# Bernstein-type Inequalities Preserved by Modified Smirnov Operator

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#### Abstract

In this paper, we consider a modified version of Smirnov operator and obtain some Bernstein-type inequalities preserved by this operator. In particular, we prove some compact generalizations of the well-known inequalities of Bernstein, Erdös and Lax, Ankeny and Rivlin and others.

**Key words and phrases**: Modified Smirnov operator, Polynomials, Bernstein inequality, Restricted zeros.

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### 1 Introduction

Let  $\mathbb{P}_n$  denote the class of polynomials  $P(z) := \sum_{j=0}^n a_j z^j$  in  $\mathbb{C}$  of degree at most  $n \in \mathbb{N}$ . Let  $\mathbb{D}$  be the open unit disk  $\{z \in \mathbb{C}; |z| < 1\}$ , such that  $\overline{\mathbb{D}}$  is the closure of  $\mathbb{D}$  and  $B(\mathbb{D})$  denotes its boundary.

Let  $P \in \mathbb{P}_n$ , then

$$\max_{z \in B(\mathbb{D})} |P'(z)| \le n \max_{z \in B(\mathbb{D})} |P(z)|$$
(1)

and

$$\max_{z \in B(\mathbb{D})} |P(Rz)| \le R^n \max_{z \in B(\mathbb{D})} |P(z)|.$$
(2)

Inequality (1) is a well-known theorem of Bernstein [4]. The inequality (2) is a simple deduction from the maximum modulus principle. In both the inequalities, the equality holds for  $P(z) = \alpha z^n, \alpha \neq 0$ .

If we restrict to a class of polynomials having no zeros in  $\mathbb{D}$ , the inequalities (1) and (2) can be sharpened. In fact, if  $P(z) \neq 0$  in  $\mathbb{D}$ , then

$$\max_{z \in B(\mathbb{D})} |P'(z)| \le \frac{n}{2} \max_{z \in B(\mathbb{D})} |P(z)| \tag{3}$$

and for R > 1,

$$\max_{z \in B(\mathbb{D})} |P(Rz)| \le \frac{R^n + 1}{2} \max_{z \in B(\mathbb{D})} |P(z)|.$$

$$\tag{4}$$

Inequality (3) was proved by Erdös and Lax [9], whereas Ankeny and Rivlin [3] used (3) to prove (4). These inequalities were further improved by Aziz and Dawood [2], where under the same hypothesis, it was proved that

$$\max_{z \in B(\mathbb{D})} |P'(z)| \le \frac{n}{2} \left\{ \max_{z \in B(\mathbb{D})} |P(z)| - \min_{z \in B(\mathbb{D})} |P(z)| \right\}$$
(5)

and for R > 1

$$\max_{z \in B(\mathbb{D})} |P(Rz)| \le \left\{ \frac{R^n + 1}{2} \right\} \max_{z \in B(\mathbb{D})} |P(z)| - \left\{ \frac{R^n - 1}{2} \right\} \min_{z \in B(\mathbb{D})} |P(z)|.$$
(6)

The equality in (3)-(6) holds for the polynomials of the form  $P(z) = \alpha z^n + \beta$ , with  $|\alpha| = |\beta|$ .

In 1930 Bernstein [5] also proved the following result:

**Theorem 1.1.** Let P(z) be a polynomial in  $\mathbb{P}_n$  having all zeros in  $\overline{\mathbb{D}}$  and p(z) be a polynomial of degree not exceeding that of P(z). If  $|p(z)| \leq |P(z)|$  on  $B(\mathbb{D})$ , then

$$|p'(z)| \le |P'(z)|$$
 for  $z \in \mathbb{C} \setminus \mathbb{D}$ .

The equality holds only if  $p = e^{i\gamma} P, \gamma \in \mathbb{R}$ .

