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ENDOMORPHISM RINGS OF TOROIDAL SOLENOIDS

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ABSTRACT. We study the endomorphism ring $\operatorname{End}(G_A)$ of a subgroup G_A of \mathbb{Q}^n defined by a non-singular $n \times n$ -matrix A with integer entries. In the case when the characteristic polynomial of A is irreducible and an extra assumption holds if n is not prime, we show that $\operatorname{End}(G_A)$ is commutative and can be identified with a subring of the number field generated by an eigenvalue of A. The obtained results can be applied to studying endomorphisms of associated toroidal solenoids and \mathbb{Z}^n -odometers. In particular, we build a connection between toroidal solenoids and S-integer dynamical systems, provide a formula for the number of periodic points of a toroidal solenoid endomorphism, and show that the linear representation group of a \mathbb{Z}^n -odometer is computable.

1. INTRODUCTION

We study the endomorphism ring of a subgroup of \mathbb{Q}^n defined by a matrix with integer entries. The group arises naturally as the character group of a toroidal solenoid. More

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precisely, let $A \in M_n(\mathbb{Z})$ be a non-singular $n \times n$ -matrix with integer entries. Denote

$$G_A = \left\{ A^k \mathbf{x} \, | \, \mathbf{x} \in \mathbb{Z}^n, \, k \in \mathbb{Z} \right\}, \quad \mathbb{Z}^n \subseteq G_A \subseteq \mathbb{Q}^n.$$

One can readily verify that G_A is a subgroup of \mathbb{Q}^n . In [S22] and [S24], we study the classification problem of groups G_A . In particular, given two matrices $A, B \in M_n(\mathbb{Z})$ with integer entries, we answer the question of when the corresponding groups G_A , G_B are isomorphic as abstract groups in terms of the matrices A, B. We cover the case n = 2 in [S22] and the case of an arbitrary n in [S24]. Groups G_A arise in connection with toroidal solenoids. Toroidal solenoids defined by non-singular matrices with integer entries were introduced by M. C. McCord in 1965 |M65|. A toroidal solenoid S_A defined by a non-singular $A \in M_n(\mathbb{Z})$ is an *n*-dimensional topological abelian group. It is compact, metrizable, and connected, but not locally connected and not path connected. Toroidal solenoids are examples of inverse limit dynamical systems. When n = 1 and A = d, $d \in \mathbb{Z}$, solenoids are called *d*-adic solenoids or Vietoris solenoids. The first examples were studied by L. Vietoris in 1927 for d = 2 [V27] and later in 1930 by van Dantzig for an arbitrary d [D30]. It is known that the first Cech cohomology group $H^1(\mathbb{S}_A, \mathbb{Z})$ of \mathbb{S}_A is isomorphic to G_{A^t} , where A^t is the transpose of A. On the other hand, since \mathbb{S}_A is a compact connected abelian group, $H^1(\mathbb{S}_A, \mathbb{Z})$ is isomorphic to the character group $\widehat{\mathbb{S}_A}$ of \mathbb{S}_A . Thus $\widehat{\mathbb{S}_A} \cong G_{A^t}$ and, using Pontryagin duality theorem, $\mathbb{S}_A \cong \widehat{G_{A^t}}$ as topological groups, where G_{A^t} is endowed with the discrete topology, the dual $\widehat{G_{A^t}}$ is endowed with the compact-open topology, and S_A is endowed with the topology of an inverse limit. Groups G_A also arise as the first cohomology groups of constant base \mathbb{Z}^n -odometers X_{A^t} defined by A^t [GPS19] and as the dimension groups of subshifts of finite type defined by A^t [BS24].

In this paper, we study the endomorphism ring $End(G_A)$ of G_A for an arbitrary n. Based on our work in [S22] and [S24], we give a general criterion for a matrix with rational entries $T \in M_n(\mathbb{Q})$ to define an endomorphism of G_A , equivalently, $T(G_A) \subseteq G_A$ (Theorem 5) as well as a more concrete description when n = 2 (Section 3.3). In the case when the characteristic polynomial of A is irreducible and an extra assumption holds if n is not prime, we prove that $T \in End(G_A)$ is either zero or T commutes with A. Moreover, the eigenvalues of T are elements of the form $a\lambda^k$, where λ is an eigenvalue of A, $k \in \mathbb{Z}$, and a is an algebraic integer of the number field $\mathbb{Q}(\lambda)$ generated by λ (Proposition 8). This implies that $End(G_A)$ is a commutative ring, and $Aut(G_A)$ is a finitely generated abelian group. As a consequence, considering a toroidal solenoid \mathbb{S}_A as a dynamical system with the automorphism σ defined by multiplication by A, it shows that every homomorphism of \mathbb{S}_A as a topological group is a morphism of the dynamical system (\mathbb{S}_A, σ) . Other noteworthy consequences include the connection between toroidal solenoids and \mathcal{S} -integer dynamical systems defined in [CEW97]. It turns out that one can consider a toroidal solenoid as a similar but more general object than an \mathcal{S} -integer dynamical system. The connection allows us to obtain a formula for the number of periodic points of an endomorphism of \mathbb{S}_A similar to the one in [CEW97]. The case of n = 2 is covered in [HL23] for a more general class of toroidal solenoids, and our formula holds in higher-dimensions for \mathbb{S}_A . We also apply our results to \mathbb{Z}^n -odometers. We recover results of [CP24] in the case when n = 2 and generalize them to higher-dimensions. We also discuss the question of [CP24] on the computability of the linear representation group of a \mathbb{Z}^n -odometer defined by an integer matrix.

2. NOTATION

```
A, B \in \mathcal{M}_n(\mathbb{Z}) non-singular
h_A \in \mathbb{Z}[x] characteristic polynomial of A
G_A = \left\{ A^k \mathbf{x} \, | \, \mathbf{x} \in \mathbb{Z}^n, \, k \in \mathbb{Z} \right\}
\mathcal{R} = \mathbb{Z}\left[\frac{1}{\det A}\right]
\mathcal{P} = \mathcal{P}(A) = \{ \text{primes } p \in \mathbb{N} \text{ dividing } \det A \}
\mathcal{P}' = \mathcal{P}'(A) = \{ p \in \mathcal{P} \mid h_A \not\equiv x^n \pmod{p} \}
t_p = multiplicity of zero in the reduction of h_A modulo p
\mathbb{Q}_p = field of p-adic numbers
\mathbb{Z}_p = \text{ring of } p \text{-adic integers}
\mathbb{F}_p = finite field with p elements
\mathbb{Q}_p^n = (\mathbb{Q}_p)^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p
\mathbb{Z}_p^n = (\mathbb{Z}_p)^n = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p
\overline{G}_{A,p} = G_A \otimes_{\mathbb{Z}} \mathbb{Z}_p
\overline{\mathbb{Q}} = algebraic closure of \mathbb{Q}
\lambda = eigenvalue of A
K = \mathbb{Q}(\lambda)
\mathbf{u} = \begin{pmatrix} u_1 & \dots & u_n \end{pmatrix}^t eigenvector of A corresponding to \lambda
\mathbb{Z}[\mathbf{u}] = \{m_1 u_1 + \cdots + m_n u_n \mid m_1, \dots, m_n \in \mathbb{Z}\}
Y_A(\mathbf{u},\lambda) = \{m_1\lambda^{k_1}u_1 + \dots + m_n\lambda^{k_n}u_n \mid m_1,\dots,m_n,k_1,\dots,k_n \in \mathbb{Z}\}
\{\lambda_1,\ldots,\lambda_n\} = eigenvalues of A
\{\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_n\} = \mathrm{embeddings} \text{ of } K \mathrm{ into } \overline{\mathbb{Q}}
M = (\sigma_1(\mathbf{u}) \ldots \sigma_n(\mathbf{u})) \in \mathcal{M}_n(\mathbb{Q})
m = (\det M)^2 \in \mathbb{Z}
\mathcal{O}_K = \text{ring of integers of } K
\mathcal{O}_K^{\times} = \text{units of } \mathcal{O}_K
\mathfrak{p} = \text{prime ideal of } \mathcal{O}_K \text{ above } p
\operatorname{val}_{\mathfrak{p}}(x) = \mathfrak{p}-adic valuation of x \in K
\mathcal{S} = a set of prime ideals of \mathcal{O}_K
\mathcal{O}_{K,\mathcal{S}} = \{x \in K \mid \operatorname{val}_{\mathfrak{p}}(x) \ge 0 \text{ for any prime ideal } \mathfrak{p} \text{ of } \mathcal{O}_K \text{ not in } \mathcal{S}\}
\mathcal{U}_{K,\mathcal{S}} = \{ x \in K \mid \operatorname{val}_{\mathfrak{p}}(x) = 0 \text{ for any prime ideal } \mathfrak{p} \text{ of } \mathcal{O}_K \text{ not in } \mathcal{S} \}
\mathcal{S}_{\lambda} = all prime ideals of \mathcal{O}_{K} dividing \lambda
K_{\mathfrak{p}} = completion of K with respect to \mathfrak{p}
\mathcal{O}_{\mathfrak{p}} = \operatorname{ring} \operatorname{of} \operatorname{integers} \operatorname{of} K_{\mathfrak{p}}
\mathcal{X}_{A,\mathfrak{p}} = \operatorname{Span}_{K} \{ \text{generalized } \lambda \text{-eigenvectors of } A, \mathfrak{p} \mid \lambda \}
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rad(n) = product of all distinct prime divisors of $n \in \mathbb{Z}$ (u, v) = greatest common divisor of $u, v \in \mathbb{Z}$ \mathbb{S}_A = toroidal solenoid defined by A (47) \widehat{G} = Hom_{cont} (G, \mathbb{T}^1) Pontryagin dual of a topological group G $X_A = \mathbb{Z}^n$ -odometer defined by A (49) $\overrightarrow{N}(X_A)$ = linear representation group of X_A (50)

3. Endomorphisms of G_A

This section presents key results on endomorphisms of G_A , derived as consequences from the proofs in [S24] that characterize isomorphisms between two groups of the form $G_A, G_B \ (B \in \mathcal{M}_n(\mathbb{Z}) \text{ is non-singular}).$

3.1. Localization and easy cases. Let $A \in M_n(\mathbb{Z})$ be a non-singular $n \times n$ -matrix with integer entries. Denote

(1)
$$G_A = \left\{ A^k \mathbf{x} \, | \, \mathbf{x} \in \mathbb{Z}^n, \, k \in \mathbb{Z} \right\}, \quad \mathbb{Z}^n \subseteq G_A \subseteq \mathbb{Q}^n,$$

where $\mathbf{x} \in \mathbb{Z}^n$ is written as a column. Denote by $\operatorname{End}(G_A)$ the endomorphism ring of G_A , consisting of all (group) homomorphisms $\phi : G_A \longrightarrow G_A$. Denote

(2)
$$\mathcal{R} = \mathcal{R}(A) = \mathbb{Z}\left[\frac{1}{\det A}\right] = \left\{\frac{k}{(\det A)^l} \mid k, l \in \mathbb{Z}\right\}.$$

If $T \in \text{End}(G_A)$, then $T \in M_n(\mathcal{R})$. Indeed, one can check that any homomorphism from G_A to G_A is given by a matrix $T \in M_n(\mathbb{Q})$. Moreover, from the definition of G_A , there exists $i \in \mathbb{N} \cup \{0\}$ such that $A^i T \in M_n(\mathbb{Z})$, hence $T \in M_n(\mathcal{R})$. Thus, $\text{End}(G_A) \subseteq M_n(\mathcal{R})$. Note that $\mathbb{Z}[A, A^{-1}] \subseteq \text{End}(G_A)$. Let

$$\mathcal{P} = \mathcal{P}(A) = \{ \text{primes } p \in \mathbb{N} \text{ dividing } \det A \},\$$

 $\mathcal{P}' = \mathcal{P}'(A) = \{ p \in \mathcal{P} \mid h_A \not\equiv x^n \pmod{p} \},$

where $h_A \in \mathbb{Z}[x]$ is the characteristic polynomial of A.

For a prime $p \in \mathbb{N}$, let \mathbb{Z}_p denote the ring of *p*-adic integers and let \mathbb{Q}_p denote the field of *p*-adic numbers. Let $\overline{G}_{A,p} = G_A \otimes_{\mathbb{Z}} \mathbb{Z}_p$, so that

$$\overline{G}_{A,p} = \left\{ A^k \mathbf{x} \, | \, \mathbf{x} \in \mathbb{Z}_p^n, \, k \in \mathbb{Z} \right\}, \quad \mathbb{Z}_p^n \subseteq \overline{G}_{A,p} \subseteq \mathbb{Q}_p^n.$$

We know that

(3)
$$\overline{G}_{A,p} \cong \mathbb{Q}_p^{t_p} \oplus \mathbb{Z}_p^{n-t_p}$$

as \mathbb{Z}_p -modules, and t_p equals the multiplicity of zero in the reduction of h_A modulo p, $0 \le t_p \le n$ [S22, p. 196, Prop. 3.8].

Clearly, for $T \in M_n(\mathbb{Q})$, if $T(G_A) \subseteq G_A$, then $T(\overline{G}_{A,p}) \subseteq \overline{G}_{A,p}$, *i.e.*, every $T \in \text{End}(G_A)$ induces an endomorphism of $\overline{G}_{A,p}$. It turns out that the converse is also true. **Lemma 1.** For $T \in M_n(\mathbb{Q})$, we have that $T \in End(G_A)$ if and only if $T \in M_n(\mathcal{R})$ and $T \in End(\overline{G}_{A,p})$ for any $p \in \mathcal{P}'$.

Proof. Let $T \in M_n(\mathbb{Q})$. By above, the conditions are necessary. We now show that they are sufficient. It follows from [F73, p. 183, Lemma 93.2] that

(4)
$$G_A = \bigcap_{p \in \mathcal{P}} (\mathcal{R}^n \cap \overline{G}_{A,p})$$

(see also [S24, p. 8, Corollary 2.4] for more detail). Also, $\overline{G}_{A,p} = \mathbb{Q}_p^n$ for any prime $p \in \mathcal{P} \setminus \mathcal{P}'$ by (3). Thus, $T(\overline{G}_{A,p}) \subseteq \overline{G}_{A,p}$ for any $p \in \mathcal{P} \setminus \mathcal{P}'$. Therefore, if $T \in M_n(\mathcal{R})$ and $T \in \operatorname{End}(\overline{G}_{A,p})$ for any $p \in \mathcal{P}'$, then $T \in \operatorname{End}(G_A)$ by (4).

Lemma 2. (1) If $\mathcal{P} = \emptyset$, equivalently, $A \in \operatorname{GL}_n(\mathbb{Z})$, then

$$\operatorname{End}(G_A) = \operatorname{M}_n(\mathbb{Z}).$$

(2) If $\mathcal{P}' = \emptyset$ and $A \notin \operatorname{GL}_n(\mathbb{Z})$, then

$$\operatorname{End}(G_A) = \operatorname{M}_n(\mathcal{R}).$$

Proof. Clearly, $\mathcal{P} = \emptyset$ if and only if det $A = \pm 1$ if and only if $G_A = \mathbb{Z}^n$, and Lemma 2 (1) is clear. Lemma 2 (2) follows from Lemma 1.

Thus, by Lemma 2, for the rest of the paper we assume $A \notin \operatorname{GL}_n(\mathbb{Z})$ and $\mathcal{P}' \neq \emptyset$.

3.2. **Eigenvectors.** In practice, to apply Lemma 1, one needs a basis for the decomposition (3). In [S24], we show that a divisible part of $\overline{G}_{A,p}$ (isomorphic to $\mathbb{Q}_p^{t_p}$) can be described by generalized eigenvectors of A and to treat a reduced part of $\overline{G}_{A,p}$ (isomorphic to $\mathbb{Z}_p^{n-t_p}$) we need a characteristic of G_A . For the reader's convenience, we recall those results and put them together in a criterion for $T \in M_n(\mathbb{Q})$ to be an endomorphism of G_A (Theorem 5 below). We first introduce notation.

Throughout the text, $\overline{\mathbb{Q}}$ denotes a fixed algebraic closure of \mathbb{Q} . Let $F \subset \overline{\mathbb{Q}}$ be a finite extension of \mathbb{Q} that contains all the eigenvalues of A. Let \mathcal{O}_F denote the ring of integers of F. Throughout the paper, $\lambda_1, \ldots, \lambda_n \in \mathcal{O}_F$ denote (not necessarily distinct) eigenvalues of A and $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ denotes a Jordan canonical basis of A. Without loss of generality, we can assume that each $\mathbf{u}_i \in (\mathcal{O}_F)^n$, $i = 1, \ldots, n$. For a prime $p \in \mathbb{N}$ let \mathfrak{p} be a prime ideal of \mathcal{O}_F above p and let $\mathcal{X}_{A,\mathfrak{p}}$ denote the span over F of vectors in $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ corresponding to eigenvalues divisible by \mathfrak{p} . Note that

$$\dim_F \mathcal{X}_{A,\mathfrak{p}} = t_p,$$

where $t_p = t_p(A)$ denotes the multiplicity of zero in the reduction \bar{h}_A modulo p of the characteristic polynomial h_A of A, $0 \leq t_p \leq n$. Indeed, $\dim_F \mathcal{X}_{A,\mathfrak{p}}$ is the number of eigenvalues (with multiplicities) of A divisible by \mathfrak{p} . One can write $h_A = (x - \lambda_1) \cdots (x - \lambda_n)$ over \mathcal{O}_F . Considering the reduction \bar{h}_A of h_A modulo \mathfrak{p} , we see that the number of eigenvalues of A divisible by \mathfrak{p} is equal to the multiplicity t_p of zero in \bar{h}_A . Equivalently,

 $\mathcal{X}_{A,\mathfrak{p}}$ is generated over F by generalized λ -eigenvectors of A for any eigenvalue λ of A divisible by \mathfrak{p} .

