Nonassociative gauge gravity theories with R-flux star products and Batalin-Vilkovisky quantization in algebraic quantum field theory

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November 8, 2024

Abstract

Nonassociative modifications of general relativity, GR, and quantum gravity, QG, models naturally arise as star product and R-flux deformations considered in string/ M-theory. Such nonassociative and noncommutative geometric and quantum information theories were formulated on phase spaces defined as cotangent Lorentz bundles enabled with nonassociative symmetric and nonsymmetric metrics and nonlinear and linear connection structures. We outline the analytic methods and proofs that corresponding geometric flow evolution and dynamical field equations can be decoupled and integrated in certain general off-diagonal forms. New classes of solutions describing nonassociative black holes, wormholes, and locally anisotropic cosmological configurations are constructed using such methods. We develop the Batalin-Vilkovisky, BV, formalism for quantizing modified gravity theories, MGTs, involving twisted star products and semi-classical models of nonassociative gauge gravity with de Sitter/affine/ Poincaré double structure groups. Such theories can be projected on Lorentz spacetime manifolds in certain forms equivalent to GR or MGTs with torsion generalizations etc. We study the properties of the classical and quantum BV operators for nonassociative phase spaces and nonassociative gauge gravity. Recent results and methods from algebraic QFT are generalized to involve nonassociative star product deformations of the anomalous master Ward identity. Such constructions are elaborated in a nonassociative BV perspective and for developing non-perturbative methods in QG.

Keywords: Nonassociative geometry and strings; quantum gauge gravity; nonassociative star products and R-flux; Batalin-Vilkovisky formalism; algebraic quantum field theories.

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1 Introduction

Quantization of gravity is one of the big open problems in modern physics. In the most general and rigorous mathematical form, and in the framework of a unification theory, the theory of quantum gravity, QG, is approached in string/ M-theory [1, 2, 3]. Certain models of nonassociative and noncommutative modified gravity theories, MGTs, are expected to arise in low-energy limits of string theories [4, 5, 6, 7]. For so-called R-flux deformations, such MGTs are constructed using the concept of twisted star product, \star , [8, 9]. We follow an approach when \star -products are defined on nonassociative phase spaces in certain forms extending Einstein's gravity, i.e. the general relativity, GR, theory [10, 11]. Nonassociative and noncommutative algebraic, geometric and topological structures are studied in modern mathematics. Various applications are considered in quantum field theories, QFTs, and used for elaborating new methods of quantization.

In a series of partner works [12, 13, 14, 15, 16], we proved that physically important nonassociative MGTs formulated on phase spaces are described by systems of nonlinear partial differential equations, PDEs, which can be decoupled and integrated in certain general off-diagonal forms. This is possible if the so-called anholonomic frame and connection deformation method, AFCDM, is applied. We constructed and studied physical properties of new classes of black holes, BHs; wormholes, WHs; and locally anisotropic cosmological solutions encoding nonassociative data. Nonassociative phase space off-diagonal solutions and respective physical models typically are not characterized by thermodynamic variables formulated in the framework of the Bekenstein-Hawking BH paradigm [17, 18]. This is because, in general, such configurations do not involve certain conventional horizons or holographic conditions. Nevertheless, we can formulate a new type of nonassociative geometric flow thermodynamics [13, 14, 15, 16]. It generalizes G. Perelman's constructions defining a new type the statistical

and geometric thermodynamics for Ricci flows of Riemannian metrics [19]. Here, we note that in our works the aim is not to formulate and proof any nonassociative variant of the Poincaré–Thurston conjecture. This is a very difficult and ambiguous problem in modern mathematics because of existence of different types of nonassociative and noncommutative calculi. In (pseudo) Riemannian geometric the differential and integral calculus is defined in a unique way. We develop certain approaches when generalized G. Perelman thermodynamic variables are defined in MGTs and used for characterizing off-diagonal solutions encoding, for instance, nonassociative data. Such models can be quantized by elaborating new methods of quantization.

Considering nontrivial nonassociative and noncommutative algebraic and geometric structures on phase spaces, we substantially modify not only the GR theory and the standard models in QFT. Such modifications request the elaboration of a new mathematical formalism and application of nonholonomic geometric and quantum information methods resulting in nonassociative and noncommutative models of QG. Here we note that the theories with a general twisted star product are non-variational but can be formulated in abstract geometric forms. For applications in modern cosmology and astrophysics, we can consider any convenient approach with variational, abstract geometric and effective field theories. The corresponding fundamental equations can be formulated in certain parametric forms (for instance, using linear decompositions on the Planck and string constants) which allows us to encode nonassociative and noncommutative data.

In a partner work [16], a nonassociative gauge gravity model with double affine or de Sitter gauge structure groups on cotangent Lorentz bundles were formulated. For projections on nonassociative phase space bases, those constructions reproduce the results on nonassociative MGTs from [12, 13, 14, 15]. Here we note that possible implications of nonassociative geometric and classical gravity theories distinguishing specific properties of off-diagonal parametric and physically important solutions were not studied in modern literature on QFT and QG. Our nonassociative and nonholonomic gauge gravity approach is motivated by the fact that using a corresponding class of nonholonomic distributions, the nonassociative Yang-Mills, YM, equations can be decoupled and solved in very general forms and then quantized using standard methods for gauge field theories. The modified YM equations can be formulated as some equivalents of nonassociative star product deformed Einstein equations. The projections of nonassociative gravitational YM equations on Lorentz spacetime manifolds transform into standard Einstein equations. Such equations which may encode, or not, certain nonassociative and noncommutative data defining certain effective sources. In a more general context, we can elaborate on nonassociative gauge gravity models with nontrivial torsion and nonmetricity fields and try to quantize such nonassociative MGTs using (non) perturbative methods formulated for quantizing gauge theories.

To quantize associative and commutative theories with local symmetries (including gauge theories and various MGTs) is convenient to use the Bechi, Rouet, Stora and Tyutin, BRST, method [20, 21, 22], see recent results and references in [23]. It is not clear how this method can be extended in general variational or nonvariational forms for nonassociative and noncommutative gauge theories. Nevertheless, we can elaborate on an abstract geometric formalism for nonassociative gauge and gravity fields and perform quantization of physically important quasi-stationary or locally anisotropic cosmological configurations. Such configurations are determined by off-diagonal solutions with corresponding types of nonlinear symmetries which are additional to the corresponding types of gauge symmetries or diffeomorphisms. This also supposes adding effective and auxiliary fields (ghosts, anti-ghosts, etc.) in nonassociative geometric form, and even an infinite number of differential and integral nonassociative and noncommutative calculi (and various types of geometries) can be elaborated. Using nonholonomic geometric methods, nonassociative geometric flow and gravitational and matter field equations can be postulated following algebraic and geometric principles, when parametric decompositions resulting in effective thermofield and quantum field and quantum evolution theories can be stated to possess a well-defined variational calculus. This refers, in particular, to the models with nonassociative star product R-flux if they are elaborated in parametric form as a nonassociative gauge gravity when the constructions are similar to standard BRST ones but with corresponding double affine or de Sitter structure groups.

A generalization of the BRST is known as the Batalin and Vilkovisky, BV, formalism [24]. We do not

provide a comprehensive list of references on further developments and applications of the BV formalism but cite [25, 26]. Those articles and references therei contain recent reviews and rigorous mathematical approaches to perturbative algebraic QFT (in brief, pAQFT). In this work, the focus is on conceptual problems for developing the BV formalism and certain methods of pAQFT to quantizing nonassociative gauge gravity theories and MGTs of type [16, 13, 15]. Our main goal is to show that the BV can be applied when the principles of locality, deformation and homology (formulated in [25, 26] for Lorentz spacetimes) are extended on nonassociative phase spaces modelled on cotangent Lorentz bundles. Here, we emphasize that the star products determined by R-flux involve non-local constructions which became effectively local for certain parametric decompositions. In such an approach, we naturally generalize the BV formalism in abstract nonassociative geometric form.

In this work, we develop a nonlinear functional approach to nonassociative gauge QFT and QG when explicit classes of generic off-diagonal parametric solutions are used for extending the BV formalism in nonholonomic form on nonassociative phase spaces. The constructions may encode certain prescribed classes of nonassociative and noncommutative algebraic and nonlinear geometric symmetries, parametric decompositions, effective gravitational polarizations and deformation of horizons (if such ones are prescribed for some classes of solutions). Such configurations define nontrivial gravitational vacuum structures when certain quasi-classical limits are used for developing perturbative and non-perturbative schemes of quantization. This goes beyond traditional schemes of quantization with certain completely determined by Lagrange or Hamilton, S-matrix methods etc. Those approaches are not related directly to the properties of physically important systems of nonlinear PDEs and the properties of off-diagonal solutions depending on various generating and integrating functions, physical constants, generating sources and corresponding (non) linear symmetries. We argue that nonassociative gauge gravity theories can be elaborated both in classical and quantum forms by reformulating the BV formalism and the mentioned three principles (of locality, deformation and homology). This is possible for pAQFTs defined on nonholonomic cotangent Lorentz bundles, i.e. on nonassociative phase spaces, endowed with parametric star product structure.

The paper is structured as follows: In section 2, we provide an introduction to (classical) nonassociative gauge gravity with star product and R-flux deformations and state the respective conditions for extracting associative and commutative gauge and gravity theories on phase spaces. We consider a general ansatz for generating quasi-stationary off-diagonal solutions in such theories and emphasize that corresponding configurations can be dualized (on a time-like coordinate) for constructing locally anisotropic cosmological models. The goal of section 3 is to study the nonassociative dynamics and nonlinear symmetries of the classical gauge de Sitter gravity and the definition of nonassociative classical BV operator and the Møller maps. Section 4 is devoted to the quantization of nonassociative gauge de Sitter gravity and related renormalization procedures using the quantum BV operator. Perturbative and non-perturbative methods for nonassociative 8-d BH configurations is analyzed. Conclusions and perspectives are considered in section 5. In the Appendix, we outline the necessary formulas for generating off-diagonal quasi-stationary solutions on nonassociative phase spaces.

2 Nonassociative gauge gravity models with star product

In this section, we outline necessary results on nonassociative MGTs and gauge gravity with twisted star products determined by R-flux deformations in string theory. Details in are provided in [12, 13, 15, 16] and references therein.

2.1 Nonassociative phase spaces modelled on cotangent Lorentz bundles

We begin with the geometry of commutative phase spaces, which can be constructed on tangent Lorentz bundles $\mathcal{M} = T\mathbf{V}$, or cotangent Lorentz bundles $\mathcal{M} = T^*\mathbf{V}$, of a Lorentzian spacetime manifold \mathbf{V} of signature (+ + + -). In this work, we shall consider only geometric and physical models defined on \mathcal{M} . Respective spacetime and momentum coordinates are denoted as $u = (x, p) = \{ u^{\alpha} = (x^i, p_a) \}$, for indices i, j, k, ... = 1, 2, 3, 4; a, b, c, ... = 5, 6, 7, 8 and $x^4 = t$, or $y^4 = t$, is a time like coordinate as for a base Lorentz spacetime. A N-connection structure is defined as a nonholonomic (equivalently, anholonomic, i.e. non-integrable) splitting, $\mathbf{N} : TT^*\mathbf{V} = hT^*\mathbf{V} \oplus cT^*\mathbf{V}$, where \oplus denotes the Whitney direct sum.¹ A metric structure $\mathbf{g} = (hg, cg) = \{ \mathbf{g}_{\alpha\beta} = (g_{ij}, \mathbf{g}^{ab}) \}$ on \mathcal{M} is of local signature (+ + + -; + + -). Here, "h" states a corresponding horizontal splitting and "c" is for a co-fiber, i.e. co-vertical splitting of the geometric objects and necessary indices on a cotangent bundle. N-connections can be introduced on phase spaces and spacetimes in arbitrary geometric form or for some additional assumptions that they are associated to certain nonlinear gauge fields; or related to certain off-diagonal terms of metrics; or to some linear N-elongations of local partial derivatives/ differentials and respective systems of reference.

If a phase space \mathcal{M} is enabled with a N-connection structure, we can introduce the concept of distinguished connection, d-connection $\mathbf{D} = (h \ D, c \ D)$. Such a d-connection is a linear connection which preserves a Nconnection structure (in our case, nonholonmic 4+4 splitting) under affine linear transports.² To apply the AFCDM for constructing exact and parametric solutions in MGTs is convenient to work with the so-called canonical d-connection \mathbf{D} . It is completely determined by the coefficients of \mathbf{g} and \mathbf{N} to be metric compatible, $\mathbf{D} \ \mathbf{g} = 0$, but contains a nontrivial d-torsion structure induced as a nonholonomic effect, when the canonical d-torsion tensor $\mathbf{T} \ = \{hh \ \mathbf{T} = 0; cc \ \mathbf{T} = 0, \text{ when } hc \ \mathbf{T} \neq 0\} \neq 0.^3$

To prove certain general decoupling and integration properties of modified Einstein equations and generating generic off-diagonal solutions encoding nonassociative or locally anisotropic data we had to develop also a nonholonomic dyadic, 2-d, shell by shell oriented decomposition formalism [12]. In such cases, we write ${}_{s}\mathcal{M}$ for a phase space ${}^{\prime}\mathcal{M}$ enabled with a conventional (2+2)+(2+2) splitting with four oriented shells s = 1, 2, 3, 4. For such s-decompositions, the N-connection is defined

$${}_{s}^{!}\mathbf{N}: {}_{s}T\mathbf{T}^{*}\mathbf{V} = {}^{1}hT^{*}V \oplus {}^{2}vT^{*}V \oplus {}^{3}cT^{*}V \oplus {}^{4}cT^{*}V, \text{ for } s = 1, 2, 3, 4.$$
(1)

In a local coordinate basis, the N-connection (1) is characterized by a corresponding set of coefficients ${}^{'}_{s}\mathbf{N} = \{ {}^{'}N_{i_{s}a_{s}}({}^{'}u)\}$ and such coefficients allow us to introduce certain N-elongated bases (N-/ s-adapted bases as linear N-operators):

Having prescribed a nonholonomic s-structure, we can express any metric or d-metric as a s-metric

$$\mathbf{g}_{s} = \{ \mathbf{g}_{\alpha_{s}\beta_{s}} \} = (h_{1} \mathbf{g}, v_{2} \mathbf{g}, c_{3} \mathbf{g}, c_{4} \mathbf{g}) \in T\mathbf{T}^{*}\mathbf{V} \otimes T\mathbf{T}^{*}\mathbf{V}$$

¹We use the duality label " " and "boldface" symbols to state the geometric constructions can be adapted to an N-connection splitting. For such N-adapted models, tensors transform into d-tensors, vectors transform into d-vectors and connections transform into d-connections, where "d" means distinguished by a N-connection h-c-splitting. Here we note that we developed in nonholonomic form for nonassociative geometry and gravity [12] an abstract (index and coordinate-free) geometric formalism when any N-connection can be defined equivalently as a nonholonomic, equivalently, anholonomic, or non-integrable distribution. Our approach generalizes for spaces with nontrivial N-connection structure the abstract and index formalism for GR [27]. Such an N-connection and adapted distortions of linear connections can be correspondingly introduced with the goal of decoupling certain systems of nonlinear PDEs.

²We note that a Levi Civita connection ∇ (in brief, LC-connection; by definition, it is metric compatible and torsionless) is not a d-connection because it is not adapted to a N-connection structure. Nevertheless, an N-adapted distortion formula $\mathbf{D} = \nabla + \mathbf{Z}$ can be defined; when \mathbf{Z} is the distortion d-tensor encoding contributions from respective torsion of \mathbf{D} , and non-metricity d-tensor, $\mathbf{Q} := \mathbf{Dg}$, for $\nabla \mathbf{g} = 0$. The abstract and coefficient formulas of d-adapted geometric objects (in general nonassociative form) can be found in [12, 13, 15].

³Such a d-torsion is different from that in the Einstein-Cartan theory or other torsions in string and gauge gravity. In our case, the data $(\mathbf{g}, \mathbf{N}, \mathbf{\hat{D}})$ can be considered as certain nonholonomic geometric variables which allow us to solve physically important systems of nonlinear PDEs. Using canonical distortion relations, all results can be re-defined equivalently for LC-configurations $(\mathbf{g}, \mathbf{\nabla})$ if it will be necessary.

when $[\mathbf{g}_{\alpha_s\beta_s}(\mathbf{s}^{\prime}u)] \mathbf{e}^{\alpha_s} \otimes_s [\mathbf{e}^{\beta_s} = \{ [\mathbf{g}_{\alpha_s\beta_s} = ([\mathbf{g}_{i_1j_1}, [\mathbf{g}_{a_2b_2}, [\mathbf{g}^{a_3b_3}, [\mathbf{g}^{a_4b_4})]\}, \text{ for } [\mathbf{e}^{\alpha_s}(2) \text{ chosen in s-adapted form. In this paper, we shall omit the bulk of shell index formulas and explicit geometric proofs of classical formulas, equations and solutions, which can be found in our partner works [12, 13, 14, 15, 16].$

A nonassociative (twisted) star product from [10, 11] can be defined in a form involving N-elongated differential operators \mathbf{e}_{i_s} (2) acting on some functions f(x,p) and q(x,p) defined on $\mathbf{e}_s \mathcal{M}$, see details in [12, 13]. Such a s-adapted star product \star_s can be computed as

$$f \star_{s} q := \cdot [\mathcal{F}_{s}^{-1}(f,q)]$$

$$= \cdot [\exp(-\frac{1}{2}i\hbar(\ \mathbf{e}_{i_{s}} \otimes \ \mathbf{e}^{i_{s}} - \ \mathbf{e}^{i_{s}} \otimes \ \mathbf{e}_{i_{s}}) + \frac{i\ell_{s}^{4}}{12\hbar}R^{i_{s}j_{s}a_{s}}(p_{a_{s}}\ \mathbf{e}_{i_{s}} \otimes \ \mathbf{e}_{j_{a}} - \ \mathbf{e}_{j_{s}} \otimes p_{a_{s}}\ \mathbf{e}_{i_{s}}))]f \otimes q$$

$$= f \cdot q - \frac{i}{2}\hbar[(\ \mathbf{e}_{i_{s}}f)(\ \mathbf{e}^{i_{s}}q) - (\ \mathbf{e}^{i_{s}}f)(\ \mathbf{e}_{i_{s}}q)] + \frac{i\ell_{s}^{4}}{6\hbar}R^{i_{s}j_{s}a_{s}}p_{a_{s}}(\ \mathbf{e}_{i_{s}}f)(\ \mathbf{e}_{j_{s}}q) + \dots$$

$$(3)$$

The antisymmetric coefficients $R^{i_s j_s a_s}$ define the nonassociative part of the star product (in brief, we can write \star or correspondingly \star_N). The string length constant ℓ characterizes the R-flux contributions from string theory. In formulas (3), the tensor product \otimes can be written also in an s-adapted form \otimes_s . Explicit computations for R-flux deformations of s-adapted geometric objects and (physical) equations, can be adapted and classified with respect to decompositions on small parameters \hbar (stating noncommutative properties) and $\kappa = \ell_s^3/6\hbar$. For parametric decompositions, all tensor products turn into usual multiplications which is very important for computing classical and quantum effects in theories encoding nonassociative data.

For any phase space ${}_{s}^{*}\mathcal{M}$, we can lift s-adapted geometric objects on the total space of a vector bundle ${}_{s}^{*}\mathcal{E}({}_{s}^{*}\mathcal{M})$ and call ${}_{s}^{*}\mathcal{E}$ a s-vector bundle, see details and references in [16]. A star product (3) deforms ${}_{s}^{*}\mathcal{E}({}_{s}^{*}\mathcal{M})$ on ${}_{s}^{*}\mathcal{M}$ into respective nonassociative ones labeled by a \star -symbol, ${}_{s}^{*}\mathcal{E}^{*}$ on ${}_{s}^{*}\mathcal{M}^{*}$. Such geometric constructions were considered in noncommutative form in [28, 29] for the so-called N-adapted Seiberg-Witten star product *, or $*_{N}$. Nevertheless, to introduce nonassociative structures is not just a formal changing of, for instance, $*_{N}$ into a \star_{s} (3) without R-flux terms. For nonassociative models with R-flux \star -deformations, $\star : \mathbf{g} \to \mathbf{g}^{\star} = (\mathbf{\breve{g}}^{\star}, \mathbf{\breve{g}}^{\star})$, the \star -metrics contain nonassociative symmetric, \breve{g}^{\star} , and nonassociative nonsymmetric, $\mathbf{\breve{g}}^{\star}$, components; the procedure of inverting metrics became nonlinear and sophisticated. In general, all components of geometric s-objects depend both on spacetime and momentum-like variables. Typically, the non-symmetry of metrics is not considered in noncommutative *-theories.

Abstract geometric and tedious index/coordinate computations of the fundamental geometric and physical objects on ${}_{s}^{*}\mathcal{M}^{\star}$ allow us to express all-important formulas for the "star" d-metrics, d-connections, d-torsions, d-curvatures, etc., into certain \hbar and κ -parametric forms which are provided in [12, 13]. Such computations can be considered for defining \star -versions of LC-connections, $\nabla \to \nabla^{\star}$; or star product deformations of arbitrary d-connections, $\mathbf{D} \to \mathbf{D}^{\star}$, or canonical s-connections, $\stackrel{[}{s}\mathbf{D} \to \stackrel{[}{s}\mathbf{D}^{\star}$, etc. Correspondingly, we can compute the parametric and s-adapted forms for a star product deformation of the Ricci tensor, or canonical s-tensor, $\mathcal{R}ic^{\star}[\mathbf{g}^{\star}, \nabla^{\star}]$ or $\widehat{\mathcal{R}}ic^{\star}[\mathbf{g}^{\star}, \mathbf{D}^{\star}]$ etc. The \hbar - and κ -parametric terms determined by \star deformations of pseudo-Riemannian metrics can be re-defined equivalently as certain effective sources encoding nonassociative/ noncommutative data from string theory.