For  $z \in \mathbb{C} \setminus \mathbb{D}$ , denoting by  $\Omega_{|z|}$  the image of the disc  $\{t \in \mathbb{C}; |t| < |z|\}$  under the mapping  $\phi(t) = \frac{t}{1+t}$ , Smirnov [13] as a generalization of Theorem 1.1 proved the following:

**Theorem 1.2.** Let p and P be polynomials possessing conditions as in Theorem 1.1, then for  $z \in \mathbb{C} \setminus \mathbb{D}$ 

$$|\mathbb{S}_{\alpha}[p](z)| \le |\mathbb{S}_{\alpha}[P](z)| \tag{7}$$

for all  $\alpha \in \overline{\Omega}_{|z|}$ , with  $\mathbb{S}_{\alpha}[p](z) := zp'(z) - n\alpha p(z)$ , where  $\alpha$  is a constant.

For  $\alpha \in \overline{\Omega}_{|z|}$  in inequality (7), the equality holds at a point  $z \in \mathbb{C} \setminus \mathbb{D}$  only if  $p = e^{i\gamma}P$ ,  $\gamma \in \mathbb{R}$ . We note that for fixed  $z \in \mathbb{C} \setminus \mathbb{D}$ , Inequality (7) can be replaced by (see for reference [7, 8])

$$\left|zp'(z) - n\frac{az}{1+az}p(z)\right| \le \left|zP'(z) - n\frac{az}{1+az}P(z)\right|,$$

where a is arbitrary from  $\overline{\mathbb{D}}$ . Equivalently for  $z \in \mathbb{C} \setminus \mathbb{D}$ 

$$|\tilde{\mathbb{S}}_a[p](z)| \le |\tilde{\mathbb{S}}_a[P](z)|,$$

where  $\tilde{\mathbb{S}}_{a}[p](z) = (1 + az)p'(z) - nap(z)$  is known as modified Smirnov operator. The modified Smirnov operator  $\tilde{\mathbb{S}}_{a}$  is more preferred in a sense than Smirnov operator  $\mathbb{S}_{\alpha}$ , because the parameter a of  $\tilde{\mathbb{S}}_{a}$  does not depend on z unlike parameter  $\alpha$  of  $\mathbb{S}_{\alpha}$ .

Marden [10] introduced a differential operator  $\mathbb{B} : \mathbb{P}_n \to \mathbb{P}_n$  of *mth* order. This operator carries a polynomial  $p \in \mathbb{P}_n$  into

$$\mathbb{B}[p](z) = \lambda_0 p(z) + \lambda_1 \frac{nz}{2} p'(z) + \dots + \lambda_m \left(\frac{nz}{2}\right)^m p^m(z),$$

where  $\lambda_0, \lambda_1, ..., \lambda_m$  are constants such that

$$u(z) = \lambda_0 + \binom{n}{1}\lambda_1 z + \dots + \binom{n}{m}\lambda_m z^m \neq 0, \quad for \quad Re(z) > \frac{n}{4}.$$
(8)

Rahman and Schmeisser [12] considered the Marden operator for m = 2 and showed that this operator preserves the inequalities between polynomials and accordingly proved the following:

**Theorem 1.3.** Let p and P be polynomials possessing conditions as in Theorem 1.1, Then

$$|\mathbb{B}[p](z)| \le |\mathbb{B}[P](z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D},$$
(9)

where the constants  $\lambda_0, \lambda_1, ..., \lambda_m$  possess condition (8). For  $z \in \mathbb{C} \setminus \mathbb{D}$  in (9), the equality holds if and only if  $p(z) = \gamma z^n$ ,  $\gamma \neq 0$ .

A variety of key papers concerning the  $\mathbb{B}$ -operator have appeared in the literature [11, 14].

In order to compare the Smirnov operator  $\mathbb{S}_{\alpha}[p](z) := zp'(z) - n\alpha p(z)$  and the Rahman's operator (with  $\lambda_2 = 0$ )  $\mathbb{B}[p](z) = \lambda_0 p(z) + \lambda_1 \frac{nz}{2} p'(z)$ , we require  $\alpha \in \overline{\Omega}_{|z|}$  in inequality (7) and in inequality (9) the root of the polynomial  $u(z) = \lambda_0 + n\lambda_1 z$  should lie in the half-plane  $Re(z) \leq \frac{n}{4}$ , that is

$$Re\left(-\frac{\lambda_0}{\lambda_1 n}\right) \leq \frac{n}{4}.$$

Compare the sets of parameters in Theorem 1.2 and Theorem 1.3, we see that in Theorem 1.2, this set(coefficient near -p(z)) is  $\mathcal{A} = \{n\alpha : \alpha \in \Omega_{|z|}\}$  and in Theorem 1.3, the set of such coefficient near -p(z) is

$$\mathcal{B} = \left\{ -\frac{2\lambda_0}{\lambda_1 n} : Re\left(-\frac{\lambda_0}{\lambda_1 n}\right) \le \frac{n}{4} \right\} = \left\{ t : Re(t) \le \frac{n}{2} \right\}.$$

