Fault-Equivalent Lowest Common Ancestors

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Abstract

Let T be a rooted tree in which a set M of vertices are marked. The lowest common ancestor (LCA) of M is the unique vertex ℓ with the following property: after failing (i.e., deleting) any single vertex x from T, the root remains connected to ℓ if and only if it remains connected to some marked vertex. In this note, we introduce a generalized notion called f-fault-equivalent LCAs (f-FLCA), obtained by adapting the above view to f failures for arbitrary $f \geq 1$. We show that there is a unique vertex set $M^* = FLCA(M, f)$ of minimal size such after the failure of any f vertices (or less), the root remains connected to some $v \in M$ iff it remains connected to some $u \in M^*$. Computing M^* takes linear time. A bound of $|M^*| \leq 2^{f-1}$ always holds, regardless of |M|, and holds with equality for some choice of T and M.

1 Introduction and Results

Consider the following motivating problem. There is an *n*-vertex tree T, rooted at a source s. In this tree, a nonempty and possibly large vertex subset of interest $M \subseteq V(T)$ is marked. We are preparing for the future failures (or faults) of at most f currently unknown vertices, which will be deleted from the tree. (A faulty vertex may or may not be marked.) After the failures $F \subseteq V(T)$ occur, we will be interested to understand whether they cause s to disconnect from all the marked vertices. Namely, we will want to answer the following question: Is there some marked vertex $v \in M$ that remains reachable from s in T - F?

However, we would like to save on memory costs, and avoid storing the entire set M. Instead, we want to preprocesses M to find and store a smaller "representative" set of vertices $M^* \subseteq V(T)$, which is equivalent to M in terms of the above question. Namely, for every fault-set $F \subseteq V(T)$ with $|F| \leq f$, all vertices in M^* are disconnected from s in T - F if and only if this is true for the original marked set M.

We now introduce some definitions to formalize the above. First, we define the *covering* relation.

Definition 1. For two vertex sets $A, B \subseteq V(T)$, we say that A covers B, and denote $A \succeq B$, if for every $b \in B$ there exists $a \in A$ which is an ancestor of b in T.

Note that the failure of $F \subseteq V(T)$ disconnects s from all of M iff $F \succcurlyeq M$. Our requirement from the representative set M^* is that this should happen iff $F \succcurlyeq M^*$, whenever $|F| \leq f$. We therefore define the *f*-fault equivalence relation.

Definition 2. For $f \ge 1$, we say that two vertex sets $M, N \subseteq V(T)$ are *f*-fault-equivalent, and denote $M \sim_f N$, if for any $F \subseteq V(T)$ with $|F| \le f$, it holds that $F \succcurlyeq M$ iff $F \succcurlyeq N$.

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Our motivating problem can now be succinctly stated as follows:

Problem 1. Given the tree T, the marked set $\emptyset \neq M \subseteq V(T)$ and a fault parameter $f \geq 1$, find a set $M^* \subseteq V(T)$ of minimal size such that $M^* \sim_f M$.

Relation to Lowest Common Ancestors. When preparing for a single vertex failure, i.e., when f = 1, a moment's reflection will show that one can always choose M^* having only one vertex: the *lowest common ancestor* (*LCA*) of all marked vertices, denoted LCA(M). Indeed, a single failed vertex v disconnects all of M from s iff $\{v\} \geq M$, namely iff v is a common ancestor of all marked vertices, which happens iff v is an ancestor of LCA(M), i.e., iff $\{v\} \geq \{\text{LCA}(M)\}$. In fact, LCA(M) is the *only* single vertex satisfying this property. This means that LCA(M) could be equivalently defined by the unique optimal solution to Problem 1 with f = 1.

Thus, letting f increase beyond 1 yields a generalized notion of LCA given by the optimal solution to Problem 1. Also, as we will show, the optimal solution M^* is always unique, and consists of LCAs of subsets of M. For these reasons, we call M^* the f-fault LCA of M, and denote it by FLCA(M, f). (So, by the above discussion, FLCA $(M, 1) = \{LCA(M)\}$.)

Results. In this note, we give a simple algorithm to compute FLCA(M, f), the optimal solution for Problem 1, and answer a natural question of interest (given our "memory savings" motivation): how small can FLCA(M, f) be? When f = 1, we saw it has size 1, regardless of how large M is. This extends to a bound of 2^{f-1} on the size of FLCA(M, f), which is worst-case optimal (i.e., for some choice of T, M, $|FLCA(M, f)| = 2^{f-1}$). Thus, when the fault parameter f is constant, we can represent any marked vertex set M (in the sense of f-fault equivalence) using only a constant number of representative vertices. Our results are summarized in the following theorem.

