# CENTRAL LIMIT THEOREM FOR NON-STATIONARY RANDOM PRODUCTS OF SL(2, R) MATRICES

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Abstract. We prove Central Limit Theorem for non-stationary random products of  $SL(2,\mathbb{R})$  matrices, generalizing the classical results by Le Page and Tutubalin that were obtained in the case of iid random matrix products.

# 1. INTRODUCTION

1.1. Historical background. The two most fundamental results in probability that are present in almost every textbook are the (strong) Law of Large Numbers (LLN) and the Central Limit Theorem (CLT). In the most basic form, if  $\{\xi_n\}$  is an iid sequence of random variables with finite expectation  $a$  and finite variance  $\sigma^2$ , the LLN claims that almost surely  $\frac{1}{n} \sum_{i=1}^n \xi_i \to a$ , and the CLT claims that  $\frac{\sum_{i=1}^{n} \xi_i - na}{\sqrt{n}\sigma}$  converges in distribution to a normal distribution  $\mathcal{N}(0, 1)$  with mean 0 and variance 1.

There are many ways to relax the assumptions in both cases. In particular, the random variables do not have to be identically distributed. For example, a non-stationary version of the LLN known as Kolmogorov's Law [\[Kol\]](#page-24-0) claims that if  $\{\xi_i\}$ is a sequence of independent random variables with  $a_i = \mathbb{E}\xi_i$ ,  $\sigma_i^2 = \text{Var}(\xi_i)$ , and  $\sum_{i=1}^{\infty}$  $\frac{\sigma_i^2}{i^2} < \infty$ , then  $\frac{\sum_{i=1}^n (\xi_i - a_i)}{n} \to 0$  almost surely. On the other hand, if for some  $\delta > 0$  the sequence  $\mathbb{E}|\xi_i|^{2+\delta}$  is uniformly bounded, then the sequence of random variables  $\frac{\sum_{i=1}^{n} (\xi_i - a_i)}{\sum_{i=1}^{n} \sigma_i^2}$  converges in distribution to  $\mathcal{N}(0, 1)$ .

There are plenty of different generalizations and forms of these statements. For example, for some of the analogs of the LLN and CLT for the sums of iid random variables in the context of random walks on groups see the survey [\[F\]](#page-23-0) and monograph [\[BQ2\]](#page-23-1), and references therein. Here we discuss random matrix products. In this case, a multiplicative version of the LLN is given by Furstenberg and Kesten [\[FurK\]](#page-23-2). A stronger result is the famous Furstenberg Theorem, which also guarantees positivity of the Lyapunov exponent:

<span id="page-0-0"></span>**Theorem 1.1** (H. Furstenberg [\[Fur\]](#page-23-3)). Let  $\{X_k, k \geq 1\}$  be independent and identically distributed random variables, taking values in  $SL(d, \mathbb{R})$ , the  $d \times d$  matrices with determinant one, let  $G_X$  be the smallest closed subgroup of  $SL(d, \mathbb{R})$  containing the support of the distribution of  $X_1$ , and assume that

 $\mathbb{E}[\log ||X_1||] < \infty.$ 

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Also, assume that  $G_X$  is not compact and is strongly irreducible, i.e. there exists no  $G_X$ -invariant finite union of proper subspaces of  $\mathbb{R}^d$ . Then there exists a positive constant  $\lambda_F$  (Lyapunov exponent) such that with probability one

$$
\lim_{n \to \infty} \frac{1}{n} \log ||X_n \dots X_2 X_1|| = \lambda_F > 0.
$$

**Remark 1.2.** In the case of random products of  $SL(2,\mathbb{R})$  matrices, the assumption that  $G_X$  is not compact and is strongly irreducible is equivalent to the assumption that there exists no measure on  $\mathbb{RP}^1$  invariant under the action of every map from  $G_X$ , see [\[AB,](#page-23-4) Lemma 3.6].

The CLT for the products of iid random matrices is also available. The initial results were obtained for matrices with positive coefficients [\[Bel\]](#page-23-5), [\[FurK\]](#page-23-2). In the case of absolutely continuous distributions it was obtained by Tutubalin [\[T1,](#page-24-1) [T2\]](#page-24-2). The requirements on regularity of distributions was relaxed by Le Page, who proved the CLT for random matrix products under the assumption of finite exponential moments  $[L]$ , see also  $[BL]$ ,  $[GR]$ ,  $[GR]$ ,  $[J]$ . Finally, the assumption on the moments of the distribution was optimized by Benoist and Quint [\[BQ1\]](#page-23-9):

<span id="page-1-0"></span>**Theorem 1.3** (Benuist, Quint, [\[BQ1\]](#page-23-9)). Let  $\{X_k, k \geq 1\}$  be independent and identically distributed random matrices in  $SL(d, \mathbb{R})$ . Assume that  $G_X$  is non-compact and strongly irreducible and

$$
\mathbb{E}\left[ (\log \|X_1\|)^2 \right] < \infty.
$$

Then there exists  $\sigma > 0$  such that the random variables

$$
\frac{\log ||X_n \dots X_1|| - n\lambda_F}{\sqrt{n}},
$$

where  $\lambda_F > 0$  is the Lyapunov exponent, converge in distribution to  $\mathcal{N}(0, \sigma^2)$ .

Notice that both Theorems [1.1](#page-0-0) and [1.3](#page-1-0) require the sequence of random matrices to be identically distributed. That requirement allows to consider a stationary measure for the random dynamics on the projective space, which is a key notion used in the proofs of both results. Nevertheless, the classical LLN and CLT for sums of real valued random variables hold without that assumption, and it is natural to expect that non-stationary versions of the LLN and CLT for random matrix products should hold as well. Indeed, the non-stationary version of the Furstenberg Theorem was recently provided in [\[GK1\]](#page-23-10), and it already found interesting applications in spectral theory [\[GK2\]](#page-23-11). The non-stationary version of the CLT for random products of  $SL(2,\mathbb{R})$  matrices is the main result of this paper.

1.2. Preliminaries and main results. Let us now provide the setting needed to state our main result. From now on, let us restrict ourselves to the case of products of  $SL(2,\mathbb{R})$  matrices.

Let  $K$  be a compact subset in the set of probability measures on the group  $SL(2, \mathbb{R})$ . We will say that the measures condition is satisfied if for every measure  $\mu \in \mathcal{K}$  there are no Borel probability measures  $\nu_1, \nu_2$  on  $\mathbb{RP}^1$  such that  $(f_A)_*\nu_1 = \nu_2$  for  $\mu$ -almost every  $A \in SL(2,\mathbb{R})$ .

Let us fix some sequence  $\{\mu_i\}_{i\in\mathbb{N}}, \mu_i \in \mathcal{K}$ , and let  $A_i \in SL(2,\mathbb{R})$  be independent matrix-valued random variables, with  $A_i$  being distributed w.r.t.  $\mu_i$ . Set

$$
T_n = A_n A_{n-1} \dots A_1,
$$

and denote

$$
(2) \t\t\t L_n = \mathbb{E} \log ||T_n||.
$$

If the measures condition is satisfied, then for any  $\{\mu_i\}_{i\in\mathbb{N}}$ ,  $\mu_i \in \mathcal{K}$ , the sequence  ${L_n}$  must grow at least linearly, i.e. the norms of the random products must grow exponentially on average, see [\[GK1,](#page-23-10) Theorem 1.5]. A related statement on exponential growth of the norms in the case of non-stationary linear cocycles over Markov chains was established by Goldsheid [\[G\]](#page-23-12). Moreover, if additionally a uniform bound on some exponential moment exists for distributions from  $K$ , the non-random sequence  ${L_n}$  describes the behavior of almost every random product, and in this sense serves as a non-stationary analog of Lyapunov exponent. Namely, almost surely one has  $\lim_{n\to\infty}\frac{1}{n}(\log||T_n||-L_n)=0$ , see [\[GK1,](#page-23-10) Theorem 1.1]. This provides a direct analog of the LLN for non-stationary random matrix products.

That compels the question whether an analog of CLT for non-identically distributed random variables must hold in this setting. Our main result provides a positive answer in dimension two:

<span id="page-2-1"></span>**Theorem 1.4.** Let  $K$  be a compact subset in the set of probability measures on the group  $SL(2,\mathbb{R})$  that satisfies the measures condition, and there exists  $\gamma \geq 9$  and  $M > 0$  such that for any  $\mu \in \mathcal{K}$  one has

$$
\mathbb{E}_{\mu}(\log ||A||)^{\gamma} < M.
$$

Then the random variables

<span id="page-2-2"></span><span id="page-2-0"></span>
$$
\frac{\log ||T_n|| - L_n}{\sqrt{\text{Var}(\log ||T_n||)}}
$$

converge in distribution to  $\mathcal{N}(0, 1)$ , with the convergence that is uniform with respect of the choice of the sequence  $\mu_1, \mu_2, \dots \in \mathcal{K}$ .

Also, there are constants  $C_1, C_2 > 0$  and an index  $n_0$  such that for all  $n \geq n_0$ and all  $\mu_1, \ldots, \mu_n \in \mathcal{K}$  one has

$$
(4) \tC_1 n \leq \text{Var}(\log ||T_n||) \leq C_2 n.
$$

- Remark 1.5. (a) The condition [\(3\)](#page-2-0) is most likely not optimal. We would expect that it should be sufficient to require  $\gamma > 2$ , compare with the version of the LLN for real valued random variables provided above. Notice that it is still more restrictive than  $\gamma = 2$  which is optimal in the iid case, compare with Theorem [1.3.](#page-1-0)
- (b) One should expect that, under suitable conditions, Theorem [1.4](#page-2-1) should hold for random  $SL(d, \mathbb{R})$  matrix products for every  $d \geq 2$ . To prove such a statement, it would be helpful to have a non-stationary analog of simplicity of the Lyapunov spectrum, see [\[GR\]](#page-23-7), [\[GM\]](#page-23-8) for the case of iid random matrix products. In the case of some specific regular distributions in  $SL(d, \mathbb{R})$  such an analog was recently established [\[AFGQ\]](#page-23-13), but a statement for a general sequence of distributions is currently not available, even if certainly expected.

We consider this paper as a "proof of concept", the demonstration that enormous amount of results on random walks on groups formulated in terms of the law of large numbers, CLT, the law of the iterated logarithms etc. can be expected to hold in the non-stationary setting, even if the notion of the stationary measure on the projective space is not defined. The key observation here is that a random dynamical system acts on the measures on the phase space by convolutions, i.e.

averaging of the push-forwards of the measure by the random dynamics, and such an action "moves" measures toward the space of measures with some specific modulus of continuity, e.g. Hölder or log-Hölder, depending on the setting, see [\[GKM,](#page-23-14) Theorem 2.8], [\[M1,](#page-24-5) Theorems 2.4 and 2.9]. For some other recent results related to non-stationary random dynamics see [\[GK3\]](#page-23-15), [\[M2\]](#page-24-6), [\[M3\]](#page-24-7).

1.3. Notations and plan of the proof of the main result. Let us introduce some notations. Let

$$
T_{(n_1,n_2)} := A_{n_2} A_{n_2-1} \dots A_{n_1+1}
$$

be the part of the product of our random matrices  $A_i$ , where the index varies from  $n_1 + 1$  to  $n_2$ . Also, denote

$$
\xi_n = \log ||T_n||, \quad \xi_{(n_1, n_2]} = \log ||T_{(n_1, n_2)}||.
$$

Note that if two intervals of indices  $(n_1, n_2]$  and  $(n'_1, n'_2]$  are disjoint, then the corresponding products  $T_{(n_1,n_2]}$  and  $T_{(n'_1,n'_2]}$  are independent, and thus so are their log-norms  $\xi_{(n_1,n_2]}$  and  $\xi_{(n'_1,n'_2]}$ .