Consider the differential inequalities from Theorem 1.2 and Theorem 1.3 for  $z \in B(\mathbb{D})$ , we have  $\mathcal{A} = \mathcal{B}$ . But for  $z \in \mathbb{C} \setminus \mathbb{D}$  we have  $\mathcal{B} \subset \mathcal{A}$ . In other words in Theorem 1.2 and Theorem 1.3 formally the same inequality was obtained but for different set of parameters. Moreover, the set of parameters in Theorem 1.2 is essentially wider than that of Theorem 1.3. Consequently,

$$\mathbb{B}[p](z) = \lambda_1 \frac{n}{2} S_{\alpha}[p](z).$$
(10)

These facts were first observed by Ganenkova and Starkov [7].

In this paper, we prove some more general results concerning the modified Smirnov operator preserving inequalities between polynomials, which in turn yields compact generalizations of some well-known polynomial inequalities.

#### 2 Auxiliary Results

Before writing our main results, we prove the following lemmas which are required for their proofs.

**Lemma 1.** Let  $P \in \mathbb{P}_n$ , and has all zeros in  $\overline{\mathbb{D}}$ . Let  $a \in B(\mathbb{D})$  be not the exceptional value for P, then all the zeros of  $\tilde{\mathbb{S}}_a[P]$  lie in  $\overline{\mathbb{D}}$ .

The above lemma is due to Genenkova and starkov [7]. Also, the next lemma is due to Aziz [1].

**Lemma 2.** If p(z) is a polynomial of degree n having all its zeros in  $|z| \le k$ , where  $k \ge 0$ , then for every  $R \ge r$  and  $rR \ge k^2$ ,

$$|p(Rz)| \ge \left(\frac{R+k}{r+k}\right)^n |p(rz)| \quad for \quad z \in B(\mathbb{D}).$$

**Lemma 3.** If  $p \in \mathbb{P}_n$  with  $|p(z)| < \mathbb{M}$  for  $z \in B(\mathbb{D})$ , Then

 $|\tilde{\mathbb{S}}_a[p](z)| \leq \mathbb{M}|\tilde{\mathbb{S}}_a[z^n]| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.$ 

*Proof.* Since  $|p(z)| < \mathbb{M}$  for  $z \in B(\mathbb{D})$ . If  $\lambda$  is a complex number with  $|\lambda| > 1$ . Then

$$|p(z)| < |\lambda \mathbb{M} z^n| \quad for \quad z \in B(\mathbb{D}).$$

Since  $\lambda \mathbb{M} z^n$  has all zeros in  $\overline{\mathbb{D}}$ , therefore by Rouche's theorem all zeros of  $p(z) - \lambda \mathbb{M} z^n$ lie in  $\overline{\mathbb{D}}$ . Hence by Lemma 1, all zeros of  $\tilde{\mathbb{S}}_a[p(z) - \lambda \mathbb{M} z^n]$  lie in  $\overline{\mathbb{D}}$ . Since  $\tilde{\mathbb{S}}_a$  is linear, it follows that  $\tilde{\mathbb{S}}_a[p](z) - \tilde{\mathbb{S}}_a[\lambda \mathbb{M} z^n]$  has all zeros in  $\overline{\mathbb{D}}$ . This gives

$$|\tilde{\mathbb{S}}_{a}[p](z)| \leq \mathbb{M}|\tilde{\mathbb{S}}_{a}[z^{n}]| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.$$
(11)

Because if this is not true, then there exist some  $z_0 \in \mathbb{C} \setminus \mathbb{D}$  such that

 $|\tilde{\mathbb{S}}_a[p](z_0)| > \mathbb{M}|\tilde{\mathbb{S}}_a[z_0^n]|.$ 

Choosing  $\lambda = \frac{\tilde{\mathbb{S}}_{a}[p](z_{0})}{\mathbb{M}[\tilde{\mathbb{S}}_{a}[z_{0}^{n}]}$ , so that  $|\lambda| > 1$ . With this choice of  $\lambda$ , we get a contradiction and hence inequality (11) is true.