Let $p \in \mathbb{N}$ be a prime and let $a \in \mathbb{Z}_p$, $a = \sum_{i=0}^{\infty} a_i p^i$, $\forall a_i \in \{0, 1, \dots, p-1\}$. For $k \ge 1$, we denote

$$a^{(k)} = a_0 + a_1 p + \dots + a_{k-1} p^{k-1} \in \mathbb{Z}.$$

Similarly, for $\mathbf{x} = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \in \mathbb{Z}_p^n, x_1, \dots, x_n \in \mathbb{Z}_p$, and $k \ge 1$, we denote

$$\mathbf{x}^{(k)} = \begin{pmatrix} x_1^{(k)} & \dots & x_n^{(k)} \end{pmatrix} \in \mathbb{Z}^n.$$

Finally, for $\mathbf{x} \in \mathbb{Z}_p^n$, we denote

$$p^{-\infty}\mathbf{x} = \{p^{-k}\mathbf{x}^{(k)} \mid k \in \mathbb{N}\} \subset \mathbb{Q}^n.$$

Lemma 3. There exists a basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ of \mathbb{Z}^n such that for any $p \in \mathcal{P}'$ there are $\alpha_{pij} \in \mathbb{Z}_p$, $i \in \{1, \ldots, t_p\}$, $j \in \{t_p + 1, \ldots, n\}$, and G_A is generated over \mathbb{Z} by

$$\{\mathbf{f}_1, \dots, \mathbf{f}_n, q^{-\infty}\mathbf{f}_1, \dots, q^{-\infty}\mathbf{f}_n, p^{-\infty}\mathbf{x}_{pi} \mid p \in \mathcal{P}', \ q \in \mathcal{P} \setminus \mathcal{P}', \ 1 \le i \le t_p\}$$

where

$$\mathbf{x}_{pi} = \mathbf{f}_i + \sum_{j=t_p+1}^n \alpha_{pij} \mathbf{f}_j.$$

Definition 4. Let $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ and $\alpha_{pij} \in \mathbb{Z}_p$ be as in Lemma 3. The set

$$M(A; \mathbf{f}_1, \dots, \mathbf{f}_n) = \{ \alpha_{pij} \in \mathbb{Z}_p \mid p \in \mathcal{P}', \ 1 \le i \le t_p < j \le n \}$$

is called a *characteristic* of G_A relative to the ordered basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ [GM81].

By conjugating A by a matrix $S \in \operatorname{GL}_n(\mathbb{Z})$, without loss of generality, we can assume that we have a characteristic of G_A relative to the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ [S24, Lemma 3.8]. The next theorem gives a necessary and sufficient criterion for $T \in \operatorname{M}_n(\mathbb{Q})$ to be an endomorphism of G_A , equivalently, $T(G_A) \subseteq G_A$. The proof follows easily from the proof of [S24, Theorem 4.3], which gives a criterion for when $T(G_A) = G_B$ for a non-singular $B \in \operatorname{M}_n(\mathbb{Z})$. Theorem 5 works well in practice, since there is an algorithm to produce a characteristic of G_A out of generalized eigenvectors of A (see [S24, Remark 4.5] and [GM81]).

Theorem 5. Let $A \in M_n(\mathbb{Z})$ be non-singular, $\mathcal{P}' \neq \emptyset$, let $F \subset \overline{\mathbb{Q}}$ be any finite extension of \mathbb{Q} that contains all the eigenvalues of A, and assume G_A has a characteristic

$$M(A; \mathbf{e}_1, \dots, \mathbf{e}_n) = \{ \alpha_{pij} \in \mathbb{Z}_p \mid p \in \mathcal{P}', \ 1 \le i \le t_p(A) < j \le n \}.$$

For $T \in M_n(\mathbb{Q})$, we have that $T(G_A) \subseteq G_A$ if and only if $T \in M_n(\mathcal{R})$, for any $p \in \mathcal{P}'$ and a prime ideal \mathfrak{p} of \mathcal{O}_F above p we have that

(5)
$$T(\mathcal{X}_{A,\mathfrak{p}}) \subseteq \mathcal{X}_{A,\mathfrak{p}},$$

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and any *j*-th column $(\gamma_{1j} \ldots \gamma_{nj})$ of $T, j \in \{t_p + 1, \ldots, n\}$, satisfies

(6)
$$\gamma_{kj} - \sum_{i=1}^{t_p} \gamma_{ij} \alpha_{pik} \in \mathbb{Z}_p \text{ for any } k \in \{t_p + 1, \dots, n\}.$$

3.3. 2-dimensional case. For $n \in \mathbb{Z}$, $n \neq \pm 1$, let $rad(n) \in \mathbb{N}$ be the product of all distinct prime divisors $p \in \mathbb{N}$ of n.

If n = 2, then there are three cases distinguished in [S22]:

(a) the characteristic polynomial $h_A \in \mathbb{Z}[x]$ of A is irreducible (equivalently, A has no rational eigenvalues),

(b) h_A is reducible (equivalently, A has eigenvalues $\lambda_1, \lambda_2 \in \mathbb{Z}$), rad (λ_1) does not divide rad (λ_2) , and rad (λ_2) does not divide rad (λ_1) ,

(c) h_A is reducible and every prime dividing one eigenvalue divides the other, *e.g.*, $rad(\lambda_2)$ divides $rad(\lambda_1)$ (denoted by $rad(\lambda_2) | rad(\lambda_1)$).

Case (a) is treated in Section 4.1 below.

Case (b). Note that if n = 2 and $\mathcal{P}' \neq \emptyset$, then det $A \neq \pm 1$, A has distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{Z}$, and hence A is diagonalizable over \mathbb{Q} . Moreover, there exists $S \in \mathrm{GL}_2(\mathbb{Z})$ such that $SAS^{-1} = M\Lambda M^{-1}$, where

(7)
$$\Lambda = \begin{pmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & u\\ 0 & v \end{pmatrix}, \ \lambda_1, \lambda_2, u, v \in \mathbb{Z}, \ (u, v) = 1, \ v \mid (\lambda_1 - \lambda_2),$$

where (u, v) = 1 means that u, v are coprime [S22, Corollary A.2]. Since $S(G_A) = G_{SAS^{-1}}$, *i.e.*, G_A , $G_{SAS^{-1}}$ are isomorphic, without loss of generality, we can assume that A itself is upper-triangular and has the form $A = M\Lambda M^{-1}$.

Theorem 6. Assume $A = M\Lambda M^{-1}$, where M, Λ are given by (7), and $\mathcal{P}' \neq \emptyset$. Assume case (b), i.e., $\operatorname{rad}(\lambda_1)$ does not divide $\operatorname{rad}(\lambda_2)$, and $\operatorname{rad}(\lambda_2)$ does not divide $\operatorname{rad}(\lambda_1)$. Then $T \in \operatorname{End}(G_A)$ if and only if

(8)
$$T = MXM^{-1}, \quad X = \operatorname{diag} \begin{pmatrix} x_1 & x_2 \end{pmatrix},$$

where $x_i \in \mathbb{Z}[\lambda_i^{-1}]$, i = 1, 2, and $\frac{x_1 - x_2}{v} \in \mathcal{R}$. In particular, $\operatorname{End}(G_A)$ is commutative, isomorphic to a subring of $\mathbb{Z}[\lambda_1^{-1}] \times \mathbb{Z}[\lambda_2^{-1}]$, and lies inside the centralizer of A in $M_2(\mathcal{R})$.

Proof. Assume $T \in \text{End}(G_A)$. Note that $t_p = 1$ for any $p \in \mathcal{P}'$. Thus, in the notation of Section 3.2, $F = \mathbb{Q}$, $\mathfrak{p} = p$, and $\mathcal{X}_{A,\mathfrak{p}}$ is a one-dimensional vector space over \mathbb{Q} generated by an eigenvector \mathbf{u} of A corresponding to an eigenvalue λ . Thus, (5) states that $T(\mathbf{u}) = x\mathbf{u}$, for some $x \in \mathbb{Q}$. In case (b), there exists a prime $p \in \mathbb{N}$ dividing λ_2 that does not divide λ_1 and there exists a prime $q \in \mathbb{N}$ dividing λ_1 that does not divide λ_2 , *i.e.*, $p, q \in \mathcal{P}'$.

Applying (5) to $F = \mathbb{Q}$, $\mathfrak{p} = p$ and $\mathfrak{p} = q$, we get that $T(\mathbf{u}_1) = x_1\mathbf{u}_1$, $T(\mathbf{u}_2) = x_2\mathbf{u}_2$ for eigenvectors $\mathbf{u}_1, \mathbf{u}_2$ of A corresponding to λ_1, λ_2 , respectively, and $x_1, x_2 \in \mathbb{Q}$. Hence, $T = MXM^{-1}$, where $X = \text{diag}\begin{pmatrix} x_1 & x_2 \end{pmatrix} \in M_2(\mathbb{Q})$. We will use Lemma 1 to show that the remaining conditions hold. One can easily check that for T and M given by (8) and (7), respectively, we have that $T \in M_2(\mathcal{R})$ if and only if $x_1, x_2, \frac{x_1-x_2}{v} \in \mathcal{R}$. Also, $T \in \text{End}(\overline{G}_{A,p})$ if and only if for any $m \in \mathbb{N} \cup \{0\}$ there exists $k_m \in \mathbb{N} \cup \{0\}$ with

(9)
$$A^{k_m}TA^{-m} \in \mathcal{M}_2(\mathbb{Z}_p).$$

Here,

$$A^{k_m}TA^{-m} = M \begin{pmatrix} x_1 \lambda_1^{k_m - m} & 0\\ 0 & x_2 \lambda_2^{k_m - m} \end{pmatrix} M^{-1}.$$

Note that any $p \in \mathcal{P}'$ does not divide $\lambda_1 - \lambda_2$ and, therefore, $M \in \mathrm{GL}_2(\mathbb{Z}_p)$. Thus, (9) holds if and only if $x_1\lambda_1^{k_m-m}, x_2\lambda_2^{k_m-m} \in \mathbb{Z}_p, p \in \mathcal{P}'$, which, together with $x_1, x_2 \in \mathcal{R}$, implies that $x_i \in \mathbb{Z}[\lambda_i^{-1}], i = 1, 2$. Similarly, one shows that (8) is sufficient for $T \in \mathrm{End}(G_A)$. \Box

Case (c) is different from cases (a) and (b) in the sense that for $T \in \text{End}(G_A)$, we have that $T(\mathbf{u})$ is *not* an eigenvector of A for *every* eigenvector \mathbf{u} of A. Namely, there exists $p \in \mathcal{P}'$ dividing λ_1 and hence (5) applied to $F = \mathbb{Q}$, $\mathfrak{p} = p$ states that $T(\mathbf{u}_1) = x_1\mathbf{u}_1$ for an eigenvector \mathbf{u}_1 of A corresponding to λ_1 . However, $T(\mathbf{u}_2)$ is not necessarily a multiple of \mathbf{u}_2 for an eigenvector \mathbf{u}_2 of A corresponding to λ_2 .

Theorem 7. Assume $A = M\Lambda M^{-1}$, where M, Λ are given by (7), and $\mathcal{P}' \neq \emptyset$. Assume case (c), i.e., $\operatorname{rad}(\lambda_2) | \operatorname{rad}(\lambda_1)$. Then $T \in \operatorname{End}(G_A)$ if and only if

(10)
$$T = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in \mathcal{M}_2(\mathcal{R}), \quad z \in \mathbb{Z}[\lambda_2^{-1}].$$

In particular, $\operatorname{End}(G_A)$ is not commutative and does not lie inside the centralizer of A in $\operatorname{M}_2(\mathcal{R})$.

Proof. Let $T \in \text{End}(G_A)$. Assume $\operatorname{rad}(\lambda_2) | \operatorname{rad}(\lambda_1)$. Note that any $p \in \mathcal{P}'$ divides λ_1 and does not divide λ_2 . Then (5) applied to $F = \mathbb{Q}$, $\mathfrak{p} = p$ states that $T(\mathbf{e}_1) = x_1\mathbf{e}_1$ for some $x_1 \in \mathbb{Q}$, since A is upper-triangular and \mathbf{e}_1 is an eigenvector of A corresponding to λ_1 (by assumption). Therefore, T is also upper-triangular. Let

(11)
$$T = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}.$$

As in the proof of Theorem 6, $T \in M_2(\mathcal{R})$ and (9) holds for T given by (11) and any $p \in \mathcal{P}'$. Taking into account that λ_2 is a unit in \mathbb{Z}_p and $M \in \mathrm{GL}_2(\mathbb{Z}_p)$, this implies $z \in \mathbb{Z}[\lambda_2^{-1}]$. Similarly, one shows that (10) is sufficient for $T \in \mathrm{End}(G_A)$. \Box

4. IRREDUCIBLE CHARACTERISTIC POLYNOMIAL

For an eigenvalue $\lambda \in \mathbb{Q}$ of A let $K = \mathbb{Q}(\lambda)$ and let S_{λ} consist of all prime ideals of the ring of integers \mathcal{O}_K of K dividing λ . (Note that $\lambda \in \mathcal{O}_K$.) We denote by $\mathcal{O}_{K,\lambda}$ the ring of S_{λ} -integers, *i.e.*,

(12) $\mathcal{O}_{K,\lambda} = \{x \in K \mid \operatorname{val}_{\mathfrak{p}}(x) \ge 0 \text{ for any prime ideal } \mathfrak{p} \text{ of } \mathcal{O}_K \text{ not in } \mathcal{S}_{\lambda}\} = \mathcal{O}_K [\lambda^{-1}]$ and

and

 $\mathcal{U}_{K,\mathcal{S}_{\lambda}} = \{ x \in K \mid \operatorname{val}_{\mathfrak{p}}(x) = 0 \text{ for any prime ideal } \mathfrak{p} \text{ of } \mathcal{O}_{K} \text{ not in } \mathcal{S}_{\lambda} \}$

is the group of \mathcal{S}_{λ} -units. In particular, $\mathcal{R} = \mathcal{O}_{\mathbb{Q},\mathcal{P}}$.

In the next proposition, we consider the generic case when the characteristic polynomial $h_A \in \mathbb{Z}[x]$ of A is irreducible. We also add an extra assumption that there exists a prime $p \in \mathbb{N}$ such that n and t_p are coprime, denoted by $(n, t_p) = 1$. It turns out that if $(n, t_p) = 1$, then $T(\mathbf{u})$ is a multiple of \mathbf{u} for any $T \in \text{End}(G_A)$ and any eigenvector \mathbf{u} of A. In particular, $\text{End}(G_A)$ is commutative. If $(n, t_p) \neq 1$ for any $p \in \mathcal{P}'$, then this is not necessarily true (see Example 5 below with a non-commutative $\text{End}(G_A)$).

Proposition 8. Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial $h_A \in \mathbb{Z}[x]$ and $\mathcal{P}' \neq \emptyset$. Assume, in addition, that there exists a prime $p \in \mathbb{N}$ with $(n, t_p) = 1$. Then there is a ring embedding $i = i(A, \lambda) : \operatorname{End}(G_A) \hookrightarrow \mathcal{O}_{K,\lambda}$, which induces a group embedding $i : \operatorname{Aut}(G_A) \hookrightarrow \mathcal{U}_{K,S_{\lambda}}$.

Proof. Let $T \in \text{End}(G_A)$ be arbitrary. It follows from [S24] that either T = 0 or Tis non-singular and preserves eigenspaces, *i.e.*, $T\mathbf{u}, \mathbf{u} \in K^n$ are both eigenvectors of Acorresponding to the same eigenvalue λ . We provide a comprehensive overview of the argument, both for the sake of completeness and because the results in [S24] are presented for isomorphisms from G_A to G_B , rather than endomorphisms. Here, $B \in M_n(\mathbb{Z})$ is another non-singular matrix. Nevertheless, the same principles apply. Let $p \in \mathcal{P}'$. By (3), $\overline{G}_{A,p} \cong \mathbb{Q}_p^{t_p} \oplus \mathbb{Z}_p^{n-t_p}$ as \mathbb{Z}_p -modules, $0 < t_p < n$. In [S24, Lemma 4.1], we show that after appropriate extension of scalars, the \mathbb{Z}_p -divisible part $D_p(A)$ of $\overline{G}_{A,p} = G_A \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is generated by eigenvectors of A corresponding to eigenvalues divisible by a prime ideal \mathfrak{p} above p. Let F be a finite Galois extension of \mathbb{Q} containing all the eigenvalues of A, *e.g.*, F is the splitting field of the characteristic polynomial h_A of A;

$$\mathbb{Q} \subset K = \mathbb{Q}(\lambda) \subseteq F \subset \overline{\mathbb{Q}}.$$

Let \mathfrak{p} be a prime ideal of the ring of integers \mathcal{O}_F of F above p. Denote by Σ the set of all distinct eigenvalues of A and let P denote the set of all $\lambda \in \Sigma$ divisible by \mathfrak{p} . Since h_A is irreducible, the cardinalities are $|\Sigma| = n$ and $|P| = t_p$. Denote

$$U_P = \bigoplus_{\lambda \in P} \operatorname{Span}_F(\mathbf{u}(\lambda)),$$

where $\mathbf{u}(\lambda) \in F^n$ is an eigenvector of A corresponding to λ . Thus, U_P is the span of all eigenvectors of A corresponding to eigenvalues divisible by \mathbf{p} and $U_P = \mathcal{X}_{A,\mathbf{p}}$ in the notation of Section 3.2. An endomorphism T of G_A induces a \mathbb{Z}_p -module endomorphism

of $G_{A,p}$ and therefore, $T(D_p(A)) \subseteq D_p(A)$. This implies $T(U_P) \subseteq U_P$. Thus, there exists a non-empty subset $S \subseteq \Sigma$ with the smallest cardinality satisfying $T(U_S) \subseteq U_S$. Denote $G = \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It is not hard to see that for any $R, V \subseteq \Sigma$ and $\sigma \in G$ we have

(13)
$$U_R \cap U_V = U_{R \cap V}, \quad \sigma(U_R) = U_{\sigma(R)}.$$

Assume $T(U_N) \subseteq U_N$ for some non-empty $N \subseteq \Sigma$ and let $\sigma \in G$. Since T is defined over \mathbb{Q} , using properties (13), we have $T(U_{\sigma(N)}) \subseteq U_{\sigma(N)}$. Hence, $T(U_{S \cap \sigma(N)}) \subseteq U_{S \cap \sigma(N)}$. Since S is the smallest with this property, either $S \cap \sigma(N) = S$ or $S \cap \sigma(N) = \emptyset$. Equivalently, $\sigma(S) \cap N = \sigma(S)$ or $\sigma(S) \cap N = \emptyset$. In particular, taking $N = \tau(S)$ for an arbitrary $\tau \in G$, either $\sigma(S) = \tau(S)$ or $\sigma(S) \cap \tau(S) = \emptyset$. Moreover, since h_A is irreducible, G acts transitively on Σ . This implies that N is a disjoint union of orbits $\sigma(S)$ of $S, \sigma \in G$, and, furthermore, there exists a subset $H \subseteq G$ depending on N such that

(14)
$$N = \bigsqcup_{\sigma \in H} \sigma(S), \quad |N| = |H| \cdot |S|.$$

Clearly, $T(U_N) \subseteq U_N$ holds for $N = \Sigma$ and also for N = P. Thus, by (14), |S| divides both n and t_p . By assumption, $(n, t_p) = 1$ and hence |S| = 1. Therefore, there exists an eigenvector \mathbf{u} (corresponding to an eigenvalue λ) of A such that $T(\mathbf{u}) = x\mathbf{u}$ for some $x \in \overline{\mathbb{Q}}$. For a fixed eigenvalue λ , we can choose $\mathbf{u} \in K^n$, $K = \mathbb{Q}(\lambda)$, and hence $x \in K$. Multiplying \mathbf{u} by an appropriate integer, without loss of generality, we can assume $\mathbf{u} \in (\mathcal{O}_K)^n$. Since h_A is irreducible, G acts transitively on the set of all eigenvalues of A, *i.e.*, there exist $\sigma_1, \ldots, \sigma_n \in G$, $\sigma_1 = \mathrm{id}$, such that $A = M\Lambda M^{-1}$, where

(15)
$$\Lambda = \operatorname{diag} \left(\sigma_1(\lambda) \quad \dots \quad \sigma_n(\lambda) \right), \quad M = \left(\sigma_1(\mathbf{u}) \quad \dots \quad \sigma_n(\mathbf{u}) \right)$$

with each $\sigma_i(\mathbf{u})$ written as a column, $i \in \{1, \ldots, n\}$. Then

(16)
$$T = MXM^{-1}, \quad X = \operatorname{diag} \left(\sigma_1(x) \quad \dots \quad \sigma_n(x) \right).$$

Thus, if λ is fixed, then T is completely determined by $x \in K$. A different choice of λ , e.g., $\sigma(\lambda)$ for some $\sigma \in G$, will result in $\sigma(x)$. We fix an eigenvalue λ of A and let Mbe given by (15). Define $i : \operatorname{End}(G_A) \longrightarrow K$ via $i(T) = x, T \in \operatorname{End}(G_A)$. By above, iis an injective ring homomorphism. Note that $M \in \operatorname{M}_n(\mathcal{O}_F)$. By [NT91, p. 4, Theorem 2], there exists a finite extension $L \subset \overline{\mathbb{Q}}$ of F and $P \in \operatorname{GL}_n(\mathcal{O}_L)$ such that PM is uppertriangular. From the definition of G_A , there exists $i \in \mathbb{N} \cup \{0\}$ such that $A^iT \in \operatorname{M}_n(\mathbb{Z})$. In particular, $\lambda^i x \in \mathcal{O}_L \cap K = \mathcal{O}_K$ and hence $x \in \mathcal{O}_K[\lambda^{-1}] = \mathcal{O}_{K,\lambda}$.

