Bernstein-type Inequalities Preserved by Modified Smirnov Operator

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Abstract

In this paper, we consider a modified version of Smirnov operator and obtain some Bernstein-type inequalities preserved by this operator. In particular, we prove some compact generalizations of the well-known inequalities of Bernstein, Erdös and Lax, Ankeny and Rivlin and others.

Key words and phrases: Modified Smirnov operator, Polynomials, Bernstein inequality, Restricted zeros.

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1 Introduction

Let \mathbb{P}_n denote the class of polynomials $P(z) := \sum_{j=0}^n a_j z^j$ in \mathbb{C} of degree at most $n \in \mathbb{N}$. Let D be the open unit disk $\{z \in \mathbb{C}; |z| < 1\}$, such that \overline{D} is the closure of D and $B(\mathbb{D})$ denotes its boundary.

Let $P \in \mathbb{P}_n$, then

$$
\max_{z \in B(\mathbb{D})} |P'(z)| \le n \max_{z \in B(\mathbb{D})} |P(z)| \tag{1}
$$

and

$$
\max_{z \in B(\mathbb{D})} |P(Rz)| \le R^n \max_{z \in B(\mathbb{D})} |P(z)|. \tag{2}
$$

Inequality (1) is a well-known theorem of Bernstein [\[4\]](#page-8-0). The inequality (2) is a simple deduction from the maximum modulus principle. In both the inequalities, the equality holds for $P(z) = \alpha z^n, \alpha \neq 0$.

If we restrict to a class of polynomials having no zeros in \mathbb{D} , the inequalities [\(1\)](#page-0-0) and [\(2\)](#page-0-1) can be sharpened. In fact, if $P(z) \neq 0$ in D, then

$$
\max_{z \in B(\mathbb{D})} |P'(z)| \le \frac{n}{2} \max_{z \in B(\mathbb{D})} |P(z)| \tag{3}
$$

and for $R > 1$,

$$
\max_{z \in B(\mathbb{D})} |P(Rz)| \le \frac{R^n + 1}{2} \max_{z \in B(\mathbb{D})} |P(z)|. \tag{4}
$$

Inequality (3) was proved by Erdös and Lax [\[9\]](#page-8-1), whereas Ankeny and Rivlin [\[3\]](#page-8-2) used (3) to prove [\(4\)](#page-1-0). These inequalities were further improved by Aziz and Dawood [\[2\]](#page-8-3), where under the same hypothesis, it was proved that

$$
\max_{z \in B(\mathbb{D})} |P'(z)| \le \frac{n}{2} \left\{ \max_{z \in B(\mathbb{D})} |P(z)| - \min_{z \in B(\mathbb{D})} |P(z)| \right\}
$$
(5)

and for $R > 1$

$$
\max_{z \in B(\mathbb{D})} |P(Rz)| \le \left\{ \frac{R^n + 1}{2} \right\} \max_{z \in B(\mathbb{D})} |P(z)| - \left\{ \frac{R^n - 1}{2} \right\} \min_{z \in B(\mathbb{D})} |P(z)|. \tag{6}
$$

The equality in [\(3\)](#page-0-2)-[\(6\)](#page-1-1) holds for the polynomials of the form $P(z) = \alpha z^n + \beta$, with $|\alpha| = |\beta|.$

In 1930 Bernstein [\[5\]](#page-8-4) also proved the following result:

Theorem 1.1. Let $P(z)$ be a polynomial in \mathbb{P}_n having all zeros in $\overline{\mathbb{D}}$ and $p(z)$ be a polynomial of degree not exceeding that of $P(z)$. If $|p(z)| \leq |P(z)|$ on $B(\mathbb{D})$, then

$$
|p'(z)| \le |P'(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.
$$

The equality holds only if $p = e^{i\gamma} P, \gamma \in \mathbb{R}$.