The next two Lemmas are given by Shah and Fatima [15].

**Lemma 4.** If  $p \in \mathbb{P}_n$ , then for  $z \in \mathbb{C} \setminus \mathbb{D}$ 

$$|\tilde{S}_{a}[p](z)| + |\tilde{S}_{a}[g](z)| \le \left\{ |\tilde{S}_{a}[E_{n}](z)| + n|a| \right\} \max_{z \in B(\mathbb{D})} |p(z)|,$$
(12)

where  $g(z) = z^n \overline{p(\frac{1}{\overline{z}})}$ .

**Lemma 5.** Let p(z) and P(z) be two polynomials such that  $\deg p(z) \leq \deg P(z) = n$ . If P(z) has all zeros in  $\mathbb{D}$  and  $|p(z)| \leq |P(z)|$  for  $z \in B(\mathbb{D})$ , then for any complex number  $\beta$  with  $\beta \in \overline{\mathbb{D}}$  and  $R \geq 1$ , we have for  $z \in B(\mathbb{D})$ 

$$\left|\tilde{S}_{a}[p](Rz) - \beta \tilde{S}_{a}[p](z)\right| \leq \left|\tilde{S}_{a}[P](Rz) - \beta \tilde{S}_{a}[P](z)\right|.$$
(13)

The result is sharp and equality holds if  $a \in \overline{\mathbb{D}}$  is not the exceptional value for the polynomial  $p(z) = e^{i\gamma}P(z)$ , where  $\gamma \in \mathbb{R}$  and P(z) is any polynomial having all the zeros in  $\overline{\mathbb{D}}$  and strict inequality holds for  $z \in \mathbb{D}$ , unless  $p(z) = e^{i\gamma}P(z)$ ,  $\gamma \in \mathbb{R}$ .

We now prove the following result which is a compact generalization of inequalities (1) and (2).

### 3 Main Results

**Theorem 3.1.** If p(z) is a polynomial of degree n, then, for every real or complex number  $\beta$ ,  $|\beta| \leq 1$  and  $R \geq 1$ ,

$$\left|\tilde{S}_{a}[p](Rz) - \beta \tilde{S}_{a}[p](z)\right| \leq \left|R^{n} - \beta\right| \left|\tilde{S}_{a}[E_{n}](z)\right| \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.$$
(14)

Equivalently for R > 1

$$\begin{aligned} \left| (1+az)[RP'(Rz) - \beta P'(z)] - na[P(Rz) - \beta P(z)] \right| \\ &\leq n |R^n - \beta| |z|^{n-1} \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}, \end{aligned}$$
(15)

where  $E_n(z) = z^n$ . The result is sharp and holds for  $p(z) = \gamma z^n$ ,  $\gamma \neq 0$ .

**Corollary 1.** For  $\beta = 0$ , a = 0 and R = 1, the inequality (15) reduces to

$$|P'(z)| \le n|z|^{n-1} \max_{z \in B(\mathbb{D})} |P(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}$$

which in particular gives inequality (1). The equality holds for  $p(z) = \gamma z^n, \ \gamma \neq 0$ .

