# AN INFINITE FAMILY OF ARTIN-SCHREIER CURVES WITH MINIMAL A-NUMBER

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ABSTRACT. Let p be an odd prime and k be an algebraically closed field with characteristic p. Booher and Cais showed that the a-number of a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover of curves  $\phi:Y\to X$  must be greater than a lower bound determined by the ramification of  $\phi$ . In this paper, we provide evidence that the lower bound is optimal by finding examples of Artin-Schreier curves that have a-number equal to its lower bound for all p. Furthermore we use formal patching to generate infinite families of Artin-Schreier curves with a-number equal to the lower bound in any characteristic.

#### 1. Introduction

Let p be an odd prime and k be an algebraically closed field with characteristic p. Let  $\phi: Y \to X$  be a smooth, projective, and connected cover of curves over k with Galois group G. Some broad questions are

- "What properties of the curve Y can be determined solely from properties of the curve X and the map  $\phi$ ?"
- "What information is needed to determine the other properties of Y?"

A classic version of this question concerns the genus of the curves, a standard numerical invariant associated to a curve. The genera of X and Y can be described as the k-dimension of  $H^0(X,\Omega^1_X)$  and  $H^0(Y,\Omega^1_Y)$ , the space of regular 1-forms on X and Y, respectively. The well known Riemann-Hurwitz formula explains how the genus of Y can be determined entirely from X and ramification information about the cover  $\phi$ ,

$$2g_Y - 2 = |G|(2g_X - 2) + \sum_{y \in \phi^{-1}(S)} \sum_{i \ge 0} (|G_i(y)| - 1),$$

where S is the branch locus of  $\phi$  and  $G_i(y)$  is the ith ramification group in lower numbering at y.

When k has characteristic p, as in this paper, there are additional invariants arising from the Frobenius automorphism. We will work with the Cartier operator (which is dual to the Frobenius on  $H^1(X,\mathcal{O}_X)$  via Serre duality). For the curve X, the Cartier operator is a  $p^{-1}$ -semilinear map  $\mathcal{C}_X:H^0(X,\Omega^1_X)\to H^0(X,\Omega^1_X)$ . As  $H^0(X,\Omega^1_X)$  is a finitely generated  $k[\mathcal{C}_X]$ -module, the structure theorem for finitely generated modules over a P.I.D. gives the following decomposition of  $k[\mathcal{C}_X]$ -modules,

(1) 
$$H^{0}(X, \Omega_{X}^{1}) = \bigoplus_{i} k[\mathcal{C}_{X}] / \mathcal{C}_{X}^{n_{i}} \oplus \bigoplus_{j} k[\mathcal{C}_{X}] / f_{j}(\mathcal{C}_{X})^{n_{j}},$$

where  $f_j(\mathcal{C}_X)$  are irreducible polynomials in  $k[\mathcal{C}_X]$  not equal to  $\mathcal{C}_X$ . (Although  $k[\mathcal{C}_X]$  is not technically a P.I.D. as its non-commutative, there exists an identical structure theorem for non-commutative P.I.D. [Jac43, Theorem 3.19].) Note that

 $\mathcal{C}_X$  acts nilpotently on the first part of (1). The *p*-rank of X, which we write as  $s_X$ , is then defined as the *k*-dimension of  $\bigoplus_j k[\mathcal{C}_X]/f_j(\mathcal{C}_X)^{n_j}$ , the second half of the decomposition in (1). Like the genus, the *p*-rank is another invariant of Y that can often be determined from X and ramification information from  $\pi$ . The Deuring-Shafarevich formula says that when G is a p-group,

$$s_Y - 1 = |G|(s_X - 1) + \sum_{y \in \phi^{-1}(S)} (d_y - 1),$$

where S is the branch locus of  $\phi$  and  $d_y$  is the unique break in the ramification filtration at y. We refer to  $d_y$  as the ramification break at y.

The a-number is an additional numerical invariant describing the structure of  $H^0(X,\Omega_X^1)$ , defined as the number of summands in  $\bigoplus_i k[\mathcal{C}_X]/\mathcal{C}_X^{n_i}$ , the first half of (1), i.e. the k-dimension of  $\ker(\mathcal{C}_X)$ . This invariant is less understood and the focus of this paper. Since the a-number is similar to the p-rank in definition and simplicity, it would be natural to attempt to find an analog of the Deuring-Sharfarevich formula. However, no such formula exists, and we can use Artin-Schreier curves to see this. Artin-Schreier curves are smooth, projective, connected covers of  $\mathbb{P}^1$  with Galois group  $\mathbb{Z}/p\mathbb{Z}$  and are a key case of p-group curves. Any Artin-Schreier curve can be defined by an equation of the form  $y^p - y = f$  where  $f \in k(x)$  is nonconstant and k(x) is the function field corresponding to  $\mathbb{P}^1$ . The Artin-Schreier curve defined by  $y^3 - y = x^7$  has a-number 4 while the Artin-Schreier curve defined by  $y^3 - y = x^7$  has a-number 3, despite both being covers of  $\mathbb{P}^1$  and being branched only over  $\infty$  with ramification break d=7. This shows that the a-number of Y cannot be determined using the same information needed to determine its genus and p-rank.

However X and the ramification of  $\phi$  still constrain the a-number of Y. Farnell and Pries [FP12] discovered a formula for the a-number of an Artin-Schreier curve dependent only on X and its ramification information for a specific congruence condition on the order of the poles of its defining equation. Elkin and Pries [EP13] found a specific formula for the a-number of hyperelliptic k-curves in characteristic 2 dependent only on its ramification information. These are specific cases where the a-number of Y can be determined given some condition on the characteristic or the ramification. In general, Booher and Cais [BC20] were able to find upper and lower bounds for the a-number of a curve Y, where  $\phi: Y \to X$  is a branched  $\mathbb{Z}/p\mathbb{Z}$ -cover, depending only on X and the ramification of  $\phi$ ,

(2) 
$$\max_{1 \le j \le p-1} \left( \sum_{Q \in S} \sum_{i=j}^{p-1} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p} - \left(1 - \frac{1}{p}\right) \frac{jd_Q}{p} \right\rfloor \right) \right) \le a_Y,$$

(3) 
$$a_Y \le pa_X + \sum_{Q \in S} \sum_{i=1}^{p-1} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor - (p-i) \left\lfloor \frac{id_Q}{p^2} \right\rfloor \right),$$

where S is the branch locus and  $d_Q$  is the ramification break over Q in lower numbering. When the branch locus contains only one point Q, we refer to the lower bound in the left side of (2) as  $L(d_Q)$ .