Now, a long product of matrices can be split into two parts (that we will later choose to be of comparable lengths): for any  $n, n'$  one has

$$
T_{n+n'} = T_{(n,n+n')}T_n;
$$

in particular, this implies

<span id="page-3-0"></span>(5)  $\xi_{n+n'} = \log ||T_n T_{(n,n+n')}|| \leq \log ||T_n|| + \log ||T_{(n,n+n')}|| = \xi_n + \xi_{(n,n+n')}$ .

The right hand side of the inequality in [\(5\)](#page-3-0) is a sum of two independent random variables; let us introduce the random variable  $R_{n,n'}$  that measures the difference between the right and left hand sides of [\(5\)](#page-3-0):

<span id="page-3-2"></span>(6) 
$$
R_{n,n'} = \log ||T_n|| + \log ||T_{(n,n+n')}|| - \log ||T_{n+n'}|| = (\xi_n + \xi_{(n,n+n')}) - \xi_{n+n'}.
$$

We start the proof of Theorem [1.4](#page-2-1) with establishing uniform moment bounds for the discrepancy  $R_{n,n'}$ ; this is done in Sec. [2,](#page-4-0) see Proposition [2.1.](#page-4-1) To do so, we have to show that it is (sufficiently) improbable that the mostly expanded vector for the product  $T_n$  is sent to the direction close to the one that is contracted by  $T_{(n,n+n')}$ . This can be reformulated in terms of the action on the projective line  $\mathbb{RP}^1$ : in these terms, it is the probability of sending a point to a given small neighbourhood. We use results from  $[M1]$ , where such estimates (log-Hölder bounds after a finite number of non-stationary iterations) were established, to obtain Lemma [2.2,](#page-4-2) providing such tail estimates.

Next, we use these estimates to establish a control on the central moments of  $\xi_n$ , using the relation

<span id="page-3-1"></span>(7) 
$$
\xi_{n+n'} = (\xi_n + \xi_{(n,n+n')}) - R_{n,n'}.
$$

To do so, we use the fact that the sum in the parenthesis is a sum of independent random variables, and the moments for  $R_{n,n'}$  are uniformly bounded, thus its addition cannot increase the moments too much. This is done in Section [3.1,](#page-7-0) see Proposition [3.1.](#page-7-1) Then, in Section [3.2](#page-9-0) (see Lemma [3.3\)](#page-9-1) we get a lower bound for the linear growth of the variances  $\text{Var}\,\xi_n$ , thus altogether establishing the conclusion [\(4\)](#page-2-2). The argument is again based on using [\(7\)](#page-3-1); a key difficulty here is to establish the arbitrarily large lower bound for the variances. The latter is Proposition [3.2,](#page-9-2) whose proof (that turned out to be surprisingly technical) is provided in Section [6.](#page-19-0)

Finally, the key step in the proof of Theorem [1.4](#page-2-1) is a bootstrapping argument, provided in Section [4.](#page-10-0) Namely, the sum of two independent random variables (for instance,  $\xi_n$  and  $\xi_{(n,n+n')})$  is closer to the Gaussian behavior than the summands separately. We introduce the quantitative way [\(27\)](#page-10-1) of measuring how close the distribution is to the Gaussian, and establish the corresponding inequality in Sec-tion [4.3.](#page-12-0) Then, we control how an additional perturbation, coming from the  $R_{n,n'}$ term, can worsen the bounds. This is done in Section [4.4.](#page-13-0)

We conclude by joining the bootstrapping estimates with the bounds established for  $\xi_n$  and  $R_{n,n'}$ , and complete the proof of Theorem [1.4](#page-2-1) in Section [5.](#page-16-0)

#### 2. MOMENT ESTIMATES FOR  $R_{n,n'}$

<span id="page-4-0"></span>In this section we provide the estimates on the moments of discrepancies  $R_{n,n'}$ defined by [\(6\)](#page-3-2).

2.1. Statements. Here is the main statement of this section:

<span id="page-4-1"></span>**Proposition 2.1.** Under the assumptions of Theorem [1.4,](#page-2-1) there exists  $C_R$ , such that for every  $n, n' \in \mathbb{N}$ , such that  $n \leq 2n'$  one has

$$
\mathbb{E}\, R_{n,n'} < C_R, \quad \mathbb{E}\, R_{n,n'}^2 < C_R, \quad \text{and} \quad \mathbb{E}\, R_{n,n'}^3 < C_R.
$$

Actually, we will show that the tails of distributions of random variables  $R_{n,n'}$ are uniformly bounded up to a linearly growing threshold. Namely, we have the following lemma:

<span id="page-4-2"></span>**Lemma 2.2.** There exist  $c, c_{\kappa} > 0$  such that

(8) 
$$
\forall n, n' \quad \forall x \leq c_{\kappa} n' \quad \mathbb{P}(R_{n,n'} > x) \leq c x^{-\gamma/2}.
$$

<span id="page-4-3"></span>In the rest of this section we prove Proposition [2.1](#page-4-1) and Lemma [2.2.](#page-4-2)

2.2. Proof of Proposition [2.1.](#page-4-1) Let us first deduce Proposition [2.1](#page-4-1) from Lemma [2.2:](#page-4-2)

Proof of Proposition [2.1.](#page-4-1) First of all, notice that it is enough to prove the estimate for  $\mathbb{E} R_{n,n'}^3$ , as it implies the other two by using Hölder's inequality.

In order to estimate  $\mathbb{E} R_{n,n'}^3$  we split it in two parts:

$$
\mathbb{E} R_{n,n'}^{3} = \mathbb{E} (R_{n,n'}^{3} \cdot \mathbf{1}_{R_{n,n'} \leq c_{\kappa} n'}) + \mathbb{E} (R_{n,n'}^{3} \cdot \mathbf{1}_{R_{n,n'} > c_{\kappa} n'})
$$

The first summand can be estimated using [\(8\)](#page-4-3):

$$
\mathbb{E}\left(R_{n,n'}^3 \cdot \mathbf{1}_{R_{n,n'}\leq c_{\kappa}n'}\right) \leq \int_0^{c_{\kappa}n'} \mathbb{P}\left(R_{n,n'}\geq x\right) \cdot 3x^2 dx \leq 1+3c \int_1^{\infty} x^{2-\frac{\gamma}{2}} dx = \text{const.}
$$

To estimate the second one, we notice that  $R_{n,n'}$  is bounded from above by the sum of  $n + n'$  independent variables  $\log ||A_i||$ ,  $i = 1, ..., n + n'$ , with 9-th moment not exceeding  $M$  by [\(3\)](#page-2-0). We apply the Hölder's inequality with exponents 3 and 3/2 and inequality [\(8\)](#page-4-3):

$$
\mathbb{E}\left(R_{n,n'}^3 \cdot \mathbf{1}_{R_{n,n'} > c_{\kappa}n'}\right) \leq \left(\mathbb{E}\left(R_{n,n'}^9\right)^{\frac{1}{3}} \cdot \mathbb{P}\left(R_{n,n'} \geq c_{\kappa}n'\right)^{\frac{2}{3}} \leq M^{\frac{1}{3}}(n+n')^3 \cdot c^{\frac{2}{3}}(c_{\kappa}n')^{-\frac{\gamma}{3}},
$$
  
and the right hand side again is bounded uniformly for  $n \leq 2n'$ .

2.3. Establishing tail estimates: proof of Lemma [2.2.](#page-4-2) We start by recalling some properties of  $SL(2,\mathbb{R})$  matrices. Let  $v \in \mathbb{R}^2$  be a vector and  $B \in SL(2,\mathbb{R})$  be a matrix. Define a function

$$
\Theta(B, v) = \log(||B|| \cdot |v|) - \log(|Bv|) = \log \frac{||B|| \cdot |v|}{|Bv|}
$$

that compares the expansion by  $B$  of the vector  $v$  with the maximum possible.

Denote by  $f_B$  the projectivization of the matrix  $B$ , namely

 $f_B: \mathbb{RP}^1 \to \mathbb{RP}^1$ , such that  $f_B \circ [\cdot] = [\cdot] \circ B$ ,

where  $[\cdot]$  is the canonical projection  $[\cdot] : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \mathbb{P}^1$ . Consider the projection [v], and denote by  $U_r([v])$  its radius r neighbourhood in  $\mathbb{RP}^1$ . Finally, consider the projections  $[e_1], [e_2]$  of the basis vectors  $e_1, e_2 \in \mathbb{R}^2$ . The following lemma allows to control the "loss of expansion"  $\Theta(B, v)$  in terms of closeness of the direction of v to preimages of given directions:

<span id="page-5-1"></span>**Lemma 2.3.** For  $x > 0$  take  $r = 2e^{-x/2}$ . If

<span id="page-5-0"></span>
$$
f_{B^{-1}}([e_1]) \notin U_r([v])
$$
 and  $f_{B^{-1}}([e_2]) \notin U_r([v])$ 

then

$$
(9) \t\t \Theta(B, v) \leq x.
$$

Proof. First of all, notice that

$$
\Theta(B, v) = \log(||B|| \cdot |v|) - \log(|Bv|) \le
$$
  
 
$$
\le \log(||B||) + \log(|v|) - \log(||B^{-1}||) - \log(|v|) \le 2\log(||B||).
$$

Hence, if  $||B|| \leq e^{x/2}$  then inequality [\(9\)](#page-5-0) holds automatically. From now on we assume that  $||B|| > e^{x/2}$ .

Recall that every matrix  $B \in SL(2,\mathbb{R})$  can be written as a product

$$
B = \mathrm{Rot}_{\beta_1} \left( \begin{array}{c} \|B\| & 0 \\ 0 & \|B\|^{-1} \end{array} \right) \mathrm{Rot}_{\beta_2},
$$

where  $\text{Rot}_{\beta}$  is a rotation by the angle  $\beta$ . Moreover, without loss of generality we can assume that  $\beta_2 = 0$  (the statement of Lemma [2.3](#page-5-1) is invariant under a precomposition of B with a rotation).

Note that at least one of the points  $f_{B^{-1}}([e_1]), f_{B^{-1}}([e_2])$  is within  $e^{-x}$  of the direction most contracted by  $B$ . Indeed,

$$
B^{-1} = \left(\begin{smallmatrix} \|B\|^{-1} & 0 \\ 0 & \|B\| \end{smallmatrix}\right) \operatorname{Rot}_{\beta_1}^{-1};
$$

the vectors  $e_1, e_2$  are orthogonal, thus there exists  $i = 1, 2$  such that for rotation preimage  $\text{Rot}_{\beta_1}^{-1}e_i$  its second coordinate is no smaller than the first one. Hence, for the vector  $B^{-1}e_i$  its second coordinate exceeds the first one at least by the factor  $||B||^2 \ge e^x$ , and hence  $B^{-1}e_i$  is at the distance at most  $e^{-x/2} < e^{-x}$  of the second coordinate axis, most contracted by  $B$  (see Fig. [1\)](#page-6-0).

Now, as the direction of the vector v is not within  $r = 2e^{-x/2}$  of the one of  $B^{-1}e_i$ , the angle between v and the second coordinate axis is at least  $\frac{r}{2}$ , and hence the absolute value of the first coordinate of the unit vector v is at least  $\sin\left(\frac{r}{2}\right) \geq \frac{2}{\pi} \cdot \frac{r}{2}$ . Applying B, we hence get a vector with length at least  $||B|| \cdot \frac{r}{\pi}$ . Thus,

$$
\Theta(B, v) \le \log \|B\| - \log \frac{r\|B\|}{\pi} = \log \frac{\pi}{r} = \frac{x}{2} \log \frac{\pi}{2} < x.
$$



<span id="page-6-0"></span>FIGURE 1. Vector v, the preimage  $B^{-1}(e_i)$  and its r-neighbourhood.

□

In what follows we will use the following statement about regularity of measures produced by actions of products of random matrices on the projective space. This is a particular case of [\[M1,](#page-24-5) Theorem 2.9].