Theorem 1. Let T be an n-vertex tree rooted at vertex s, $M \subseteq V(T)$ be a non-empty set of marked vertices, and $f \ge 1$. The following hold:

- 1. There is a unique set FLCA(M, f) having minimal size among all the f-fault-equivalent sets to M, namely among $\{N \subseteq V(T) \mid N \sim_f M\}$.
- 2. It holds that $|\operatorname{FLCA}(M, f)| \leq 2^{f-1}$, and this bound is tight. That is, for some choice of T and M, this holds with equality.
- 3. There is an O(n) time algorithm to compute FLCA(M, f) given T, M and f. Further, after O(n) time for preprocessing T, one can compute FLCA(M, f) within O(|M|) time.

Edge Faults. We remark that considering failures of *edges* instead of vertices, or even allowing a mixture of failing vertices and edges, does not change our results regarding FLCA(M, f). To state this explicitly: FLCA(M, f) is the unique vertex set $M^* \subseteq V(T)$ having minimal size such that for every $F \subseteq V(T) \cup E(T)$ of size $|F| \leq f$, in T - F it holds that the root s is connected to some $v \in M$ iff it is connected to some $u \in M^*$. This is due to the fact that, in terms of connectivity to the root, the failure of an edge in a tree has the same effect as the failure of its lower endpoint.

Aggregation. It is easy to prove that the function $\varphi(\cdot) = \text{FLCA}(\cdot, f)$ admits the following nice aggregation property: $\varphi(A \cup B) = \varphi(A \cup \varphi(B))$. Such aggregation properties are often exploited for efficient computations. As a "toy example", suppose the marked set M is revealed to us over time in batches M_1, M_2, M_3, \ldots . Then we can save on memory in the time between batches t and t+1, only (at most 2^{f-1}) vertices in $M_t^* = \text{FLCA}(M_1 \cup \cdots \cup M_t, f)$. When M_{t+1} arrives, we can use the aggregation property and compute M_{t+1}^* as $\text{FLCA}(M_t^* \cup M_{t+1}, f)$.

Potential Applications. The notions and results presented in this note were developed during research on fault-tolerant graph data structures, but eventually did not make their way into the final solutions. Still, the author believes they could be of potential use in the field of fault-tolerant graph structures and algorithms, and hopes such applications would be found in the future.

2 Proof of Theorem 1

The proof is by analyzing the following algorithm for computing FLCA(M, f). The notation T_v stands for the subtree of T rooted at vertex v.

Algorithm 1 Algorithm \mathcal{A} for computing FLCA(M, f)Input: Rooted tree T, non-empty vertex set $M \subseteq V(T)$, integer $f \ge 1$ Output: Vertex set $\mathcal{A}(T, M, f) \subseteq V(T)$ 1: $\ell \leftarrow \text{LCA}(M)$ 2: if $\ell \in M$ then return $\{\ell\}$ 3: $u_1, \ldots, u_d \leftarrow$ the children of ℓ with $M \cap V(T_{u_i}) \neq \emptyset$ 4: if d > f then return $\{\ell\}$ 5: $M_1, \ldots, M_d \leftarrow M \cap V(T_{u_1}), \ldots, M \cap V(T_{u_k})$ 6: return $\bigcup_{i=1}^d \mathcal{A}(T, M_i, f - d + 1)$ \triangleright Note: $1 \le f - d + 1 \le f - 1$

We divide the proof into several claims regarding algorithm \mathcal{A} . All of them are proved by strong induction on f. We denote the output as $M^* = \mathcal{A}(T, M, f)$, and in case Line 6 is executed, we also denote $M_i^* = \mathcal{A}(T, M_i, f - d + 1)$.

Throughout, we will use extensively the following easy-to-observe properties of the covering relation \succeq from Definition 1, without explicitly stating them. The notation T[u, v] stands for the (unique) tree path between vertices u and v.

Observation 2. The covering relation \succ from Definition 1 has the following properties:

- 1. It is reflexive and transitive. (Namely, \geq is a preorder.)
- 2. $A \succcurlyeq B \iff B \subseteq \bigcup_{a \in A} V(T_a) \iff \forall b \in B, A \cap T[s, b] \neq \emptyset.$
- 3. If $A \succeq B$, then for every $B' \subseteq B$ it holds that $A \succeq B'$ and $\{LCA(A)\} \succeq \{LCA(B')\}$.
- 4. If $A_i \geq B_i$ for all *i*, then $\bigcup_i A_i \geq \bigcup_i B_i$.
- 5. Assume $A \succeq B$ and $B \subseteq V(T_v)$. Let A' be any subset of A obtained by removing some vertices lying outside of $T_v \cup T[s, v]$. Then $A' \succeq B$.