In some sense, the bounds in (2) and (3) are an analog to the Deuring-Shaferavich for the a-number. However, because they are not an exact formula, work needs to be done to show that they are the optimal bounds. Experimentally, the bounds

seem to be optimal. A randomly generated curve over  $\mathbb{F}_p$  with a fixed ramification break, d, has a high chance of having an a-number equal to the lower bound, which suggests (2) is optimal. The authors of  $[\mathrm{Abn}+22]$  provides some harder evidence that the lower bound is optimal for small p, as they exhibit curves over  $\mathbb{F}_3$  and  $\mathbb{F}_5$  with any size ramification break that attains an a-number equal to the lower bound in (2). In this paper, we provide additional evidence that the bound in (2) is optimal.

**Theorem 3.12.** For any prime p and any positive  $d \equiv -1 \pmod{p^2}$  there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  branched at one point with ramification break d and a-number equal to L(d).

**Theorem 3.13.** For any prime p and any positive  $d \equiv p-1 \pmod{p^2}$  there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  branched at one point with ramification break d and a-number equal to L(d).

**Theorem 3.14.** For any prime  $p \leq 23$  and any positive  $d \equiv -1 \pmod{p}$ , there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  branched at one point with ramification break d and a-number equal to L(d).

In Theorem 3.12 and 3.13, we generate families of curves with arbitrarily large ramification break in any finite characteristic that have a-number equal to the lower bound in (2). We do this by exhibiting small curves with minimal a-number and then using formal patching to build an infinite family of curves by combining the smaller curves together inductively. Note that our method generates curves of arbitrarily large ramification breaks for any prime p, as opposed to just p=3 and p=5. This serves as evidence that the bounds are optimal.

Remark 1.1. The specific method we use can only produce curves with ramification break  $d \equiv -1 \pmod{p}$ . We only tackle two of p many possible congruence classes in this paper due to the difficulty in finding curves that have a-number equal to the lower bound for all p that are easy to analyze. We produce curves for all of the congruence classes for small p in Theorem 3.14 by using the MAGMA computational algebra system [BCP97] to find examples.

**Remark 1.2.** In Section 3, we form covers with a-number equal to the lower bound with a single branch point by combining curves with the same branch point. However, we could also construct covers with multiple branch points having minimal a-number. To do so, combine the examples with a single branch point that we give in this paper using the formal patching arguments in [BP20; Abn+22].

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#### 2. SMALL ARTIN-SCHREIER CURVES

2.1. Numerical invariants of Artin-Schreier curves. Fix an odd prime p. Let k be an algebraically closed field with characteristic p. An Artin-Schreier curve is a finite morphism of smooth, projective, connected curves  $X \to \mathbb{P}^1$  with Galois group  $\mathbb{Z}/p\mathbb{Z}$ . Any Artin-Schreier curve can be defined by an equation of the form  $y^p - y = f(x)$  where  $f(x) \in k(x)$  is nonconstant, and k(x) is the function field of  $\mathbb{P}^1$ . There are several numerical invariants associated to an Artin-Schreier curve.

namely the genus, p-rank, and a-number. These invariants can be viewed as properties of the Cartier operator and the space of regular 1-forms,  $H^0(X, \Omega_X^1)$ .

The Cartier operator,  $\mathcal{C}_X: H^0(X,\Omega^1_X) \to H^0(X,\Omega^1_X)$ , is a semi-linear operator with the following properties:

(4) 
$$C_X(f^p\alpha + \beta) = fC_X(\alpha) + C_X(\beta),$$

$$C_X(x^{p-1}dx) = dx,$$

$$C_X(x^ndx) = 0 \text{ if } n \not\equiv -1 \pmod{p}.$$

The genus, g, of X is equal to the dimension of  $H^0(X, \Omega_X^1)$ . The p-rank is equal to the dimension of the image of  $\mathcal{C}_X^g$ . The a-number, written as a(X), is equal to the dimension of the kernel of the Cartier operator.

An Artin-Schreier curve with branch locus S and  $D = \{d_Q\}_{Q \in S}$ , the set of ramification breaks over S, was found to have lower bound L(D) [BC20],

$$L(D) = \max_{1 \le j \le p-1} \left( \sum_{Q \in S} \sum_{i=j}^{p-1} \left( \left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p} - \left(1 - \frac{1}{p}\right) \frac{jd_Q}{p} \right\rfloor \right) \right).$$

We will only examine curves branched at one point. So in this case  $D = \{d\}$  is a singleton set and we denote the lower bound as L(d). We will use the following simplified formula for L(d) in calculations.

**Lemma 2.1.** Let  $\pi: Y \to X$  be a finite morphism of smooth, projective, and geometrically connected curves over a perfect field with odd characteristic p and Galois with group  $\mathbb{Z}/p\mathbb{Z}$  branched at a single point. Let  $d \in \mathbb{N}$  be the ramification break over that point. Then,

$$L(d) = \sum_{i=\frac{p+1}{2}}^{p-1} \left\lfloor \frac{id}{p} \right\rfloor - \left\lfloor \frac{id}{p} - \left(1 - \frac{1}{p}\right) \frac{(p+1)d}{2p} \right\rfloor.$$

*Proof.* [Abn+22, Cor 2.15].

2.2. Computations with Artin-Schreier curves. Let p be an odd prime and let k be an algebraically closed field with characteristic p. For an Artin-Schreier curve X defined by  $y^p - y = f$  such that  $f \in k[x]$  has a pole of order d at infinity, the set  $\mathcal{B}_X$ , defined by

(5) 
$$\mathcal{B}_X := \left\{ y^i x^j dx : 0 \le i \le p - 2, 0 \le j \le \left\lceil \frac{(p - i - 1)d}{p} \right\rceil - 2 \right\},$$

is a basis for  $H^0(X, \Omega_X^1)$  [BC20, Lemma 3.7]. We will examine the specific cases  $d = p^2 - 1$  and  $d = p^2 + 1$ , so the following lemmas will be computationally helpful.

