

Unbiased Approximations for Stationary Distributions of McKean-Vlasov SDEs

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Abstract

We consider the development of unbiased estimators, to approximate the stationary distribution of McKean-Vlasov stochastic differential equations (MVSDEs). These are an important class of processes, which frequently appear in applications such as mathematical finance, biology and opinion dynamics. Typically the stationary distribution is unknown and indeed one cannot simulate such processes exactly. As a result one commonly requires a time-discretization scheme which results in a discretization bias and a bias from not being able to simulate the associated stationary distribution. To overcome this bias, we present a new unbiased estimator taking motivation from the literature on unbiased Monte Carlo. We prove the unbiasedness of our estimator, under assumptions. In order to prove this we require developing ergodicity results of various discrete time processes, through an appropriate discretization scheme, towards the invariant measure. Numerous numerical experiments are provided, on a range of MVSDEs, to demonstrate the effectiveness of our unbiased estimator. Such examples include the Currie-Weiss model, a 3D neuroscience model and a parameter estimation problem.

Key words: McKean-Vlasov SDE, Unbiased Approximation, Stationary Distributions, Euler-Maruyama discretization

1 Introduction

The focus of this article is on McKean-Vlasov stochastic differential equations (SDEs), which are SDEs whose coefficients depend not only on the state of the process but also on its distribution. In particular, we focus on the following McKean-Vlasov [22] stochastic differential equation (MVSDE), with a fixed initial condition $X_0 = x_0 \in \mathbb{R}^d$, given as

$$dX_t = a(X_t, \bar{\xi}_1(X_t, \mu_t)) dt + b(X_t, \bar{\xi}_2(X_t, \mu_t)) dW_t, \quad (1.1)$$

where for $j \in \{1, 2\}$,

$$\bar{\xi}_j(X_t, \mu_t) = \int_{\mathbb{R}^d} \xi_j(X_t, x) \mu_t(dx),$$

where $\{W_t\}_{t \geq 0}$ is a standard d -dimensional Brownian motion, for $d \geq 1$. Furthermore, $\xi_j : \mathbb{R}^{2d} \rightarrow \mathbb{R}$, $a : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ is the associated drift term, $b : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ is the diffusion coefficient and finally μ_t is the law of the diffusion process X_t . In contrast to classical SDEs, the distribution of the MVSDE (1.1) solves a nonlinear Fokker-Planck equation, which is a partial differential equation. McKean-Vlasov processes have proven to be very useful for inference problems, in a wide range of useful applications. These applications include, but are not limited to, stochastic filtering, financial mathematics, opinion dynamics and flocking processes. Well-known examples of MVSDEs include the Kalman-Bucy filter, the stochastic Cucker-Smale flocking dynamics and the stochastic Hegselmann-Krause model [6, 12, 13, 24]. Normally the solution of a MVSDE is approximated through an interacting particle system, where it is well-known that empirical law of the particle system converges to the law of the MVSDE, in the infinite particle limit.

We are interested in the simulation of the stationary distribution, or invariant measure, of the SDE, call it π , which is assumed to exist, but is typically unknown. There has been a limited literature on providing methods to approximation π . This includes both developing analysis (with particular assumptions stated) [9, 11], and applying Monte Carlo methods given motivation to a Bayesian framework through important sampling [4]. However, despite these recent developments, there is still an issue related to these approximations methods, in that there remains a time-discretization bias, through numerically solving (1.1). Therefore, we motivate this work through the question of whether one can attain an unbiased scheme, to approximate invariant measures. In the context of Monte Carlo methods, unbiased estimation has been a recent hot topic largely due to important works by Rhee and Glynn. Specifically the authors produced an unbiased estimator related to SDEs through a randomized multilevel telescoping sum identity. We refer the reader to these various works [14, 23, 26].

The notion of randomized multilevel Monte Carlo (MLMC) methods is based upon a hierarchy of time-discretized SDEs becoming increasingly more precise in terms of discretization (i.e. an adaptive step-size). Then given a probability distribution on the amount of time-discretization it is possible to obtain unbiased and sometimes finite variance, finite expected cost estimators associated to ordinary SDEs [2, 15, 17, 26] and using non-randomized MLMC for some classes of MVSDE problems; [1, 4, 5]. In terms of simulating invariant measures associated to regular SDEs several works have appeared including [8, 27]. The main idea of this article is to appropriately adapt and analyze methodology from [1, 17] in the context of producing truly unbiased estimation from the stationary distribution of a MVSDE model.

1.1 Contributions

The main contributions of this article are provided below:

- We develop a first approximation scheme for unbiased estimates of the stationary distribution of MVSDEs. This is based on the notion of randomization of MLMC, which utilizes two Euler-Maruyama discretizations of MVSDEs.
- We prove a number of ergodicity results related to the Euler-Maruyama discretizations of the MVSDEs. In particular we demonstrate exponential ergodicity of both the discretized equations, and the unbiased estimator. To the best of the authors knowledge, these are the first set of results in the literature for the discretized setting.
- We provide and present a main theorem which demonstrates that, our estimator is unbiased. This result relies heavily on the previous ergodicity results.
- Numerical experiments are provided, to demonstrate the robustness of the proposed unbiased approximation scheme. We test this on a range of MVSDEs motivated through different applications. These include a parameter estimation problem, an Ornstein-Uhlenbeck process, the Currie-Weiss Model and a more challenging 3D neuron model.

The outline of this paper is as follows. In Section 2 we present our unbiased methodology. Section 3 houses our mathematical results with some discussion on how some simulation parameters of the method can be chosen. Numerical experiments will be provided in Section 4, to verify our theoretical findings. We will test our methodology on a range of MVSDEs which include a toy Ornstein-Uhlenbeck process, 3D neuron model (motivated from neuroscience) and the Currie-Weiss Model. We conclude with some final remarks in Section 5. Finally we defer the proofs of most of our results to the appendix.

2 Method

Our methodology for unbiased estimation is now described. Below, we will use the convention that the time-discretized dynamics of the SDE (1.1) is iterated over unit time. This is simply a convention and any $\mathcal{O}(1)$ time could be used. The reason for such iteration shall be explained later on.

2.1 Discretization for MVSDE

We denote by $P_{\mu_{t-1},t}(x_{t-1}, dx_t)$ the conditional law of X_t (as given in (1.1)) given \mathcal{F}_{t-1} (the natural filtration of the process), for $t \geq 1$; that is, the transition kernel over unit time. In most cases of practical interest, μ_t and the dynamics $P_{\mu_{t-1},t}(x_{t-1}, dx_t)$ are difficult to work with. For instance the transition kernel cannot be simulated in many problems. We introduce a time-discretization over a regular grid of spacing $\Delta_l = 2^{-l}$, $l \in \mathbb{N}_0$. We will use the Euler-Maruyama method associated to (1.1) and denote the law at any time $t \in \{0, \Delta_l, 2\Delta_l, \dots\}$ as μ_t^l . That is, we now consider the approximation for $k \in \mathbb{N}_0$:

$$\begin{aligned} X_{k\Delta_l} &= X_{(k-1)\Delta_l} + a \left(X_{(k-1)\Delta_l}, \bar{\xi}_1(X_{(k-1)\Delta_l}, \mu_{(k-1)\Delta_l}^l) \right) + \\ & b \left(X_{(k-1)\Delta_l}, \bar{\xi}_2(X_{(k-1)\Delta_l}, \mu_{(k-1)\Delta_l}^l) \right) [W_{k\Delta_l} - W_{(k-1)\Delta_l}] \end{aligned} \quad (2.1)$$

where $X_0 = x_0$ and $\mu_0^l = \delta_{\{x_0\}}$. Associated to (2.1), we denote by $P_{\mu_{t-1},t}^l(x_{t-1}, dx_t)$ the conditional law of X_t , $t \in \mathbb{N}$, given \mathcal{F}_{t-1} for $t \geq 1$; that is, the transition kernel over unit time induced by (2.1). It should be remarked that in many cases, (2.1) cannot be simulated exactly as the expectations associated to $\mu_{(k-1)\Delta_l}^l$ cannot be computed even if one knows $\mu_{(k-1)\Delta_l}^l$, which is again unlikely. We denote by π^l stationary distribution of $P_{\mu_{t-1},t}^l(x_{t-1}, dx_t)$ which is assumed to exist. More precisely, we shall make some assumptions later on in the article which ensure that

$$\lim_{t \rightarrow \infty} \mathcal{W}_2(\mu_t^l, \pi^l) = 0,$$

where \mathcal{W}_2 is the Wasserstein-2 distance (which will be defined later on). We note that the mathematical details here are minimized to help readability of this section of the article.

We now consider a method that will be used to provide a Monte Carlo based approximation of the law μ_t^l . The approach we present is a simple discretized method in Algorithm 1 from [28]. In Algorithm 1, the notation $\mathcal{N}_d(\kappa, \Sigma)$ denotes the d -dimensional Gaussian distribution with mean κ and covariance matrix Σ . I_d is the $d \times d$ identity matrix and $\overset{\text{ind}}{\sim}$ denotes independently distributed as. Algorithm 1 can be used to approximate expectations w.r.t. μ_t^l and indeed on the grid in-between time $t-1$ and t . Algorithm 1 is given in the form that we need it later on in the article. In our main method it will be critical that for two discretizations, we will need to be able to sample from a dependent coupling of a pair of laws μ_t^l, μ_t^{l-1} ; this is presented in Algorithm 2.

Let $l_* \in \mathbb{N}_0$ be given and $\{N_l\}_{l \geq l_*}$, be an increasing sequence of non-negative integers such that $\lim_{l \rightarrow \infty} N_l = \infty$. Our objective will be first to generate Algorithm 1 sequentially in t for $l = l_*$ and N_{l_*} samples. Then using the particle system that we have generated, to *plug-in all the required laws* to a simulation of the Markov kernel $P_{\mu_{t-1},t}^{l, N_l}$; we denote the associated stationary distribution of $P_{\mu_{t-1},t}^{l, N_l}$ as Π^l .

Note that in effect here, one runs Algorithm 1 independently of $P_{\mu_{t-1},t}^{l, N_l}$ and the latter kernel, conditional on the output of Algorithm 1 is a conventional Euler-Maruyama method. Subsequently for $l > l_*$, we will be running Algorithm 2 sequentially in t with N_l and N_{l-1} samples and then simulating a coupling of $P_{\mu_{t-1},t}^{l, N_l}$ and $P_{\mu_{t-1},t}^{l-1, N_{l-1}}$ conditional on the simulation of Algorithm 2, that is, *plugging-in all the required laws* and generating Algorithm 2 independently of all other randomness. The stationary distribution of $P_{\mu_{t-1},t}^{l, N_l}$ is still Π^l .

2.2 Overall Strategy

We now present our main strategy as given in [17]; see also [8, 27] for a related approaches, which are not easily adapted to this context. We suppose that $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a functional of interest and that for every $l \in \mathbb{N}_0$, $\Pi^l(\varphi) = \int_{\mathbb{R}^d} \varphi(x) \Pi^l(dx)$ is finite. Let \mathbb{P}_L be any positive probability mass function on

Algorithm 1 Approximating the Laws when starting with a particle approximation at time $t - 1$, $t \in \mathbb{N}$.

1. Input $l \in \mathbb{N}_0$ the level of discretization, $N \in \mathbb{N}$ the number of particles, $t \in \{1, \dots, T\}$. If $t = 1$ set $\mu_0^N(dx) = \delta_{\{x_0\}}(dx)$ otherwise input an empirical measure $\mu_{t-1}^N(dx) = \frac{1}{N} \sum_{i=1}^N \delta_{\{X_{t-1}^i\}}(dx)$. Set $k = 1$.
2. For $i \in \{1, \dots, N\}$ generate:

$$X_{t-1+k\Delta_l}^i = X_{t-1+(k-1)\Delta_l}^i + a \left(X_{t-1+(k-1)\Delta_l}^i, \bar{\xi}_1(X_{t-1+(k-1)\Delta_l}^i, \mu_{t-1+(k-1)\Delta_l}^N) \right) + b \left(X_{t-1+(k-1)\Delta_l}^i, \bar{\xi}_2(X_{t-1+(k-1)\Delta_l}^i, \mu_{t-1+(k-1)\Delta_l}^N) \right) \left[W_{t-1+k\Delta_l}^i - W_{t-1+(k-1)\Delta_l}^i \right]$$

where

$$\begin{aligned} \bar{\xi}_m(X_{t-1+(k-1)\Delta_l}^i, \mu_{t-1+(k-1)\Delta_l}^N) &= \frac{1}{N} \sum_{j=1}^N \xi_m(X_{t-1+(k-1)\Delta_l}^i, X_{t-1+(k-1)\Delta_l}^j) \quad m \in \{1, 2\} \\ \mu_{t-1+(k-1)\Delta_l}^N(dx) &= \frac{1}{N} \sum_{j=1}^N \delta_{\{X_{t-1+(k-1)\Delta_l}^j\}}(dx) \\ \left[W_{t-1+k\Delta_l}^i - W_{t-1+(k-1)\Delta_l}^i \right] &\stackrel{\text{ind}}{\sim} \mathcal{N}_d(0, \Delta_l I_d). \end{aligned}$$

Set $k = k + 1$, if $k = \Delta_l^{-1} + 1$ go to step 3. otherwise go to the start of step 2..