For $z \in \mathbb{C} \setminus \mathbb{D}$, denoting by $\Omega_{|z|}$ the image of the disc $\{t \in \mathbb{C}; |t| < |z|\}$ under the mapping $\phi(t) = \frac{t}{1+t}$, Smirnov [\[13\]](#page-8-5) as a generalization of Theorem [1.1](#page-1-2) proved the following:

Theorem 1.2. Let p and P be polynomails possessing conditions as in Theorem [1.1,](#page-1-2) then for $z \in \mathbb{C} \setminus \mathbb{D}$

$$
|\mathbb{S}_{\alpha}[p](z)| \le |\mathbb{S}_{\alpha}[P](z)| \tag{7}
$$

for all $\alpha \in \overline{\Omega}_{|z|}$, with $\mathbb{S}_{\alpha}[p](z) := zp'(z) - n\alpha p(z)$, where α is a constant.

For $\alpha \in \overline{\Omega}_{|z|}$ in inequality [\(7\)](#page-1-3), the equality holds at a point $z \in \mathbb{C} \setminus \mathbb{D}$ only if $p =$ $e^{i\gamma}P, \gamma \in \mathbb{R}$. We note that for fixed $z \in \mathbb{C} \setminus \mathbb{D}$, Inequality [\(7\)](#page-1-3) can be replaced by (see for reference [\[7](#page-8-6), [8\]](#page-8-7))

$$
\left| z p'(z) - n \frac{az}{1 + az} p(z) \right| \le \left| z P'(z) - n \frac{az}{1 + az} P(z) \right|,
$$

where a is arbitrary from \mathbb{D} . Equivalently for $z \in \mathbb{C} \setminus \mathbb{D}$

$$
|\tilde{\mathbb{S}}_a[p](z)| \leq |\tilde{\mathbb{S}}_a[P](z)|,
$$

where $\tilde{S}_a[p](z) = (1 + az)p'(z) - nap(z)$ is known as modified Smirnov operator. The modified Smirnov operator $\tilde{\mathbb{S}}_a$ is more preferred in a sense than Smirnov operator \mathbb{S}_α , because the parameter a of $\tilde{\mathbb{S}}_a$ does not depend on z unlike parameter α of $\tilde{\mathbb{S}}_\alpha$.

Marden [\[10\]](#page-8-8) introduced a differential operator $\mathbb{B}: \mathbb{P}_n \to \mathbb{P}_n$ of mth order. This operator carries a polynomial $p \in \mathbb{P}_n$ into

$$
\mathbb{B}[p](z) = \lambda_0 p(z) + \lambda_1 \frac{nz}{2} p'(z) + \ldots + \lambda_m \left(\frac{nz}{2}\right)^m p^m(z),
$$

where $\lambda_0, \lambda_1, ..., \lambda_m$ are constants such that

$$
u(z) = \lambda_0 + {n \choose 1} \lambda_1 z + \dots + {n \choose m} \lambda_m z^m \neq 0, \quad \text{for} \quad Re(z) > \frac{n}{4}.\tag{8}
$$

Rahman and Schmeisser [\[12\]](#page-8-9) considered the Marden operator for $m = 2$ and showed that this operator preserves the inequalities between polynomials and accordingly proved the following:

Theorem 1.3. Let p and P be polynomials possessing conditions as in Theorem [1.1,](#page-1-2) Then

$$
|\mathbb{B}[p](z)| \le |\mathbb{B}[P](z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}, \tag{9}
$$

where the constants $\lambda_0, \lambda_1, ..., \lambda_m$ possess condition [\(8\)](#page-2-0). For $z \in \mathbb{C} \setminus \mathbb{D}$ in [\(9\)](#page-2-1), the equality holds if and only if $p(z) = \gamma z^n, \ \gamma \neq 0$.

A variety of key papers concerning the B-operator have appeared in the literature $[11, 14]$ $[11, 14]$.

In order to compare the Smirnov operator $\mathbb{S}_{\alpha}[p](z) := zp'(z) - n\alpha p(z)$ and the Rahman's operator (with $\lambda_2 = 0$) $\mathbb{B}[p](z) = \lambda_0 p(z) + \lambda_1 \frac{nz}{2}$ $\frac{iz}{2}p'(z)$, we require $\alpha \in \overline{\Omega}_{|z|}$ in inequality [\(7\)](#page-1-3) and in inequality [\(9\)](#page-2-1) the root of the polynomial $u(z) = \lambda_0 + n\lambda_1 z$ should lie in the half-plane $Re(z) \leq \frac{n}{4}$ $\frac{n}{4}$, that is