<span id="page-6-1"></span>**Theorem 2.4** ([\[M1\]](#page-24-5)). Let  $T_{(n,m)} = A_m A_{m-1} \ldots A_{n+1}$  be a product of random  $SL(2,\mathbb{R})$  matrices whose distributions satisfy the measures condition. Then there exists a positive constant C and  $0 < \kappa < 1$ , such that for every pair of points  $p_1, p_2 \in \mathbb{RP}^1$  we have the following:

$$
\forall n < m \,\forall r > \kappa^{m-n} \quad \mathbb{P}\left(f_{T_{(n,m]}}(p_1) \in U_r(p_2)\right) < C|\log(r)|^{-\frac{\gamma}{2}}.
$$

Combining Lemma [2.3](#page-5-1) and Theorem [2.4,](#page-6-1) we are now ready to prove Lemma [2.2.](#page-4-2)

Proof of Lemma [2.2.](#page-4-2) Recall that

$$
R_{n,n'} = \log(||T_n||) + \log(||T_{(n,n+n'}||) - \log(||T_{n+n'}||).
$$

There exists a unit vector  $u \in \mathbb{R}^2$  (that depends on  $T_n$ ), such that

$$
||T_n|| = |T_n u|.
$$

Set  $v = T_n u$  and  $B = T_{(n,n+n')}$ . Note, that the discrepancy  $R_{n,n'}$  is bounded from above by  $\Theta(B, v)$ :

$$
R_{n,n'} = \log(|T_n u|) + \log(||T_{(n,n+n']}||) - \log(||T_{n+n'}||) \le
$$
  
\$\leq \log(||T\_{(n,n+n']}|| \cdot |T\_n u|) - \log(||T\_{n+n'} u||) = \Theta(T\_{(n,n+n')}, T\_n u) = \Theta(B, v).

Applying Lemma [2.3](#page-5-1) we obtain that for every  $x > 0$ 

$$
\mathbb{P}(R_{n,n'} > x) \leq \mathbb{P}(\Theta(B, v) > x) \leq
$$
  
\$\leq \mathbb{P}(f\_{B^{-1}}([e\_1]) \in U\_r([v])) + \mathbb{P}(f\_{B^{-1}}([e\_2]) \in U\_r([v])),\$

where  $r = 2e^{-x/2}$ . The random product

$$
T_{(n,n+n')}^{-1} = A_{n+1}^{-1} A_{n+2}^{-1} \dots A_{n+n'}^{-1}
$$

is formed by the matrices whose distributions satisfy the measures condition. Ac-cording to Theorem [2.4](#page-6-1) there exist constants  $C, \kappa < 1$ , such that for every  $[v] \in \mathbb{RP}^1$ and every  $r > \kappa^{n'}$  we have

$$
(10) \qquad \mathbb{P}\left(f_{B^{-1}}([e_1]) \in U_r([v])\right) + \mathbb{P}\left(f_{B^{-1}}([e_2]) \in U_r([v])\right) < 2C|\log(r)|^{-\frac{\gamma}{2}}.
$$

Substituting  $r = 2e^{-x/2}$  and taking  $c_{\kappa} = -\log(\kappa)$ , we obtain the desired estimate  $(8)$ .

# <span id="page-7-4"></span><span id="page-7-2"></span>3. MOMENTS GROWTH FOR  $\xi_n$

In this section we establish upper bounds on the moments of  $|\xi_n - \mathbb{E} \xi_n|$ , and also establish linear growth of the variance of  $\xi_n$ .

<span id="page-7-0"></span>3.1. **Moments upper bounds for**  $\xi_n$ **.** Here we prove the following statement:

<span id="page-7-1"></span>Proposition 3.1. Under the assumptions of Theorem [1.4](#page-2-1) there exists a constant  $C_{\xi} < \infty$ , such that for any  $n \in \mathbb{N}$  and any  $\mu_1, \ldots, \mu_n \in \mathcal{K}$  the following holds:

(11) 
$$
\mathbb{E} |\xi_n - \mathbb{E} \xi_n| < C_\xi \sqrt{n},
$$

(12) 
$$
\mathbb{E} |\xi_n - \mathbb{E} \xi_n|^2 < C_{\xi} n,
$$

and

(13) 
$$
\mathbb{E} |\xi_n - \mathbb{E} \xi_n|^3 < C_{\xi} n^{\frac{3}{2}}.
$$

*Proof of Proposition [3.1.](#page-7-1)* First, let us establish the upper bound  $(12)$  for the variances  $\text{Var}\,\xi_n$ . We will use the decomposition

(14) 
$$
\xi_{n+n'} = (\xi_n + \xi_{(n,n+n')}) - R_{n,n'};
$$

the triangle inequality for the  $L_2$ -norm (applied to the centred variables) then implies

<span id="page-7-6"></span><span id="page-7-5"></span><span id="page-7-3"></span>
$$
\sqrt{\text{Var}\,\xi_{n+n'}} \le \sqrt{\text{Var}(\xi_n + \xi_{(n,n+n')})} + \sqrt{\text{Var}\,R_{n,n'}},
$$

Due to Proposition [2.1](#page-4-1) and due to the independence of  $\xi_n$  and  $\xi_{(n,n+n')}$ , we have

(15) 
$$
\sqrt{\text{Var}\,\xi_{n+n'}} \leq \sqrt{\text{Var}\,\xi_n + \text{Var}\,\xi_{(n,n+n')}} + \sqrt{C_R},
$$

We will now recurrently construct a sequence  $c_n$ , such that for any  $n \in \mathbb{N}$  and any  $\mu_1, \ldots, \mu_n \in \mathcal{K}$ ,

(16) Var ξ<sup>n</sup> ≤ c<sup>n</sup> · n.

The existence of  $c_1$  is guaranteed by the uniform moments condition [\(3\)](#page-2-0). Now, to construct  $c_m$  with  $m > 1$ , take  $n = \lfloor \frac{m}{2} \rfloor$  and  $n' = m - n$ . Next, let us divide [\(15\)](#page-7-3) construct  $c_m$  w<br>by  $\sqrt{m}$ : we get

$$
\sqrt{\frac{\text{Var}\,\xi_m}{m}} \leq \sqrt{\frac{\text{Var}\,\xi_n}{n} \cdot \frac{n}{m} + \frac{\text{Var}\,\xi_{(n,n+n')}}{n'} \cdot \frac{n'}{m}} + \frac{\sqrt{C_R}}{\sqrt{m}} \leq \sqrt{\max(c_n, c'_n)} + \frac{\sqrt{C_R}}{\sqrt{m}}.
$$

Hence, it suffices to take  $c_m$  to be defined by the relation

(17) 
$$
\sqrt{c_m} = \sqrt{\max(c_n, c'_n)} + \frac{\sqrt{C_R}}{\sqrt{m}}
$$

For such sequence, it is easy to check by induction that for all  $m = 2^k + 1, \ldots, 2^{k+1}$ we have

.

$$
\sqrt{c_m} \le \sqrt{c_1} + \sum_{j=0}^k \frac{\sqrt{C_R}}{\sqrt{2^j}};
$$

which in turn implies a uniform bound

$$
\sqrt{c_m} \le \sqrt{c_1} + \frac{\sqrt{C_R}}{1 - \frac{1}{\sqrt{2}}},
$$

thus concluding the proof of [\(12\)](#page-7-2).

This also implies  $(11)$ : indeed, due to the Hölder inequality,

$$
\mathbb{E} |\xi_n - \mathbb{E} \xi_n| \leq \sqrt{\text{Var} \xi_n}.
$$

Finally, let us prove [\(13\)](#page-7-5). Consider the centred random variables

$$
\widetilde{\xi}_n = \xi_n - \mathbb{E} \xi_n, \quad \widetilde{\xi}_{(n,n+n')} = \xi_{(n,n+n')} - \mathbb{E} \xi_{(n,n+n')}, \quad \widetilde{R}_{n,n'} = R_{n,n'} - \mathbb{E} R_{n,n'}.
$$

Again applying the decomposition  $(14)$ , and using the  $L_3$ -triangle inequality, we get

(18) 
$$
\sqrt[3]{\mathbb{E}|\tilde{\xi}_{n+n'}|^3} \leq \sqrt[3]{\mathbb{E}|\tilde{\xi}_{n} + \tilde{\xi}_{(n,n+n')}|}^3 + \sqrt[3]{\mathbb{E}|\tilde{R}_{n,n'}|}^3
$$

The second summand in the right hand side does not exceed  $\sqrt[3]{C_R}$  due to Proposition [2.1.](#page-4-1) To estimate the first one, note that for any

<span id="page-8-1"></span>
$$
\forall a, b \in \mathbb{R} \quad |a+b|^3 \le |a|^3 + |b|^3 + 3(|a|^2 \cdot |b| + |a| \cdot |b|^2);
$$

thus,

$$
\mathbb{E}\left|\widetilde{\xi}_n+\widetilde{\xi}_{(n,n+n']}\right|^3\leq E\left|\widetilde{\xi}_n\right|^3+\mathbb{E}\left|\widetilde{\xi}_{(n,n+n')}\right|^3+3C_{\xi}^2(n(n')^{1/2}+n^{1/2}n').
$$

Taking a cubic root and using inequality  $\sqrt[3]{a+b} \leq \sqrt[3]{a} + \sqrt[3]{b}$ , we get

<span id="page-8-0"></span>
$$
(19) \qquad \sqrt[3]{\mathbb{E}\left|\widetilde{\xi}_n + \widetilde{\xi}_{(n,n+n')}\right|^3} \le \sqrt[3]{\mathbb{E}\left|\widetilde{\xi}_n\right|^3 + \mathbb{E}\left|\widetilde{\xi}_{(n,n+n')}\right|^3} + \sqrt[3]{6C_{\xi}^2 \cdot m^{3/2}}.
$$

Again, for an arbitrary  $m > 1$  set the indices  $n = \lfloor \frac{m}{2} \rfloor$  and  $n' = m - n$ . Substituting [\(19\)](#page-8-0) into [\(18\)](#page-8-1) and dividing by  $\sqrt{m}$ , we get (20)

<span id="page-8-3"></span>
$$
\sqrt[3]{\frac{\mathbb{E}\left|\widetilde{\xi}_m\right|^3}{m^{3/2}}} \leq \sqrt[3]{\frac{\mathbb{E}\left|\widetilde{\xi}_n\right|^3}{n^{3/2}} \cdot \left(\frac{n}{m}\right)^{3/2} + \frac{\mathbb{E}\left|\widetilde{\xi}_{(n,n+n')} \right|^3}{(n')^{3/2}} \cdot \left(\frac{n'}{m}\right)^{3/2}} + \left(6C_{\xi} + \frac{\sqrt[3]{C_R}}{\sqrt{m}}\right)
$$

As previously, we are going to find a sequence  $c_n$  such that for any  $n \in \mathbb{N}$  and any  $\mu_1, \ldots, \mu_n \in \mathcal{K}$ ,

(21) 
$$
\mathbb{E}|\tilde{\xi}_n|^3 \leq c_n \cdot n^{3/2}.
$$

Substituting  $(21)$  for  $n, n'$  into the right hand side of  $(20)$ , we bound it from above by

<span id="page-8-2"></span>
$$
\sqrt[3]{\left(\frac{n}{m}\right)^{3/2} + \left(\frac{n'}{m}\right)^{3/2}} \cdot \max\left(\sqrt[3]{c_n}, \sqrt[3]{c_{n'}}\right) + \left(6C_{\xi} + \frac{\sqrt[3]{C_R}}{\sqrt{m}}\right)
$$

Now, the factor before  $\max(\sqrt[3]{c_n}, \sqrt[3]{c_n})$  is bounded away from 1: it doesn't exceed

$$
\sqrt[3]{\left(\frac{n}{m}\right)^{3/2}+\left(\frac{n'}{m}\right)^{3/2}}\leq\sqrt[3]{\max\left(\left(\frac{n}{m}\right)^{1/2},\left(\frac{n'}{m}\right)^{1/2}\right)}\leq\left(\frac{2}{3}\right)^{1/6}<1.
$$

Hence, it suffices to take all  $c_n = \tilde{C}^3$ , where the constant  $\tilde{C}$  is chosen sufficiently large, so that  $\mathbb{E}|\tilde{\xi}_1^3| \leq \tilde{C}^3$  and that

$$
\widetilde{C} \le \left(\frac{2}{3}\right)^{1/6} \cdot \widetilde{C} + (6C_{\xi} + \sqrt[3]{C_R}).
$$

Indeed, inequality [\(20\)](#page-8-3) then becomes an inductive proof of [\(21\)](#page-8-2).