**Lemma 2.2.** For odd prime p and integers  $0 \le i \le p-2$ , the following two equations hold:

$$\left\lceil \frac{(p-i-1)(p^2-1)}{p} \right\rceil - 2 = p^2 - (1+i)p - 2,$$

$$\left\lceil \frac{(p-i-1)(p^2+1)}{p} \right\rceil - 2 = p^2 - (1+i)p - 1.$$

*Proof.* Elementary.

**Lemma 2.3.** If p is an odd prime, then

$$L(p^2+1) = L(p^2-1) = \left(\frac{p-1}{2}\right)\frac{p^2-1}{2}.$$

*Proof.* We only prove  $L(p^2+1)=\left(\frac{p-1}{2}\right)\frac{p^2-1}{2}$ . The proof showing  $L(p^2-1)=\left(\frac{p-1}{2}\right)\frac{p^2-1}{2}$  is similar. Setting  $d=p^2+1$  in Lemma 2.1 and simplifying gives

$$L(p^2+1) = \sum_{i=\frac{p+1}{2}}^{p-1} ip + \left\lfloor \frac{i}{p} \right\rfloor - \left( ip + \frac{-p^2+1}{2} + \left\lfloor -\frac{-2ip + p^2 - 1}{2p^2} \right\rfloor \right).$$

Since  $\frac{p+1}{2} \le i \le p-1$ , the range for the numerator of the floor term will be  $0 < -(-2ip+p^2-1) < 2p^2$ . Hence  $\lfloor \frac{i}{p} \rfloor = \lfloor -\frac{-2ip+p^2-1}{2p^2} \rfloor = 0$ . Simplifying gives

$$L(p^2 - 1) = \sum_{i = \frac{p+1}{2}}^{p-1} \frac{p^2 - 1}{2} = \left(\frac{p-1}{2}\right) \frac{p^2 - 1}{2}.$$

**Lemma 2.4.** If p is an odd prime, then

$$L(p-1) = \frac{(p-1)^2}{4}.$$

*Proof.* The proof of Lemma 2.4 is similar to the proof of Lemma 2.3  $\Box$ 

**Lemma 2.5.** If p is an odd prime and d is a positive integer,

$$L(d + p^2) = L(d) + L(p^2 + 1).$$

*Proof.* Plugging  $d + p^2$  into Lemma 2.1 gives

$$\begin{split} L(d+p^2) &= \sum_{i=\frac{p+1}{2}}^{p-1} \left\lfloor \frac{i(d+p^2)}{p} \right\rfloor - \left\lfloor \frac{i(d+p^2)}{p} - \left(1 - \frac{1}{p}\right) \frac{(p+1)(d+p^2)}{2p} \right\rfloor \\ &= \sum_{i=\frac{p+1}{2}}^{p-1} \left\lfloor \frac{id}{p} \right\rfloor - \left\lfloor \frac{id}{p} - \left(1 - \frac{1}{p}\right) \frac{(p+1)d}{2p} \right\rfloor + \frac{p^2 - 1}{2} \\ &= L(d) + \left(\frac{p-1}{2}\right) \frac{p^2 - 1}{2}. \end{split}$$

Hence, with Lemma 2.3,  $L(d + p^2) = L(d) + L(p^2 + 1)$ 

We will use the lexicographic ordering on  $\mathcal{B}_X$  with y > x. i.e. for differentials  $y^i x^j dx$  and  $y^a x^b dx$ , if  $y^i x^j dx > y^a x^b dx$  then either i > a or i = a and j > b. For any element  $\alpha \in \mathcal{B}_X$ , define  $\mathcal{B}_\alpha$  to be the set of basis vectors smaller than  $\alpha$  under the lexicographic order. We also denote the span of the image of a set under the Cartier operator with  $\operatorname{Span}(\mathcal{C}_X(A))$ .

$$B_{\alpha} := \{ \beta \in \mathcal{B}_X : \beta < \alpha \}$$
  
$$\operatorname{Span}(\mathcal{C}_X(A)) := \operatorname{Span}(\{\mathcal{C}_X(a) : a \in A\}).$$

The defining equation  $y^p - y = f$  with  $f \in k[x]$  can be used to rewrite differentials in  $H^0(X, \Omega_X^1)$ . For  $y^m x^n dx \in H^0(X, \Omega_X^1)$ , the differential can be rewritten  $y^m x^n dx = (y^p - f)^m x^n dx$ . This equivalence is useful in calculations with the Cartier Operator since the rewritten form works well with the properties in (4) to

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determine what the Cartier of any differential is. The following theorems concern cases where f is a binomial and thus the following fact will be useful.

**Lemma 2.6.** Fix odd prime p, distinct positive integers  $d, e \not\equiv 0 \pmod{p}$  and nonzero  $a, b \in k$ . Let  $X \to \mathbb{P}^1$  be the Artin-Schreier cover defined by the equation  $y^p - y = ax^d + bx^e$ . A differential  $y^m x^n dx \in H^0(X, \Omega^1_X)$  can be expressed as

$$y^{m}x^{n}dx = \sum_{i=0}^{m} \sum_{j=0}^{m-i} {m \choose i} {m-i \choose j} a^{m-i-j}b^{j}y^{pi}x^{(m-i-j)d+je+n}dx.$$

*Proof.* Since  $y^m x^n dx \in H^0(X, \Omega_X^1)$ , we can use the equation  $y^p - y = ax^d + bx^e$  to substitute  $y^m x^n dx = (y^p - (ax^d + bx^e))^m x^n dx$ . The binomial theorem then gives the desired formula.

2.3. Exhibiting curves with minimal a-number. In this section, we exhibit curves for all odd primes p with some fixed ramification break d such that their a-number is equal to the lower bound L(d). These curves will be the base curves that combine to get an infinite family of curves in section 3.2.

