CENTRAL LIMIT THEOREM FOR NON-STATIONARY RANDOM PRODUCTS OF $SL(2,\mathbb{R})$ MATRICES

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ABSTRACT. We prove Central Limit Theorem for non-stationary random products of $SL(2,\mathbb{R})$ matrices, generalizing the classical results by Le Page and Tutubalin that were obtained in the case of iid random matrix products.

1. INTRODUCTION

1.1. **Historical background.** The two most fundamental results in probability that are present in almost every textbook are the (strong) Law of Large Numbers (LLN) and the Central Limit Theorem (CLT). In the most basic form, if $\{\xi_n\}$ is an iid sequence of random variables with finite expectation a and finite variance σ^2 , the LLN claims that almost surely $\frac{1}{n} \sum_{i=1}^{n} \xi_i \to a$, and the CLT claims that $\frac{\sum_{i=1}^{n} \xi_i - na}{\sqrt{n\sigma}}$ converges in distribution to a normal distribution $\mathcal{N}(0,1)$ with mean 0 and variance 1.

There are many ways to relax the assumptions in both cases. In particular, the random variables do not have to be identically distributed. For example, a non-stationary version of the LLN known as *Kolmogorov's Law* [Kol] claims that if $\{\xi_i\}$ is a sequence of independent random variables with $a_i = \mathbb{E}\xi_i$, $\sigma_i^2 = \text{Var}(\xi_i)$, and $\sum_{i=1}^{\infty} \frac{\sigma_i^2}{i^2} < \infty$, then $\frac{\sum_{i=1}^{n} (\xi_i - a_i)}{n} \to 0$ almost surely. On the other hand, if for some $\delta > 0$ the sequence $\mathbb{E}|\xi_i|^{2+\delta}$ is uniformly bounded, then the sequence of random variables $\frac{\sum_{i=1}^{n} (\xi_i - a_i)}{\sum_{i=1}^{n} \sigma_i^2}$ converges in distribution to $\mathcal{N}(0, 1)$.

There are plenty of different generalizations and forms of these statements. For example, for some of the analogs of the LLN and CLT for the sums of iid random variables in the context of random walks on groups see the survey [F] and monograph [BQ2], and references therein. Here we discuss random matrix products. In this case, a multiplicative version of the LLN is given by Furstenberg and Kesten [FurK]. A stronger result is the famous Furstenberg Theorem, which also guarantees positivity of the Lyapunov exponent:

Theorem 1.1 (H. Furstenberg [Fur]). Let $\{X_k, k \ge 1\}$ be independent and identically distributed random variables, taking values in $SL(d, \mathbb{R})$, the $d \times d$ matrices with determinant one, let G_X be the smallest closed subgroup of $SL(d, \mathbb{R})$ containing the support of the distribution of X_1 , and assume that

 $\mathbb{E}[\log \|X_1\|] < \infty.$

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Also, assume that G_X is not compact and is strongly irreducible, i.e. there exists no G_X -invariant finite union of proper subspaces of \mathbb{R}^d . Then there exists a positive constant λ_F (Lyapunov exponent) such that with probability one

$$\lim_{n \to \infty} \frac{1}{n} \log \|X_n \dots X_2 X_1\| = \lambda_F > 0.$$

Remark 1.2. In the case of random products of $SL(2, \mathbb{R})$ matrices, the assumption that G_X is not compact and is strongly irreducible is equivalent to the assumption that there exists no measure on \mathbb{RP}^1 invariant under the action of every map from G_X , see [AB, Lemma 3.6].

The CLT for the products of iid random matrices is also available. The initial results were obtained for matrices with positive coefficients [Bel], [FurK]. In the case of absolutely continuous distributions it was obtained by Tutubalin [T1, T2]. The requirements on regularity of distributions was relaxed by Le Page, who proved the CLT for random matrix products under the assumption of finite exponential moments [L], see also [BL], [GR], [GM], [J]. Finally, the assumption on the moments of the distribution was optimized by Benoist and Quint [BQ1]:

Theorem 1.3 (Benuist, Quint, [BQ1]). Let $\{X_k, k \ge 1\}$ be independent and identically distributed random matrices in $SL(d, \mathbb{R})$. Assume that G_X is non-compact and strongly irreducible and

(1)
$$\mathbb{E}\left[(\log \|X_1\|)^2\right] < \infty$$

Then there exists $\sigma > 0$ such that the random variables

$$\frac{\log \|X_n \dots X_1\| - n\lambda_F}{\sqrt{n}},$$

where $\lambda_F > 0$ is the Lyapunov exponent, converge in distribution to $\mathcal{N}(0, \sigma^2)$.

Notice that both Theorems 1.1 and 1.3 require the sequence of random matrices to be identically distributed. That requirement allows to consider a stationary measure for the random dynamics on the projective space, which is a key notion used in the proofs of both results. Nevertheless, the classical LLN and CLT for sums of real valued random variables hold without that assumption, and it is natural to expect that non-stationary versions of the LLN and CLT for random matrix products should hold as well. Indeed, the non-stationary version of the Furstenberg Theorem was recently provided in [GK1], and it already found interesting applications in spectral theory [GK2]. The non-stationary version of the CLT for random products of $SL(2, \mathbb{R})$ matrices is the main result of this paper.

1.2. Preliminaries and main results. Let us now provide the setting needed to state our main result. From now on, let us restrict ourselves to the case of products of $SL(2,\mathbb{R})$ matrices.

Let \mathcal{K} be a compact subset in the set of probability measures on the group $\mathrm{SL}(2,\mathbb{R})$. We will say that **the measures condition** is satisfied if for every measure $\mu \in \mathcal{K}$ there are no Borel probability measures ν_1 , ν_2 on \mathbb{RP}^1 such that $(f_A)_*\nu_1 = \nu_2$ for μ -almost every $A \in \mathrm{SL}(2,\mathbb{R})$.

Let us fix some sequence $\{\mu_i\}_{i\in\mathbb{N}}, \mu_i\in\mathcal{K}$, and let $A_i\in\mathrm{SL}(2,\mathbb{R})$ be independent matrix-valued random variables, with A_i being distributed w.r.t. μ_i . Set

$$T_n = A_n A_{n-1} \dots A_1,$$

and denote

(2)
$$L_n = \mathbb{E} \log \|T_n\|$$

If the measures condition is satisfied, then for any $\{\mu_i\}_{i\in\mathbb{N}}, \mu_i \in \mathcal{K}$, the sequence $\{L_n\}$ must grow at least linearly, i.e. the norms of the random products must grow exponentially on average, see [GK1, Theorem 1.5]. A related statement on exponential growth of the norms in the case of non-stationary linear cocycles over Markov chains was established by Goldsheid [G]. Moreover, if additionally a uniform bound on some exponential moment exists for distributions from \mathcal{K} , the non-random sequence $\{L_n\}$ describes the behavior of almost every random product, and in this sense serves as a non-stationary analog of Lyapunov exponent. Namely, almost surely one has $\lim_{n\to\infty} \frac{1}{n} (\log ||T_n|| - L_n) = 0$, see [GK1, Theorem 1.1]. This provides a direct analog of the LLN for non-stationary random matrix products.

That compels the question whether an analog of CLT for non-identically distributed random variables must hold in this setting. Our main result provides a positive answer in dimension two:

Theorem 1.4. Let \mathcal{K} be a compact subset in the set of probability measures on the group $SL(2,\mathbb{R})$ that satisfies the measures condition, and there exists $\gamma \geq 9$ and M > 0 such that for any $\mu \in \mathcal{K}$ one has

(3)
$$\mathbb{E}_{\mu}(\log \|A\|)^{\gamma} < M.$$

Then the random variables

$$\frac{\log \|T_n\| - L_n}{\sqrt{\operatorname{Var}(\log \|T_n\|)}}$$

converge in distribution to $\mathcal{N}(0,1)$, with the convergence that is uniform with respect of the choice of the sequence $\mu_1, \mu_2, \dots \in \mathcal{K}$.

Also, there are constants $C_1, C_2 > 0$ and an index n_0 such that for all $n \ge n_0$ and all $\mu_1, \ldots, \mu_n \in \mathcal{K}$ one has

(4)
$$C_1 n \le \operatorname{Var}(\log \|T_n\|) \le C_2 n.$$

- **Remark 1.5.** (a) The condition (3) is most likely not optimal. We would expect that it should be sufficient to require $\gamma > 2$, compare with the version of the LLN for real valued random variables provided above. Notice that it is still more restrictive than $\gamma = 2$ which is optimal in the iid case, compare with Theorem 1.3.
- (b) One should expect that, under suitable conditions, Theorem 1.4 should hold for random SL(d, ℝ) matrix products for every d ≥ 2. To prove such a statement, it would be helpful to have a non-stationary analog of simplicity of the Lyapunov spectrum, see [GR], [GM] for the case of iid random matrix products. In the case of some specific regular distributions in SL(d, ℝ) such an analog was recently established [AFGQ], but a statement for a general sequence of distributions is currently not available, even if certainly expected.

We consider this paper as a "proof of concept", the demonstration that enormous amount of results on random walks on groups formulated in terms of the law of large numbers, CLT, the law of the iterated logarithms etc. can be expected to hold in the non-stationary setting, even if the notion of the stationary measure on the projective space is not defined. The key observation here is that a random dynamical system acts on the measures on the phase space by convolutions, i.e. averaging of the push-forwards of the measure by the random dynamics, and such an action "moves" measures toward the space of measures with some specific modulus of continuity, e.g. Hölder or log-Hölder, depending on the setting, see [GKM, Theorem 2.8], [M1, Theorems 2.4 and 2.9]. For some other recent results related to non-stationary random dynamics see [GK3], [M2], [M3].

1.3. Notations and plan of the proof of the main result. Let us introduce some notations. Let

$$T_{(n_1,n_2]} := A_{n_2} A_{n_2-1} \dots A_{n_1+1}$$

be the part of the product of our random matrices A_i , where the index varies from $n_1 + 1$ to n_2 . Also, denote

$$\xi_n = \log ||T_n||, \quad \xi_{(n_1, n_2]} = \log ||T_{(n_1, n_2]}||.$$

Note that if two intervals of indices $(n_1, n_2]$ and $(n'_1, n'_2]$ are disjoint, then the corresponding products $T_{(n_1,n_2]}$ and $T_{(n'_1,n'_2]}$ are independent, and thus so are their log-norms $\xi_{(n_1,n_2]}$ and $\xi_{(n'_1,n'_2]}$.