We have obtained the desired upper bound  $\mathbb{E}|\tilde{\xi}_n|^3 \leq \tilde{C}^3 \cdot n^{3/2}$ , thus concluding the proof of the proposition.

<span id="page-9-0"></span>3.2. Linear growth of variances. In this section we will prove inequality  $(4)$ , i.e. we will show that there are constants  $C_1, C_2 > 0$  and an index  $n_0$  such that for all  $n \geq n_0$  and all  $\mu_1, \ldots, \mu_n \in \mathcal{K}$  one has

$$
C_1 n \leq \text{Var}(\log ||T_n||) \leq C_2 n.
$$

We start by observing that estimate [\(12\)](#page-7-2) guarantees that for every  $n \in \mathbb{N}$ 

$$
\text{Var}(\xi_n) < C_{\xi} n,
$$

so we only need to establish the lower bound. The proof of the lower bound can be split into two statements:

<span id="page-9-2"></span>**Proposition 3.2.** Under the assumptions of Theorem [1.4,](#page-2-1) for any  $c > 0$  there exists  $n_1 \in \mathbb{N}$  such that for any  $n \geq n_1$  and any collection of distributions  $\mu_1, \ldots, \mu_n \in \mathcal{K}$ one has

$$
\text{Var}_{\mu_1,\ldots,\mu_n} \xi_n \geq c.
$$

We will provide the proof of Proposition [3.2](#page-9-2) in Section [6.](#page-19-0)

<span id="page-9-1"></span>**Lemma 3.3.** Under the assumptions of Theorem [1.4,](#page-2-1) assume that  $\text{Var} \xi_n$  becomes arbitrarily large:

<span id="page-9-3"></span>(22)  $\forall c \quad \exists n_1 : \quad \forall n \geq n_1 \quad \forall \mu_1, \dots, \mu_n \in \mathcal{K}$   $\text{Var}_{\mu_1, \dots, \mu_n} \xi_n \geq c.$ 

Then there exists  $C_1 > 0$  and  $n_0$  such that  $\text{Var} \xi_n \geq C_1 n$  for all  $n \geq n_0$ .

Proof. Recall that

<span id="page-9-5"></span><span id="page-9-4"></span>
$$
\xi_{n+n'} = \xi_n + \xi_{(n,n+n')} - R_{n,n'}.
$$

Cauchy-Schwarz inequality implies that

(23) 
$$
\sqrt{\text{Var}\,\xi_{n+n'}} \ge \sqrt{\text{Var}(\xi_n + \xi_{(n,n+n')})} - \sqrt{\text{Var}\,R_{n,n'}}.
$$

Take  $n_1$  such that [\(22\)](#page-9-3) holds with  $c = 16C_R$ , and let  $\varepsilon := \frac{C_R}{2n_1}$ , where  $C_R$  is defined in Proposition [2.1.](#page-4-1) Then, for every  $n = n_1, \ldots, 2n_1 - 1$  and any  $\mu_1, \ldots, \mu_n \in \mathcal{K}$  one has

(24) 
$$
\sqrt{\text{Var}\,\xi_n} \ge \sqrt{\varepsilon(n+1)} + 3\sqrt{C_R}.
$$

We claim that in that case the estimate [\(24\)](#page-9-4) holds for every  $n \geq n_1$ . In order to show that we will proceed by induction. Indeed, let  $n \geq 2n_1$  be the first number for which [\(24\)](#page-9-4) is not yet established; decompose it as  $n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$ . Then, each of the variances  $\text{Var} \xi_{\lfloor \frac{n}{2} \rfloor}$ ,  $\text{Var} \xi_{\lceil \frac{n}{2} \rceil}$  in [\(23\)](#page-9-5) is bounded from below by  $\sqrt{\varepsilon(\lfloor n/2 \rfloor + 1)} +$  $3\sqrt{C_R}$ , and hence

$$
\sqrt{\operatorname{Var}\xi_{\lfloor \frac{n}{2}\rfloor} + \operatorname{Var}\xi_{\left(\lfloor \frac{n}{2}\rfloor, n\right]}} - \sqrt{\operatorname{Var}R_{\lfloor \frac{n}{2}\rfloor, \lceil \frac{n}{2}\rceil}} \ge
$$
\n
$$
\geq \sqrt{2} \cdot (\sqrt{\varepsilon(\lfloor n/2\rfloor + 1)} + 3\sqrt{C_R}) - \sqrt{C_R} \ge
$$
\n
$$
\geq \sqrt{\varepsilon(n+1)} + (3\sqrt{2} - 1)\sqrt{C_R},
$$

where we have used  $2(\lfloor n/2 \rfloor + 1) \ge n+1$ . As  $3\sqrt{2}-1 > 3$ , this proves the induction step. In particular, for every  $n \geq n_1$  we have  $\text{Var} \xi_n \geq \varepsilon n$ .

Proposition [3.2](#page-9-2) and Lemma [3.3](#page-9-1) together prove [\(4\)](#page-2-2).

# <span id="page-10-2"></span>Corollary 3.4. There exists  $C_3$  such that

• for all  $n \ge n_0$  and any  $\mu_1, \ldots, \mu_n \in \mathcal{K}$ , the normalized variable  $\eta_n = \frac{\xi_n - \mathbb{E}\xi_n}{\sqrt{\text{Var}\xi_n}}$ satisfies

$$
\mathbb{E}|\eta_n|^3 < C_3.
$$

• For all  $n, n' \geq n_0$  and any  $\mu_1, \ldots, \mu_{n+n'} \in \mathcal{K}$ , the variance of the normalized sum

$$
\eta_{n,n'} = \frac{\theta_{n,n'} - \mathbb{E}\,\theta_{n,n'}}{\sqrt{\text{Var}\,\theta_{n,n'}}}, \quad \text{where} \quad \theta_{n,n'} := \xi_n + \xi_{(n,n+n')}.
$$

satisfies

(26) E |ηn,n′ | <sup>3</sup> < C3.

## 4. Bootstrapping: distance to the Gaussian distribution

<span id="page-10-0"></span>This section is devoted to the bootstrapping arguments that allow to show the convergence to Gaussian law.

4.1. Preliminaries. Let  $\xi$ ,  $\xi'$  be two independent random variables with finite third moment and comparable variances: for a given constant  $C > 1$ , we have

$$
C^{-1} < \frac{\text{Var}\,\xi}{\text{Var}\,\xi'} < C.
$$

We will provide a value  $N'_{\rho}(\xi)$ , measuring quantitatively non-Gaussianity of the law of  $\xi$ , such that (under appropriate assumptions) it will be smaller for the sum  $\xi + \xi'$ than for the summands separately.

**Definition 4.1.** For a random variable  $\xi$  we denote by  $\varphi_{\xi}(t)$  its *characteristic* function:

$$
\varphi_{\xi}(t) = \mathbb{E}e^{it\xi}.
$$

Now, let

<span id="page-10-1"></span>(27) 
$$
N_{\rho}(\eta) = \sup_{0 < |t| < \rho} \frac{\left| \log \left( \varphi_{\eta}(t) e^{t^2/2} \right) \right|}{|t|^3}, \quad N_{\rho}'(\xi) = N_{\rho} \left( \frac{\xi - \mathbb{E} \xi}{\sqrt{\text{Var} \xi}} \right).
$$

Then  $\eta \sim \mathcal{N}(0, 1)$  if and only if  $N_\rho(\eta) = 0$  for all  $\rho > 0$  (as the distribution of a random variable is uniquely determined by its characteristic function). Note also that  $N_{\rho}(\eta)$  might be infinite if the corresponding characteristic function  $\varphi_n$  vanishes somewhere on  $[-\rho, \rho]$ . Finally, the logarithm here is a function of a complex variable (as the characteristic function might be, and most often is, non-real). As soon as  $\varphi_{\eta}(t)$  doesn't vanish on  $[-\rho, \rho]$ , we define the composition log  $(\varphi_{\eta}(t)e^{t^2/2})$  by a continuous extension, starting with the value  $log 1 = 0$  at  $t = 0$ .

4.2. Initial estimates. To start a bootstrapping argument, one needs some initial bounds, in this case, for the norms  $N'_{\rho}(\xi)$  for some  $\rho > 0$ .

<span id="page-11-3"></span>**Lemma 4.2.** Let  $X$  be a random variable with

<span id="page-11-1"></span>
$$
\mathbb{E} X = 0, \quad \text{Var } X = 1, \quad \mathbb{E} |X|^3 < C_X.
$$

Then its characteristic function satisfies

(28) 
$$
\left|\varphi_X(t) - \left(1 - \frac{t^2}{2}\right)\right| \leq C_X \cdot |t|^3 \text{ for all } t \in \mathbb{R}.
$$

*Proof.* Note that it suffices to establish the estimate for the second derivative  $\varphi''(t)$ : for all  $t \in \mathbb{R}$ ,

(29) 
$$
|\varphi''_X(t) + 1| \leq C_X|t|.
$$

Indeed, integrating [\(29\)](#page-11-0) two times then suffices to obtain [\(28\)](#page-11-1):

<span id="page-11-0"></span>
$$
\varphi_X(t) - \left(1 - \frac{t^2}{2}\right) = \int_0^t dt_1 \int_0^{t_1} (\varphi''_X(t_2) + 1) dt_2.
$$

<span id="page-11-2"></span>Now, let us rewrite the estimated expression in [\(29\)](#page-11-0):

(30) 
$$
|\varphi''_X(t) + 1| = |\mathbb{E}(X^2(e^{itX} - 1))| \leq \mathbb{E}(X^2|e^{itX} - 1|) \leq \mathbb{E}(X^2 \cdot |tX|);
$$

here we have used  $\text{Var } X = 1$  for the first equality and the fact that  $e^{ix}$  is a 1-Lipschitz function for the last inequality. Finally, the right hand side of [\(30\)](#page-11-2) can be rewritten as

$$
|t| \cdot \mathbb{E} |X|^3 < C_X |t|,
$$

thus completing the proof.

Joining this with the estimate from Corollary [3.4,](#page-10-2) we get the initial bound lemma.