**Remark 1.** For  $\beta = 0$  and R > 1, the inequality (14) reduces to a result due to Fatima and Shah [6].

**Theorem 3.2.** Let  $P \in \mathbb{P}_n$  and  $Q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$ , then for every real or complex  $\beta$  with  $|\beta| \leq 1$  and R > 1,

$$\begin{split} |\tilde{S}_a[P](Rz) - \beta \tilde{S}_a[P](z)| + |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)| \\ \leq \left\{ |R^n - \beta| \tilde{S}_a[E_n](z) + n|1 - \beta||a| \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.$$
(16)

Equivalently

$$\left| (1+az)[RP'(Rz) - \beta P'(z)] - na[P(Rz) - \beta P(z)] + (1+az)[RQ'(Rz) - \beta Q'(z)] - na[Q(Rz) - \beta Q(z)] \right|$$
  
$$\leq \left\{ n|R^n - \beta||z|^{n-1} + n|1 - \beta||a| \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.$$
(17)

**Corollary 2.** If P(z) is a polynomial of degree n, then for  $\beta = 1$ , a = 0 and  $R \ge 1$  in inequality (17), we get

$$\left| RP'(Rz) - P'(z) \right| + \left| RQ'(Rz) - Q'(z) \right| \le n(R^n - 1)|z|^{n-1} \max_{z \in B(\mathbb{D})} |P(z)|,$$

where  $E_n(z) = z^n$ . The result is best possible and the equality holds for  $p(z) = \gamma z^n$ ,  $\gamma \neq 0$ . Theorem 3.2 includes a result due to Rahman [12] as a special case.

**Remark 2.** If we take  $\beta = 0$  and R = 1 in (16), then the inequality reduces to Lemma 4

$$\left|\tilde{S}_a[P](z)\right| + \left|\tilde{S}_a[Q](z)\right| \le \left[\tilde{S}_a[E_n](z) + n|a|\right] \max_{z \in B(\mathbb{D})} |P(z)|.$$

**Theorem 3.3.** Let  $P \in \mathbb{P}_n$  such that P(z) is a polynomial of degree n which does not vanish in  $\mathbb{D}$  and  $Q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$ , then for every real or complex  $\beta$  with  $|\beta| \leq 1$  and R > 1

$$|\tilde{S}_{a}[P](Rz) - \beta \tilde{S}_{a}[P](z)| \leq \left\{ \frac{|R^{n} - \beta|\tilde{S}_{a}[E_{n}](z) + n|1 - \beta||a|}{2} \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.$$
(18)

Equivalently

$$|(1+az)[RP'(Rz) - \beta P'(z)] - na[P(Rz) - \beta P(z)]$$

$$\leq \left\{ \frac{|R^n - \beta|n|z|^{n-1} + n|1 - \beta||a|}{2} \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D},$$
(19)

where  $E_n(z) = z^n$ . The result is best possible and the equality holds for  $p(z) = \gamma z^n$ ,  $\gamma \neq 0$ . Corollary 3. For  $\beta = 0$ , a = 0 and R = 1, the inequality (19) reduces to

$$|P'(z)| \le \frac{n}{2} |z|^{n-1} \max_{z \in B(\mathbb{D})} |P(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}$$

which in particular gives inequality (3). The equality holds for  $p(z) = \gamma z^n$ ,  $\gamma \neq 0$ .

**Remark 3.** For  $\beta = 0$ , R = 1, the inequality (18) reduces to a result due to Shah and Fatima [15]

$$|\tilde{S}_a[P](z)| \le \frac{1}{2} \left\{ \tilde{S}_a[E_n](z) + n|a| \right\} \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

# 4 Proofs of the theorems

Proof of Theorem 3.1. For R = 1, the result is trivial. Henceforth, we assume R > 1. If

$$\max_{z \in B(\mathbb{D})} |p(z)| = M$$

then

$$p(z)| < M$$
 for  $z \in B(\mathbb{D})$ .

Equivalently for every  $\lambda$  with  $|\lambda| > 1$ , we have

$$|p(z)| < |M\lambda z^n| \quad \text{for} \quad z \in B(\mathbb{D}).$$
(20)

Therefore by Rouche's theorem, it follows that all the zeros of  $F(z) = p(z) + M\lambda z^n$  lie in  $\mathbb{D}$ . By Lemma 1, it follows that all the zeros of  $\tilde{S}_a[F](z)$  lie in  $\mathbb{D}$ .

So, all the zeros of  $\tilde{S}_a[F](z) = \tilde{S}_a[p(z) + M\lambda z^n]$  lie in  $\mathbb{D}$ . Therefore, all the zeros of  $\tilde{S}_a[p(z)] + M\lambda \tilde{S}_a[E_n](z)$  lie in  $\mathbb{D}$ , where  $E_n(z) = z^n$ .