2.2 Associative and commutative gauge gravity on phase spaces

We outline necessary results on a model of gauge gravity theory on \mathcal{M} when the gauge structure group is $\mathcal{G}r = (SO(4,1), SO(4,1))$. In this theory, the de Sitter group SO(4,1) may encode consequent nonlinear extensions of the affine structure group Af(4,1) and the Poincaré group ISO(3,1), see details and physical motivation in [28, 29], for commutative gauge gravity and supergravity, and a recent paper [16], for nonassociative generalizations. The commutative geometric parts of such phase space models involve commutative nonholonomic (co) vector bundle spaces

$${}^{\mathsf{I}}\mathcal{E}({}^{\mathsf{I}}\mathcal{M}) := \left({}^{\mathsf{I}}\mathcal{E} = h \, {}^{\mathsf{I}}\mathcal{E} \oplus c \, {}^{\mathsf{I}}\mathcal{E}, \, {}^{\mathsf{I}}\mathcal{G}r = SO(4,1) \oplus \, {}^{\mathsf{I}}SO(4,1), \, {}^{\mathsf{I}}\pi = (h\pi, c\pi), \, {}^{\mathsf{I}}\mathcal{M}\right),\tag{4}$$

which can be associated to respective tangent bundles $T(\ \mathcal{M})$ and co-tangent bundles $T^*(\ \mathcal{M})$. Such d- and s-vector bundles are enabled with N-adapted projections π and $\ \pi$, when brief notations like $\ \mathcal{E}$ or $\mathcal{E} = \mathcal{E}(\mathcal{M})$ are used. In the above formula, the group $\ SO(4,1)$ is isomorphic to SO(4,1) but may have different parameterizations corresponding to different types of spacetime coordinates and co-fiber momentum-like variables.

A canonical de Sitter gauge gravitational connection on \mathcal{E} is introduced as a 1-form

$${}^{'}\widehat{\mathcal{A}} = \begin{bmatrix} {}^{'}\widehat{\mathcal{A}}\frac{\alpha}{\beta} & l_{0}^{-1} {}^{'}\chi\frac{\alpha}{\beta} \\ l_{0}^{-1} {}^{'}\chi\frac{\beta}{\beta} & 0 \end{bmatrix},$$
(5)

where l_0 is a dimensional constant which is used because of different physical dimensions of $\hat{\mathcal{A}}$ - and $\hat{\mathcal{X}}$ -fields. In the nonholonomic gauge gravitational potential (5), $\hat{\mathcal{X}}^{\underline{\alpha}} = \hat{\mathcal{X}}^{\underline{\alpha}}_{\alpha} \mathbf{e}^{\alpha}$ are for 8×8 matrices $\hat{\mathcal{X}}^{\underline{\alpha}}_{\alpha}(\mathbf{u})$ subjected to the condition that $\mathbf{g}_{\alpha\beta} = \hat{\mathcal{X}}^{\underline{\alpha}}_{\alpha} \hat{\mathcal{X}}^{\underline{\beta}}_{\beta} \hat{\eta}_{\underline{\alpha}\underline{\beta}}$, where the 8-d dubbing of Minkovski metric can be written $\hat{\eta}_{\underline{\alpha}\underline{\beta}} = diag(1, 1, 1, -1, 1, 1, 1, -1)$ in any point $\mathbf{u} \in \hat{\mathcal{M}}$. For $\hat{\mathcal{A}}^{\underline{\alpha}}_{\underline{\beta}} = \hat{\mathcal{A}}^{\underline{\alpha}}_{\underline{\beta}\gamma} \mathbf{e}^{\gamma} = \hat{\mathcal{A}}^{\underline{\alpha}}_{\underline{\beta}\gamma} \mathbf{e}^{\gamma_s}$, with s-adapted \mathbf{e}^{γ_s} (2), the coefficients transform as $\hat{\mathcal{A}}^{\underline{\alpha}}_{\underline{\beta}\gamma} = \hat{\mathcal{X}}^{\underline{\alpha}}_{\alpha} \hat{\mathcal{X}}^{\beta}_{\underline{\beta}} \hat{\Gamma}^{\alpha}_{\beta\gamma} + \hat{\mathcal{X}}^{\underline{\alpha}}_{\alpha} \mathbf{e}_{\gamma}(\hat{\mathcal{X}}^{\beta}_{\underline{\beta}})$. Such formulas determined by the N-, or s-adapted coefficients of a canonical d-/s-connection $\hat{s} \hat{\mathbf{D}} = \{\hat{\Gamma}^{\alpha_s}_{\beta_s\gamma_s}\}$. Similar constructions can be done for an arbitrary linear connection $\hat{\Gamma}^{\alpha}_{\beta\gamma}$, as in the metric-affine geometry or a LC-connection $\hat{\nabla}$ for pseudo-Riemannian models. Using the "hat-connection" $\hat{s} \hat{\mathbf{D}}$, we can prove general decoupling and integration properties of various modified Einstein equations in MGTs.

A d- / s-adapted metric structure $|\mathbf{g}_{\alpha\beta} \approx |\mathbf{g}_{\alpha\beta\varsigma}|_{s}$ allows to define respective Hodge d- /s-operators $*\approx$ $*_N \approx *_s$ (we shall omit N- or s-labels for simplicity) and the absolute differential operator $|d| \approx |_s d$ and skew product \wedge on $|_s \mathcal{M}$. Defining in s-adapted geometric form the curvature of (5), $|_s \widehat{\mathcal{F}} = |_s d|_s \widehat{\mathcal{A}} + |_s \widehat{\mathcal{A}} \wedge |_s \widehat{\mathcal{A}}$, and using $*_s$, we derive in abstract geometrical form the commutative gauge gravitational equations on $|_s \mathcal{E}$,

$${}^{\scriptscriptstyle i}_{s}d(\ast_{s}{}^{\scriptscriptstyle i}_{s}\widehat{\mathcal{F}}) + {}^{\scriptscriptstyle i}_{s}\widehat{\mathcal{A}} \wedge (\ast_{s}{}^{\scriptscriptstyle i}_{s}\widehat{\mathcal{F}}) - (\ast_{s}{}^{\scriptscriptstyle i}_{s}\widehat{\mathcal{F}}) \wedge {}^{\scriptscriptstyle i}_{s}\widehat{\mathcal{A}} = -\lambda {}^{\scriptscriptstyle i}_{s}\widehat{\mathcal{J}}.$$
(6)

The source in (6) is also parameterized in s-adapted form,

$${}_{s}^{'}\widehat{\mathcal{J}} = \begin{bmatrix} {}_{s}^{'}\widehat{\mathcal{J}}\frac{\alpha}{\beta} & -l_{0}{}_{s}^{'}t\frac{\alpha}{\beta} \\ -l_{0}{}_{s}^{'}t\underline{\beta} & 0 \end{bmatrix}, \text{ where } {}_{s}^{'}\widehat{\mathcal{J}}\frac{\alpha}{\beta} = {}^{'}\widehat{\mathcal{J}}\frac{\alpha}{\beta\gamma_{s}} {}^{'}\mathbf{e}^{\gamma_{s}}$$
(7)

is identified to zero for the model with LC-connection. It is induced nonholonomically for the canonical dconnection but considered as a spin density if elaborated on phase spaces which are similar to the Riemann-Cartan theory. The 1-form $\frac{l}{s}t^{\underline{\alpha}} = \frac{l}{t_{\alpha_s}} e^{\alpha_s}$ is a phase space analogue of the energy-momentum tensor for the matter. In above formulas, the constant λ can be related to the gravitational constant l^2 in 8-d extending by analogy the 4-d formulas in GR. Other constants on the phase space (from string gravity etc.) are introduced as in 4-d theories, for instance, we consider $l^2 = 2l_0^2\lambda$, $\lambda_1 = -3/l_0$. Of course, we can re-define equivalently (6) in various other forms but the hat variables have the priority to allow a general decoupling and integration of physically important systems of nonlinear PDEs.

Using (5) in s-adapted form, we can define a canonical gauge s-operator ${}^{'}\widehat{\mathcal{D}}_{\alpha_s} := {}^{'}\widehat{\mathbf{D}}_{\alpha_s} + {}^{'}\widehat{\mathcal{A}}_{\alpha_s}$ and write the nonassociative Yang-Mills equations (6) in the form

$${}^{\scriptscriptstyle |}\widehat{\mathcal{D}}_{\alpha_s}{}^{\scriptscriptstyle |}\widehat{\mathcal{F}}^{\alpha_s\beta_s} = {}^{\scriptscriptstyle |}\widehat{\mathbf{D}}_{\alpha_s}{}^{\scriptscriptstyle |}\widehat{\mathcal{F}}^{\alpha_s\beta_s} + [{}^{\scriptscriptstyle |}\widehat{\mathcal{A}}_{\alpha_s},{}^{\scriptscriptstyle |}\widehat{\mathcal{F}}^{\alpha_s\beta_s}] = -\lambda {}^{\scriptscriptstyle |}\widehat{\mathcal{J}}^{\beta_s}.$$
(8)

In these formulas, [A, B] = AB - BA is the commutator on the Lie algebra of the chosen gauge group ${}^{}\mathcal{G}r$. The gauge gravitation fields on phase space satisfy also the Bianchi identity (which is equivalent to the Jacobi identity):

$$[\ ^{'}\widehat{\mathcal{D}}_{\mu_{s}}, [\ ^{'}\widehat{\mathcal{D}}_{\nu_{s}}, \ ^{'}\widehat{\mathcal{D}}_{\alpha_{s}}]] + [\ ^{'}\widehat{\mathcal{D}}_{\alpha_{s}}, [\ ^{'}\widehat{\mathcal{D}}_{\mu_{s}}, \ ^{'}\widehat{\mathcal{D}}_{\nu_{s}}]] + [\ ^{'}\widehat{\mathcal{D}}_{\nu_{s}}, [\ ^{'}\widehat{\mathcal{D}}_{\alpha_{s}}, \ ^{'}\widehat{\mathcal{D}}_{\mu_{s}}]] = 0, \tag{9}$$

considered for matrix operators with values in Lie algebra.

We emphasize that the YM-like gravitational equations (6) can be postulated or derived in abstract geometric form as in [27] but using the canonical de Sitter gauge gravitational connection (5) instead of similar constructions involving the LC-connection and standard YM connections with the de Sitter structure group. Such abstract geometric formulations can be used for the non-variational models when the structure group is chosen to be of affine or Poincaré type. In the last cases, the Killing metric form is degenerated and the total bundle gauge theory is not variational. Nevertheless, we can elaborate a variational model by introducing an effective constant *a* instead of the term $l_0^{-1} \, \, \chi_{\underline{\beta}}$ in (5). In such cases, an effective Lagrangian can be defined as in the YM theory which allows a variational proof of field equations of type (8). Projecting such equations for the LC-connection, or the canonical d-connection, on a base spacetime Lorentz manifold, the constant *a* disappears and the (affine) YM gauge equations transform into standard Einstein equations. Such a proof is provided for noncommutative generalizations in [28] (see in that paper the references on previous works with more details on N-adapted variational proofs etc.). For this work (involving nonholonomic generalizations of the BV method), it is enough to use the abstract geometric definition of YM-like equations. In the next subsection, we show that this approach can be extended for nonassociative gauge gravity theories by using distortions of connections and star product deformations.

2.3 Nonassociative star product deformation of de Sitter gauge gravity

It is a tedious technical task to compute in explicit frame index or coordinate forms of star product deformations of geometric and physical objects on phase spaces [11, 12]. Non-trivial N-connection structures make the formalism more sophisticated. The Convention 2 from [12] was formulated with the aim to compute such \star_s -deformations in s-adapted form or using abstract geometric methods. Applying on (co) vector/tangent bundles on corresponding phase spaces and gauge geometric s-objects the twisted star product operator \star_s (3), we can define corresponding nonassociative s-objects in gauge gravity theory. For instance, in a phase space,

$$\star_s: \mathbf{g}_s \to \mathbf{g}_s^{\star} = (\breve{g}_s^{\star}, \breve{g}_s^{\star}); \quad {}_s \widehat{\mathbf{D}} = \; {}_s^{} \nabla + \; {}_s \widehat{\mathbf{Z}} \to \; {}_s^{} \widehat{\mathbf{D}}^{\star} = \; {}_s^{} \nabla^{\star} + \; {}_s^{} \widehat{\mathbf{Z}}^{\star}; * \to \breve{*}, \tag{10}$$

when the Hodge operator $\check{*}$ on ${}_{s}^{\prime}\mathcal{M}^{\star}$ and the parametric deformations can be defined using the symmetric part \check{g}_{s}^{\star} of the star product deformed s-metric.

The (co) vector bundles on ${}_{s}^{*}\mathcal{M}$, and respective geometric s-objects subjected to \star_{s} -deformations define certain nonassociative bundle spaces, ${}_{s}^{*}\mathcal{E}({}_{s}^{*}\mathcal{M}) \rightarrow {}_{s}^{*}\mathcal{E}^{*}({}_{s}^{*}\mathcal{M}^{*}) = ({}_{s}^{*}\mathcal{E}^{*}, {}_{s}^{*}\mathcal{R}, {}_{s}^{*}\mathcal{M}^{*})$, for double structure group ${}^{i}\mathcal{G}r$ preserved as in (4). Here we note that we can elaborate on more general classes of nonassociative gauge models when, for instance, ${}^{i}\mathcal{G}r$ transforms into some quantum groups (with group deforms which are additional to \star_{s} -deformations). Such quantum group theories involve various assumptions on algebraic structure and request new physical motivations compared to the "nonassociative string theory R-flux deformation philosophy". The procedure of general decoupling and integration for such quantum gauge gravitational models is more sophisticated. In this work, we extend the research program outlined in [12, 13, 15, 16] for nonassociative gravitational and matter field theories with gauge groups when the structure of such groups and corresponding algebras are not subjected to quantum deformations. For such theories, we can apply in direct form the AFCDM and generate exact and parametric solutions of physically important systems of nonlinear PDEs.

In abstract and nonholonomic s-adapted geometric forms, we define and compute nonassociative deformations of type: $\star_s : \ \hat{\mathcal{A}} \to \ _s^{} \hat{\mathcal{A}}^{\star}$, with $\ \mathbf{g}_{\alpha\beta}$ identified to \breve{g}_s^{\star} if we consider zero powers of parameters \hbar and κ from (3). We can prescribe also the nonholonomic structure for the same $\ \chi^{\underline{\alpha}}$ and $\ \mathbf{e}^{\gamma_s}$ from (5), with s-adapted deformations $\star_s : \ \hat{\mathcal{A}}^{\underline{\alpha}}_{\beta\gamma_s} \to \ \hat{\mathcal{A}}^{\underline{\star}\underline{\alpha}}_{\beta\gamma_s}$. This allows us to compute nonassociative star product deformations:

$$\star_s: \ _s^{'}\widehat{\mathcal{F}} = \ _s^{'}d \ _s^{'}\widehat{\mathcal{A}} + \ _s^{'}\widehat{\mathcal{A}} \ ^s \wedge^{\star} \ _s^{'}\widehat{\mathcal{A}} \to \ _s^{'}\widehat{\mathcal{F}}^{\star} = \ _s^{'}d \ _s^{'}\widehat{\mathcal{A}}^{\star} + \ _s^{'}\widehat{\mathcal{A}}^{\star} \ ^s \wedge^{\star} \ _s^{'}\widehat{\mathcal{A}}^{\star}, \tag{11}$$

with respective deformation of s-adapted anti-symmetric operator $\wedge \to {}^{s} \wedge^{\star}$; $[A, B] \to [A^{\star}, B^{\star}]^{\star}$. For generalized

sources of the YM equations, $\star_s : \ \ \widehat{\mathcal{J}}^{\beta_s} \to \ \ \widehat{\mathcal{J}}^{\star\beta_s}$; and

$${}_{s}^{'}\widehat{\mathcal{D}} = {}_{s}^{'}\widehat{\mathbf{D}} + {}_{s}^{'}\widehat{\mathcal{A}} \rightarrow {}_{s}^{'}\widehat{\mathcal{D}}^{\star} = {}_{s}^{'}\widehat{\mathbf{D}}^{\star} + {}_{s}^{'}\widehat{\mathcal{A}}^{\star}, {}_{s}^{'}\widehat{\mathcal{J}} \rightarrow {}_{s}^{'}\widehat{\mathcal{J}}^{\star}.$$

This allows us to \star_s -deform the YM type equations for the de Sitter phase space gravity (6) and formulate their nonassociative versions,

$${}^{^{}}d(\breve{\ast}{}^{^{'}}\widehat{\mathcal{F}}{}^{^{\star}}) + {}^{^{'}}\widehat{\mathcal{A}}{}^{^{\star}} \wedge (\ast {}^{^{'}}\widehat{\mathcal{F}}{}^{^{\star}}) - (\ast {}^{^{'}}\widehat{\mathcal{F}}{}^{^{\star}}) \wedge {}^{^{'}}\widehat{\mathcal{A}}{}^{^{\star}} = -\lambda {}^{^{'}}\widehat{\mathcal{J}}{}^{^{\star}}.$$

This equation can be written also in nonholonomic s-adapted coefficient form as a \star_s -deformation of (8),

$$[\widehat{\mathcal{D}}_{\alpha_s}^{\star}]\widehat{\mathcal{F}}^{\star\alpha_s\beta_s} = [\widehat{\mathbf{D}}_{\alpha_s}^{\star}]\widehat{\mathcal{F}}^{\star\alpha_s\beta_s} + [\widehat{\mathcal{A}}_{\alpha_s}^{\star}, \widehat{\mathcal{F}}^{\star\alpha_s\beta_s}]^{\star} = -\lambda [\widehat{\mathcal{J}}^{\star\beta_s}].$$
(12)

The above nonassociative YM equations (12) are subjected additionally to the conditions of \star_s -deformed Bianchi identities (9) involving a nonzero "Jacobiator", which is typical for nonassociative theories. Such values can be computed as induced (effective) ones by choosing respective s-adapted \hbar and κ parametric decompositions. They reflect both the nonassociative and nonholonomic structure of such phase space theories. We can consider certain analogies with nonholonomic mechanics when the dynamical equations and conservation laws are supplemented by additional non-integrable constraints, Lagrange multiples, modified conservation laws etc.

We emphasize that the nonassociative and noncommutative theories determined by a general twisted product [8, 10, 11] are generic non-variational. So, the nonassociative gauge gravitational equations (12) are, in general, non-variational and nonlocal. Such issues related to nonassociative R-flux gravitational models are discussed in our partner works [13, 14, 15, 16, 48]. Here we note these important three points: 1) Effective variational theories encoding nonassociative data can be formulated after considering \hbar and κ parametric decompositions of the geometric and physical s-objects in (12). 2) Such physically important equations can be derived in low-energy limits of string theory (with corresponding nonholonomic re-parameterizations of terms with R-fluxes); or as star product deformations of any variational or non-variational associative and commutative YM-like equations (6) or (8). 3) To postulate or derive in abstract geometric form nonassociative gauge gravitational field equations using the nonassociative gauge gravitational potential ${}_{s}\hat{\mathcal{A}}^{\star}$ (11) is also possible. This approach is preferred for generic non-variational nonassociative theories. It allows an abstract nonassociative geometric generalization of the mathematical methods and results on BV and AQFT theory from [25, 26, 32, 37, 38, 40, 41, 42, 43, 45, 46]. Such methods are very powerful and important in elaborating new models of QG including nonassociative nonvariational and nonlocal contributions from string and M-theory. In this work, we study such possibilities for certain general and physically important classes of off-diagonal solutions of nonassociative gauge gravitational equations (8). This first step is important for elaborating physical applications of the BV formalism to generic nonlinear theories like GR and various modifications. Such results are important for elaborating QG models in the conditions a general nonlinear functional analysis theory does not exist and can't be formulated in a unique way for generic nonassociative non-variational theories.

Finally, we note that an effective or matter field source $\widehat{\mathcal{J}}^{*\beta_s}$ (7) can be correspondingly parameterized and physically motivated [16] in some forms that the nonassociative YM equations (12) present certain alternatives or gauge like generalizations of the nonassociative star product deformed Einstein equations. In nonassociative vacuum form, such models were considered in [10, 11] and, in s-adapted form, with extensions to certain classes of nontrivial sources and off-diagonal solutions, in [12, 13, 15].

2.4 Quasi-stationary off-diagonal solutions in nonassociative gauge gravity

To study the physical implications of nonassociativity in the framework of gauge gravity theory it is important to derive certain physically important solutions of the gravitational field equations (12). This is a very difficult technical problem because the nonassociative YM equations consist of a strongly coupled system of nonlinear PDEs. Surprisingly, the AFCDM allows to construction of such solutions (see the first examples for cosmological solitonic hierarchies in [16]). This is possible if we parameterize and project the nonassociative YM equations on the base phase space in a form which is equivalent to nonassociative modified Einstein equations studied in our partner works. In this subsection, we have shown how such nonholonomic projections can be defined in certain forms the general decoupling and integration properties of (nonassociative) gravitational equations which was proved in [12]. This can be used for constructing physically important solutions.