Now, a long product of matrices can be split into two parts (that we will later choose to be of comparable lengths): for any n, n' one has

$$T_{n+n'} = T_{(n,n+n']}T_n;$$

in particular, this implies

(5) $\xi_{n+n'} = \log ||T_n T_{(n,n+n')}|| \le \log ||T_n|| + \log ||T_{(n,n+n')}|| = \xi_n + \xi_{(n,n+n')}.$

The right hand side of the inequality in (5) is a sum of two independent random variables; let us introduce the random variable $R_{n,n'}$ that measures the difference between the right and left hand sides of (5):

(6)
$$R_{n,n'} = \log ||T_n|| + \log ||T_{(n,n+n']}|| - \log ||T_{n+n'}|| = (\xi_n + \xi_{(n,n+n']}) - \xi_{n+n'}.$$

We start the proof of Theorem 1.4 with establishing uniform moment bounds for the discrepancy $R_{n,n'}$; this is done in Sec. 2, see Proposition 2.1. To do so, we have to show that it is (sufficiently) improbable that the mostly expanded vector for the product T_n is sent to the direction close to the one that is contracted by $T_{(n,n+n')}$. This can be reformulated in terms of the action on the projective line \mathbb{RP}^1 : in these terms, it is the probability of sending a point to a given small neighbourhood. We use results from [M1], where such estimates (log-Hölder bounds after a finite number of non-stationary iterations) were established, to obtain Lemma 2.2, providing such tail estimates.

Next, we use these estimates to establish a control on the central moments of ξ_n , using the relation

(7)
$$\xi_{n+n'} = \left(\xi_n + \xi_{(n,n+n']}\right) - R_{n,n'}.$$

To do so, we use the fact that the sum in the parenthesis is a sum of independent random variables, and the moments for $R_{n,n'}$ are uniformly bounded, thus its addition cannot increase the moments too much. This is done in Section 3.1, see Proposition 3.1. Then, in Section 3.2 (see Lemma 3.3) we get a lower bound for the linear growth of the variances $\operatorname{Var} \xi_n$, thus altogether establishing the conclusion (4). The argument is again based on using (7); a key difficulty here is to establish the arbitrarily large lower bound for the variances. The latter is Proposition 3.2, whose proof (that turned out to be surprisingly technical) is provided in Section 6. Finally, the key step in the proof of Theorem 1.4 is a bootstrapping argument, provided in Section 4. Namely, the sum of two independent random variables (for instance, ξ_n and $\xi_{(n,n+n']}$) is closer to the Gaussian behavior than the summands separately. We introduce the quantitative way (27) of measuring how close the distribution is to the Gaussian, and establish the corresponding inequality in Section 4.3. Then, we control how an additional perturbation, coming from the $R_{n,n'}$ term, can worsen the bounds. This is done in Section 4.4.

We conclude by joining the bootstrapping estimates with the bounds established for ξ_n and $R_{n,n'}$, and complete the proof of Theorem 1.4 in Section 5.

2. Moment estimates for $R_{n,n'}$

In this section we provide the estimates on the moments of discrepancies $R_{n,n'}$ defined by (6).

2.1. Statements. Here is the main statement of this section:

Proposition 2.1. Under the assumptions of Theorem 1.4, there exists C_R , such that for every $n, n' \in \mathbb{N}$, such that $n \leq 2n'$ one has

$$\mathbb{E} R_{n,n'} < C_R, \quad \mathbb{E} R_{n,n'}^2 < C_R, \quad and \quad \mathbb{E} R_{n,n'}^3 < C_R.$$

Actually, we will show that the tails of distributions of random variables $R_{n,n'}$ are uniformly bounded up to a linearly growing threshold. Namely, we have the following lemma:

Lemma 2.2. There exist $c, c_{\kappa} > 0$ such that

(8)
$$\forall n, n' \quad \forall x \le c_{\kappa} n' \quad \mathbb{P}\left(R_{n,n'} > x\right) \le c x^{-\gamma/2}.$$

In the rest of this section we prove Proposition 2.1 and Lemma 2.2.

2.2. **Proof of Proposition 2.1.** Let us first deduce Proposition 2.1 from Lemma 2.2:

Proof of Proposition 2.1. First of all, notice that it is enough to prove the estimate for $\mathbb{E} R^3_{n,n'}$, as it implies the other two by using Hölder's inequality.

In order to estimate $\mathbb{E} R^3_{n,n'}$ we split it in two parts:

$$\mathbb{E} R_{n,n'}^3 = \mathbb{E} \left(R_{n,n'}^3 \cdot \mathbf{1}_{R_{n,n'} \leq c_{\kappa} n'} \right) + \mathbb{E} \left(R_{n,n'}^3 \cdot \mathbf{1}_{R_{n,n'} > c_{\kappa} n'} \right).$$

The first summand can be estimated using (8):

$$\mathbb{E}\left(R_{n,n'}^3 \cdot \mathbf{1}_{R_{n,n'} \le c_\kappa n'}\right) \le \int_0^{c_\kappa n'} \mathbb{P}\left(R_{n,n'} \ge x\right) \cdot 3x^2 \,\mathrm{d}x \le 1 + 3c \int_1^\infty x^{2-\frac{\gamma}{2}} \,\mathrm{d}x = \mathrm{const}$$

To estimate the second one, we notice that $R_{n,n'}$ is bounded from above by the sum of n + n' independent variables $\log ||A_i||$, i = 1, ..., n + n', with 9-th moment not exceeding M by (3). We apply the Hölder's inequality with exponents 3 and 3/2 and inequality (8):

$$\mathbb{E}\left(R_{n,n'}^3 \cdot \mathbf{1}_{R_{n,n'} > c_{\kappa}n'}\right) \leq \left(\mathbb{E}\,R_{n,n'}^9\right)^{\frac{1}{3}} \cdot \mathbb{P}\left(R_{n,n'} \ge c_{\kappa}n'\right)^{\frac{2}{3}} \leq M^{\frac{1}{3}}(n+n')^3 \cdot c^{\frac{2}{3}}(c_{\kappa}n')^{-\frac{\gamma}{3}},$$

and the right hand side again is bounded uniformly for $n \leq 2n'$.

2.3. Establishing tail estimates: proof of Lemma 2.2. We start by recalling some properties of $SL(2, \mathbb{R})$ matrices. Let $v \in \mathbb{R}^2$ be a vector and $B \in SL(2, \mathbb{R})$ be a matrix. Define a function

$$\Theta(B, v) = \log(\|B\| \cdot |v|) - \log(|Bv|) = \log \frac{\|B\| \cdot |v|}{|Bv|}$$

that compares the expansion by B of the vector v with the maximum possible.

Denote by f_B the projectivization of the matrix B, namely

 $f_B : \mathbb{RP}^1 \to \mathbb{RP}^1$, such that $f_B \circ [\cdot] = [\cdot] \circ B$,

where $[\cdot]$ is the canonical projection $[\cdot] : \mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^1$. Consider the projection [v], and denote by $U_r([v])$ its radius r neighbourhood in \mathbb{RP}^1 . Finally, consider the projections $[e_1], [e_2]$ of the basis vectors $e_1, e_2 \in \mathbb{R}^2$. The following lemma allows to control the "loss of expansion" $\Theta(B, v)$ in terms of closeness of the direction of v to preimages of given directions:

Lemma 2.3. For x > 0 take $r = 2e^{-x/2}$. If

$$f_{B^{-1}}([e_1]) \notin U_r([v])$$
 and $f_{B^{-1}}([e_2]) \notin U_r([v])$

then

(9)
$$\Theta(B,v) \le x$$

Proof. First of all, notice that

$$\begin{split} \Theta(B,v) &= \log(\|B\| \cdot |v|) - \log(|Bv|) \leq \\ &\leq \log(\|B\|) + \log(|v|) - \log(\|B^{-1}\|) - \log(|v|) \leq 2\log(\|B\|). \end{split}$$

Hence, if $||B|| \leq e^{x/2}$ then inequality (9) holds automatically. From now on we assume that $||B|| > e^{x/2}$.

Recall that every matrix $B \in SL(2, \mathbb{R})$ can be written as a product

$$B = \operatorname{Rot}_{\beta_1} \begin{pmatrix} \|B\| & 0\\ 0 & \|B\|^{-1} \end{pmatrix} \operatorname{Rot}_{\beta_2},$$

where $\operatorname{Rot}_{\beta}$ is a rotation by the angle β . Moreover, without loss of generality we can assume that $\beta_2 = 0$ (the statement of Lemma 2.3 is invariant under a precomposition of *B* with a rotation).

Note that at least one of the points $f_{B^{-1}}([e_1]), f_{B^{-1}}([e_2])$ is within e^{-x} of the direction most contracted by B. Indeed,

$$B^{-1} = \begin{pmatrix} \|B\|^{-1} & 0\\ 0 & \|B\| \end{pmatrix} \operatorname{Rot}_{\beta_1}^{-1};$$

the vectors e_1 , e_2 are orthogonal, thus there exists i = 1, 2 such that for rotation preimage $\operatorname{Rot}_{\beta_1}^{-1} e_i$ its second coordinate is no smaller than the first one. Hence, for the vector $B^{-1}e_i$ its second coordinate exceeds the first one at least by the factor $\|B\|^2 \ge e^x$, and hence $B^{-1}e_i$ is at the distance at most $e^{-x/2} < e^{-x}$ of the second coordinate axis, most contracted by B (see Fig. 1).

Now, as the direction of the vector v is not within $r = 2e^{-x/2}$ of the one of $B^{-1}e_i$, the angle between v and the second coordinate axis is at least $\frac{r}{2}$, and hence the absolute value of the first coordinate of the unit vector v is at least $\sin\left(\frac{r}{2}\right) \geq \frac{2}{\pi} \cdot \frac{r}{2}$. Applying B, we hence get a vector with length at least $||B|| \cdot \frac{r}{\pi}$. Thus,

$$\Theta(B, v) \le \log \|B\| - \log \frac{r\|B\|}{\pi} = \log \frac{\pi}{r} = \frac{x}{2} \log \frac{\pi}{2} < x.$$



FIGURE 1. Vector v, the preimage $B^{-1}(e_i)$ and its r-neighbourhood.

In what follows we will use the following statement about regularity of measures produced by actions of products of random matrices on the projective space. This is a particular case of [M1, Theorem 2.9].