Now for any  $\beta \in \mathbb{C}$ ,  $|\beta| \leq 1$ , by using the application of Lemma 1, it follows that all the zeros of

$$\hat{S}_{a} \{ [F](Rz) - \beta[F](z) \} = (1 + az) \{ RF'(Rz) - \beta F'(z) \} - na \{ F(Rz) - \beta F(z) \} 
= (1 + az) RF'(Rz) - naF(Rz) - \beta \{ (1 + az)F'(z) - (na)F(z) \} 
= \tilde{S}_{a}[F](Rz) - \beta \tilde{S}_{a}[F](z)$$

lie in  $\mathbb{D}$  for every a such that  $a \in B(\mathbb{D})$  is not the exceptional value of F.

Since,

$$\tilde{S}_a[F](z) = \tilde{S}_a[p](z) + \lambda \tilde{S}_a[E_n](z)M$$

and

$$\tilde{S}_a[F](Rz) = \tilde{S}_a[p](Rz) + \lambda R^n \tilde{S}_a[E_n](z) M.$$

Therefore, all the zeros of

$$\begin{split} \tilde{S}_a\left\{[F](Rz) - \beta[F](z)\right\} &= \tilde{S}_a[F](Rz) - \beta \tilde{S}_a[F](z) \\ &= \tilde{S}_a[p](Rz) + \lambda R^n \tilde{S}_a[E_n](z)M - \beta \left\{\tilde{S}_a[p](z) + \lambda \tilde{S}_a[E_n](z)M\right\} \\ &= \tilde{S}_a[p](Rz) - \beta \tilde{S}_a[p](z) + \lambda [R^n - |\beta|] \tilde{S}_a[E_n](z)M \end{split}$$

lie in  $\mathbb{D}$  for R > 1,  $|\lambda| > 1$ . This implies

$$|\tilde{S}_a[p](Rz) - \beta \tilde{S}_a[p](z)| \le |R^n - \beta| |\tilde{S}_a[E_n](z)| M \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}, \ R > 1.$$
(21)

If inequality (21) is not true, then there is some point  $z_0 \in \mathbb{C} \setminus \mathbb{D}$  such that

$$|\tilde{S}_a[p](Rz_0) - \beta \tilde{S}_a[p](z_0)| > |R^n - \beta| |\tilde{S}_a[E_n](z_0)| M \quad for \quad z_0 \in \mathbb{C} \setminus \mathbb{D}, \ R > 1.$$
(22)

Take

$$\lambda = -\frac{\tilde{S}_a[p](Rz_0) - \beta \tilde{S}_a[p](z_0)}{\{R^n - \beta\} \tilde{S}_a[E_n](z_0)M},$$

such that  $\lambda \in \mathbb{C} \setminus \mathbb{D}$  and with such choice of  $\lambda$  we have for  $z_0 \in \mathbb{C} \setminus \mathbb{D}$ 

 $\tilde{S}_a \{ [F](Rz_0) - \beta[F](z_0) \} = 0$ 

which is a contradiction. Hence, we get

$$|\tilde{S}_a[p](Rz) - \beta \tilde{S}_a[p](z)| \le |R^n - \beta| |\tilde{S}_a[E_n](z)| \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}.$$
(23)

Proof of Theorem 3.2. Let

 $\max_{z \in B(\mathbb{D})} |p(z)| = M$ 

then  $|p(z)| \leq M$  for  $z \in \mathbb{D}$ . Using Rouche's theorem, it follows that for every real or complex number  $\alpha$  with  $|\alpha| > 1$ ,  $F(z) = P(z) + \alpha M$  does not vanish in  $\mathbb{D}$ . Using the Theorem 3.1 and lemma 5, on the polynomial F(z), we get for every real or complex number  $\beta$  with  $|\beta| \leq 1$ 

$$\left| \begin{split} \tilde{S}_{a}[P(Rz) - \beta \tilde{S}_{a}[P](z) + n\alpha(1-\beta)|a|M| \\ \leq \left| \tilde{S}_{a}[Q](Rz) - \beta \tilde{S}_{a}[Q](z) + \alpha(R^{n}-\beta)\tilde{S}_{a}[E_{n}](z)M \right| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}, \end{split}$$

where  $Q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$ . Choosing the argument of  $\alpha$  in R.H.S of above inequality such that

$$\begin{split} &|\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z) + \alpha (R^n - \beta) \tilde{S}_a[E_n](z)M| \\ &= |\alpha| |(R^n - \beta)| \tilde{S}_a[E_n](z)M - |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)|. \end{split}$$