It is well-known that $\mathcal{U}_{K,S}$ is a finitely generated abelian group. Therefore, by Proposition 8, $\operatorname{Aut}(G_A)$ is also finitely-generated.

Corollary 9. Under the assumptions of Proposition 8, $End(G_A)$ is a commutative ring, and $Aut(G_A)$ is a finitely generated abelian group.

Clearly, $A^i \in \operatorname{End}(G_A)$ for any $i \in \mathbb{Z}$. Since $i(A^i) = \lambda^i$, this implies that $\mathbb{Z}[\lambda^{\pm 1}] = \mathbb{Z}[\lambda, \frac{1}{\lambda}] \subseteq i(\operatorname{End}(G_A))$ and $i(\operatorname{End}(G_A))$ (equivalently, $\operatorname{End}(G_A)$) is a $\mathbb{Z}[\lambda^{\pm 1}]$ -module (equivalently, a $\mathbb{Z}[t^{\pm 1}]$ -module via $\lambda \mapsto t, t$ is a variable). Thus,

$$\mathbb{Z}[\lambda^{\pm 1}] \subseteq i(\operatorname{End}(G_A)) \subseteq \mathcal{O}_{K,\lambda} = \mathcal{O}_K[\lambda^{-1}].$$

Moreover, under the assumptions of Proposition 8, $\operatorname{End}(G_A)$ is a finitely-generated $\mathbb{Z}[\lambda^{\pm 1}]$ module. Indeed, we have that

$$\mathbb{Z}[\lambda] \subseteq Y \subseteq \mathcal{O}_K, \quad Y = \imath(\operatorname{End}(G_A)) \cap \mathcal{O}_K.$$

It is well-known that both \mathcal{O}_K , $\mathbb{Z}[\lambda]$ are finitely-generated \mathbb{Z} -modules of rank n and therefore so is Y. Let $s = [Y : \mathbb{Z}[\lambda]]$, and let $\gamma_1, \ldots, \gamma_s \in Y$ be representatives of $Y/\mathbb{Z}[\lambda]$. Let $T \in \text{End}(G_A)$, $i(T) = x \in \mathcal{O}_{K,\lambda}$, so that $y = \lambda^i x \in \mathcal{O}_K$ for some $i \in \mathbb{N} \cup \{0\}$. Hence, $y \in Y$ and $y = \gamma + a$ for some $\gamma \in \{\gamma_1, \ldots, \gamma_s\}$ and $a \in \mathbb{Z}[\lambda]$. Then,

$$x = \lambda^{-i}y = \gamma\lambda^{-i} + a\lambda^{-i}, \quad a\lambda^{-i} \in \mathbb{Z}[\lambda^{\pm 1}],$$

i.e., 1, $\gamma_1, \ldots, \gamma_s$ generate $i(\operatorname{End}(G_A))$ over $\mathbb{Z}[\lambda^{\pm 1}]$. This proves the following

Corollary 10. Under the assumptions of Proposition 8, $i(\text{End}(G_A))$ (equivalently, $\text{End}(G_A)$) is a finitely-generated $\mathbb{Z}[\lambda^{\pm 1}]$ -module. If $\mathcal{O}_K = \mathbb{Z}[\lambda]$, then

$$i(\operatorname{End}(G_A)) = \mathbb{Z}[\lambda^{\pm 1}] = \mathcal{O}_{K,\lambda}.$$

Corollary 11. Under the assumptions of Proposition 8,

$$\operatorname{End}(G_A) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}(\lambda).$$

4.1. 2-dimensional case. The approach in the proof of Proposition 8 can be made more precise. We demonstrate it in the case n = 2. Assume $A \in M_2(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial $h_A \in \mathbb{Z}[x]$ and $\lambda \in \overline{\mathbb{Q}}$ is a root of h_A . Also, let $\mathcal{P}' \neq \emptyset$, equivalently, there exists a prime $p \in \mathbb{N}$ that divides det A and does not divide Tr A. In the notation of the proof of Proposition 8, we have that

(17)
$$A = M \begin{pmatrix} \lambda & 0 \\ 0 & \sigma(\lambda) \end{pmatrix} M^{-1}, \quad M = \begin{pmatrix} \mathbf{u} & \sigma(\mathbf{u}) \end{pmatrix},$$
$$X = \begin{pmatrix} x & 0 \\ 0 & \sigma(x) \end{pmatrix}, \quad T(x) = MXM^{-1},$$

where $\mathbf{u} \in (\mathcal{O}_K)^2$ is an eigenvector of A corresponding to λ written as a column, $K = \mathbb{Q}(\lambda)$ is a quadratic extension of \mathbb{Q} , $x \in K$, and $\sigma \in \operatorname{Gal}(K/\mathbb{Q})$ is the only non-trivial element. Moreover, there exist a finite extension L of K and $P \in \operatorname{GL}_2(\mathcal{O}_L)$ such that

(18)
$$PM = \begin{pmatrix} 1 & u \\ 0 & v \end{pmatrix}, \quad PTP^{-1} = \begin{pmatrix} x & w \\ 0 & \sigma(x) \end{pmatrix}, \quad w(x) = \frac{u(\sigma(x) - x)}{v},$$

where $u, v \in \mathcal{O}_L$, $\sigma(\lambda) - \lambda = vv'$ for some $v' \in \mathcal{O}_L$, and the ideal generated by u and v in \mathcal{O}_L is \mathcal{O}_L , denoted by $(u, v) = \mathcal{O}_L$. This follows from the fact that for any number field K there exists a finite extension L of K such that every ideal of \mathcal{O}_K becomes principal

in \mathcal{O}_L (see *e.g.*, [NT91, p. 4, Theorem 2] and [S22, Corollary A.2]). In particular, if \mathcal{O}_K is a principal ideal domain, then one can take L = K. Denote by \mathcal{S}' the set of all prime ideals of \mathcal{O}_L lying above all primes in \mathcal{P} .

Proposition 12. Assume $A \in M_2(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial $h_A \in \mathbb{Z}[x]$ and $\mathcal{P}' \neq \emptyset$. Then

(19)
$$i(\operatorname{End}(G_A)) = \{ x \in \mathcal{O}_{K,\lambda} \, | \, T(x) = MXM^{-1} \in \operatorname{M}_2(\mathcal{R}) \}.$$

Let $[\mathcal{O}_K : \mathbb{Z}[\lambda]] = l_1 l_2$, where $l_1, l_2 \in \mathbb{N}$, $\operatorname{rad}(l_1)$ divides det A, $(l_2, \det A) = 1$. Let $K = \mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}$ is square-free, and let $\{1, \omega\}$ be the integral basis of \mathcal{O}_K with $\omega = (1 + \sqrt{d})/2$ if $d \equiv 1 \pmod{4}$ and $\omega = \sqrt{d}$ otherwise. Then $i(\operatorname{End}(G_A))$ is generated over $\mathbb{Z}[\lambda^{\pm 1}]$ by $\{1, \alpha \omega\}$, where $\alpha \in \mathbb{N}$ divides l_2 . In particular, α is the smallest natural number such that

(20)
$$\frac{\alpha(\sigma(\omega) - \omega)}{v} \in \mathcal{O}_{L,\mathcal{S}'}.$$

Proof. Let $x \in \mathcal{O}_{K,\lambda}$ and let $T = MXM^{-1}$, where M and X are given by (17). It follows from the definition of $\mathcal{O}_{K,\lambda}$ that $x = y\lambda^{-i}$ for some $y \in \mathcal{O}_K$ and $i \in \mathbb{N} \cup \{0\}$. Moreover, $T = T(x) \in \operatorname{End}(G_A)$ if and only if $T \in M_2(\mathcal{R})$. Indeed, the necessary part follows from Section 3.1. To prove the sufficient part, By Lemma 1, it is enough to show that $T \in \operatorname{End}(\overline{G}_{A,p})$ for any $p \in \mathcal{P}'$. For any $p \in \mathcal{P}'$ there exists a prime ideal \mathfrak{p} of \mathcal{O}_L above p such that \mathfrak{p} divides λ and \mathfrak{p} does not divide $\sigma(\lambda)$. Let $L_{\mathfrak{p}}$ denote the completion of Lwith respect to \mathfrak{p} with its ring of integers $\mathcal{O}_{\mathfrak{p}}$. We have that $T(\overline{G}_{A,p}) \subseteq \overline{G}_{A,p} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{p}}$. Note that

$$V = \left\{ (PAP^{-1})^k \mathbf{x} \, | \, \mathbf{x} \in \mathcal{O}_{\mathfrak{p}}^2, \, k \in \mathbb{Z} \right\} = \left\{ \begin{pmatrix} \alpha & \beta \end{pmatrix}^t \, | \, \alpha \in L_{\mathfrak{p}}, \, \beta \in \mathcal{O}_{\mathfrak{p}} \right\} = L_{\mathfrak{p}} \oplus \mathcal{O}_{\mathfrak{p}},$$

since $P \in \operatorname{GL}_2(\mathcal{O}_L)$. Now it is clear that $PTP^{-1}V \subseteq V$, since PTP^{-1} has the form (18) and $\sigma(x) = \sigma(y)\sigma(\lambda)^{-i} \in \mathcal{O}_{\mathfrak{p}}$. Indeed, $\sigma(y) \in \mathcal{O}_K$, $\mathcal{O}_K \hookrightarrow \mathcal{O}_L \hookrightarrow \mathcal{O}_{\mathfrak{p}}$ and $\sigma(\lambda)$ is a unit in $\mathcal{O}_{\mathfrak{p}}$, since \mathfrak{p} does not divide $\sigma(\lambda)$. This shows that T defined by (17) with $x \in \mathcal{O}_{K,\lambda}$ belongs to $\operatorname{End}(G_A)$ if and only if $T \in M_2(\mathcal{R})$, *i.e.*, (19) holds.

We have that $x = y\lambda^{-i}$, $y \in \mathcal{O}_K$, and hence $y = a + b\omega$ for $a, b \in \mathbb{Z}$. Since $\mathbb{Z}[\lambda^{\pm 1}] \subseteq i(\operatorname{End}(G_A))$, $x \in i(\operatorname{End}(G_A))$ if and only if $b\omega \in i(\operatorname{End}(G_A))$ if and only if $T(b\omega) \in \operatorname{M}_2(\mathcal{R})$ by (19) if and only if $w(b\omega) \in \mathcal{O}_{L,S'}$ by (18) if and only if

(21)
$$\frac{b(\sigma(\omega) - \omega)}{v} \in \mathcal{O}_{L,S'},$$

since $(u, v) = \mathcal{O}_L$ by assumption. It is well-known and one can also easily check that $\sigma(\lambda) - \lambda = \pm [\mathcal{O}_K : \mathbb{Z}[\lambda]](\sigma(\omega) - \omega)$. Since v divides $\sigma(\lambda) - \lambda$ by (18) and l_1 is a unit in $\mathcal{O}_{L,\mathcal{S}'}$, (21) holds for $b = l_2$. Also, the set

$$I = \{ b \in \mathbb{Z} \mid b\omega \in i(\operatorname{End}(G_A)) \}$$

is an ideal of \mathbb{Z} , $l_2 \in I$, and therefore $I = (\alpha)$ is generated by the smallest $\alpha \in \mathbb{N}$ such that $\alpha \in I$. In particular, α divides l_2 .

Corollary 13. Assume $A \in M_2(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial $h_A \in \mathbb{Z}[x]$ and $\mathcal{P}' \neq \emptyset$. If $\operatorname{rad}[\mathcal{O}_K : \mathbb{Z}[\lambda]]$ divides det A, then

$$i(\operatorname{End}(G_A)) = \mathcal{O}_{K,\lambda}.$$

Proof. If $\operatorname{rad}[\mathcal{O}_K : \mathbb{Z}[\lambda]]$ divides det A, then in the notation of Proposition 12, $l_2 = 1$ and hence $\alpha = 1$. Then, by Proposition 12,

$$i(\operatorname{End}(G_A)) = \left\{ c + d\omega \,|\, c, d \in \mathbb{Z}[\lambda^{\pm 1}] \right\} = \mathcal{O}_{K,\lambda}.$$

Remark 14. Note that if $D = m^2 \cdot d$ is the discriminant of h_A , where $m \in \mathbb{N}$ and $d \in \mathbb{Z}$ is square-free, then

$$[\mathcal{O}_K : \mathbb{Z}[\lambda]] = \begin{cases} m, & d \equiv 1 \pmod{4} \\ \frac{m}{2}, & \text{otherwise.} \end{cases}$$

Example 1. In this example we show that the endomorphism ring of G_A does not determine G_A up to an isomorphism, *i.e.*, there exist $A, B \in M_n(\mathbb{Z})$ such that $\operatorname{End}(G_A) \cong \operatorname{End}(G_B)$ as rings, but $G_A \not\cong G_B$ as groups. Consider a quadratic number field $K = \mathbb{Q}(\lambda)$ defined by a root $\lambda \in \overline{\mathbb{Q}}$ of an irreducible polynomial $h = x^2 - x + 13$ [LMFDB, Number field 2.0.51.1]. Since the class group of K has order 2, there are two $\operatorname{GL}_2(\mathbb{Z})$ -conjugacy classes of matrices $[A], [B], A, B \in M_2(\mathbb{Z})$, with A corresponding to the trivial ideal \mathcal{O}_K and B corresponding to the non-trivial ideal $I = \mathbb{Z}[3, \lambda + 1]$, a generator of the class group. For example,

$$A = \begin{pmatrix} 0 & 1 \\ -13 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 3 \\ -5 & 2 \end{pmatrix}.$$

Both A and B share the same characteristic polynomial h with eigenvalues λ_1 , λ_2 , so that $\operatorname{Tr} A = \operatorname{Tr} B = 1$, det $A = \det B = 13$, the discriminant $D = -3 \cdot 17$ is square-free, $\lambda_{1,2} = \frac{1 \pm \sqrt{D}}{2}$, $\mathcal{R} = \mathcal{R}(A) = \mathcal{R}(B) = \{r13^s \mid r, s \in \mathbb{Z}\}$. Moreover, A, B are conjugated by a matrix from $\operatorname{GL}_2(\mathbb{Q})$, but there is no matrix $S \in \operatorname{GL}_2(\mathbb{Z})$ such that $A = SBS^{-1}$. By Corollary 10 and Remark 14, $i(\operatorname{End}(G_A)) = i(\operatorname{End}(G_B)) = \mathcal{O}_{K,\lambda}$, where *i* is defined by the choice of λ (λ_1 or λ_2). Thus, $\operatorname{End}(G_A) \cong \operatorname{End}(G_B)$ as rings. However, $G_A \ncong G_B$ as groups. Indeed, assume $G_A \cong G_B$. By [S22, p. 207, Corollary 6.3], $G_A \cong G_B$ if and only if there exists $T \in \operatorname{GL}_2(\mathcal{R})$ such that $A = TBT^{-1}$, where $T = MXN^{-1}$, $X = \operatorname{diag} \begin{pmatrix} x & \sigma(x) \end{pmatrix}$ for some $x \in K$, $A = M\Lambda M^{-1}$, $B = N\Lambda N^{-1}$, $\Lambda = \operatorname{diag} (\lambda & \sigma(\lambda))$,

$$M = \begin{pmatrix} 1 & 1 \\ \lambda & \sigma(\lambda) \end{pmatrix}, \quad N = \begin{pmatrix} 3 & 3 \\ \lambda + 1 & \sigma(\lambda) + 1 \end{pmatrix}.$$

In particular, det $T = N_{K/\mathbb{Q}}(x) \cdot \det M \cdot (\det N)^{-1} = N_{K/\mathbb{Q}}(x) \cdot 3^{-1}$, where $N_{K/\mathbb{Q}}(x)$ denotes the norm of x. Since $T \in \operatorname{GL}_2(\mathcal{R})$, det $T \in \mathcal{R}^{\times}$, which implies

(22)
$$N_{K/\mathbb{Q}}(x) = \pm 13^k \cdot 3, \quad k \in \mathbb{Z}.$$

It is known that $(13) = I_1 \cdot I_2$, where I_1, I_2 are principal ideals of \mathcal{O}_K and $(3) = I^2$ [SageMath]. Thus, it follows from (22) that $(x) = I_1^s \cdot I_2^t \cdot I$ for some $s, t \in \mathbb{Z}$, which implies that I is principal. This is a contradiction, since I has order 2 in the class group of K and hence it is not principal. Thus, $G_A \not\cong G_B$.