3. Output all the required laws $\mu_{t-1+\Delta_l}^N, \dots, \mu_t^N$.
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$\mathbb{N}_{l_*} := \{l_*, l_* + 1, \dots\}$. Let $\{\xi_l\}_{l \in \mathbb{N}_{l_*}}$ be any sequence of independent random variables, such that

$$\begin{aligned} \mathbb{E}[\xi_{l_*}] &= \mathbb{E}^P[\Pi^{l_*}(\varphi)] \\ \mathbb{E}[\xi_l] &= \mathbb{E}^P[\Pi^l(\varphi) - \Pi^{l-1}(\varphi)] =: \mathbb{E}^P[[\Pi^l - \Pi^{l-1}](\varphi)] \quad l \in \{l_* + 1, l_* + 2, \dots\} \end{aligned}$$

where \mathbb{E}^P is the expectation w.r.t. the law associated to the simulated systems in Algorithm 1 and Algorithm 2. Now, let L be a random variable with probability \mathbb{P}_L that is independent of the sequence $\{\xi_l\}_{l \in \mathbb{N}_{l_*}}$ then

$$\hat{\pi}(\varphi) = \frac{\xi_L}{\mathbb{P}_L(L)}, \quad (2.2)$$

will be shown to be an unbiased estimator of $\pi(\varphi)$; see [23, 26] for the initial statement and proof. Note however, that we have an additional level of complexity than the original papers as Π^l are random measures. Moreover, if

$$\sum_{l \in \mathbb{N}_{l_*}} \frac{\mathbb{E}[\xi_l^2]}{\mathbb{P}_L(l)} < +\infty, \quad (2.3)$$

the estimator $\hat{\pi}(\varphi)$ has finite variance. There is also the independent sum-estimator, which can be better than this estimator and has been described in [26]. The main challenge is then to construct the sequence $\{\xi_l\}_{l \in \mathbb{N}_{l_*}}$.

Typically, one will run $M \in \mathbb{N}$ independent replicates of (2.2) and use the average

$$\hat{\pi}(\varphi)_{\text{avg}} := \frac{1}{M} \sum_{i=1}^M \hat{\pi}(\varphi)^i, \quad (2.4)$$

Algorithm 2 Approximating the Consecutive Laws when starting with a particle approximation at time $t - 1$, $t \in \mathbb{N}$.

1. Input $l \in \mathbb{N}_0$ the level of discretization, $(N_l, N_{l-1}) \in \mathbb{N}^2$, $N_{l-1} < N_l$, the number of particles, $t \in \{1, \dots, T\}$. If $t = 1$ set $\mu_0^{l, N_l}(dx) = \tilde{\mu}_0^{l-1, N_{l-1}}(dx) = \delta_{\{x_0\}}(dx)$ otherwise input a pair of empirical measures $\mu_{t-1}^{l, N_l}(dx) = \frac{1}{N_l} \sum_{i=1}^{N_l} \delta_{\{X_{t-1}^{l, i}\}}(dx)$, $\tilde{\mu}_{t-1}^{l-1, N_{l-1}}(dx) = \frac{1}{N_{l-1}} \sum_{i=1}^{N_{l-1}} \delta_{\{\tilde{X}_{t-1}^{l-1, i}\}}(dx)$. Set $k = 1$.
2. For $i \in \{1, \dots, N_l\}$ generate:

$$X_{t-1+k\Delta_l}^{l, i} = X_{t-1+(k-1)\Delta_l}^{l, i} + a \left(X_{t-1+(k-1)\Delta_l}^{l, i}, \bar{\xi}_1(X_{t-1+(k-1)\Delta_l}^{l, i}, \mu_{t-1+(k-1)\Delta_l}^{l, N_l}) \right) + b \left(X_{t-1+(k-1)\Delta_l}^{l, i}, \bar{\xi}_2(X_{t-1+(k-1)\Delta_l}^{l, i}, \mu_{t-1+(k-1)\Delta_l}^{l, N_l}) \right) \left[W_{t-1+k\Delta_l}^i - W_{t-1+(k-1)\Delta_l}^i \right]$$

where

$$\begin{aligned} \bar{\xi}_m(X_{t-1+(k-1)\Delta_l}^{l, i}, \mu_{t-1+(k-1)\Delta_l}^{l, N_l}) &= \frac{1}{N_l} \sum_{j=1}^{N_l} \xi_m(X_{t-1+(k-1)\Delta_l}^{l, i}, X_{t-1+(k-1)\Delta_l}^{l, j}) \quad m \in \{1, 2\} \\ \mu_{t-1+(k-1)\Delta_l}^{l, N_l}(dx) &= \frac{1}{N_l} \sum_{j=1}^{N_l} \delta_{\{X_{t-1+(k-1)\Delta_l}^{l, j}\}}(dx). \end{aligned}$$

Set $k = k + 1$, if $k = \Delta_l^{-1} + 1$ go to step 3. otherwise go to the start of step 2..

3. For $i \in \{1, \dots, N_{l-1}\}$ compute:

$$\begin{aligned} \tilde{X}_{t-1+k\Delta_{l-1}}^{l-1, i} &= \tilde{X}_{t-1+(k-1)\Delta_{l-1}}^{l-1, i} + a \left(\tilde{X}_{t-1+(k-1)\Delta_{l-1}}^{l-1, i}, \bar{\xi}_1(\tilde{X}_{t-1+(k-1)\Delta_{l-1}}^{l-1, i}, \tilde{\mu}_{t-1+(k-1)\Delta_{l-1}}^{l-1, N_{l-1}}) \right) + \\ &b \left(\tilde{X}_{t-1+(k-1)\Delta_{l-1}}^{l-1, i}, \bar{\xi}_2(\tilde{X}_{t-1+(k-1)\Delta_{l-1}}^{l-1, i}, \tilde{\mu}_{t-1+(k-1)\Delta_{l-1}}^{l-1, N_{l-1}}) \right) \left[W_{t-1+k\Delta_{l-1}}^i - W_{t-1+(k-1)\Delta_{l-1}}^i \right] \end{aligned}$$

where

$$\begin{aligned} \bar{\xi}_m(\tilde{X}_{t-1+(k-1)\Delta_{l-1}}^{l-1, i}, \tilde{\mu}_{t-1+(k-1)\Delta_{l-1}}^{l-1, N_{l-1}}) &= \frac{1}{N_{l-1}} \sum_{j=1}^{N_{l-1}} \xi_m(\tilde{X}_{t-1+(k-1)\Delta_{l-1}}^{l-1, i}, \tilde{X}_{t-1+(k-1)\Delta_{l-1}}^{l-1, j}) \quad m \in \{1, 2\} \\ \tilde{\mu}_{t-1+(k-1)\Delta_{l-1}}^{l-1, N_{l-1}}(dx) &= \frac{1}{N_{l-1}} \sum_{j=1}^{N_{l-1}} \delta_{\{\tilde{X}_{t-1+(k-1)\Delta_{l-1}}^{l-1, j}\}}(dx) \end{aligned}$$

and the increments of the Brownian motion $\left[W_{t-1+k\Delta_{l-1}}^i - W_{t-1+(k-1)\Delta_{l-1}}^i \right]$ were generated in step 2.. Set $k = k + 1$, if $k = \Delta_{l-1}^{-1} + 1$ go to step 4. otherwise go to the start of step 3..

4. Output all the required laws $\mu_{t-1+\Delta_l}^{l, N_l}, \dots, \mu_t^{l, N_l}$, $\tilde{\mu}_{t-1+\Delta_l}^{l-1, N_{l-1}}, \dots, \tilde{\mu}_t^{l-1, N_{l-1}}$.
-

where $\hat{\pi}(\varphi)^i$ represents the i -th independent replicate of the estimate.

To continue with our discussion we will need a positive probability mass-function \mathbb{P}_p on \mathbb{N}_0 and a sequence of non-decreasing, non-negative integers $\{I_p\}_{p \in \mathbb{N}_0}$ with $\lim_{p \rightarrow \infty} I_p = \infty$.

2.2.1 Approximation of $\Pi^{l_*}(\varphi)$

Throughout the section $l \in \mathbb{N}_{l_*}$ is fixed. Our method for constructing ξ_{l_*} is detailed in Algorithm 3.

Algorithm 3 Simulation of ξ_{l_*} .

1. Input N_{l_*} .
2. Generate $P \sim \mathbb{P}_p$.
3. Generate Algorithm 1 with N_{l_*} particles, sequentially until time I_p where the empirical measures at any time $t \in \{1, \dots, I_p\}$ have been obtained from time $t - 1$ and the case $t = 0$ has been specified in Algorithm 1.
4. For $t \in \{1, \dots, I_p\}$ generate $U_t^{l_*} | u_{t-1}^{l_*}$ using $P_{\mu, t-1}^{l_*}(u_{t-1}^{l_*}, \cdot)$ where $\mu = \mu_{t-1}^{l_*, N_{l_*}}$, all of the laws $\mu_{t-1}^{l_*, N_{l_*}}, \mu_{t-1+\Delta_{l_*}}^{l_*, N_{l_*}}, \dots, \mu_{t-\Delta_{l_*}}^{l_*, N_{l_*}}$ needed are obtained in Step 3. and $u_0^{l_*} = x_0$.
5. If $p = 0$ return

$$\xi_{l_*} = \frac{1}{\mathbb{P}_p(p)} \frac{1}{I_p} \sum_{t=1}^{I_p} \varphi(u_t^{l_*})$$

otherwise return

$$\xi_{l_*} = \frac{1}{\mathbb{P}_p(p)} \left\{ \frac{1}{I_p} \sum_{t=1}^{I_p} \varphi(u_t^{l_*}) - \frac{1}{I_{p-1}} \sum_{t=1}^{I_{p-1}} \varphi(u_t^{l_*}) \right\}.$$

The approach as developed in Algorithm 3 is a simple adaptation of the method in [17] in the context here, except for that method one does not have to feed the empirical measures into any simulation as we have done here. Note that in practice one would run Step 3. and Step 4. concurrently, that is at each time they are simulated at the same Δ_{l_*} order increments, for computational efficiency. However, for clarity of presentation we have de-coupled the two steps. The key property of the estimator, that will help to ensure that our final estimator is unbiased is that we will show almost surely

$$\mathbb{E}[\xi_{l_*} | \mathcal{L}] = \Pi^{l_*}(\varphi). \quad (2.5)$$

where \mathcal{L} is the filtration generated by Algorithm 1 and Algorithm 2 along with L generated from \mathbb{P}_L (independently of all other random variables). The property in (2.5) is intrinsically based on the convergence of $\mathbb{E}[\frac{1}{I_p} \sum_{t=1}^{I_p} \varphi(u_t^{l_*}) | \mathcal{L}]$. The details are in the proof of our main result, but we try to give some intuition here.

2.2.2 Approximation of $[\Pi^l - \Pi^{l-1}](\varphi)$

Our objective is now to provide, for $l \in \{l_* + 1, l_* + 2, \dots\}$ fixed, an estimator of $[\Pi^l - \Pi^{l-1}](\varphi)$, such that

$$\mathbb{E}[[\widehat{\Pi^l - \Pi^{l-1}}](\varphi) | \mathcal{L}] = [\Pi^l - \Pi^{l-1}](\varphi). \quad (2.6)$$

One could simply use the method outlined above, independently, for Π_l and Π_{l-1} and independently for each $l \in \{l_* + 1, l_* + 2, \dots\}$. However, this is unlikely to provide an estimator that can achieve (2.3) and hence the variance of such an approach is infinite and not useful in practice. We therefore present an alternative method.

To describe the simulation of ξ_l , for $l \in \{l_* + 1, l_* + 2, \dots\}$, we will need the method given in Algorithm 4. The algorithm as stated is essentially a synchronous coupling of the simulation of an Euler-Maruyama

time-discretization. The main difference is that one has to approximate the unknown laws, which for the purposes of Algorithm 4 this is assumed to be given. Our method for simulating ξ_t is given in Algorithm 5. The method in Algorithm 5 helps one to achieve the property (2.6) as will be proved later on. The essential point is that the differences $\mathbb{E}[\frac{1}{I_p} \sum_{t=1}^{I_p} \varphi(u_t^l) | \mathcal{L}] - \Pi^l(\varphi)$ will converge almost surely to zero and this permits the unbiasedness that we need. The coupling that is achieved in Algorithm 2 and Algorithm 4 will help to yield a finite variance estimator.

Algorithm 4 Simulation of a Coupling of $P_{\mu,t}^l(u_{t-1}, \cdot)$ and $P_{\bar{\mu},t}^{l-1}(\bar{u}_{t-1}, \cdot)$.