<span id="page-11-6"></span>**Lemma 4.3.** There exists  $\rho_0 > 0$ , such that for any  $n \geq n_0$  the value  $N_{\rho_0}(\eta_n)$  is well-defined and satisfied

$$
N_{\rho_0}(\eta_m) < 3C_3
$$
 and  $\rho_0^3 N_{\rho_0}(\eta_m) < \frac{1}{100}$ ,

where  $C_3$  is the constant defined in Corollary [3.4.](#page-10-2)

*Proof.* Applying Lemma [4.2,](#page-11-3) and multiplying its conclusion by  $e^{\frac{t^2}{2}}$ , we get

(31) 
$$
\left| e^{\frac{t^2}{2}} \varphi_{\eta_n}(t) - e^{\frac{t^2}{2}} \left( 1 - \frac{t^2}{2} \right) \right| \leq C_3 |t|^3 \cdot e^{\frac{t^2}{2}}
$$

Now,

<span id="page-11-4"></span>
$$
e^{\frac{t^2}{2}}\left(1-\frac{t^2}{2}\right) = 1 + o(|t|^3),
$$

thus for sufficiently small  $\rho$  one has

(32) 
$$
\forall |t| \le \rho \qquad \left| e^{\frac{t^2}{2}} \left( 1 - \frac{t^2}{2} \right) - 1 \right| \le \frac{1}{2} C_3 |t|^3,
$$

as well as

<span id="page-11-5"></span>
$$
\forall |t|\leq \rho \qquad e^{\frac{t^2}{2}}\leq e^{\frac{\rho^2}{2}}\leq \frac{3}{2},
$$

$$
\Box
$$

hence the right hand side of [\(31\)](#page-11-4) can be replaced by  $\frac{3}{2}C_3|t|^3$ . Joining this with [\(32\)](#page-11-5), we get

$$
\forall |t| \le \rho \qquad \left| e^{\frac{t^2}{2}} \varphi_{\eta_n}(t) - 1 \right| \le \left( \frac{1}{2} + \frac{3}{2} \right) \cdot C_3 |t|^3 = 2C_3 |t|^3.
$$

Taking

$$
\rho_0:=\min\left(\rho,\sqrt[3]{\frac{1}{300C_3}}\right),
$$

we ensure that  $2C_3\rho_0^3 < \frac{1}{100}$ , and hence that log is  $\frac{3}{2}$ -Lipschitz in the (complex) disc  $U_{2C_3\rho_0^3}(1)$ . Therefore,

$$
\forall |t| \leq \rho_0 \quad \left| \log e^{\frac{t^2}{2}} \varphi_{\eta_n}(t) \right| \leq \frac{3}{2} \cdot 2C_3 |t|^3,
$$

which implies the desired

$$
N_{\rho_0}(\eta_n) \le 3C_3.
$$

<span id="page-12-0"></span>4.3. Sum of two independent variables. The following is the first step of the bootstrapping argument, estimating the decrease of  $N'$ -values for the sum of two independent random variables. Notice that in addition to the decrease by a linear factor, the parameter  $\rho$  (describing the size of the domain) gets increased.

<span id="page-12-2"></span>**Lemma 4.4.** For any C there exists  $\lambda < 1$  and  $L > 1$  such that if for some  $\rho > 0$ for some independent random variables  $\xi, \xi'$  one has

$$
C^{-1} < \frac{\text{Var}\,\xi}{\text{Var}\,\xi'} < C
$$

and values  $N'_{\rho}(\xi), N'_{\rho}(\xi')$  are finite, then

$$
N'_{L\rho}(\xi + \xi') \le \lambda \cdot \max(N'_{\rho}(\xi), N'_{\rho}(\xi')).
$$

Proof. Let

$$
\eta = \frac{\xi - \mathbb{E}\,\xi}{\sqrt{\text{Var}\,\xi}}, \quad \eta' = \frac{\xi' - \mathbb{E}\,\xi'}{\sqrt{\text{Var}\,\xi'}}, \quad \eta'' = \frac{(\xi + \xi') - (\mathbb{E}\,(\xi + \xi'))}{\sqrt{\text{Var}(\xi + \xi')}}.
$$

Also, denote

$$
c = \sqrt{\frac{\operatorname{Var}\xi}{\operatorname{Var}\xi + \operatorname{Var}\xi'}}, \quad c' = \sqrt{\frac{\operatorname{Var}\xi'}{\operatorname{Var}\xi + \operatorname{Var}\xi'}};
$$

then, one has

$$
\eta'' = c\eta + c'\eta',
$$

with the coefficients that satisfy

$$
c^2 + (c')^2 = 1
$$
,  $c, c' \le \sqrt{\frac{C}{C+1}} < 1$ .

By definition, we have for any  $L$ 

$$
N'_{L\rho}(\xi + \xi') = N_{L\rho}(\eta'') = N_{L\rho}(c\eta + c'\eta').
$$

Now, for the characteristic functions we have

<span id="page-12-1"></span>
$$
\varphi_{c\eta+c'\eta'}(t) = \varphi_{\eta}(ct) \cdot \varphi_{\eta'}(c't),
$$

and as  $c^2 + (c')^2 = 1$ , we have  $(33)$  $t^{2} \varphi_{c\eta+c'\eta'}(t) = e^{(ct)^{2}} \varphi_{\eta}(ct) \cdot e^{(c't)^{2}} \varphi_{\eta'}(c't).$  □

If  $cL$ ,  $c'L \leq 1$ , taking the logarithm of [\(33\)](#page-12-1) and dividing by  $|t|^3$ , we get

$$
N_{L\rho}(\eta'') = \sup_{0 < |t| < L\rho} \frac{\left| \log \left( e^{t^2} \varphi_{\eta''}(t) \right) \right|}{|t|^3} \leq
$$
\n
$$
\leq \sup_{0 < |t| < L\rho} \frac{\left| \log \left( e^{(ct)^2} \varphi_{\eta}(ct) \right) \right|}{|t|^3} + \sup_{0 < |t| < L\rho} \frac{\left| \log \left( e^{(c't)^2} \varphi_{\eta'}(c't) \right) \right|}{|t|^3}
$$

Making the  $ct$  and  $c't$  variable change in the first and second expressions respectively in the right hand side, we obtain

$$
N_{L\rho}(\eta'') \le c^3 N_{cL\rho}(\eta) + (c')^3 N_{c'L\rho}(\eta') \le
$$
  
 
$$
\le (c^3 + (c')^3) \cdot \max(N_{\rho}(\eta), N_{\rho}(\eta')) \le
$$
  
 
$$
\le \max(c, c') \cdot \max(N_{\rho}(\eta), N_{\rho}(\eta')).
$$

Taking 
$$
L = \sqrt{\frac{C+1}{C}}
$$
 and  $\lambda = \frac{1}{L}$  concludes the proof.

<span id="page-13-0"></span>4.4. Correction by an additional term. The expression [\(7\)](#page-3-1) for  $\xi_{n+n'}$ , besides the sum of two independent random variables

$$
\theta_{n,n'} = \xi_n + \xi_{(n,n+n')},
$$

has an additional term

<span id="page-13-1"></span>
$$
R_{n,n'} = \theta_{n,n'} - \xi_{n+n'}.
$$

Due to this, an extra (and possibly non-independent) term is added to the normalized random variable: for

(34) 
$$
X = \frac{\theta_{n,n'} - \mathbb{E}\,\theta_{n,n'}}{\sqrt{\text{Var}\,\theta_{n,n'}}}, \quad Y = \frac{\xi_{n+n'} - \mathbb{E}\,\xi_{n+n'}}{\sqrt{\text{Var}\,\xi_{n+n'}}}
$$

this term is the difference

$$
r = r_{n,n'} = Y - X
$$

We are going to analyze and control its influence. First, note that  $\mathbb{E} r = 0$ , and  $\mathbb{E}|r|^3$  satisfies an upper bound:

<span id="page-13-2"></span>**Lemma 4.5.** There exists  $Q_r < \infty$ , such that for every  $n, n' \geq n_0$ , satisfying

<span id="page-13-3"></span>
$$
\frac{n}{2} \le n' \le 2n,
$$

and any  $\mu_1, \ldots, \mu_{n+n'} \in \mathcal{K}$ , we have

(35) 
$$
\mathbb{E}|r_{n,n'}|^3 < Q_r(n+n')^{-\frac{3}{2}}.
$$

*Proof.* Note that  $r_{n,n'}$  can be expressed as

(36) 
$$
r = r_{n,n'} = \frac{\sqrt{\text{Var }\theta_{n,n'}} - \sqrt{\text{Var }\xi_{n,n'}}}{\sqrt{\text{Var }\xi_{n,n'}}} \cdot X - \frac{1}{\sqrt{\text{Var }\xi_{n,n'}}}(R_{n,n'} - \mathbb{E } R_{n,n'}).
$$

Now, the  $L_3$ -norms of both X and  $R_{n,n'}$  are uniformly bounded (see Proposition [3.1](#page-7-1)) and Corollary [3.4\)](#page-10-2). On the other hand, from the Cauchy-Schwartz inequality one has

$$
\left| \sqrt{\text{Var} \, \theta_{n,n'}} - \sqrt{\text{Var} \, \xi_{n+n'}} \right| \leq \sqrt{\text{Var} \, R_{n,n'}},
$$

and hence both coefficients admit upper bounds as  $\frac{\text{const}}{\sqrt{n+n'}}$ 

Our next step is to control the influence of such a "small" change by  $r$  on the non-Gaussianity value  $N<sub>o</sub>$ :

<span id="page-14-3"></span>**Lemma 4.6.** Let  $X, Y$  be two random variables with

<span id="page-14-1"></span>
$$
\mathbb{E} X = \mathbb{E} Y = 0, \quad \text{Var } X = \text{Var } Y = 1, \quad \mathbb{E} X^3, \mathbb{E} Y^3 < C_X.
$$

Assume that for  $r = Y - X$  one has  $\mathbb{E}|r|^3 < C_r$ . Then for any  $t \in \mathbb{R}$  one has

(37) 
$$
|\varphi_X(t) - \varphi_Y(t)| \leq C_r^{\frac{1}{3}} C_X^{\frac{2}{3}} \cdot |t|^3
$$

Proof. Let us take the second derivative of the difference of the characteristic functions:

<span id="page-14-2"></span>(38) 
$$
(\varphi_X - \varphi_Y)''(t) = -\mathbb{E}(X^2 e^{itX} - Y^2 e^{itY})
$$
  
= 
$$
-\mathbb{E}(X^2 (e^{itX} - 1) - Y^2 (e^{itY} - 1)),
$$

where the second equality follows from  $\text{Var } X = \text{Var } Y$ . Now, note that it suffices to obtain an estimate

(39) 
$$
|(\varphi_X - \varphi_Y)''(t)| \leq 3C_r^{\frac{1}{3}} C_X^{\frac{2}{3}} \cdot |t|,
$$

as again we can integrate two times:

<span id="page-14-0"></span>
$$
\varphi_X(t) - \varphi_Y(t) = \int_0^t dt_1 \int_0^{t_1} (\varphi_X - \varphi_Y)''(t_2) dt_2,
$$

and integrating [\(39\)](#page-14-0) two times, we get the desired [\(37\)](#page-14-1).

In order to obtain the estimate [\(39\)](#page-14-0), let us decompose the right hand side of [\(38\)](#page-14-2):

$$
|\varphi''_X(t) - \varphi''_Y(t)| \le \mathbb{E} |(X^2 - Y^2)(e^{itX} - 1)| + \mathbb{E} (Y^2|e^{itY} - e^{itX}|).
$$

To estimate the first summand, we note that  $Y^2 - X^2 = r(X + Y)$ , hence it does not exceed

$$
\mathbb{E} |(X^2 - Y^2)(e^{itX} - 1)| \leq \mathbb{E} (|r| \cdot (|X| + |Y|) \cdot |tX|).
$$

The expectation of the products  $|r| \cdot |X|^2$  and  $|r| \cdot |X||Y|$  can be estimated using the Hölder inequality: each does not exceed  $C_r^{1/3} C_X^{2/3}$ . We thus get

$$
\mathbb{E} |(X^2 - Y^2)(e^{itX} - 1)| \leq 2C_r^{1/3} C_X^{2/3} \cdot |t|.
$$

In the same way,

$$
\mathbb{E}\left(Y^2|e^{itY}-e^{itX}|\right)\leq \mathbb{E}\left(Y^2|t(X-Y)|\right)=|t|\cdot \mathbb{E}\left(|r|\cdot Y^2\right),
$$

and the right hand side does not exceed  $C_r^{1/3} C_X^{2/3} |t|$ . Adding these estimates together, we obtain the desired  $(39)$ .