2.4.1 Projections on phase spaces and effective parametric MGTs

We can formulate a nonassociative gauge gravity model when the nonassociative YM equations (12) are equivalent to certain nonassociative star product modifications of the Einstein equations on the phase spaces. This is possible if we consider instead of the de Sitter structure group SO(4,1), the affine structure group Af(4,1). Then we construct a gauge potential which is similar to (5). We consider some constants $\chi_0^{\underline{\alpha}}$ instead of $\frac{1}{s}\chi^{\underline{\alpha}}$ in the last line of the matrix $\frac{1}{s}\hat{\mathcal{A}}^*$, which takes values into the double Poincaré-Lie algebra,

$${}^{\scriptscriptstyle \perp}_{s}\widehat{\mathcal{A}}^{\star} \to {}^{\scriptscriptstyle \perp}_{s}\widehat{\mathcal{A}}^{\star}_{[P]} = \begin{bmatrix} {}^{\scriptscriptstyle \perp}_{s}\widehat{\mathcal{A}}^{\star\underline{\alpha}}_{\ \underline{\beta}} & l_{0}^{-1} {}^{\scriptscriptstyle \perp}_{s}\chi^{\underline{\alpha}} \\ \chi^{\underline{\alpha}}_{0} & 0 \end{bmatrix}.$$
(13)

The constants χ_0^{α} from (13) can be fixed to be zero at the end of computations. Then, we can introduce such constants in the source (7) when further \star_s -transforms result into

$${}_{s}^{\top}\widehat{\mathcal{J}}^{\star} = \begin{bmatrix} {}_{s}^{\top}\widehat{\mathcal{J}}_{\underline{\beta}}^{\star\underline{\alpha}} = 0 & -l_{0} {}_{s}^{\top}t^{\star\underline{\alpha}} \\ {}_{\alpha}^{\underline{\alpha}} & 0 \end{bmatrix}, \text{ with } {}_{s}^{\top}t^{\star\underline{\alpha}} = \chi^{\underline{\alpha}\beta_{s}} {}^{\top}\mathcal{J}_{\alpha_{s}\beta_{s}}^{\star} {}^{\dagger}\mathbf{e}^{\alpha_{s}}.$$
(14)

In these formulas, $\mathcal{J}_{\alpha_s\beta_s}^{\star}$ is the star product deformation of the effective energy-momentum tensor extended on phase space and written in s-adapted form $\mathcal{J}_{\alpha_s\beta_s}$ on $\mathcal{J}_{\alpha_s\beta_s}^{\star}$. Such nonholonomic nonassociative or commutative sources are considered in nonassociative gravity and nonassociative geometric flow theories, see details and references in [12, 13, 15].

For nonassociative gauge potentials and sources, respectively, of type (13) and (14), the projections on ${}_{s}\mathcal{M}^{\star}$ of nonassociative YM equations (12) transform into nonassociative s-adapted canonical gravitational equations considered in our partner works,

$${}^{'}\widehat{\mathbf{R}}ic^{\star}_{\alpha_{s}\beta_{s}} = {}^{'}\mathcal{J}^{\star}_{\alpha_{s}\beta_{s}}.$$
(15)

In the vacuum case with $\mathcal{J}_{\alpha_s\beta_s}^{\star} = 0$ and ${}_{s}\hat{\mathbf{D}}^{\star} \to {}_{s}^{\star}\nabla^{\star}$, the system of nonlinear PDEs (15) are just the vacuum gravitational equations for nonassociative and noncommutative gravity studied in [10, 11]. More than that, both equations (12) and (15) transform into standard Einstein equations in GR if we construct the affine potential (13) for the standard LC-connection ∇ on a pseudo-Riemannian spacetime base. The above equations can be proven in s-adapted forms using a tedious calculus considered in [28, 29, 16] for nonholonomic commutative phase spaces and in noncommutative gauge gravity with Seiberg-Witten product.

2.4.2 Off-diagonal parametric quasi-stationary solutions

The system of nonlinear PDEs (15) can be decoupled and integrated in certain very general forms when the coefficients off-diagonal metrics and star-modified connections depend on all spacetime and momentum-like coordinates [12]. Such solutions can be generated in a more simple form for quasi-stationary configurations with a Killing symmetry (this is enough for the purposes of this work). For instance, we can use a time-like vector $\mathbf{e}_4 = \partial_4 = \partial_t$ (with $x^4 = y^4 = t$) and introduce such nonholonomic parameterizations of (effective) sources (14):

$$\mathcal{J}^{\star\alpha_s}_{\beta_s}(\mathbf{u}) = diag[\mathcal{J}^{\star}(x^{k_1})\delta^{j_1}_{i_1}, \mathcal{J}^{\star}(x^{k_1}, x^3)\delta^{a_2}_{b_2}, \mathcal{J}^{\star}(x^{k_2}, p_5)\delta^{b_3}_{a_3}, \mathcal{J}^{\star}(x^{k_3}, p_7)\delta^{b_4}_{a_4}].$$
(16)

In this formula, $k_1 = 1, 2; k_2 = 1, 2, 3, 4; k_3 = 1, 2, ...6$; we can consider variants when $x^3 \to x^4$ (for locally anisotropic cosmological configurations) or when $p_5 \to p_6$, or when $p_7 \to p_8$, for generating other classes of

off-diagonal solutions. In a more general case, we can generate solutions for arbitrary shell parametrization ${}_{s}\mathcal{J}^{\star}(x^{k_{s-1}}, p_{a_s})$, for s = 3, 4. Any class of locally anisotropic cosmological solutions, with Killing symmetry $\mathbf{e}_3 = \partial_3$, of nonassociative modified Einstein equations (15) can be lifted on respective co-vector bundles for generating solutions of nonassociative YM equations (12), see details in [16]. The goal of this section is to show how the AFCDM can be used for generating quasi-stationary solutions of the systems of nonlinear PDEs (15) and (12). Here, we also note that the \star -labels of the shell components in (16) state that such effective sources encode in parametric form⁴ nonassociative data for R-flux and matter field deformations as explained in details in [12, 13, 15]. In this work, we shall consider that ${}_{s}\mathcal{J}^{\star}$ are certain generating sources which can be prescribed following physical arguments, for instance, to model nonassociative BH, or WH, or locally anisotropic cosmological configurations.

Let us consider a prime ${}_{s}{}^{*}\mathbf{g}({}^{u})$ on ${}_{s}{}^{*}\mathcal{M}$. Geometrically, it can be an arbitrary s-metric, or taken as an important solution of some (modified) Einstein equations which allows applications in modern physics and information theory. Our goal is to construct a family of target ${}_{s}{}^{*}\mathbf{g}(x^{k_3}, p_7)$ defining a solution of (12),

In (17), the η -polarization functions (i.e. phase space gravitational polarization functions, which can be also considered as generating functions) define star product deformations. To elaborate on quantum models, we can consider quantum perturbative or non-perturbative deformations, for instance, of some solutions in GR or a MGTs extended on phase spaces. Here we emphasize that parameterizations (16) and (17) used for (2) and (3) allows to transform (15) into a system of nonlinear PDEs with general decoupling and integration properties [12]. After a class of solutions is defined in a certain general form by respective generating functions and generating sources and integration functions, we can impose certain boundary or Cauchy conditions. This allows us to elaborate on physical models with polarization of physical constants or vacuum gravitational and phase space configurations, deformation of horizons (for certain conditions) etc.

The ansatz for generating quasi-stationary solutions of nonassociative gravitational YM equations (12) with η -polarization functions and fixed energy parameter $p_8 = E_0$ can be parameterized (see details in [12, 13, 15]) using such quadratic line elements on phase space:

$$d\hat{s}^{2} = g_{\alpha_{s}}(\hbar,\kappa,x^{i_{3}},p_{7})(\mathbf{e}^{a_{s}})^{2} = g_{i_{1}}(\hbar,\kappa,x^{k_{1}})(dx^{i_{1}})^{2} + g_{a_{2}}(\hbar,\kappa,x^{i_{1}},y^{3})[\mathbf{e}^{a_{2}}(\hbar,\kappa,x^{i_{1}},y^{3})]^{2} + [g^{a_{3}}(\hbar,\kappa,x^{i_{2}},p_{6})[\ \mathbf{e}_{a_{3}}(\hbar,\kappa,x^{i_{2}},p_{6})]^{2} + [g^{a_{4}}(\hbar,\kappa,x^{i_{3}},p_{7})[\ \mathbf{e}_{a_{4}}(\hbar,\kappa,x^{i_{3}},p_{7})]^{2}$$
(18)

$$= \eta_{i_1}(\ \ u) \mathring{g}_{i_1}(\ \ u) (dx^{i_1})^2 + \eta_{a_2}(\ \ u) \mathring{g}_{a_2}(\ \ u) (\mathbf{e}^{a_2})^2 + \ \ \eta^{a_3}(\ \ u) (\ \ \mathbf{e}_{a_3})^2 + \ \ \eta^{a_4}(\ \ u) (\ \ \mathbf{e}_{a_4})^2,$$

where

$$\begin{split} \mathbf{e}^{a_2} &= dy^{a_2} + N_{k_1}^{a_2}(\hbar,\kappa,x^{i_1},y^3) dx^{k_1} = dy^{a_2} + \eta_{k_1}^{a_2}(\ \ 'u) \mathring{N}_{k_1}^{a_2}(\ \ 'u) dx^{k_1}, \\ \mathbf{e}_{a_3} &= dp_{a_3} + \ \ N_{a_3k_2}(\hbar,\kappa,x^{i_2},p_5) dx^{k_2} = dp_{a_3} + \ \ \eta_{a_3k_2}(\ \ 'u) \ \ \ \mathring{N}_{a_3k_2}(\ \ 'u) dx^{k_2}, \\ \mathbf{e}_{a_4} &= dp_{a_4} + \ \ \ N_{a_4k_3}(\hbar,\kappa,\ \ \ 'x^{i_3},p_7) d\ \ \ 'x^{k_3} = dp_{a_4} + \ \ \ \eta_{a_4k_3}(\ \ 'u) \ \ \ \mathring{N}_{a_4k_3}(\ \ 'u) dx^{k_3}. \end{split}$$

In Appendix, we provide explicit formulas for off-diagonal solutions with s-coefficients depending on generating and integration functions and on generating sources ${}^{*}_{s}\mathcal{J}^{*}(16)$. Such values can be chosen in different forms allowing to construct physically important exact or parametric solutions.

Any quasi-stationary solution (18) can be written in some off-diagonal functional forms with labels stating certain basic properties defined by corresponding classes of generation functions and (effective) sources and

⁴We consider the small parameters \hbar and κ ; we can use also other type physical constants like the gravitational one G, with extensions to higher dimensions, a BH mass, M, a cosmological constant, Λ , an electric charge, e, etc.; for simplicity, we shall write only $\Im^{*\alpha_s}_{\beta_s}(\hbar,\kappa, \neg u)$ or $\Im^*_{\beta_s}(\hbar,\kappa, \neg u)$ assuming that (if necessary) we can introduce and emphasize another types of physically important constants; this allows us to define explicit classes of effective sources and s-metrics encoding various types of parametrical dependencies; using polarization functions, we can write $\neg_{\alpha_s}(\hbar,\kappa, \neg u)$ and $\neg_{\alpha_{s-1}}^{i_s}(\hbar,\kappa, \neg u)$

integration functions/ constants,

$$[\mathfrak{Cqs}] = {}^{'}_{s} \mathbf{g}^{\star}[solit, horiz, polarizc, sg, regular, geomflow, therm, kinetic, ramif, filament, anhol, ...]$$

$$= {}^{'}_{\alpha\beta} g^{\star}_{\alpha\beta}(\hbar, \kappa; x^{i}, x^{3}p_{a}; \partial_{4}, \partial_{8}; {}^{'}_{s}\mathcal{J}^{\star}, {}^{'}_{s}\Lambda; {}^{'}_{g} g_{\alpha\beta}, {}^{'}_{s}\eta \sim {}^{'}\zeta_{\alpha_{s}}(1 + \kappa {}^{'}\chi_{\alpha_{s}}), ..)d {}^{'}u^{\alpha} \otimes d {}^{'}u^{\beta}, \text{ or}$$

$$[\mathfrak{g}^{\star}[\mathfrak{Clacs}] = {}^{'}_{g_{\alpha\beta}}(\hbar, \kappa; x^{i}, t, p_{a}; \partial_{3}, \partial_{8}; {}^{'}_{s}\mathcal{J}^{\star}, {}^{'}_{s}\Lambda; {}^{'}_{g} g_{\alpha\beta}, {}^{'}_{s}\eta \sim {}^{'}\zeta_{\alpha_{s}}(1 + \kappa {}^{'}\chi_{\alpha_{s}}), ...)d {}^{'}u^{\alpha} \otimes d {}^{'}u^{\beta}$$

$$(19)$$

In (19), we consider such abstract labels for quasi-stationary configurations in $[g^{\star}[\mathfrak{Cqs}]]$: vacuum polarizations, for instance, as solitonic hierarchies, *solit*; deformations of horizons (creation and disappearance), *horiz*; polarization of constants, *polarizc*; singularities, *sg*; and regularizations, *regular*; geometric flows evolution, *geomflow*, thermodynamic properties, *therm*; kinetic properties, *kinetic*; ramification, *ramif*; filaments, *filament*; non-holonomic constraints *anhol*; and other possible variants which are determined by generating and integration functions and generating sources. Corresponding nonlinear symmetries relating effective generating sources to shell effective cosmological constants; and when Horava-Lifshitz and Finsler-Lagrange-Hamilton structures [33] can be modelled on (nonassociative) phase spaces, in GR, or other type MGTs.

In explicit form, nonassociative off-diagonal solutions $[\mathbf{g}^*[\mathfrak{Cqs}]]$ can be generated by prescribing a corresponding class of integration functions (A.5) for s-metrics (A.1). We also need additional assumptions on the type polarization functions and nonlinear symmetries (A.3). For more special conditions on parametric decompositions, we can generate regular BH configurations on phase space as in (A.11). Here we note that physical properties of nonassociative phase space WH and black ellipsoid, BE, and BH solutions with singularities were studied in [12, 13, 14, 15]. The (nonassociative) vacuum structure in gauge gravity and projections on phase spaces may encode solitonic hierarchies branching of solutions and filament structures as we explained in [16] for locally anisotropic cosmological solutions of type $[\mathbf{g}^*[\mathfrak{Clacs}]]$ (19). To generate such nonassociative accelerating cosmological models we can use the time-like duality symmetries as we explain in (A.7).

We conclude this sections with such remarks:

- 1. The modified YM equations for nonassociative gauge gravity (12) and, for phase space projections, (15) are characterized by generic nonlinear solutions $\mathbf{g}^*[\mathfrak{Cqs}]$ or $\mathbf{g}^*[\mathfrak{Clacs}]$ as in (19) defining a very rich geometric structure. Any solution is characterized by a nonassociative geometric flow thermodynamics, and respective nonlinear symmetries. Such nonassociative theories with general twisted star products are not variational, see details in [13, 14, 15]. To elaborate on new methods of quantization and constructing respective QG gravity models and formulating a generalized nonassociative BV formalism, we have to apply advanced geometric methods of the nonassociative geometric flow theory in certain forms correlated to the AFCDM.
- 2. For parametric decompositions, the corresponding gauge gravity and phase gravity equations and their generic off-diagonal solutions 'g^{*}[Cqs] or 'g^{*}[Clacs] can be formulated in the framework of some effective variational theories on phase spaces with respective effective Lagrange or Hamilton densities. In this work, such densities encode nonassociative data. This approach allows us to develop in nonholonomic form (for nonassociative gauge gravity theories) the BV formalism using rigorous mathematical results from pAQFT [24, 25, 26].

3 The BV formalism for the nonassociative classical gauge gravity theory

In this section, we consider phase spaces modelled by quasi-stationary off-diagonal solutions $[g^{\star}[\mathfrak{Cqs}]]$ (19) of the YM equations (12) for a classical nonassociative gauge gravity theory with star product. The main goals are to generalize for such configurations the principles of locality and homology [25, 26] to construct nonassociative gauge models, which will then be used to quantize using nonassociative and nonholonomic deformation principles.

3.1 Geometric preliminaries and kinematic structure

We choose a region ${}^{\prime}\mathcal{U} \subset {}^{\prime}_{s}\mathcal{M}$ which is time-oriented and globally hyperbolic both for projections on the Lorentz spacetime manifold and for typical cofibers. This means that both h- and c-components are Cauchy surfaces. In brief, we write ${}^{\prime}\mathcal{U}_{\mathfrak{Cqs}}^{\star}$ if such a phase space region is defined by a ${}^{\prime}\mathbf{g}^{\star}[\mathfrak{Cqs}]$ with corresponding parametric dependencies. Respectively, we write ${}^{\prime}\mathcal{E}_{\mathfrak{Cqs}}^{\star} = {}^{\prime}_{s}\mathcal{E}^{\star}({}^{\prime}\mathcal{U}_{\mathfrak{Cqs}}^{\star})$ (4) and omit star-labels for configurations with $[00] = [\hbar^{0}, \kappa^{0}]$, when ${}^{\prime}\mathcal{U}_{\mathfrak{Cqs}}$ are determined by a ${}^{\prime}\mathbf{g}[\mathfrak{Cqs}]$ which, for instance, contains dependencies on parameters depending BHs, WHs or other types physically important solutions in an associative and commutative gravity theory. By choosing an explicit quasi-stationary solution (A.1) or (A.11), we decide what kind of s-objects define our (nonassociative) model, for instance, using certain classes of s-tensors, s-connections, scalar fields etc.⁵

Next, we extend to nonassociative phase spaces the important notion of spacetime support of a functional (which typically encodes *localization* properties of observables and *additivity*). For the goals of this work (working with quasi-stationary phase space configurations subjected to star product deformations, for simplicity, in linear parametric form up to $[\hbar^1, \kappa^1]$), we define the *phase space support of a functional* as

$$\sup p \, 'F^{\star}_{\mathfrak{Cqs}} = \{ \, 'u \in \, '\mathcal{U}_{\mathfrak{Cqs}} \, | \, \forall \text{ neighborhoods of } \, 'u \exists \, '\varphi_1^{\star}, \, '\varphi_2^{\star} \in \, '\mathcal{E}_{\mathfrak{Cqs}}, \sup p \, '\varphi_2^{\star} \subset \, '\mathcal{U}_{\mathfrak{Cqs}}$$
such that
$$\, 'F^{\star}_{\mathfrak{Cqs}}(\, '\varphi_1^{\star} + \, '\varphi_2^{\star}) \neq \, 'F^{\star}_{\mathfrak{Cqs}}(\, '\varphi_1^{\star}) \}.$$

A functional $F_{\mathfrak{Cgs}}^{\star}$ is *additive* on phase space configuration \mathfrak{Cqs} if

Then, a functional $F^{\star}_{\mathfrak{Cas}}$ is *local* if it can be written in the form

$${}^{\scriptscriptstyle \mathsf{T}} F^{\star}_{\mathfrak{Cqs}}({}^{\scriptscriptstyle \mathsf{T}} \varphi^{\star}) = \int_{{}^{\scriptscriptstyle \mathsf{U}}_{\mathfrak{Cqs}}} {}^{\scriptscriptstyle \mathsf{T}} \omega^{\star} (j_u^k({}^{\scriptscriptstyle \mathsf{T}} \varphi^{\star})) {}^{\scriptscriptstyle \mathsf{T}} \delta {}^{\scriptscriptstyle \mathsf{T}} \mu_{\mathfrak{Cqs}}({}^{\scriptscriptstyle \mathsf{T}} u),$$
(20)

where ω^* is a function on the jet bundle over $\mathcal{U}_{\mathfrak{Cqs}}$, subjected to star product deformations up to $[\hbar^1, \kappa^1]$, and $j_u^k(\neg \varphi^*) = (\neg u, \neg \varphi^*(\neg u), \neg \partial \neg \varphi^*(\neg u), \ldots)$, with derivatives up to order k, is the k-jet of $\neg \varphi^*$ at the point $\neg u$,

⁵We shall use also such conventions: $\mathcal{E}_{\mathfrak{E}_{q\mathfrak{s}}}^{C}$ denote the space of smooth compactly supported sections of $\mathcal{E}_{\mathfrak{E}_{q\mathfrak{s}}}$; for complexifications of topological duals of $\mathcal{E}_{\mathfrak{E}_{q\mathfrak{s}}}$ and $\mathcal{E}_{\mathfrak{E}_{q\mathfrak{s}}}^{C}$, which are equipped with the strong topology, we write respectively $\mathcal{E}_{\mathfrak{E}_{q\mathfrak{s}}}$; we denote by $sec(\mathcal{E}_{\mathfrak{E}_{q\mathfrak{s}}})^*$ the space of smooth section of the dual bundle $(\mathcal{E}_{\mathfrak{E}_{q\mathfrak{s}}})^*$; then, $(\mathcal{E}_{\mathfrak{E}_{q\mathfrak{s}}})^*$ denotes the complexification of the space of sections of $(\mathcal{E}_{\mathfrak{E}_{q\mathfrak{s}}})^*$ tensored with the bundle of densities over $\mathcal{U}_{\mathfrak{E}_{q\mathfrak{s}}}^*$; and we write $(\mathcal{E}_{\mathfrak{E}_{q\mathfrak{s}}})^n$ for the complexification of the space of sections of the *n*-fold exterior tensor product of a star product deformed bundle, seen as a s-vector bundle over $(\mathcal{U}_{\mathfrak{E}_{q\mathfrak{s}}})^n$. We have to introduce an "abuse of notations" comparing to [27, 25, 12, 16] because the procedure of quantization of nonlinear systems encoding nonassociative data depends in an explicit form on the type of quasi-stationary configurations $\mathfrak{C}_{q\mathfrak{s}}$. To make the notation system more simple we shall denote (if not ambiguous), for instance, the elements of $\mathcal{E}_{\mathfrak{e}_{q\mathfrak{s}}}^*$ by φ^* even such elements carry s-indices (which can be invoked when it becomes necessary).

up to $[\hbar^1, \kappa^1]$. Here we note that our approach involves quasi-stationary off-diagonal configurations defined as parametric solutions for \mathfrak{Cqs} .