1. Input $l \in \{l_* + 1, \dots\}$, $t \in \mathbb{N}$, the empirical laws $\mu_{t-1}^{l, N_l}, \mu_{t-1+\Delta_l}^{l, N_l}, \dots, \mu_{t-\Delta_l}^{l, N_l}, \mu_{t-1}^{l-1, N_{l-1}}, \mu_{t-1+\Delta_{l-1}}^{l-1, N_{l-1}}, \dots, \mu_{t-\Delta_{l-1}}^{l-1, N_{l-1}}$ and $(u_{t-1}, \bar{u}_{t-1}) \in \mathbb{R}^{2d}$.

2. For $k \in \{1, \dots, \Delta_l^{-1}\}$ run the dynamics

$$X_{t-1+k\Delta_l} = X_{t-1+k\Delta_l} + a \left(X_{t-1+(k-1)\Delta_l}, \bar{\xi}_1(X_{t-1+(k-1)\Delta_l}, \mu_{t-1+(k-1)\Delta_l}^{l, N_l}) \right) + b \left(X_{t-1+(k-1)\Delta_l}, \bar{\xi}_2(X_{t-1+(k-1)\Delta_l}, \mu_{t-1+(k-1)\Delta_l}^{l, N_l}) \right) [W_{t-1+k\Delta_l} - W_{t-1+(k-1)\Delta_l}]$$

where $X_{t-1} = u_{t-1}$ and for $k \in \{1, \dots, \Delta_l^{-1}\}$, $[W_{t-1+k\Delta_l} - W_{t-1+(k-1)\Delta_l}] \stackrel{\text{ind}}{\sim} \mathcal{N}(0, \Delta_l I_d)$. Set $U_t = x_t$.

3. For $k \in \{1, \dots, \Delta_{l-1}^{-1}\}$ run the dynamics

$$X_{t-1+k\Delta_{l-1}} = X_{t-1+k\Delta_{l-1}} + a \left(X_{t-1+(k-1)\Delta_{l-1}}, \bar{\xi}_1(X_{t-1+(k-1)\Delta_{l-1}}, \mu_{t-1+(k-1)\Delta_{l-1}}^{l-1, N_{l-1}}) \right) + b \left(X_{t-1+(k-1)\Delta_{l-1}}, \bar{\xi}_2(X_{t-1+(k-1)\Delta_{l-1}}, \mu_{t-1+(k-1)\Delta_{l-1}}^{l-1, N_{l-1}}) \right) [W_{t-1+k\Delta_{l-1}} - W_{t-1+(k-1)\Delta_{l-1}}]$$

where $X_{t-1} = \bar{u}_{t-1}$ and for $k \in \{1, \dots, \Delta_{l-1}^{-1}\}$, $[W_{t-1+k\Delta_{l-1}} - W_{t-1+(k-1)\Delta_{l-1}}]$ is determined from the simulation in Step 2. Set $\bar{U}_t = x_t$.

4. Return (u_t, \bar{u}_t) .
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2.2.3 Final Methodology and Estimator

We now consolidate the above discussion by summarizing our proposed methodology to unbiasedly estimate $\pi(\varphi)$ and this is presented in Algorithm 6. As implied by (2.4) Algorithm 6 can be run M -times on parallel. The choice of $\{N_l\}_{l \geq l_*}$, $\{I_p\}_{p \in \mathbb{N}_0}$, \mathbb{P}_L and \mathbb{P}_P is discussed in Section 3.

The approach that we have considered as stated previously, follows that of [17] but as also mentioned, there are alternatives based on [8, 27]. These previous papers are rather dependent upon the notion that the simulated Markov kernel (i.e. the $P_{\mu,t}^l$ in our notation) is time-homogenous. This is critical when adopting the methodology of [14] which those works use and as is clear from our context we do not have this property. Therefore we have concentrated upon the ideas in [17].

An alternative idea is to use a double randomization that focusses upon the systems generated in Algorithms 1 and 2. In principle we expect that it is possible to do this, but we expect that the resulting mathematical analysis is more complicated and the addition of an extra chain (i.e. the approaches in Algorithms 3 and 5) does not add a significant cost versus using Algorithms 1 and 2 on their own; hence we have proceeded with Algorithm 6.

Algorithm 5 Simulation of ξ_l .

1. Input $l \in \{l_* + 1, \dots\}$, (N_l, N_{l-1}) .
2. Generate $P \sim \mathbb{P}_p$.
3. Generate Algorithm 2 with (N_l, N_{l-1}) particles, sequentially until time I_p where the empirical measures at any time $t \in \{1, \dots, I_p\}$ have been obtained from time $t-1$ and the case $t=0$ has been specified in Algorithm 2.
4. For $t \in \{1, \dots, I_p\}$ generate $(U_t^l, \bar{U}_t^{l-1} | (u_{t-1}^l, \bar{u}_{t-1}^{l-1}))$ from the coupling of $P_{\mu, t}^l(u_{t-1}^l, \cdot)$ and $P_{\bar{\mu}, t}^{l-1}(\bar{u}_{t-1}^{l-1}, \cdot)$ given in Algorithm 4 where $\mu = \mu_{t-1}^{l, N_l}$, $\bar{\mu} = \mu_{t-1}^{l-1, N_{l-1}}$, all of the laws $\mu_{t-1}^{l, N_l}, \mu_{t-1+\Delta_l}^{l, N_l}, \dots, \mu_{t-\Delta_l}^{l, N_l}, \mu_{t-1}^{l-1, N_{l-1}}, \mu_{t-1+\Delta_{l-1}}^{l-1, N_{l-1}}, \dots, \mu_{t-\Delta_{l-1}}^{l-1, N_{l-1}}$, needed are obtained in Step 3. and $u_0^l = \bar{u}_0^{l-1} = x_0$.

5. If $p=0$ return

$$\xi_l = \frac{1}{\mathbb{P}_p(p)} \left\{ \frac{1}{I_p} \sum_{t=1}^{I_p} \varphi(u_t^l) - \frac{1}{I_p} \sum_{t=1}^{I_p} \varphi(\bar{u}_t^{l-1}) \right\}$$

otherwise return

$$\xi_l = \frac{1}{\mathbb{P}_p(p)} \left\{ \left[\frac{1}{I_p} \sum_{t=1}^{I_p} \varphi(u_t^l) - \frac{1}{I_p} \sum_{t=1}^{I_p} \varphi(\bar{u}_t^{l-1}) \right] - \left[\frac{1}{I_{p-1}} \sum_{t=1}^{I_{p-1}} \varphi(u_t^l) - \frac{1}{I_{p-1}} \sum_{t=1}^{I_{p-1}} \varphi(\bar{u}_t^{l-1}) \right] \right\}.$$

Algorithm 6 Unbiased estimator $\widehat{\pi}(\varphi)$.

Input: \mathbb{P}_L .

1. Sample $L \sim \mathbb{P}_L$.
2. If $L = l_*$, generate ξ_{l_*} using Algorithm 3 and return

$$\widehat{\pi}(\varphi) = \frac{\xi_{l_*}}{\mathbb{P}_L(l_*)}.$$

3. If $L > l_*$, generate ξ_l using Algorithm 5 and return

$$\widehat{\pi}(\varphi) = \frac{\xi_l}{\mathbb{P}_L(l)}.$$

3 Theoretical Results

3.1 Notation

Denote by $\mathcal{C}_b(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ the set of \mathbb{R}^{d_2} valued bounded continuous functions whose domain is \mathbb{R}^{d_1} and equip it with the norm $\|f\| = \sup_{x \in \mathbb{R}^{d_1}} |f(x)|$. Denote by $\mathcal{C}_b^1(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ the set of continuously differentiable functions with domain \mathbb{R}^{d_1} and values in \mathbb{R}^{d_2} whose partial derivatives of order 1 are bounded functions. For a function $f \in \mathcal{C}_b^1(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ define the seminorm $|f|_1 = \max_{i \in \{1, \dots, d_1\}} \sup_{x \in \mathbb{R}^{d_1}} |\partial_{x_i} f(x)|$ and for $f \in \mathcal{C}_b(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}) \cap \mathcal{C}_b^1(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ define the norm $\|f\|_1 = \max(\|f\|, |f|_1)$. Denote by $\mathcal{C}^{\text{Lip}}(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ the set of Lipschitz continuous functions $f : \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_2}$. Define the seminorm $|f|_{\text{Lip}}$ and the norm $\|f\|_{\text{Lip}}$ for

$f \in \mathcal{C}_b^{\text{Lip}}(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ by

$$|f|_{\text{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}, \quad \|f\|_{\text{Lip}} := \max(\|f\|, |f|_{\text{Lip}}).$$

For the function a denote by $\nabla_1 a(x, y)$ the gradient of the function $x \mapsto a(x, y)$ and by $\nabla_2 a(x, y)$ the gradient of the function $y \mapsto a(x, y)$, similarly for the functions b, ξ_1, ξ_2 .

3.2 Assumptions

In order for us to proceed we require a number of assumptions for our theory. We state the following assumptions.

(A1) The functions $a \in \mathcal{C}_b^1(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^d)$, $b \in \mathcal{C}^{\text{Lip}}(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^{d \times d})$, $\xi_1, \xi_2 \in \mathcal{C}^{\text{Lip}}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}) \cap \mathcal{C}_b(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$.

(A2) The following inequality holds

$$- \sup_{x \in \mathbb{R}^{d+1}} \sup_{|y|=1} y^\top \nabla_1 a(x) y > 2 \|\nabla_2 a\| \|\xi_1\|_{\text{Lip}} + 2|b|_{\text{Lip}}^2 (1 + \|\xi_2\|_{\text{Lip}})^2.$$

Let us now discuss the importance of each assumption above. It is well known that under condition **(A1)** a unique strong solution exists for the SDE (1.1), with details in [21]. The assumptions **(A1-2)** are needed to guarantee the existence of the invariant measure for both the continuous McKean-Vlasov SDE and the discretized SDE and to ensure the stability of the Euler Scheme. To show the existence of the invariant measure of the McKean-Vlasov SDE (1.1) we utilize [29, Theorem 3.1] which guarantees geometric ergodicity. which in turn requires one to verify conditions (H1), (H2'), and (H3) of [29].

3.3 Unbiasedness of the Estimator

Our main result is the unbiasedness of the estimator. Note that we require l_* to be large enough and this is assumed. The proof of this result depends itself on numerous ergodicity results that are established in Appendix A.

Theorem 3.1. *Assume (A1-2). Then for any $\varphi \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}) \cap \mathcal{C}^{\text{Lip}}(\mathbb{R}^d, \mathbb{R})$ we have*

$$\mathbb{E}[\widehat{\pi}(\varphi)] = \pi(\varphi).$$

Proof. The proof is completed in several simple computations as given below:

$$\begin{aligned} \mathbb{E}[\widehat{\pi}(\varphi)] &= \mathbb{E} \left[\sum_{l=1}^{\infty} \sum_{p=1}^{\infty} \left[\frac{1}{I_p} \sum_{t=1}^{M_p} \varphi(U_t^l) - \frac{1}{M_p} \sum_{t=1}^{M_p} \varphi(\bar{U}_t^{l-1}) \right] - \left[\frac{1}{I_{p-1}} \sum_{t=1}^{I_{p-1}} \varphi(U_t^l) - \frac{1}{I_{p-1}} \sum_{t=1}^{I_{p-1}} \varphi(\bar{U}_t^{l-1}) \right] \right] \\ &= \lim_{L \rightarrow \infty} \lim_{p \rightarrow \infty} \mathbb{E} \left[\frac{1}{I_p} \sum_{t=1}^{M_p} \varphi(U_t^l) \right] \\ &= \lim_{L \rightarrow \infty} \mathbb{E} \left[\lim_{p \rightarrow \infty} \mathbb{E} \left[\frac{1}{I_p} \sum_{i=1}^{I_p} \varphi(U_i^l) \middle| \mathcal{L} \right] \right] \\ &= \lim_{L \rightarrow \infty} \mathbb{E}[\Pi^L(\varphi)] \\ &= \pi(\varphi), \end{aligned}$$

where we have used Theorem A.5 to go from line three to line four and again to go the final line and the interchanges of limits and integrals are justifiable by the bounded convergence theorem. \square

3.4 Discussion

Theorem 3.1 gives us unbiasedness, but little else to help us choose the parameters in the method. We conjecture that, based on theory in [1, 17], that the variance is upper-bounded by a term that is as below

$$\mathcal{O} \left(\sum_{l=l_*}^{\infty} \sum_{p=0}^{\infty} \frac{1}{\mathbb{P}_L(l)\mathbb{P}_P(p)} \left\{ \frac{\Delta_l}{I_p} \left(1 + \frac{1}{N_l} \right) \right\} \right). \quad (3.1)$$

Note that to achieve this bound, we expect that we need some ergodicity properties of the Markov kernel $P_{\mu,t}^l$ with convergence rates that are l -independent. We would expect again that this might only occur under iteration as we have done in this paper. The expected cost is

$$\mathcal{O} \left(\sum_{l=l_*}^{\infty} \sum_{p=0}^{\infty} \mathbb{P}_L(l)\mathbb{P}_P(p)I_p\Delta_l^{-1}N_l^2 \right).$$

In this case it is difficult to choose $\{N_l\}_{l \geq l_*}$, $\{I_p\}_{p \in \mathbb{N}_0}$, \mathbb{P}_L and \mathbb{P}_P so that both (3.1) and the expected cost is finite. One can choose $N_l = l$, $I_p = 2^p$, $\mathbb{P}_L(l) \propto 2^{-l}(l+1)\log_2(l+2)^2$, and $\mathbb{P}_P(p) \propto 2^{-p}(p+1)\log_2(p+2)^2$ and the expression in (3.1) is finite.