<span id="page-14-6"></span>**Proposition 4.7.** In the assumptions of Lemma [4.6,](#page-14-3) let  $K_r := C_r^{1/3} C_X^{2/3}$  be the factor that appears in its conclusion. Assume additionally that for some  $\rho > 0$  one  $\mathit{has}$ 

$$
\rho^3 N_\rho(X) \le \frac{1}{100}
$$

and

(41) 
$$
K_r \rho^3 e^{\frac{\rho^2}{2}} \le \frac{1}{100}.
$$

Then

<span id="page-14-5"></span><span id="page-14-4"></span>
$$
N_{\rho}(Y) \le N_{\rho}(X) + 2K_r e^{\frac{\rho^2}{2}}.
$$

*Proof.* Note first that due to [\(40\)](#page-14-4), for any  $|t| \leq \rho$  we have

$$
\left|\log\left(\varphi_X(t)e^{\frac{t^2}{2}}\right)\right| \le \frac{1}{100}
$$

and hence

$$
\left|\varphi_X(t)e^{\frac{t^2}{2}} - 1\right| \le \frac{1}{50}
$$

(as the exponent function is 2-Lipschitz in  $U_{1/100}(0)$ ).

At the same time, the conclusion of Lemma [4.6](#page-14-3) and the assumption [\(41\)](#page-14-5) imply that for any  $|t| \leq \rho$  one has

$$
\left|\varphi_X(t)e^{\frac{t^2}{2}} - \varphi_Y(t)e^{\frac{t^2}{2}}\right| = \left|\varphi_X(t) - \varphi_Y(t)\right| \cdot e^{\frac{t^2}{2}} \le K_r \rho^3 \cdot e^{\frac{\rho^2}{2}} \le \frac{1}{100},
$$

hence altogether

$$
\left|\varphi_Y(t)e^{\frac{t^2}{2}}-1\right|\leq \left|\varphi_X(t)e^{\frac{t^2}{2}}-\varphi_Y(t)e^{\frac{t^2}{2}}\right|+\left|\varphi_X(t)e^{\frac{t^2}{2}}-1\right|\leq \frac{1}{100}+\frac{1}{50}\leq \frac{1}{25}.
$$

Finally, the logarithm function is 2-Lipschitz in  $U_{\frac{1}{25}}(1)$ , and thus for such t

(42) 
$$
\left| \log \left( \varphi_Y(t) e^{\frac{t^2}{2}} \right) \right| \le \left| \log \left( \varphi_X(t) e^{\frac{t^2}{2}} \right) \right| + 2 \cdot \left| \varphi_Y(t) e^{\frac{t^2}{2}} - \varphi_X(t) e^{\frac{t^2}{2}} \right|
$$
  

$$
\le N_\rho(X) |t|^3 + 2K_r e^{\frac{t^2}{2}} |t|^3 \le (N_\rho(X) + 2K_r e^{\frac{t^2}{2}}) \cdot |t|^3.
$$

This implies the desired

$$
N_{\rho}(Y) \le N_{\rho}(X) + 2K_r e^{\frac{\rho^2}{2}}.
$$

We will now apply Proposition [4.7](#page-14-6) to the random variables  $X$  and  $Y$ , given by [\(34\)](#page-13-1), that occur in our study of random matrix products. Namely, denote for any  $n, n' \geq n_0$ 

(43) 
$$
\eta_n = \frac{\xi_n - \mathbb{E}\,\xi_n}{\sqrt{\text{Var}\,\xi_n}}, \quad \theta_{n,n'} = \xi_n + \xi_{(n,n+n')}, \quad \eta_{n,n'} = \frac{\theta_{n,n'} - \mathbb{E}\,\theta_{n,n'}}{\sqrt{\text{Var}\,\theta_{n,n'}}}.
$$

<span id="page-15-0"></span>**Corollary 4.8.** In the assumptions of Theorem [1.4,](#page-2-1) there exists a constant  $K$  such that that if for some  $\rho > 0$ ,  $n, n' \ge n_0$ ,  $\frac{n}{2} \le n' \le 2n$ , one has

(44) 
$$
\rho^3 N_{\rho}(\eta_{n,n'}) \le \frac{1}{100}
$$

(45) 
$$
\rho^3 \frac{K}{\sqrt{n+n'}} e^{\frac{\rho^2}{2}} \le \frac{1}{100},
$$

then

<span id="page-15-2"></span><span id="page-15-1"></span>
$$
N_{\rho}(\eta_{n+n'}) \le N_{\rho}(\eta_{n,n'}) + 2\frac{K}{\sqrt{n+n'}}e^{\frac{\rho^2}{2}}.
$$

Proof. First, recall that due to Corollary [3.4](#page-10-2) one has

$$
\mathbb{E} |\eta_{n,n'}|^3 < C_3, \quad \mathbb{E} |\eta_{n+n'}|^3 < C_3,
$$

and hence while applying Proposition [4.7](#page-14-6) to  $X, Y$ , given by [\(34\)](#page-13-1), one can take  $C_X = C_3.$ 

Now, apply Lemma [4.5:](#page-13-2) from its conclusion [\(35\)](#page-13-3) see that the value  $K_r = C_r^{\frac{1}{3}} C_X^{\frac{2}{3}}$ in Proposition [4.7](#page-14-6) is bounded from above by

$$
K_r = C_r^{\frac{1}{3}} C_3^{\frac{2}{3}} \le \frac{Q_r^{\frac{1}{3}}}{\sqrt{n+n'}} \cdot C_3^{\frac{2}{3}} = \frac{K}{\sqrt{n+n'}},
$$

where

$$
K := Q_r^{\frac{1}{3}} C_3^{\frac{2}{3}}
$$

.

The conclusion now immediately follows from Proposition [4.7,](#page-14-6) applied to the random variables  $X, Y$ , given by [\(34\)](#page-13-1). □

### <span id="page-16-2"></span>5. Proof of the main result

<span id="page-16-0"></span>Joining the results of the previous sections, we obtain the following proposition:

<span id="page-16-1"></span>**Proposition 5.1.** In the assumptions of Theorem [1.4,](#page-2-1) there exist sequences  $\rho_n \rightarrow$  $\infty$ ,  $\delta_n \to 0$ , such that for any  $n \geq n_0$  and any  $\mu_1, \ldots, \mu_n \in \mathcal{K}$  $(46)$  N  $'_{\rho_n}(\xi_n) \leq \delta_n.$ 

$$
\int_{0}^{2\pi} \rho_n(\mathbf{S}u) = \int_{0}^{2\pi} u \cdot \mathbf{S}u
$$

Prior to proving it, note that our main result follows from it almost immediately:

Proof of Theorem [1.4.](#page-2-1) Assume that the assumptions of Theorem [1.4](#page-2-1) are satisfied. Due to Proposition [5.1,](#page-16-1) for any  $\rho > 0$  we have

$$
\lim_{n \to \infty} N'_{\rho}(\xi_n) = \lim_{n \to \infty} N_{\rho}(\eta_n) = 0,
$$

where

$$
\eta_n = \frac{\xi_n - \mathbb{E}\,\xi_n}{\sqrt{\text{Var}\,\xi_n}}.
$$

In particular, the characteristic functions  $\varphi_{\eta_n}(t)$  of the normalized variables converge uniformly on compact sets to  $e^{-\frac{t^2}{2}}$ . As the weak convergence of random variables is equivalent (Lévy's continuity theorem) to the pointwise convergence of their characteristic functions, we have the desired weak convergence

$$
\eta_n = \frac{\xi_n - \mathbb{E}\,\xi_n}{\sqrt{\text{Var}\,\xi_n}} \to \mathcal{N}(0,1), \quad n \to \infty.
$$

Moreover, this convergence is actually uniform in the choice of the sequence of measures  $\mu_1, \ldots, \mu_n, \cdots \in \mathcal{K}$ .

*Proof of Proposition [5.1.](#page-16-1)* We construct the sequence  $(\rho_n, \delta_n)_{n \geq n_0}$  so that the desired property  $(46)$  can be established by induction on n. Namely, we have the following

<span id="page-16-3"></span>**Lemma 5.2.** Let the sequence  $(\rho_n, \delta_n)_{n \geq n_0}$  be chosen in such a way that the following conditions hold:

- As  $n \to \infty$ , one has  $\rho_n \to \infty$  and  $\delta_n \to 0$ .
- For some  $n_1 \geq 2n_0$ , we have

$$
\rho_n = \rho'_0, \quad \delta_n = 3C_3 \quad \text{for all} \quad n = n_0, \dots, n_1,
$$

where

$$
\rho_0' = \min\left(\rho_0, \left(\frac{3C_3}{100}\right)^{1/3}\right)
$$

and  $\rho_0$  is given by Lemma [4.3.](#page-11-6)

• For every  $m > 2n_0$ , taking  $n = \lceil \frac{m}{2} \rceil$  and  $n' = m - n$ , one has

<span id="page-17-0"></span>(47) 
$$
\rho_m \leq L \min(\rho_n, \rho_{n'}),
$$

<span id="page-17-2"></span>(48) 
$$
\rho_m^3 \delta_m \le \frac{1}{100},
$$

<span id="page-17-1"></span>(49) 
$$
\left(\lambda \max(\delta_n, \delta_{n'}) + 2 \frac{Ke^{\frac{\rho_m^2}{2}}}{\sqrt{m}}\right) \leq \delta_m,
$$

where constants L and  $\lambda$  are defined by the conclusion of Lemma [4.4](#page-12-2) for  $C = 2\frac{C_2}{C_1}$  with  $C_1, C_2$  given by [\(4\)](#page-2-2).

Then the conclusion of Proposition [5.1](#page-16-1) holds for this sequence.

*Proof.* The proof of [\(46\)](#page-16-2) is by induction. Namely, for  $m = n_0, \ldots, n_1$  the conclusion follows from the choice of  $\rho'_0$  and Lemma [4.3.](#page-11-6) Let us make the induction step: for  $m > n_1$  let  $n = \lceil \frac{m}{2} \rceil$  and  $n' = m - n$ . Due to the induction assumption,

$$
N'_{\rho_n}(\xi_n) \le \delta_n, \quad N'_{\rho_n}(\xi_{(n,n+n')}) \le \delta_{n'},
$$

and thus due to Lemma [4.4](#page-12-2) and the inequality [\(47\)](#page-17-0)

$$
N'_{\rho_m}(\theta_{n,n'})=N'_{\rho_m}(\xi_n+\xi_{(n,n+n')})\leq \lambda \max(\delta_n,\delta_{n'}).
$$

Now, let us apply Corollary [4.8](#page-15-0) for  $\rho_m$ ,  $n, n'$ . First, check that the assumptions of Corollary [4.8](#page-15-0) are satisfied. Indeed,

$$
N_{\rho_m}(\eta_{n,n'}) = N'_{\rho_m}(\theta_{n,n'}) \le \lambda \max(\delta_n, \delta_{n'}) \le \delta_m
$$

due to [\(49\)](#page-17-1); multiplying by  $\rho_m^3$  and applying [\(48\)](#page-17-2), we get

$$
\rho_m^3 N'_{\rho_m}(\theta_{n,n'}) \le \frac{1}{100},
$$

what proves  $(44)$ . Next,  $(45)$  follows again from  $(49)$  and  $(48)$ :

$$
\frac{K}{\sqrt{m}}\rho_m^3e^{\frac{\rho_m^2}{2}}\leq \rho_m^3\delta_m\leq \frac{1}{100}.
$$

Corollary [4.8](#page-15-0) is applicable, and hence (again using [\(49\)](#page-17-1)) we get

$$
N'_{\rho_m}(\xi_m) = N_{\rho_m}(\eta_m) \le \left(\lambda \max(\delta_n, \delta_{n'}) + 2\frac{Ke^{\frac{\rho_m^2}{2}}}{\sqrt{m}}\right) \le \delta_m.
$$

The induction step is complete.