We denote by ${}^{'}_{loc}\mathcal{F}^{\star}_{\mathfrak{Cqs}}$ the spaces of compactly supported smooth local functions on ${}^{'}\mathcal{E}^{\star}_{\mathfrak{Cqs}}$. Respectively, the commutative algebra ${}^{'}\mathcal{F}_{\mathfrak{Cqs}}$ of multilocal functionals is defined as the completion of ${}^{'}_{loc}\mathcal{F}_{\mathfrak{Cqs}}$ with respect of the point wise product ${}^{'}\mathcal{F}_{\mathfrak{Cqs}} \cdot {}^{'}\mathcal{G}_{\mathfrak{Cqs}}({}^{'}\varphi) = {}^{'}\mathcal{F}_{\mathfrak{Cqs}}({}^{'}\varphi) \cdot \mathcal{G}_{\mathfrak{Cqs}}({}^{'}\varphi)$. Star product deformations result in a nonassociative ${}^{'}_{loc}\mathcal{F}^{\star}_{\mathfrak{Cqs}}$. We can introduce also regular functionals ${}^{'}_{reg}\mathcal{F}_{\mathfrak{Cqs}}$ and say that ${}^{'}\mathcal{F}_{\mathfrak{Cqs}} \in {}^{'}_{reg}\mathcal{F}_{\mathfrak{Cqs}}$ if all the derivatives ${}^{'}\mathcal{F}^{(n)}_{\mathfrak{Cqs}}({}^{'}\varphi)$ are smooth, when for all ${}^{'}\varphi \subset {}^{'}\mathcal{E}_{\mathfrak{cqs}}$, $n \in \mathbb{N}$ we have ${}^{'}\mathcal{F}^{(n)}_{\mathfrak{Cqs}}({}^{'}\varphi) \in ({}^{'}\mathcal{E}_{\mathfrak{cqs}}({}^{'}\mathcal{U}_{\mathfrak{cqs}})^{n})^{!}$. Similarly, star product deformations define ${}^{'}_{req}\mathcal{F}^{\star}_{\mathfrak{Cqs}}$.

3.2 Nonassociative dynamics and nonlinear and linear symmetries

For a general nonassociative star product (3), we are not able to formulate a unique variational derivation of physically important systems of PDEs in nonassociative geometric flow and gravitational theories, see details in [13, 14, 15]. Such equations can be postulated in abstract geometric form and then we can consider nonholonomic parametric deformations for a class of solutions and construct some nonholonomic Lagrangians or Hamiltonians [32]. In \mathfrak{Cqs} , various types of non-trivial compact solutions can be found. To get around such obstructions with can consider an effective Lagrangian density for a $\mathcal{U}_{\mathfrak{Cqs}}$ with a cutoff function $f \in \mathcal{U}_{\mathfrak{Cqs}} := \mathcal{C}^{\infty}(\mathcal{U}_{\mathfrak{Cqs}}, \mathbb{R})$. This way, we can define all relevant s-objects and, for instance, nonassociative deformations of the Euler-Lagrange derivative in a way which is independent of f.

3.2.1 Effective *-deformed Lagrangians and Euler-Lagrange s-operators

A nonassociative star product (3) deformation of a generalized Lagrangian L on a fixed configuration $\mathcal{U}_{\mathfrak{Cqs}} \subset {}^{'}_{s}\mathcal{M}$, i.e. $\star : L \to L^{\star}$, for a map $L : \mathcal{U}_{\mathfrak{Cqs}} \to {}^{'}_{loc}\mathcal{F}_{\mathfrak{Cqs}}$ defined by three properties:

- 1. additivity, when $L^{\star}(\varphi_{1}^{\star}+\varphi_{2}^{\star}+\varphi_{3}^{\star}) = L^{\star}(\varphi_{1}^{\star}+\varphi_{2}^{\star}) L^{\star}(\varphi_{2}^{\star}) + L^{\star}(\varphi_{2}^{\star}+\varphi_{3}^{\star})$ for $\varphi_{1}^{\star}, \varphi_{2}^{\star}, \varphi_{3}^{\star} \in U^{\star}_{\mathfrak{Cqs}}$, for $U^{\star}_{\mathfrak{Cqs}} \to U^{\star}_{\mathfrak{Cqs}}$, with $\sup p \varphi_{1}^{\star} \cap \sup p \varphi_{3}^{\star} = \emptyset$;
- 2. support, when $\sup p({}^{\!\!\!} L^{\star} ({}^{\!\!\!} \varphi^{\star})) \subseteq \sup p({}^{\!\!\!} \varphi^{\star})$ is defined by star product deformation of $\sup p({}^{\!\!\!} L({}^{\!\!\!} \varphi)) \subseteq \sup p({}^{\!\!\!} \varphi);$
- 3. covariance, when for a local Minkowski spacetime and then for a cofiber, we consider an isometry group ${}^{\mathcal{P}} = (P_{+}^{\uparrow}, {}^{\mathcal{P}}_{+}^{\uparrow})$, where P_{+}^{\uparrow} is the proper orthocronous Poincaré group, we require the property ${}^{\mathcal{L}}L^{\star}({}^{\mathcal{I}}f)(\rho^{*}{}^{\mathcal{I}}\varphi^{\star}) = {}^{\mathcal{L}}L^{\star}(\rho_{*}{}^{\mathcal{I}}f)({}^{\mathcal{I}}\varphi^{\star})$, for every $\rho \in {}^{\mathcal{P}}$.

The abstract geometric formalism [27, 12] can be extended to the spaces of all generalized Lagrangians $\mathcal{L} = \{ L \}$, when we assume that the Lagrangians satisfy the condition $\mathcal{L}^* = \mathcal{L}$, where * is an involution which is different from the star product. Such an involution is not just the complex conjugation, but for graded geometries, it also swaps the order of factors. Then all geometric s-objects are subjected to star product deformations, $\star : \mathcal{L} \to \mathcal{L}^*$, which allow to work with nonassociative generalized Lagrangians \mathcal{L}^* .

Even in nonassociative geometry with general twist products, we are not able to formulate a general and uniquely defined variational calculus, the abstract geometric formalism allows us to perform BV classical and quantum constructions for any class of quasi-stationary configurations \mathfrak{Cqs} . The equivalence classes of $\exists f$ on the space of generating and integration functions (A.5) are related via nonlinear symmetries (A.3). In certain effective parametric forms, we can always define corresponding actions $\exists S^*(\exists L^*)$ considering equivalence classes of Lagrangians $\exists L_1^*$ and $\exists L_2^*$, which are equivalent, $\exists L_1^* \sim \exists L_2^*$, if $\sup p(\exists L_1 - \exists L_2)(\exists f) \subset \sup p(d \exists f), \forall \exists f \in \exists \tilde{\mathcal{U}}_{\mathfrak{Cqs}}$. In brief, we shall write $\exists S^*$ instead of $\exists S_{\mathfrak{Cqs}}^*$, or $\exists S^*(\exists L_{\mathfrak{Cqs}}^*)$, if such simplifications do not result in ambiguities. Using canonical nonholonomic geometric variables (10) and (11), two important examples of nonassociative generalized Lagrangians are written in the form:

$${}^{'}_{\varphi}L^{\star}({}^{'}f)[{}^{'}\varphi^{\star}] = \frac{1}{2} \int_{{}^{'}\mathcal{U}_{\mathfrak{Cqs}}} \left({}^{'}\widehat{\mathbf{D}}_{\alpha_{s}}^{\star}{}^{'}\varphi^{\star}{}^{'}\widehat{\mathbf{D}}^{\star\alpha_{s}}{}^{'}\varphi^{\star} - m^{2}({}^{'}\varphi^{\star})^{2} \right) {}^{'}f \, \delta^{8}\mu, \text{ free scalar field };$$

$${}^{'}_{gr}L^{\star}({}^{'}f)[{}^{'}_{s}\widehat{\mathcal{A}}^{\star}] = -\frac{1}{2} \int_{{}^{'}\mathcal{U}_{\mathfrak{Cqs}}} {}^{'}f \, tr({}^{'}\widehat{\mathcal{F}}^{\star} \wedge (\ast {}^{'}\widehat{\mathcal{F}}^{\star})), \text{ nonassociative gauge gravitational field }, \quad (21)$$

where trace, tr, is in the Killing metric over the Lie algebra of the gauge group; the effective mass m includes distortions of connections; and the measure $\delta^8 \mu$ is defined by a chosen \mathfrak{Cqs} .

For any $L^{\star} \in \mathcal{L}^{\star}$, $\varphi^{\star} \in \mathcal{E}_{\mathfrak{Cqs}}^{\star}$, we define a functional

$$|\delta L^{\star}(\varphi_1)[\varphi^{\star}] := L^{\star}(f)[\varphi^{\star} + \varphi_1] - L^{\star}(f)[\varphi^{\star}],$$

when $\delta' L^*$ acts on $\mathcal{U}_{\mathfrak{Cqs}} \times \mathcal{E}_{\mathfrak{Cqs}}^*$ and $f \equiv 1$ on $\sup p' \varphi_1^*$ and such a map does not depend on choice of f. This functional allows to introduce in effective form the *Euler-Lagrange derivative* of S^* ,

 $S_{ol}\mathcal{E}_{\mathfrak{Cqs}}^{\star}$ denotes the spaces of solutions defined as the zero locus of the 1-form $d S^{\star}$, $S_{ol}\mathcal{E}_{\mathfrak{Cqs}}^{\star} \subset \mathcal{E}_{\mathfrak{Cqs}}^{\star}$; $S_{ol}\mathcal{F}_{\mathfrak{Cqs}}$ denotes the space of off-shell functionals in the space of functionals on $S_{ol}\mathcal{E}_{\mathfrak{Cqs}}^{\star}$.

In above formulas, we can identify the space of solutions of ${}^{-}S^{\star}({}^{-}\varphi^{\star}) \equiv 0$ to be equivalent with the class of parametric solutions for ${}^{-}\mathcal{E}^{\star}_{\mathfrak{Cqs}} = {}^{-}_{s}\mathcal{E}^{\star}({}^{-}\mathcal{U}^{\star}_{\mathfrak{Cqs}})$ (4). Here we also note that in all formulas involving ${}^{-}\varphi^{\star}$ or ${}^{-}\varphi$, we can introduce abstract indices $\check{\alpha}$, with inverse hat, for (nonassociative) fields labeling degree of freedom of corresponding scalar fields, any type of (gravitational) gauge fields etc. So, we can write ${}^{-}\varphi^{\star} = \{{}^{-}\varphi^{\star}_{\check{\alpha}}\}$ and use notations like $\frac{{}^{-i}\delta S^{\star}}{{}^{-i}\varphi^{\star}_{\check{\alpha}}({}^{-i}u)}$ for $\frac{{}^{-i}\delta L^{\star}({}^{-i}f)}{{}^{-i}\delta {}^{-i}\varphi^{\star}_{\check{\alpha}}({}^{-i}u)}$ evaluated at ${}^{-i}f \equiv 1$.

3.2.2 Nonassociative configuration phase spaces' symmetries and Noether theorem

A specific type of nonassociative nonlinear symmetries (A.3) characterize the quasi-stationary off-diagonal solutions $[g^{\star}[\mathfrak{Cqs}]]$ (A.1). Introducing effective Lagrangians and actions as in (22), we impose other types of symmetries for nonassociative parametric gravitational and matter field interactions. Let us analyse how additional symmetries characterize respective effective systems.

Geometrically, we study additional symmetries defined as a vector field $\mathcal{X}^{\star}(u)$ on $\mathcal{E}_{\mathfrak{Cqs}}^{\star}$ such that

$${}^{}\partial^{\star}_{\,\,,\mathcal{X}}{}^{\,\,\prime}S^{\star} \equiv 0, \text{ where } {}^{}\partial^{\star}_{\,\,,\mathcal{X}}{}^{\,\,\prime}S^{\star} := \int \frac{{}^{\,\,\prime}\delta L^{\star}({}^{\,\,\prime}f)}{{}^{\,\,\prime}\delta{}^{\,\,\prime}\varphi^{\star}} {}^{\,\,\prime}\mathcal{X}^{\star}, \,\,{}^{\prime}f \equiv 1 \text{ on } \sup p \,\,{}^{\,\,\prime}\mathcal{X}^{\star}.$$

In abstract geometric form,

$${}^{\prime}\mathcal{X}^{\star} = \int {}^{\prime}\mathcal{X}^{\star}({}^{\prime}u) \frac{{}^{\prime}\delta}{{}^{\prime}\delta^{+}\varphi^{\star}} \in \Gamma(T {}^{\prime}\mathcal{E}^{\star}_{\mathfrak{Cqs}})$$
(23)

s-vectors for configurations ${}^{\mathsf{g}}\mathsf{s}[\mathfrak{Cqs}]$ involving s-adapted frames. The value ${}^{\mathsf{d}}\mathsf{s}_{\mathcal{X}} {}^{\mathsf{s}}S^{\mathsf{s}}$ is just the insertion of 1-form ${}^{\mathsf{d}}\mathsf{s}^{\mathsf{s}}$ into a vector field ${}^{\mathsf{s}}\mathcal{X}^{\mathsf{s}}$.

For our research, we can focus on local symmetries of a nonassociative system which can be expressed as $\mathcal{X}^* = I + \omega^* \rho^*(\mathsf{T}\xi)$, where I is a symmetry that vanishes identically on $\mathsf{T}_{Sol} \mathcal{E}^*_{\mathfrak{Cqs}}$ (22), ω^* is a local function defied as a map $\mathcal{E}^*_{\mathfrak{Cqs}} \to \mathcal{U}_{\mathfrak{Cqs}}$ and ρ^* determine infinitesimal symmetries, when $\mathsf{T}\xi \in \mathsf{T}_{\mathfrak{cqs}}$. Applying such formulas, the multiplication with an element of $\Gamma(T \mathcal{E}^*_{\mathfrak{Cqs}})$, and $\rho^* : \mathsf{T}_{\mathfrak{cqs}} \to \Gamma(T \mathcal{E}^*_{\mathfrak{cqs}})$ is a double Lie-algebra morphism defined by a given local action σ^* of Lie d-algebra $\mathsf{T}_{\mathfrak{cqs}}$ on $\mathsf{T}_{Sol} \mathcal{E}^*_{\mathfrak{cqs}}$ when

In these formulas, we use ${}^{}\mathfrak{g}_{c}$ with a subscript "c" stating that the maps are on corresponding spaces of smooth compactly supported sections over nonassociative deformed vector bundles over ${}^{}\mathcal{U}_{\mathfrak{Cqs}}$ and when the action ${}^{}\sigma^{*}$ on ${}^{'}_{Sol}\mathcal{E}^{*}_{\mathfrak{Cqs}}$ is defined to be local. For linear parametric deformations, we can consider nonassociative gauge gravity models with ${}^{}\rho^{*} \simeq \rho^{*}$ when nonassociativity is encoded into functionals like F^{*} , or ${}^{'}\varphi^{*}$ and ${}^{}\sigma^{*}$.

For any nonassociative gauge model defined by a quasi-stationary solution of (12), the presence of local symmetries implies that the effective equations of motion ${}^{'}_{Sol} \mathcal{E}^{\star}_{\mathfrak{Cqs}}$ (22) have orbits of the action ${}^{'}\sigma^{\star}$ (i.e. have redundancies). This is formulated mathematically as the second Noether theorem:

$$\int \frac{\delta^{\dagger} S^{\star}}{\delta^{\dagger} \varphi^{\star}_{\check{\alpha}}} \mathcal{X}^{\star}_{\check{\alpha}}(u) \delta^{8} \mu(u) = \int \varphi^{\star}_{\check{\beta}}(Q^{\check{\beta}}_{\check{\alpha}})^{*} \frac{\delta^{\dagger} S^{\star}}{\delta^{\dagger} \varphi^{\star}_{\check{\alpha}}} \delta^{8} \mu = 0,$$
(25)

for * denoting the formal adjoint of a differential operator obtained using integration by parts. Formulas (25) state the condition to be a symmetry for any local and compactly supported s-vector can be expressed as a differential s-operator

$${}^{\mathsf{'}}\mathcal{X}^{\star}_{\check{\alpha}}({}^{\mathsf{'}}u)[{}^{\mathsf{'}}\varphi^{\star}] = Q^{\check{\beta}}_{\check{\alpha}}({}^{\mathsf{'}}\varphi^{\star}) {}^{\mathsf{'}}\varphi^{\star}_{\check{\beta}}({}^{\mathsf{'}}u) = a({}^{\mathsf{'}}u)[{}^{\mathsf{'}}\varphi^{\star}] + b^{\alpha}({}^{\mathsf{'}}u)[{}^{\mathsf{'}}\varphi^{\star}] {}^{\mathsf{'}}\widehat{\mathbf{D}}^{\star}_{\alpha} {}^{\mathsf{'}}\varphi^{\star}({}^{\mathsf{'}}u) + \dots$$

The effective equations of motion with $\frac{|\delta' S^*|}{|\delta' \varphi_{\alpha}^*}$ and encoding nonassociative data are not all independent and related both to linear and nonlinear symmetries (A.3) stated for quasi-stationary solutions $|\mathbf{g}^*[\mathfrak{Cqs}]]$. For a general nonassociative gauge gravity theory with twisted star product, it is not possible to define a unique variational calculus and prove a general form of second Noether theorem (25) as in [36]. Such nonholonomic constructions can be performed in abstract geometric form with further parametric decompositions. This allows to define the space $|_{inv} \mathcal{F}^*_{\mathfrak{Cqs}}$ of functionals on the solution space $|_{Sol} \mathcal{E}^*_{\mathfrak{Cqs}}$ (22) that are invariant on the actions of symmetries $|\rho^*|$ encoding in parametric form off-diagonal solutions of gauge gravitational equations with nonassociative data.

3.3 Ghosts and the BV complex of nonassociative quasi-stationary solutions

Any class of quasi-stationary or locally anisotropic solutions $\mathbf{g}^{\star}[\mathfrak{Clacs}]$ or $\mathbf{g}^{\star}[\mathfrak{Cqs}]$ (19) may involve singularities, nonassociative data, gravitational polarizations, solitonic hierarchies etc. In the presence of local symmetries, the equations of motion (22) have redundancies when the Cauchy problem is not well posed. In certain cases, a redundancy can be removed by taking the quotient by the action of infinitesimal symmetries (25), which does not solve the issues related to the existence of nonlinear symmetries (A.3) and the non-variational properties of general twisted star products. We can approach such problems following the guiding idea of homology when (instead of taking quotients) we go to a large class of nonholonomic geometries with distortion of connections. For certain classes of nonassociative physically important systems of PDEs, we can prove general decoupling and integration properties and select equivalent classes of solutions with physically important properties, which are better behaved and can be used as a first step towards quantization. The goal

of this subsection is to characterize the space $_{inv}\mathcal{F}_{\mathfrak{Cqs}}^{\star}$ of functionals on $_{Sol}\mathcal{E}_{\mathfrak{Cqs}}^{\star}$ in a way that will facilitate quantization. Here we note that nonholonomic methods for deformation quantization of GR and Finsler-like MGT on phase spaces was elaborated in [34, 35]. In this work, those nonholonomic geometric constructions are extended for nonassociative theories with twisted star product.

3.3.1 Nonassociative local symmetries and ghosts

The homological interpretation of the associative and commutative parts of the space $\mathcal{F}_{\mathfrak{Cqs}}^{\star}$ (of functionals on the solution space $\mathcal{F}_{Sol}\mathcal{E}_{\mathfrak{Cqs}}^{\star}$ (22)) denoted respectively $\mathcal{F}_{\mathfrak{Cqs}}$ and $\mathcal{E}_{\mathfrak{Cqs}}$, is similar to that reviewed in section 2.3 of [25] and formulated in rigorous mathematical form in [37]. Those proofs can not be extended for general twist products when a unique variational formulation of physical models is not possible. Nevertheless, we can study possible physical implications of nonlinear and linear symmetries of respective classes of solutions that can be modified for parametric deformations on phase spaces to encode nonassociative geometric data using a respective abstract and nonholonomic geometric formalism and R-flux deformations.

We consider the Lie d-algebra ${}^{\mathsf{g}}_{c}$ on ${}^{\mathsf{g}}_{Sol} \mathcal{E}^{\mathsf{g}}_{\mathfrak{cqs}}$ characterizing the infinitesimal local symmetries as we explained in (24). Considering such symmetries as s-adapted derivations on functionals that are themselves compactly supported and considered for respective classes of solutions, we can work directly with ${}^{\mathsf{g}}$ (dropping the condition of compact support for symmetries on phase space and further deformations). For our purposes, we shall work with the space of symmetry-invariant variables defined by certain functionals ${}^{\mathsf{B}}{}^{\mathsf{K}}$ such that ${}^{\mathsf{I}}\partial_{{}^{\mathsf{I}}}\rho^{\mathsf{K}}({}^{\mathsf{I}}\xi)$ ${}^{\mathsf{B}}{}^{\mathsf{K}} =$ 0; $\forall {}^{\mathsf{I}}\xi \in {}^{\mathsf{I}}\mathfrak{g}$. For physical applications, such conditions can be satisfied in linear parametric form. Here we note that the spaces of invariants under the action of a Lie algebra as in the above formula and respective homological algebra and homology groups constructions can be characterized using the Chevalley-Eilenberg complex. We cite [37, 25] for precise definitions and emphasize that in this paper we work on phase spaces with d-algebras, i.e. couples of algebras corresponding to h- and c-spaces, which are star-product deformed in parametric form. Advanced topological homologic methods which are very sophisticate for researchers in theoretical physics are not considered in this work.