Remark 3.1. *We note our ergodicity results assume a discretized MVSDE based on the Euler-Maruyama discretization. It is important to highlight these results could be potentially extended to higher-order discretization methods which have more favorable error rates. Examples of this would include splitting order schemes, which have proven to be successful.*

4 Numerical Experiments

In this section we consider testing our unbiased estimator to approximate the invariant measures for a selection of different MVSDEs. In particular we will consider three models to test, which include the Curie-Weiss model, an OU process and a model taking motivation for mathematical neuroscience. We will demonstrate the effectiveness of our approximation method through different plots such as comparing the MSE to the number of Monte Carlo samples M and approximating the density of the invariant measure. One of our numerical experiments is a parameter estimation problem. Before discussing each model, we provide a brief overview on our simulation setup.

4.1 Simulation Setting

For $M \in \mathbb{M}$ we denote the by $\hat{\pi}_M$ the estimator described in (2.4) run with M independent simulations. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$. We estimate the mean squared error (MSE) corresponding to M and φ we run 50 independent runs $\{\hat{\pi}_M^k\}_{k=1}^{50}$ of the method and calculate the MSE given as

$$\text{MSE} = \frac{1}{50} \sum_{k=1}^{50} (\hat{\pi}_M^k(\varphi) - \pi(\varphi))^2$$

We set $l_* = 3$, $l_{\max} = 10$, $p_{\max} = 7$ and

$$\begin{aligned} \mathbb{P}_L(l) &\propto 2^{-l}(l+1)\log(l+2)\mathbb{1}_{\{l_{\min} \leq l \leq l_{\max}\}}, \\ \mathbb{P}_P(p) &\propto 2^{-p}(p+1)\log(p+2)^2\mathbb{1}_{\{0 \leq p \leq p_{\max}\}}, \end{aligned}$$

and where $N_l = \mathcal{O}(l)$. We denote by $K_h(x) = \mathcal{N}(0, h^2)$ the density of the normal distribution with standard deviation h . The density p of π is estimated using kernel density estimation (KDE) by

$$p(x) \approx \int_{\mathbb{R}^d} K_h(x-y) d\hat{\pi}_M(y) = \hat{\pi}_M(K_h(x-\cdot)),$$

for an appropriate choice of $h \in \mathbb{R}$.

4.2 Curie-Weiss Model

Our first model we test our unbiased methodology on is the following one-dimensional SDE

$$dX_t = \beta(-X_t^3 + X_t + K\mathbb{E}[X_t])dt + \sigma dW_t, \quad (4.1)$$

with the initial condition $X_0 = x_0 \in \mathbb{R}$, where $\beta, K, \sigma > 0$ and W_t is a standard Brownian motion. It is well-known the invariant distribution π of this model is absolutely continuous with respect to the Lebesgue measure and has the density

$$p(x) = C \exp\left(-\frac{\beta x^4}{2} + \beta x^2\right),$$

where C is the reciprocal of the normalizing constant. For this example, we set $\beta = 1, K = 0.25, \sigma = 1, x_0 = 1$ and $\varphi(x) = x^2$. We numerically approximate $C \approx 0.2401$ and $\pi(\varphi) \approx 0.8935$. We approximate $\pi(\varphi)$ using our method to and evaluate $\hat{\pi}_M(\varphi)$. Figure 1 shows the MSE $\mathbb{E}[(\hat{\pi}_M(\varphi) - \pi(\varphi))^2]$ and the average running time corresponding to the runs. Furthermore we approximate the density p using our method and KDE. Figure 1 demonstrates that unbiased estimator works well, as the KDE attains high accuracy of the stationary distribution, and the rates are favourable which are coincide with the discussion on the cost in Section 3.

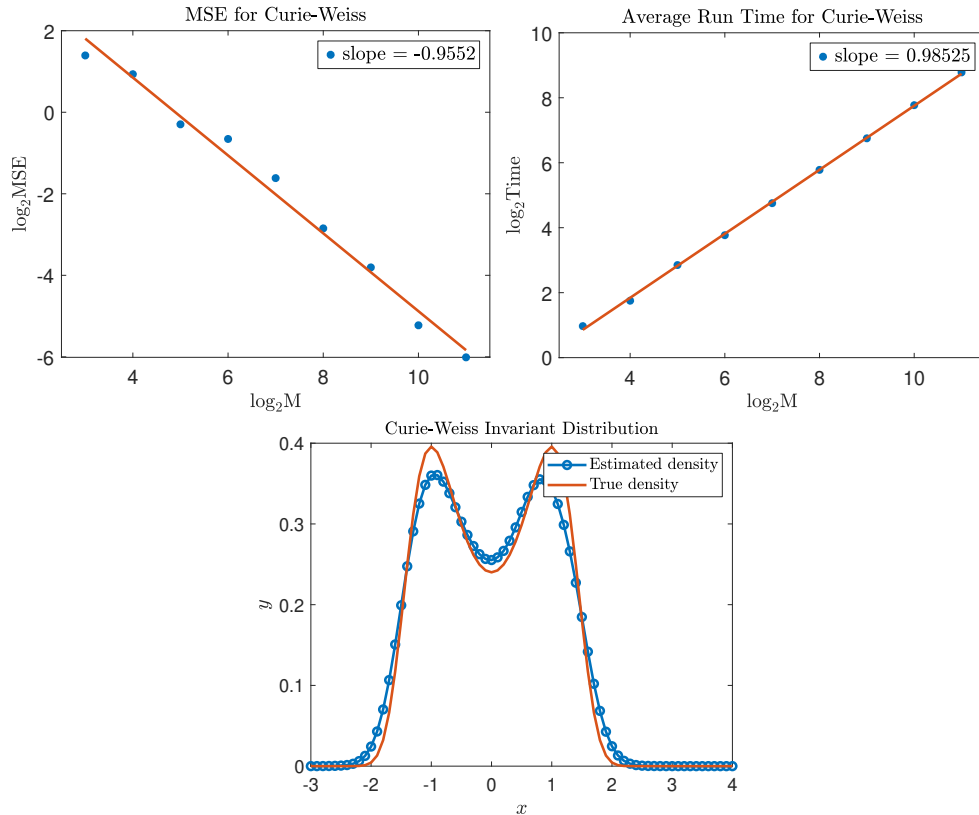


Figure 1: Numerical simulations for the Curie-Weiss model (4.1). Top left: MSE approximation of ϕ . Top right: Meeting time for Curie-Weiss model. Bottom: comparison of exact and approximated invariant distribution.

4.3 Parameter Estimation

Let $\{p_\theta(x, y)\}_{\theta \in \Theta}$ be a parameterized family of probability densities on $\mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$. Given a $y \in \mathbb{R}^{d_y}$ consider the problem of finding the Maximum Likelihood Estimator (MLE)

$$\theta^* = \operatorname{argmax}_{\theta \in \Theta} p_\theta(y) = \operatorname{argmax}_{\theta \in \Theta} \int p_\theta(x, y) dx.$$

[18] shows that the MLE is the limit $\theta^* = \lim_{t \rightarrow \infty} \theta_t$ with θ_t being the solution of the following McKean-Vlasov SDE

$$\begin{cases} d\theta_t = \left(\int_{\mathbb{R}^d} \nabla_\theta \log p_{\theta_t}(x, y) d\mu_t(x) \right) dt, \\ dX_t = \nabla_x \log p_{\theta_t}(X_t, y) dt + \sqrt{2} dW_t. \end{cases} \quad (4.2)$$

where μ_t is the law of the process X_t and W_t is a standard Brownian motion. Furthermore, the law μ_t of the process X_t defined in (4.2) is absolutely continuous with the Lebesgue measure and $\lim_{t \rightarrow \infty} d\mu_t/dx = p_{\theta^*}(\cdot|y)$ where $p_{\theta^*}(x|y) = p_{\theta^*}(x, y)/p_{\theta^*}(y)$ is the posterior. To apply our method we discretize the system as follows: For each $N, l \in \mathbb{N}$ we consider

$$\begin{cases} \theta_{(k+1)\Delta_l} = \theta_{k\Delta_l} + \left(\int_{\mathbb{R}^d} \nabla_\theta \log p_{\theta_{k\Delta_l}}(x, y) d\mu_{k\Delta_l}^N(x) \right) \Delta_l + \Delta_l (B_{(k+1)\Delta_l} - B_{k\Delta_l}), \\ X_{(k+1)\Delta_l}^i = X_{k\Delta_l}^i + \nabla_x \log p_{\theta_{k\Delta_l}}(X_{k\Delta_l}^i, y) \Delta_l + \sqrt{2} (W_{(k+1)\Delta_l} - W_{k\Delta_l}), \end{cases} \quad (4.3)$$

where $\mu_{k\Delta_l}^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_{k\Delta_l}^i}$, $i \in \{1, \dots, N\}$, $k \in \Delta_l^{-1} \mathbb{N}_0$, and B_t is a standard Brownian motion independent of W_t .

We consider the toy example considered in [18]. For $\theta \in \mathbb{R}$ let $\bar{\theta} \in \mathbb{R}^d$ be the vector whose all components are equal to θ . Let $p_\theta(x, y) = \mathcal{N}(y; x, I_d) \mathcal{N}(x; \bar{\theta}, I_d)$. Let $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, the likelihood $p_\theta(y) = \mathcal{N}(y; \bar{\theta}, 2I_d)$, the posterior $p_\theta(x|y) = \mathcal{N}(x; \frac{y+\bar{\theta}}{2}, \frac{1}{2}I_d)$, and the MLE has the closed form $\theta^* = \frac{1}{d} \sum_{i=1}^d y_i$. We set $d = 10$ and apply our method to the vector (θ_t, X_t) . Denote by $\hat{\pi}$ the estimate our method returns for the invariant distribution of (θ_t, X_t) and define the function $\varphi : (\theta, x) \mapsto \theta$. Our estimate of the MLE θ^* is $\hat{\pi}(\varphi)$. Figure 2 shows the convergences rate as function of the number of independent samples M and the average running time. Figure 2 shows the estimated posterior of the 10th component of the process X_t in (4.3). The results obtained for this experiment, follow similarly to that for the Currie-Weiss Model, where we obtain two rate which are approximately -1 and 1, when comparing M to the MSE and the average run time. By average run time we mean the total sum of all the M runs.

4.4 3D Neuron Model

Our final model we test is inspired from the work of [3], which develops a non globally Lipschitz MV-SDE to model neuron activity. It is a 3D neuron model which has a specific form of for the drift term and diffusion coefficient. For this model we assume the SDE takes the general form,

$$dX_t = a(t, x, \mu) dt + b(t, x, \mu) dW_t, \quad (4.4)$$

where the drift term, and diffusion coefficient, have the following representation

$$\begin{aligned} a(t, x, \mu) &:= \begin{pmatrix} x_1 - (x_1)^3/3 - x_2 + I - \int_{\mathbb{R}^3} J(x_1 - V_{rev}) z_3 d\mu(z) \\ c(x_1 + a - bx_2) \\ a_r \frac{T_{max}(1-x_3)}{1 + \exp(-\lambda(x_1 - V_T))} - a_d x_3 \end{pmatrix} \\ b(t, x, \mu) &:= \begin{pmatrix} b_{ext} & 0 & - \int_{\mathbb{R}^3} b_J(x_1 - V_{rev}) z_3 d\mu(z) \\ 0 & 0 & 0 \\ 0 & b_{32}(x) & 0 \end{pmatrix} \end{aligned}$$