$$
Re\left(-\frac{\lambda_0}{\lambda_1 n}\right) \leq \frac{n}{4}.
$$

Compare the sets of parameters in Theorem [1.2](#page-1-4) and Theorem [1.3,](#page-2-2) we see that in Theorem [1.2,](#page-1-4) this set(coefficient near $-p(z)$) is $\mathcal{A} = \{n\alpha : \alpha \in \Omega_{|z|}\}\$ and in Theorem [1.3,](#page-2-2) the set of such coefficient near $-p(z)$ is

$$
\mathcal{B} = \left\{ -\frac{2\lambda_0}{\lambda_1 n} : Re\left(-\frac{\lambda_0}{\lambda_1 n}\right) \leq \frac{n}{4} \right\} = \left\{ t : Re(t) \leq \frac{n}{2} \right\}.
$$

Consider the differential inequalities from Theorem [1.2](#page-1-4) and Theorem [1.3](#page-2-2) for $z \in B(\mathbb{D})$, we have $\mathcal{A} = \mathcal{B}$. But for $z \in \mathbb{C} \setminus \mathbb{D}$ we have $\mathcal{B} \subset \mathcal{A}$. In other words in Theorem [1.2](#page-1-4) and Theorem [1.3](#page-2-2) formally the same inequality was obtained but for different set of parameters. Moreover, the set of parameters in Theorem [1.2](#page-1-4) is essentially wider than that of Theorem [1.3.](#page-2-2) Consequently,

$$
\mathbb{B}[p](z) = \lambda_1 \frac{n}{2} S_\alpha[p](z). \tag{10}
$$

These facts were first observed by Ganenkova and Starkov [\[7\]](#page-8-6).

In this paper, we prove some more general results concerning the modified Smirnov operator preserving inequalities between polynomials, which in turn yields compact generalizations of some well-known polynomial inequalities.

2 Auxiliary Results

Before writing our main results, we prove the following lemmas which are required for their proofs.

Lemma 1. Let $P \in \mathbb{P}_n$, and has all zeros in $\overline{\mathbb{D}}$. Let $a \in B(\mathbb{D})$ be not the exceptional value for P, then all the zeros of $\tilde{\mathbb{S}}_a[P]$ lie in $\overline{\mathbb{D}}$.

The above lemma is due to Genenkova and starkov [\[7\]](#page-8-6). Also, the next lemma is due to Aziz $|1|$.

Lemma 2. If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \geq 0$, then for every $R \ge r$ and $rR \ge k^2$,

$$
|p(Rz)| \ge \left(\frac{R+k}{r+k}\right)^n |p(rz)| \quad \text{for} \quad z \in B(\mathbb{D}).
$$

Lemma 3. If $p \in \mathbb{P}_n$ with $|p(z)| < M$ for $z \in B(\mathbb{D})$, Then

 $|\tilde{\mathbb{S}}_a[p](z)| \leq \mathbb{M} |\tilde{\mathbb{S}}_a[z^n]| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.$

Proof. Since $|p(z)| < M$ for $z \in B(\mathbb{D})$. If λ is a complex number with $|\lambda| > 1$. Then

$$
|p(z)| < |\lambda \mathbb{M} z^n| \quad for \quad z \in B(\mathbb{D}).
$$

Since $\lambda \mathbb{M} z^n$ has all zeros in $\overline{\mathbb{D}}$, therefore by Rouche's theorem all zeros of $p(z) - \lambda \mathbb{M} z^n$ lie in $\overline{\mathbb{D}}$. Hence by Lemma [1,](#page-3-0) all zeros of $\tilde{\mathbb{S}}_a[p(z)-\lambda \mathbb{M}z^n]$ lie in $\overline{\mathbb{D}}$. Since $\tilde{\mathbb{S}}_a$ is linear, it follows that $\tilde{\mathbb{S}}_a[p](z) - \tilde{\mathbb{S}}_a[\lambda \mathbb{M} z^n]$ has all zeros in $\overline{\mathbb{D}}$. This gives

$$
|\tilde{\mathbb{S}}_a[p](z)| \le \mathbb{M}|\tilde{\mathbb{S}}_a[z^n]| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.\tag{11}
$$