Example 2. In this example, we show that the condition "rad[$\mathcal{O}_K : \mathbb{Z}[\lambda]$] divides det A" in Corollary 13 is not necessary for $i(\operatorname{End}(G_A)) = \mathcal{O}_{K,\lambda}$. Here, in the notation of Proposition 12, $l_2 \neq 1$ and $\alpha = 1$. Let

$$A = \begin{pmatrix} -1 & 3\\ 3 & 2 \end{pmatrix}, \quad h_A(x) = x^2 - x - 11,$$

 $D = 3^2 \cdot 5$ is the discriminant of h_A . Hence, by Remark 14, m = 3, d = 5, $\omega = \frac{1+\sqrt{5}}{2}$, $\lambda = \frac{1+3\sqrt{5}}{2}$, $[\mathcal{O}_K : \mathbb{Z}[\lambda]] = 3$, and $l_2 = 3$. Note that \mathcal{O}_K is a principal ideal domain, hence, in the notation of Section 4.1, L = K,

$$M = \begin{pmatrix} 1 & 1 \\ \omega & \sigma(\omega) \end{pmatrix}, \quad PM = \begin{pmatrix} 1 & 1 \\ 0 & \sigma(\omega) - \omega \end{pmatrix}, \quad P \in \mathrm{GL}_2(\mathcal{O}_K),$$

and $v = \sigma(\omega) - \omega$. Thus, (20) holds for $\alpha = 1$ and $i(\operatorname{End}(G_A)) = \mathcal{O}_{K,\lambda} = \mathcal{O}_K[\lambda^{-1}].$

Example 3. In this example, we demonstrate how (19) can be used to determine $\operatorname{End}(G_A)$ for a rational canonical form $A \in M_2(\mathbb{Z})$ of a monic irreducible quadratic polynomial $h_A = x^2 + \beta x + \gamma \in \mathbb{Z}[x]$. By Lemma 2, $\operatorname{End}(G_A) = M_2(\mathbb{Z})$ if $\gamma = \pm 1$, $\operatorname{End}(G_A) = M_2(\mathcal{R})$ if $\gamma \neq \pm 1$ and $\mathcal{P}' = \emptyset$. Assume, $\mathcal{P}' \neq \emptyset$, equivalently, $\operatorname{rad}(\gamma)$ does not divide $\operatorname{rad}(\beta)$. Then, Proposition 12 can be applied. Let

$$A = \begin{pmatrix} 0 & -\gamma \\ 1 & -\beta \end{pmatrix}, \quad M = \begin{pmatrix} -\gamma & -\gamma \\ \lambda & \sigma(\lambda) \end{pmatrix}, \quad A = M \begin{pmatrix} \lambda & 0 \\ 0 & \sigma(\lambda) \end{pmatrix} M^{-1}.$$

For $x = b\omega$, $b \in \mathbb{Z}$, one can check that $T(x) = MXM^{-1} \in M_2(\mathcal{R})$ if and only if l_2 divides b. Therefore, by Proposition 12, $i(\operatorname{End}(G_A))$ is generated by $\{1, l_2\omega\}$ as a $\mathbb{Z}[\lambda^{\pm 1}]$ -module. More precisely, if $T_0 = MX_0M^{-1}$ with $X_0 = \operatorname{diag}(l_2\omega \ l_2\sigma(\omega))$, then

$$\operatorname{End}(G_A) = \left\{ \sum_i b_i A^{m_i} + \sum_j c_j A^{n_j} T_0 \mid \forall b_i, m_i, c_j, n_j \in \mathbb{Z}, \, i, j \in \mathbb{N} \right\},\$$

where each sum has finitely many non-zero terms. Since 1 and $l_2\omega$ are $\mathbb{Z}[\lambda^{\pm 1}]$ -dependent, End(G_A) is a finitely generated $\mathbb{Z}[\lambda^{\pm 1}]$ -module of rank 1.

5. Character groups, solenoids, and S-integer dynamical systems

5.1. Character groups. In this section, we describe the Pontryagin dual \widehat{G}_A of G_A . Here, G_A is considered as a topological group endowed with the discrete topology and \widehat{G}_A is a topological group with the underlying space consisting of continuous group homomorphisms from G_A to a circle \mathbb{T}^1 endowed with the compact-open topology. Let $A \in M_n(\mathbb{Z})$ be non-singular and let $h_A \in \mathbb{Z}[t]$ be the characteristic polynomial of A. Let $h_A = h_1 h_2 \cdots h_s$, where $h_1, \ldots, h_s \in \mathbb{Z}[t]$ are irreducible of degrees n_1, \ldots, n_s , respectively.

Lemma 15. [S24, Lemma 8.1] G_A is dense in \mathbb{R}^n endowed with the standard topology if and only if $h_i(0) \neq \pm 1$ for all $i \in \{1, 2, \ldots, s\}$.

In the case when G_A is not dense in \mathbb{R}^n , by Lemma 15, there exist $f_1, f_2 \in \mathbb{Z}[t]$ such that $h_A = f_1 f_2$ and $f_2(0) = \pm 1$. Let $g_2 \in \mathbb{Z}[t]$ be of maximal degree such that there exists $g_1 \in \mathbb{Z}[t]$ with $h_A = g_1 g_2, g_2(0) = \pm 1$. In other words, if $h_A = h_1 h_2 \cdots h_s$, where $h_1, \ldots, h_s \in \mathbb{Z}[t]$ are irreducible, $h_i(0) \neq \pm 1$ for all $i \in \{1, \ldots, t\}$ and $h_j(0) = \pm 1$ for all $j \in \{t + 1, \ldots, s\}, 1 \leq t < s$, then $g_1 = h_1 \cdots h_t$ and $g_2 = h_{t+1} \cdots h_s$. Then there exists $S \in \operatorname{GL}_n(\mathbb{Z})$ such that

(23)
$$SAS^{-1} = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix},$$

where $A_1 \in M_k(\mathbb{Z})$ has characteristic polynomial g_1 and $A_2 \in M_{n-k}(\mathbb{Z})$ has characteristic polynomial g_2 [N72, p. 50, Thm. III.12]. Thus, G_{A_1} is dense in \mathbb{R}^k endowed with the standard topology, det $A_2 = \pm 1$, and $G_{A_2} = \mathbb{Z}^{n-k}$. Thus, the natural exact sequence

$$0 \longrightarrow G_{A_1} \longrightarrow G_{SAS^{-1}} \longrightarrow G_{A_2} \longrightarrow 0$$

splits, so that

$$S(G_A) = G_{SAS^{-1}} \cong G_{A_1} \oplus \mathbb{Z}^{n-k}, \quad G_A \cong G_{A_1} \oplus \mathbb{Z}^{n-k}.$$

Therefore,

(24)
$$\widehat{G}_A \cong \widehat{G}_{A_1} \oplus \widehat{\mathbb{Z}}^{n-k} \cong \widehat{G}_{A_1} \oplus \mathbb{T}^{n-k}.$$

Therefore, to study the character group \widehat{G}_A , it is enough to consider the case when G_A is dense in \mathbb{R}^n endowed with the standard topology. For $\mathbf{y} \in \mathbb{R}^n$, denote by $\Lambda(\mathbf{y}) \in \widehat{\mathbb{R}^n}$ the character of \mathbb{R}^n given by

$$\Lambda(\mathbf{y})(\mathbf{x}) = e^{2\pi i \mathbf{y} \cdot \mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^n,$$

where $\mathbf{y} \cdot \mathbf{x}$ is the standard dot product of vectors in \mathbb{R}^n . We consider $\Lambda(\mathbf{y})$ as a character of G_A via the restriction. Note that if G_A is endowed with the topology τ induced from the standard topology on \mathbb{R}^n and (G_A, τ) is dense in \mathbb{R}^n , then $\mathbb{R}^n \cong \widehat{\mathbb{R}^n} \cong (\widehat{G_A}, \tau)$ with the isomorphism given by $\mathbf{y} \mapsto \Lambda(\mathbf{y}), \mathbf{y} \in \mathbb{R}^n$. However, if G_A is endowed with the discrete topology, then the structure of $\widehat{G_A}$ is more complicated and is given by a quotient of an adèle ring of K (see Theorem 19 below).

In the next lemma, we give a description of \widehat{G}_A for an arbitrary non-singular $A \in M_n(\mathbb{Z})$. For $\mathbf{y} \in \mathbb{Q}_p^n$, we will denote by $\{\mathbf{y}\}_p$ the "fractional" part of \mathbf{y} , *i.e.*, $\mathbf{y} = \{\mathbf{y}\}_p + \mathbf{y}_1$, where $\mathbf{y}_1 \in \mathbb{Z}_p^n$.

Lemma 16. Let $A \in M_n(\mathbb{Z})$ be non-singular and let \widehat{G}_A denote the Pontryagin dual of G_A , where G_A is endowed with the discrete topology. Consider the following map:

$$\psi: \mathbb{R}^n \times \widehat{G_A/\mathbb{Z}^n} \longrightarrow \widehat{G_A}, \quad \psi(\theta, \chi') = \Lambda(-\theta)\chi', \quad \theta \in \mathbb{R}^n,$$

where $\chi' \in \widehat{G_A/\mathbb{Z}^n}$ is considered as a character of G_A trivial on \mathbb{Z}^n . If G_A is dense in \mathbb{R}^n with respect to the standard topology on \mathbb{R}^n , then the map $\mathbb{Z}^n \longrightarrow \widehat{G_A/\mathbb{Z}^n}$ given by $\mathbf{m} \mapsto \Lambda(\mathbf{m})$ is an embedding and ψ induces a group isomorphism

(25)
$$\left(\mathbb{R}^n \times \widehat{G_A/\mathbb{Z}^n}\right) / \mathbb{Z}^n \cong \widehat{G_A}$$

where \mathbb{Z}^n is embedded into the product as $\mathbf{m} \mapsto (\mathbf{m}, \Lambda(\mathbf{m}))$. Moreover,

$$\widehat{G_A/\mathbb{Z}^n} \cong \prod_{p \mid \det A} \mathbb{Z}_p^{t_p}$$

and for every $\chi' \in \widehat{G_A/\mathbb{Z}^n}$ there exist $\mathbf{v}_p \in \mathbb{Z}_p^n$, $p \mid \det A$, such that

$$\chi'(\mathbf{x}) = \prod_{p \mid \det A} e^{2\pi i \{\mathbf{x} \cdot \mathbf{v}_p\}_p}$$

Proof. As in [C08], for $\chi \in \widehat{G}_A$, let $\chi(\mathbf{e}_1) = e^{-2\pi i \theta_1}, \ldots, \chi(\mathbf{e}_n) = e^{-2\pi i \theta_n}$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of \mathbb{R}^n . Define $\chi' \in \widehat{G}_A$ via

(26)
$$\theta = (\theta_1 \quad \dots \quad \theta_n) \in \mathbb{R}^n, \quad \chi' = \Lambda(\theta)\chi.$$

Then χ' is trivial on \mathbb{Z}^n , *i.e.*, $\chi' \in \widehat{G_A/\mathbb{Z}^n}$, and hence ψ is onto. We now find the kernel of ψ . Let $\chi = \Lambda(-\theta)\chi'$ be trivial on G_A . Then, $\Lambda(-\theta)$ is trivial on \mathbb{Z}^n , since χ' is trivial on \mathbb{Z}^n by assumption. Hence, $\theta \in \mathbb{Z}^n$ and $\chi' = \Lambda(\theta)$. Finally, if $\Lambda(\mathbf{m})$ is trivial on G_A and G_A is dense in \mathbb{R}^n in the standard topology, then $\Lambda(\mathbf{m})$ is trivial on \mathbb{R}^n , since $\Lambda(\mathbf{m})$ is a continuous character of \mathbb{R}^n in the standard topology. This implies that $\mathbf{m} = \mathbf{0}$ and hence $\mathbf{m} \mapsto \Lambda(\mathbf{m})$ defines an embedding of \mathbb{Z}^n into $\widehat{G_A/\mathbb{Z}^n}$.

We know that there is an isomorphism

$$\psi_A: \prod_{p \mid \det A} \overline{G}_{A,p} / \mathbb{Z}_p^n \xrightarrow{\sim} G_A / \mathbb{Z}^n$$

induced by $\psi_A(\mathbf{v}_p) = {\mathbf{v}_p}_p, \mathbf{v}_p \in \overline{G}_{A,p}$ [S24, Lemma 3.2]. Let $\chi' \in \widehat{G_A/\mathbb{Z}^n}$. Then

(27)
$$\widehat{G_A/\mathbb{Z}^n} \cong \prod_{p \mid \det A} \widehat{\overline{G}_{A,p}/\mathbb{Z}_p^n}, \quad \chi' = \prod_{p \mid \det A} \chi'_p, \quad \forall \chi'_p \in \widehat{\overline{G}_{A,p}/\mathbb{Z}_p^n}.$$

We now fix a prime p dividing det A and describe a character χ'_p . By [S22, Lemma 2.10], $\overline{G}_{A,p}/\mathbb{Z}_p^n \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{t_p}$ and it is well-known that $\widehat{\mathbb{Q}_p/\mathbb{Z}_p} \cong \mathbb{Z}_p$. Thus,

$$\widehat{\overline{G}_{A,p}/\mathbb{Z}_p^n} \cong (\widehat{\mathbb{Q}_p/\mathbb{Z}_p})^{t_p} \cong \mathbb{Z}_p^{t_p}.$$

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Tracing the maps, one can show that each character χ'_p of $\overline{G}_{A,p}$ trivial on \mathbb{Z}_p^n is determined by $\mathbf{v}_p \in \mathbb{Z}_p^n$ via

$$\chi'_p(\mathbf{x}) = e^{2\pi i \{\mathbf{x} \cdot \mathbf{v}_p\}_p}, \quad \mathbf{x} \in \overline{G}_{A,p}.$$

5.2. G_A as a subgroup of a number field. So far, we have considered G_A as a subset of \mathbb{Q}^n . We now show that when the characteristic polynomial of A is irreducible, one can consider G_A as a subset of a number field $\mathbb{Q}(\lambda)$, where $\lambda \in \overline{\mathbb{Q}}$ is an eigenvalue of A.

For the rest of the section we will fix an eigenvalue $\lambda \in \overline{\mathbb{Q}}$ of A and a corresponding eigenvector $\mathbf{u} \in (\overline{\mathbb{Q}})^n$. Let $K = \mathbb{Q}(\lambda)$ and, without loss of generality, we can assume that $\mathbf{u} = (u_1 \ldots u_n) \in (\mathcal{O}_K)^n$, where \mathcal{O}_K denotes the ring of integers of K. Let S_λ consist of (all) prime ideals of \mathcal{O}_K dividing λ . (Note that $\lambda \in \mathcal{O}_K$.) Recall that $\mathcal{O}_{K,\lambda}$ denotes the ring of S_{λ} -integers, *i.e.*,

 $\mathcal{O}_{K,\lambda} = \{x \in K \mid \operatorname{val}_{\mathfrak{p}}(x) \ge 0 \text{ for any prime ideal } \mathfrak{p} \text{ of } \mathcal{O}_K \text{ not in } \mathcal{S}_{\lambda}\} = \mathcal{O}_K [\lambda^{-1}].$

Assume $h_A \in \mathbb{Z}[t]$ is irreducible. Then $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on all the eigenvalues $\lambda_1 = \lambda, \ldots, \lambda_n$ of A, *i.e.*, $\lambda_i = \sigma_i(\lambda)$ for embeddings $\sigma_1 = \operatorname{id}, \sigma_2, \ldots, \sigma_n \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of K into $\overline{\mathbb{Q}}$. Then $\mathbf{u}_i = \sigma_i(\mathbf{u})$ is an eigenvector of A corresponding to λ_i , $i \in \{1, \ldots, n\}$. For $\mathbf{x} \in G_A$, since $\mathbf{u}_1 = \mathbf{u}, \ldots, \mathbf{u}_n$ are linearly independent over $\overline{\mathbb{Q}}$, $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{u}_i$ for some $x_1, \ldots, x_n \in \overline{\mathbb{Q}}$. Since $\mathbf{x} \in \mathbb{Q}^n$, $\sigma_i(\mathbf{x}) = \mathbf{x}$ and hence $x_i = \sigma_i(x_1)$ for all i. Note that $x_1 \in K$. Indeed, since $\mathbf{u}_1 = \mathbf{u} \in K^n$, for any $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/K)$, we have that $\sigma(\mathbf{u}_1) = \mathbf{u}_1$, $\sigma(\mathbf{x}) = \mathbf{x}$, and hence $\sigma(x_1) = x_1$ and $x_1 \in K$. Thus, the projection μ along \mathbf{u} defines an injective homomorphism

(28)
$$\mu: G_A \hookrightarrow K, \quad \mu(\mathbf{x}) = x_1, \quad \mathbf{x} = \sum_{i=1}^n \sigma_i(x_1 \mathbf{u}), \quad \sigma_1 = \mathrm{id}.$$

To prove our main result in this section, Theorem 19 below, we will need the following lemma.

Lemma 17. Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial $h_A \in \mathbb{Z}[x]$. Then, $\mathcal{O}_{K,\lambda} \subseteq \mu(G_A)$, where μ is given by (28).

Proof. Let $x_1 \in \mathcal{O}_{K,\lambda}$ and let $\mathbf{x} = \sum_i \sigma_i(x_1\mathbf{u}_1)$. We need to show that $\mathbf{x} \in G_A$. By construction, $\mathbf{x} \in \mathbb{Q}^n$. Moreover, $\mathbf{x} \in \mathcal{R}^n$, since for any $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any $\mathfrak{p} \in \mathcal{S}_{\lambda}$, $\sigma(\mathfrak{p})$ is a prime ideal of $\mathbb{Q}(\sigma(\lambda))$ dividing $\sigma(\lambda)$. Thus, $\operatorname{val}_{\mathfrak{q}} \sigma_i(x_1\mathbf{u}_1) \geq 0$ for any prime ideal \mathfrak{q} of the splitting field L of h_A not dividing det A. In other words, in the "denominators" of $\sigma_i(x_1\mathbf{u}_1)$'s there are only prime ideals dividing λ_i 's and hence in the denominators of $\mathbf{x} \in \mathbb{Q}^n$ we only have primes $p \in \mathbb{N}$ dividing det A. Thus, by (4), we only need to show that $\mathbf{x} \in \overline{G}_{A,p}$ for any $p \in \mathcal{P}$ under the embedding induced by $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$.