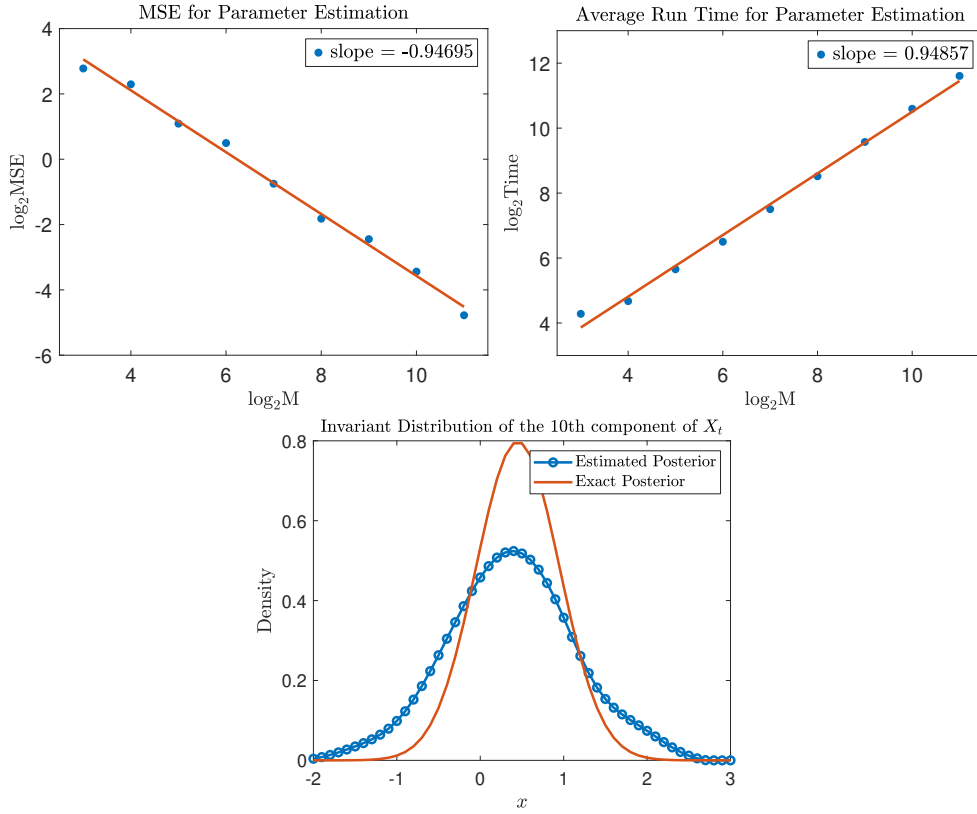


Figure 2: Numerical simulations for parameter estimation example (4.2). Top left: MSE approximation of ϕ . Top right: Meeting time for parameter estimation example. Bottom: comparison of exact and approximated posterior distribution.

with

$$b_{32}(x) := \mathbb{1}_{\{x_3 \in (0,1)\}} \sqrt{a_r \frac{T_{max}(1-x_3)}{1 + \exp(-\lambda(x_1 - V_T))} + a_d x_3 \Gamma \exp(-\Lambda/(1 - (2x_3 - 1)^2))},$$

where $T = 2$ is chosen as the final time. Finally we set an initial condition and parameter values as

$$X_0 \sim \mathcal{N} \left(\begin{pmatrix} V_0 \\ w_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} \sigma_{V_0} & 0 & 0 \\ 0 & \sigma_{w_0} & 0 \\ 0 & 0 & \sigma_{y_0} \end{pmatrix} \right),$$

where the parameters have the values

$$\begin{array}{cccccccc} V_0 = 0 & \sigma_{V_0} = 0.4 & a = 0.7 & b = 0.8 & c = 0.08 & I = 0.5 & b_{ext} = 0.5 \\ w_0 = 0.5 & \sigma_{w_0} = 0.4 & V_{rev} = 1 & a_r = 1 & a_d = 1 & T_{max} = 1 & \lambda = 0.2 \\ y_0 = 0.3 & \sigma_{y_0} = 0.05 & J = 1 & b_J = 0.2 & V_T = 2 & \Gamma = 0.1 & \Lambda = 0.5. \end{array}$$

Figure 3 shows the estimated marginals of the invariant distribution of the process X_t , while also presenting a comparison of the the number of samples M and the the average run time, which attains a similar slope of approximately -1 .

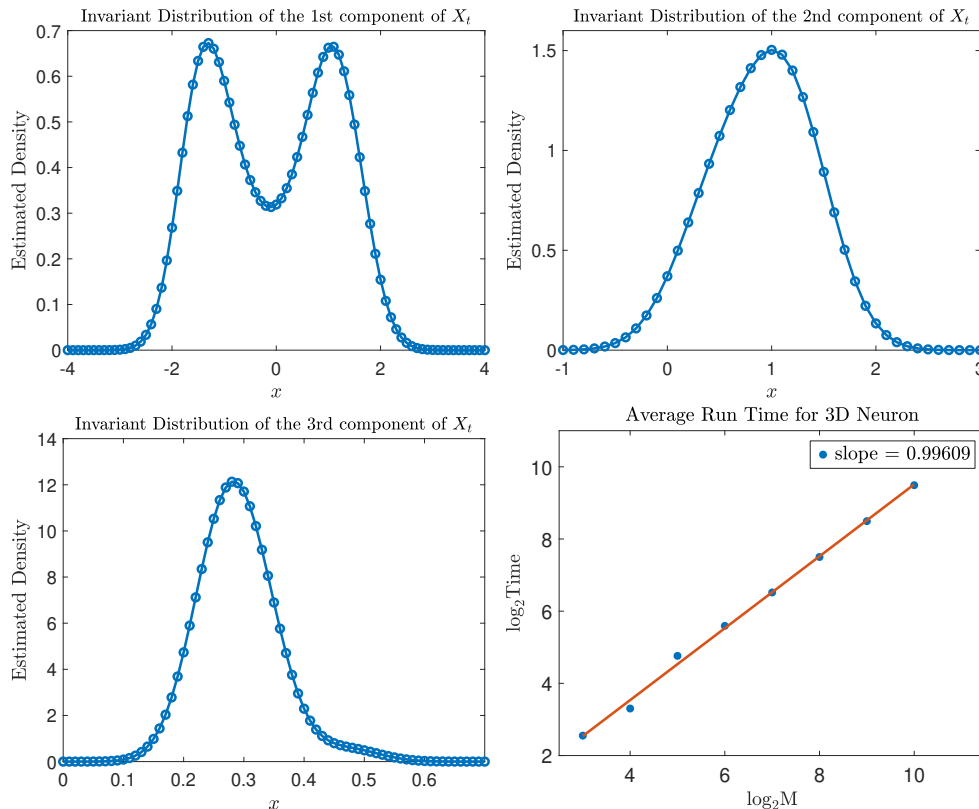


Figure 3: Numerical simulations for the 3D neuron model (4.4). Top left: approximated Invariant distribution for 1st component. Top right: approximated Invariant distribution for 2nd component. Bottom left: approximated Invariant distribution for 3rd component. Bottom right: Meeting time for parameter estimation example.

5 Conclusion

McKean-Vlasov stochastic differential equations (MVSDEs) are an important class of processes, that are used in a range of applications. A recent study of these processes has been on inference, related to either parameter estimation, or approximations of their corresponding invariant measure. The focus of this work was to develop a method that is able to unbiasedly approximate invariant measure of MVSDEs, which are commonly subject to a bias resulting from a discretization scheme. We consider an Euler-Maruyama discretization and present an unbiased algorithm motivated from the unbiased Monte Carlo algorithms, that exploit variance reduction techniques. We were firstly able to demonstrate the ergodicity of various processes we consider to an invariant measure at a geometric rate. To the best of our knowledge these are the first such results, in the discrete-time setting. From this we proved that our estimator is unbiased. We presented various numerical experiments to verify our theory on a range of MVSDEs. These includes a Currie-Weiss model, a 3D neuron model and a parameter estimation problem. Our motivation was to consider MVSDEs which omitted an invariant measure, and some where an approximation was required.

In terms of future work there are a number of interesting directions one can take.

- A first direction is to consider higher-order discretization schemes, which have more favourable strong and weak error rates. Examples of this would be splitting order schemes, such as BAOAB

and UBU. This has been explored in the following works [8, 25], which also used to develop unbiased estimator for sampling.

- A second direction would be the consideration of neural MVSEs [30], which are motivated from the recent directions of neural SDEs and neural ODEs, used for approximations within deep-learning. This is a recent field, with considerable potential in diffusion models. Presenting new schemes at handling various biases would prove useful.
- One could consider the application of such unbiased methods to McKean-Vlasov stochastic partial differential equations (MVSPDEs). This is very much an open direction as very little work has been conducted, both numerically and theoretically.
- Finally it would be of interest to verify if our unbiased estimator is of finite variance. This computation is not so trivial, and goes beyond the work of this article. We envision the proof procedure would follow similarly to that in [1].

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A Ergodicity Results

A.1 Notation

For a matrix $A \in \mathbb{R}^{d \times d}$ define the norm $\|A\| = \sup_{x \in \mathbb{R}^d \setminus \{0\}} \frac{|Ax|}{|x|}$ which is equal to the absolute value of the largest eigenvalue of A . Denote by $\mathcal{P}_0(\mathbb{R}^d)$ the set of probability measures on \mathbb{R}^d , and for $i \in \mathbb{N}$ define the set of probability measures with finite i -th second moment as $\mathcal{P}_i(\mathbb{R}^d) = \{\mu \in \mathcal{P}_0(\mathbb{R}^d) : \int |x|^i \mu(dx) < \infty\}$. In order for us to characterize the notion of ergodicity, we require ergodicity with respect to a metric. One metric we will consider for this work is the i -Wasserstein distance, which is provided in the following definition. For a random variable X defined on a probability space with probability measure \mathbb{P} we denote by $\mathcal{L}_X = \mathbb{P} \circ X^{-1}$ the law of X . Let μ be any sigma-finite measure on the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $\mathcal{B}(\mathbb{R}^d)$ are the Borel sets, and let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be μ -integrable, then we write $\mu(\varphi) = \int_{\mathbb{R}^d} \varphi(x) \mu(dx)$.

Definition A.1 (i -Wasserstein distance). *The i -Wasserstein distance between two measures $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ is defined as*

$$\mathcal{W}_i(\mu, \nu) = \left(\inf_{\gamma \in \Gamma(\mu, \nu)} \int_{\mathbb{R}^{2d}} |x - y|^i \gamma(d(x, y)) \right)^{1/i},$$

where

$$\Gamma(\mu, \nu) = \left\{ \gamma \in \mathcal{P}_0(\mathbb{R}^d \times \mathbb{R}^d) : \int_{A \times \mathbb{R}^d} \gamma(d(x, y)) = \mu(A), \int_{\mathbb{R}^d \times A} \gamma(d(x, y)) = \nu(A) \right\},$$

is the set of couplings of μ and ν .

Definition A.2 (\mathcal{M} distance). *Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$, define the \mathcal{M} distance between the two measures μ, ν by*

$$\mathcal{M}(\mu, \nu) = \sup_{\|f\|_{\text{Lip}} \leq 1} \left| \int_{\mathbb{R}^d} f(x) \mu(dx) - \int_{\mathbb{R}^d} f(x) \nu(dx) \right|.$$

We utilize the metric \mathcal{M} and the metrics \mathcal{W}_i in the proofs below. The \mathcal{M} and \mathcal{W}_1 metrics are related by the inequality $\mathcal{M}(\mu, \nu) \leq \mathcal{W}_1(\mu, \nu)$ which follow from the Kantorovich duality:

$$\mathcal{W}_1(\mu, \nu) = \sup_{|f|_{\text{Lip}} \leq 1} \left| \int_{\mathbb{R}^d} f(x) \mu(dx) - \int_{\mathbb{R}^d} f(x) \nu(dx) \right|.$$

Jensen's inequality guarantees that $\mathcal{W}_1(\mu, \nu) \leq \mathcal{W}_2(\mu, \nu)$. For random variables X, Y defined on the same probability space we have $\mathcal{W}_2(\mathcal{L}_X, \mathcal{L}_Y) \leq \mathbb{E}[|X - Y|^2]^{1/2}$. The spaces $(\mathcal{P}_i(\mathbb{R}^d), \mathcal{W}_i)$ are complete metric spaces. Throughout the proofs below, we use the symbol C for generic constants and its value may change from one line to another. Dependencies on various model and simulation parameters will be stated as needed.

A.2 Outline of the Results and Structure

In the appendix we prove several results associated to various processes. We recall that the original continuous-time process $\{X_t\}_{t \geq 0}$ is governed by the dynamics (1.1). We will then denote by $\{\tilde{X}_t\}_{t \in \{0, \Delta_l, \dots\}}$ as the exact Euler-Maruyama time-discretization as featured in (2.1). We will also have to analyze the interacting particle system as described in Algorithm 1 which is denoted as $\{\tilde{X}_t^i\}_{(i,t) \in \{1, \dots, N\} \times \{0, \Delta_l, \dots\}}$. Note that N is fixed here. Finally we will consider $\{\bar{X}_t\}_{t \in \{0, \Delta_l, \dots\}}$ which the Euler-Maruyama time-discretization of the SDE (1.1), except that we plug-in the laws of the SDE approximated by $\{\tilde{X}_t^i\}_{(i,t) \in \{1, \dots, N\} \times \{0, \Delta_l, \dots\}}$.

We now prove a series of results which are needed for our main results in Section 3 in the main text. We begin with Proposition A.1 which essentially implies that there is unique stationary distribution of the processes $\{X_t\}_{t \geq 0}$ and is summarized in Theorem A.1. We then turn to the exact Euler-Maruyama time-discretization $\{\tilde{X}_t\}_{t \in \{0, \Delta_l, \dots\}}$ for which we prove, in Theorem A.2, existence of a unique stationary distribution π^l . Theorem A.3 shows that π^l converges to π as l -grows in 2-Wasserstein distance. In Theorem A.4 we give a convergence theorem for the empirical measures $\mu_t^{l,N}$ associated to the particle system $\{\tilde{X}_t^i\}_{(i,t) \in \{1, \dots, N\} \times \{0, \Delta_l, \dots\}}$, in terms of the convergence in N and t in expected 2-Wasserstein distance with π^l . Theorem A.5 shows that $\{\bar{X}_t\}_{t \in \{0, \Delta_l, \dots\}}$ has a unique stationary distribution Π^l as t grows and a rather important law of large numbers on a time-discrete grid (used in the proof of Theorem 3.1). Corollary A.2 considers the case that $N = N_l$ and a convergence in 2-Wasserstein distance of Π^l to π . The results should be read in order and proofs of later results rely on earlier ones.