To complete the proof of Proposition [5.1,](#page-16-1) it remains to construct the sequences

$$
\rho_m \to \infty, \quad \delta_m \to 0
$$

that satisfy the assumptions of Lemma [5.2.](#page-16-3) Roughly speaking, the contraction with the factor  $\lambda$  effectively allows to bring  $\delta_m$  to zero as the additional term  $\frac{2Ke^{\frac{\rho_m^2}{2}}}{\sqrt{m}}$ tends to zero. It suffices to make the radii  $\rho_m$  increase extremely slow, so that the exponent  $e^{\frac{\rho_m^2}{2}}$  would not break this asymptotic vanishing.

We will choose  $\rho_m$  so that

$$
\rho_m \le \frac{1}{2} \sqrt{\log m};
$$

$$
\sqcup
$$

such a restriction already allows to note for checking [\(48\)](#page-17-2), [\(49\)](#page-17-1) that

<span id="page-18-2"></span>
$$
\frac{Ke^{\frac{\rho_m^2}{2}}}{m^{1/2}} < \frac{Km^{1/8}}{m^{1/2}} < \frac{K}{m^{1/4}}
$$

We let

(50) 
$$
\delta_m = Am^{-\beta}, \quad m > n_1,
$$

where the constant A is chosen so that at  $m = n_1$  this value coincides with  $3C_3$ ,

(51) 
$$
A = 3C_3 \cdot n_1^{\beta},
$$

and the (sufficiently small) power  $\beta > 0$  and the (sufficiently large) initial index  $n_1$ are yet to be fixed.

Now, choose the exponent  $\beta > 0$  sufficiently small so that

<span id="page-18-0"></span>
$$
\lambda \cdot 2^{\beta} < 1, \quad \beta < \frac{1}{4},
$$

and fix  $\lambda' \in (2^{\beta} \lambda, 1)$ .

Then, for all sufficiently large  $n_1$  the condition [\(49\)](#page-17-1) holds and can be proved by induction. Indeed, in the left hand side the first summand is

$$
\lambda \max(\delta_n, \delta_{n'}) \le \lambda \cdot A \left(\frac{m-1}{2}\right)^{-\beta} = 2^{-\beta} \lambda \cdot Am^{-\beta} \cdot \left(\frac{m-1}{m}\right)^{-\beta} < \lambda' \delta_m.
$$

The second summand is at most  $K \cdot m^{-\frac{1}{4}}$ , thus it suffices to check for  $m > n_1$  the inequality

<span id="page-18-1"></span>
$$
\lambda' A m^{-\beta} + K m^{-\frac{1}{4}} < A m^{-\beta},
$$

or, equivalently,

(52) 
$$
(1 - \lambda')Am^{-\beta} > Km^{-\frac{1}{4}}.
$$

As  $\beta < \frac{1}{4}$ , it suffices to check it for  $m = n_1$  (recall that [\(51\)](#page-18-0) is used to determine A for given  $n_1$  and  $\beta$ ). Indeed, [\(52\)](#page-18-1) holds once  $n_1$  is sufficiently large to ensure

$$
(1 - \lambda')3C_3 > Kn_1^{-\frac{1}{4}}
$$

We fix a sufficiently large  $n_1$  so that [\(52\)](#page-18-1) holds, fix the corresponding A (defined by [\(51\)](#page-18-0)) and the sequence  $(\delta_m)$ , defined for  $m > n_1$  by [\(50\)](#page-18-2). Then, we use [\(48\)](#page-17-2) and [\(47\)](#page-17-0) to choose the sequence  $(\rho_m)$ . Namely, for  $m > n_1$  we let

(53) 
$$
\rho_m = \min\left(L\min(\rho_n, \rho_{n'}), \frac{1}{2}\sqrt{\log m}, (100\delta_m)^{-1/3}\right)
$$

Then the inequality  $\rho_m \leq (100\delta_m)^{-1/3}$  implies [\(48\)](#page-17-2), and [\(47\)](#page-17-0) is satisfied automatically. Finally, as

.

<span id="page-18-3"></span>
$$
\min\left(\frac{1}{2}\sqrt{\log m}, (100\delta_m)^{-1/3}\right) \to \infty, \quad m \to \infty,
$$

the sequence  $(\rho_m)$  defined by [\(53\)](#page-18-3) also tends to infinity; actually, it will coincide with  $\frac{1}{2}\sqrt{\log m}$  for all sufficiently large m.

For the sequences  $(\rho_m, \delta_m)$ , the conditions of Lemma [5.2](#page-16-3) are satisfied, and this completes the proof of our main result.  $\Box$ 

#### 6. Unboundedness of Variance: proof of Proposition [3.2](#page-9-2)

<span id="page-19-0"></span>In this section we prove Proposition [3.2,](#page-9-2) i.e. show that under the assumptions of Theorem [1.4](#page-2-1) variances of  $\xi_n$  become arbitrarily large.

In order to do so, we will assume that  $n$  is quite large, and will decompose the full product  $A_n \dots A_1$  into a several "long" groups  $D_{m+1}, \dots, D_1$ , between which some "short" compositions are applied:

$$
A_n \dots A_1 = D_{m+1}(B_{\mu_{n_0,m}} \dots B_{\mu_{1,m}}) D_m \dots D_2(B_{\mu_{n_0,1}} \dots B_{\mu_{1,1}}) D_1.
$$

We will show that (for an appropriate choice of lengths) even conditionally to all  $D_1, \ldots, D_{m+1}$ , the distribution of the log-norm of the product (with high probability) has sufficiently high variance. At the same time, dividing by the product of norms of  $D_j$ , we get the composition

$$
\frac{D_{m+1}}{\|D_{m+1}\|}(B_{\mu_{n_0,m}}\ldots B_{\mu_{1,m}})\frac{D_m}{\|D_m\|}\ldots\frac{D_2}{\|D_2\|}(B_{\mu_{n_0,1}}\ldots B_{\mu_{1,1}})\frac{D_1}{\|D_1\|},
$$

where all the quotients  $\frac{D_j}{\|D_j\|}$  are almost rank-one matrices.

Therefore, we first consider the variance of a distribution of images of a given vector under random linear maps of rank one. In this case it is easier to show that the variance grows, see Lemma [6.1](#page-19-1) and Lemma [6.6](#page-21-0) below. By continuity, if one replaces random rank one linear maps by random linear maps of large norm, and uses the fact that for a matrix  $D \in SL(2,\mathbb{R})$  with large norm,  $\frac{D}{\|D\|}$  is close to a linear map of rank one and norm one, then a lower bound on variances still holds, see Lemma [6.7](#page-22-0) and Corollary [6.8.](#page-22-1) Finally, we can complete the proof of Proposition [3.2](#page-9-2) by applying the fact that with large probability a composition of a long enough sequence of random  $SL(2,\mathbb{R})$  matrices has a large norm.

Let us now realize this strategy.

Let  $Y \subseteq GL_2(\mathbb{R})$  be the space of all linear maps  $\mathbb{R}^2 \to \mathbb{R}^2$  of norm 1 and of rank 1. Notice that Y is homeomorphic to the torus  $\mathbb{T}^2$ ; indeed, it follows from the fact that any such map can be represented as a composition of an orthogonal projection to a one-dimensional subspace and a rotation.

<span id="page-19-1"></span>**Lemma 6.1.** There exist  $\varepsilon_0 > 0$  and  $n_0 \in \mathbb{N}$  such that for any non-zero vector  $v \in \mathbb{R}^2$ , any  $p \in Y$ , and any  $\mu_1, \mu_2, \ldots, \mu_{n_0} \in \mathcal{K}$  we have

$$
\text{Var}\log |p \circ (B_{\mu_{n_0}} \dots B_{\mu_1})v| \ge \varepsilon_0.
$$

To prove Lemma [6.1](#page-19-1) we will use a statement from [GK] that was called Atom Dissolving Theorem there. We will start with a couple of definitions.

**Definition 6.2.** Denote by  $\mathfrak{M}_{\mathfrak{a}\mathfrak{x}}(\nu)$  the weight of a maximal atom of a probability measure  $\nu$ . In particular, if  $\nu$  has no atoms, then  $\mathfrak{M}(\nu) = 0$ .

**Definition 6.3.** Let X be a metric compact. For a measure  $\mu$  on the space of homeomorphisms  $Homeo(X)$ , we say that there is

- no finite set with a deterministic image, if there are no two finite sets  $F, F' \subset X$  such that  $f(F) = F'$  for  $\mu$ -a.e.  $f \in \text{Homeo}(X)$ ;
- no measure with a deterministic image, if there are no two probability measures  $\nu, \nu'$  on X such that  $f_*\nu = \nu'$  for  $\mu$ -a.e.  $f \in \text{Homeo}(X)$ .

The following statement is a general statement for non-stationary dynamics, ensuring the "dissolving of atoms": decrease of the probability of a given point being sent to any particular point.

<span id="page-20-0"></span>**Theorem 6.4** (Atoms Dissolving Theorem 2.8 from [GK]). Let  $\mathbf{K}_X$  be a compact set of probability measures on  $Homeo(X)$ .

• Assume that for any  $\mu \in \mathbf{K}_X$  there is no finite set with a deterministic image. Then for any  $\varepsilon > 0$  there exists n such that for any probability measure  $\nu$  on X and any sequence  $\mu_1, \ldots, \mu_n \in \mathbf{K}_X$  we have

$$
\mathfrak{Mar}\left(\mu_{n} * \cdots * \mu_1 * \nu\right) < \varepsilon.
$$

In particular, for any probability measure  $\nu$  on X and any sequence  $\mu_1, \mu_2, \ldots \in \mathbf{K}_X$  we have

$$
\lim_{n\to\infty} \mathfrak{Mar}\left(\mu_n * \cdots * \mu_1 * \nu\right) = 0.
$$

• If, moreover, for any  $\mu \in \mathbf{K}_X$  there is no measure with a deterministic image, then the convergence is exponential and uniform over all sequences  $\mu_1, \mu_2, \ldots$  from  $K^{\mathbb{N}}$  and all probability measures  $\nu$ . That is, there exists  $\lambda < 1$  such that for any n, any v and any  $\mu_1, \mu_2, \dots \in \mathbf{K}_X$ 

$$
\mathfrak{Mar}\left(\mu_{n} * \cdots * \mu_1 * \nu\right) < \lambda^n.
$$

In the proof below we will only be using the first part of Theorem [6.4.](#page-20-0)

Proof of Lemma [6.1.](#page-19-1) Due to Theorem [6.4](#page-20-0) and our assumptions regarding the measures from K, there exists  $n' \in \mathbb{N}$  such that for any  $\mu_1, \mu_2, \ldots, \mu_{n'} \in \mathcal{K}$  we have

$$
\mathfrak{Mar}\left(\mu_{n} * \cdots * \mu_1 * \nu\right) < \frac{1}{2}
$$

for any probability measure  $\nu$  on  $\mathbb{RP}^1$ . To prove Lemma [6.1](#page-19-1) it is enough to choose  $n_0 = n' + 1.$ 

Since

$$
\operatorname{Var} \log |p \circ (B_{\mu_n} \dots B_{\mu_1}) v| = \operatorname{Var} \log \left| p \circ (B_{\mu_n} \dots B_{\mu_1}) \frac{v}{|v|} \right|,
$$

without loss of generality we can assume that  $|v| = 1$  and, slightly abusing the notation, consider it an element of  $\mathbb{RP}^1$ . For given  $p \in Y$ ,  $v \in \mathbb{R}^2$ ,  $|v| = 1$ ,  $\{\mu_1, \mu_2, \ldots, \mu_{n_0}\} \in \mathcal{K}^{n_0}$  consider the probability distribution  $\chi$  on  $[0, +\infty)$  of the random images  $|p \circ (B_{\mu_{n_0}} \dots B_{\mu_1}) v|.$ 

<span id="page-20-1"></span>**Lemma 6.5.** The function  $\Phi : \mathbb{RP}^1 \times Y \times \mathcal{K}^{n_0} \to \mathbb{R} \cup {\infty}$  defined by

$$
\Phi(v, p, \mu_1, \mu_2, \dots, \mu_{n_0}) = \begin{cases} \infty, & \text{if } \chi(\{0\}) > 0; \\ \text{Var} \log |p \circ (B_{\mu_{n_0}} \dots B_{\mu_1}) v|, & \text{if } \chi(\{0\}) = 0, \end{cases}
$$

is lower semicontinuous.