For further geometric constructions, we use a s-vector co-bundle ${}_{s}^{!}\mathcal{E}({}_{s}^{!}\mathcal{M})$ on respective phase space ${}_{s}^{!}\mathcal{M}$, when a star product (3) deforms such spaces into respective nonassociative ones labelled by a \star -symbol, ${}_{s}^{!}\mathcal{E}^{\star}$ on ${}_{s}^{!}\mathcal{M}^{\star}$, see details in [16]. A graded s-adapted phase s-vector co-bundle is by definition ${}_{s}^{!}\overline{\mathcal{E}} := {}_{s}^{!}\mathcal{E} \oplus {}^{!}\mathfrak{g}[1]$, and, in \star -deformed form, ${}_{s}^{!}\overline{\mathcal{E}}^{\star} := {}_{s}^{!}\mathcal{E}^{\star} \oplus {}^{!}\mathfrak{g}[1]$, when the functionals on ${}^{!}\mathfrak{g}[1]$ are identified with the exterior d-algebra over the duals ${}^{!}\mathfrak{g}'$, i.e. $\wedge {}^{!}\mathfrak{g}'$, for ${}^{!}\mathfrak{g} = (h {}^{!}\mathfrak{g}, c {}^{!}\mathfrak{g})$. The ghost phase space fields are introduced as evaluation functionals

$${}^{\mathsf{I}}\xi^{I}({}^{\mathsf{I}}u) = {}^{\mathsf{I}}c^{I}({}^{\mathsf{I}}u)[{}^{\mathsf{I}}\xi], \tag{26}$$

where an abstract index I is used. Such ghosts are related to the graded d-algebra of the Chevalley-Eilenberg complex

$$\mathcal{C}_{s}^{\mathsf{'}}\mathcal{E} := \mathcal{C}_{[ml]}^{\infty}(\ _{s}^{\mathsf{'}}\overline{\mathcal{E}}, \mathbb{C}) = (\wedge \ '\mathfrak{g}'\widehat{\otimes} \ '\mathcal{F}_{\mathfrak{Cqs}}, \ _{ce}^{\mathsf{'}}\gamma); \text{ and, for star product deformations,}$$

$$\mathcal{C}_{s}^{\mathsf{'}}\mathcal{E}^{\star} := \mathcal{C}_{[ml]}^{\infty}(\ _{s}^{\mathsf{'}}\overline{\mathcal{E}}^{\star}, \mathbb{C}) = (\wedge \ '\mathfrak{g}'\widehat{\otimes} \ '\mathcal{F}_{\mathfrak{Cqs}}^{\star}, \ _{ce}^{\mathsf{'}}\gamma),$$

$$(27)$$

where [ml] refers to the space of multilocal functionals on $\frac{1}{s}\overline{\mathcal{E}}$. In (27), $\widehat{\otimes}$ is the appropriately completed tensor product and the grading of $\mathcal{C}_{s}\mathcal{E}$ is called the pure ghost number $\#pg = (\#_{h} + \#_{c})pg$, defined as a sum of h-and c-ghosts.

Using the complex (27), we introduce the nonassociative Chevalley-Eilenberg differential $\downarrow_{ce} \gamma$, defined by

$$\begin{pmatrix} {}^{}_{ce}\gamma \ {}^{\prime}\mathcal{B}^{\star} \end{pmatrix} ({}^{\prime}\varphi^{\star}, {}^{\prime}\xi) := \partial_{{}^{\prime}\rho({}^{\prime}\xi)} \ {}^{\prime}\mathcal{B}^{\star} ({}^{\prime}\varphi^{\star}), \text{ for } {}^{\prime}\xi \in {}^{\prime}\mathfrak{g}', \text{ or } \\ {}^{}_{ce}\gamma \ {}^{\prime}\mathcal{B}^{\star} = \partial_{{}^{\prime}\rho({}^{\prime}c)} \ {}^{\prime}\mathcal{B}^{\star}, \text{ in terms of evaluation functionals, i.e. ghosts.}$$

In these formulas, $\mathcal{B}^{\star} \in \mathcal{F}^{\star}_{\mathfrak{Cqs}}$ and $\mathcal{B}^{\star} \in \mathcal{C}^{\infty}_{[ml]}(\mathcal{B}^{\star}, \mathfrak{g})$; and $\mathcal{B}^{\star}_{ce}\gamma$ encodes the action ρ of the gauges d-algebra \mathfrak{g} on $\mathcal{F}^{\star}_{\mathfrak{Cqs}}$. We express for the ghosts field (26)

$${}_{ce}^{}\gamma \ c = -\frac{1}{2} [\ c,\ c] \text{ and } {}_{ce}^{}\gamma \ \omega^{\star}(\ \xi_1,\ \xi_2) = \ \omega^{\star}([\ \xi_1,\ \xi_2]) \in \wedge^2 \ \mathfrak{g}'$$

for any form $\omega^{\star} \in [\mathfrak{g}']$. Because $[\mathfrak{g}'] \mathcal{B}^{\star} \equiv 0$ if $\mathcal{B}^{\star} \in [\mathfrak{g}']_{\mathfrak{Cqs}}$, we can consider that the zero holonomy group $H^0(\mathcal{C}_s \mathcal{E}^{\star})$ characterizes the nonassociative gauge invariant functionals. Such formulas generalize on (nonassociative) phase space the constructions from section 2.3.2 in [25].

3.3.2 Nonassociative classical BV complex

Let us characterize the nonassociative geometric properties of the space ${}_{inv}^{*}\mathcal{F}_{\mathfrak{Cqs}}^{*}$. We consider s-vector fields on ${}_{s}^{*}\overline{\mathcal{E}}^{*} := {}_{s}^{*}\mathcal{E}^{*} \oplus {}^{*}\mathfrak{g}[1]$ instead of s-vector fields on ${}_{s}^{*}\mathcal{E}^{*}$. We call ${}_{s}^{*}\overline{\mathcal{E}}^{*}$ an extended configuration phase space (equivalently, graded phase space) involving a nonholonomic splitting (with corresponding nonholonomic hand c-splitting). For quasi-stationary configurations, we take ${}_{s}^{*}\mathcal{E}^{*}$ as ${}_{Sol}^{*}\mathcal{E}_{\mathfrak{cqs}}^{*}$ and write ${}_{Sol}^{*}\overline{\mathcal{E}}^{*}$. Now, we are able to construct respective nonholonomic s-adapted, ${}^{*}\mathcal{BV}$, and nonassociative, ${}^{*}\mathcal{BV}^{*}$, BV complexes when $\star : {}^{*}\mathcal{BV} \to {}^{*}\mathcal{BV}^{*}$. We may distinguish such complexes by respective labels like ${}_{Sol}^{*}\mathcal{BV}^{*}$ or ${}^{*}\mathcal{BV}_{\mathfrak{cqs}}^{*}$. In abstract geometric form, a nonassociative BV complex is defined by an underlying d-algebra of multilocal polyvector fields (which can be s-adapted) on ${}_{s}^{*}\overline{\mathcal{E}}^{*}$. This space is modelled as the space of multilocal compactly supported functional on the graded nonholonomic s-adapted manifold (generalized cotangent s-vector bundle)

$$T^*[-1] {}_s^{\dagger} \overline{\mathcal{E}}^{\star} \equiv {}_s^{\dagger} \mathcal{E}^{\star}[0] \oplus {}^{\dagger} \mathfrak{g}[\mathbf{1}] \oplus ({}_s^{\dagger} \mathcal{E}^{\star})^{!}[-\mathbf{1}] \oplus {}^{\dagger} \mathfrak{g}^{!}[-\mathbf{2}],$$
(28)

where labels (with ! and possible additionally ones with Sol, or \mathfrak{Cqs}) are explained in footnote 5. In explicit form, the elements of ${}^{\!\!\!/}\mathcal{BV}^{\star}$ are multilocal functionals of the nonassociative field multiplets ${}^{\!\!\!/}\varphi^{\star} = \{ {}^{\!\!\!/}\varphi^{\star}_{\tilde{\alpha}} \}$ and of corresponding antifields ${}^{\!\!\!/}_{\frac{1}{2}}\varphi^{\star} = \{ {}^{\!\!\!/}_{\frac{1}{2}}\varphi^{\star}_{\tilde{\alpha}} \}$ as defined for formulas (23). In a more general context, an index $\check{\alpha}$ runs through all the physical and ghost indices on nonassociative phase spaces, with possible nonholonomic dyadic or h- and c-splitting of phase space. We use the convention that the phase anti-field are right derivatives $\frac{{}^{\!\!/}_{\tau}\delta}{\delta^{\!\!\!/}\varphi_{\tilde{\alpha}}}$, with for graded manifolds are different from the left derivatives $\frac{{}^{\!\!\!/}_{\ell}\delta}{\delta^{\!\!\!/}\varphi_{\tilde{\alpha}}}$. For phase and spacetime indices, such derivatives can be defined in N-adapted or s-adapted forms and then subjected to star product deformations.⁶

The complex \mathcal{BV}^* can be considered as the space of graded multivector fields quipped with a generalized Schouten bracket (which is an antibracket),

$$\star: \{ \ \mathcal{X}, \ \mathcal{Y}\} \to \{ \ \mathcal{X}^{\star}, \ \mathcal{Y}^{\star}\} := \sum_{\check{\alpha}} \left(\langle \frac{{}_{i}^{\dagger} \delta^{} \mathcal{X}^{\star}}{\delta^{} \varphi_{\check{\alpha}}}, \frac{{}_{l}^{\dagger} \delta^{} \mathcal{Y}^{\star}}{\delta^{}_{\pm} \varphi_{\check{\alpha}}^{\star}} \rangle + \langle \frac{{}_{i}^{\dagger} \delta^{} \mathcal{X}^{\star}}{\delta^{}_{\pm} \varphi_{\check{\alpha}}^{\star}}, \frac{{}_{l}^{\dagger} \delta^{} \mathcal{Y}^{\star}}{\delta^{} \varphi_{\check{\alpha}}} \rangle \right)$$

To define variational configurations, we use linear parametric deformations. Using such an antibracket, we can introduce locally a right derivation $\delta_S \rightarrow \delta_S^*$, when

$$\begin{split} ^{\dagger} \delta_{S}^{\star} \,^{\dagger} \mathcal{X}^{\star} &= \{ {}^{\dagger} \mathcal{X}^{\star}, \, {}^{\dagger} L^{\star} ({}^{\dagger} f) \}, \, {}^{\dagger} f \equiv 1 \text{ on } \sup p \,^{\dagger} \mathcal{X}, \, {}^{\dagger} \mathcal{X} \in \,^{\dagger} \mathcal{V}^{\star}; \\ &= \{ {}^{\dagger} \mathcal{X}^{\star}, \, {}^{\dagger} S^{\star} ({}^{\dagger} f) \}. \end{split}$$

⁶Both in h- and c-forms, the d-algebra has two gradings: the ghost numbers $\#gh = (\#_h + \#_c)gh$, for the main phase space gradings, and the antifield numbers $\#af = (\#_h + \#_c)af$. In this work, we consider a gauge gravity theory on phase space with a structure d-group 'g when the R-flux deformations result in effective variational model with distorted connections and sources but preserving the ghost and antighost number and 'g. For such conditions, the functionals of physical fields on (nonassociative) phase spaces have both numbers equal to 0; and, for functionals of ghosts, #af = 0 and #gh = #pg, when, for the "pure" ghost grading, a phase ghost 'c has #pg = 1. All s-vector fields have a non-zero antifield number computed $\#af(\ _{\pm}^{i}\varphi_{\alpha}^{\star}) = 1 + \#pg(\ '\varphi_{\alpha}^{\star}),$ and #gh = -#af.

Because we work with linear parametric decompositions, for any $\gamma \mathcal{X}^*$, we can find an effective action Θ^* that $\gamma \mathcal{X}^* = \{\mathcal{X}^*, \Theta^*\}$. Using such effective decompositions, we define the *nonassociative classical BV differential* as

$$\aleph^{\star} = \{\bullet, \ |S^{\star} + \ |\Theta^{\star}\} := \{\bullet, \ |S_{ext}^{\star}\}, \text{ with extended action } \ |S_{ext}^{\star}.$$
(29)

The nilpotent property, $(\ \aleph^{\star})^2 = 0$, results in the nonassociative classical master equation (nCME), which modulo terms vanishing in the limit of a constant $\ f$ is written in the form

$$\{ L_{ext}^{\star}(f), L_{ext}^{\star}(f) \} = 0,$$
(30)

for an effective ${}^{\perp}L_{ext}^{\star}$ used for constructing ${}^{\perp}S_{ext}^{\star}$.⁷ Here we note that the operator ${}^{\perp}\aleph^{\star}$ increases the ghost number by one (it is of order 1 in #gh) and can be expressed as ${}^{\perp}\aleph^{\star} = {}^{\perp}\delta^{\star} + {}^{\perp}\gamma^{\star}$, where the extension of ${}^{\perp}\delta_{S}^{\star}$ is denoted ${}^{\perp}\delta^{\star}$ (an operator of order -1 in #af) and the extension of ${}^{\perp}_{ce}\gamma$ is denoted ${}^{\perp}\gamma^{\star}$ (an operator of order 0).

Above formulas (29) and (30) define a nonassociative variant of the Koszul-Tate complex (${}^{\prime}\mathcal{BV}_{\mathfrak{Cqs}}^{\star}, {}^{\prime}\delta^{\star}$) which is a resolution for nonlinear and linear symmetries considered in our nonassociative gauge gravity theory. It would be noted a resolution if the (non) linear symmetries were not independent. ${}^{\prime}\mathcal{BV}_{\mathfrak{Cqs}}^{\star}$ has a simpler algebraic structure when the quotients/ spaces of orbits and nonlinear symmetries are resolved than in the case of ${}^{\prime}_{inv}\mathcal{F}_{\mathfrak{Cqs}}^{\star}$.

To introduce the gravitational gauge fixing we use a star product automorphism acting on s-adapted generators in the form

$${}^{\scriptscriptstyle |}\alpha_{\psi}^{\star}({}^{\scriptscriptstyle |}_{\ddagger}\phi_{\check{\alpha}}^{\star}({}^{\scriptscriptstyle |}u)) := \frac{{}^{\scriptscriptstyle |}\delta^{\star}{}^{\scriptscriptstyle |}\psi^{\star}({}^{\scriptscriptstyle |}f)}{{}^{\scriptscriptstyle |}\delta{}^{\scriptscriptstyle |}\varphi_{\check{\alpha}}({}^{\scriptscriptstyle |}u)} \text{ and } {}^{\scriptscriptstyle |}\alpha_{\psi}^{\star}({}^{\scriptscriptstyle |}\phi^{I}({}^{\scriptscriptstyle |}u)) = {}^{\scriptscriptstyle |}\phi^{I}({}^{\scriptscriptstyle |}u),$$

where ${}^{!}f({}^{!}u) = 1$, see formulas (26), and a gauge fixing phase space fermion ${}^{!}\psi_{M}^{*}({}^{!}f)$ is stated as a fixed generalized Lagrangian of ghost number -1. Similarly to [38, 25], we can choose ${}^{!}\alpha_{\psi}^{*}$ such that #af = 0, and in a form that ${}^{!}\alpha_{\psi}^{*}$ lives the star product antibracket invariant when the R-flux deformed action results into hyperbolic equations. Here we note that a Lorenz-like gauge for nonassociative gauge gravity theory (12) we need to extend the BV complex with antighosts ${}^{!}\overline{C}_{\star} = \{ {}^{!}\overline{C}_{\star}^{I} \}$, of degree -1, and so-called Nakanishi-Lautrup field ${}^{!}B_{\star} = \{ {}^{!}B_{\star}^{I} \}$, degree 0, which form a trivial pair when ${}^{!}\aleph^{*}|_{c}\overline{C}_{\star}^{I} = i {}^{!}B_{\star}^{I}$ and ${}^{!}\aleph^{*}|_{c}B_{\star}^{I} = 0$. Such operators act on an extended nonassociative configuration space ${}^{!}_{s}\overline{\mathcal{E}}^{*} \equiv {}^{!}_{s}\mathcal{E}^{*}[0] \oplus {}^{!}\mathfrak{g}[\mathfrak{o}] \oplus {}^{!}\mathfrak{g}[-\mathfrak{1}]$, when the gauge fixing fermion defined and computed

$$\psi^{\star}(\ {}^{\scriptscriptstyle \mathsf{f}} f) = i \int_{M} \left(\frac{\alpha}{2} \kappa(\ {}^{\scriptscriptstyle \mathsf{T}} \overline{C}_{\star},\ {}^{\scriptscriptstyle \mathsf{t}} B_{\star}) + < \ {}^{\scriptscriptstyle \mathsf{T}} \overline{C}_{\star}, \, \breve{\ast} \ {}^{\scriptscriptstyle \mathsf{t}} d(\breve{\ast} \ {}^{\scriptscriptstyle \mathsf{t}} \widehat{\mathcal{A}}^{\star}) > \ {}^{\scriptscriptstyle \mathsf{t}} \delta \ {}^{\scriptscriptstyle \mathsf{t}} \mu_{\mathfrak{Cqs}}(\ {}^{\scriptscriptstyle \mathsf{t}} u), \right)$$

with a measure as in (20) and $\widehat{\mathcal{A}}^{\star}$ (13), see details in section 2.3 of [25].

We conclude that for gauge-fixed theories, gradings are convenient to be redefined considering that #ta is the total antifield number (1 for the antifield generators and 0 for fields). Using the decomposition $\aleph^* = \delta^* + \gamma^*$ and for a total action S^*_{ext} , where S^* denotes the #ta = 0 term, and when $\Theta^* := S^*_{ext} - S^*$ as in formulas (29). Expressing

$$\delta^{\star} = \{\cdot, S^{\star}\} \text{ and } \gamma^{\star} = \{\cdot, \Theta^{\star}\},$$

where the differential δ^* acts trivially both on fields and antifields (giving $\delta^* + \varphi^*_{\alpha} = \frac{\delta + S^*}{\delta + \varphi^*_{\alpha}}$), we derive nonassociative gauge-fixed equations of motion which are hyperbolic equations of motion of S^* . This is true for any solution of type $\delta_{Sol} \mathcal{E}^*_{cqs}$ and respective nonlinear symmetries.

⁷For associative and commutative configurations on a Lorentz manifold, the formula (30) transforms into the CME (11) in [25], where another system of notations is used. In this work, we have to consider a star product deformed classical BV differential (29) with an "abuse" of notations, like ' \aleph^* and ' Θ^* , because we elaborate our theory on nonassociative phase spaces which requires more sophisticated geometric and index-type notations.

3.4 Linearization and nonassociative classical BV operator and the Møller maps

On nonassociative phase spaces with parametric decompositions, we can split and linearize respectively the extended action S_{ext}^{\star} in a form that

$${}^{+}S_{0}^{\star} = {}^{+}S_{00}^{\star} + {}^{+}\Theta_{0}^{\star}, \text{ the quadratic term in (anti) fields, } \#ta({}^{+}S_{00}^{\star}) = 0, \#ta({}^{+}\Theta_{0}^{\star}) = 1;$$

$${}^{+}V_{0}^{\star} = {}^{+}V_{0}^{\star}, \text{ the interacton term };$$

$${}^{+}S_{00}^{\star} + {}^{+}V_{0}^{\star}, \text{ the total antifield independent part of the action.}$$

$$(31)$$

Using above formulas we can defined the linearized nonassociative differentials (respective BRST and BV type):

$$^{\prime}\gamma^{\star} ^{\prime}\mathcal{F} := \{ ^{\prime}\mathcal{F}, ^{\prime}\Theta_{0}^{\star} \} \text{ and } ^{\prime}\aleph_{0}^{\star} = ^{\prime}\delta_{0}^{\star} + ^{\prime}\gamma_{0}^{\star},$$
(32)

where ${}^{\dagger}\delta_0^{\star}({}^{\dagger}_{\dagger}\varphi_{\star}^{\check{\alpha}}) = -\frac{{}^{\dagger}\delta_{00}}{\delta_{}\varphi_{\star}^{\star}}$. Such operators allow us to introduce two important nonassociative differential operators, ${}^{\dagger}P_{\star}^{\check{\alpha}\check{\beta}}({}^{\dagger}u)$ and ${}^{\dagger}K_{\star\check{\beta}}^{\check{\alpha}}$, when

$$\frac{{}^{!}_{l}\delta^{-}S^{\star}_{00}}{\delta^{-}\varphi^{\star}_{\check{\alpha}}}({}^{-}\varphi^{\star}_{\check{\alpha}}) := {}^{-}P^{\check{\alpha}\check{\beta}}_{\star}({}^{-}u)({}^{-}\varphi^{\star}_{\check{\beta}}), \text{ in brief } = {}^{-}P_{\star}{}^{-}\varphi^{\star}; \text{ and } \frac{{}^{!}_{r}\delta^{!}_{l}\delta^{-}\Theta^{\star}_{0}}{\delta^{-}\varphi^{\star}_{\check{\alpha}}({}^{-}u_{1})\delta^{+}_{+}\varphi^{\check{\beta}}_{\star}({}^{-}u_{1})}({}^{-}\varphi^{\star}_{\check{\alpha}}) := {}^{-}K^{\check{\alpha}}_{\star\check{\beta}}({}^{-}u)({}^{-}\varphi^{\star}_{\check{\beta}}).$$

On nonassociative phase space, we can assume a gauge fixing in such a way that ${}^{P_{\star}}$ is Green hyperbolic for any *h*- and *c*-component as for (associative and commutative) gauge and gravity theories was shown in [38]. We introduce a double Green function of motion operator ${}^{P_{\star}}$ in canonical form (with hat d-operators) by ${}^{g^{\star}}[\mathfrak{Clacs}]$, when ${}^{i}\widehat{\Delta}_{\star}^{A/R} = (h \Delta_{\star}^{A/R}, c \Delta_{\star}^{A/R})$, where *A* and *B* mean respectively "advanced" and "retarded". This allows to define a nonassociative Pauli-Jordan function

$$^{\downarrow}\widehat{\triangle}_{\star} = {}^{\downarrow}\widehat{\triangle}_{\star}^{R} - {}^{\downarrow}\widehat{\triangle}_{\star}^{A} \tag{33}$$

and prove in abstract geometric form such properties (nonassociative generalizations of [38, 39, 40]): For any $\hat{\Delta}_{\star}$ being a retarded, advanced or causal propagator corresponding to $P_{\star} \varphi^{\star} = 0$, there are satisfied the consistency conditions

$$\sum_{\breve{\beta}} \left[(-1)^{|\breve{\alpha}|} K^{\breve{\alpha}}_{\star\breve{\beta}}(|u')|^{\breve{\alpha}} \widehat{\Delta}^{\star\breve{\beta}}_{\star}(|u',|u) + K^{\breve{\gamma}}_{\star\breve{\beta}}(|u|)|^{\breve{\alpha}} \widehat{\Delta}^{\star\breve{\alpha}}_{\star}(|u',|u) \right] = 0, \tag{34}$$

which are determined for a S_{00}^{\star} being invariant under γ_0^{\star} , and when the nCME (30) are satisfied.