We start with an auxiliary lemma, stating that the output M^* must lie between $\ell = \text{LCA}(M)$ and M in terms of covering.

Lemma 3. $\{\ell\} \geq M^* \geq M$.

Proof. If $M^* = \{\ell\}$ this is trivial. Otherwise, the algorithm must have executed Line 6. By the induction hypothesis, $\{\text{LCA}(M_i)\} \succeq M_i^* \succeq M_i$ for all $i = 1, \ldots, d$. As $\{\ell\} \succeq \{\text{LCA}(M_i)\}$, we

deduce that

$$\{\ell\} \succcurlyeq \bigcup_{i=1}^{d} M_i^* = M^*$$
$$\succcurlyeq \bigcup_{i=1}^{d} M_i = M$$

as required.

We now turn to prove the first item of Theorem 1, by the following Claim 4 and Claim 5.

Claim 4 (*f*-Fault Equivalence). $M^* \sim_f M$.

Proof. Let $F \subseteq V(T)$ with $|F| \leq f$. We should prove that $F \succcurlyeq M^* \iff F \succcurlyeq M$.

 (\Longrightarrow) Follows immediately from Lemma 3.

(\Leftarrow) If $F \geq \{\ell\}$ then $F \geq M^*$ by Lemma 3 and we are done. Assume now that $F \neq \{\ell\}$, i.e. $F \cap T[s,\ell] = \emptyset$. As $F \geq M$, it follows that $\ell \notin M$. Hence, the condition of Line 2 is not satisfied, and Line 3 must have been executed. Each subtree T_{u_i} must intersect F, since otherwise, the subset M_i of M could not have been covered by F (because $F \cap T[s,\ell] = \emptyset$). Because T_{u_1}, \ldots, T_{u_d} are disjoint, we see that $d \leq |F| \leq f$. This means that the condition of Line 4 is not satisfied, hence Line 6 must have been executed.

Let $F_i = F \cap V(T_{u_i})$. Note that $F_i \geq M_i$, because $F \geq M_i$ and all vertices in $F - F_i$ lie outside of $T_{u_i} \cup T[s, u_i]$. Also, each of the d-1 disjoint subtrees $\{T_{u_j}\}_{j \neq i}$ contains one vertex from $F - F_i$, and thus $|F_i| \leq |F| - (d-1) \leq f - d + 1$. Since $M_i \sim_{f-d+1} M_i^*$ holds by the induction hypothesis, we obtain that $F_i \geq M_i^*$. We conclude that

$$F \succcurlyeq \bigcup_{i=1}^{d} F_i \succcurlyeq \bigcup_{i=1}^{d} M_i^* = M^*.$$

Claim 5 (Minimality and Uniqueness). If $N \subseteq V(T)$ and $N \sim_f M$, then $|N| \ge |M^*|$, and equality holds iff $N = M^*$.

Proof. Consider first the case where $M^* = \{\ell\}$. Then, as M is non-empty, N also cannot be empty, i.e. $|N| \ge 1 = |M^*|$. If equality holds, then N contains a single vertex v. Trivially, $\{v\} \ge N$. Since $N \sim_f M$ (and $f \ge 1$) we obtain $\{v\} \ge M$. Thus, v is a common ancestor of all vertices in M, so it must be an ancestor of $\ell = \text{LCA}(M)$. However, it cannot be a strict ancestor of ℓ , as then we would have $\{\ell\} \ge M$ and $\{\ell\} \ge \{v\} = N$, contradicting the assumption that $N \sim_f M$. Thus $v = \ell$, so $N = M^*$.

It remains to consider the case where $M^* \neq \{\ell\}$. Then the algorithm must have executed Line 6, and the conditions of Line 2 and Line 4 were not satisfied. Hence, $U = \{u_1, \ldots, u_d\} \succeq M$ and $d \leq f$. Since $N \sim_f M$ we get that $U \succeq N$. Therefore, letting $N_i = N \cap V(T_{u_i})$, we have $N = \biguplus_{i=1}^d N_i$ (where \uplus denotes disjoint union).