We fix an arbitrary $p \in \mathcal{P}$. Even though h_A is irreducible over \mathbb{Q} , it might not be irreducible over \mathbb{Q}_p . By the definition of t_p , we have $h_A(x) \equiv f(x)x^{t_p} \pmod{p}$, where $f \in \mathbb{F}_p[x], f(0) \neq 0$. Therefore, by Hensel's lemma, $h_A = h_1h_2$, where $h_1, h_2 \in \mathbb{Z}_p[t]$,

 $h_1 \equiv f \pmod{p}, h_2 \equiv x^{t_p} \pmod{p}, \text{ and } p \text{ does not divide } h_1(0).$ Let L be a finite Galois extension of \mathbb{Q} containing all eigenvalues of A, e.g., L is the splitting field of h_A . Let \mathfrak{q} be a prime ideal of L lying above p. Without loss of generality, we can assume that \mathfrak{q} divides $\lambda_1 = \lambda, \ldots, \lambda_{t_p}$ in \mathcal{O}_L , so that $\lambda_1, \ldots, \lambda_{t_p}$ are all roots of h_2 in $\overline{\mathbb{Q}}_p$. Then, for any prime ideal \mathfrak{p} of \mathcal{O}_K dividing λ , $\sigma_i(\mathfrak{p})$ is not divisible by \mathfrak{q} for any $t_p < i \leq n$. Indeed, clearly \mathfrak{q} does not divide any $\sigma_i(\mathfrak{p})$ if \mathfrak{p} lies above a prime $p' \in \mathbb{N}$ not equal to p. If \mathfrak{p} lies above p and $\sigma_j(\mathfrak{p})$ is divisible by \mathfrak{q} for some $j > t_p$, then \mathfrak{q} divides $\sigma_j(\lambda)$ and we have a contradiction with deg $h_2 = t_p$. This implies that $x_j = \sigma_j(x_1) \in \mathcal{O}_{\mathfrak{q}}$ for any $j > t_p$, since $x_1 \in \mathcal{O}_{K,\lambda}$ by assumption. Let $\mathbf{x} = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_1 = \sum_{i=1}^{t_p} \sigma_i(x_1 \mathbf{u}_1), \mathbf{y}_2 = \sum_{i=t_p+1}^n \sigma_i(x_1 \mathbf{u}_1)$. Let $G_p = \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Since any $\sigma \in G_p$ permutes roots of h_1 (respectively, h_2), we have that $\sigma(\mathbf{y}_i) = \mathbf{y}_i, i = 1, 2$. Hence, $\mathbf{y}_2 \in (\mathcal{O}_q)^n \cap \mathbb{Q}_p^n$, since $\mathbf{u}_1 \in \mathcal{O}_K \subseteq \mathcal{O}_L$. Thus, $\mathbf{y}_2 \in \mathbb{Z}_p^n \subseteq \overline{G}_{A,p}$. Then, $\mathbf{y}_1 \in \operatorname{Span}_{L_{\mathfrak{q}}}(\mathbf{u}_1, \ldots, \mathbf{u}_{t_p}) \cap \mathbb{Q}_p^n$, where $\operatorname{Span}_{L_{\mathfrak{q}}}(\mathbf{u}_1, \ldots, \mathbf{u}_{t_p}) \cap \mathbb{Q}_p^n = D_p(A) \subseteq \overline{G}_{A,p}$ by [S24, Lemma 4.1]. This shows that $\mathbf{x} \in \overline{G}_{A,p}$ for any $p \in \mathcal{P}$.

We now describe the image of μ inside K. Let $M = (\sigma_1(\mathbf{u}) \dots \sigma_n(\mathbf{u})) \in M_n(\overline{\mathbb{Q}})$ be as in (15), where $\sigma_1, \dots, \sigma_n$ are all embeddings of K into $\overline{\mathbb{Q}}$ and $\sigma_1 = \mathrm{id}$.

Lemma 18. Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial $h_A \in \mathbb{Z}[x]$. Let $\mathbf{w} = (w_1 \dots w_n)^t \in (\overline{\mathbb{Q}})^n$ be the 1st column of $(M^t)^{-1}$, that is \mathbf{w} is an eigenvector of A^t corresponding to λ . Denote

$$Y_{A^{t}}(\mathbf{w},\lambda) = \{m_{1}\lambda^{k_{1}}w_{1} + \dots + m_{n}\lambda^{k_{n}}w_{n} \,|\, m_{1},\dots,m_{n},k_{1},\dots,k_{n} \in \mathbb{Z}\}.$$

Then $Y_{A^t}(\mathbf{w}, \lambda)$ is a $\mathbb{Z}[\lambda^{\pm 1}]$ -submodule of K, $\mu(G_A) = Y_{A^t}(\mathbf{w}, \lambda)$, and $G_A \cong Y_{A^t}(\mathbf{w}, \lambda)$.

Proof. By definition, $\mathbf{w} \cdot \mathbf{u} = 1$, where we assume $\mathbf{u} \in K^n$. Since $A^t \mathbf{w} = \lambda \mathbf{w}$ and A^t has integer entries, there is an eigenvector $\mathbf{w}' \in K^n$ of A^t corresponding to λ . Thus, $\mathbf{w} = \alpha \mathbf{w}'$ for some $\alpha \in \overline{\mathbb{Q}}$. Therefore,

$$\mathbf{w} \cdot \mathbf{u} = \alpha \mathbf{w}' \cdot \mathbf{u} = 1.$$

Since $\mathbf{w}' \cdot \mathbf{u} \in K$, this implies that $\alpha \in K$, $\mathbf{w} \in K^n$, and $Y_{A^t}(\mathbf{w}, \lambda) \subseteq K$. Clearly, $Y_{A^t}(\mathbf{w}, \lambda)$ is a $\mathbb{Z}[\lambda^{\pm 1}]$ -submodule of K. Moreover,

$$\mathbb{Z}[\mathbf{w}] = \{m_1 w_1 + \dots + m_n w_n \,|\, m_1, \dots, m_n \in \mathbb{Z}\}$$

is a $\mathbb{Z}[\lambda]$ -module, since A^t has integer entries. Then, any $y \in Y_{A^t}(\mathbf{w}, \lambda)$ has the form $y = u\lambda^k$ for some $u \in \mathbb{Z}[\mathbf{w}]$ and $k \in \mathbb{Z}$.

Let $\mathbf{x} \in G_A$, $\mu(\mathbf{x}) = x_1$. Then, by the definition of μ , $\mathbf{x} = \sum_i \sigma_i(x_1\mathbf{u})$. It can be easily verified that $(M^t)^{-1} = (\sigma_1(\mathbf{w}) \dots \sigma_n(\mathbf{w}))$. By the definition of G_A and (15), $\mathbf{x} \in G_A$ if and only if there exists $k \in \mathbb{Z}$ and $\mathbf{m} \in \mathbb{Z}^n$ such that

$$\mathbf{x} = A^k \mathbf{m} = M \Lambda^k M^{-1} \mathbf{m} = M \begin{pmatrix} x_1 & \sigma_2(x_1) & \dots & \sigma_n(x_1) \end{pmatrix}^t$$

if and only if $x_1 \in Y_{A^t}(\mathbf{w}, \lambda)$.

Denote

(29)
$$Y_{A}(\mathbf{u},\lambda) = \{m_{1}\lambda^{k_{1}}u_{1} + \dots + m_{n}\lambda^{k_{n}}u_{n} \mid m_{1},\dots,m_{n},k_{1},\dots,k_{n} \in \mathbb{Z}\},\$$

an analogue of $Y_{A^t}(\mathbf{w}, \lambda)$ for A and \mathbf{u} . Then $Y_A(\mathbf{u}, \lambda)$ is a $\mathbb{Z}[\lambda^{\pm 1}]$ -submodule of $\mathcal{O}_{K,\lambda}$, since $u_1, \ldots, u_n \in \mathcal{O}_K$ by assumption. Let $m = (\det M)^2 \in \mathbb{Z}$ be the discriminant of the lattice $\mathbb{Z}[\mathbf{u}]$. Then we have the diagram

where the "down" isomorphisms are given by μ applied to G_{A^t} , G_A .

5.3. Character group of G_A via adèles. We will use the notation introduced in [T67]. Let S_{∞} denote the set of all infinite places of K. Denote

$$\begin{array}{ll} \operatorname{complex} \mathfrak{p} \in \mathcal{S}_{\infty} & \Lambda_{\mathfrak{p}}(\xi) = -2\Re(\xi) & \xi \in K_{\mathfrak{p}} = \mathbb{C} \\ \operatorname{real} \mathfrak{p} \in \mathcal{S}_{\infty} & \Lambda_{\mathfrak{p}}(\xi) = -\xi & \xi \in K_{\mathfrak{p}} = \mathbb{R} \\ \operatorname{finite} \mathfrak{p} & \Lambda_{\mathfrak{p}}(\xi) = \{\operatorname{Tr}_{K_{\mathfrak{p}}/\mathbb{Q}_{p}} \xi\}_{p} & \xi \in K_{\mathfrak{p}}. \end{array}$$

Recall that S_{λ} denotes the set of all finite places (prime ideals) of K dividing λ in \mathcal{O}_K . Note that $S_{\infty} \cup S_{\lambda}$ is a finite set. Denote

$$\mathbb{A}_{K,\lambda} = \prod_{\mathfrak{p}\in\mathcal{S}_{\infty}\cup\mathcal{S}_{\lambda}} K_{\mathfrak{p}}.$$

It is an object of the same nature as the adèle ring \mathbb{A}_K of K, consisting of all elements $(\ldots, \eta_{\mathfrak{p}}, \ldots)$, where \mathfrak{p} runs through all the places of K, $\eta_{\mathfrak{p}} \in K_{\mathfrak{p}}$ for any \mathfrak{p} and $\eta_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ for all but finitely many \mathfrak{p} . It is known that the character group of K endowed with the discrete topology is isomorphic to the quotient of \mathbb{A}_K by K, where K is embedded into \mathbb{A}_K diagonally via $\xi \mapsto (\ldots, \xi_{\mathfrak{p}}, \ldots)$, each $\xi_{\mathfrak{p}} = \xi$ [T67]. From Section 5.2, if the characteristic polynomial of A is irreducible, then G_A can be considered as a subset of K via μ in (28). Each $\eta = (\ldots, \eta_{\mathfrak{p}}, \ldots) \in \mathbb{A}_{K,\lambda}$ defines a character χ of K via

(31)
$$\chi(\eta)(\xi) = \prod_{\mathfrak{p}\in\mathcal{S}_{\infty}\cup\mathcal{S}_{\lambda}} e^{2\pi i\Lambda_{\mathfrak{p}}(\eta_{\mathfrak{p}}\xi)}, \quad \xi\in K,$$

hence η defines a character of G_A via restriction and isomorphism μ in (28):

(32)
$$\chi(\eta)(\mathbf{x}) = \chi(\eta)(x_1), \quad x_1 = \mu(\mathbf{x}) \in K, \quad \mathbf{x} \in G_A.$$

We have a diagonal embedding of K into $\mathbb{A}_{K,\lambda}$ via $\xi \mapsto (\xi, \ldots, \xi)$. We will denote by ξ a general element of K, by \mathbf{x} a general element of G_A , and by x_1 a general element of $\mu(G_A)$. In what follows, we will show that \widehat{G}_A is isomorphic to a quotient of $\mathbb{A}_{K,\lambda}$ by $Y_A(\mathbf{u},\lambda)$ defined by (29). **Theorem 19.** Let $A \in M_n(\mathbb{Z})$ be non-singular with an irreducible characteristic polynomial $h_A \in \mathbb{Z}[x]$. Let $\lambda \in \overline{\mathbb{Q}}$ be an eigenvalue of A and let $K = \mathbb{Q}(\lambda)$. The map $\phi : \mathbb{A}_{K,\lambda} \to \widehat{G}_A$ given by $(\ldots, \eta_{\mathfrak{p}}, \ldots) \mapsto \chi$,

(33)
$$\chi(\mathbf{x}) = \prod_{\mathfrak{p}\in\mathcal{S}_{\infty}\cup\mathcal{S}_{\lambda}} e^{2\pi i\Lambda_{\mathfrak{p}}(\eta_{\mathfrak{p}}\mu(\mathbf{x}))}, \quad \mathbf{x}\in G_A,$$

is onto. Moreover, ker $\phi = Y_A(\mathbf{u}, \lambda)$, so that

$$\widehat{G}_A \cong \mathbb{A}_{K,\lambda}/Y_A(\mathbf{u},\lambda).$$

We divide the proof of Theorem 19 into parts proved in Lemmas 20 - 23 below. Namely, ϕ is onto (Lemma 20), $Y_A(\mathbf{u}, \lambda) \subseteq \ker \phi$ (Lemma 21), if $\eta \in \mathbb{A}_{K,\lambda}$ is trivial on $\mathcal{O}_{K,\lambda}$, then $\eta \in K$ (Lemma 22), and $\ker \phi \subseteq Y_A(\mathbf{u}, \lambda)$ (Lemma 23).

Lemma 20. ϕ is onto.

Proof of Lemma 20. Let \mathcal{S} denote the set of all finite places of K dividing all $p \in \mathcal{P}$, *i.e.*, all prime ideals of \mathcal{O}_K lying above all $p \in \mathcal{P}$ as opposed to \mathcal{S}_{λ} , the set of prime ideals of \mathcal{O}_K dividing λ . Let $\mathbb{A}_{K,\mathcal{S}} = \prod_{\mathfrak{p} \in \mathcal{S}_{\infty} \cup \mathcal{S}} K_{\mathfrak{p}}$. Similar to (33), we also have a natural homomorphism $\psi : \mathbb{A}_{K,\mathcal{S}} \to Y_{A^t}(\mathbf{w},\lambda) \cong \widehat{G}_A$ given by $(\ldots, \eta_{\mathfrak{p}}, \ldots) \mapsto \chi$,

(34)
$$\chi(x_1) = \prod_{\mathfrak{p} \in \mathcal{S}_{\infty} \cup \mathcal{S}} e^{2\pi i \Lambda_{\mathfrak{p}}(\eta_{\mathfrak{p}} x_1)}, \quad x_1 \in Y_{A^t}(\mathbf{w}, \lambda)$$

(see Lemma 18). We now show that ψ is onto. Indeed, by Lemma 16, for any $\chi \in \widehat{G}_A$ there exist $\theta \in \mathbb{R}^n$ and $\mathbf{v}_p \in \mathbb{Z}_p^n$, $p \in \mathcal{P}$, such that

(35)
$$\chi(\mathbf{x}) = e^{-2\pi i \mathbf{x} \cdot \boldsymbol{\theta}} \prod_{p \in \mathcal{P}} e^{2\pi i \{\mathbf{x} \cdot \mathbf{v}_p\}_p}, \quad \mathbf{x} \in G_A.$$

Let $n = r_1 + 2r_2$, where r_1 is the number of real roots of h_A and r_2 is the number of pairs of conjugate complex roots of h_A . Without loss of generality, we can assume that $\mathcal{S}_{\infty} = \{\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}\}$. For $\theta \in \mathbb{R}^n$, denote $\eta_{\mathfrak{p}} = \sigma_j(\mathbf{u}) \cdot \theta \in K_{\mathfrak{p}}$, where $\mathfrak{p} \in \mathcal{S}_{\infty}$ corresponds to $\sigma_j, j \in \{1, \ldots, r_1 + r_2\}$. Then,

(36)
$$e^{-2\pi i \mathbf{x} \cdot \theta} = \prod_{\mathfrak{p} \in \mathcal{S}_{\infty}} e^{2\pi i \Lambda_{\mathfrak{p}}(\eta_{\mathfrak{p}} x_1)}, \quad x_1 = \mu(\mathbf{x}).$$

Recall that $\mathbf{x} = \sum_{i=1}^{n} \sigma_i(x_1 \mathbf{u})$. Then

(37)
$$\{\mathbf{x} \cdot \mathbf{v}_p\}_p = \sum_{\mathfrak{p}|p} \{ \operatorname{Tr}_{K_{\mathfrak{p}}/\mathbb{Q}_p}(\mathbf{v}_p \cdot \mathbf{u}x_1) \}_p = \sum_{\mathfrak{p}|p} \Lambda_{\mathfrak{p}}(\eta_{\mathfrak{p}}x_1), \quad \eta_{\mathfrak{p}} = \mathbf{v}_p \cdot \mathbf{u},$$

which together with (36), shows that ψ given by (34) is onto. Denote

 $\mathcal{R}[\mathbf{u}] = \operatorname{Span}_{\mathcal{R}}(u_1, \dots, u_n), \ \mathbb{Z}[\mathbf{u}] = \operatorname{Span}_{\mathbb{Z}}(u_1, \dots, u_n), \ \mathbb{Z}_p[\mathbf{u}] = \mathbb{Z}[\mathbf{u}] \otimes_{\mathbb{Z}} \mathbb{Z}_p,$

where $\mathcal{R}[\mathbf{u}] \subset K$, $\mathbb{Z}[\mathbf{u}] \subset \mathcal{O}_K$, and $\mathbb{Z}_p[\mathbf{u}] \subset \mathcal{O}_p$. We claim that $\mathcal{R}[\mathbf{u}] \subseteq \ker \psi$, where $\mathcal{R}[\mathbf{u}]$ is embedded diagonally into $\mathbb{A}_{K,S}$. Indeed, if $\eta = \mathbf{r} \cdot \mathbf{u} \in \mathcal{R}[\mathbf{u}]$, $\mathbf{r} \in \mathcal{R}^n$, then

$$\sum_{\mathfrak{p}\in\mathcal{S}_{\infty}\cup\mathcal{S}}\Lambda_{\mathfrak{p}}(\eta x_{1})=-\mathbf{r}\cdot\mathbf{x}+\sum_{p\in\mathcal{P}}\{\mathbf{r}\cdot\mathbf{x}\}_{p}=0$$

by [T67, Lemma 4.1.5], since $\mathbf{r} \cdot \mathbf{x} \in \mathcal{R}$ (*i.e.*, $\mathbf{r} \cdot \mathbf{x} \in \mathbb{Q}$ and has only primes from \mathcal{P} in the denominators) for any $\mathbf{x} \in G_A$. Hence,

$$(\ldots,\eta,\ldots)\mapsto \chi(x_1)=\prod_{\mathfrak{p}\in\mathcal{S}_\infty\cup\mathcal{S}}e^{2\pi i\Lambda_\mathfrak{p}(\eta x_1)}=1$$

for any $x_1 \in Y_{A^t}(\mathbf{w}, \lambda)$. One can show that any element from $\mathbb{A}_{K,S}$ is equivalent to an element $\eta = (\dots, \eta_{\mathfrak{p}}, \dots)$ from $\Omega_1 = \prod_{\mathfrak{p} \in S_{\infty}} K_{\mathfrak{p}} \prod_{p \in \mathcal{P}} \prod_{\mathfrak{p} \mid p} \mathbb{Z}_p[\mathbf{u}]$ modulo $\mathcal{R}[\mathbf{u}]$, where for each $p \in \mathcal{P}$ there exists $\eta_p \in \mathbb{Z}_p[\mathbf{u}]$ such that $\eta_{\mathfrak{p}} = \eta_p$ for any $\mathfrak{p}|p$. It also follows from (37).