Theorem 2.4 ([M1]). Let $T_{(n,m]} = A_m A_{m-1} \dots A_{n+1}$ be a product of random $SL(2,\mathbb{R})$ matrices whose distributions satisfy the measures condition. Then there exists a positive constant C and $0 < \kappa < 1$, such that for every pair of points $p_1, p_2 \in \mathbb{RP}^1$ we have the following:

$$\forall n < m \ \forall r > \kappa^{m-n} \quad \mathbb{P}\left(f_{T_{(n,m]}}(p_1) \in U_r(p_2)\right) < C |\log(r)|^{-\frac{\gamma}{2}}$$

Combining Lemma 2.3 and Theorem 2.4, we are now ready to prove Lemma 2.2.

Proof of Lemma 2.2. Recall that

$$R_{n,n'} = \log(||T_n||) + \log(||T_{(n,n+n']}||) - \log(||T_{n+n'}||).$$

There exists a unit vector $u \in \mathbb{R}^2$ (that depends on T_n), such that

$$||T_n|| = |T_n u|.$$

Set $v = T_n u$ and $B = T_{(n,n+n']}$. Note, that the discrepancy $R_{n,n'}$ is bounded from above by $\Theta(B, v)$:

$$\begin{aligned} R_{n,n'} &= \log(|T_n u|) + \log(||T_{(n,n+n']}||) - \log(||T_{n+n'}||) \leq \\ &\leq \log(||T_{(n,n+n']}|| \cdot |T_n u|) - \log(||T_{n+n'} u||) = \Theta(T_{(n,n+n']}, T_n u) = \Theta(B, v). \end{aligned}$$

Applying Lemma 2.3 we obtain that for every x > 0

$$\mathbb{P}(R_{n,n'} > x) \le \mathbb{P}(\Theta(B, v) > x) \le \\ \le \mathbb{P}(f_{B^{-1}}([e_1]) \in U_r([v])) + \mathbb{P}(f_{B^{-1}}([e_2]) \in U_r([v])),$$

where $r = 2e^{-x/2}$. The random product

$$T_{(n,n+n']}^{-1} = A_{n+1}^{-1} A_{n+2}^{-1} \dots A_{n+n'}^{-1}$$

is formed by the matrices whose distributions satisfy the measures condition. According to Theorem 2.4 there exist constants $C, \kappa < 1$, such that for every $[v] \in \mathbb{RP}^1$ and every $r > \kappa^{n'}$ we have

(10)
$$\mathbb{P}\left(f_{B^{-1}}([e_1]) \in U_r([v])\right) + \mathbb{P}\left(f_{B^{-1}}([e_2]) \in U_r([v])\right) < 2C |\log(r)|^{-\frac{1}{2}}.$$

Substituting $r = 2e^{-x/2}$ and taking $c_{\kappa} = -\log(\kappa)$, we obtain the desired estimate (8).

3. Moments growth for ξ_n

In this section we establish upper bounds on the moments of $|\xi_n - \mathbb{E}\xi_n|$, and also establish linear growth of the variance of ξ_n .

3.1. Moments upper bounds for ξ_n . Here we prove the following statement:

Proposition 3.1. Under the assumptions of Theorem 1.4 there exists a constant $C_{\xi} < \infty$, such that for any $n \in \mathbb{N}$ and any $\mu_1, \ldots, \mu_n \in \mathcal{K}$ the following holds:

(11)
$$\mathbb{E}\left|\xi_n - \mathbb{E}\xi_n\right| < C_{\xi}\sqrt{n},$$

(12)
$$\mathbb{E} |\xi_n - \mathbb{E} \xi_n|^2 < C_{\xi} n,$$

and

(13)
$$\mathbb{E} |\xi_n - \mathbb{E} \xi_n|^3 < C_{\xi} n^{\frac{3}{2}}.$$

Proof of Proposition 3.1. First, let us establish the upper bound (12) for the variances $\operatorname{Var} \xi_n$. We will use the decomposition

(14)
$$\xi_{n+n'} = \left(\xi_n + \xi_{(n,n+n']}\right) - R_{n,n'};$$

the triangle inequality for the L_2 -norm (applied to the centred variables) then implies

$$\sqrt{\operatorname{Var}\xi_{n+n'}} \le \sqrt{\operatorname{Var}(\xi_n + \xi_{(n,n+n']})} + \sqrt{\operatorname{Var}R_{n,n'}}.$$

Due to Proposition 2.1 and due to the independence of ξ_n and $\xi_{(n,n+n']}$, we have

(15)
$$\sqrt{\operatorname{Var}\xi_{n+n'}} \le \sqrt{\operatorname{Var}\xi_n + \operatorname{Var}\xi_{(n,n+n']}} + \sqrt{C_R}$$

We will now recurrently construct a sequence c_n , such that for any $n \in \mathbb{N}$ and any $\mu_1, \ldots, \mu_n \in \mathcal{K}$,

(16)
$$\operatorname{Var}\xi_n \le c_n \cdot n.$$

The existence of c_1 is guaranteed by the uniform moments condition (3). Now, to construct c_m with m > 1, take $n = \lfloor \frac{m}{2} \rfloor$ and n' = m - n. Next, let us divide (15) by \sqrt{m} : we get

$$\sqrt{\frac{\operatorname{Var}\xi_m}{m}} \le \sqrt{\frac{\operatorname{Var}\xi_n}{n} \cdot \frac{n}{m} + \frac{\operatorname{Var}\xi_{(n,n+n']}}{n'} \cdot \frac{n'}{m}} + \frac{\sqrt{C_R}}{\sqrt{m}} \le \sqrt{\max(c_n, c_n')} + \frac{\sqrt{C_R}}{\sqrt{m}}.$$

Hence, it suffices to take c_m to be defined by the relation

(17)
$$\sqrt{c_m} = \sqrt{\max(c_n, c'_n)} + \frac{\sqrt{C_R}}{\sqrt{m}}$$

For such sequence, it is easy to check by induction that for all $m = 2^k + 1, \ldots, 2^{k+1}$ we have

$$\sqrt{c_m} \le \sqrt{c_1} + \sum_{j=0}^k \frac{\sqrt{C_R}}{\sqrt{2^j}};$$

which in turn implies a uniform bound

$$\sqrt{c_m} \le \sqrt{c_1} + \frac{\sqrt{C_R}}{1 - \frac{1}{\sqrt{2}}},$$

thus concluding the proof of (12).

This also implies (11): indeed, due to the Hölder inequality,

$$\mathbb{E}\left|\xi_n - \mathbb{E}\,\xi_n\right| \le \sqrt{\operatorname{Var}\,\xi_n}.$$

Finally, let us prove (13). Consider the centred random variables

$$\widetilde{\xi}_n = \xi_n - \mathbb{E}\xi_n, \quad \widetilde{\xi}_{(n,n+n']} = \xi_{(n,n+n']} - \mathbb{E}\xi_{(n,n+n']}, \quad \widetilde{R}_{n,n'} = R_{n,n'} - \mathbb{E}R_{n,n'}.$$

Again applying the decomposition (14), and using the L_3 -triangle inequality, we get

(18)
$$\sqrt[3]{\mathbb{E}\left|\widetilde{\xi}_{n+n'}\right|^3} \le \sqrt[3]{\mathbb{E}\left|\widetilde{\xi}_n + \widetilde{\xi}_{(n,n+n')}\right|^3} + \sqrt[3]{\mathbb{E}\left|\widetilde{R}_{n,n'}\right|^3}$$

The second summand in the right hand side does not exceed $\sqrt[3]{C_R}$ due to Proposition 2.1. To estimate the first one, note that for any

$$\forall a, b \in \mathbb{R} \quad |a+b|^3 \le |a|^3 + |b|^3 + 3(|a|^2 \cdot |b| + |a| \cdot |b|^2);$$

thus,

$$\mathbb{E}\left|\widetilde{\xi}_{n}+\widetilde{\xi}_{(n,n+n')}\right|^{3} \leq E\left|\widetilde{\xi}_{n}\right|^{3}+\mathbb{E}\left|\widetilde{\xi}_{(n,n+n')}\right|^{3}+3C_{\xi}^{2}(n(n')^{1/2}+n^{1/2}n').$$

Taking a cubic root and using inequality $\sqrt[3]{a+b} \leq \sqrt[3]{a} + \sqrt[3]{b}$, we get

(19)
$$\sqrt[3]{\mathbb{E}} \left| \widetilde{\xi}_n + \widetilde{\xi}_{(n,n+n')} \right|^3 \leq \sqrt[3]{\mathbb{E}} \left| \widetilde{\xi}_n \right|^3 + \mathbb{E} \left| \widetilde{\xi}_{(n,n+n')} \right|^3 + \sqrt[3]{6C_{\xi}^2 \cdot m^{3/2}}.$$

Again, for an arbitrary m > 1 set the indices $n = \lfloor \frac{m}{2} \rfloor$ and n' = m - n. Substituting (19) into (18) and dividing by \sqrt{m} , we get (20)

$$\sqrt[3]{\frac{\mathbb{E}\,|\tilde{\xi}_{m}|^{3}}{m^{3/2}}} \leq \sqrt[3]{\frac{\mathbb{E}\,|\tilde{\xi}_{n}|^{3}}{n^{3/2}} \cdot \left(\frac{n}{m}\right)^{3/2}} + \frac{\mathbb{E}\,\left|\tilde{\xi}_{(n,n+n')}\right|^{3}}{(n')^{3/2}} \cdot \left(\frac{n'}{m}\right)^{3/2} + \left(6C_{\xi} + \frac{\sqrt[3]{C_{R}}}{\sqrt{m}}\right)$$

As previously, we are going to find a sequence c_n such that for any $n \in \mathbb{N}$ and any $\mu_1, \ldots, \mu_n \in \mathcal{K}$,

(21)
$$\mathbb{E} \,|\widetilde{\xi}_n|^3 \le c_n \cdot n^{3/2}.$$

Substituting (21) for n, n' into the right hand side of (20), we bound it from above by ______

$$\sqrt[3]{\left(\frac{n}{m}\right)^{3/2} + \left(\frac{n'}{m}\right)^{3/2}} \cdot \max\left(\sqrt[3]{c_n}, \sqrt[3]{c_{n'}}\right) + \left(6C_{\xi} + \frac{\sqrt[3]{C_R}}{\sqrt{m}}\right)$$

Now, the factor before max $(\sqrt[3]{c_n}, \sqrt[3]{c_{n'}})$ is bounded away from 1: it doesn't exceed

$$\sqrt[3]{\left(\frac{n}{m}\right)^{3/2} + \left(\frac{n'}{m}\right)^{3/2}} \le \sqrt[3]{\max\left(\left(\frac{n}{m}\right)^{1/2}, \left(\frac{n'}{m}\right)^{1/2}\right)} \le \left(\frac{2}{3}\right)^{1/6} < 1.$$

Hence, it suffices to take all $c_n = \tilde{C}^3$, where the constant \tilde{C} is chosen sufficiently large, so that $\mathbb{E} |\tilde{\xi}_1^3| \leq \tilde{C}^3$ and that

$$\widetilde{C} \le \left(\frac{2}{3}\right)^{1/6} \cdot \widetilde{C} + (6C_{\xi} + \sqrt[3]{C_R}).$$

Indeed, inequality (20) then becomes an inductive proof of (21).