For associative and commutative configurations, we can define an interacting quantum BV operator [41] by taking a free one and twisting it with quantum Møller map [42, 43]. Then, it is possible to prove that the resulting method is local. Nonassociative theories with twister star product (3) introduce a generic nonlocal structure both for classical and quantum models. The advantage of the cited mathematical formulas is that it works for nonlocal operators on phase spaces. In this subsection, we consider such constructions for nonassociative classical gauge gravity models which in parametric form can be defined by an effective action $S^* = S_0^* + V^*$,. Here, the interaction term V^* is a star product deformation of local compactly supported functions.

We assume that S^* and necessary functionals $\mathcal{F}_{\mathfrak{Cqs}}^*$ are determined on the solution space $\mathcal{F}_{Sol}\mathcal{E}_{\mathfrak{Cqs}}^*$ (22)), with respective non-trivial (non) linear symmetries. We define a nonassociative generalization of the maps from [42, 43] in the form:

In above formulas, the maps are inverted as formal power series on small λ when $r_{\lambda V}^{\star}$ goes from certain nonassociative interactions to a fee theory. One holds the so-called intertwining relation:

$$[{}^{+}r^{\star}_{\lambda V} ({}^{+}\mathcal{F}^{\star}_{\mathfrak{C}\mathfrak{q}\mathfrak{s}}), {}^{+}r^{\star}_{\lambda V} ({}^{+}\mathcal{G}^{\star}_{\mathfrak{C}\mathfrak{q}\mathfrak{s}})] = {}^{-}r^{\star}_{\lambda V} [{}^{+}\mathcal{F}^{\star}_{\mathfrak{C}\mathfrak{q}\mathfrak{s}}, {}^{+}\mathcal{G}^{\star}_{\mathfrak{C}\mathfrak{q}\mathfrak{s}}] {}^{+}r^{\star}_{\lambda V},$$

where, respectively, [., .] and $[., .]_V$ are the free and the interacting Poisson brackets (as for the Pielers bracket). For instance, we have

$$[{}^{\mathsf{T}}\mathcal{F}^{\star}, {}^{\mathsf{T}}\mathcal{G}^{\star}] = \sum_{\check{\alpha},\check{\beta}} \langle \frac{{}^{\mathsf{T}}\delta^{\mathsf{T}}\mathcal{F}^{\star}}{\delta^{\mathsf{T}}\varphi_{\check{\alpha}}^{\star}}, {}^{\mathsf{T}}\widehat{\Delta}_{\star\check{\alpha}\check{\beta}} \frac{{}^{\mathsf{T}}\delta^{\mathsf{T}}\mathcal{G}^{\star}}{\delta^{\mathsf{T}}\varphi_{\check{\beta}}^{\star}} \rangle,$$
(36)

for ${}^{+}\mathcal{F}^{\star}$, ${}^{+}\mathcal{G}^{\star} \in {}^{+}\mathcal{B}\mathcal{V}^{\star}_{\mathfrak{Cqs}}$. To close ${}^{+}\mathcal{B}\mathcal{V}^{\star}_{\mathfrak{Cqs}}$ under such brackets we have to extend the constructions to a large space, for instance, considering microcausal functions on $T^{*}[-1]_{s}^{+}\overline{\mathcal{E}}^{\star}$ (28). This allows us to work with functionals that are smooth and compactly supported satisfying conditions similar to (14) - (16) in [25]. The effective potential ${}^{+}V^{\star}_{(1)}({}^{+}\varphi^{\star})$ in (35) is defined in a form that ${}^{+}r^{\star-1}_{\lambda V}$ maps the solutions encoded into free equations of motion into the solutions generated by the interacting equations of motion, i.e.:

$${}^{\prime}r_{\lambda V}^{\star-1}\frac{\delta {}^{\prime}S_{0}^{\star}}{\delta {}^{\prime}\varphi^{\star}} = {}^{\prime}r_{\lambda V}^{\star-1}({}^{\prime}P_{\star}{}^{\prime}\varphi^{\star}) = {}^{\prime}P_{\star}{}^{\prime}\varphi^{\star} + {}^{\prime}\lambda {}^{\prime}P_{\star} \circ {}^{\prime}\widehat{\bigtriangleup}_{\star}^{R}{}^{\prime}V_{(1)}^{\star}({}^{\prime}\varphi^{\star}) = {}^{\prime}P_{\star}{}^{\prime}\varphi^{\star} + {}^{\prime}\lambda {}^{\prime}V_{(1)}^{\star}({}^{\prime}\varphi^{\star}).$$

The formulas (35) can be generalized to the case when nonassociative gauge symmetries are present. In such cases, ${}^{'}S_{0}^{\star}$ has two terms then the first one (${}^{'}S_{00}^{\star}$, it does not depend on antifields) defines ${}^{'}P_{\star}$ and ${}^{'}\widehat{\Delta}_{\star}^{R}$. We can consider

Using such operators, we can formulate and prove such results for $nCME(S^*)$, i.e. the classical master equations for the nonassociative gauge gravity with star product deformation (3):

which allows to write the classical BV operator (29) of the nonassociative gauge gravity theory as

$$^{\mathsf{'}}\aleph^{\star} = {^{\mathsf{'}}}r_{\lambda V}^{\star-1} \circ {^{\mathsf{''}}\aleph_0^{\star}} \circ {^{\mathsf{''}}}r_{\lambda V}^{\star}. \tag{38}$$

Details on proofs of formulas (37) and (38) in associative and commutative forms are provided in [25] (for the Theorem 2.15 and Corollary 2.16 and appendix of that work) and [41]. Nonassociative generalizations can be obtained using the abstract geometric formalism for nonholonomic parametric deformations on phase spaces.

4 BV-scheme and quantization of nonassociative gauge de Sitter gravity

The goal of this section is to show how the BV formalism [20, 21, 23, 24] can be applied for quantizing quasiparametric off-diagonal solutions. We cite also recent mathematical developments [25, 37, 40, 26, 39, 41, 42, 43] using certain methods for quantizing the classical nonassociative gauge gravity theories from the previous two sections. The twisted star product (3) can be written in the form $\star = \star_{\hbar} + \star_{\kappa}$, where \star_{\hbar} includes terms with \hbar^0 and \hbar^1 and \star_{κ} encodes terms with are proportional to κ with possible mutiplications on terms which are also \hbar^0 - and \hbar^1 -parametric. The quantization scheme is elaborated using the Planck constant \hbar and when the string parameter κ defines nonassociative R-flux deformations. For simplicity, we consider only the κ -linear terms when the higher order contributions can be computed recurrently.

4.1 Free nonassociative gauge gravitational fields

We can construct a quantized algebra of free nonassociative filed using deformation quantization, DQ, encoding also κ -parametric R-flux deformation of the classical s-algebras $(\ _{\mu c} \mathcal{BV}^{\star}_{\mathfrak{Eqs}}, [\cdot, \cdot])$. Such noncommutative gauge gravity theories were elaborated in [28, 29] when the DQ formalism for GR and MGTs was studied in N-adapted form in [34, 35]. In our approach, we equip the nonassociative phase space of formal power series $\ _{\mu c} \mathcal{BV}^{\star}_{\mathfrak{Eqs}}[[\hbar, \kappa]]$ with a nonassociative star product $\star = \star_{\hbar} + \star_{\kappa}$ (3), when the noncommutative component \star_{\hbar} is used for defining the quantum s-operator product of quantum observables as in [43, 25, 42, 41]. Here we note that the meaning of the string constant κ differs from that of \hbar used for quantization.

For any s-operators \mathcal{F}^{\star} and \mathcal{G}^{\star} , we can consider such a deformation of the point wise product,

where the multiplication s-operator is considered on associative and commutative phase space as

$${}^{\scriptscriptstyle |}m({}^{\scriptscriptstyle |}\mathcal{F}^{\star}\otimes_{s}{}^{\scriptscriptstyle |}\mathcal{G}^{\star})({}^{\scriptscriptstyle |}\varphi^{\star})={}^{\scriptscriptstyle |}\mathcal{F}^{\star}({}^{\scriptscriptstyle |}\varphi^{\star})\cdot_{s}{}^{\scriptscriptstyle |}\mathcal{G}^{\star}({}^{\scriptscriptstyle |}\varphi^{\star}),$$

and a functional differential s-operator is defined $D_W^{\star} := \frac{1}{2} \sum_{\check{\alpha},\check{\beta}} \langle W_{\check{\alpha}\check{\beta}}^{\star}, \frac{V_{i}\delta}{\delta \varphi_{\check{\alpha}}^{\star}} \otimes_s \frac{V_{i}\delta}{\delta \varphi_{\check{\beta}}^{\star}} \rangle$. In this formula, we use $W_{\check{\alpha}\check{\beta}}^{\star}$ as the phase space R-flux deformation of so called 2-point function of a Hardmard state. In [44, 39], similar details are provided on how to chose such an W-operator to be positive definite, satisfy the appropriate wave conditions and when

$$W^{\star}_{\check{\alpha}\check{\beta}} = rac{i}{2} \ \hat{\bigtriangleup}_{\star\check{\alpha}\check{\beta}} + \ H^{\star}_{\check{\alpha}\check{\beta}}$$

In this formula, ${}^{'}H^{\star}_{\check{\alpha}\check{\beta}}$ is a symmetric bi-solution for ${}^{'}P_{\star}$. The nonholonomic s-adapted structure can be prescribed in such a form that additionally to the standard properties there satisfied also the consistency condition (see (34)) for the symmetric part:

$$\sum_{\breve{\beta}} \left[(-1)^{|\ '\varphi^{\star}_{\breve{\alpha}}|\ '} K^{\breve{\alpha}}_{\star\breve{\beta}}(\ 'u') \ 'H^{*\breve{\beta}\breve{\gamma}}_{\star}(\ 'u',\ 'u) + \ 'K^{\breve{\gamma}}_{\star\breve{\beta}}(\ 'u) \ 'H^{*\breve{\alpha}\breve{\beta}}_{\star}(\ 'u',\ 'u) \right] = 0.$$
(39)

So, γ_0^{\star} is a right derivation with respect to the star product if the conditions (39) are satisfied. Here we note that δ_0^{\star} is also a right derivation with respect to \star since $W_{\check{\alpha}\check{\beta}}^{\star}$ is a solution for the linearized parametric nonassociative phase space equations of motion s-operator P_{\star} .

4.2 Interacting nonassociative gauge gravitational fields

In this subsection, we consider the space $_{reg}^{}\mathcal{BV}_{\mathfrak{Cqs}}^{\star}$ of regular functional for which the derivatives at every point are smooth compactly supported functions on $T^{*}[-1]_{s}^{+}\overline{\mathcal{E}}^{\star}$ (28).

4.2.1 Time-ordered products and Peierls bracket

For any \mathcal{F}^{\star} , the time-ordering operator, \mathcal{T}^{\star} , is defined using the formula for an internal kernel, $K_{\varkappa,\breve{e}}^{\star}$,

$${}^{\mathsf{T}}\mathcal{T}^{\star}{}^{\mathsf{T}}\mathcal{F}^{\star}({}^{\mathsf{T}}\varphi^{\star}) := e^{\frac{\hbar}{2}} {}^{\mathcal{T}}\mathcal{T}^{\star}_{\mathsf{T}}\hat{\Delta}_{\star}^{F}}, \text{ for } {}^{\mathsf{T}}\mathcal{D}_{K}^{\star} := \sum_{\check{\alpha},\check{\beta}} \langle {}^{\mathsf{T}}K_{\check{\alpha}\check{\beta}}^{\star}, \frac{}{\delta} {}^{\mathsf{T}}\varphi_{\check{\alpha}}^{\star} \otimes_{s} \frac{}{\delta} {}^{\mathsf{T}}\varphi_{\check{\beta}}^{\star} \rangle, \tag{40}$$

is defined to include parametric κ -terms from \star_{κ} and $\widehat{\}\widehat{\bigtriangleup}^{F}_{\star} = \frac{i}{2} \left(\widehat{\}\widehat{\bigtriangleup}^{R}_{\star} + \widehat{\}\widehat{\bigtriangleup}^{A}_{\star} \right) + \widehat{\}H_{\star}$. We state that formally \mathcal{T}^{\star} is a s-operator of convolution corresponding to oscillating Gaussian measure with covariance $i\hbar \widehat{\}\widehat{\bigtriangleup}^{F}_{\star}$ when,

$$\mathcal{T}^{\star} \mathcal{F}^{\star}(\mathcal{\varphi}^{\star}) = \int \mathcal{F}^{\star}(\mathcal{\varphi}^{\star} - \mathcal{\varphi}^{\star}_{1}) \mathcal{\delta} \mathcal{\mu}_{i\hbar} \mathcal{A}_{\star}^{F}(\mathcal{\varphi}^{\star}_{1})$$

In such a formula, $\delta' \mu_{i\hbar} \cap \widehat{\Delta}_{\star}^{F}(\varphi_{1}^{\star})$ is chosen to correspond to some $\delta' \mu_{\mathfrak{Cqs}}(u)$. Using such formulas, we define the time-ordered product τ_{\star} on $\Gamma_{reg} \mathcal{BV}_{\mathfrak{Cqs}}^{\star}[[\hbar, \kappa]]$ as

$$\mathcal{F}_1^{\star} \cdot_{\mathcal{T}^{\star}} \,\,^{\vee} \mathcal{F}_2^{\star} := \,\,^{\vee} \mathcal{T}^{\star} (\,\,^{\vee} \mathcal{T}^{\star - 1} \,\,^{\vee} \mathcal{F}_1^{\star} \cdot \,\,^{\vee} \mathcal{T}^{\star - 1} \,\,^{\vee} \mathcal{F}_2^{\star}).$$

We note that $\cdot_{\mathcal{T}^{\star}}$ consists a time-ordered version of $\star = \star_{\hbar} + \star_{\kappa}$ (3) when

Let us consider as in AQFT a QFT model of assigning s-algebras of observables

$$\mathcal{U}^{\star}(\mathcal{O}^{\star}) = (\mathcal{U}^{\star}_{uc} \mathcal{BV}^{\star}_{\mathfrak{Cas}}[[\hbar,\kappa]], \star_{\hbar} + \star_{\kappa}) \subset \mathcal{U}^{\star}_{s} \mathcal{M}^{\star}$$

on a given nonassociative phase space. In parametric form, we approximate $\mathcal{U}^*(\mathcal{O}^*) \approx \mathcal{U}^*_{\mathfrak{Cqs}}(\mathcal{O}^*)$ for a compact phase space region $\mathcal{O}^* = {}^{'}_{s}\mathcal{O}^* = (h \mathcal{O}^*, c \mathcal{O}^*)$ defined by a $[\mathfrak{g}^*[\mathfrak{Cqs}]]$ (19), when a phase space differential \mathcal{O}^* and star product * are assigned of such chain of complexes in s-algebras of regions. Here we note that the decoupling properties of nonassociative gauge gravity equations does not depend on the existence, or not, of certain Cauchy hypersurfaces in ${}^{'}_{s}\mathcal{M}^*$. We can consider such a hypersurface in the form that every in-extendible causal curve intersects it exactly once (with a conventional splitting on h- and v-parts) in any neighborhood ${}^{'}_{s}\mathcal{N}^* \subset {}^{'}_{s}\mathcal{M}^*$. Above stated conditions allow us to extend on nonassociative phase spaces two important axioms considered of associative and commutative spacetimes in section 3.2.2 of [25]:

- Axiom 1 on weak causality on nonassociative phase space ${}^{!}{}_{s}\mathcal{M}^{\star}$: For any h- and c-components which are of spacelike signature, the commutator $[{}^{!}\mathcal{U}^{\star}({}^{!}\mathcal{O}_{1}^{\star}), {}^{!}\mathcal{U}^{\star}({}^{!}\mathcal{O}_{1}^{\star})] = {}^{!}d^{\star} {}^{!}\mathcal{X}^{\star}$ for some ${}^{!}\mathcal{X}^{\star} \in {}^{!}\mathcal{U}^{\star}({}^{!}\mathcal{O}^{\star})$ when ${}^{!}\mathcal{O}^{\star}$ contains both ${}^{!}\mathcal{O}^{\star}_{1}$ and ${}^{!}\mathcal{O}^{\star}_{2}$.
- Axiom 2 on time-slicing of nonassociative phase space ${}^{!}_{s}\mathcal{M}^{\star}$: For any ${}^{!}_{s}\mathcal{N}^{\star}$ of a Cauchy hypersurface in the region ${}^{!}\mathcal{O}^{\star} \subset {}^{!}_{s}\mathcal{M}^{\star}$, the map ${}^{!}\mathcal{U}^{\star}({}^{!}\mathcal{N}^{\star})$ and ${}^{!}\mathcal{U}^{\star}({}^{!}\mathcal{O}^{\star})$ are quasi-isomorphic (i.e. are isomorphic on the level of cohomology groups of corresponding h- and c-components).

$$i\hbar[\ \mathcal{F}_{2}^{\star},\ \mathcal{F}_{1}^{\star}]_{\star} = \mathcal{F}_{2}^{\star} \star \mathcal{F}_{1}^{\star} - \mathcal{F}_{1}^{\star} \star \mathcal{F}_{2}^{\star} = \mathcal{F}_{2}^{\star} \star \mathcal{F}_{+}^{\star}(\mathcal{F}_{1}^{\star}) - \mathcal{F}_{-}^{\star}(\mathcal{F}_{1}^{\star}) \star \mathcal{F}_{2}^{\star}$$
$$= \mathcal{F}_{2}^{\star} \mathcal{F}_{\tau}^{\star} \mathcal{F}_{+}^{\star}(\mathcal{F}_{1}^{\star}) - \mathcal{F}_{-}^{\star}(\mathcal{F}_{1}^{\star}) \mathcal{F}_{\tau}^{\star} \mathcal{F}_{2}^{\star} \mod \mathrm{Im} \mathcal{I}d^{\star},$$

considering modulo the image od of d^* . This allows us to express a quantum nonassociative version of the Peierls bracket (36), when

$$i\hbar[\mathcal{F}_2^\star, \mathcal{F}_1^\star]_\star = \mathcal{F}_2^\star \cdot_{\mathcal{T}^\star} \mathfrak{M}_0^\star \mathcal{F}_3^\star \mod \hbar^2, \operatorname{Im} \mathcal{U}^\star.$$

For $d^{\star} \mathcal{F}_2^{\star} = 0$, we can introduce an antibracket in the right side,

$$i\hbar[\ \mathcal{F}_2^{\star},\ \mathcal{F}_1^{\star}] = \ \mathcal{R}_0^{\star}(\ \mathcal{F}_2^{\star},\ \mathcal{F}_3^{\star}) + i\hbar\{\ \mathcal{F}_2^{\star},\ \mathcal{F}_3^{\star}\} \bmod \hbar^2, \operatorname{Im} \ d^{\star}.$$

So, we can consider

$$[\mathcal{F}_2^{\star}, \mathcal{F}_1^{\star}] = \{ \mathcal{F}_2^{\star}, \mathcal{F}_3^{\star} \} \mod \hbar, \operatorname{Im} d^{\star}$$

as the intrinsic definition of the Peierls bracket for a given antibracket and the time-ordered product in a nonassociative phase space theory satisfying the Axiom 2.