We now consider the restriction of ψ from $\mathbb{A}_{K,S}$ to $\mathbb{A}_{K,\lambda}$ and show that the restriction is also onto $Y_{A^t}(\mathbf{w},\lambda) \cong \widehat{G}_A$. This is true, because even though x_1 might not be in \mathcal{O}_q for any prime ideal \mathfrak{q} not in \mathcal{S}_λ , but $\operatorname{val}_{\mathfrak{q}} x_1$ is bounded from below by a constant that does not depend on x_1 . Indeed, by the previous paragraph, without loss of generality, we can assume that $\chi \in Y_{A^t}(\mathbf{w},\lambda)$ is defined by $\eta \in \Omega_1$. Let L be a finite Galois extension of \mathbb{Q} containing all eigenvalues of A, e.g., L is the splitting field of h_A . Recall that $x_1 = \mu(\mathbf{x}) = a\lambda^{-k}(\det M)^{-1}$, where $a \in \mathcal{O}_L$, $k \in \mathbb{N} \cup \{0\}$, $\det M \neq 0 \in \mathcal{O}_L$ by Lemma 18. For any prime ideal \mathfrak{q} of \mathcal{O}_L not dividing λ lying above a prime $q \in \mathbb{N}$, there exists $k_{\mathfrak{q}} \in \mathbb{N} \cup \{0\}$ such that $q^{k_{\mathfrak{q}}}(\det M)^{-1} \in \mathcal{O}_{\mathfrak{q}}$. Since there are finitely many prime ideals \mathfrak{q} of \mathcal{O}_L lying above q, by taking the maximum among all $k_{\mathfrak{q}}$, we can assume that there exists $k_q \in \mathbb{N} \cup \{0\}$ such that $q^{k_q}(\det M)^{-1} \in \mathcal{O}_{\mathfrak{q}}$ for any \mathfrak{q} above q. Then, $p^{k_p}x_1 \in \mathcal{O}_{\mathfrak{p}}$ for any x_1 and any prime ideal \mathfrak{p} of K above $p \in \mathcal{P}$ not dividing λ , *i.e.*, $\mathfrak{p} \notin \mathcal{S}_{\lambda}$. We now write each $\eta_p \in \mathbb{Z}_p[\mathfrak{u}]$ as $\eta_p = a_p + p^{k_p}\mu_p$ for $a_p \in \mathbb{Z}[\mathfrak{u}]$ and $\mu_p \in \mathbb{Z}_p[\mathfrak{u}]$. By the Chinese Remainder Theorem, there exists $a \in \mathbb{Z}[\mathfrak{u}]$ such that $\eta_p - a \in p^{k_p}\mathbb{Z}_p[\mathfrak{u}]$ for any $p \in \mathcal{P}$. Then $\eta - a$ defines a character of $Y_{A^t}(\mathfrak{w},\lambda)$ as follows:

(38)
$$(\dots, \eta_p - a, \dots) \mapsto \chi(x_1) = \prod_{\mathfrak{p} \in \mathcal{S}_{\infty} \cup \mathcal{S}} e^{2\pi i \Lambda_{\mathfrak{p}}((\eta_p - a)x_1)} = \prod_{\mathfrak{p} \in \mathcal{S}_{\infty} \cup \mathcal{S}_{\lambda}} e^{2\pi i \Lambda_{\mathfrak{p}}((\eta_p - a)x_1)}$$

Indeed, $\Lambda_{\mathfrak{p}}((\eta_p - a)x_1) = 0$ for any \mathfrak{p} not in \mathcal{S}_{λ} above a prime $p \in \mathcal{P}$, since $(\eta_p - a)x_1 \in \mathcal{O}_{\mathfrak{p}}$. This shows that every character of $Y_{A^t}(\mathbf{w}, \lambda)$, equivalently, of G_A , comes from an element from $\mathbb{A}_{K,\lambda}$, hence ϕ is onto.

Lemma 21. $Y_A(\mathbf{u}, \lambda) \subseteq \ker \phi$.

Proof. There exists $k \in \mathbb{N}$ such that $\lambda^k x_1 \in \mathcal{O}_p$ for any $x_1 \in \mathbb{Z}[\mathbf{w}]$ and any $\mathfrak{p} \in \mathcal{S}_{\lambda}$. Since multiplication by a power of λ is an isomorphism of $Y_{A^t}(\mathbf{w}, \lambda)$, by precomposing every character of $Y_{A^t}(\mathbf{w}, \lambda)$ with the isomorphism, without loss of generality, we can assume that x_1 itself lies in \mathcal{O}_p , *i.e.*, $\mathbb{Z}[\mathbf{w}] \subset \mathcal{O}_p$ for any $\mathfrak{p} \in \mathcal{S}_{\lambda}$. Also, since ker ϕ is a $\mathbb{Z}[\lambda^{\pm 1}]$ module, it is enough to show that $\mathbb{Z}[\mathbf{u}] \subseteq \ker \phi$. Let $\eta = \mathbf{s} \cdot \mathbf{u} \in \mathbb{Z}[\mathbf{u}], \mathbf{s} \in \mathbb{Z}^n$. By the

proof of the previous lemma, $\mathcal{R}[\mathbf{u}] \subseteq \ker \psi$. Thus, for $\eta \in \mathbb{Z}[\mathbf{u}]$ we have that

(39)
$$0 = \sum_{\mathfrak{p}\in\mathcal{S}_{\infty}\cup\mathcal{S}}\Lambda_{\mathfrak{p}}(\eta x_{1}) = \sum_{\mathfrak{p}\in\mathcal{S}_{\infty}\cup\mathcal{S}_{\lambda}}\Lambda_{\mathfrak{p}}(\eta x_{1}) + \sum_{p\in\mathcal{P}}\sum_{\mathfrak{p}\mid p,\,\mathfrak{p}\notin\mathcal{S}_{\lambda}}\Lambda_{\mathfrak{p}}(\eta x_{1}), \quad \forall x_{1}\in Y_{A^{t}}(\mathbf{w},\lambda).$$

Denote $T_p = \sum_{\mathfrak{p}|p,\mathfrak{p}\notin S_{\lambda}} \Lambda_{\mathfrak{p}}$. From (39), it is enough to show that $T_p(\eta x_1) = 0$ for any $x_1 \in Y_{A^t}(\mathbf{w},\lambda)$ and $p \in \mathcal{P}$. We have that

(40)
$$\{\mathbf{s} \cdot \mathbf{x}\}_p = \sum_{\mathfrak{p}|p} \Lambda_{\mathfrak{p}}(\eta x_1) = \sum_{\mathfrak{p}|p, \mathfrak{p} \in \mathcal{S}_{\lambda}} \Lambda_{\mathfrak{p}}(\eta x_1) + T_p(\eta x_1), \quad \forall x_1 \in Y_{A^t}(\mathbf{w}, \lambda).$$

Since $x_1 \in \mathbb{Z}[\mathbf{w}]$ if and only if $\mathbf{x} \in \mathbb{Z}^n$, we have that $\{\mathbf{s} \cdot \mathbf{x}\}_p = 0$ for any $x_1 \in \mathbb{Z}[\mathbf{w}]$. In addition, $\Lambda_{\mathfrak{p}}(\eta x_1) = 0$ for any $x_1 \in \mathbb{Z}[\mathbf{w}]$ and any $\mathfrak{p} \in S_{\lambda}$, as follows from our assumption. Therefore, from (40), $T_p(\eta x_1) = 0$ for any $x_1 \in \mathbb{Z}[\mathbf{w}]$. Since λ is a unit in the ring of integers of $K_{\mathfrak{p}}$ for any \mathfrak{p} that does not divide λ , multiplication by λ is a \mathbb{Z}_p -module automorphism of $\mathbb{Z}_p[\mathbf{w}]$, hence $T_p(\eta x_1) = 0$ for any $x_1 \in Y_{A^t}(\mathbf{w}, \lambda)$.

Lemma 22. If $\eta \in \mathbb{A}_{K,\lambda}$ is trivial on $\mathcal{O}_{K,\lambda}$, then $\eta \in K$.

Proof. Let $\eta = (\ldots, \eta_{\mathfrak{p}}, \ldots) \in \ker \phi$, *i.e.*,

(41)
$$\sum_{\mathfrak{p}\in\mathcal{S}_{\infty}\cup\mathcal{S}_{\lambda}}\Lambda_{\mathfrak{p}}(\eta_{\mathfrak{p}}x_{1})=0, \quad \forall x_{1}\in Y_{A^{t}}(\mathbf{w},\lambda).$$

Note that (41) holds for any $x_1 \in \mathcal{O}_{K,\lambda}$, since $\mathcal{O}_{K,\lambda} \subseteq Y_{A^t}(\mathbf{w},\lambda)$ by Lemma 17 and Lemma 18. By the Chinese Remainder Theorem, there exists $a \in \mathcal{O}_{K,\lambda}$ such that $\eta_{\mathfrak{p}} - a \in \mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathcal{S}_{\lambda}$. Let $\xi = \eta - a$ with $\xi_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathcal{S}_{\lambda}$. Then

(42)
$$\sum_{\mathfrak{p}\in\mathcal{S}_{\infty}\cup\mathcal{S}_{\lambda}}\Lambda_{\mathfrak{p}}(\xi_{\mathfrak{p}}x_{1})=0, \quad \forall x_{1}\in\mathcal{O}_{K,\lambda}$$

since $a \in \mathcal{O}_{K,\lambda}$ defines a trivial character on $\mathcal{O}_{K,\lambda}$. Moreover, $\Lambda_{\mathfrak{p}}(\xi_{\mathfrak{p}}x_1) = 0$ for any $x_1 \in \mathcal{O}_K$ and $\mathfrak{p} \in \mathcal{S}_{\lambda}$, hence

(43)
$$\sum_{\mathfrak{p}\in\mathcal{S}_{\infty}}\Lambda_{\mathfrak{p}}(\xi_{\mathfrak{p}}x_{1})=0, \quad \forall x_{1}\in\mathcal{O}_{K}$$

As in [T67], we denote by $\overset{\infty}{\xi}$ the projection of ξ onto $\prod_{\mathfrak{p}\in\mathcal{S}_{\infty}} K_{\mathfrak{p}}$. Since u_1, \ldots, u_n is a basis of K as a \mathbb{Q} -vector space, $\overset{\infty}{u_1}, \ldots, \overset{\infty}{u_n}$ is a basis of $\prod_{\mathfrak{p}\in\mathcal{S}_{\infty}} K_{\mathfrak{p}}$ as an *n*-dimensional \mathbb{R} -vector space. Let $\overset{\infty}{\xi} = \sum_{i=1}^{n} \alpha_i \overset{\infty}{u_i}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Applying (43) to each $x_1 = u_i \in \mathcal{O}_K$, we get that $MM^t \alpha \in \mathbb{Z}^n$ for $\alpha = (\alpha_1 \ldots \alpha_n)^t$ with M given by (15). One can check that $MM^t \in M_n(\mathbb{Z})$ and hence $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$. Thus, $\overset{\infty}{\xi} = (b, \ldots, b)$ for $b = \sum_{i=1}^{n} \alpha_i u_i, b \in K$. We now show that $\xi = b$, equivalently, $\xi_{\mathfrak{p}} = 0$ for any $\mathfrak{p} \in \mathcal{S}_{\lambda}$, so that $\eta = \xi + a = b + a \in K$. Indeed, there exists $l \in \mathbb{N}$ such that $lb \in \mathcal{O}_K$. Denote

$$\kappa = l(\xi - b)$$
, so that $\widetilde{\kappa} = (0, \dots, 0)$, $\kappa_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathcal{S}_{\lambda}$, and

$$\sum_{\mathfrak{p} \in \mathcal{S}_{\lambda}} \Lambda_{\mathfrak{p}}(\kappa_{\mathfrak{p}} x_1) = 0, \quad \forall x_1 \in \mathcal{O}_{K,\lambda}.$$

Then, $\sum_{\mathfrak{p}|p} \Lambda_{\mathfrak{p}}(\kappa_{\mathfrak{p}}x_1) = 0$, equivalently, $\sum_{\mathfrak{p}|p} \operatorname{Tr}_{K_{\mathfrak{p}}/\mathbb{Q}_p}(\kappa_{\mathfrak{p}}x_1) \in \mathbb{Z}_p$ for any $x_1 \in \mathcal{O}_{K,\lambda}$ and any $p \in \mathcal{P}$. Since the image of $\mathcal{O}_{K,\lambda}$ under the embedding $K \hookrightarrow K_{\mathfrak{p}}$ generates $K_{\mathfrak{p}}$ over \mathbb{Z}_p , we have that $\sum_{\mathfrak{p}|p} \operatorname{Tr}_{K_{\mathfrak{p}}/\mathbb{Q}_p}(\kappa_{\mathfrak{p}}u_i) = 0$ for any $p \in \mathcal{P}$ and $i \in \{1, \ldots, n\}$. It gives a system of linear equations for $\kappa_{\mathfrak{p}}$ with matrix M^t , which is non-singular. Therefore, each $\kappa_{\mathfrak{p}} = 0$.

Lemma 23. ker $\phi \subseteq Y_A(\mathbf{u}, \lambda)$.

Proof. Let $\eta \in \ker \phi$. By Lemma 17 and Lemma 18, $\mathcal{O}_{K,\lambda} \subseteq \mu(G_A)$, $\mu(G_A) = Y_{A^t}(\mathbf{w}, \lambda)$, so that η is trivial on $\mathcal{O}_{K,\lambda}$. Hence, by Lemma 22, $\eta \in K$ embedded diagonally into $\mathbb{A}_{K,\lambda}$. By [T67, Lemma 4.1.5], we have that

(44)
$$\sum_{\mathfrak{p}\in\mathcal{S}_{\infty}\cup\mathcal{S}_{\lambda}}\Lambda_{\mathfrak{p}}(\eta x_{1}) + \sum_{p\in\mathcal{P}}\sum_{\mathfrak{p}\mid p,\,\mathfrak{p\notin\mathcal{S}}_{\lambda}}\Lambda_{\mathfrak{p}}(\eta x_{1}) + \sum_{q\notin\mathcal{P},\,\mathfrak{q}\mid q}\Lambda_{\mathfrak{q}}(\eta x_{1}) = 0, \quad \forall x_{1}\in Y_{A^{t}}(\mathbf{w},\lambda).$$

where $\mathfrak{p}, \mathfrak{q}$ are prime ideals of \mathcal{O}_K , and $p, q \in \mathbb{N}$ are prime numbers. Since $\eta \in \ker \phi$, we have that $\sum_{\mathfrak{p} \in S_\infty \cup S_\lambda} \Lambda_{\mathfrak{p}}(\eta x_1) = 0$ for any $x_1 \in Y_{A^t}(\mathbf{w}, \lambda)$. Then from (44), for any $p \in \mathcal{P}$ and any $q \notin \mathcal{P}$ we have that

$$T_p(\eta x_1) \equiv \sum_{\mathfrak{p}|p, \mathfrak{p} \notin \mathcal{S}_{\lambda}} \Lambda_{\mathfrak{p}}(\eta x_1) = 0, \quad T_q(\eta x_1) \equiv \sum_{\mathfrak{q}|q} \Lambda_{\mathfrak{q}}(\eta x_1) = 0, \quad \forall x_1 \in Y_{A^t}(\mathbf{w}, \lambda).$$

Since h_A is irreducible by assumption, $K = \text{Span}_{\mathbb{Q}}(u_1, \ldots, u_n)$. Let $\eta = \mathbf{t} \cdot \mathbf{u}, \mathbf{t} \in \mathbb{Q}^n$. Then $T_q(\eta x_1) = {\text{Tr}_{K/\mathbb{Q}}(\eta x_1)}_q = {\mathbf{t} \cdot \mathbf{x}}_q = 0$ for any $\mathbf{x} \in G_A$. Since $\mathbb{Z}^n \subseteq G_A$, this implies that $\mathbf{t} \in \mathcal{R}^n$. Note that there exists $k \in \mathbb{N} \cup \{0\}$ such that

(45)
$$\lambda^k \eta x_1 \in \mathcal{O}_{\mathfrak{p}}, \quad \forall x_1 \in \mathbb{Z}[\mathbf{w}], \, \forall \mathfrak{p} \in \mathcal{S}_{\lambda}.$$

Also, note that $T_p(\lambda^k \eta x_1) = 0$ for any $x_1 \in Y_{A^t}(\mathbf{w}, \lambda)$, since $Y_{A^t}(\mathbf{w}, \lambda)$ is a $\mathbb{Z}[\lambda^{\pm 1}]$ -module. Let $\lambda^k \eta = \mathbf{s} \cdot \mathbf{u}$ for some $\mathbf{s} \in \mathbb{Q}^n$. Note that $x_1 \in \mathbb{Z}[\mathbf{w}]$ if and only if $\mathbf{x} \in \mathbb{Z}^n$. We have that

$$\{\mathbf{s}\cdot\mathbf{x}\}_p = \sum_{\mathfrak{p}\mid p} \Lambda_{\mathfrak{p}}(\lambda^k \eta x_1) = \sum_{\mathfrak{p}\mid p, \,\mathfrak{p}\in\mathcal{S}_{\lambda}} \Lambda_{\mathfrak{p}}(\lambda^k \eta x_1) + T_p(\lambda^k \eta x_1), \quad \forall x_1 \in Y_{A^t}(\mathbf{w}, \lambda),$$

and therefore, $\{\mathbf{s} \cdot \mathbf{x}\}_p = 0$ for $\mathbf{x} \in \mathbb{Z}^n$. This implies that $\mathbf{s} \in (\mathbb{Z}_{(p)})^n$ for any $p \in \mathcal{P}$, where $\mathbb{Z}_{(p)}$ consists of all rational numbers $a/b \in \mathbb{Q}$ such that (b,p) = 1. We now have that $\lambda^k \eta = \mathbf{s} \cdot \mathbf{u}$ and $\eta = \mathbf{t} \cdot \mathbf{u}$, hence $\mathbf{s} = (A^k)^t \mathbf{t}$, since u_1, \ldots, u_n are linearly independent over \mathbb{Q} . Thus, $\mathbf{s} \in (\mathcal{R} \cap_{p \in \mathcal{P}} \mathbb{Z}_{(p)})^n$ and therefore $\mathbf{s} \in \mathbb{Z}^n$ and $\lambda^k \eta \in \mathbb{Z}[\mathbf{u}]$. Hence, $\eta \in \lambda^{-k} \mathbb{Z}[\mathbf{u}] \subset Y_A(\mathbf{u}, \lambda)$. This shows that ker $\phi \subseteq Y_A(\mathbf{u}, \lambda)$.

We now find the fundamental domain \mathcal{F} of the action of $\Gamma = \ker \phi$ on $\mathbb{A}_{K,\lambda}$. We will use the result in Section 6.2 below to count the number of periodic points of a continuous

endomorphism of toroidal solenoid \mathbb{S}_A . As in the proof of Lemma 20, every element of $\mathbb{A}_{K,\lambda}$ is equivalent modulo Γ to an element of

$$\Omega = \prod_{\mathfrak{p}\in\mathcal{S}_{\infty}} K_{\mathfrak{p}} \times \prod_{p\in\mathcal{P}} \prod_{\mathfrak{p}\in\mathcal{S}_{\lambda},\,\mathfrak{p}\mid p} \mathbb{Z}_{p}[\mathbf{u}],$$

so that $\mathcal{F} \subseteq \Omega$. Moreover, by Lemma 21, $\mathbb{Z}[\mathbf{u}] \subseteq \ker \phi$ and hence

(46)
$$\mathcal{F} = [0,1)^n \times \prod_{p \in \mathcal{P}} \prod_{\mathfrak{p} \in \mathcal{S}_{\lambda}, \mathfrak{p} \mid p} \mathbb{Z}_p[\mathbf{u}],$$

where $\prod_{\mathfrak{p}\in\mathcal{S}_{\infty}}K_{\mathfrak{p}}$ is considered as an *n*-dimensional \mathbb{R} -vector space with respect to the basis $\tilde{u}_{1}^{\infty},\ldots,\tilde{u}_{n}^{\infty}$. Since \mathcal{F} has an interior, Γ is discrete.