A.3 Ergodicity

Proposition A.1. *Assume (A1-2). Then there exist constants $C_j < +\infty$, $j \in \{1, \dots, 4\}$ with $C_3 > C_4$ such that for any $(x, y, \mu, \nu) \in (\mathbb{R}^d)^2 \times \mathcal{P}_2(\mathbb{R}^d)^2$*

$$|b(x, \bar{\xi}_2(x, \mu)) - b(y, \bar{\xi}_2(y, \nu))| \leq C_1|x - y| + C_1\mathcal{M}(\mu, \nu), \quad (\text{A.1})$$

$$|a(x, \bar{\xi}_1(x, \mu)) - a(y, \bar{\xi}_1(y, \nu))| \leq C_2|x - y| + C_2\mathcal{M}(\mu, \nu), \quad (\text{A.2})$$

$$\begin{aligned} & 2\langle a(x, \bar{\xi}_1(x, \mu)) - a(y, \bar{\xi}_1(y, \nu)), x - y \rangle + |b(x, \bar{\xi}_2(x, \mu)) - b(y, \bar{\xi}_2(y, \nu))|^2 \\ & \leq -C_3|x - y|^2 + C_4\mathcal{M}(\mu, \nu)^2, \end{aligned} \quad (\text{A.3})$$

$$\sup_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} |a(0, \bar{\xi}_1(0, \mu))| < \infty, \quad \sup_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} |b(0, \bar{\xi}_2(0, \mu))| < \infty. \quad (\text{A.4})$$

Proof. For the first inequality, using the assumption that b is Lipschitz and the triangular inequality we have

$$\begin{aligned} |b(x, \bar{\xi}_2(x, \mu)) - b(y, \bar{\xi}_2(y, \nu))| & \leq |b|_{\text{Lip}}|x - y| + |b|_{\text{Lip}} \left| \int_{\mathbb{R}^d} \xi_2(x, z) \mu(dz) - \int_{\mathbb{R}^d} \xi_2(y, z) \mu(dz) \right| \\ & \quad + |b|_{\text{Lip}} \left| \int_{\mathbb{R}^d} \xi_2(y, z) \mu(dz) - \int_{\mathbb{R}^d} \xi_2(y, z) \nu(dz) \right|. \end{aligned}$$

The second term in the line is bounded by $|\xi_2|_{\text{Lip}}|x-y|$ and the third term is bounded by $\|\xi_2\|_{\text{Lip}}\mathcal{M}(\mu, \nu)$. Thus inequality (A.1) holds with $C_1 = |b|_{\text{Lip}}(1 + \|\xi_2\|_{\text{Lip}})$. Inequality (A.2) follows analogously. For (A.3) we write

$$\begin{aligned} & \langle a(x, \bar{\xi}_1(x, \mu)) - a(y, \bar{\xi}_1(y, \nu)), x - y \rangle \\ &= \langle a(x, \bar{\xi}_1(x, \mu)) - a(y, \bar{\xi}_1(x, \mu)), x - y \rangle + \langle a(y, \bar{\xi}_1(x, \mu)) - a(y, \bar{\xi}_1(y, \nu)), x - y \rangle. \end{aligned}$$

Let $A = \sup_{x \in \mathbb{R}^{d+1}} \sup_{|y|=1} y^\top \nabla_1 a(x) y$ and $B = \|\nabla_2 a\|$. The first term is bounded above by $A|x-y|^2$. For the second term, we use the Cauchy inequality and calculations similar to the one used to prove the first inequality

$$\begin{aligned} \langle a(y, \bar{\xi}_1(x, \mu)) - a(y, \bar{\xi}_1(y, \nu)), x - y \rangle &\leq |x-y| |a(y, \bar{\xi}_1(x, \mu)) - a(y, \bar{\xi}_1(y, \nu))| \\ &\leq B \|\xi_1\|_{\text{Lip}} |x-y| (|x-y| + \mathcal{M}(\mu, \nu)) \\ &\leq \frac{3}{2} B \|\xi_1\|_{\text{Lip}} |x-y|^2 + \frac{1}{2} B \|\xi_1\|_{\text{Lip}} \mathcal{M}(\mu, \nu)^2. \end{aligned}$$

Therefore

$$\begin{aligned} & 2 \langle a(x, \bar{\xi}_1(x, \mu)) - a(y, \bar{\xi}_1(y, \nu)), x - y \rangle + |b(x, \bar{\xi}_2(x, \mu)) - b(y, \bar{\xi}_2(y, \nu))|^2 \\ &\leq (2A + 3B \|\xi_1\|_{\text{Lip}} + 2C_1^2) |x-y|^2 + (B \|\xi_1\|_{\text{Lip}} + 2C_1^2) \mathcal{M}(\mu, \nu)^2, \end{aligned}$$

and (A.3) follows by Assumption (A2). Inequality (A.4) follows from the continuity of the function a and the boundedness of the function $\bar{\xi}_1$

$$\sup_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} a(0, \bar{\xi}_1(0, \mu)) \leq \sup_{|x| \leq \|\xi_1\|} a(0, x) < \infty.$$

□

Proposition A.1 verifies the conditions needed for [29, Theorem 3.1], which we now state.

Theorem A.1. *Assume (A1-2). Then there exists a unique $\pi \in \mathcal{P}_2(\mathbb{R}^d)$ such that*

$$\lim_{t \rightarrow \infty} \mathcal{W}_2(\mu_t, \pi) = 0.$$

Let $\Delta_l > 0$, unless explicitly stated the processes below in this section depend implicitly on Δ_l . Define $\Delta_l W_t = W_{t+\Delta_l} - W_t$. Recall the discrete process $\{\tilde{X}_t\}_{t \in \{0, \Delta_l, \dots\}}$

$$\tilde{X}_{t+\Delta_l} = \tilde{X}_t + a(\tilde{X}_t, \bar{\xi}_1(\tilde{X}_t, \mu_t^l)) \Delta + b(\tilde{X}_t, \bar{\xi}_1(\tilde{X}_t, \mu_t^l)) \Delta W_t,$$

where μ_t^l is the law of \tilde{X}_t . We follow the proof of Theorem 3.1 in [29] but in a discrete manner to show that the process $\{\tilde{X}_t\}_{t \in \{0, \Delta_l, \dots\}}$ has a unique invariant measure. Define

$$\Delta^* = \min \left\{ \frac{C_3 - C_4}{2C_2}, \frac{C_3}{4C_2^2} \right\}.$$

Δ^* will serve as a threshold for the values of Δ_l for which the following proofs will be valid.

Theorem A.2. *Assume (A1-2) and that $\Delta_l < \Delta^*$. Then there exists a unique $\pi^l \in \mathcal{P}_2(\mathbb{R}^d)$ such that*

$$\lim_{t \rightarrow \infty} \mathcal{W}_2(\mu_t^l, \pi^l) = 0.$$

Moreover, π^l is independent of μ_0^l and if $\mu_0^l = \pi^l$ then $\tilde{X}_t \sim \pi^l$ for every $t \in \{0, \Delta_l, \dots\}$.

Proof. Let $s \in \{\Delta_l, 2\Delta_l, \dots\}$. Let $\{\tilde{Y}_t\}_{t \in \{0, \Delta_l, \dots\}}$ be the discrete process defined by $\tilde{Y}_0 \sim \mu_s^l$ and

$$\tilde{Y}_{t+\Delta_l} = \tilde{Y}_t + a(\tilde{Y}_t, \bar{\xi}_1(\tilde{Y}_t, \nu_t^l))\Delta_l + b(\tilde{Y}_t, \bar{\xi}_1(\tilde{Y}_t, \nu_t^l))\Delta_l W_t,$$

where we remark that $\mathbb{E}[|\tilde{X}_0 - \tilde{Y}_0|^2] = \mathcal{W}_2(\mu_0^l, \mu_s^l)^2$ and that ν_t^l is the law of \tilde{Y}_t . As \tilde{Y}_t is discrete and follows the same iteration as \tilde{X}_t we have $\nu_t^l = \mu_{s+t}^l$. For every $t \in \{0, \Delta_l, \dots\}$ we have

$$\begin{aligned} \mathbb{E}[|\tilde{X}_{t+\Delta_l} - \tilde{Y}_{t+\Delta_l}|^2] &= \mathbb{E}[|\tilde{X}_t - \tilde{Y}_t|^2] + \mathbb{E}[|a(\tilde{X}_t, \bar{\xi}_1(\tilde{X}_t, \mu_t^l)) - a(\tilde{Y}_t, \bar{\xi}_1(\tilde{Y}_t, \nu_t^l))|^2]\Delta_l^2 \\ &\quad + \mathbb{E}[2\langle a(\tilde{X}_t, \bar{\xi}_1(\tilde{X}_t, \mu_t^l)) - a(\tilde{Y}_t, \bar{\xi}_1(\tilde{Y}_t, \nu_t^l)), \tilde{X}_t - \tilde{Y}_t \rangle \\ &\quad + |b(\tilde{X}_t, \bar{\xi}_2(\tilde{X}_t, \mu_t^l)) - b(\tilde{Y}_t, \bar{\xi}_2(\tilde{Y}_t, \nu_t^l))|^2]\Delta_l \\ &\leq (1 - C_3\Delta_l + C_2\Delta_l^2)\mathbb{E}[|\tilde{X}_t - \tilde{Y}_t|^2] + (C_4\Delta_l + C_2\Delta_l^2)\mathcal{W}_2(\mu_t^l, \nu_t^l)^2 \\ &\leq (1 - C_3\Delta_l + C_2\Delta_l^2 + C_4\Delta_l + C_2\Delta_l^2)\mathbb{E}[|\tilde{X}_t - \tilde{Y}_t|^2], \end{aligned}$$

where the constants C_2, C_3, C_4 are as in Proposition A.1. As $\Delta_l < \Delta^*$ we have

$$\epsilon = C_3 - C_2\Delta_l - C_4 - C_2\Delta_l > 0.$$

Therefore for every $t \in \{0, \Delta_l, \dots\}$

$$\begin{aligned} \mathbb{E}[|\tilde{X}_t - \tilde{Y}_t|^2] &= \mathbb{E}[|\tilde{X}_0 - \tilde{Y}_0|^2] \prod_{k=0}^{t/\Delta_l-1} \frac{\mathbb{E}[|\tilde{X}_{(k+1)\Delta_l} - \tilde{Y}_{(k+1)\Delta_l}|^2]}{\mathbb{E}[|\tilde{X}_{k\Delta_l} - \tilde{Y}_{k\Delta_l}|^2]} \\ &\leq \mathcal{W}_2(\mu_0^l, \nu_0^l)^2 (1 - \epsilon\Delta_l)^{t/\Delta_l} \\ &\leq \mathcal{W}_2(\mu_0^l, \nu_0^l)^2 e^{-\epsilon t}. \end{aligned} \tag{A.5}$$

This implies that

$$\mathcal{W}_2(\mu_t^l, \mu_{t+s}^l)^2 \leq \mathcal{W}_2(\mu_0^l, \mu_s^l)^2 e^{-\epsilon t} \leq 4 \sup_{s \in \{0, \Delta_l, \dots\}} \mathbb{E}[|\tilde{X}_s|^2] e^{-\epsilon t}. \tag{A.6}$$

To bound $\mathbb{E}[|\tilde{X}_t|^2]$ we follows similar calculations as follows

$$\begin{aligned} \mathbb{E}[|\tilde{X}_{t+\Delta_l}|^2] &= \mathbb{E}[|\tilde{X}_t|^2] + \Delta_l \mathbb{E}[2\langle a(\tilde{X}_t, \bar{\xi}_1(\tilde{X}_t, \mu_t^l)), \tilde{X}_t \rangle + |b(\tilde{X}_t, \bar{\xi}_1(\tilde{X}_t, \mu_t^l))|^2] \\ &= \mathbb{E}[|\tilde{X}_t|^2] + \Delta_l \mathbb{E}[2\langle a(\tilde{X}_t, \bar{\xi}_1(\tilde{X}_t, \mu_t^l)) - a(0, \bar{\xi}_1(0, \mu_t^l)), \tilde{X}_t \rangle + |b(\tilde{X}_t, \bar{\xi}_1(\tilde{X}_t, \mu_t^l)) - b(0, \bar{\xi}_1(0, \mu_t^l))|^2] \\ &\quad + \Delta_l \mathbb{E}[2\langle b(\tilde{X}_t, \bar{\xi}_2(\tilde{X}_t, \mu_t^l)), b(0, \bar{\xi}_2(0, \mu_t^l)) \rangle - |b(0, \bar{\xi}_2(0, \mu_t^l))|^2] + \Delta_l \mathbb{E}[\langle a(0, \bar{\xi}_1(0, \mu_t^l)), \tilde{X}_t \rangle] \\ &\leq (1 - C_3\Delta_l)\mathbb{E}[|\tilde{X}_t|^2] + C\Delta_l \mathbb{E}[|\tilde{X}_t|] + C\Delta_l \\ &\leq (1 - C_3\Delta_l/2)\mathbb{E}[|\tilde{X}_t|^2] + C\Delta_l, \end{aligned} \tag{A.7}$$

where we used the boundedness of $a(0, \bar{\xi}_1(0, \mu_t^l))$ and $b(0, \bar{\xi}_1(0, \mu_t^l))$, the Cauchy inequality, and the inequality

$$x \leq \frac{C}{2C_3} + \frac{C_3}{2C} x^2.$$

Iterating inequality (A.7) yields

$$\mathbb{E}[|\tilde{X}_t|^2] \leq (1 - C_3\Delta_l/2)^{t/\Delta_l} \mathbb{E}[|\tilde{X}_0|^2] + C\Delta_l \sum_{k=0}^{\infty} (1 - C_3\Delta_l/2)^k \leq e^{-C_3 t/2} \mathbb{E}[|\tilde{X}_0|^2] + \frac{2C}{C_3}. \tag{A.8}$$

Therefore the sequence $\{\mu_t^l\}_{t \in \{0, \Delta_l, \dots\}}$ is a Cauchy sequence on the complete metric space $(\mathcal{P}_2, \mathcal{W}_2)$, thus there exists an invariant measure π^l that satisfies $\lim_{t \rightarrow \infty} \mathcal{W}_2(\mu_t^l, \pi^l) = 0$. Taking $s \rightarrow \infty$ in the first half of inequality (A.6) shows that if $\mu_0^l = \pi^l$ then $\tilde{X}_t \sim \pi^l$.