We have obtained the desired upper bound $\mathbb{E} |\tilde{\xi}_n|^3 \leq \tilde{C}^3 \cdot n^{3/2}$, thus concluding the proof of the proposition.

3.2. Linear growth of variances. In this section we will prove inequality (4), i.e. we will show that there are constants $C_1, C_2 > 0$ and an index n_0 such that for all $n \ge n_0$ and all $\mu_1, \ldots, \mu_n \in \mathcal{K}$ one has

$$C_1 n \le \operatorname{Var}(\log \|T_n\|) \le C_2 n.$$

We start by observing that estimate (12) guarantees that for every $n \in \mathbb{N}$

$$\operatorname{Var}(\xi_n) < C_{\xi} n_{\xi}$$

so we only need to establish the lower bound. The proof of the lower bound can be split into two statements:

Proposition 3.2. Under the assumptions of Theorem 1.4, for any c > 0 there exists $n_1 \in \mathbb{N}$ such that for any $n \ge n_1$ and any collection of distributions $\mu_1, \ldots, \mu_n \in \mathcal{K}$ one has

$$\operatorname{Var}_{\mu_1,\ldots,\mu_n} \xi_n \ge c$$

We will provide the proof of Proposition 3.2 in Section 6.

Lemma 3.3. Under the assumptions of Theorem 1.4, assume that $\operatorname{Var} \xi_n$ becomes arbitrarily large:

(22)
$$\forall c \ \exists n_1 : \ \forall n \ge n_1 \ \forall \mu_1, \dots, \mu_n \in \mathcal{K} \ \operatorname{Var}_{\mu_1, \dots, \mu_n} \xi_n \ge c$$

Then there exists $C_1 > 0$ and n_0 such that $\operatorname{Var} \xi_n \geq C_1 n$ for all $n \geq n_0$.

Proof. Recall that

$$\xi_{n+n'} = \xi_n + \xi_{(n,n+n']} - R_{n,n'}.$$

Cauchy-Schwarz inequality implies that

(23)
$$\sqrt{\operatorname{Var}\xi_{n+n'}} \ge \sqrt{\operatorname{Var}(\xi_n + \xi_{(n,n+n']})} - \sqrt{\operatorname{Var}R_{n,n'}}$$

Take n_1 such that (22) holds with $c = 16C_R$, and let $\varepsilon := \frac{C_R}{2n_1}$, where C_R is defined in Proposition 2.1. Then, for every $n = n_1, \ldots, 2n_1 - 1$ and any $\mu_1, \ldots, \mu_n \in \mathcal{K}$ one has

(24)
$$\sqrt{\operatorname{Var}\xi_n} \ge \sqrt{\varepsilon(n+1)} + 3\sqrt{C_R}$$

We claim that in that case the estimate (24) holds for every $n \ge n_1$. In order to show that we will proceed by induction. Indeed, let $n \ge 2n_1$ be the first number for which (24) is not yet established; decompose it as $n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$. Then, each of the variances $\operatorname{Var} \xi_{\lfloor \frac{n}{2} \rfloor}$, $\operatorname{Var} \xi_{\lceil \frac{n}{2} \rceil}$ in (23) is bounded from below by $\sqrt{\varepsilon(\lfloor n/2 \rfloor + 1)} + 3\sqrt{C_R}$, and hence

$$\begin{split} \sqrt{\operatorname{Var} \xi_{\lfloor \frac{n}{2} \rfloor} + \operatorname{Var} \xi_{\left\lfloor \lfloor \frac{n}{2} \rfloor, n \right]}} &- \sqrt{\operatorname{Var} R_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}} \geq \\ &\geq \sqrt{2} \cdot \left(\sqrt{\varepsilon(\lfloor n/2 \rfloor + 1)} + 3\sqrt{C_R} \right) - \sqrt{C_R} \geq \\ &\geq \sqrt{\varepsilon(n+1)} + (3\sqrt{2} - 1)\sqrt{C_R}, \end{split}$$

where we have used $2(\lfloor n/2 \rfloor + 1) \ge n+1$. As $3\sqrt{2} - 1 > 3$, this proves the induction step. In particular, for every $n \ge n_1$ we have $\operatorname{Var} \xi_n \ge \varepsilon n$.

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Proposition 3.2 and Lemma 3.3 together prove (4).

Corollary 3.4. There exists C_3 such that

• for all $n \ge n_0$ and any $\mu_1, \ldots, \mu_n \in \mathcal{K}$, the normalized variable $\eta_n = \frac{\xi_n - \mathbb{E} \xi_n}{\sqrt{\operatorname{Var} \xi_n}}$ satisfies

(25)
$$\mathbb{E} |\eta_n|^3 < C_3.$$

• For all $n, n' \ge n_0$ and any $\mu_1, \ldots, \mu_{n+n'} \in \mathcal{K}$, the variance of the normalized sum

$$\eta_{n,n'} = \frac{\theta_{n,n'} - \mathbb{E}\,\theta_{n,n'}}{\sqrt{\operatorname{Var}\,\theta_{n,n'}}}, \quad where \quad \theta_{n,n'} := \xi_n + \xi_{(n,n+n']},$$

satisfies

(26)
$$\mathbb{E} |\eta_{n,n'}|^3 < C_3.$$

4. BOOTSTRAPPING: DISTANCE TO THE GAUSSIAN DISTRIBUTION

This section is devoted to the bootstrapping arguments that allow to show the convergence to Gaussian law.

4.1. **Preliminaries.** Let ξ , ξ' be two independent random variables with finite third moment and comparable variances: for a given constant C > 1, we have

$$C^{-1} < \frac{\operatorname{Var} \xi}{\operatorname{Var} \xi'} < C.$$

We will provide a value $N'_{\rho}(\xi)$, measuring quantitatively non-Gaussianity of the law of ξ , such that (under appropriate assumptions) it will be smaller for the sum $\xi + \xi'$ than for the summands separately.

Definition 4.1. For a random variable ξ we denote by $\varphi_{\xi}(t)$ its *characteristic function*:

$$\varphi_{\xi}(t) = \mathbb{E}e^{it\xi}.$$

Now, let

(27)
$$N_{\rho}(\eta) = \sup_{0 < |t| < \rho} \frac{\left| \log \left(\varphi_{\eta}(t) e^{t^2/2} \right) \right|}{|t|^3}, \quad N_{\rho}'(\xi) = N_{\rho} \left(\frac{\xi - \mathbb{E}\xi}{\sqrt{\operatorname{Var}\xi}} \right).$$

Then $\eta \sim \mathcal{N}(0, 1)$ if and only if $N_{\rho}(\eta) = 0$ for all $\rho > 0$ (as the distribution of a random variable is uniquely determined by its characteristic function). Note also that $N_{\rho}(\eta)$ might be infinite if the corresponding characteristic function φ_{η} vanishes somewhere on $[-\rho, \rho]$. Finally, the logarithm here is a function of a complex variable (as the characteristic function might be, and most often is, non-real). As soon as $\varphi_{\eta}(t)$ doesn't vanish on $[-\rho, \rho]$, we define the composition $\log \left(\varphi_{\eta}(t)e^{t^2/2}\right)$ by a continuous extension, starting with the value $\log 1 = 0$ at t = 0. 4.2. Initial estimates. To start a bootstrapping argument, one needs some initial bounds, in this case, for the norms $N'_{\rho}(\xi)$ for some $\rho > 0$.

Lemma 4.2. Let X be a random variable with

$$\mathbb{E} X = 0$$
, $\operatorname{Var} X = 1$, $\mathbb{E} |X|^3 < C_X$.

Then its characteristic function satisfies

(28)
$$\left|\varphi_X(t) - \left(1 - \frac{t^2}{2}\right)\right| \le C_X \cdot |t|^3 \text{ for all } t \in \mathbb{R}.$$

Proof. Note that it suffices to establish the estimate for the second derivative $\varphi''(t)$: for all $t \in \mathbb{R}$,

(29)
$$|\varphi_X''(t) + 1| \le C_X |t|.$$

Indeed, integrating (29) two times then suffices to obtain (28):

$$\varphi_X(t) - \left(1 - \frac{t^2}{2}\right) = \int_0^t dt_1 \int_0^{t_1} (\varphi_X''(t_2) + 1) dt_2.$$

Now, let us rewrite the estimated expression in (29):

(30)
$$|\varphi_X''(t) + 1| = |\mathbb{E} (X^2 (e^{itX} - 1))| \le \mathbb{E} (X^2 |e^{itX} - 1|) \le \mathbb{E} (X^2 \cdot |tX|);$$

here we have used $\operatorname{Var} X = 1$ for the first equality and the fact that e^{ix} is a 1-Lipschitz function for the last inequality. Finally, the right hand side of (30) can be rewritten as

$$|t| \cdot \mathbb{E} |X|^3 < C_X |t|,$$

thus completing the proof.

Joining this with the estimate from Corollary 3.4, we get the initial bound lemma.

Lemma 4.3. There exists $\rho_0 > 0$, such that for any $n \ge n_0$ the value $N_{\rho_0}(\eta_n)$ is well-defined and satisfied

$$N_{\rho_0}(\eta_m) < 3C_3 \quad and \quad \rho_0^3 N_{\rho_0}(\eta_m) < \frac{1}{100},$$

where C_3 is the constant defined in Corollary 3.4.