6. TOROIDAL SOLENOIDS

In this section, we apply our results concerning groups G_A and their endomorphisms to the case of toroidal solenoids. Let \mathbb{T}^n denote a torus considered as a quotient of \mathbb{R}^n by its subgroup \mathbb{Z}^n . A matrix $A \in M_n(\mathbb{Z})$ induces a map $A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$, $A([\mathbf{x}]) = [A\mathbf{x}], [\mathbf{x}] \in \mathbb{T}^n$, $\mathbf{x} \in \mathbb{R}^n$. Consider the inverse system $(M_j, f_j)_{j \in \mathbb{N}}$, where $f_j : M_{j+1} \longrightarrow M_j$, $M_j = \mathbb{T}^n$ and $f_j = A$ for all $j \in \mathbb{N}$. The inverse limit \mathbb{S}_A of the system is called a (*toroidal*) solenoid. As a set, \mathbb{S}_A is a subset of $\prod_{j=1}^{\infty} M_j$ consisting of points $(z_j) \in \prod_{j=1}^{\infty} M_j$ such that $z_j \in M_j$ and $f_j(z_{j+1}) = z_j$ for $\forall j \in \mathbb{N}$, *i.e.*,

(47)
$$\mathbb{S}_A = \left\{ (z_j) \in \prod_{j=1}^{\infty} \mathbb{T}^n \mid z_j \in \mathbb{T}^n, \ A(z_{j+1}) = z_j, \ j \in \mathbb{N} \right\}.$$

Endowed with the natural group structure and the induced topology from the Tychonoff (product) topology on $\prod_{j=1}^{\infty} \mathbb{T}^n$, \mathbb{S}_A is an *n*-dimensional topological abelian group. It is compact, metrizable, and connected, but not locally connected and not path connected. The map $\sigma : \mathbb{S}_A \to \mathbb{S}_A$ induced by multiplication by A, $(z_j) \mapsto (A(z_j))$ is an automorphism of \mathbb{S}_A as a topological group, and the pair (\mathbb{S}_A, σ) is a dynamical system. The endomorphism ring $\operatorname{End}(\mathbb{S}_A, \sigma)$ of the dynamical system (\mathbb{S}_A, σ) consists of endomorphisms of \mathbb{S}_A as a topological group that commute with σ . It is known that $\widehat{G_{A^t}} \cong \mathbb{S}_A$. Indeed, G_{A^t} is a direct limit of groups $(A^t)^{-j}\mathbb{Z}^n$, $j \in \mathbb{N} \cup \{0\}$. Here, each $(A^t)^{-j}\mathbb{Z}^n$ is isomorphic to \mathbb{Z}^n and the maps $(A^t)^{-j}\mathbb{Z}^n \to (A^t)^{-(j+1)}\mathbb{Z}^n$ are inclusions. Applying the functor $\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{T}^1)$ to the system, we obtain the inverse limit of groups

$$\operatorname{Hom}_{\mathbb{Z}}((A^t)^{-j}\mathbb{Z}^n, \mathbb{T}^1) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{T}^1) \cong \mathbb{T}^n$$

with the maps f_j as above that defines \mathbb{S}_A . This gives an isomorphism of topological groups $\widehat{G}_{A^t} \cong \mathbb{S}_A$, where G_{A^t} is endowed with the discrete topology and \widehat{G}_{A^t} is endowed with the compact-open topology. Moreover, since G_{A^t} is a locally compact abelian group, it follows from the Pontryagin duality theorem that the map between the rings $\operatorname{End}(G_{A^t})$ and $\operatorname{End}(\widehat{G}_{A^t})$ given by $\phi \mapsto (\chi \mapsto \chi \circ \phi), \phi \in \operatorname{End}(G_{A^t}), \chi \in \widehat{G}_{A^t}$, is a ring isomorphism between the opposite ring $\operatorname{End}(G_{A^t})^{op}$ of $\operatorname{End}(G_{A^t})$ and $\operatorname{End}(\widehat{G_{A^t}})$. Thus, we have a ring isomorphism $\operatorname{End}(\mathbb{S}_A) \cong \operatorname{End}(G_{A^t})^{op}$, under which σ corresponds to multiplication by A^t on G_{A^t} . Therefore, $\operatorname{End}(\mathbb{S}_A, \sigma)$ is isomorphic to the subring of $\operatorname{End}(G_{A^t})^{op}$ consisting of $T \in \operatorname{M}_n(\mathbb{Q}) \cap \operatorname{End}(G_{A^t})$ that commute with A^t . By Corollary 9, under the assumptions of Proposition 8, $\operatorname{End}(G_{A^t})$ is commutative and, in particular, every endomorphism of G_{A^t} commutes with A^t . This implies that every endomorphism of \mathbb{S}_A commutes with σ and hence is an endomorphism of the dynamical system (\mathbb{S}_A, σ) . Thus, we have the following

Proposition 24. Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial, $\mathcal{P}' \neq \emptyset$, and there exists a prime $p \in \mathbb{N}$ with $(n, t_p) = 1$. Then $\text{End}(\mathbb{S}_A)$ is a commutative ring and, in particular, $\text{End}(\mathbb{S}_A, \sigma) = \text{End}(\mathbb{S}_A)$.

6.1. S-integer dynamical systems. In [CEW97], the authors introduce the so-called S-integer dynamical system (X, α) , a dual object to the group of S-integers $\mathcal{O}_{K,S}$ in a number field K and an element $\xi \in \mathcal{O}_{K,S}$. Groups of the form G_A arise topologically as character groups of toroidal solenoids. Thanks to the description of endomorphisms of G_A in Proposition 8 and the description of G_A as a subset of a number field in Lemma 18, one can see a connection between toroidal solenoids and S-integer dynamical systems.

Definition 25. Let K be a number field and let S be a set of prime ideals of the ring of integers \mathcal{O}_K of K. Let $\xi \in K$, $\xi \neq 0$, $\xi \in \mathcal{O}_{K,S}$, where $\mathcal{O}_{K,S}$ is the ring of S-integers of K defined as

$$\mathcal{O}_{K,\mathcal{S}} = \{x \in K \mid \operatorname{val}_{\mathfrak{p}}(x) \ge 0 \text{ for any prime ideal } \mathfrak{p} \text{ of } \mathcal{O}_K \text{ not in } \mathcal{S}\}.$$

Let $X \cong \widetilde{\mathcal{O}_{K,S}}$ be the (Pontryagin) dual to the discrete (countable) group $\mathcal{O}_{K,S}$ and let $\alpha : X \to X$ be a continuous group endomorphism, the dual to the monomorphism $\hat{\alpha} : \mathcal{O}_{K,S} \to \mathcal{O}_{K,S}$ given by $\hat{\alpha}(x) = \xi x$. A pair $(X, \alpha) = (X^{(K,S)}, \alpha^{(K,S,\xi)})$ is called an *S*-integer dynamical system [CEW97].

Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial and let $\lambda \in \overline{\mathbb{Q}}$ be an eigenvalue of $A, K = \mathbb{Q}(\lambda)$. It is known that $\mathbb{S}_A \cong \widehat{G_{A^t}}$ as topological groups, where G_{A^t} is endowed with the discrete topology. By (30), we know that $G_{A^t} \cong Y_A(\mathbf{u}, \lambda)$, where $\mathbf{u} = (u_1 \dots u_n) \in (\mathcal{O}_K)^n$ is an eigenvector of A corresponding to λ , and $Y_A(\mathbf{u}, \lambda)$ is a $\mathbb{Z}[\lambda^{\pm 1}]$ -submodule of $\mathcal{O}_{K,\lambda}$ given by

(48)
$$Y_A(\mathbf{u},\lambda) = \{m_1\lambda^{k_1}u_1 + \dots + m_n\lambda^{k_n}u_n \mid m_1,\dots,m_n,k_1,\dots,k_n \in \mathbb{Z}\}$$

This implies that $\mathbb{S}_A \cong Y_A(\mathbf{u}, \lambda)$, a description of \mathbb{S}_A in the spirit of Definition 25. Assume in addition that $\mathcal{P}' \neq \emptyset$ and that there exists a prime $p \in \mathbb{N}$ with $(n, t_p) = 1$. Then under the isomorphism $G_{A^t} \cong Y_A(\mathbf{u}, \lambda)$ given by (28) applied to A^t , an endomorphism α of \mathbb{S}_A is dual to an endomorphism $\hat{\alpha}(y) = xy, y \in Y_A(\mathbf{u}, \lambda)$, where $x = i(T), T \in \text{End}(G_{A^t})$, via (16) and the proof of Proposition 8. Recall that $\mathcal{O}_{K,\lambda} = \mathcal{O}_K[\lambda^{-1}]$. It is a ring of \mathcal{S} -integers with \mathcal{S} consisting of prime ideals of \mathcal{O}_K dividing λ . Thus, in general, \mathbb{S}_A is not an \mathcal{S} -integer dynamical system, since it corresponds to a subring $Y_A(\mathbf{u}, \lambda)$ of a ring of \mathcal{S} -integers $\mathcal{O}_{K,\lambda}$. However, we show below that similar results can be proved

regarding toroidal solenoids. This suggests that there might be a more general object that encompasses both S-integer dynamical systems and toroidal solenoids corresponding to matrices with irreducible characteristic polynomials satisfying the condition $(n, t_p) = 1$.

Theorem 26. Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial and $\mathcal{P}' \neq \emptyset$. Assume, in addition, that there exists a prime $p \in \mathbb{N}$ with $(n, t_p) = 1$. Let $\lambda \in \overline{\mathbb{Q}}$ be an eigenvalue of A. Then \mathbb{S}_A is isomorphic to \widehat{Y} , where Yis a $\mathbb{Z}[\lambda^{\pm 1}]$ -submodule of $\overline{\mathbb{Q}}$ generated by u_1, \ldots, u_n , where $\mathbf{u} = (u_1 \ldots u_n) \in (\overline{\mathbb{Q}})^n$ is an eigenvector of A corresponding to λ , and Y is endowed with the discreet topology. If $K = \mathbb{Q}(\lambda)$ and $\mathbf{u} \in (\mathcal{O}_K)^n$, then $Y \subseteq \mathcal{O}_{K,\lambda}$, and

 $\operatorname{Span}_{\mathbb{Z}}(u_1,\ldots,u_n) \subseteq Y \subseteq \operatorname{Span}_{\mathcal{R}}(u_1,\ldots,u_n),$

where $\mathcal{R} = \mathbb{Z}\left[\frac{1}{\det A}\right]$. Moreover, under the isomorphism, each endomorphism α of \mathbb{S}_A is dual to an endomorphism $\hat{\alpha}(y) = xy, y \in Y, x \in \mathcal{O}_{K,\lambda}$, of Y.

Corollary 27. Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial and $\mathcal{P}' \neq \emptyset$. Assume, in addition, that there exists a prime $p \in \mathbb{N}$ with $(n, t_p) = 1$. If there exist an eigenvalue $\lambda \in \overline{\mathbb{Q}}$ with a corresponding eigenvector $\mathbf{u} =$ $(u_1 \ldots u_n) \in (\mathcal{O}_K)^n$ of A, $K = \mathbb{Q}(\lambda)$, such that $\operatorname{Span}_{\mathbb{Z}}(u_1, \ldots, u_n) = \mathbb{Z}[\lambda] = \mathcal{O}_K$, then $(\mathbb{S}_A, \alpha), \alpha \in \operatorname{End}(\mathbb{S}_A)$, is an \mathcal{S}_{λ} -integer dynamical system with \mathcal{S}_{λ} consisting of prime ideals of \mathcal{O}_K dividing λ .

6.2. Ergodicity and periodic points of endomorphisms of \mathbb{S}_A . We now apply the characterization of \mathbb{S}_A via Theorem 19 to count numbers of periodic points of a continuous endomorphism $T \in \text{End}(\mathbb{S}_A)$ of \mathbb{S}_A .

Proposition 28. Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial. Let $\lambda \in \overline{\mathbb{Q}}$ be an eigenvalue of A, $K = \mathbb{Q}(\lambda)$. Let $\mathbf{w} = \begin{pmatrix} w_1 & \dots & w_n \end{pmatrix} \in (\mathcal{O}_K)^n$ be an eigenvector of A^t corresponding to λ . We have an isomorphism

$$\mathbb{S}_A \cong \mathbb{A}_{K,\lambda}/Y_{A^t}(\mathbf{w},\lambda),$$

where

$$Y_{A^t}(\mathbf{w},\lambda) = \{m_1\lambda^{k_1}w_1 + \dots + m_n\lambda^{k_n}w_n \mid m_1,\dots,m_n,k_1,\dots,k_n \in \mathbb{Z}\},\$$

and $Y_{A^t}(\mathbf{w}, \lambda)$ is embedded diagonally into $\mathbb{A}_{K,\lambda}$.

Proof. Since $\mathbb{S}_A \cong \widehat{G_{A^t}}$ (see the 1st paragraph of Section 6), the proposition follows from Theorem 19 applied to A^t .

Let $T \in \text{End}(\mathbb{S}_A)$ be a continuous endomorphism of \mathbb{S}_A . By definition, the set $F_k(T)$ of points of period $k \geq 1$ of T is

$$F_k(T) = \{ x \in \mathbb{S}_A \mid T^k(x) = x \}.$$

Lemma 29. Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial and $\mathcal{P}' \neq \emptyset$. Assume, in addition, that there exists a prime $p \in \mathbb{N}$ with $(n, t_p) = 1$. Then $T \in \text{End}(\mathbb{S}_A)$ is ergodic if and only if each eigenvalue of T is not a root of unity. *Proof.* It follows from [CEW97, Theorem 4.2], Proposition 8, and Theorem 26.

Let $T : \mathbb{S}_A \longrightarrow \mathbb{S}_A$ be a continuous homomorphism. Since $\mathbb{S}_A \cong \widehat{G}_{A^t}$ as topological groups, T induces a homomorphism (denoted by the same letter by abuse of notation) $T : \widehat{G}_{A^t} \longrightarrow \widehat{G}_{A^t}$. Since G_{A^t} is locally compact with respect to the discrete topology, we have that $\widehat{\widehat{G}_{A^t}} \cong G_{A^t}$ and the induced dual $T : G_{A^t} \longrightarrow G_{A^t}$ is a homomorphism. By Proposition 8, $\xi = i(T) \in \mathcal{O}_{K,\lambda}$.

Proposition 30. Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial and $\mathcal{P}' \neq \emptyset$. Assume, in addition, that there exists a prime $p \in \mathbb{N}$ with $(n, t_p) = 1$. Let $T \in \text{End}(\mathbb{S}_A)$ be an ergodic continuous endomorphism of \mathbb{S}_A (see Lemma 29 above). Then the number $|F_k(T)|$ of points of period $k \geq 1$ of T is finite and

$$|F_k(T)| = \prod_{\mathfrak{p} \in S_{\lambda} \cup S_{\infty}} |\xi^k - 1|_{\mathfrak{p}}, \quad \xi = \imath(T) \in \mathcal{O}_{K,\lambda}.$$

Proof. We adapt [CEW97, Lemma 5.1] and [CEW97, Lemma 5.2] to our case. By Proposition 28, Theorem 19 and (46), the fundamental domain of the action of Γ on $\mathbb{A}_{K,\lambda}$ applied to $\widehat{G_{A^t}}$, has the form

$$\mathcal{F} = [0,1)^n \times \prod_{p \in \mathcal{P}} \prod_{\mathfrak{p} \in \mathcal{S}_{\lambda}, \, \mathfrak{p} \mid p} \mathbb{Z}_p[\mathbf{w}],$$

where $\mathbf{w} \in (\mathcal{O}_K)^n$ is an eigenvector of A^t corresponding to an eigenvalue λ . Let $T \in \text{End}(G_{A^t})$ be the induced dual of T (see the paragraph above the statement of the proposition). By Proposition 8, $T(\mathbf{w}) = \xi \mathbf{w}$ for some $\xi \in \mathcal{O}_{K,\lambda}$. Thus, the induced action of T on $\mathbb{A}_{K,\lambda}$ is $T(\eta) = \xi \eta$, $\eta \in \mathbb{A}_{K,\lambda}$. Since $\mu(\mathcal{F}) \neq 0$, by [CEW97, Lemma 5.1], we have

$$|F_k(T)| = |\ker(T^k - \mathrm{id})| = \mu((T^k - \mathrm{id})\mathcal{F})/\mu(\mathcal{F}) = \prod_{\mathfrak{p} \in \mathcal{S}_\lambda \cup \mathcal{S}_\infty} |\xi^k - 1|_{\mathfrak{p}}.$$

Remark 31. Our formula for $|F_k(T)|$ in Proposition 30 is consistent with earlier results from [HL23] and [M08]. Recall that G_A is a subgroup of \mathbb{Q}^n of rank n. In [HL23], the authors provide a formula for $|F_k(T)|$, when T is a continuous endomorphism of the dual group \widehat{G} of a subgroup G of \mathbb{Q}^2 of rank 2. In [M08], the author presents a more general formula for when \widehat{G} is a finite-dimensional compact abelian group. Our approach differs from both works. In particular, we describe \mathbb{S}_A using an adèle ring. In [M08], the formula involves several global fields K_1, \ldots, K_n , sets of finite places \mathcal{P}_i in K_i , and $\xi_i \in K_i$ [M08, Theorem 1.1]. In our case of $\widehat{G} = \mathbb{S}_A$, under the assumptions in Proposition 30, the number of global fields n is 1, with $K_1 = \mathbb{Q}(\lambda)$ and \mathcal{P}_1 consisting of prime ideals of the ring of integers of K_1 that do not divide λ . The correspondence between [M08, Theorem 1.1] and Proposition 30 is established through the product formula.