**Proposition 2.7.** Let p be an odd prime. If  $f(x) \in k[x]$  has degree p-1, then the Artin-Schreier cover  $X \to \mathbb{P}^1$  defined by  $y^p - y = f(x)$  has a-number equal to L(p-1).

*Proof.* By Lemma 2.4, we get the lower bound

$$L(p-1) = \frac{(p-1)^2}{4}.$$

This is the a-number of any Artin-Schreier cover defined by  $y^p - y = f(x)$  such that  $f(x) \in k[x]$  has degree p-1. [FP12, Theorem 3.9]

**Lemma 2.8.** Let p be an odd prime and  $d=p^2+1$ . Let  $X\to \mathbb{P}^1$  be the Artin-Schreier cover defined by the equation  $y^p-y=-x^d-x^{d/2+p}$ . Let  $0\le k<\frac{p-1}{2}$  and  $0\le l\le \frac{d(\frac{p-2}{2}-k)+1}{p}$ . Then there is  $y^mx^ndx\in \mathcal{B}_X$  with  $m<\frac{p-1}{2}$  such that the largest term of  $\mathcal{C}_X(y^mx^ndx)$  is  $y^kx^ldx$ .

*Proof.* Let a be the largest integer such that  $a(d-2p) \leq lp+(p-1)$ . Set m=a+1+k and n=lp+(p-1)-b, where b is the largest element of  $\{a(d-2p), \frac{2a+1}{2}d-p, (a+1)d\}$  such that  $n\geq 0$ . Check using (5) that  $y^mx^ndx\in\mathcal{B}_X$ . Using Lemma 2.6, we write

(6) 
$$y^m x^n dx = \sum_{i=0}^m \sum_{j=0}^{m-i} {m \choose i} {m-i \choose j} y^{pi} x^{(m-i)d-j(d/2-p)+n} dx.$$

Recall from (4) that  $C_X(y^{pa}x^bdx) \neq 0$  if and only if  $b \equiv -1 \pmod p$ . Finding the largest term of  $\mathcal{C}_X(y^mx^ndx)$ , then, is equivalent to finding the largest  $0 \leq i \leq m$  such that there is  $(m-i)d-j(\frac{d}{2}-p)+n \equiv -1 \pmod p$  with j maximized under these conditions. Substituting m and n then gives  $(a+1+k-i)d-j(\frac{d}{2}-p)-b+lp+(p-1) \equiv \pmod p$ , so it suffices to find the largest i such that there is j with  $(a+1+k-i)d-j(\frac{d}{2}-p)-b=0$ . This occurs with i=k and  $j=\frac{(a+1)d}{d/2-p}$ . With this choice of i and j, we get the term  $y^{pk}x^{lp+(p-1)}dx$  and  $\mathcal{C}_X(y^{pk}x^{lp+(p-1)}dx)=y^kx^ldx$ .

**Lemma 2.9.** Let p be an odd prime and  $d = p^2 + 1$ . Let  $X \to \mathbb{P}^1$  be the Artin-Schreier cover defined by the equation  $y^p - y = -x^d - x^{d/2+p}$ . For all  $\alpha = y^m x^n dx \in \mathcal{B}_X$  with  $m \ge \frac{p-1}{2}$ ,  $C_X(\alpha) \notin \operatorname{Span}(\mathcal{C}_X(\mathcal{B}_\alpha))$ .

Proof. Fix  $\alpha = y^m x^n dx$  with  $m \geq \frac{p-1}{2}$ . Using Lemma 2.6,  $\alpha$  can be rewritten using to the formula in (6). For a fixed  $0 \leq i \leq m$ , any  $0 \leq j \leq m-i$  gives  $(m-i)d-j(\frac{d}{2}-p)+n$  a distinct value. Hence each summand given by a pair (i,j) has a unique combination of exponents and does not combine with any other summand. Every pair (i,j) such that  $m-\frac{p-1}{2} \leq i \leq m$  and  $0 \leq j \leq 1$  gives  $(m-i)d-j(\frac{d}{2}-p)+n$  a distinct class  $\pmod{p}$ . Hence there is always a summand from (6) with exponent of the x term congruent to  $-1 \pmod{p}$ . Hence every basis element  $y^m x^n dx$  with  $m \geq \frac{p-1}{2}$  has  $\mathcal{C}_X(y^m x^n dx) \neq 0$ . Choose  $0 \leq i \leq m$  to be the largest integer such that there exists  $0 \leq j \leq m-i$ 

Choose  $0 \le i \le m$  to be the largest integer such that there exists  $0 \le j \le m-i$  with  $(m-i)d-j(\frac{d}{2}-p)+n$ . Choose j to be as large as possible. This exists since  $\mathcal{C}_X(\alpha)$  is nonzero. Let  $\beta \in \mathcal{B}_\alpha$  with  $\beta = y^k x^l dx$ , so either k < m or k = m and l < n. Assume by way of contradiction that the coefficient of  $y^{pi} x^{(m-i)d-j(\frac{d}{2}-p)+n} dx$  in the expanded form of  $\beta$  is nonzero. This implies that  $0 \le i \le k$  and there exists  $0 \le g \le k-i$  and  $(k-i)d-g(\frac{d}{2}-p)+l=(m-i)d-j(\frac{d}{2}-p)+n$ . Solving for j gives  $j(\frac{d}{2}-p)=d(m-k)+g(\frac{d}{2}-p)+n-l$ . If m=k then using (5) we get bounds

$$0 \le n, l \le p^2 - \left(1 + \frac{p-1}{2}\right)p - 2 = \frac{p^2 - p}{2} - 2,$$

which implies  $n-l \leq \frac{p^2-1}{2}-2$ . However since j is an integer,  $(\frac{d}{2}-p)$  must divide n-l. This implies l=n, a contradiction. Now assume m>k. From (5), we get  $l \leq p^2-(1+k)p-2$ . So for any m and k we get  $j \geq (d(m-k)+n-l)/(\frac{d}{2}-p)>1$ . Now observe that  $(m-(i+1))d-(j-2)(\frac{d}{2}-p)+n\equiv -1\pmod{p}$  and  $0\leq i+1\leq m$  and  $0\leq j-2\leq m-(i+1)$ . Hence the maximality of i is violated, since (i+1) satisfies the necessary condition. By way of contradiction, the coefficient of  $y^{pi}x^{(m-i)d-j(\frac{d}{2}-p)+n}dx$  is 0 in the expanded form for any differential in  $\mathcal{B}_{\alpha}$ . Hence for all  $\alpha=y^mx^ndx\in\mathcal{B}_X$  with  $m\geq \frac{p-1}{2}$ ,  $\mathcal{C}_X(\alpha)\not\in \mathrm{Span}(\mathcal{C}_X(\mathcal{B}_{\alpha}))$ .