Proof. Applying Lemma 4.2, and multiplying its conclusion by $e^{\frac{t^2}{2}}$, we get

(31)
$$\left| e^{\frac{t^2}{2}} \varphi_{\eta_n}(t) - e^{\frac{t^2}{2}} \left(1 - \frac{t^2}{2} \right) \right| \le C_3 |t|^3 \cdot e^{\frac{t^2}{2}}$$

Now,

$$e^{\frac{t^2}{2}}\left(1-\frac{t^2}{2}\right) = 1 + o(|t|^3),$$

thus for sufficiently small ρ one has

(32)
$$\forall |t| \le \rho \qquad \left| e^{\frac{t^2}{2}} \left(1 - \frac{t^2}{2} \right) - 1 \right| \le \frac{1}{2} C_3 |t|^3$$

as well as

$$\forall |t| \le \rho \qquad e^{\frac{t^2}{2}} \le e^{\frac{\rho^2}{2}} \le \frac{3}{2},$$

hence the right hand side of (31) can be replaced by $\frac{3}{2}C_3|t|^3$. Joining this with (32), we get

$$\forall |t| \le \rho \qquad \left| e^{\frac{t^2}{2}} \varphi_{\eta_n}(t) - 1 \right| \le \left(\frac{1}{2} + \frac{3}{2}\right) \cdot C_3 |t|^3 = 2C_3 |t|^3.$$

Taking

$$\rho_0 := \min\left(\rho, \sqrt[3]{\frac{1}{300C_3}}\right),\,$$

we ensure that $2C_3\rho_0^3 < \frac{1}{100}$, and hence that log is $\frac{3}{2}$ -Lipschitz in the (complex) disc $U_{2C_3\rho_0^3}(1)$. Therefore,

$$\forall |t| \le \rho_0 \quad \left| \log e^{\frac{t^2}{2}} \varphi_{\eta_n}(t) \right| \le \frac{3}{2} \cdot 2C_3 |t|^3,$$

which implies the desired

$$N_{\rho_0}(\eta_n) \le 3C_3.$$

4.3. Sum of two independent variables. The following is the first step of the bootstrapping argument, estimating the decrease of N'-values for the sum of two independent random variables. Notice that in addition to the decrease by a linear factor, the parameter ρ (describing the size of the domain) gets increased.

Lemma 4.4. For any C there exists $\lambda < 1$ and L > 1 such that if for some $\rho > 0$ for some independent random variables ξ, ξ' one has

$$C^{-1} < \frac{\operatorname{Var} \xi}{\operatorname{Var} \xi'} < C$$

and values $N'_{\rho}(\xi), N'_{\rho}(\xi')$ are finite, then

$$N_{L\rho}'(\xi+\xi') \le \lambda \cdot \max(N_{\rho}'(\xi), N_{\rho}'(\xi')).$$

Proof. Let

$$\eta = \frac{\xi - \mathbb{E}\,\xi}{\sqrt{\mathrm{Var}\,\xi}}, \quad \eta' = \frac{\xi' - \mathbb{E}\,\xi'}{\sqrt{\mathrm{Var}\,\xi'}}, \quad \eta'' = \frac{(\xi + \xi') - (\mathbb{E}\,(\xi + \xi'))}{\sqrt{\mathrm{Var}(\xi + \xi')}},$$

Also, denote

$$c = \sqrt{\frac{\operatorname{Var}\xi}{\operatorname{Var}\xi + \operatorname{Var}\xi'}}, \quad c' = \sqrt{\frac{\operatorname{Var}\xi'}{\operatorname{Var}\xi + \operatorname{Var}\xi'}};$$

then, one has

$$\eta'' = c\eta + c'\eta',$$

with the coefficients that satisfy

$$c^{2} + (c')^{2} = 1, \quad c, c' \le \sqrt{\frac{C}{C+1}} < 1.$$

By definition, we have for any L

$$N'_{L\rho}(\xi + \xi') = N_{L\rho}(\eta'') = N_{L\rho}(c\eta + c'\eta').$$

Now, for the characteristic functions we have

$$\varphi_{c\eta+c'\eta'}(t) = \varphi_{\eta}(ct) \cdot \varphi_{\eta'}(c't),$$

and as $c^{2} + (c')^{2} = 1$, we have (33) $e^{t^{2}}\varphi_{c\eta+c'\eta'}(t) = e^{(ct)^{2}}\varphi_{\eta}(ct) \cdot e^{(c't)^{2}}\varphi_{\eta'}(c't).$ If $cL, c'L \leq 1$, taking the logarithm of (33) and dividing by $|t|^3$, we get

$$N_{L\rho}(\eta'') = \sup_{0 < |t| < L\rho} \frac{\left| \log \left(e^{t^2} \varphi_{\eta''}(t) \right) \right|}{|t|^3} \le \\ \le \sup_{0 < |t| < L\rho} \frac{\left| \log \left(e^{(ct)^2} \varphi_{\eta}(ct) \right) \right|}{|t|^3} + \sup_{0 < |t| < L\rho} \frac{\left| \log \left(e^{(c't)^2} \varphi_{\eta'}(c't) \right) \right|}{|t|^3}$$

Making the ct and c't variable change in the first and second expressions respectively in the right hand side, we obtain

$$N_{L\rho}(\eta'') \le c^3 N_{cL\rho}(\eta) + (c')^3 N_{c'L\rho}(\eta') \le \le (c^3 + (c')^3) \cdot \max(N_{\rho}(\eta), N_{\rho}(\eta')) \le \le \max(c, c') \cdot \max(N_{\rho}(\eta), N_{\rho}(\eta')).$$

Taking $L = \sqrt{\frac{C+1}{C}}$ and $\lambda = \frac{1}{L}$ concludes the proof.

4.4. Correction by an additional term. The expression (7) for $\xi_{n+n'}$, besides the sum of two independent random variables

$$\theta_{n,n'} = \xi_n + \xi_{(n,n+n']},$$

has an additional term

$$R_{n,n'} = \theta_{n,n'} - \xi_{n+n'}.$$

Due to this, an extra (and possibly non-independent) term is added to the normalized random variable: for

(34)
$$X = \frac{\theta_{n,n'} - \mathbb{E}\,\theta_{n,n'}}{\sqrt{\operatorname{Var}\,\theta_{n,n'}}}, \quad Y = \frac{\xi_{n+n'} - \mathbb{E}\,\xi_{n+n'}}{\sqrt{\operatorname{Var}\,\xi_{n+n'}}}$$

this term is the difference

$$r = r_{n,n'} = Y - X$$

We are going to analyze and control its influence. First, note that $\mathbb{E} r = 0$, and $\mathbb{E} |r|^3$ satisfies an upper bound:

Lemma 4.5. There exists $Q_r < \infty$, such that for every $n, n' \ge n_0$, satisfying

$$\frac{n}{2} \le n' \le 2n,$$

and any $\mu_1, \ldots, \mu_{n+n'} \in \mathcal{K}$, we have

(35)
$$\mathbb{E} |r_{n,n'}|^3 < Q_r (n+n')^{-\frac{3}{2}}.$$

Proof. Note that $r_{n,n'}$ can be expressed as

(36)
$$r = r_{n,n'} = \frac{\sqrt{\operatorname{Var}\theta_{n,n'}} - \sqrt{\operatorname{Var}\xi_{n+n'}}}{\sqrt{\operatorname{Var}\xi_{n+n'}}} \cdot X - \frac{1}{\sqrt{\operatorname{Var}\xi_{n+n'}}} (R_{n,n'} - \mathbb{E}R_{n,n'}).$$

Now, the L_3 -norms of both X and $R_{n,n'}$ are uniformly bounded (see Proposition 3.1 and Corollary 3.4). On the other hand, from the Cauchy-Schwartz inequality one has

$$\left|\sqrt{\operatorname{Var}\theta_{n,n'}} - \sqrt{\operatorname{Var}\xi_{n+n'}}\right| \le \sqrt{\operatorname{Var}R_{n,n'}},$$

and hence both coefficients admit upper bounds as $\frac{\text{const}}{\sqrt{n+n'}}$.

Our next step is to control the influence of such a "small" change by r on the non-Gaussianity value N_{ρ} :

Lemma 4.6. Let X, Y be two random variables with

$$\mathbb{E} X = \mathbb{E} Y = 0$$
, $\operatorname{Var} X = \operatorname{Var} Y = 1$, $\mathbb{E} X^3$, $\mathbb{E} Y^3 < C_X$.

Assume that for r = Y - X one has $\mathbb{E} |r|^3 < C_r$. Then for any $t \in \mathbb{R}$ one has

(37)
$$|\varphi_X(t) - \varphi_Y(t)| \le C_r^{\frac{1}{3}} C_X^{\frac{4}{3}} \cdot |t|^3$$

Proof. Let us take the second derivative of the difference of the characteristic functions:

(38)
$$(\varphi_X - \varphi_Y)''(t) = -\mathbb{E} \left(X^2 e^{itX} - Y^2 e^{itY} \right)$$

= $-\mathbb{E} \left(X^2 (e^{itX} - 1) - Y^2 (e^{itY} - 1) \right),$

where the second equality follows from $\operatorname{Var} X = \operatorname{Var} Y$. Now, note that it suffices to obtain an estimate

(39)
$$|(\varphi_X - \varphi_Y)''(t)| \le 3C_r^{\frac{1}{3}}C_X^{\frac{1}{3}} \cdot |t|,$$

as again we can integrate two times:

$$\varphi_X(t) - \varphi_Y(t) = \int_0^t dt_1 \int_0^{t_1} (\varphi_X - \varphi_Y)''(t_2) dt_2,$$

and integrating (39) two times, we get the desired (37).

In order to obtain the estimate (39), let us decompose the right hand side of (38):

$$|\varphi_X''(t) - \varphi_Y''(t)| \le \mathbb{E} |(X^2 - Y^2)(e^{itX} - 1)| + \mathbb{E} (Y^2 | e^{itY} - e^{itX} |).$$

To estimate the first summand, we note that $Y^2 - X^2 = r(X + Y)$, hence it does not exceed

$$\mathbb{E}|(X^2 - Y^2)(e^{itX} - 1)| \le \mathbb{E}(|r| \cdot (|X| + |Y|) \cdot |tX|).$$

The expectation of the products $|r| \cdot |X|^2$ and $|r| \cdot |X||Y|$ can be estimated using the Hölder inequality: each does not exceed $C_r^{1/3}C_X^{2/3}$. We thus get

$$\mathbb{E}|(X^2 - Y^2)(e^{itX} - 1)| \le 2C_r^{1/3}C_X^{2/3} \cdot |t|.$$

In the same way,

$$\mathbb{E}\left(Y^2|e^{itY} - e^{itX}|\right) \le \mathbb{E}\left(Y^2|t(X - Y)|\right) = |t| \cdot \mathbb{E}\left(|r| \cdot Y^2\right),$$

and the right hand side does not exceed $C_r^{1/3} C_X^{2/3} |t|$. Adding these estimates together, we obtain the desired (39).