Example 4. In this example, we demonstrate how Proposition 30 can be used to count periodic points of a toroidal solenoid endomorphism when n > 2. Let n = 3, let $A \in M_3(\mathbb{Z})$

be a matrix with characteristic polynomial $h_A = x^3 - x^2 + 2x - 6 \in \mathbb{Z}[x]$. One can check that h_A is irreducible, det A = 6, $\mathcal{P} = \mathcal{P}' = \{2, 3\}$, so that the hypotheses on A in Proposition 30 hold. Let λ be a root of h_A , $K = \mathbb{Q}(\lambda)$. It is known that K is not Galois over \mathbb{Q} , \mathcal{O}_K is a principal ideal domain, and $\mathcal{O}_K = \mathbb{Z}[\lambda]$ [LMFDB, Number field 3.1.808.1]. Also, the ideals of \mathcal{O}_K generated by 2, 3 have the following prime decompositions: $2\mathcal{O}_K = \mathfrak{p}_1^2\mathfrak{p}_2$ and $3\mathcal{O}_K = \mathfrak{q}_1\mathfrak{q}_2$, where $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2$ are prime ideals of \mathcal{O}_K . There is a choice of λ such that $\lambda \mathcal{O}_K = \mathfrak{p}_1\mathfrak{q}_2$, $\mathfrak{p}_1 = (2 - \lambda)\mathcal{O}_K$, $\mathfrak{p}_2 = (5\lambda^2 + 4\lambda + 17)\mathcal{O}_K$, $\mathfrak{q}_1 = (\lambda^2 - \lambda - 1)\mathcal{O}_K$, $\mathfrak{q}_2 = (\lambda^2 + \lambda + 3)\mathcal{O}_K$ [SageMath]. By Corollary 10, End(G_{A^t}) $\cong \mathbb{Z}[\lambda^{\pm 1}]$. We pick an arbitrary element $\xi = (2 + \lambda - \lambda^2)\lambda^{-3} \in \mathbb{Z}[\lambda^{\pm 1}]$. It defines an endomorphism $\widehat{T} = (2I + A^t - (A^t)^2)(A^t)^{-3}$ of G_{A^t} , and its dual T is a continuous ergodic endomorphism of \mathbb{S}_A . One can compute $(\xi - 1)\mathcal{O}_K = \mathfrak{p}_1^{-2}\mathfrak{q}_2^{-3}\mathfrak{a}^2$, where \mathfrak{a} is a prime ideal of \mathcal{O}_K above 13 with the norm $N(\mathfrak{a}) = 13$. Since $\mathcal{S}_\lambda = \{\mathfrak{p}_1, \mathfrak{q}_2\}$, by Proposition 30, the number of fixed points of T is given by

$$|F_1(T)| = \prod_{\mathfrak{p} \in \mathcal{S}_{\lambda} \cup \mathcal{S}_{\infty}} |\xi - 1|_{\mathfrak{p}} = N(\mathfrak{a})^2 = 169.$$

Similarly, $(\xi^2 - 1)\mathcal{O}_K = \mathfrak{p}_1^{-4}\mathfrak{q}_2^{-6}\mathfrak{a}^2\mathfrak{b}$, where \mathfrak{b} is a prime ideal of \mathcal{O}_K above 229 with the norm $N(\mathfrak{b}) = -229$. By Proposition 30, the number of points of period 2 of T is given by

$$|F_2(T)| = \prod_{\mathfrak{p} \in \mathcal{S}_{\lambda} \cup \mathcal{S}_{\infty}} |\xi^2 - 1|_{\mathfrak{p}} = |N(\mathfrak{a})^2 N(\mathfrak{b})| = 169 \times 229.$$

Corollary 32. Assume $A \in M_n(\mathbb{Z})$ is non-singular with an irreducible characteristic polynomial and $\mathcal{P}' \neq \emptyset$. Assume, in addition, that there exists a prime $p \in \mathbb{N}$ with $(n, t_p) = 1$. Let $T \in \text{End}(\mathbb{S}_A)$, $\xi = i(T) \in \mathcal{O}_{K,\lambda}$ (see Proposition 8). Assume ξ is not a root of unity. Then the growth rate of the number of periodic points exists and is given by

$$p^+(\xi) = p^-(\xi) = h_{top}(\xi).$$

Here,

$$p^+(\xi) = \lim \sup_{k \to \infty} \frac{1}{k} \log |F_k(\xi)|, \qquad p^-(\xi) = \lim \inf_{k \to \infty} \frac{1}{k} \log |F_k(\xi)|,$$

and $h_{top}(\xi)$ is the topological entropy of ξ .

Proof. Follows from the proof of [CEW97, Theorem 6.1] and Proposition 30.

7. Endomorphisms of \mathbb{Z}^n -odometers

 \mathbb{Z}^n -odometer is a dynamical system consisting of a topological space X and an action of the group \mathbb{Z}^n on X (by homeomorphisms). Consider a decreasing sequence of finite-index subgroups of \mathbb{Z}^n

$$G = \mathbb{Z}^n \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

and the natural maps $\pi_i: G/G_{i+1} \longrightarrow G/G_i, i \in \mathbb{N}$. The associated \mathbb{Z}^n -odometer is the inverse limit

(49)
$$X = \lim_{\longleftarrow} \left(G/G_i \right)$$

together with the natural action of \mathbb{Z}^n . For the sequence

$$G_i = \left\{ A^i \mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^n \right\}, \quad i \in \mathbb{N},$$

denote by X_A the corresponding odometer. In [CP24], the authors study the linear representation group of X_A denoted by $\vec{N}(X_A)$. By [CP24, Lemma 2.6], $\vec{N}(X_A)$ consists of $T \in \operatorname{GL}_n(\mathbb{Z})$ (*i.e.*, $T \in \operatorname{M}_n(\mathbb{Z})$ with det $T = \pm 1$) such that for any $m \in \mathbb{N} \cup \{0\}$ there exists $k_m \in \mathbb{N} \cup \{0\}$ with

(50)
$$A^{-m}TA^{k_m} \in \mathcal{M}_n(\mathbb{Z}).$$

By taking the transpose of the condition, one can see that it is equivalent to the condition that T^t defines an endomorphism of G_{A^t} .

Lemma 33. $T \in \vec{N}(X_A)$ if and only if $T^t \in \text{End}(G_{A^t}) \cap \text{GL}_n(\mathbb{Z})$.

Therefore, our results can be applied to T^t . In particular, we can provide an alternate proof of [CP24, Theorem 3.3] for the case when n = 2 and also generalize some parts of it to an arbitrary n.

Lemma 34. Let $A \in M_n(\mathbb{Z})$ be non-singular, let $F \subset \overline{\mathbb{Q}}$ be any finite extension of \mathbb{Q} that contains all the eigenvalues of A, and let $\mathcal{P}'(A) \neq \emptyset$. Let $T \in GL_n(\mathbb{Z})$. Then $T(G_A) \subseteq G_A$ if and only if

$$T(\mathcal{X}_{A,\mathfrak{p}}) \subseteq \mathcal{X}_{A,\mathfrak{p}}$$

for any $p \in \mathcal{P}'$ and a prime ideal \mathfrak{p} of \mathcal{O}_F above p.

Proof. Follows from Theorem 5, since (6) holds for any matrix $T \in M_n(\mathbb{Z})$.

Corollary 35. Let $A \in M_n(\mathbb{Z})$ be non-singular, let $F \subset \overline{\mathbb{Q}}$ be any finite extension of \mathbb{Q} that contains all the eigenvalues of A, and let $\mathcal{P}'(A) \neq \emptyset$. Let $T \in GL_n(\mathbb{Z})$. Then $T \in \vec{N}(X_A)$ if and only if

$$T(\mathcal{X}_{A^t,\mathfrak{p}}) \subseteq \mathcal{X}_{A^t,\mathfrak{p}}$$

for any $p \in \mathcal{P}'$ and a prime ideal \mathfrak{p} of \mathcal{O}_F above p.

Proof. Follows from Lemma 33 and Lemma 34.

Proposition 36. Let $A \in M_n(\mathbb{Z})$ be non-singular.

- (1) If $\mathcal{P}(A) = \emptyset$ or $\mathcal{P}'(A) = \emptyset$, then $\vec{N}(X_A) = \mathrm{GL}_n(\mathbb{Z})$.
- (2) Assume $\mathcal{P}'(A) \neq \emptyset$. If the characteristic polynomial $h_A \in \mathbb{Z}[x]$ is irreducible and there exists a prime $p \in \mathbb{N}$ with $(n, t_p) = 1$, then $\vec{N}(X_A)$ is the centralizer of A in $\operatorname{GL}_n(\mathbb{Z})$. Moreover, $\vec{N}(X_A)$ is isomorphic to a subgroup of the group of units in the ring of integers of a number field $K = \mathbb{Q}(\lambda)$ generated by an eigenvalue $\lambda \in \overline{\mathbb{Q}}$ of A. Therefore, as such, $\vec{N}(X_A)$ is a finitely generated abelian group and when n = 2 and h_A has no real roots, then $\vec{N}(X_A)$ is finite.

Here, the condition $\mathcal{P}(A) = \emptyset$ is equivalent to det $A = \pm 1$. Recall that

$$\mathcal{P}'(A) = \{ p \in \mathcal{P}(A), \ h_A \not\equiv x^n \pmod{p} \}.$$

Thus, the condition $\mathcal{P}'(A) = \emptyset$ in the case when n = 2 is equivalent to the fact that every prime dividing det A also divides $\operatorname{Tr} A$, $\operatorname{rad}(\det A)$ divides trace A in the notation of [CP24]. In the case when n = 2, the conditions that the characteristic polynomial $h_A \in \mathbb{Z}[x]$ is irreducible is equivalent to the fact that A has no rational eigenvalues and the condition that there exists a prime $p \in \mathbb{N}$ with $(n, t_p) = 1$ holds automatically when $\mathcal{P}'(A) \neq \emptyset$.

Proof of Proposition 36. Statement (1) follows from Lemma 2. Statement (2) follows from Proposition 8, since by above, for any $T \in \vec{N}(X_A)$, $T^t \in \text{End}(A^t)$. Moreover, since $T \in \text{GL}_n(\mathbb{Z})$ by the definition of $\vec{N}(X_A)$, in the notation of the proof of Proposition 8, $x = i(A^t)(T^t)$ is a unit in the ring of integers \mathcal{O}_K of K, *i.e.*, $x \in \mathcal{O}_K^{\times}$. It is well-known that the group of units \mathcal{O}_K^{\times} of \mathcal{O}_K is finitely generated and when n = 2 and h_A has no real roots, then \mathcal{O}_K^{\times} is finite. \Box

Recall that if n = 2, $A \notin \operatorname{GL}_2(\mathbb{Z})$, and $\mathcal{P}'(A) \neq \emptyset$, there are three cases distinguished in [S22]:

(a) $h_A \in \mathbb{Z}[x]$ is irreducible (equivalently, A has no rational eigenvalues),

(b) h_A is reducible (equivalently, A has eigenvalues $\lambda_1, \lambda_2 \in \mathbb{Z}$), rad (λ_1) does not divide rad (λ_2) , and rad (λ_2) does not divide rad (λ_1) ,

(c) h_A is reducible and every prime $p \in \mathbb{N}$ dividing one eigenvalue, divides the other, *e.g.*, $\operatorname{rad}(\lambda_1)$ divides $\operatorname{rad}(\lambda_2)$ (denoted by $\operatorname{rad}(\lambda_1) | \operatorname{rad}(\lambda_2)$).

Case (a) is covered by Proposition 36 (2). The remaining two cases are covered in the next proposition (*c.f.*, [CP24, Theorem 3.3 (b)]).

Proposition 37. Let $A \in M_2(\mathbb{Z})$ be non-singular, $\mathcal{P}(A) \neq \emptyset$, and $\mathcal{P}'(A) \neq \emptyset$. Assume h_A is reducible. Then A has distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{Z}$.

(1) Assume that $rad(\lambda_1)$ does not divide $rad(\lambda_2)$, and $rad(\lambda_2)$ does not divide $rad(\lambda_1)$.

- If there exists a matrix $M \in M_2(\mathbb{Z})$ diagonalizing A^t with det M dividing 2, then $\vec{N}(X_A) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- Otherwise, $\vec{N}(X_A) = \{\pm id\}.$

(2) Assume $\operatorname{rad}(\lambda_1) |\operatorname{rad}(\lambda_2) \text{ or } \operatorname{rad}(\lambda_2)| \operatorname{rad}(\lambda_1)$, then $\vec{N}(X_A)$ is isomorphic to the group of lower-triangular matrices in $\operatorname{GL}_2(\mathbb{Z})$.

Proof. Recall that $T \in \vec{N}(X_A)$ if and only if $T^t \in \text{End}(G_{A^t}) \cap \text{GL}_n(\mathbb{Z})$ by Lemma 33. Then (1) follows from Theorem 6 applied to A^t and T^t . Indeed, for T^t given by (8), we have that $T \in \text{GL}_2(\mathbb{Z})$ if and only if $x_1 = \pm 1$, $x_2 = \pm 1$. Moreover, one can check that

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 T^t has integer coefficients if and only if $v \in \mathbb{Z}$ divides $x_1 - x_2$ in \mathbb{Z} . Since $x_1 - x_2 = 0$ or $x_1 - x_2 = \pm 2$, if v divides 2, then $\vec{N}(X_A) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and otherwise, $\vec{N}(X_A) = \{\pm id\}$. Similarly, (2) follows from Theorem 7 applied to A^t and T^t .

Remark 38. In [CP24], the authors conclude that $\vec{N}(X_A)$ is computable when n = 2 and pose the question whether the group $\vec{N}(X_A)$ is computable when n > 2. In [EHO19, p. 750, Main Algorithm 2], the authors show that the centralizer of an element in $\operatorname{GL}_n(\mathbb{Z})$ is computable. Thus, if assumptions in (1), (2) of Proposition 36 hold (the assumptions are computable), then $\vec{N}(X_A)$ is also computable by Proposition 36 and [EHO19, p. 750, Main Algorithm 2].

Example 5. [S22, Example 10]. Let $A \in M_n(\mathbb{Z})$ be non-singular with an irreducible characteristic polynomial $h_A \in \mathbb{Z}[x]$. By Proposition 36 (2), if there exists t_p with $(n, t_p) = 1$, then $\vec{N}(X_A)$ is the centralizer of A in $GL_n(\mathbb{Z})$ and $\vec{N}(X_A)$ is abelian. In this example, we show that this is not always the case. Here n = 4 and $t_p = 2$, so the condition $(n, t_p) = 1$ in Proposition 36 (2) does not hold.

Let $h(x) = x^4 - 2x^3 + 21x^2 - 20x + 5$, irreducible over \mathbb{Q} , and let $\lambda \in \overline{\mathbb{Q}}$ be a root of $h, K = \mathbb{Q}(\lambda)$. By [LMFDB], $\mathcal{O}_K = \mathbb{Z}[\lambda], K$ is Galois over $\mathbb{Q}, \operatorname{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let

$$\mathbf{u} = \begin{pmatrix} 1 & \lambda & \lambda^2 & \lambda^3 \end{pmatrix}^t, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 20 & -21 & 2 \end{pmatrix},$$

so that **u** is an eigenvector of A corresponding to λ , and A has characteristic polynomial $h_A(x) = h(x) = x^4 - 2x^3 + 21x^2 - 20x + 5$. Also, det A = 5, $\mathcal{P}(A) = \mathcal{P}'(A) = \{5\}$, $t_5 = 2$. By [SageMath], $(5) = \mathfrak{p}_1^2 \mathfrak{p}_2^2$, where $\mathfrak{p}_1, \mathfrak{p}_2$ are prime ideals of $\mathbb{Z}[\lambda], \mathfrak{p}_1 = (\lambda)$, and there exists $g \in \operatorname{Gal}(K/\mathbb{Q})$ of order 2 such that $g(\mathfrak{p}_i) = \mathfrak{p}_i, i = 1, 2$. In the notation of [S24, Theorem 4.3], $\mathcal{X}_{A,\mathfrak{p}_1} = \operatorname{Span}_K(\mathbf{u}, g(\mathbf{u}))$. We have that

$$\mathbf{u} = \begin{pmatrix} 1 & \lambda & \lambda^2 & \lambda^3 \end{pmatrix}^t, \quad g(\mathbf{u}) = \begin{pmatrix} 1 & g(\lambda) & g(\lambda^2) & g(\lambda^3) \end{pmatrix}^t,$$

where

$$g(\lambda) = -4\lambda^3 + 6\lambda^2 - 81\lambda + 40,$$

$$g(\lambda^2) = -4\lambda^3 + 5\lambda^2 - 80\lambda + 20,$$

$$g(\lambda^3) = 75\lambda^3 - 114\lambda^2 + 1520\lambda - 770.$$

Then $g(\mathbf{u}) = L\mathbf{u}$, where $L \in \mathrm{GL}_4(\mathbb{Z})$ and

$$L = \begin{pmatrix} 1 & 0 & 0 & 0\\ 40 & -81 & 6 & -4\\ 20 & -80 & 5 & -4\\ -770 & 1520 & -114 & 75 \end{pmatrix}.$$

Since $g(\mathbf{u})$ is not a multiple of \mathbf{u} (they are eigenvectors corresponding to two distinct eigenvalues $g(\lambda)$, λ , respectively), we see that \mathbf{u} is not an eigenvector of L. Therefore, A and L do not commute. On the other hand, $L \in \vec{N}(X_A)$. Indeed, $\operatorname{Gal}(K/\mathbb{Q})$ acts transitively on the prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ above 5, so there exists $g' \in \operatorname{Gal}(K/\mathbb{Q})$ such that $g'(\mathfrak{p}_1) = \mathfrak{p}_2$. By above, $L(\mathcal{X}_{A,\mathfrak{p}_1}) \subseteq \mathcal{X}_{A,\mathfrak{p}_1}$ and applying g', we get $L(\mathcal{X}_{A,\mathfrak{p}_2}) \subseteq \mathcal{X}_{A,\mathfrak{p}_2}$. By [S24, Theorem 4.3], $L(G_A) \subseteq G_A$. On the other hand, for any unit $\xi \in \mathcal{O}_K^{\times}$, let $T = T(\xi)$ be given by (15) and (16), *i.e.*,

(51)
$$T = MXM^{-1}, \quad X = \operatorname{diag} \begin{pmatrix} \sigma_1(\xi) & \sigma_2(\xi) & \sigma_3(\xi) & \sigma_4(\xi) \end{pmatrix},$$

where M is a matrix diagonalizing A and $\operatorname{Gal}(K/\mathbb{Q}) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. Then, by construction, $T \in \operatorname{GL}_4(\mathbb{Q})$ and $\det T = \pm 1$. Since $\mathcal{O}_K = \mathbb{Z}[\lambda], \xi = \sum_{i=0}^3 a_i \lambda^i$ with $a_0, a_1, a_2, a_3 \in \mathbb{Z}$. This implies that $T = \sum_{i=0}^3 a_i A^i$ and hence $T \in \operatorname{GL}_4(\mathbb{Z})$ and $T \in \vec{N}(X_A)$. Therefore, $i^{-1}(\mathcal{O}_K^{\times}) \subseteq \vec{N}(X_A)$. However, $L \notin i^{-1}(\mathcal{O}_K^{\times})$ and does not commute with the image.

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