Therefore

$$\begin{split} &|\tilde{S}_a[P(Rz) - \beta \tilde{S}_a[P](z)| - n|\alpha||1 - \beta||a|M\\ &\leq |\alpha||R^n - \beta|\tilde{S}_a[E_n](z)M - |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)|. \end{split}$$

This implies

$$\begin{split} &|\tilde{S}_a[P(Rz) - \beta \tilde{S}_a[P](z)| + |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)| \\ &\leq |\alpha| \left\{ |(R^n - \beta)| \tilde{S}_a[E_n](z) + n|1 - \beta||a| \right\} M. \end{split}$$

Now, letting  $|\alpha| \to 1$ , we get

$$\begin{split} &|\tilde{S}_{a}[P(Rz) - \beta \tilde{S}_{a}[P](z)| + |\tilde{S}_{a}[Q](Rz) - \beta \tilde{S}_{a}[Q](z)| \\ &\leq \Big\{ |(R^{n} - \beta)|\tilde{S}_{a}[E_{n}](z) + n|1 - \beta||a| \Big\} \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}. \end{split}$$
(24)

Proof of Theorem 3.3. Let

$$\max_{z \in B(\mathbb{D})} |p(z)| = M$$

then for every real or complex number  $\beta$  with  $|\beta| \leq 1$  and R > 1, we have from inequality (24)

$$\begin{split} |\tilde{S}_a[P(Rz) - \beta \tilde{S}_a[P](z)| + |\tilde{S}_a[Q](Rz) - \beta \tilde{S}_a[Q](z)| \\ \leq \Big\{ |(R^n - \beta)|\tilde{S}_a[E_n](z) + n|1 - \beta||a| \Big\} \max_{z \in B(\mathbb{D})} |p(z)| \quad for \quad z \in \mathbb{C} \setminus \mathbb{D}. \end{split}$$
(25)

Also from the Lemma 5, we have

$$\left|\tilde{S}_{a}[P](Rz) - \beta \tilde{S}_{a}[P](z)\right| \leq \left|\tilde{S}_{a}[Q](Rz) - \beta \tilde{S}_{a}[Q](z)\right|.$$
(26)

Adding  $\left| \tilde{S}_a[P](Rz) - \beta \tilde{S}_a[P](z) \right|$  on the both sides of the inequality (26), we get

$$2\left\{\left|\tilde{S}_{a}[P](Rz) - \beta\tilde{S}_{a}[P](z)\right|\right\} \leq |\tilde{S}_{a}[P(Rz) - \beta\tilde{S}_{a}[P](z)| + |\tilde{S}_{a}[Q](Rz) - \beta\tilde{S}_{a}[Q](z)|.$$

Using the inequality (24) in above inequality, we get

$$2\left\{\left|\tilde{S}_a[P](Rz) - \beta\tilde{S}_a[P](z)\right|\right\} \le \left\{\left|(R^n - \beta)|\tilde{S}_a[E_n](z) + n|1 - \beta||a|\right\}\max_{z \in B(\mathbb{D})}|p(z)|.$$

Therefore,

$$\left|\tilde{S}_{a}[P](Rz) - \beta \tilde{S}_{a}[P](z)\right| \leq \left\{\frac{|(R^{n} - \beta)|\tilde{S}_{a}[E_{n}](z) + n|1 - \beta||a|}{2}\right\} \max_{z \in B(\mathbb{D})} |p(z)|$$
for  $z \in \mathbb{C} \setminus \mathbb{D}$ .

# 5 Declaration

**Conflicts of interest:** On behalf of authors, the corresponding author states that there is no conflict of interest.

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