To prove uniqueness and independence of the initial distribution of the process, let $\tilde{\nu}$ be an invariant measure and define the processes \tilde{Y}_t as above but with $\tilde{Y}_0 \sim \tilde{\nu}$ and suppose that $\mu_0^l = \pi^l$. By invariance of π^l we have $\tilde{Y}_t \sim \tilde{\nu}$ for all $t \in \{0, \delta_l, \dots\}$. From inequality (A.5) we have

$$\mathcal{W}_2(\pi^l, \tilde{\nu})^2 \leq \mathcal{W}_2(\pi^l, \tilde{\nu})^2 e^{-\epsilon t},$$

and the claim follows by taking $t \rightarrow \infty$. \square

Corollary A.1. *Assume (A1-2). Let $\bar{\Delta} < \Delta^*$ and $\Delta_l < \bar{\Delta}$. The bounds in (A.5), (A.6), and (A.8) depend only on $\bar{\Delta}$, $\mathbb{E}[|\tilde{X}_0|^2]$, and the norms of the coefficients a, b, ξ_1, ξ_2 .*

Theorem A.3. *Assume (A1-2). Then we have*

$$\lim_{l \rightarrow \infty} \mathcal{W}_2(\pi^l, \pi) = 0.$$

Proof. Without loss of generality assume $\Delta_l < \Delta^*$. First, by [29, Theorem 3.1] we have $\sup_{s \geq 0} \mathbb{E}[|X_s|^2] < \infty$. Second, by using the inequality $|x + y|^2 \leq 2|x|^2 + 2|y|^2$, Lipschitz properties of the functions a and b , Proposition A.1, Cauchy inequality, Ito isometry, and Fubini's Theorem, we have that for $t > s > 0$:

$$\mathbb{E}[|X_t - X_s|^2] \leq C \mathbb{E} \left[\left| \int_s^t (|X_u| + 1) du \right|^2 \right] + C \mathbb{E} \left[\int_s^t (|X_u| + 1)^2 du \right] < C(t - s), \quad (\text{A.9})$$

with C independent of t and s . Write

$$X_{t+\Delta_l} - \tilde{X}_{t+\Delta_l} = A + B,$$

where

$$\begin{aligned} A := & X_t - \tilde{X}_t + (a(X_t, \bar{\xi}_1(X_t, \mu_t)) - a(\tilde{X}_t, \bar{\xi}_1(\tilde{X}_t, \mu_t^l)))\Delta_l \\ & + (b(X_t, \bar{\xi}_1(X_t, \mu_t)) - b(\tilde{X}_t, \bar{\xi}_2(\tilde{X}_t, \mu_t^l)))\Delta_l W_t, \end{aligned}$$

and

$$\begin{aligned} B := & \int_t^{t+\Delta_l} (a(X_u, \bar{\xi}_1(X_u, \mu_u)) - a(X_t, \bar{\xi}_1(X_t, \mu_t))) du \\ & + \int_t^{t+\Delta_l} (b(X_u, \bar{\xi}_2(X_u, \mu_u)) - b(X_t, \bar{\xi}_2(X_t, \mu_t))) dW_u. \end{aligned}$$

Following the proof of Theorem A.2 there exists $\epsilon > 0$ independent of Δ_l that satisfies

$$\mathbb{E}[|A|^2] \leq (1 - \epsilon\Delta_l)\mathbb{E}[|X_t - \tilde{X}_t|^2].$$

Using (A.9) and Ito isometry we have

$$\mathbb{E}[|B|^2] \leq C\Delta_l \sup_{0 \leq s \leq \Delta_l} \mathbb{E}[|X_{t+s} - X_t|^2] \leq C\Delta_l^2.$$

Using (A.9) and Itô isometry we have that $\mathbb{E}[\langle A, B \rangle]$ is equal to

$$\begin{aligned}
& \mathbb{E} \left[\left((b(X_t, \bar{\xi}_1(X_t, \mu_t)) - b(\tilde{X}_t, \bar{\xi}_2(\tilde{X}_t, \mu_t^l))) \Delta_l W_t \right)^\top \int_t^{t+\Delta_l} (b(X_u, \bar{\xi}_2(X_u, \mu_u)) - b(X_t, \bar{\xi}_2(X_t, \mu_t))) dW_u \right] \\
& \leq \mathbb{E} \left[\left| (b(X_t, \bar{\xi}_1(X_t, \mu_t)) - b(\tilde{X}_t, \bar{\xi}_2(\tilde{X}_t, \mu_t^l))) \right| \int_t^{t+\Delta_l} |b(X_u, \bar{\xi}_2(X_u, \mu_u)) - b(X_t, \bar{\xi}_2(X_t, \mu_t))| du \right] \\
& \leq C \mathbb{E} \left[\left(|X_t - \tilde{X}_t| + \mathcal{W}_1(\mu_t, \mu_t^l) \right) \int_t^{t+\Delta_l} (|X_u - X_t| + \mathcal{W}_1(\mu_u, \mu_t)) du \right] \\
& \leq C \Delta_l \mathbb{E}[|X_t - \tilde{X}_t|^2]^{1/2} \sup_{0 \leq s \leq \Delta_l} \mathbb{E}[|X_{t+s} - X_t|^2]^{1/2} \\
& \leq C \Delta_l^{3/2} \mathbb{E}[|X_t - \tilde{X}_t|^2]^{1/2} \\
& \leq \frac{\epsilon \Delta_l}{2} \mathbb{E}[|X_t - \tilde{X}_t|^2] + C \frac{\Delta_l^2}{2\epsilon}.
\end{aligned}$$

To deduce the last three lines we used the inequalities

$$\mathcal{W}_1(\mu_t, \mu_t^l) \leq \mathbb{E}[|X_t - \tilde{X}_t|^2]^{1/2}, \quad \mathcal{W}_1(\mu_u, \mu_t) \leq \mathbb{E}[|X_u - X_t|^2]^{1/2}, \quad x \leq \frac{\epsilon}{2C\sqrt{\Delta_l}} x^2 + \frac{C\sqrt{\Delta_l}}{2\epsilon}.$$

Therefore

$$\begin{aligned}
\mathbb{E}[|X_{t+\Delta_l} - \tilde{X}_{t+\Delta_l}|^2] & \leq \mathbb{E}[|A|^2] + 2\mathbb{E}[\langle A, B \rangle] + \mathbb{E}[|B|^2] \\
& \leq (1 - \epsilon \Delta_l / 2) \mathbb{E}[|X_t - \tilde{X}_t|^2] + C \Delta_l^2.
\end{aligned}$$

Iterating this last inequality yields

$$\mathbb{E}[|X_t - \tilde{X}_t|^2] \leq (1 - \epsilon \Delta_l / 2)^{t/\Delta_l} \mathbb{E}[|X_0 - \tilde{X}_0|^2] + C \Delta_l^2 \sum_{k=0}^{\infty} (1 - \epsilon \Delta_l / 2)^k \leq C e^{-\epsilon t/2} + C \Delta_l.$$

Using $\mathcal{W}_2(\mu_t, \mu_t^l) \leq \mathbb{E}[|X_t - \tilde{X}_t|^2]$ and taking $t \rightarrow \infty$ then $l \rightarrow \infty$ proves the claim. \square

Consider $(i, k) \in \{1, \dots, N\} \times \mathbb{N}_0$

$$\tilde{X}_{(k+1)\Delta_l}^i = \tilde{X}_{k\Delta_l}^i + a(\tilde{X}_{k\Delta_l}^i, \bar{\xi}_1(\tilde{X}_{k\Delta_l}^i, \mu_{k\Delta_l}^{l,N})) \Delta_l + b(\tilde{X}_{k\Delta_l}^i, \bar{\xi}_1(\tilde{X}_{k\Delta_l}^i, \mu_{k\Delta_l}^{l,N})) \Delta_l W_{k\Delta_l}^i, \quad (\text{A.10})$$

where $\mu_{k\Delta_l}^{l,N} = \frac{1}{N} \sum_{j=1}^N \delta_{\tilde{X}_{k\Delta_l}^j}$, $\tilde{X}_0^i = x_0$, $i \in \{1, \dots, N\}$ and $\{W_{k\Delta_l}^i\}_{i \in \{1, \dots, N\}}$ are independent standard Brownian motions.

Theorem A.4. Assume (A1-A2) and $\Delta_l < \Delta^*$. Then we have

$$\lim_{\substack{N \rightarrow \infty \\ t \rightarrow \infty}} \mathbb{E}[\mathcal{W}_2(\mu_t^{l,N}, \pi^l)^2] = 0.$$

Proof. Consider the system

$$\tilde{X}_{t+\Delta_l}^i = \tilde{X}_t^i + a(\tilde{X}_t^i, \bar{\xi}_1(\tilde{X}_t^i, \mu_t^l)) \Delta_l + b(\tilde{X}_t^i, \bar{\xi}_1(\tilde{X}_t^i, \mu_t^l)) \Delta_l W_t^i,$$

with $\tilde{X}_0^i \stackrel{\text{ind}}{\sim} \mu_0^l$ for $i \in \{1, \dots, N\}$. Following the calculations in Theorem A.2 we can show that

$$\sup_{t \geq 0} \mathbb{E}[|\tilde{X}_t^i|^2] < \infty, \quad \sup_{t \geq 0} \mathbb{E}[|\tilde{X}_t^i|^2] < \infty,$$

and

$$\begin{aligned}
\mathbb{E}[|\tilde{X}_{t+\Delta_l}^i - \tilde{X}_{t+\Delta_l}^i|^2] &= \mathbb{E}[|\tilde{X}_t^i - \tilde{X}_t^i|^2] + \mathbb{E}[|a(\tilde{X}_t^i, \bar{\xi}_1(\tilde{X}_t^i, \mu_t^{l,N})) - a(\tilde{X}_t^i, \bar{\xi}_1(\tilde{X}_t^i, \mu_t^l))|^2] \Delta_l^2 \\
&\quad + \mathbb{E}[2\langle a(\tilde{X}_t^i, \bar{\xi}_1(\tilde{X}_t^i, \mu_t^{l,N})) - a(\tilde{X}_t^i, \bar{\xi}_1(\tilde{X}_t^i, \mu_t^l)), \tilde{X}_t^i - \tilde{X}_t^i \rangle \\
&\quad + |b(\tilde{X}_t^i, \bar{\xi}_2(\tilde{X}_t^i, \mu_t^{l,N})) - b(\tilde{X}_t^i, \bar{\xi}_2(\tilde{X}_t^i, \mu_t^l))|^2] \Delta_l \\
&\leq (1 - C_3 \Delta_l + C_2 \Delta_l^2) \mathbb{E}[|\tilde{X}_t^i - \tilde{X}_t^i|^2] + (C_4 \Delta_l + C_2 \Delta_l^2) \mathbb{E}[\mathcal{M}(\mu_t^{l,N}, \mu_t^l)^2]. \\
&\leq (1 - C_3 \Delta_l + C_2 \Delta_l^2 + C_4 \Delta_l + C_2 \Delta_l^2) \mathbb{E}[|\tilde{X}_t^i - \tilde{X}_t^i|^2].
\end{aligned}$$