**Proposition 2.10.** Let p be an odd prime. If  $d = p^2 + 1$ , the Artin-Schreier cover  $X \to \mathbb{P}^1$  defined by  $y^p - y = -x^d - x^{d/2+p}$  has a-number equal to L(d).

*Proof.* An upperbound for the a-number can be found by computing a lower bound for the dimension of the image of the Cartier operator by rank-nullity. Fix  $\alpha = y^{\frac{p-1}{2}}dx$ . By Lemma 2.11, there are  $\sum_{k=0}^{(p-3)/2} \left\lfloor \frac{d(\frac{p-2}{2}-k)+p+1}{p} \right\rfloor$  many  $\omega \in \mathcal{B}_X$  such that  $\omega < \alpha$  and  $\mathcal{C}_X(\omega)$  have distinct largest terms. Thus the rank of  $\mathcal{C}_X(B_\alpha)$  is at least  $\sum_{k=0}^{(p-3)/2} \left\lfloor \frac{d(\frac{p-2}{2}-k)+p+1}{p} \right\rfloor$ . Using Lemma 2.12, we get the lower bound  $\dim(\operatorname{Span}(\mathcal{C}_X(\mathcal{B}_X))) \geq \dim(\operatorname{Span}(\mathcal{C}_X(\mathcal{B}_\alpha))) + |\{\omega \in \mathcal{B}_X : \omega \geq y^{\frac{p-1}{2}}dx\}|$ . Denote this lower bound as R(X):

$$R(X) = \sum_{k=0}^{(p-3)/2} \left\lfloor \frac{d(\frac{p-2}{2} - k) + p + 1}{p} \right\rfloor + \sum_{k=(p-1)/2}^{p-2} (p^2 - (1+k)p).$$

An upperbound for the a-number can be found by subtracting this lower bound from the genus. From (5) and Lemma 2.2, note that the genus can be written as

$$g_X = \sum_{k=0}^{p-2} (p^2 - (1+k)p).$$

$$a(X) \le \sum_{k=0}^{p-2} (p^2 - (1+k)p) - R(X)$$

$$\le \sum_{k=0}^{(p-3)/2} (p^2 - (1+k)p) - \left\lfloor \frac{d(\frac{p-2}{2} - k) + p + 1}{p} \right\rfloor$$

$$\le \sum_{k=(p+1)/2}^{p-1} \left( \frac{-p^2 + 3}{2} + \left\lfloor \frac{-k+1}{p} \right\rfloor \right)$$

Since  $(p+1)/2 \le k \le p-1$ , we get that  $\lfloor \frac{-k+1}{p} \rfloor = -1$ . Thus simplifying gives

$$a(X) \le \sum_{k=\frac{p+1}{2}}^{p-1} \frac{p^2 - 1}{2} = \left(\frac{p-1}{2}\right) \frac{p^2 - 1}{2} = L(d).$$

Since  $a(X) \ge L(d)$  [BC20, Theorem 1.1], we get a(X) = L(d).

The case for  $d = p^2 - 1$  is proven with a similar approach to the case for  $d = p^2 + 1$ .

**Lemma 2.11.** Let p be an odd prime and  $d = p^2 - 1$ . Let  $X \to \mathbb{P}^1$  be the Artin-Schreier cover defined by the equation  $y^p - y = -x^d - x^{d/2}$ . Let  $0 \le k < \frac{p-1}{2}$  and  $0 \le l \le \frac{d(\frac{p-2}{2}-k)-p+1}{p}$ . Then there is  $y^m x^n dx \in \mathcal{B}_X$  with  $m < \frac{p-1}{2}$  such that the largest term of  $\mathcal{C}_X(y^m x^n dx)$  is  $y^k x^l dx$ .

*Proof.* The proof of Lemma 2.11 is similar to the proof of Lemma 2.8.  $\Box$ 

**Lemma 2.12.** Let p be an odd prime and  $d := p^2 - 1$ . Let  $X \to \mathbb{P}^1$  be the Artin-Schreier cover defined by the equation  $y^p - y = -x^d - x^{d/2}$ . If  $\alpha = y^m x^n dx \in \mathcal{B}_X$  has  $m \ge \frac{p-1}{2}$ , then  $\mathcal{C}_X(\alpha) \notin \operatorname{Span}(\mathcal{C}_X(\mathcal{B}_\alpha))$ .

*Proof.* The proof of Lemma 2.12 is similar to the proof of Lemma 2.9.  $\Box$ 

**Proposition 2.13.** Let p be an odd prime. If  $d = p^2 - 1$ , the Artin-Schreier cover  $X \to \mathbb{P}^1$  defined by  $y^p - y = -x^d - x^{d/2}$  has a-number equal to L(d).

*Proof.* The proof of Proposition 2.13 is similar to the proof of Proposition 2.10.  $\Box$ 

**Example 2.14.** The Artin-Schreier covers  $X \to \mathbb{P}^1$  defined by  $y^{11} - y = -x^{122} - x^{72}$  and  $Y \to \mathbb{P}^1$  defined by  $y^{11} = -x^{120} - x^{60}$  both have the *a*-number a(X) = a(Y) = 300, which is equal to the lower bound, L(120) = L(122) = 300.

**Remark 2.15.** Experimentally, it seems that an Artin-Schreier curves defined by  $y^p - y = -x^{np^2-1} - x^{\frac{np^2+(n-1)p-1}{2}}$  for some  $n \in \mathbb{N}$  have a-number equal to the lower bound  $L(np^2-1)$ . A technique similar to the ones presented in Propositions 2.13 and 2.10 might be able to show this, but we present a conceptual approach to generating an infinite family with a-number equal to the lower bound in section 3.