4.2.2 Interactions and the renormalization problem

We model nonassociative phase space interactions (31) as in the classical master equations (37) assuming that $V^* \in {}^{-}_{reg} \mathcal{BV}^*_{\mathfrak{Cqs}}$. The quantum observable of the free theory associated with this nonassociative phase space potential as $T^* V^*$. This is a quantization map (in our approach, it is nonholonomic s-adapted and determined by a configuration \mathfrak{Cqs} and respective parametric deformations) which involves also a normal ordering $T^* V^* \equiv V^*$:

Considering formal power series on a small λ as in (35), we define the formal S-matrix

$${}^{\scriptscriptstyle -}\mathcal{S}^{\star}({}^{\scriptscriptstyle -}\lambda {}^{\scriptscriptstyle -}V^{\star}) := e^{i {}^{\scriptscriptstyle -}\lambda {}^{\scriptscriptstyle +} {}^{\scriptscriptstyle -}V^{\star}:/\hbar} = {}^{\scriptscriptstyle -}\mathcal{T}^{\star}(e^{i {}^{\scriptscriptstyle -}\lambda {}^{\scriptscriptstyle -}V^{\star}/\hbar}),$$

where $\mathcal{S}^{\star}(\lambda : V^{\star}) \in \mathcal{S}^{\star}(\hbar) = \mathcal{S}^{\star}(\hbar) [[\lambda, \kappa]]$, explicit computations involve κ -linear terms. The formulas can be recurrently extended as power series on κ if we consider such string R-flux deformations. Interacting nonholonomic s-fields are considered as elements of $\mathcal{S}^{\star}(\hbar, \lambda, \kappa)$ given and computed as

when $r_0^{\star}(\mathcal{F}^{\star}) =: \mathcal{F}^{\star}$; for $\lambda = 0$. Using the s-operator (42), we can define the interacting nonassociative star product

$${}^{\scriptscriptstyle \mathsf{I}}\mathcal{F}_1^\star \star_s^{int} {}^{\scriptscriptstyle \mathsf{I}}\mathcal{F}_2^\star := {}^{\scriptscriptstyle \mathsf{I}}r_V^{\star-1}({}^{\scriptscriptstyle \mathsf{I}}r_V^\star({}^{\scriptscriptstyle \mathsf{I}}\mathcal{F}_1^\star) \star_s {}^{\scriptscriptstyle \mathsf{I}}r_V^\star({}^{\scriptscriptstyle \mathsf{I}}\mathcal{F}_2^\star))$$

which encode in parametric form nonassociative quantum regularizations of the constants in (3).

On nonassociative phase spaces, we face the same problem of quantization as on typical Lorentz manifolds when quantum interactions and observables are local but not regular. For details on the corresponding renormalization problem, we cite [25], section 3.2.4. Here we also note that our goal is to elaborate on models of physical interactions which are usually local and encoding in effective parametric forms certain nonassociative data. Both on base spacetime Lorentz manifold and for co-fiber constructions the time-order product $\cdot_{\mathcal{T}^*}$ is not well defined on local and nonlinear functionals because of singularities of $\[]\hat{\Delta}^F_*$. So, the renormalization problem of nonassociative gauge gravitational fields (in our gauge models) is then to extend the effective parametric functional $\[]S^*$ to local arguments by using time-ordering products on nonassociative phase spaces,

$$\mathcal{S}^{\star} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{F}^{\star}_{n}(\mathcal{V}^{\star}, ..., \mathcal{V}^{\star}), \text{ for}$$
$$\mathcal{F}^{\star}_{n}(\mathcal{F}^{\star}_{1}, ..., \mathcal{F}^{\star}_{n}) := \mathcal{F}^{\star}_{1} \mathcal{T}^{\star} ... \mathcal{T}^{\star} \mathcal{F}^{\star}_{n}$$

In these formulas, the s-adapted time-ordered product of n local functionals is well defined by choosing pairwise disjointed supports respectively defined on base and typical fibers. To construct a causal perturbation theory on phase space we can extend ${}^{\vee}\mathcal{F}_n^{\star}$ to arbitrary local functional following the Epstein and Glaser process. In this case, the causal factorization property is stated as

$${}^{\scriptscriptstyle \mathsf{T}}\mathcal{F}_n^\star({}^{\scriptscriptstyle \mathsf{T}}F_1^\star \otimes \ldots \otimes {}^{\scriptscriptstyle \mathsf{T}}F_n^\star) = {}^{\scriptscriptstyle \mathsf{T}}\mathcal{F}_n^\star({}^{\scriptscriptstyle \mathsf{T}}F_1^\star \otimes \ldots \otimes {}^{\scriptscriptstyle \mathsf{T}}F_k^\star) \star_s {}^{\scriptscriptstyle \mathsf{T}}\mathcal{F}_{n-j}^\star({}^{\scriptscriptstyle \mathsf{T}}F_{k+1}^\star \otimes \ldots \otimes {}^{\scriptscriptstyle \mathsf{T}}F_n^\star)$$

if the supports of $F_1^{\star}, ..., F_k^{\star}$ are later than the supports of $F_{k+1}^{\star}, ..., F_n^{\star}$.

4.2.3 Renormalized nonassociative QME and quantum BV operator

We can apply the BV-formalism elaborated for the pAQFT [41, 25] by extending on nonassociative phase spaces the conditions that the free classical nonassociative BV s-operator \aleph_0^* from (see formulas (29) and (38))

is defined

$${}^{\mathsf{'}}\mathfrak{N}_{0}^{\star}(e^{i}: {}^{\mathsf{'}}V^{\star}:/\hbar}) = {}^{\mathsf{'}}\mathcal{T}^{\star}(e^{i}: {}^{\mathsf{'}}V^{\star}/\hbar} - i\hbar {}^{\mathsf{'}}\widehat{\Delta}_{\star}e^{i}: {}^{\mathsf{'}}V^{\star}/\hbar})$$

$$= {}^{\mathsf{'}}\mathcal{T}^{\star}(e^{i}: {}^{\mathsf{'}}V^{\star}/\hbar}(\frac{i}{\hbar}\{{}^{\mathsf{'}}V^{\star}, {}^{\mathsf{'}}S_{0}^{\star}\} + \frac{i}{2\hbar}\{{}^{\mathsf{'}}V^{\star}, {}^{\mathsf{'}}V^{\star}\} + {}^{\mathsf{'}}\widehat{\Delta}_{\star}({}^{\mathsf{'}}V^{\star}))) = 0.$$

$$(43)$$

Such computation is performed using the identity satisfied by ${}^{\mathsf{T}}\mathcal{T}^{\star}(40), {}^{\mathsf{d}}_{0}({}^{\mathsf{T}}\mathcal{T}^{\star}F^{\star}) = {}^{\mathsf{T}}\mathcal{T}^{\star}({}^{\mathsf{d}}_{0}{}^{\mathsf{T}}F^{\star}-i\hbar{}^{\mathsf{d}}_{\Delta_{\star}}F^{\star}).$ This follows from the consistency conditions (39) which for the decomposition ${}^{\mathsf{d}}_{0} = {}^{\mathsf{d}}_{0}^{\star} + {}^{\mathsf{d}}_{0}^{\star}(32)$, result in ${}^{\mathsf{T}}\mathcal{T}^{\star} \circ {}^{\mathsf{d}}_{0} = {}^{\mathsf{d}}_{0}^{\star} \circ {}^{\mathsf{T}}\mathcal{T}^{\star}$. In above formulas, the nonassociative BV Laplacian is defined and computed as

$$\widehat{\triangle}_{\star} \mathcal{X}^{\star} = (-1)^{(1+\#gh(\mathcal{X}^{\star}))} \sum_{\check{\alpha}} \int \frac{\frac{1}{r} \delta^2 \mathcal{X}^{\star}}{\delta \varphi_{\check{\alpha}}^{\star} \delta \frac{1}{\sharp} \varphi_{\check{\alpha}}^{\star}} \delta^{8} \mu$$

Using the classical master equations nCME (30) and, for symmetry reasons, setting $\widehat{\Delta}_{\star} : S_0^{\star} = 0$, when

$$\aleph_{0}^{\star}(e^{i}_{\mathcal{T}}^{': \mathcal{T}} V^{\star}; / \hbar) = \frac{i}{\hbar} e^{i}_{\mathcal{T}}^{': \mathcal{T}} V^{\star}; / \hbar}_{\mathcal{T}} \cdot_{\mathcal{T}^{\star}} \mathcal{T}^{\star} \left(\frac{1}{2} \{ -S_{0}^{\star} + -V^{\star}, -S_{0}^{\star} + -V^{\star} \} - i\hbar \widehat{\bigtriangleup}_{\star} (-S_{0}^{\star} + -V^{\star}) \right),$$

we write (43) as nonassociative quantum master equation, nQME:

$$\frac{1}{2} \{ {}^{\prime}S_0^{\star} + {}^{\prime}V^{\star}, {}^{\prime}S_0^{\star} + {}^{\prime}V^{\star} \} = i\hbar \, {}^{\prime}\widehat{\Delta}_{\star} ({}^{\prime}S_0^{\star} + {}^{\prime}V^{\star}).$$
(44)

This equation can be considered as a condition on V^* stating the locality of the *nonassociative quantum* BV*s-operator* $\widehat{\aleph}_0^*$ (in general, a nonassociative star product defines a nonlocal structure but it can distinguished in parametric form as local ones on base and co-fiber spaces). In the free theories, we define and compute parametrically

$${}^{\scriptscriptstyle |}\widetilde{\aleph}_0^\star := ({}^{\scriptscriptstyle |}\mathcal{T}^\star)^{-1} \circ {}^{\scriptscriptstyle |}\aleph_0^\star \circ {}^{\scriptscriptstyle |}\mathcal{T}^\star = {}^{\scriptscriptstyle |}\aleph_0^\star - i\hbar {}^{\scriptscriptstyle |}\widehat{\triangle}_\star$$

$$\tag{45}$$

which follows from (43) and (44). We omit here the s-labels for the geometric s-objects which have to be introduced on a nonholonomic phase space determined by s-adapted off-diagonal solutions in nonassociative gauge gravity.

The nonassociative quantum BV s-operator (45) can be generalized on regular functionals for the interacting nonassociative gauge fields (see formulas (42) and (38))

$$\widehat{\aleph}^{\star} = \ {}^{}^{} r_{\lambda V}^{\star-1} \circ \ \widehat{\aleph}_{0}^{\star} \circ \ {}^{}^{} r_{\lambda V}^{\star}.$$

This is a nonassociative quantum twist of the free classical BV s-operator by a non-local map involving both the free and the quantum interacting theories. In the classical limit, we obtain the formulas (37). Nevertheless, the operator $\widehat{\Re}^{\star}$ is local and characterizes nonassociative quantum gauge invariant observables. This follows form the property that assuming nQME, we can compute

$$\hat{\aleph}^{\star} F^{\star} = e^{i \cdot V^{\star} / \hbar} \mathcal{T}^{\star} \hat{\aleph}_{0}^{\star} (e^{i \cdot V^{\star} / \hbar} \mathcal{T}^{\star} F^{\star})$$

$$= \{ F^{\star}, S_{0}^{\star} + V^{\star} \} - i\hbar \hat{\Delta}_{\star} (F^{\star}) = \hat{\aleph}_{0}^{\star} - i\hbar \hat{\Delta}_{\star} (F^{\star}).$$

$$(46)$$

This operator is also nilpotent by definition.

We can extend the nQME and $\widehat{\aleph}^*$ to local s-observables by replacing $\cdot_{\mathcal{T}^*}$ with the renormalized timeordered product on nonasociative phase spaces by generalizing in abstract nonholonomic geometric form the results stated by Theorem 3.4 in [41]. In parametric effective form, the associative product \cdot_r on $\mathcal{T}_r(\mathcal{F}^*)$ is given by

defined as $|\mathcal{T}_r = (\bigoplus_n |\mathcal{T}_r^n) \circ |\beta^*$, where $|\beta^*$ is the inversion of multiplication for $|\mathcal{T}_r|_{\mathcal{F}_{loc}^*} = id$, so : $|V^* := |V^*$.

The τ_{τ^*} defined by (47) is an associative and commutative product and we can use it in place of τ_{τ^*} and define the renormalized parametric nQME and the quantum nonassiative BV s-operators using formulas (43) and (45). For applications, we can consider the terms proportional to κ^0 and κ^1 (the linear ones on κ encoding nonassociative data in effective quantum form). We can simplify such formulas considering a nonassociative phase space generalization of the anomalous Master Ward Identity

$$^{\mathsf{N}} {}^{\mathsf{N}} (e^{i : \mathsf{V}^{\star}:/\hbar}) \equiv \{ e^{i : \mathsf{V}^{\star}:/\hbar}, \mathsf{S}^{\mathsf{N}}_{0} \} = \frac{i}{\hbar} e^{i : \mathsf{V}^{\star}:/\hbar} \cdot_{\mathcal{T}_{r}} \left(\frac{1}{2} \{ \mathsf{S}^{\mathsf{N}}_{0} + \mathsf{V}^{\mathsf{N}}, \mathsf{S}^{\mathsf{N}}_{0} + \mathsf{V}^{\mathsf{N}} \} |_{\mathcal{T}_{r}} - i\hbar \mathsf{S}^{\mathsf{N}}_{\mathsf{N}} \right),$$
(48)

where $\widehat{\Delta}^V_{\star}$ is identified with the anomaly term. For spacetime bases, similar details are provided in [39, 45]), when the necessary formulas and proofs can be extended in parametric and abstract nonassociative geometric forms. Therefore, the renormalized quantum nonassociative master equation corresponding to (48) are

$$\frac{1}{2} \{ {}^{\vee}S_0^{\star} + {}^{\vee}V^{\star}, {}^{\vee}S_0^{\star} + {}^{\vee}V^{\star} \}_{{}^{\vee}\mathcal{T}_r} - i\hbar {}^{\vee}\widehat{\Delta}_{\star}^V = 0.$$
(49)

Replacing in this formula $\widehat{\Delta}^{V}_{\star}(F^{\star}) := \frac{d}{d\lambda} \widehat{\Delta}^{V+\lambda}_{\star}F^{\star}|_{\lambda=0}$ and considering that the renormalized nQME hold, we can write such a master equation using the renormalized nonassociative BV s-operator from (46),

$$\widehat{\aleph}^{\star} F^{\star} = \{ F^{\star}, S_0^{\star} + V^{\star} \} - i\hbar \widehat{\bigtriangleup}^V (F^{\star})$$

So, using the renormalized time ordered product $\cdot_{\mathcal{T}_r^{\star}}$, we obtained an anomaly (which is local on respective hand c-components and of order $O(\hbar, \kappa)$) via $\widehat{} \widehat{\Delta}^V_{\star}({}^{+}F^{\star})$ instead of $\widehat{} \widehat{\Delta}_{\star}({}^{+}F^{\star})$. In the renormalized nonassociative gauge gravity theory, $\widehat{} \widehat{\Delta}^V_{\star}$ can be well-defined on local s-vector fields, in contrast to $\widehat{} \widehat{\Delta}_{\star}$.

In section 4 of [25] and in [46] (in the variant with local S-matrices and generating C^* -algebra) possible approaches towards a non-perturbative formulation of the BV-formalism elaborated for the pAQFT are analyzed. In a general formalism with nonassociative twisted star products, the formulation of a general variational formalism is impossible (except effective models with parametric decompositions). This may consist a program of research and a series of future works on nonassociative pAQFT using effective S-matrices and nonassociative Schwinger-Dyson equations. Here we note that using off-diagonal parametric solutions in nonassociative gauge gravity, with nonlinear symmetries (A.2) and gravitational polarizations ${}^{\downarrow}_{s}\eta {}^{\downarrow}_{g\alpha_{s}} \sim {}^{\downarrow}\zeta_{\alpha_{s}}(1 + \kappa {}^{\downarrow}\chi_{\alpha_{s}}) {}^{\downarrow}_{g\alpha_{s}}$, we can define and compute ${}^{\downarrow}\widehat{\Delta}^{V}_{\star}$ as a respective distortion of ${}^{\downarrow}\widehat{\Delta}_{\star}$, for functionals ${}^{\downarrow}V^{\star}[{}^{\downarrow}_{s}\eta] \sim {}^{\downarrow}V^{\star}[{}^{\downarrow}_{s}\chi]$. Such anomaly terms and their distortions can be computed using nonlinear symmetries (A.3). So, our approach with nonholonomic deformations and generating classical and off-diagonal solutions is generic non-perturbative and allow to select well-defined, for instance, quasi-stationary nonassociative configurations resulting in renormalized nonassociative BV s-operators. In this work, we do not consider nonassociative and noncommutative extensions of non-perturbative quantum methods with local S-matrices and generating C^* -algebras [25, 46] because it is not clear how such a formalism can be elaborated in a general form for nonlocal and non-variational theories encoding nonassociativity. For explicit classes of physically important solutions with parametric R-flux nonassociativity existing in the low energy limit of string/M-theory, an effective variational formulation of corresponding models is possible. This motivates the goals of this work to elaborate on a geometric and quantum BV formalism for some general classes of off-diagonal solutions with parametric nonassociative data.

4.3 BV quantization of nonassociative 8-d modified BH configurations

In this subsection, we consider an example of how the classical and quantum BV schemes from sections 3 and 4 can be applied for quantizing nonassociative BH solutions. The BV formalism is used in explicit

form for quantizing the quasi-stationary off-diagonal R-flux deformations of regular phase spaces with Dymnikova backgrounds resulting in 8-d s-metrics of type (A.11).⁸ All formulas are provided in abstract geometric form which allows an "economic" geometric formulation and straightforward application of the BV method for nonassociative gauge gravity theories.

4.3.1 Generating data for nonassociative star product deformed of Dymnikova BHs

We parameterize the quasi-stationary solutions for off-diagonal deformations of primary metrics (A.8) to the target ones (A.11) in the form

$${}^{\scriptscriptstyle \mathsf{J}}_{s} \mathbf{\breve{g}} = [{}^{\scriptscriptstyle \mathsf{J}}_{g\alpha_s}, {}^{\scriptscriptstyle \mathsf{J}}_{i_{s-1}}] \rightarrow {}^{\scriptscriptstyle \mathsf{J}}_{\eta} \mathbf{\breve{g}} = {}^{\scriptscriptstyle \mathsf{J}}_{s} \eta [\breve{r}_0, \breve{r}_g, {}^{\scriptscriptstyle \mathsf{J}}_{s}\Lambda] {}^{\scriptscriptstyle \mathsf{J}}_{g\alpha_s} [\breve{r}_0, \breve{r}_g] \sim {}^{\scriptscriptstyle \mathsf{J}}_{\zeta\alpha_s} [\breve{r}_0, \breve{r}_g, {}^{\scriptscriptstyle \mathsf{J}}_{s}\Lambda] (1 + \kappa {}^{\scriptscriptstyle \mathsf{J}}_{\alpha_s} [\breve{r}_0, \breve{r}_g, {}^{\scriptscriptstyle \mathsf{J}}_{s}\Lambda]) {}^{\scriptscriptstyle \mathsf{J}}_{g\alpha_s} [\breve{r}_0, \breve{r}_g].$$
(50)

In these formulas, \check{r}_0 and \check{r}_g are physical constants for the prime metric's phase space defined by the generalized Dymnikova BH solution, see (A.9) and (A.10). The quasi-stationary target metrics $\[\]_{\eta} \check{g} = \{\]_{\eta} \check{g}_{\alpha_s}\}$ (A.11) are of type (17) involving generating functions (A.2) parameterized in the form:

$$\psi \simeq \psi(\hbar, \kappa, \breve{r}_0, \breve{r}_g, {}^{1}_{1}\Lambda, x^{k_1}), \eta_4 \simeq \eta_4(\hbar, \kappa, \breve{r}_0, \breve{r}_g, {}^{1}_{2}\Lambda, x^{k_1}, y^3),$$

$$[\eta^6 \simeq [\eta^6(\hbar, \kappa, \breve{r}_0, \breve{r}_g, {}^{1}_{3}\Lambda, x^{i_2}, p_5), [\eta^8 \simeq [\eta^8(\hbar, \kappa, \breve{r}_0, \breve{r}_g, {}^{4}_{4}\Lambda, x^{i_2}, p_5, p_7)].$$

$$(51)$$

The effective shell cosmological constants, ${}_{s}^{\prime}\Lambda$, from (51) are related to effective matter sources encoding nonassociative data, ${}_{s}^{\prime}\mathcal{J}^{\star}$, via nonlinear symmetries of type (A.3). For such formulas, the generating functions, generating source and effective cosmological constants transform respectively as

$$[\eta_4, \ \ \eta^6, \ \ \eta^8, \ \ _s\mathcal{J}^\star] \longleftrightarrow [\ \ _s\Psi, \ \ _s\mathcal{J}^\star] \longleftrightarrow [\ \ _s\Phi^\star, \ \ _s\Lambda], \tag{52}$$

with dependencies on respective sets of physical constants $[\hbar, \kappa, \check{r}_0, \check{r}_g, {}_s\Lambda]$. We can introduce also other constants and integration functions stating, for instance, certain ellipsoidal symmetries with an eccentricity ϵ , or certain constants defining toroid/ cylindric off-diagonal deformation etc. We emphasize that the nonlinear symmetries (A.3) and (52) are not gauge-like symmetries considered for gravitational gauge theories. They reflect (nonassociative) parametric properties of certain classes of quasi-stationary solutions of modified Einstein equations and their lifts or equivalents in total spaces written as YM equations.

4.3.2 The BV formalism for nonassociative BH solutions

Let us explain the main steps for performing BV quantization of ${}^{\dagger}\eta \ddot{\mathbf{g}}$ (A.11) determined by (nonassociative) data (51) and (52). The geometric quantization can be formulated in a non-perturbative parametric form using gravitational η -polarizations and nonlinear symmetries transforming generating sources ${}_{s}\mathcal{J}^{\star}$ into effective cosmological constants ${}^{\flat}_{s}\Lambda$. For certain classes of well-defined (as effective relativistic theories) off-diagonal configurations with asymptotic quasi-classical limits (selected by respective data $[\hbar, \kappa, \check{r}_0, \check{r}_g, {}^{\flat}_s\Lambda]$), we can elaborate on perturbative schemes on \hbar and linearized on κ and possible recurrent higher order terms. We omit in this section cumbersome perturbative formulas involving k-linear decompositions of type (51).

