimensional cases.
- As shown in Figure 11(1), simplexoids x₁y₁y₂, x₁y₂y₃, x₁y₁y₃, x₂y₁y₂, x₂y₂y₃, x₁y₁y₃ are joined together to form a two-dimensional sphere S₂. Vertices x and y are inside S₂.
- As shown in Figure 11(2), simplexoids xy_1y_2, xy_2y_3 and xy_1y_3 form a bridge B_1 , which effect a partition of S_2 into 2 disjoint segments.
- As shown in Figure 11(3), there is another bridge B_2 with internal vertex y. Its vertics of attachment includes both x_1 and x_2 . By definition, B_1 and B_2 are skew. On the other hand, since there are no hanging simplexoid in the graph, bridge B_2 must include simplexoids $\{yx_1y_1, yx_1y_2, yx_1y_3, yx_2y_1, yx_2y_2, yx_2y_3, yy_1y_2, yy_2y_3, yy_1y_3\}$.
- As shown in Figure 11(4), let $A = \{x, x_1, x_2\}$ and $B = \{y, y_1, y_2, y_3\}$, At this point, $K_{3,4}^3 = S_2 \cup B_1 \cup B_2 = (A, B)$ is a complete bipartite 3-uniform-topological hypergraph.

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