#### 3. An Infinite Family of Curves via Patching

3.1. **Notation and background.** Notation 3.1 will be used for the entirety of this section.

Notation 3.1. Fix an odd prime p and let k be an algebraically closed field with characteristic p. Let R=k[[t]] be the ring of formal power series over k and K=k((t)) be the field of fractions of R. Set  $U=\operatorname{Spec}(k[[u]])$  and  $V=\operatorname{Spec}(k[[v]])$ . For a positive integer e, set  $\Omega^e_{uv}=k[[u,v,t]]/(uv-t^e)$  and let  $S^e_{uv}=\operatorname{Spec}(\Omega^e_{uv})$ . A relative curve (or R-curve) is a flat finitely presented morphism  $f:X\to\operatorname{Spec}(R)$  of relative dimension 1. Let  $P^e_R$  be an R-curve whose generic fibre is isomorphic to  $\mathbb{P}^1_k$  and whose special fibre consists of projective lines  $P_u$  and  $P_v$  meeting transversally at a point p0 where p1 where p2 and p3 and p3.

An Artin-Schreier curve can be defined by an equation of the form  $y^p - y = f(x)$  where  $f(x) \in k(x)$  is nonconstant, and k(x) is the function field of  $\mathbb{P}^1$ . We often define a smooth projective curve and its function field by describing its affine parts. A morphism of curves is a *cover* if it is finite and generically separable. For a point u on X, the  $germ\ \hat{X}_u$  of the curve X at u is the spectrum of the complete local ring of functions of X at u. Let  $\phi: Y \to X$  be a  $\mathbb{Z}/p\mathbb{Z}$ -cover of curves branched at the point u. Suppose  $\eta \in \phi^{-1}(u)$ , then the ramification break of  $\phi$  at  $\eta$  is the integer  $d = \operatorname{val}(q(\pi_{\eta}) - \pi_{\eta})$ , where q is a generator of  $\mathbb{Z}/p\mathbb{Z}$  and  $\pi_{\eta}$  is a uniformizer of Y at  $\eta$ .

This section closely follows the ideas presented in [Pri03]. We use the technique of formal patching to glue together curves with known ramification breaks and anumbers to form a curve with a larger ramification break and known anumber. This technique, pioneered by Harbater and Stevenson [HS99], has been used in a similar way to exhibit curves with specific newton polygons [BP20] and large conductors [Pri03].

**Definition 3.2.** A thickening problem of covers for  $(X, \mathbb{S})$  consists of the following:

- (1) A cover  $f: Y \to X$  of geometrically connected reduced projective k-curves,
- (2) For each  $s \in \mathbb{S}$ , a Noetharian normal complete local domain  $R_s$  with  $R \subseteq R_s$  such that t is in the maximal ideal of  $R_s$  and a finite generically separable  $R_s$ -algebra  $A_s$ ,
- (3) For each  $s \in \mathbb{S}$ , a pair of k-algebra isomorphisms  $F_s : R_s/(t) \to \hat{O}_{X,s}$  and  $E_s : A_s/(t) \to \hat{O}_{Y,s}$  compatible with the inclusion morphisms.

**Definition 3.3.** A thickening of X is a projective normal R-curve  $X^*$  such that  $X_k^* \simeq X$ . We call a thickening problem G-Galois if f and the inclusion  $R \subseteq A_s$  are G-Galois and  $F_s$  is compatible with the G-Galois action for all  $s \in S$ . We call it relative if the problem has a thickening  $X^*$  of X that is a trivial deformation of X away from S such that the pullback of  $X^*$  over the complete local ring at a point  $s \in S$  is isomorphic to  $R_s$ .

**Definition 3.4.** A solution to a thickening problem of covers is a cover  $f^*: Y^* \to X^*$  of projective normal R-curves, where  $X^*$  is a thickening of X, whose closed fibre is isomorphic to f, whose pullback to the formal completion of  $X^*$  along  $X' = X - \mathbb{S}$  is a trivial deformation of the restriction of f over X', and whose pullback over the complete local ring at a point  $s \in \mathbb{S}$  is isomorphic to  $R_s \subseteq A_s$  with all isomorphisms being compatible.

**Theorem 3.5.** Every G-Galois thickening problem for covers has a G-Galois solution. If the thickening problem is relative then the solution is unique.

Proof. [HS99, Theorem 4].  $\Box$ 

## 3.2. Patching curves with a specific congruence.

**Proposition 3.6.** Following Notation 3.1, let  $\phi_1: X \to U$ ,  $\phi_2: Y \to V$  be  $\mathbb{Z}/p\mathbb{Z}$ -Galois covers of normal connected germs of curves with ramification breaks  $j_1$  and  $j_2$  such that  $j_1 + j_2 \equiv 0 \pmod{p}$ . Let  $e = j_1 + j_2$ . Then there exists  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $\phi_R: W_R \to S_{uv}^e$  of irreducible germs of R-curves with the properties:

- (1) The cover  $\phi_R$  has one branch point,  $b_R$ .
- (2) The pullbacks of the special fibre of  $\phi_R$  to U and V are isomorphic to  $\phi_1$  and  $\phi_2$  away from  $b_R$ .
- (3) The generic fibre  $\phi_K : W_K \to S^e_{uv,K}$  of  $\phi_R$  is a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover of normal irreducible germs of curves whose branch locus is  $b_K := b_R \times_R K$ .
- (4) The cover  $\phi_K$  has ramification break e-1 over the branch point.

*Proof.* We adapt the proof from [Pri03, Proposition 2.3.4]. There exists automorphisms  $A_u$  and  $A_v$  of fixing the closed points of U and V such that  $A_u^*\phi_1$  and  $A_v^*\phi_2$  can be given by the equations  $x^p-x=u^{j_1}$  and  $y^p-y=v^{j_2}$  respectively (see [Art67, §10.4]). Denote the transformed covers as  $\phi_1'$  and  $\phi_2'$  respectively. We may suppose the Galois action of  $\phi_1'$  maps  $x\mapsto x+1$  and the Galois action of  $\phi_2'$  maps  $y\mapsto y+a$  for some  $a\in\mathbb{F}_p^\times$ .