Because if this is not true, then there exist some $z_0 \in \mathbb{C} \setminus \mathbb{D}$ such that

 $|\tilde{\mathbb{S}}_a[p](z_0)| > M |\tilde{\mathbb{S}}_a[z_0^n]|.$

Choosing $\lambda = \frac{\tilde{S}_a[p](z_0)}{\mathbb{M}(\tilde{S}_a[x_0])}$ $\frac{\mathbb{S}_a[p](z_0)}{\mathbb{M}[\tilde{\mathbb{S}}_a[z_0]^n]}$, so that $|\lambda| > 1$. With this choice of λ , we get a contradiction and hence inequality (11) is true. \Box

The next two Lemmas are given by Shah and Fatima [\[15\]](#page-8-12).

Lemma 4. If $p \in \mathbb{P}_n$, then for $z \in \mathbb{C} \setminus \mathbb{D}$

$$
|\tilde{S}_a[p](z)| + |\tilde{S}_a[g](z)| \le \left\{ |\tilde{S}_a[E_n](z)| + n|a| \right\} \max_{z \in B(\mathbb{D})} |p(z)|,\tag{12}
$$

where $g(z) = z^n p(\frac{1}{\overline{z}})$ $\frac{1}{\bar{z}}$.

Lemma 5. Let $p(z)$ and $P(z)$ be two polynomials such that $\deg p(z) \leq \deg P(z) = n$. If $P(z)$ has all zeros in $\mathbb D$ and $|p(z)| \leq |P(z)|$ for $z \in B(\mathbb D)$, then for any complex number β with $\beta \in \overline{\mathbb{D}}$ and $R \geq 1$, we have for $z \in B(\mathbb{D})$

$$
\left| \tilde{S}_a[p](Rz) - \beta \tilde{S}_a[p](z) \right| \le \left| \tilde{S}_a[P](Rz) - \beta \tilde{S}_a[P](z) \right|.
$$
 (13)

The result is sharp and equality holds if $a \in \overline{D}$ is not the exceptional value for the polynomial $p(z) = e^{i\gamma} P(z)$, where $\gamma \in \mathbb{R}$ and $P(z)$ is any polynomial having all the zeros in $\overline{\mathbb{D}}$ and strict inequality holds for $z \in \mathbb{D}$, unless $p(z) = e^{i\gamma} P(z)$, $\gamma \in \mathbb{R}$.

We now prove the following result which is a compact generalization of inequalities [\(1\)](#page-0-0) and [\(2\)](#page-0-1).

3 Main Results

Theorem 3.1. If $p(z)$ is a polynomial of degree n, then, for every real or complex number $\beta, |\beta| \leq 1$ and $R \geq 1$,

$$
\left|\tilde{S}_a[p](Rz) - \beta \tilde{S}_a[p](z)\right| \le \left|R^n - \beta\right| \left|\tilde{S}_a[E_n](z)\right| \max_{z \in B(\mathbb{D})} |p(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \tag{14}
$$

Equivalently for $R > 1$

$$
\left| (1+az)[RP'(Rz) - \beta P'(z)] - na[P(Rz) - \beta P(z)] \right|
$$

\n
$$
\leq n|R^{n} - \beta||z|^{n-1} \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}, \tag{15}
$$

where $E_n(z) = z^n$. The result is sharp and holds for $p(z) = \gamma z^n$, $\gamma \neq 0$.

Corollary 1. For $\beta = 0$, $a = 0$ and $R = 1$, the inequality [\(15\)](#page-4-0) reduces to

$$
|P'(z)| \le n|z|^{n-1} \max_{z \in B(\mathbb{D})} |P(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}
$$

which in particular gives inequality [\(1\)](#page-0-0). The equality holds for $p(z) = \gamma z^n, \gamma \neq 0$.

Remark 1. For $\beta = 0$ and $R > 1$, the inequality [\(14\)](#page-4-1) reduces to a result due to Fatima and Shah [\[6\]](#page-8-13).