Proposition 4.7. In the assumptions of Lemma 4.6, let $K_r := C_r^{1/3} C_X^{2/3}$ be the factor that appears in its conclusion. Assume additionally that for some $\rho > 0$ one has

(40)
$$\rho^3 N_{\rho}(X) \le \frac{1}{100}$$

and

(41)
$$K_r \rho^3 e^{\frac{\rho^2}{2}} \le \frac{1}{100}.$$

Then

$$N_{\rho}(Y) \le N_{\rho}(X) + 2K_r e^{\frac{\rho^2}{2}}.$$

Proof. Note first that due to (40), for any $|t| \leq \rho$ we have

$$\left|\log\left(\varphi_X(t)e^{\frac{t^2}{2}}\right)\right| \le \frac{1}{100}$$

and hence

$$\left|\varphi_X(t)e^{\frac{t^2}{2}} - 1\right| \le \frac{1}{50}$$

(as the exponent function is 2-Lipschitz in $U_{1/100}(0)$).

At the same time, the conclusion of Lemma 4.6 and the assumption (41) imply that for any $|t| \le \rho$ one has

$$\left|\varphi_X(t)e^{\frac{t^2}{2}} - \varphi_Y(t)e^{\frac{t^2}{2}}\right| = \left|\varphi_X(t) - \varphi_Y(t)\right| \cdot e^{\frac{t^2}{2}} \le K_r \rho^3 \cdot e^{\frac{\rho^2}{2}} \le \frac{1}{100},$$

hence altogether

$$\left|\varphi_{Y}(t)e^{\frac{t^{2}}{2}} - 1\right| \leq \left|\varphi_{X}(t)e^{\frac{t^{2}}{2}} - \varphi_{Y}(t)e^{\frac{t^{2}}{2}}\right| + \left|\varphi_{X}(t)e^{\frac{t^{2}}{2}} - 1\right| \leq \frac{1}{100} + \frac{1}{50} \leq \frac{1}{25}.$$

Finally, the logarithm function is 2-Lipschitz in $U_{\frac{1}{25}}(1)$, and thus for such t

(42)
$$\left| \log \left(\varphi_Y(t) e^{\frac{t^2}{2}} \right) \right| \leq \left| \log \left(\varphi_X(t) e^{\frac{t^2}{2}} \right) \right| + 2 \cdot \left| \varphi_Y(t) e^{\frac{t^2}{2}} - \varphi_X(t) e^{\frac{t^2}{2}} \right|$$

 $\leq N_\rho(X) |t|^3 + 2K_r e^{\frac{t^2}{2}} |t|^3 \leq (N_\rho(X) + 2K_r e^{\frac{t^2}{2}}) \cdot |t|^3.$

This implies the desired

$$N_{\rho}(Y) \le N_{\rho}(X) + 2K_r e^{\frac{\rho^2}{2}}.$$

We will now apply Proposition 4.7 to the random variables X and Y, given by (34), that occur in our study of random matrix products. Namely, denote for any $n, n' \ge n_0$

(43)
$$\eta_n = \frac{\xi_n - \mathbb{E}\,\xi_n}{\sqrt{\operatorname{Var}\,\xi_n}}, \quad \theta_{n,n'} = \xi_n + \xi_{(n,n+n']}, \quad \eta_{n,n'} = \frac{\theta_{n,n'} - \mathbb{E}\,\theta_{n,n'}}{\sqrt{\operatorname{Var}\,\theta_{n,n'}}}.$$

Corollary 4.8. In the assumptions of Theorem 1.4, there exists a constant K such that that if for some $\rho > 0$, $n, n' \ge n_0$, $\frac{n}{2} \le n' \le 2n$, one has

(44)
$$\rho^3 N_{\rho}(\eta_{n,n'}) \le \frac{1}{100}$$

(45)
$$\rho^3 \frac{K}{\sqrt{n+n'}} e^{\frac{\rho^2}{2}} \le \frac{1}{100},$$

then

$$N_{\rho}(\eta_{n+n'}) \le N_{\rho}(\eta_{n,n'}) + 2\frac{K}{\sqrt{n+n'}}e^{\frac{\rho^2}{2}}.$$

Proof. First, recall that due to Corollary 3.4 one has

$$\mathbb{E} |\eta_{n,n'}|^3 < C_3, \quad \mathbb{E} |\eta_{n+n'}|^3 < C_3,$$

and hence while applying Proposition 4.7 to X, Y, given by (34), one can take $C_X = C_3$.

Now, apply Lemma 4.5: from its conclusion (35) see that the value $K_r = C_r^{\frac{1}{3}} C_X^{\frac{2}{3}}$ in Proposition 4.7 is bounded from above by

$$K_r = C_r^{\frac{1}{3}} C_3^{\frac{2}{3}} \le \frac{Q_r^{\frac{1}{3}}}{\sqrt{n+n'}} \cdot C_3^{\frac{2}{3}} = \frac{K}{\sqrt{n+n'}}$$

where

$$K := Q_r^{\frac{1}{3}} C_3^{\frac{1}{3}}$$

The conclusion now immediately follows from Proposition 4.7, applied to the random variables X, Y, given by (34).

5. Proof of the main result

Joining the results of the previous sections, we obtain the following proposition:

Proposition 5.1. In the assumptions of Theorem 1.4, there exist sequences $\rho_n \to \infty$, $\delta_n \to 0$, such that for any $n \ge n_0$ and any $\mu_1, \ldots, \mu_n \in \mathcal{K}$ (46) $N'_{\rho_n}(\xi_n) \le \delta_n.$

Proof of Theorem 1.4. Assume that the assumptions of Theorem 1.4 are satisfied. Due to Proposition 5.1, for any $\rho > 0$ we have

$$\lim_{n \to \infty} N'_{\rho}(\xi_n) = \lim_{n \to \infty} N_{\rho}(\eta_n) = 0,$$

where

$$\eta_n = \frac{\xi_n - \mathbb{E}\,\xi_n}{\sqrt{\operatorname{Var}\,\xi_n}}.$$

In particular, the characteristic functions $\varphi_{\eta_n}(t)$ of the normalized variables converge uniformly on compact sets to $e^{-\frac{t^2}{2}}$. As the weak convergence of random variables is equivalent (Lévy's continuity theorem) to the pointwise convergence of their characteristic functions, we have the desired weak convergence

$$\eta_n = \frac{\xi_n - \mathbb{E}\,\xi_n}{\sqrt{\operatorname{Var}\,\xi_n}} \to \mathcal{N}(0, 1), \quad n \to \infty.$$

Moreover, this convergence is actually uniform in the choice of the sequence of measures $\mu_1, \ldots, \mu_n, \cdots \in \mathcal{K}$.

Proof of Proposition 5.1. We construct the sequence $(\rho_n, \delta_n)_{n \ge n_0}$ so that the desired property (46) can be established by induction on n. Namely, we have the following

Lemma 5.2. Let the sequence $(\rho_n, \delta_n)_{n \ge n_0}$ be chosen in such a way that the following conditions hold:

- As $n \to \infty$, one has $\rho_n \to \infty$ and $\delta_n \to 0$.
- For some $n_1 \ge 2n_0$, we have

$$\rho_n = \rho'_0, \quad \delta_n = 3C_3 \quad for \ all \quad n = n_0, \dots, n_1,$$

where

$$\rho_0' = \min\left(\rho_0, \left(\frac{3C_3}{100}\right)^{1/3}\right)$$

and ρ_0 is given by Lemma 4.3.

• For every $m > 2n_0$, taking $n = \lceil \frac{m}{2} \rceil$ and n' = m - n, one has

(47)
$$\rho_m \le L \min(\rho_n, \rho_{n'}),$$

(48)
$$\rho_m^3 \delta_m \le \frac{1}{100},$$

(49)
$$\left(\lambda \max(\delta_n, \delta_{n'}) + 2 \frac{K e^{\frac{\rho_m^2}{2}}}{\sqrt{m}}\right) \le \delta_m,$$

where constants L and λ are defined by the conclusion of Lemma 4.4 for $C = 2\frac{C_2}{C_1}$ with C_1, C_2 given by (4).

Then the conclusion of Proposition 5.1 holds for this sequence.

Proof. The proof of (46) is by induction. Namely, for $m = n_0, \ldots, n_1$ the conclusion follows from the choice of ρ'_0 and Lemma 4.3. Let us make the induction step: for $m > n_1$ let $n = \lceil \frac{m}{2} \rceil$ and n' = m - n. Due to the induction assumption,

$$N'_{\rho_n}(\xi_n) \le \delta_n, \quad N'_{\rho_n}(\xi_{(n,n+n']}) \le \delta_{n'},$$

and thus due to Lemma 4.4 and the inequality (47)

$$N'_{\rho_m}(\theta_{n,n'}) = N'_{\rho_m}(\xi_n + \xi_{(n,n+n']}) \le \lambda \max(\delta_n, \delta_{n'}).$$

Now, let us apply Corollary 4.8 for ρ_m, n, n' . First, check that the assumptions of Corollary 4.8 are satisfied. Indeed,

$$N_{\rho_m}(\eta_{n,n'}) = N'_{\rho_m}(\theta_{n,n'}) \le \lambda \max(\delta_n, \delta_{n'}) \le \delta_m$$

due to (49); multiplying by ρ_m^3 and applying (48), we get

$$\rho_m^3 N'_{\rho_m}(\theta_{n,n'}) \le \frac{1}{100}$$

what proves (44). Next, (45) follows again from (49) and (48):

$$\frac{K}{\sqrt{m}}\rho_m^3 e^{\frac{\rho_m^2}{2}} \leq \rho_m^3 \delta_m \leq \frac{1}{100}.$$

Corollary 4.8 is applicable, and hence (again using (49)) we get

$$N_{\rho_m}'(\xi_m) = N_{\rho_m}(\eta_m) \le \left(\lambda \max(\delta_n, \delta_{n'}) + 2\frac{Ke^{\frac{\rho_m^2}{2}}}{\sqrt{m}}\right) \le \delta_m$$

The induction step is complete.