Let $\kappa > 0$ such that

$$C_3 - C_2 \Delta_l - (1 + \kappa)(C_4 + C_2 \Delta_l) > 0,$$

such a κ exists because $\Delta_l < \Delta^*$. Using the following inequality

$$(x + y)^2 \leq (1 + \kappa)x^2 + \left(1 + \frac{1}{\kappa}\right)y^2,$$

we have that for any $f \in \mathcal{C}^{\text{Lip}}(\mathbb{R}^d, \mathbb{R}) \cap \mathcal{C}_b(\mathbb{R}^d, \mathbb{R})$

$$\begin{aligned}
\mathbb{E}[|\mu_t^{l,N}(f) - \mu_t^l(f)|^2] &\leq (1 + \kappa) \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=1}^N \{f(\tilde{X}_t^i) - f(\tilde{X}_t^i)\} \right|^2 \right] + \left(1 + \frac{1}{\kappa}\right) \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=1}^N \{f(\tilde{X}_t^i) - \mathbb{E}[f(\tilde{X}_t^i)]\} \right|^2 \right] \\
&\leq (1 + \kappa) \|f\|_{\text{Lip}}^2 \sup_i \mathbb{E}[|\tilde{X}_t^i - \tilde{X}_t^i|^2] + \left(1 + \frac{1}{\kappa}\right) \frac{1}{N} \mathbb{E} \left[(f(\tilde{X}_t^1) - \mathbb{E}[f(\tilde{X}_t^1)])^2 \right] \\
&\leq (1 + \kappa) \|f\|_{\text{Lip}}^2 \sup_i \mathbb{E}[|\tilde{X}_t^i - \tilde{X}_t^i|^2] + 4 \left(1 + \frac{1}{\kappa}\right) \|f\|^2 \frac{1}{N}.
\end{aligned}$$

In the above calculation, for the first term after the first inequality we used the inequality

$$\left(\frac{1}{N} \sum_{i=1}^N x_i \right)^2 \leq \frac{1}{N} \sum_{i=1}^N x_i^2,$$

and for the second term we use the fact that the random variables \tilde{X}_t^i are i.i.d.. Now we define a sequence of bounded Lipschitz random functions f_n that satisfy the following inequalities

$$\mathcal{M}(\mu_t^{l,N}, \mu_t^l) - \frac{1}{n} \leq \mu_t^{l,N}(f_n) - \mu_t^l(f_n) \leq \mathcal{M}(\mu_t^{l,N}, \mu_t^l), \quad \|f_n\|_{\text{Lip}} \leq 1.$$

Since $\mathcal{M}(\mu_t^{l,N}, \mu_t^l) \leq 2$ we have by dominated convergence

$$\mathbb{E}[\mathcal{M}(\mu_t^{l,N}, \mu_t^l)^2] = \lim_{n \rightarrow \infty} \mathbb{E}[|\mu_t^{l,N}(f_n) - \mu_t^l(f_n)|^2] \leq (1 + \kappa) \sup_i \mathbb{E}[|\tilde{X}_t^i - \tilde{X}_t^i|^2] + 4 \left(1 + \frac{1}{\kappa}\right) \frac{1}{N}.$$

Letting $\epsilon = C_3 - C_2 \Delta_l - (1 + \kappa)(C_4 + C_2 \Delta_l)$, we have

$$\sup_i \mathbb{E}[|\tilde{X}_{t+\Delta}^i - \tilde{X}_{t+\Delta}^i|^2] \leq (1 - \epsilon \Delta) \sup_i \mathbb{E}[|\tilde{X}_t^i - \tilde{X}_t^i|^2] + \frac{C \Delta_l}{N}.$$

Consequently

$$\begin{aligned}
\sup_i \mathbb{E}[|\tilde{X}_t^i - \tilde{X}_t^i|^2] &\leq (1 - \epsilon \Delta)^{t/\Delta_l} \sup_i \mathbb{E}[|\tilde{X}_0^i - \tilde{X}_0^i|^2] + \frac{C \Delta_l}{N} \sum_{k=0}^{\infty} (1 - \epsilon \Delta_l)^k \\
&< e^{-\epsilon t} \sup_i \mathbb{E}[|\tilde{X}_0^i - \tilde{X}_0^i|^2] + \frac{C}{\epsilon N}.
\end{aligned}$$

Noticing that $\mathcal{W}_2(\mu_t^{l,N}, \mu_t^l)^2 \leq \sup_i \mathbb{E}[|\tilde{X}_t^i - \tilde{X}_t^i|^2]$ proves that

$$\lim_{\substack{N \rightarrow \infty \\ t \rightarrow \infty}} \mathbb{E}[\mathcal{W}_2(\mu_t^{l,N}, \mu_t^l)^2] = 0.$$

Finally, the Theorem statement follows from the inequality

$$\mathcal{W}_2(\mu_t^{l,N}, \pi^l)^2 \leq 2\mathcal{W}_2(\mu_t^{l,N}, \mu_t^l)^2 + 2\mathcal{W}_2(\mu_t^l, \pi^l)^2,$$

and using Theorem A.2. □

For $k \in \mathbb{N}_0$ set

$$\bar{X}_{(k+1)\Delta_l} = \bar{X}_{k\Delta_l} + a(\bar{X}_{k\Delta_l}, \bar{\xi}_1(\bar{X}_{k\Delta_l}, \mu_{k\Delta_l}^{l,N}))\Delta_l + b(\bar{X}_{k\Delta_l}, \bar{\xi}_1(\bar{X}_{k\Delta_l}, \mu_{k\Delta_l}^{l,N}))\Delta_l B_{k\Delta_l}, \quad (\text{A.11})$$

where the empirical measures have been plugged in from the system (A.10), $\bar{X}_0 = x_0$ and $B_{k\Delta_l}$ is a standard Brownian motion independent of all random variables. By conditioning on \mathcal{L} we can follow the same strategy of Theorem A.2 and show that there exists a unique invariant (random) measure Π^l for the process defined in (A.11). Furthermore, we have

$$\mathcal{W}_2(\Pi^l, \mathcal{L}_{\bar{X}_t})^2 \leq \mathcal{W}_2(\Pi^l, \mathcal{L}_{\bar{X}_0})^2 e^{-\epsilon t}. \quad (\text{A.12})$$

with ϵ a constant independent of \mathcal{L} , which implies $\mathcal{W}_2(\Pi^l, \mathcal{L}_{\bar{X}_t}) \rightarrow 0$ both a.s. and in \mathbb{L}_2 as $t \rightarrow \infty$. Next, we show that a law of large numbers holds.

Theorem A.5. *Assume (A1-2) and $\Delta_l < \Delta^*$. Then there exists a unique $\Pi^l \in \mathcal{P}_2(\mathbb{R}^d)$ such that*

$$\mathcal{W}_2(\Pi^l, \mathcal{L}_{\bar{X}_t}) \xrightarrow[t \rightarrow \infty]{\text{a.s. and } \mathbb{L}_2} 0$$

and

$$\mathcal{W}_2(\Pi^l, \pi^l) \xrightarrow[N \rightarrow \infty]{\mathbb{L}_2} 0. \quad (\text{A.13})$$

In addition, for any $\varphi \in \mathcal{C}^{\text{Lip}}(\mathbb{R}^d, \mathbb{R})$

$$\mathbb{E} \left[\frac{1}{I} \sum_{t=1}^I \varphi(\bar{X}_t) \middle| \mathcal{L} \right] \xrightarrow[I \rightarrow \infty]{\text{a.s. and } \mathbb{L}_2} \Pi^l(\varphi). \quad (\text{A.14})$$

Proof. The existence and uniqueness of Π^l is established above and we defer the proof of (A.13) at the end. For $\varphi \in \mathcal{C}^{\text{Lip}}(\mathbb{R}^d, \mathbb{R})$ we have almost surely

$$\left| \mathbb{E} \left[\frac{1}{I} \sum_{t=1}^I \varphi(\bar{X}_t) \middle| \mathcal{L} \right] - \Pi^l(\varphi) \right| \leq \frac{1}{I} \sum_{t=1}^I \left| \mathbb{E} \left[\frac{1}{I} \sum_{t=1}^I \varphi(\bar{X}_t) \middle| \mathcal{L} \right] - \Pi^l(\varphi) \right| \leq \frac{|\varphi|_{\text{Lip}}}{I} \sum_{t=1}^I \mathcal{W}_2(\mathcal{L}_{\bar{X}_t}, \Pi^l),$$

and one can conclude (A.14) by Cesaro averages.

For (A.13), recall the discrete-time process in (A.11) and define the process $k \in \mathbb{N}_0$

$$\bar{Z}_{(k+1)\Delta_l} = \bar{Z}_{k\Delta_l} + a(\bar{X}_{k\Delta_l}, \bar{\xi}_1(\bar{Z}_{k\Delta_l}, \mu_{k\Delta_l}^l))\Delta_l + b(\bar{Z}_{k\Delta_l}, \bar{\xi}_1(\bar{Z}_{k\Delta_l}, \mu_{k\Delta_l}^l))\Delta_l B_{k\Delta_l},$$

$\bar{Z}_0 = x_0$. Following the calculations of Theorem A.2 we have that

$$\sup_{N \in \mathbb{N}} \sup_{s \in \{0, \Delta_l, \dots\}} \mathbb{E}[|\bar{X}_s|^2] < \infty, \quad \sup_{N \in \mathbb{N}} \sup_{s \in \{0, \Delta_l, \dots\}} \mathbb{E}[|\bar{Z}_s|^2] < \infty,$$

and

$$\mathbb{E}[|\bar{X}_{t+\Delta_l} - \bar{Z}_{t+\Delta_l}|^2] \leq (1 - C_3\Delta_l + C_2\Delta_l^2)\mathbb{E}[|\bar{X}_t - \bar{Z}_t|^2] + (C_4\Delta_l + C_2\Delta_l^2)\mathbb{E}[\mathcal{M}(\mu_t^{l,N}, \mu_t^l)^2], \quad (\text{A.15})$$

Let $\zeta > 0$ by Theorem A.4 there exist $s_1 \in \{0, \Delta_l, \dots\}$ and $A \in \mathbb{N}$ such that

$$\mathbb{E}[\mathcal{M}(\mu_t^{l,N}, \mu_t^l)^2] \leq \mathbb{E}[\mathcal{W}_2(\mu_t^{l,N}, \mu_t^l)^2] < \zeta,$$

for all $t > s_1$ and $N > A$. Let $s_2 \in \{0, \Delta_l, \dots\}$ satisfy $e^{(-C_3+C_2\Delta_l)s_2} < \zeta$ and let $s = \max(s_1, s_2)$. For every $N > A$ and $t > 2s$ the inequality (A.15) implies

$$\begin{aligned} \mathbb{E}[|\bar{X}_t - \bar{Z}_t|^2] &\leq (1 - C_3\Delta_l + C_2\Delta_l^2)^{(t-s)/\Delta_l} \mathbb{E}[|\bar{X}_s - \bar{Z}_s|^2] \\ &\quad + (C_4\Delta_l + C_2\Delta_l^2) \sum_{k=0}^{(t-s)/\Delta_l} (1 - C_3\Delta_l + C_2\Delta_l^2)^k \mathbb{E}[\mathcal{M}(\mu_{t-k\Delta_l}^{l,N}, \mu_{t-k\Delta_l}^l)^2] \\ &\leq C e^{(-C_3+C_2\Delta_l)(t-s)} + \zeta (C_4\Delta_l + C_2\Delta_l^2) \sum_{k=0}^{\infty} (1 - C_3\Delta_l + C_2\Delta_l^2)^k \\ &\leq C\zeta, \end{aligned} \quad (\text{A.16})$$

with C independent of Δ_l . Therefore

$$\lim_{\substack{N \rightarrow \infty \\ t \rightarrow \infty}} \mathbb{E}[|\bar{X}_t - \bar{Z}_t|^2] = 0. \quad (\text{A.17})$$

Finally, we have that

$$\mathbb{E}[\mathcal{W}_2(\Pi^l, \pi^l)^2] \leq 3\mathbb{E}[\mathcal{W}_2(\Pi^l, \mathcal{L}_{\bar{X}_t}^l)^2] + 3\mathbb{E}[\mathcal{W}_2(\mathcal{L}_{\bar{X}_t}^l, \mu_t^l)^2] + 3\mathcal{W}_2(\mu_t^l, \pi^l)^2.$$

The first term approaches 0 as $t, N \rightarrow \infty$ by (A.12), the second term approaches 0 using (A.17), and the third term approaches 0 by Theorem A.2. \square

Finally, using Theorem A.5 we have the following corollary when we allow $N = N_l$ (recall that N_l is defined in the main text) to grow with l .

Corollary A.2. *Assume (A1-2). Then we have that*

$$\mathcal{W}_2(\Pi^l, \pi) \xrightarrow[l \rightarrow \infty]{\mathbb{L}_2} 0.$$

Proof. By the triangular inequality we have

$$\mathcal{W}_2(\Pi^l, \pi) \leq \mathcal{W}_2(\Pi^l, \pi^l) + \mathcal{W}_2(\pi^l, \pi).$$

By using (A.16) we have $\mathcal{W}_2(\Pi^l, \pi^l) \xrightarrow[l \rightarrow \infty]{\mathbb{L}_2} 0$, and using Theorem A.3 we have $\mathcal{W}_2(\pi^l, \pi) \xrightarrow[l \rightarrow \infty]{} 0$. \square

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