Theorem 3.2. Let $P \in \mathbb{P}_n$ and $Q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$ $\frac{1}{z}$), then for every real or complex β with $|\beta| \leq 1$ and $R > 1$,

$$
|\tilde{S}_a[P](Rz) - \beta \tilde{S}_a[P](z)| + |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)|
$$

\n
$$
\leq \left\{ |R^n - \beta|\tilde{S}_a[E_n](z) + n|1 - \beta||a| \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \tag{16}
$$

Equivalently \mathbf{I}

$$
\left| (1+az)[RP'(Rz) - \beta P'(z)] - na[P(Rz) - \beta P(z)] \right|
$$

+
$$
(1+az)[RQ'(Rz) - \beta Q'(z)] - na[Q(Rz) - \beta Q(z)] \right|
$$

$$
\leq \left\{ n|R^n - \beta||z|^{n-1} + n|1 - \beta||a| \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \tag{17}
$$

Corollary 2. If $P(z)$ is a polynomial of degree n, then for $\beta = 1$, $a = 0$ and $R \ge 1$ in inequality [\(17\)](#page-4-2), we get

$$
\left| RP'(Rz) - P'(z) \right| + \left| RQ'(Rz) - Q'(z) \right| \le n(R^n - 1)|z|^{n-1} \max_{z \in B(\mathbb{D})} |P(z)|,
$$

where $E_n(z) = z^n$. The result is best possible and the equality holds for $p(z) = \gamma z^n$, $\gamma \neq 0$. Theorem [3.2](#page-4-3) includes a result due to Rahman [\[12\]](#page-8-9) as a special case.

Remark 2. If we take $\beta = 0$ and $R = 1$ in [\(16\)](#page-4-4), then the inequality reduces to Lemma [4](#page-3-2)

$$
\left|\tilde{S}_a[P](z)\right| + \left|\tilde{S}_a[Q](z)\right| \le \left[\tilde{S}_a[E_n](z) + n|a|\right] \max_{z \in B(\mathbb{D})} |P(z)|.
$$

Theorem 3.3. Let $P \in \mathbb{P}_n$ such that $P(z)$ is a polynomial of degree n which does not vanish in \mathbb{D} and $Q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$ $\frac{1}{\overline{z}}$), then for every real or complex β with $|\beta| \leq 1$ and $R > 1$

$$
\begin{aligned} |\tilde{S}_a[P](Rz) - \beta \tilde{S}_a[P](z)| \\ &\leq \left\{ \frac{|R^n - \beta|\tilde{S}_a[E_n](z) + n|1 - \beta||a|}{2} \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \end{aligned} \tag{18}
$$

Equivalently

$$
|(1+az)[RP'(Rz) - \beta P'(z)] - na[P(Rz) - \beta P(z)]
$$

\n
$$
\leq \left\{ \frac{|R^n - \beta|n|z|^{n-1} + n|1 - \beta||a|}{2} \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}, \qquad (19)
$$

where $E_n(z) = z^n$. The result is best possible and the equality holds for $p(z) = \gamma z^n$, $\gamma \neq 0$. Corollary 3. For $\beta = 0$, $a = 0$ and $R = 1$, the inequality [\(19\)](#page-5-0) reduces to

$$
|P'(z)| \leq \frac{n}{2}|z|^{n-1} \max_{z \in B(\mathbb{D})} |P(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}
$$

which in particular gives inequality [\(3\)](#page-0-2). The equality holds for $p(z) = \gamma z^n$, $\gamma \neq 0$.

Remark 3. For $\beta = 0$, $R = 1$, the inequality [\(18\)](#page-5-1) reduces to a result due to Shah and Fatima [\[15\]](#page-8-12)

$$
|\tilde{S}_a[P](z)| \leq \frac{1}{2} \left\{ \tilde{S}_a[E_n](z) + n|a| \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.
$$

4 Proofs of the theorems

Proof of Theorem [3.1.](#page-4-5) For $R = 1$, the result is trivial. Henceforth, we assume $R > 1$. If

$$
\max_{z \in B(\mathbb{D})} |p(z)| = M,
$$

then

$$
|p(z)| < M \quad \text{for} \quad z \in B(\mathbb{D}).
$$

Equivalently for every λ with $|\lambda| > 1$, we have

$$
|p(z)| < |M\lambda z^n| \quad \text{for} \quad z \in B(\mathbb{D}).\tag{20}
$$

Therefore by Rouche's theorem, it follows that all the zeros of $F(z) = p(z) + M\lambda z^n$ lie in D. By Lemma [1,](#page-3-0) it follows that all the zeros of $\tilde{S}_a[F](z)$ lie in D.