*Proof.* Notice that  $\chi$  depends continuously on  $(v, p, \mu_1, \mu_2, \dots, \mu_{n_0})$  in weak-\* topology.

Let us consider the cases when  $\chi(\{0\}) > 0$  and when  $\chi(\{0\}) = 0$  separately.

Assume first that  $\chi(\{0\}) > 0$ . We want to show that given  $M > 0$ , for any sufficiently small perturbation  $\chi'$  of  $\chi$  we have  $Var \log \chi' > M$ . Notice that the measures condition implies that  $\chi$  cannot be concentrated exclusively at  $0 \in \mathbb{R}$ . Hence for some  $\tau > 0$  we have  $\chi[\tau, +\infty) > 0$ . If  $\chi'$  is a probability distribution that is sufficiently close to  $\chi$ , then  $\chi'[\tau/2, +\infty)$  is not less than  $\frac{1}{2}\chi[\tau, +\infty)$ , and the  $\chi'$ -weight of a small neighborhood of the origin is at least  $\frac{1}{2}\chi(\{0\})$ . Choosing that neighborhood small enough guarantees that  $Var \log \chi' > M$ .

Assume now that  $\chi(\{0\}) = 0$  and  $\text{Var} \log \chi < \infty$ . Then

$$
Var \log \chi = \lim_{T \to \infty} Var \left[ (\log \chi)|_{[-T,T]} \right],
$$

so for any  $\varepsilon > 0$ , for some large enough  $T > 0$  we have

$$
\operatorname{Var}\left[ (\log \chi)|_{[-T,T]} \right] > \operatorname{Var}\log \chi - \frac{\varepsilon}{2}.
$$

Therefore, for any  $\chi'$  that is sufficiently close to  $\chi$  we have

$$
\mathrm{Var} \log \chi' \geq \mathrm{Var} \left[ ( \log \chi')|_{[-2T,2T]} \right] \geq \mathrm{Var} \left[ ( \log \chi)|_{[-T,T]} \right] - \frac{\varepsilon}{2} > \mathrm{Var} \log \chi - \varepsilon.
$$

The case when  $\chi({0}) = 0$  and Var log  $\chi = \infty$  can be treated similarly.

The space  $\mathbb{RP}^1 \times Y \times \mathcal{K}^{n_0}$  is compact. Hence, Lemma [6.5](#page-20-1) implies that it is enough to show that  $\Phi > 0$  to ensure that for some  $\varepsilon_0 > 0$  we have  $\Phi \geq \varepsilon_0 > 0$ .

Suppose this is not the case, and for some unit vector v, a linear map  $p \in Y$ , and  $\mu_1, \ldots, \mu_{n_0} \in \mathcal{K}$  we have

$$
Var \log |p \circ (B_{\mu_{n_0}} \dots B_{\mu_1})v| = 0.
$$

Then for some  $d \geq 0$  with probability 1 we have  $|p \circ (B_{\mu_{n_0}} \dots B_{\mu_1})v| = d$ . That means that  $B_{\mu_{n_0}} \dots B_{\mu_1} v$  has to belong to  $L = \{u \mid |p(u)| = d\}$ , which is a line (if  $d = 0$ ) or a union of two lines (if  $d > 0$ ). This implies that  $\mu_1 \times \mu_2 \times \ldots \times \mu_{n_0-1}$ -almost surely the image  $B_{\mu_{n_0-1}} \dots B_{\mu_1} v$  must belong to the set  $\cap_{A \in \text{supp}(\mu_{n_0})} A^{-1}(L)$ . Since



FIGURE 2. The set  $L$  and its preimages

the measure  $\mu_{n_0}$  must satisfy the measure condition, this intersection must consists at most of four points, whose projectivization gives at most two points on  $\mathbb{RP}^1$ . But this would imply that if  $\nu$  is an atomic measure on  $\mathbb{RP}^1$  at the point corresponding to the initial vector v, then  $\mu_{n_0-1} * \cdots * \mu_1 * \nu$  is a measure supported on at most two points, which contradicts the choice of  $n_0$  above. This completes the proof of Lemma [6.1.](#page-19-1)  $\Box$ 

<span id="page-21-0"></span>**Lemma 6.6.** For any  $m \in \mathbb{N}$ , any  $\{\mu_{1,i}, \ldots, \mu_{n_0,i}\}_{i=1,\ldots,m} \in \mathcal{K}^{mn_0}$ , and any  $\{p_1, \ldots, p_{m+1}\} \in Y^{m+1}$  we have

Var log  $||p_{m+1}(B_{\mu_{n_0,m}}\dots B_{\mu_{1,m}})p_m \dots p_2(B_{\mu_{n_0,1}}\dots B_{\mu_{1,1}})p_1|| \geq \varepsilon_0 m.$ 

*Proof.* As each  $p_i$  is a unit norm rank 1 matrix, it can be written as

$$
p_j = v_j \otimes \ell_j
$$
, where  $v_j \in \mathbb{R}^2$ ,  $\ell_j \in (\mathbb{R}^2)^*$ ,  $|v_j| = |\ell_j| = 1$ .

Now, let

$$
\tilde{B}_j := B_{\mu_{n_0,j}} \dots B_{\mu_{1,j}}
$$

be the j-th intermediate product. Then for the product

$$
T=p_{m+1}\tilde{B}_m p_m \dots p_2 \tilde{B}_1 p_1
$$

one has for any  $v \in \mathbb{R}^2$ 

<span id="page-22-2"></span>
$$
T(v) = v_{m+1} \cdot \ell_{m+1}(\tilde{B}_m v_m) \cdot \cdots \cdot \ell_2(\tilde{B}_1 v_1) \cdot \ell_1(v),
$$

and hence

(54) 
$$
\log ||T|| = \sum_{j=1}^{m} \log |\ell_{j+1}(\tilde{B}_j v_j)|.
$$

Right hand side of  $(54)$  is a sum of m independent random variables, and the variance of each of them is at least  $\varepsilon_0$  due to Lemma [6.1.](#page-19-1) Thus, the variance of  $\log ||T||$  is at least  $m\varepsilon_0$ .

<span id="page-22-0"></span>**Lemma 6.7.** There exists a neighborhood U of the compact  $\mathcal{K}^{n_0 m} \times Y^{m+1}$  in  $\mathcal{K}^{n_0 m} \times$  $Mat_2(\mathbb{R})^{m+1}$  such that for any

$$
\bar{\mu}\times\{D_j\}_{j=1,\ldots,m+1}\in U,
$$

where  $\bar{\mu} = {\mu_{1,i}, \ldots, \mu_{n_0,i}}_{i=1,\ldots,m}$  and  $D_j \in Mat_2(\mathbb{R})$ , we have

$$
\text{Var}\log \|D_{m+1}(B_{\mu_{n_0,m}}\ldots B_{\mu_{1,m}})D_m\ldots D_2(B_{\mu_{n_0,1}}\ldots B_{\mu_{1,1}})D_1\| \geq \frac{\varepsilon_0 m}{2}
$$

*Proof.* On  $\mathcal{K}^{n_0 m} \times Y^{m+1}$  this variance is bounded from below by  $\varepsilon_0 m$  due to Lemma [6.6.](#page-21-0) As the set  $\mathcal{K}^{n_0 m} \times Y^{m+1}$  is compact, and the variance is a lowersemicontinuous function of a distribution, there exists a neighbourhood  $U$  of this compact on which the variance is at least  $\frac{m\varepsilon_0}{2}$ . □

<span id="page-22-1"></span>**Corollary 6.8.** There exists Q such that for any  $D_1, \ldots, D_{m+1} \in SL(2, \mathbb{R})$  with  $||D_j|| \geq Q, j = 1, \ldots, m + 1$  one has

$$
\text{Var}\log \|D_{m+1}(B_{\mu_{n_0,m}}\dots B_{\mu_{1,m}})D_m\dots D_2(B_{\mu_{n_0,1}}\dots B_{\mu_{1,1}})D_1\| \geq \frac{\varepsilon_0 m}{2}
$$

Proof.

(55) 
$$
\log ||D_{m+1}(B_{\mu_{n_0,m}}\dots B_{\mu_{1,m}})D_m \dots D_2(B_{\mu_{n_0,1}}\dots B_{\mu_{1,1}})D_1|| =
$$
  
= 
$$
\log ||\tilde{D}_{m+1}(B_{\mu_{n_0,m}}\dots B_{\mu_{1,m}})\tilde{D}_m \dots \tilde{D}_2(B_{\mu_{n_0,1}}\dots B_{\mu_{1,1}})\tilde{D}_1|| + \sum_{j=1}^m \log ||D_j||,
$$

where  $\tilde{D}_j := \frac{D_j}{\| D_j \|}$  $\frac{D_j}{\|D_j\|}$ . On the other hand, as  $||D|| \to \infty$  for  $D \in SL(2,\mathbb{R})$ , one has  $\frac{D}{\|D\|} \to Y$ , so it suffices to choose Q sufficiently large to ensure that

$$
(\{\mu_{i,k}\}_{1\leq i\leq n_0, 1\leq k\leq m}, (\tilde{D}_1, \ldots, \tilde{D}_{m+1})) \in U
$$

once  $||D_j|| \ge Q$  for all  $j = 1, ..., m + 1$ , where U is provided by Lemma [6.7.](#page-22-0) □

*Proof of Proposition [3.2.](#page-9-2)* First, fix  $n_0$  and  $\varepsilon_0$  given by Lemma [6.1.](#page-19-1) Then, choose and fix m such that  $\frac{m\varepsilon_0}{4} > c$ .

Now, take a sufficiently large  $Q$  provided by Corollary [6.8.](#page-22-1) It follows from [\[G,](#page-23-12) Theorem 2.2] that for a sufficiently large  $n_2$  one has

$$
\forall n' \geq n_2 \quad \forall \mu_1, \ldots, \mu_{n'} \in \mathcal{K} \quad \mathbb{P}_{\mu_1, \ldots, \mu_{n'}}(\|A_{n'} \ldots A_1\| \geq Q) \geq 1 - \frac{1}{2(m+1)}.
$$

Now, take  $n_3 := n_2(m+1) + n_0m$ . Then, for any  $n \ge n_3$  and any  $\mu_1, \ldots, \mu_n \in \mathcal{K}$ we can decompose the product  $A_n \dots A_1$  as

$$
A_n \dots A_1 = D_{m+1} \tilde{B}_m D_m \dots D_2 \tilde{B}_1 D_1,
$$

where each  $D_j$  is a product of at least  $n_2$  matrices  $A_i$ , and each  $\tilde{B}_j$  is a product of  $n_0$  of  $A_i$ 's.

This implies that with the probability at least  $\frac{1}{2}$  one has  $||D_j|| \ge Q$  for all j, and hence the variance of the distribution conditional to such  $D_j$  is at least  $\frac{m\varepsilon_0}{2}$ . Thus, we finally have

$$
\text{Var}\,\xi_n \geq \mathbb{E}_{D_1,...D_{m+1}} \text{Var}(\xi_n \mid D_1,... D_{m+1}) \geq \frac{1}{2} \cdot \frac{m\varepsilon_0}{2} = \frac{m\varepsilon_0}{4} > c.
$$

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