We now observe that, for any $F' \subseteq V(T)$ with $|F'| \leq f - d + 1$, it holds that

$$F' \succcurlyeq M_i \iff F' \cup (U - \{u_i\}) \succcurlyeq M \iff F' \cup (U - \{u_i\}) \succcurlyeq N \iff F' \succcurlyeq N_i$$

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where the middle ' \iff ' holds as $|F' \cup (U - \{u_i\})| \leq f$ and $N \sim_f M$. This means that $N_i \sim_{f-d+1} M_i$. The induction hypothesis thus yields that $|N_i| \geq |M_i^*|$, and equality holds iff $N_i = M_i^*$. We deduce that

$$|N| = \sum_{i=1}^{d} |N_i| \ge \sum_{i=1}^{d} |M_i^*|.$$
(1)

Now, as $M_i \subseteq V(T_{u_i})$, it follows (by Lemma 3 for $M_i^* = \mathcal{A}(T, M_i, f - d + 1)$) that $M_i^* \subseteq V(T_{u_i})$. Hence, the union returned in Line 6 is disjoint, i.e. $M^* = \biguplus_{i=1}^d M_i^*$. Thus, the right-hand-side of Eqn. (1) is equal to $|M^*|$, so we have shown that $|N| \ge |M^*|$. Furthermore, in light of Eqn. (1), $|N| = |M^*|$ can hold only if for all $i = 1, \ldots, d$ we have $|N_i| = |M_i^*|$, and thus also $N_i = M_i^*$. So in this case,

$$N = \biguplus_{i=1}^{d} N_i = \biguplus_{i=1}^{d} M_i^* = M^*$$

as required.

Next, we prove the second item of Theorem 1.

Claim 6 (Size Bound). $|M^*| \leq 2^{f-1}$. Further, for some choice of T and M, equality holds.

Proof. If $M^* = \{\ell\}$ then the inequality is trivial. Otherwise, Line 6 must have been executed, so $M^* = \bigcup_{i=1}^d M_i^*$. Using the induction hypothesis, we obtain

$$|M^*| \le \sum_{i=1}^d |M^*_i| \le d \cdot 2^{(f-d+1)-1} = \frac{d}{2^{d-1}} \cdot 2^{f-1} \le 2^{f-1}$$

For a case where equality holds, consider T being a full binary of height at least f - 1, with all of its leaves marked as M. Then it is easy to verify that $M^* = \mathcal{A}(T, M, f)$ is the set of all 2^{f-1} vertices with depth f - 1.

Finally, we prove the third and last item of Theorem 1.

Claim 7 (Implementation). The tree T can be preprocessed in O(n) time so that queries (M, f) can be answered with FLCA(M, f) within O(|M|) time.

Proof. Algorithm \mathcal{A} can be implemented in O(n) time by dynamic programming on the tree T. As an improvement, we show after O(n) time for preprocessing T, one can answer queries (M, f) by computing FLCA(M, f) within O(|M|) time. To this end, We build two classical data structures for (pairwise) LCA and level ancestor queries:

lca(u, v): returns the lowest common ancestor of vertices u and v

anc(u, l) : returns the ancestor v of u such that depth(v) = l (or undefined if depth(u) < l)

This requires O(n) time, and queries can be answered in O(1) time [BF00, BF04]. Additionally, we create a lookup table D for vertices, where D[v] stores a "switch bit" initialized to zero, and a pointer initialized to a null value. This concludes the preprocessing, taking O(n) time. Given a query (M, f) with $M = \{v_1, \ldots, v_{|M|}\}$, we now explain the implementation details for executing $\mathcal{A}(T, M, f)$ in O(|M|) time.

Computing ℓ takes O(|M|) time by initializing $\ell \leftarrow lca(v_1, v_2)$ and updating $\ell \leftarrow lca(\ell, v_i)$ for $i = 3, \ldots, |M|$. Computing u_1, \ldots, u_d and M_1, \ldots, M_d is most of the work. We aim to store

 u_1, \ldots, u_d in a linked list L, and to store each M_i in a linked list pointed to from $D[u_i]$. We sequentially process each $v_i \in M$ as follows. First, we compute $u \leftarrow \operatorname{anc}(v_i, \operatorname{depth}(\ell) + 1)$. Next, we do a lookup to D[u]. If the switch bit is 0, we (i) change it to 1, (ii) change the pointer of D[u]to the head of a new linked list of length 1 that stores v_i , and (iii) add u to the linked list L. If the switch bit is 1, we just add v to the linked list pointed by D[u] (that was created when the switch was turned on). Overall, this takes O(|M|) time.

To invoke the recursive calls, We scan the linked list L containing u_1, \ldots, u_d (where this is their order in L). When processing u_i , we "clean up" the lookup table entry $D[u_i]$ by copying the pointer to the linked list containing M_i , then reverting $D[u_i]$ to its original state (switch bit 0 and null pointer). We can now execute the recursive calls $\mathcal{A}(T, M_i, f - d + 1)$. This ensures that the lookup table D returns to its cleaned state after the current query (M, f) is answered, allowing us to treat any future query (M', f') in the same manner.

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