Let  $\phi_R: W_R \to S_{uv}^e$  be the cover defined by the equation

(7) 
$$z^p - z = (u^{j_1} + av^{j_2} + d_0t)$$

for some  $d_0 \in \Omega_{uv}^e$ . Note that after reducing  $\pmod{(v,t)}$ , the equation becomes  $z^p-z=u^{j_1}$ , which is identical to the equations defining  $\phi_1'$ . So the normalization of the reduction is isomorphic to  $\phi_1'$ . Likewise the normalization of the reduction  $\pmod{(u,t)}$  is isomorphic to  $\phi_2'$ . The Galois action of  $\phi_R$  sends  $z\mapsto z+1$ , which reduces to the Galois action on  $\phi_1'$  and  $\phi_2'$  respectively. Hence the pullbacks of the normalization of the special fibre of  $\phi_R$  to U and V are isomorphic to  $\phi_1'$  and  $\phi_2'$  as covers away from the branch point. Set

$$d_0 = \frac{(u + a_1 t^{j_2})^e - u^e - a t^{j_2 e}}{u^{j_2} t}$$

with  $a_1 = \sqrt[e]{a}$ . Note that  $d_0 \in \Omega^e_{uv}$ .

Plugging  $d_0$  into (7) and simplifying gives

(8) 
$$z^p - z = u^{-j_2} (u + a_1 t^{j_2})^e.$$

Note that u has no zero or pole in  $\Omega^e_{uv,K}$ . Thus  $\phi_K$  has a unique branch point given by  $u=-a_1t^{j_2}$  and  $v=-t^{j_1}/a_1$ . This K-point specializes to the branch point  $(u,v)=(\infty,\infty)$  of  $\phi_k$ . Thus  $\phi_R$  is branched at only one R-point with this  $d_0$ . Note that (8) is irreducible in R since the right hand side is not of the form  $\alpha^p-\alpha$ . Hence  $W_R$  and  $W_K$  are irreducible. From equation (8),  $u^{-j_2}(u+a_1t^{j_2})^e$  is an eth power of a uniformizer. Hence the largest that the ramification break can be is e. By Proposition 3.7, the lower ramification break must be greater than e-1. Since p cannot divide the ramification break, the lower ramification break on the generic fibre must be e-1.

**Proposition 3.7.** Given covers  $\phi_1$  and  $\phi_2$  with ramification breaks  $j_1$  and  $j_2$ , the ramification break on the generic fibre of a flat deformation of these two covers,  $\phi$ , will have ramification break  $j \geq j_1 + j_2 - 1$ .

*Proof.* For an Artin-Schreier curve, the genus and ramification break are related by the formula  $g = \frac{p-1}{2}(j-1)$ , a corollary of the Riemann-Roch theorem. The genus of the special fiber of the deformation is  $g_{\phi} = \frac{p-1}{2}((j_1-1)+(j_2-1))+\alpha$ , where  $\alpha$  is some nonnegative integer, contributed by the singularity. Note that the genus is constant in a flat family of curves (see Corollary 9.9.10 in [Har97]). From the above equality we then get the inequality  $j_{\phi} - 1 \ge (j_1 - 1) + (j_2 - 1)$ , which gives the fact.

**Proposition 3.8.** Let  $\phi_1: X_1 \to \mathbb{P}^1$ ,  $\phi_2: X_2 \to \mathbb{P}^1$  be  $\mathbb{Z}/p\mathbb{Z}$ -Galois covers branched at only one point with ramification breaks  $j_1$  and  $j_2$  such that  $j_1 + j_2 \equiv 0 \pmod{p}$ . Then there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover of curves  $\phi:Y\to\mathbb{P}^1_k$  with the following properties:

- (1)  $\phi$  has exactly one branch point.
- (2) There is a single ramification point of  $\phi$  above the branch point and its ramification break is  $e = j_1 + j_2 - 1$ .
- (3) Y is smooth and connected.

*Proof.* We adapt the proof of Theorem 2.3.7 from [Pri03]. From Notation 3.1, label  $X^* = P_R^e$ ,  $\mathbb{S} = \{b\}$ , and  $G = \mathbb{Z}/p\mathbb{Z}$ . There exists ramified points  $\eta_1 \in \phi_1^{-1}(u)$  and  $\eta_2 \in \phi_2^{-1}(v)$ . Consider the  $\mathbb{Z}/p\mathbb{Z}$ -Galois covers of germs of curves  $\hat{\phi}_1: \hat{X}_{\eta_1} \to U$ and  $\hat{\phi}_2: \hat{Y}_{\eta_2} \to V$ . Apply Proposition 3.6 to  $\hat{\phi}_1$  and  $\hat{\phi}_2$  to get  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $\hat{\phi}_R:W_R\to S^e_{uv}$  which is ramified at one point with ramification break e. Note that  $\hat{\phi}_R$  corresponds to an inclusion of rings. Consider the cover  $\phi_k$  of the special fibre of  $P_R^e$  which restricts to  $\phi_1$  over  $P_u$  and to  $\phi_2$  over  $P_v$ . Form a relative G-Galois thickening problem using  $\phi_k$  and  $\hat{\phi}_R$  and the isomorphisms from the pullbacks of the special fibre of  $\hat{\phi}_R$  to  $\phi_1$  and  $\phi_2$ . By Theorem 3.5, this problem has a solution, a  $\mathbb{Z}/p\mathbb{Z}$  cover  $\phi_R: Y_R \to P_R^e$ . The closed fibre of  $\phi_R$  is isomorphic to  $\hat{\phi}_R$  and  $\phi_R$ is isomorphic to the trivial deformation away from the closed point. Hence,  $Y_K$  is smooth because  $W_K$  and the trivial deformation are smooth.