To complete the proof of Proposition 5.1, it remains to construct the sequences

$$\rho_m \to \infty, \quad \delta_m \to 0$$

that satisfy the assumptions of Lemma 5.2. Roughly speaking, the contraction with the factor λ effectively allows to bring δ_m to zero as the additional term $\frac{2Ke^{\frac{\rho_m^2}{2}}}{\sqrt{m}}$ tends to zero. It suffices to make the radii ρ_m increase extremely slow, so that the exponent $e^{\frac{\rho_m^2}{2}}$ would not break this asymptotic vanishing.

We will choose ρ_m so that

$$\rho_m \le \frac{1}{2}\sqrt{\log m};$$

such a restriction already allows to note for checking (48), (49) that

$$\frac{Ke^{\frac{\rho_m^2}{2}}}{m^{1/2}} < \frac{Km^{1/8}}{m^{1/2}} < \frac{K}{m^{1/4}}$$

We let

(50)
$$\delta_m = Am^{-\beta}, \quad m > n_1,$$

where the constant A is chosen so that at $m = n_1$ this value coincides with $3C_3$,

$$(51) A = 3C_3 \cdot n_1^\beta,$$

and the (sufficiently small) power $\beta > 0$ and the (sufficiently large) initial index n_1 are yet to be fixed.

Now, choose the exponent $\beta > 0$ sufficiently small so that

$$\lambda \cdot 2^{\beta} < 1, \quad \beta < \frac{1}{4},$$

and fix $\lambda' \in (2^{\beta}\lambda, 1)$.

Then, for all sufficiently large n_1 the condition (49) holds and can be proved by induction. Indeed, in the left hand side the first summand is

$$\lambda \max(\delta_n, \delta_{n'}) \le \lambda \cdot A\left(\frac{m-1}{2}\right)^{-\beta} = 2^{-\beta}\lambda \cdot Am^{-\beta} \cdot \left(\frac{m-1}{m}\right)^{-\beta} < \lambda' \delta_m$$

The second summand is at most $K \cdot m^{-\frac{1}{4}}$, thus it suffices to check for $m > n_1$ the inequality

$$\lambda' A m^{-\beta} + K m^{-\frac{1}{4}} < A m^{-\beta},$$

or, equivalently,

(52)
$$(1-\lambda')Am^{-\beta} > Km^{-\frac{1}{4}}.$$

As $\beta < \frac{1}{4}$, it suffices to check it for $m = n_1$ (recall that (51) is used to determine A for given n_1 and β). Indeed, (52) holds once n_1 is sufficiently large to ensure

$$(1 - \lambda') 3C_3 > K n_1^{-\frac{1}{4}}$$

We fix a sufficiently large n_1 so that (52) holds, fix the corresponding A (defined by (51)) and the sequence (δ_m) , defined for $m > n_1$ by (50). Then, we use (48) and (47) to choose the sequence (ρ_m) . Namely, for $m > n_1$ we let

(53)
$$\rho_m = \min\left(L\min(\rho_n, \rho_{n'}), \frac{1}{2}\sqrt{\log m}, (100\delta_m)^{-1/3}\right).$$

Then the inequality $\rho_m \leq (100\delta_m)^{-1/3}$ implies (48), and (47) is satisfied automatically. Finally, as

$$\min\left(\frac{1}{2}\sqrt{\log m}, (100\delta_m)^{-1/3}\right) \to \infty, \quad m \to \infty,$$

the sequence (ρ_m) defined by (53) also tends to infinity; actually, it will coincide with $\frac{1}{2}\sqrt{\log m}$ for all sufficiently large m.

For the sequences (ρ_m, δ_m) , the conditions of Lemma 5.2 are satisfied, and this completes the proof of our main result.

6. UNBOUNDEDNESS OF VARIANCE: PROOF OF PROPOSITION 3.2

In this section we prove Proposition 3.2, i.e. show that under the assumptions of Theorem 1.4 variances of ξ_n become arbitrarily large.

In order to do so, we will assume that n is quite large, and will decompose the full product $A_n \ldots A_1$ into a several "long" groups D_{m+1}, \ldots, D_1 , between which some "short" compositions are applied:

$$A_n \dots A_1 = D_{m+1}(B_{\mu_{n_0,m}} \dots B_{\mu_{1,m}}) D_m \dots D_2(B_{\mu_{n_0,1}} \dots B_{\mu_{1,1}}) D_1$$

We will show that (for an appropriate choice of lengths) even conditionally to all D_1, \ldots, D_{m+1} , the distribution of the log-norm of the product (with high probability) has sufficiently high variance. At the same time, dividing by the product of norms of D_i , we get the composition

$$\frac{D_{m+1}}{\|D_{m+1}\|} (B_{\mu_{n_0,m}} \dots B_{\mu_{1,m}}) \frac{D_m}{\|D_m\|} \dots \frac{D_2}{\|D_2\|} (B_{\mu_{n_0,1}} \dots B_{\mu_{1,1}}) \frac{D_1}{\|D_1\|},$$

where all the quotients $\frac{D_j}{\|D_j\|}$ are almost rank-one matrices.

Therefore, we first consider the variance of a distribution of images of a given vector under random linear maps of rank one. In this case it is easier to show that the variance grows, see Lemma 6.1 and Lemma 6.6 below. By continuity, if one replaces random rank one linear maps by random linear maps of large norm, and uses the fact that for a matrix $D \in SL(2, \mathbb{R})$ with large norm, $\frac{D}{\|D\|}$ is close to a linear map of rank one and norm one, then a lower bound on variances still holds, see Lemma 6.7 and Corollary 6.8. Finally, we can complete the proof of Proposition 3.2 by applying the fact that with large probability a composition of a long enough sequence of random $SL(2, \mathbb{R})$ matrices has a large norm.

Let us now realize this strategy.

Let $Y \subseteq GL_2(\mathbb{R})$ be the space of all linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ of norm 1 and of rank 1. Notice that Y is homeomorphic to the torus \mathbb{T}^2 ; indeed, it follows from the fact that any such map can be represented as a composition of an orthogonal projection to a one-dimensional subspace and a rotation.

Lemma 6.1. There exist $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that for any non-zero vector $v \in \mathbb{R}^2$, any $p \in Y$, and any $\mu_1, \mu_2, \ldots, \mu_{n_0} \in \mathcal{K}$ we have

$$\operatorname{Var} \log |p \circ (B_{\mu_{n_0}} \dots B_{\mu_1})v| \ge \varepsilon_0.$$

To prove Lemma 6.1 we will use a statement from [GK] that was called *Atom Dissolving Theorem* there. We will start with a couple of definitions.

Definition 6.2. Denote by $\mathfrak{Max}(\nu)$ the weight of a maximal atom of a probability measure ν . In particular, if ν has no atoms, then $\mathfrak{Max}(\nu) = 0$.

Definition 6.3. Let X be a metric compact. For a measure μ on the space of homeomorphisms Homeo(X), we say that there is

- no finite set with a deterministic image, if there are no two finite sets $F, F' \subset X$ such that f(F) = F' for μ -a.e. $f \in \text{Homeo}(X)$;
- no measure with a deterministic image, if there are no two probability measures ν, ν' on X such that $f_*\nu = \nu'$ for μ -a.e. $f \in \text{Homeo}(X)$.

The following statement is a general statement for non-stationary dynamics, ensuring the "dissolving of atoms": decrease of the probability of a given point being sent to any particular point. **Theorem 6.4** (Atoms Dissolving Theorem 2.8 from [GK]). Let \mathbf{K}_X be a compact set of probability measures on Homeo(X).

• Assume that for any $\mu \in \mathbf{K}_X$ there is no finite set with a deterministic image. Then for any $\varepsilon > 0$ there exists n such that for any probability measure ν on X and any sequence $\mu_1, \ldots, \mu_n \in \mathbf{K}_X$ we have

$$\mathfrak{Max}\left(\mu_n*\cdots*\mu_1*\nu\right)<\varepsilon.$$

In particular, for any probability measure ν on X and any sequence $\mu_1, \mu_2, \ldots \in \mathbf{K}_X$ we have

$$\lim_{n\to\infty} \mathfrak{Max}\left(\mu_n * \cdots * \mu_1 * \nu\right) = 0.$$

If, moreover, for any μ ∈ K_X there is no measure with a deterministic image, then the convergence is exponential and uniform over all sequences μ₁, μ₂,... from K^N and all probability measures ν. That is, there exists λ < 1 such that for any n, any ν and any μ₁, μ₂,... ∈ K_X

$$\mathfrak{Max}\left(\mu_n*\cdots*\mu_1*\nu\right)<\lambda^n.$$

In the proof below we will only be using the first part of Theorem 6.4.

Proof of Lemma 6.1. Due to Theorem 6.4 and our assumptions regarding the measures from \mathcal{K} , there exists $n' \in \mathbb{N}$ such that for any $\mu_1, \mu_2, \ldots, \mu_{n'} \in \mathcal{K}$ we have

$$\mathfrak{Max}\left(\mu_n*\cdots*\mu_1*\nu\right)<\frac{1}{2}$$

for any probability measure ν on \mathbb{RP}^1 . To prove Lemma 6.1 it is enough to choose $n_0 = n' + 1$.

Since

$$\operatorname{Var} \log |p \circ (B_{\mu_n} \dots B_{\mu_1})v| = \operatorname{Var} \log \left| p \circ (B_{\mu_n} \dots B_{\mu_1}) \frac{v}{|v|} \right|,$$

without loss of generality we can assume that |v| = 1 and, slightly abusing the notation, consider it an element of \mathbb{RP}^1 . For given $p \in Y$, $v \in \mathbb{R}^2$, |v| = 1, $\{\mu_1, \mu_2, \ldots, \mu_{n_0}\} \in \mathcal{K}^{n_0}$ consider the probability distribution χ on $[0, +\infty)$ of the random images $|p \circ (B_{\mu_{n_0}} \ldots B_{\mu_1})v|$.

Lemma 6.5. The function $\Phi : \mathbb{RP}^1 \times Y \times \mathcal{K}^{n_0} \to \mathbb{R} \cup \{\infty\}$ defined by

$$\Phi(v, p, \mu_1, \mu_2, \dots, \mu_{n_0}) = \begin{cases} \infty, & \text{if } \chi(\{0\}) > 0; \\ \operatorname{Var} \log \left| p \circ (B_{\mu_{n_0}} \dots B_{\mu_1}) v \right|, & \text{if } \chi(\{0\}) = 0, \end{cases}$$

is lower semicontinuous.