So, all the zeros of $\tilde{S}_a[F](z) = \tilde{S}_a[p(z) + M\lambda z^n]$ lie in \mathbb{D} . Therefore, all the zeros of $\tilde{S}_a[p(z)] + M\lambda \tilde{S}_a[E_n](z)$ lie in \mathbb{D} , where $E_n(z) = z^n$.

Now for any $\beta \in \mathbb{C}$, $|\beta| \leq 1$, by using the application of Lemma [1,](#page-3-0) it follows that all the zeros of

$$
\tilde{S}_a \{ [F](Rz) - \beta [F](z) \} = (1 + az) \{ RF'(Rz) - \beta F'(z) \} - na \{ F(Rz) - \beta F(z) \}
$$
\n
$$
= (1 + az) RF'(Rz) - naF(Rz) - \beta \{ (1 + az) F'(z) - (na) F(z) \}
$$
\n
$$
= \tilde{S}_a [F](Rz) - \beta \tilde{S}_a [F](z)
$$

lie in $\mathbb D$ for every a such that $a \in B(\mathbb D)$ is not the exceptional value of F.

Since,

$$
\tilde{S}_a[F](z) = \tilde{S}_a[p](z) + \lambda \tilde{S}_a[E_n](z)M
$$

and

$$
\tilde{S}_a[F](Rz) = \tilde{S}_a[p](Rz) + \lambda R^n \tilde{S}_a[E_n](z)M.
$$

Therefore, all the zeros of

$$
\tilde{S}_a \{ [F](Rz) - \beta [F](z) \} = \tilde{S}_a [F](Rz) - \beta \tilde{S}_a [F](z)
$$
\n
$$
= \tilde{S}_a[p](Rz) + \lambda R^n \tilde{S}_a [E_n](z)M - \beta \left\{ \tilde{S}_a[p](z) + \lambda \tilde{S}_a [E_n](z)M \right\}
$$
\n
$$
= \tilde{S}_a[p](Rz) - \beta \tilde{S}_a[p](z) + \lambda [R^n - |\beta|] \tilde{S}_a [E_n](z)M
$$

lie in $\mathbb D$ for $R > 1, |\lambda| > 1$. This implies

$$
|\tilde{S}_a[p](Rz) - \beta \tilde{S}_a[p](z)| \le |R^n - \beta||\tilde{S}_a[E_n](z)|M \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}, \ R > 1. \tag{21}
$$

If inequality [\(21\)](#page-6-0) is not true, then there is some point $z_0 \in \mathbb{C} \setminus \mathbb{D}$ such that

$$
|\tilde{S}_a[p](Rz_0) - \beta \tilde{S}_a[p](z_0)| > |R^n - \beta||\tilde{S}_a[E_n](z_0)|M \quad \text{for} \quad z_0 \in \mathbb{C} \setminus \mathbb{D}, \ R > 1. \tag{22}
$$

Take

$$
\lambda = -\frac{\tilde{S}_a[p](Rz_0) - \beta \tilde{S}_a[p](z_0)}{\{R^n - \beta\} \tilde{S}_a[E_n](z_0)M},
$$

such that $\lambda \in \mathbb{C} \setminus \mathbb{D}$ and with such choice of λ we have for $z_0 \in \mathbb{C} \setminus \mathbb{D}$

$$
\tilde{S}_a\{[F](Rz_0) - \beta[F](z_0)\} = 0
$$

which is a contradiction. Hence, we get

$$
|\tilde{S}_a[p](Rz) - \beta \tilde{S}_a[p](z)| \le |R^n - \beta||\tilde{S}_a[E_n](z)| \max_{z \in B(\mathbb{D})} |p(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \tag{23}
$$