Choose a subring  $O \subseteq R$  finitely generated over k with  $O \neq k$ , such that  $\phi_R$  can be defined over  $\operatorname{Spec}(O)$ . Note that such a subring exists because  $\phi_R$  is defined using only finitely many elements of R. Since k is algebraically closed, there are infinitely many k-points of Spec(O). Let L be the set of k-points, x, of Spec(O) such that  $\phi_x$ is not a  $\mathbb{Z}/p\mathbb{Z}$  cover of smooth connected curves. Note that the closure  $\overline{L} \neq \operatorname{Spec}(O)$ since  $Y_K$  is smooth and irreducible [Sta24, Lemma 055G]. Let  $\alpha \in \text{Spec}(O) \setminus L$  be a k-point and  $\phi := \phi_{\alpha} : Y \to X_k$  be the fibre over  $\alpha$ . The map  $\phi$  inherits properties 1 and 2 from  $\phi_R$  and is smooth and connected by construction.

**Proposition 3.9.** In the notation of Proposition 3.8, the special fiber of  $\phi_R$  is a cover of stable curves where the a-number of the cover is  $a(X_1) + a(X_2)$ .

*Proof.* Away from the reduction of the branch point, in the special fiber the cover  $Y_s$  is the disjoint union of  $X_1$  and  $X_2$  with the ramification points removed. The genus of  $Y_s$  is the sum of the genera of  $X_1$  and  $X_2$  with a contribution from the singularity caused by glueing  $X_1$  and  $X_2$ . This is the same setup as Proposition 3.7.

However, we know that

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$$g(X_1) + g(X_2) = \frac{p-1}{2}((j_1 - 1) + (j_2 - 1)) = \frac{p-1}{2}(j_1 + j_2 - 1) = g(Y_R)$$

so the singularity makes no contribution. Thus the singularity is an ordinary double point, so the generalized Jacobian is the product of the Jacobians of  $X_1$  and  $X_2$  and has no toric part. In particular, the a-number is  $a(X_1) + a(X_2)$ .

**Proposition 3.10.** Let S be irreducible with generic point  $\eta$ , and  $\pi: X \to S$  be a smooth family of projective curves over S. Then for any point  $s \in S$ ,  $a(X_{\eta}) \leq a(X_s)$ .

*Proof.* The Cartier operator can be viewed as a map of  $\mathcal{O}_{X/S}$ -modules  $F_*\Omega^1_{X/S} \to \Omega^1_{X/S}$ , whose kernel is a coherent sheaf  $\mathscr{F}$  such that for  $s \in S$ 

$$a(X_s) = \dim_{k(s)} H^0(X_s, \mathscr{F}_s).$$

Note that  $\mathscr{F}$  is flat over X and hence over S as it is a kernel of a map of locally free sheaves. By the semicontinuity theorem [Har97, Theorem III.12.8], the a-number is an upper semi-continuous function on S, i.e.  $a(X_n) \leq a(X_s)$ .

**Proposition 3.11.** For a prime p, if there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  with ramification break  $d \equiv -1 \pmod{p}$  with a-number equal to L(d), then for any  $e \geq d$  with  $e \equiv d \pmod{p^2}$  there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  branched at one point with ramification break e and a-number equal to L(e).

Proof. We prove Proposition 3.11 by induction. By assumption there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  with ramification break  $d \equiv -1 \pmod{p}$  with a-number equal to L(d). Assume  $X_1 \to \mathbb{P}^1$  is a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover of curves with ramification break  $e \geq d$  with  $e \equiv d \pmod{p^2}$  and a-number  $a(X_1) = L(e)$ . Let  $X_2 \to \mathbb{P}^1$  be the Artin-Schreier cover defined by  $y^p - y = -x^{p^2+1} - x^{\frac{p^2+1}{2}+p}$ . By Proposition 2.10,  $X_2$  has a-number  $a(X_2) = L(p^2+1)$ . Using Proposition 3.8, there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  with ramification break  $e + p^2$  and, by Proposition 3.9 and 3.10, a-number  $a(X) = a(X_1) + a(X_2)$ . By Lemma 2.5,  $L(e+p^2) = L(e) + L(p^2+1) = a(X_1) + a(X_2)$ . Hence the a-number of X is equal to the lowerbound,  $a(X) = L(e+p^2)$ . Hence, by induction, for any integer  $e \geq d$  with  $e \equiv d \pmod{p^2}$  there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  with ramification break e and e-number equal to e-number equal equal to e-number equal equa

**Theorem 3.12.** For any odd prime p and any positive  $d \equiv -1 \pmod{p^2}$  there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  branched at one point with ramification break d and a-number equal to L(d).

*Proof.* By Proposition 2.13, the cover defined by  $y^p - y = -x^{p^2-1} - x^{\frac{p^2-1}{2}}$  has a-number equal to  $L(p^2-1)$ . By Proposition 3.11, for any  $d > p^2 - 1$  with  $d \equiv -1 \pmod{p^2}$ , there exists  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  with ramification break d and a-number equal to L(d).

**Theorem 3.13.** For any odd prime p and any positive  $d \equiv p-1 \pmod{p^2}$  there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  branched at one point with ramification break d and a-number equal to L(d).

*Proof.* Choose  $f \in k[x]$  such that  $\deg(f) = p - 1$ . By Proposition 2.7, the cover defined by  $y^p - y = f$  has a-number equal to L(p-1). By Proposition 3.11, for

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any d > p-1 with  $d \equiv p-1 \pmod{p^2}$ , there exists  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  with ramification break d and a-number equal to L(d).

**Theorem 3.14.** For any odd prime  $p \leq 23$  and any positive  $d \equiv -1 \pmod{p}$ , there exists a  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  branched at one point with ramification break d and a-number equal to L(d).

Proof. Let d be a positive integer with  $d \equiv -1 \pmod{p}$ . Let  $1 \leq k \leq p$  such that  $kp-1 \equiv d \pmod{p^2}$ . By explicit computation, we found a polynomial f with degree kp-1 such that the cover defined by  $y^p-y=f(x)$  has a-number equal to L(kp-1). A list of the polynomials and the MAGMA code are attached in the arXiv submission. By Proposition 3.11, since d > kp-1 and  $d \equiv kp-1 \pmod{p^2}$ , there exists  $\mathbb{Z}/p\mathbb{Z}$ -Galois cover  $X \to \mathbb{P}^1$  with ramification break d and a-number equal to L(d).

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