Proof. Notice that χ depends continuously on $(v, p, \mu_1, \mu_2, \dots, \mu_{n_0})$ in weak-* topology.

Let us consider the cases when $\chi(\{0\}) > 0$ and when $\chi(\{0\}) = 0$ separately.

Assume first that $\chi(\{0\}) > 0$. We want to show that given M > 0, for any sufficiently small perturbation χ' of χ we have $\operatorname{Var}\log\chi' > M$. Notice that the measures condition implies that χ cannot be concentrated exclusively at $0 \in \mathbb{R}$. Hence for some $\tau > 0$ we have $\chi[\tau, +\infty) > 0$. If χ' is a probability distribution that is sufficiently close to χ , then $\chi'[\tau/2, +\infty)$ is not less than $\frac{1}{2}\chi[\tau, +\infty)$, and the χ' -weight of a small neighborhood of the origin is at least $\frac{1}{2}\chi(\{0\})$. Choosing that neighborhood small enough guarantees that $\operatorname{Var}\log\chi' > M$. Assume now that $\chi(\{0\}) = 0$ and $\operatorname{Var} \log \chi < \infty$. Then

$$\operatorname{Var}\log\chi = \lim_{T \to \infty} \operatorname{Var}\left[(\log\chi)|_{[-T,T]} \right],$$

so for any $\varepsilon > 0$, for some large enough T > 0 we have

$$\operatorname{Var}\left[(\log \chi)|_{[-T,T]}\right] > \operatorname{Var}\log \chi - \frac{\varepsilon}{2}$$

Therefore, for any χ' that is sufficiently close to χ we have

$$\operatorname{Var}\log\chi' \geq \operatorname{Var}\left[(\log\chi')|_{[-2T,2T]}\right] \geq \operatorname{Var}\left[(\log\chi)|_{[-T,T]}\right] - \frac{\varepsilon}{2} > \operatorname{Var}\log\chi - \varepsilon.$$

The case when $\chi(\{0\}) = 0$ and $\operatorname{Var} \log \chi = \infty$ can be treated similarly.

The space $\mathbb{RP}^1 \times Y \times \mathcal{K}^{n_0}$ is compact. Hence, Lemma 6.5 implies that it is enough to show that $\Phi > 0$ to ensure that for some $\varepsilon_0 > 0$ we have $\Phi \ge \varepsilon_0 > 0$.

Suppose this is not the case, and for some unit vector v, a linear map $p \in Y$, and $\mu_1, \ldots, \mu_{n_0} \in \mathcal{K}$ we have

$$\operatorname{Var}\log\left|p\circ(B_{\mu_{n_0}}\ldots B_{\mu_1})v\right|=0.$$

Then for some $d \ge 0$ with probability 1 we have $|p \circ (B_{\mu_{n_0}} \dots B_{\mu_1})v| = d$. That means that $B_{\mu_{n_0}} \dots B_{\mu_1}v$ has to belong to $L = \{u \mid |p(u)| = d\}$, which is a line (if d = 0) or a union of two lines (if d > 0). This implies that $\mu_1 \times \mu_2 \times \dots \times \mu_{n_0-1}$ -almost surely the image $B_{\mu_{n_0-1}} \dots B_{\mu_1}v$ must belong to the set $\cap_{A \in \text{supp}(\mu_{n_0})} A^{-1}(L)$. Since



FIGURE 2. The set L and its preimages

the measure μ_{n_0} must satisfy the measure condition, this intersection must consists at most of four points, whose projectivization gives at most two points on \mathbb{RP}^1 . But this would imply that if ν is an atomic measure on \mathbb{RP}^1 at the point corresponding to the initial vector v, then $\mu_{n_0-1} * \cdots * \mu_1 * \nu$ is a measure supported on at most two points, which contradicts the choice of n_0 above. This completes the proof of Lemma 6.1.

Lemma 6.6. For any $m \in \mathbb{N}$, any $\{\mu_{1,i}, \ldots, \mu_{n_0,i}\}_{i=1,\ldots,m} \in \mathcal{K}^{mn_0}$, and any $\{p_1, \ldots, p_{m+1}\} \in Y^{m+1}$ we have

Var log $||p_{m+1}(B_{\mu_{n_0,m}}\dots B_{\mu_{1,m}})p_m\dots p_2(B_{\mu_{n_0,1}}\dots B_{\mu_{1,1}})p_1|| \ge \varepsilon_0 m.$

Proof. As each p_i is a unit norm rank 1 matrix, it can be written as

$$p_j = v_j \otimes \ell_j$$
, where $v_j \in \mathbb{R}^2$, $\ell_j \in (\mathbb{R}^2)^*$, $|v_j| = |\ell_j| = 1$.

Now, let

$$\tilde{B}_j := B_{\mu_{n_0,j}} \dots B_{\mu_{1,j}}$$

be the j-th intermediate product. Then for the product

$$T = p_{m+1} \dot{B}_m p_m \dots p_2 \dot{B}_1 p_1$$

one has for any $v \in \mathbb{R}^2$

$$T(v) = v_{m+1} \cdot \ell_{m+1}(\tilde{B}_m v_m) \cdot \dots \cdot \ell_2(\tilde{B}_1 v_1) \cdot \ell_1(v),$$

and hence

(54)
$$\log \|T\| = \sum_{j=1}^{m} \log |\ell_{j+1}(\tilde{B}_j v_j)|.$$

Right hand side of (54) is a sum of m independent random variables, and the variance of each of them is at least ε_0 due to Lemma 6.1. Thus, the variance of $\log ||T||$ is at least $m\varepsilon_0$.

Lemma 6.7. There exists a neighborhood U of the compact $\mathcal{K}^{n_0m} \times Y^{m+1}$ in $\mathcal{K}^{n_0m} \times Mat_2(\mathbb{R})^{m+1}$ such that for any

$$\bar{\mu} \times \{D_j\}_{j=1,\dots,m+1} \in U,$$

where $\bar{\mu} = \{\mu_{1,i}, ..., \mu_{n_0,i}\}_{i=1,...,m}$ and $D_j \in Mat_2(\mathbb{R})$, we have

$$\operatorname{Var} \log \|D_{m+1}(B_{\mu_{n_0,m}} \dots B_{\mu_{1,m}}) D_m \dots D_2(B_{\mu_{n_0,1}} \dots B_{\mu_{1,1}}) D_1\| \ge \frac{\varepsilon_0 m}{2}$$

Proof. On $\mathcal{K}^{n_0m} \times Y^{m+1}$ this variance is bounded from below by $\varepsilon_0 m$ due to Lemma 6.6. As the set $\mathcal{K}^{n_0m} \times Y^{m+1}$ is compact, and the variance is a lower-semicontinuous function of a distribution, there exists a neighbourhood U of this compact on which the variance is at least $\frac{m\varepsilon_0}{2}$.

Corollary 6.8. There exists Q such that for any $D_1, \ldots, D_{m+1} \in SL(2, \mathbb{R})$ with $||D_j|| \ge Q, j = 1, \ldots, m+1$ one has

Var log
$$||D_{m+1}(B_{\mu_{n_0,m}}\dots B_{\mu_{1,m}})D_m\dots D_2(B_{\mu_{n_0,1}}\dots B_{\mu_{1,1}})D_1|| \ge \frac{\varepsilon_0 m}{2}$$

Proof.

(55)
$$\log \|D_{m+1}(B_{\mu_{n_0,m}}\dots B_{\mu_{1,m}})D_m\dots D_2(B_{\mu_{n_0,1}}\dots B_{\mu_{1,1}})D_1\| =$$

= $\log \|\tilde{D}_{m+1}(B_{\mu_{n_0,m}}\dots B_{\mu_{1,m}})\tilde{D}_m\dots \tilde{D}_2(B_{\mu_{n_0,1}}\dots B_{\mu_{1,1}})\tilde{D}_1\| + \sum_{j=1}^m \log \|D_j\|,$

where $\tilde{D}_j := \frac{D_j}{\|D_j\|}$. On the other hand, as $\|D\| \to \infty$ for $D \in \mathrm{SL}(2,\mathbb{R})$, one has $\frac{D}{\|D\|} \to Y$, so it suffices to choose Q sufficiently large to ensure that

$$(\{\mu_{i,k}\}_{1 \le i \le n_0, 1 \le k \le m}, (\tilde{D}_1, \dots, \tilde{D}_{m+1})) \in U$$

once $||D_j|| \ge Q$ for all j = 1, ..., m + 1, where U is provided by Lemma 6.7. \Box

Proof of Proposition 3.2. First, fix n_0 and ε_0 given by Lemma 6.1. Then, choose and fix m such that $\frac{m\varepsilon_0}{4} > c$.

Now, take a sufficiently large Q provided by Corollary 6.8. It follows from [G, Theorem 2.2] that for a sufficiently large n_2 one has

$$\forall n' \ge n_2 \quad \forall \mu_1, \dots, \mu_{n'} \in \mathcal{K} \quad \mathbb{P}_{\mu_1, \dots, \mu_{n'}} (\|A_{n'} \dots A_1\| \ge Q) \ge 1 - \frac{1}{2(m+1)}.$$

Now, take $n_3 := n_2(m+1) + n_0 m$. Then, for any $n \ge n_3$ and any $\mu_1, \ldots, \mu_n \in \mathcal{K}$ we can decompose the product $A_n \ldots A_1$ as

$$A_n \dots A_1 = D_{m+1} B_m D_m \dots D_2 B_1 D_1,$$

where each D_j is a product of at least n_2 matrices A_i , and each \tilde{B}_j is a product of n_0 of A_i 's.

This implies that with the probability at least $\frac{1}{2}$ one has $||D_j|| \ge Q$ for all j, and hence the variance of the distribution conditional to such D_j is at least $\frac{m\varepsilon_0}{2}$. Thus, we finally have

$$\operatorname{Var} \xi_n \geq \mathbb{E}_{D_1, \dots, D_{m+1}} \operatorname{Var}(\xi_n \mid D_1, \dots, D_{m+1}) \geq \frac{1}{2} \cdot \frac{m\varepsilon_0}{2} = \frac{m\varepsilon_0}{4} > c.$$

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