 \Box

Proof of Theorem [3.2.](#page-4-3) Let

 $\max_{z \in B(\mathbb{D})} |p(z)| = M$

then $|p(z)| \leq M$ for $z \in \mathbb{D}$. Using Rouche's theorem, it follows that for every real or complex number α with $|\alpha| > 1$, $F(z) = P(z) + \alpha M$ does not vanish in \mathbb{D} . Using the Theorem [3.1](#page-4-5) and lemma [5,](#page-3-3) on the polynomial $F(z)$, we get for every real or complex number β with $|\beta| \leq 1$

$$
\begin{aligned}\n\left| \tilde{S}_a[P(Rz) - \beta \tilde{S}_a[P](z) + n\alpha (1 - \beta)|a|M \right| \\
&\le \left| \tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z) + \alpha (R^n - \beta) \tilde{S}_a[E_n](z)M \right| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D},\n\end{aligned}
$$

where $Q(z) = z^n p(\frac{1}{z})$ $\frac{1}{\bar{z}}$. Choosing the argument of α in R.H.S of above inequality such that

$$
\begin{aligned} |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z) + \alpha (R^n - \beta) \tilde{S}_a[E_n](z)M| \\ &= |\alpha| |(R^n - \beta) |\tilde{S}_a[E_n](z)M - |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)|. \end{aligned}
$$

Therefore

$$
|\tilde{S}_a[P(Rz) - \beta \tilde{S}_a[P](z)| - n|\alpha||1 - \beta||a|M
$$

\n
$$
\leq |\alpha||R^n - \beta|\tilde{S}_a[E_n](z)M - |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)|.
$$

This implies

$$
|\tilde{S}_a[P(Rz) - \beta \tilde{S}_a[P](z)| + |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)|
$$

\n
$$
\leq |\alpha| \left\{ |(R^n - \beta)|\tilde{S}_a[E_n](z) + n|1 - \beta||a| \right\} M.
$$

Now, letting $|\alpha|\to 1,$ we get

$$
|\tilde{S}_a[P(Rz) - \beta \tilde{S}_a[P](z)| + |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)|
$$

\n
$$
\leq \left\{ |(R^n - \beta)|\tilde{S}_a[E_n](z) + n|1 - \beta||a| \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \tag{24}
$$

Proof of Theorem [3.3.](#page-5-2) Let

$$
\max_{z \in B(\mathbb{D})} |p(z)| = M,
$$

then for every real or complex number β with $|\beta| \leq 1$ and $R > 1$, we have from inequality [\(24\)](#page-7-0)

$$
|\tilde{S}_a[P(Rz) - \beta \tilde{S}_a[P](z)| + |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)|
$$

\n
$$
\leq \left\{ |(R^n - \beta)|\tilde{S}_a[E_n](z) + n|1 - \beta||a| \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \tag{25}
$$

Also from the Lemma [5,](#page-3-3) we have

$$
\left| \tilde{S}_a[P](Rz) - \beta \tilde{S}_a[P](z) \right| \le \left| \tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z) \right|.
$$
 (26)

 $\text{Adding } \left| \right.$ $\tilde{S}_a[P](Rz) - \beta \tilde{S}_a[P](z)$ on the both sides of the inequality [\(26\)](#page-7-1), we get

$$
2\left\{ \left| \tilde{S}_a[P](Rz) - \beta \tilde{S}_a[P](z) \right| \right\} \leq |\tilde{S}_a[P(Rz) - \beta \tilde{S}_a[P](z)| + |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)|.
$$

Using the inequality [\(24\)](#page-7-0) in above inequality, we get

$$
2\left\{ \left| \tilde{S}_a[P](Rz) - \beta \tilde{S}_a[P](z) \right| \right\} \le \left\{ \left| (R^n - \beta) \right| \tilde{S}_a[E_n](z) + n|1 - \beta||a| \right\} \max_{z \in B(\mathbb{D})} |p(z)|.
$$

Therefore,

$$
\left|\tilde{S}_a[P](Rz) - \beta \tilde{S}_a[P](z)\right| \le \left\{ \frac{|(R^n - \beta)|\tilde{S}_a[E_n](z) + n|1 - \beta||a|}{2} \right\} \max_{z \in B(\mathbb{D})} |p(z)|
$$

for $z \in \mathbb{C} \setminus \mathbb{D}$.

 \Box

5 Declaration

Conflicts of interest: On behalf of authors, the corresponding author states that there is no conflict of interest.

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