For a ${}_{\eta} \breve{\mathbf{g}}$ (A.11) we can construct a parametric nonassociative gravitational gauge potential ${}_{s} \widehat{\mathcal{A}}_{[P]}^{\star} \rightarrow {}_{\eta} \widehat{\mathcal{A}}_{[P]}^{\star}$ (13) as a R-flux deformation of (5). Such an effective gauge potential contains all data on quasi-stationary deformations (via η -polarizations) and allows to define an effective gauge gravitational action (21) when

$${}^{\scriptscriptstyle \perp}_{\eta}L^{\star} = \; {}^{\scriptscriptstyle \perp}_{gr}L^{\star}(\; {}^{\scriptscriptstyle \perp}f)[\;\; {}^{\scriptscriptstyle \perp}_{\eta}\widehat{\mathcal{A}}^{\star}] = -\frac{1}{2}\int_{\;\; {}^{\scriptscriptstyle \perp}_{\eta}\mathcal{U}_{\mathfrak{Cqs}}}\; {}^{\scriptscriptstyle \perp}f \;\; tr(\; {}^{\scriptscriptstyle \perp}_{\eta}\widehat{\mathcal{F}}^{\star} \wedge (\ast \; {}^{\scriptscriptstyle \perp}_{\eta}\widehat{\mathcal{F}}^{\star})),$$

⁸Similar nonassociative BV constructions can be performed for any nonassociative BH and WH solutions which may include, or not, singularities, off-diagonal deformations etc. Such solutions were constructed and studied in classical form [12, 13, 14, 15, 16]. It is not possible to elaborate on their BV quantization (on hundred of pages) in this work.

for ${}_{\eta}\widehat{\mathcal{F}}^{\star}$ computed as the nonassociative strength defined by ${}_{\eta}\widehat{\mathcal{A}}^{\star}$, and when the measure $\delta^{8}\mu$ is defined by a chosen ${}_{\eta}\overset{\cdot}{\mathbf{g}} \in \mathfrak{Cqs}$ for a η -deformed region ${}_{\eta}^{\prime}\mathcal{U}_{\mathfrak{Cqs}}$. Necessary formulas from subsection 3.2.1 can be applied in abstract nonassociative form for geometric objects with η -labels when ${}_{\eta}L^{\star} \in {}^{\prime}\mathcal{L}^{\star}$ and ${}^{\prime}\varphi^{\star} = {}_{\eta}^{\prime}\widehat{\mathcal{A}}^{\star} \in {}^{\prime}\mathcal{E}_{\mathfrak{Cqs}}^{\star}$. For such quasi-stationary configurations, an effective variational calculus can be defined in parametric form even though a unique nonassociative differential and integral calculus can't be defined for a general twist product. Then, we can introduce in functional effective form the Euler-Lagrange derivative of a corresponding effective action ${}_{\eta}^{\prime}S^{\star}$, see (22) with ${}^{\prime}\mathcal{E}_{\eta}^{\star} \subset {}^{\prime}\mathcal{E}_{\mathfrak{Cqs}}^{\star}$, when

 ${}^{!}_{Sol}\mathcal{E}^{\star}_{\eta}$ denotes the spaces of solutions defined as the zero locus of the 1-form $d_{\eta} S^{\star}$, ${}^{!}_{Sol}\mathcal{E}^{\star}_{\eta} \subset {}^{!}\mathcal{E}^{\star}_{\eta}$; ${}^{!}_{Sol}\mathcal{F}_{\eta}$ denotes the space of off-shell functionals in the space of functionals on ${}^{!}_{Sol}\mathcal{E}^{\star}_{\eta}$.

So, for a more special case of parametric quasi-stationary configurations (53) with corresponding effective Lagrangians and actions, the nonassociative classical BV differential for off-diagonal phase space Dymnikova BHs is computed as

$${}^{\scriptscriptstyle +}_{\eta} \aleph^{\star} = \{ \bullet, {}^{\scriptscriptstyle +}_{\eta} S^{\star} + {}^{\scriptscriptstyle +}_{\eta} \Theta^{\star} \} := \{ \bullet, {}^{\scriptscriptstyle +}_{\eta} S^{\star}_{ext} \}, \text{ with extended action } {}^{\scriptscriptstyle +}_{\eta} S^{\star}_{ext}.$$
(54)

These functional equations depend on certain subclasses of generating and integration data (A.5) and (A.6) defining nonassociative off-diagonal BH solutions. The nilpotent property, $(\ _{\eta}\aleph^{\star})^{2} = 0$, allows to define the *nonassociative classical master equation* (nCME) for such BHs, which modulo terms vanishing in the limit of a constant $\ _{f}$ is written in the form $\{\ _{\eta}L_{ext}^{\star}(\ f),\ _{\eta}L_{ext}^{\star}(\ f)\} = 0$. For such nCME, it is used an effective $\ _{\eta}L_{ext}^{\star}(\ f)$ is used for defining a respective $\ _{\eta}S_{ext}^{\star}$.

The operator ${}^{\dagger}_{\eta} \aleph^{\star}$ from (54) also increases the ghost number by one and can be expressed as a sum, ${}^{\dagger}_{\eta} \aleph^{\star} = {}^{\dagger}_{\eta} \delta^{\star} + {}^{\dagger}_{\eta} \gamma^{\star}$, for an extension of ${}^{\dagger}_{\eta} \delta^{\star}_{S}$ denoted ${}^{\dagger}_{\eta} \delta^{\star}$. Defining ${}^{\dagger}_{\eta} \Theta^{\star} := {}^{\dagger}_{\eta} S^{\star}_{ext} - {}^{\dagger}_{\eta} S^{\star}$ as in formulas (29), we can express

$${}^{\scriptscriptstyle \mid}_{\eta}\delta^{\star} = \{\cdot, \; {}^{\scriptscriptstyle \mid}_{\eta}S^{\star}\} \text{ and } {}^{\scriptscriptstyle \mid}_{\eta}\gamma^{\star} = \{\cdot, \; {}^{\scriptscriptstyle \mid}_{\eta}\Theta^{\star}\}$$

In above formulas, the differential $\frac{1}{\eta}\delta^{\star}$ acts trivially both on fields and antifields. Using $\frac{1}{\eta}\delta^{\star}\frac{1}{\sharp\eta}\widehat{\mathcal{A}}^{\star} = \frac{\frac{1}{\delta}\frac{1}{\eta}S^{\star}}{\frac{1}{\delta}\frac{1}{\sharp\eta}\widehat{\mathcal{A}}^{\star}}$, we derive nonassociative gauge-fixed equations of motion which are hyperbolic equations of motion of $\frac{1}{\eta}S^{\star}$.

On nonassociative phase space Dymnikova background (A.8) and for further parametric decompositions to an off-diagonal solution $\[\]_{\eta} \breve{\mathbf{g}}$ (A.11), and respective $\[\]_{\eta} \[\]_{\mu} \[\]_{\mu} \[\]_{\eta} \[\]_{\mu} \[\$

$${}^{\dagger}_{\eta}S_{0}^{\star} = {}^{\dagger}_{\eta}\breve{S}_{00}^{\star} + {}^{\dagger}_{\eta}\Theta_{0}^{\star}, \text{ the quadratic term in (anti) fields, } \#ta({}^{\dagger}_{\eta}S_{00}^{\star}) = 0, \#ta({}^{\dagger}_{\eta}\Theta_{0}^{\star}) = 1;$$

 ${}^{\dagger}_{\eta}V^{\star} = {}^{\dagger}\breve{V}_{0}^{\star} + {}^{\dagger}_{\eta}\Theta^{\star}, \text{ the interacton term ;}$
 ${}^{\prime}S^{\star} = {}^{\dagger}_{\eta}\breve{S}_{00}^{\star} + {}^{\dagger}\breve{V}_{0}^{\star}, \text{ the total antifield independent part of the action.}$

We can chose such nonholonomic s-adapted distributions, when ${}^{\dagger}_{\eta} \check{S}^{\star}_{00} = {}^{\dagger}_{\eta} \check{S}_{00}$ and ${}^{\dagger} \check{V}^{\star}_{0} = {}^{\dagger} \check{V}_{0}$ are defined by s-adapted classical decompositions of the primary s-metric (A.8) and its associative quantum deformations; and when the ${}^{\dagger}_{\eta} \Theta^{\star}_{0}$ and ${}^{\dagger}_{\eta} \Theta^{\star}$ are for quantum nonassociative deformations. For the target nonassociative BH solutions, the formulas (32) transform into parametric linearized s-operators of BRST and BV type,

$${}^{\downarrow}_{\eta}\gamma^{\star} {}^{\vee}\mathcal{F} := \{ {}^{\vee}\mathcal{F}, {}^{\downarrow}_{\eta}\Theta_{0}^{\star} \} \text{ and } {}^{\downarrow}_{\eta}\aleph_{0}^{\star} = {}^{\downarrow}_{\eta}\delta_{0}^{\star} + {}^{\downarrow}_{\eta}\gamma_{0}^{\star} \}$$

where ${}^{\dagger}_{\eta}\delta^{\star}_{0}({}^{\dagger}_{\sharp\eta}\widehat{\mathcal{A}}^{\star\check{\alpha}}_{[P]}) = -\frac{{}^{\dagger}_{\eta}\delta^{\dagger}_{\eta}S^{\star}_{00}}{\delta^{\dagger}_{\eta}\widehat{\mathcal{A}}^{\star}_{\check{\alpha}[P]}}$ and respective nonassociative differential operators, ${}^{\dagger}_{\eta}P^{\check{\alpha}\check{\beta}}_{\star\check{\beta}}({}^{\dagger}u)$ and ${}^{\dagger}_{\eta}K^{\check{\alpha}}_{\star\check{\beta}}$ are defined from

$$\frac{\frac{1}{l}\delta^{-1}S_{00}^{\star}}{\delta^{-1}_{\eta}\widehat{\mathcal{A}}_{\check{\alpha}[P]}^{\star}}({}^{+}_{\eta}\widehat{\mathcal{A}}_{\check{\alpha}[P]}^{\star}) := {}^{+}_{\eta}P_{\star}^{\check{\alpha}\check{\beta}}({}^{-1}u)({}^{+}_{\eta}\widehat{\mathcal{A}}_{\check{\beta}[P]}^{\star}), \text{ in brief }, = {}^{+}_{\eta}P_{\star} {}^{-1}_{\eta}\widehat{\mathcal{A}}_{\check{\beta}[P]}^{\star}; \text{ and}$$
$$\frac{{}^{+}_{r}\delta_{l}^{+}\delta^{-1}_{-\eta}\Theta_{0}^{\star}}{\delta_{-\eta}^{+}\widehat{\mathcal{A}}_{\check{\alpha}[P]}^{\star}({}^{-1}u_{1})}({}^{+}_{\eta}\widehat{\mathcal{A}}_{\check{\alpha}[P]}^{\star}) := {}^{+}_{\eta}K_{\star\check{\beta}}^{\check{\alpha}}({}^{-1}u)({}^{+}_{\eta}\widehat{\mathcal{A}}_{\check{\beta}[P]}^{\star}).$$

These functional equations depend on respective classes of generating data (A.5) and (A.6).

We chose nonholonomic configurations on nonassociative phase space and assume a gauge fixing in the total nonassociative vector bundle in such a way that ${}^{\dagger}_{\eta}P_{\star}$ is Green hyperbolic for any *h*- and *c*-component as for (associative and commutative) gauge and gravity theories. The corresponding double Green function is defined by the motion operator ${}^{\dagger}_{\eta}P_{\star}$ determined in canonical form (with hat d-operators) by ${}^{\dagger}_{\eta}\breve{g}$ as a particular case of (33). This allows us to define the respective nonassociative Pauli-Jordan functionals

$${}^{\downarrow}_{\eta}\widehat{\Delta}_{\star} = {}^{\downarrow}_{\eta}\widehat{\Delta}^{R}_{\star} - {}^{\downarrow}_{\eta}\widehat{\Delta}^{A}_{\star}.$$
(55)

We need additional assumptions to prescribe respective generating data for the nonassociative BHs to transform such functionals into certain Pauli-Jordan functions.

Using the function (55), we can express in terms of η -polarization functions the formulas (37) and (38), for nonassociative off-diagonal BH deformations. Then, we define and compute the corresponding classical BV operator (29),

Above s-operators allow us to formulate the $nCME(\frac{1}{\eta}S^*)$ (i.e. the classical master equations for nonassociative star product deformation (3) of generalized Dymnikova BHS in gauge gravity):

$$\int_{\eta} r_{\lambda V}^{\star-1} (\{ \mathcal{X}^{\star}, \mathcal{A}_{\eta} S_{0}^{\star}\}) = \{ \mathcal{A}_{V} r_{\lambda V}^{\star-1} (\mathcal{X}^{\star}), \mathcal{A}_{0} S_{0}^{\star} + \mathcal{A}_{\eta} V^{\star} \} - \int \frac{1}{r} \delta^{-1} \mathcal{X}^{\star}}{\delta^{-1} \mathcal{A}_{\tilde{\alpha}[P]}} (\mathcal{A}_{0} r_{\lambda V}^{\star-1} (\mathcal{A}_{\eta} \widehat{\mathcal{A}}_{[P]}^{\star})) \mathcal{A}_{\star \check{\alpha} \check{\beta}} (\mathcal{A}_{v} \mathcal{A}_{v} \mathcal{A}$$

Here we emphasize that such functional equations depend on respective classes of generating data (A.5) and (A.6). This can be chosen in certain forms allowing to construct well-defined physical solutions of (57) involving respective s-operators (55) and (56).

4.3.3 Nonholonomic BV scheme and quantization of nonassociative BH solutions

The nonassociative BV s-operator for phase space BH R-flux deformations (56) can be extended in quantum normalized form using the formula (43),

$${}^{\scriptscriptstyle +}_{\eta} \aleph_0^{\star}(e^{i \cdot : \cdot \cdot \cdot \eta} V^{\star : / \hbar}) = \frac{i}{\hbar} e^{i \cdot : \cdot \cdot \eta} V^{\star : / \hbar} \cdot_{\mathcal{T}^{\star}} \cdot \mathcal{T}^{\star} \left(\frac{1}{2} \{ \cdot \cdot \eta S_0^{\star} + \cdot \cdot \eta V^{\star}, \cdot \cdot \eta S_0^{\star} + \cdot \eta V^{\star} \} - i\hbar \cdot \eta \widehat{\Delta}_{\star} (\cdot \cdot \eta S_0^{\star} + \cdot \eta V^{\star}) \right),$$

when ${}_{\eta} \Delta_{\star} {}_{\eta} S_0^{\star} = 0$. Using the classical master equations nCME (30), we obtain for phase space R-flux deformed BHs the nonassociative quantum master equation, nQME:

$$\frac{1}{2} \{ \neg_{\eta} S_0^{\star} + \neg_{\eta} V^{\star}, \neg_{\eta} S_0^{\star} + \neg_{\eta} V^{\star} \} = i\hbar \neg_{\eta} \widehat{\bigtriangleup}_{\star} (\neg_{\eta} S_0^{\star} + \neg_{\eta} V^{\star}).$$

This equation can be considered as a condition on the R-flux deformed effective potential ${}^{\dagger}_{\eta}V^{\star}$ stating the parametric locality of the corresponding *nonassociative quantum BV s-operator*,

$${}^{\scriptscriptstyle |}_{\eta}\widehat{\aleph}^{\star}_{0} := ({}^{\scriptscriptstyle |}\mathcal{T}^{\star})^{-1} \circ {}^{\scriptscriptstyle |}_{\eta}\aleph^{\star}_{0} \circ {}^{\scriptscriptstyle |}\mathcal{T}^{\star} = {}^{\scriptscriptstyle |}_{\eta}\aleph^{\star}_{0} - i\hbar {}^{\scriptscriptstyle |}_{\eta}\widehat{\bigtriangleup}_{\star}.$$

Such a s-operator can be generalized on regular functionals using formulas (57), ${}_{\eta}\widehat{\aleph}^{\star} = {}_{\eta}r_{\lambda V}^{\star-1} \circ {}_{\eta}\widehat{\aleph}_{0}^{\star} \circ {}_{\eta}r_{\lambda V}^{\star}$. This allows to compute in parametric local form (assuming nQME):

The functional character of such nonassociative quantum BV s-operators and and nQME allows us to define various types of non-perturbative classical and quantum deformations determined by respective generating data encoding nonassociative R-flux deformations.

In above quantum formulas, we can chose nonholonomic s-adapted distributions to separate the terms proportional to κ^0 and κ^1 , when the κ -linear ones encode nonassociative data in effective quantum form. So, for phase space R-flux deformed Dymnikova BHs, one holds such a nonassociative generalization of the anomalous Master Ward Identity:

$${}^{\downarrow}_{\eta} \aleph_{0}^{\star} (e^{i : -\frac{i}{\eta}V^{\star}:/\hbar}) \equiv \{ e^{i : -\frac{i}{\eta}V^{\star}:/\hbar}, \quad {}^{\downarrow}_{\eta} S_{0}^{\star} \} = \frac{i}{\hbar} e^{i : -\frac{i}{\eta}V^{\star}:/\hbar} \cdot \tau_{r}^{\star} \left(\frac{1}{2} \{ -\frac{i}{\eta} S_{0}^{\star} + -\frac{i}{\eta} V^{\star}, -\frac{i}{\eta} S_{0}^{\star} + -\frac{i}{\eta} V^{\star} \} \right) \cdot \tau_{r} - i\hbar + \frac{i}{\eta} \widehat{\Delta}_{\star}^{V} \right).$$

$$(58)$$

The s-operator $\[\] \eta \widehat{\bigtriangleup}^V_{\star}$ is identified with the anomaly term.

Then, the renormalized quantum nonassociative master equation corresponding to (48), (49) and (58) are

$$\frac{1}{2} \{ \neg_{\eta} S_0^{\star} + \neg_{\eta} V^{\star}, \neg_{\eta} S_0^{\star} + \neg_{\eta} V^{\star} \} \neg_{\mathcal{T}_r} - i\hbar \neg_{\eta} \widehat{\Delta}_{\star}^V = 0.$$

Assuming the renormalized nQME hold, we can write such a master equation using the renormalized nonassociative BV s-operator from (46),

$${}^{\scriptscriptstyle \perp}_{\eta}\widehat{\aleph}^{\star}{}^{\scriptscriptstyle \perp}F^{\star} = \{ {}^{\scriptscriptstyle \perp}F^{\star}, {}^{\scriptscriptstyle \perp}_{\eta}S^{\star}_{0} + {}^{\scriptscriptstyle \perp}_{\eta}V^{\star} \} - i\hbar {}^{\scriptscriptstyle \perp}_{\eta}\widehat{\bigtriangleup}^{V}_{\star}({}^{\scriptscriptstyle \perp}F^{\star}).$$

In this subsection, we use the same renormalized time ordered product $\cdot_{\mathcal{T}_r^{\star}}$ as we computed the anomaly in subsection 4.2.3 but considering $\ \ _{\eta} \widehat{\bigtriangleup}_{\star}^V(\ F^{\star})$ instead of $\ \ _{\eta} \widehat{\bigtriangleup}_{\star}(\ F^{\star})$.

The above formulas allow us to construct quantum versions of nonassociative BHs and analyze possible implications both in perturbative and non-perturbative forms. For instance, we conclude that nonassociative R-flux contributions transform certain classical 4-d BH configurations (for instance, the Dymnikova nonsingular metric) into 8-d phase space quasi-stationary configurations. For small parametric deformations, such extradimension quantum BH solutions describe new QG phenomena defined by the renormalized nonassociative BV s-operator. Such effects may exist also in associative and commutative forms for ellipsoidal phase space configurations but described in a different form by nonholonomic BV operators. So, nonassociativity "enrich" the landscape of QG and various nonassociative and locally anisotropic effects with quantum and quantum gravitational polarizations can be computed as additional off-diagonal deformations of BH metrics.

5 Conclusions and perspectives

In this paper, we have focused on conceptual problems and elaborating new geometric methods using the Batalin-Vilkovisky, BV, formalism [20, 21, 22, 23, 24] for quantizing nonassociative and noncommutative gauge gravity theories [16, 28, 29]. Such theories are defined by twisted star products [8, 9] and R-flux parametric deformations considered in string and nonassociative gravity [10, 11]. Corresponding quantum gravity, QG, models encode nonassociative and noncommutative data for generic off-diagonal solutions of modified gravitational Yang-Mills, YM, or nonassociative Einstein-Dirac-Maxwell, EDM, equations [12, 13, 14, 15, 16, 47]. For projections on phase spaces (modelled as cotangent Lorentz bundles), such solutions describe nonassociative classical EDM and YM quasi-stationary systems, or locally anisotropic cosmological models. The main goal of this work is to elaborate on quantum models of nonassociative geometric flows, gravity and matter field theories. Quantization of such models is performed using generalized BV schemes and advanced mathematic methods from the algebraic quantum field theory, AQFT, [25, 37, 40, 26, 32, 38, 41, 42, 43, 46]. This paper is the first one in the literature which connects general classes of parametric off-diagonal solutions to QG and BV quantization of nonassociative MGTs. It is a natural development of a recent author's work [48] on nonassociative QG with Gorrof-Sagnotti terms. In that paper, rigorous mathematical methods of nonassociative BV quantization have not been considered.

Nonassociative star products and R-flux deformations substantially modify the classical general relativity, GR, and modified gravity theories, MGTs, by introducing nonlocal configurations on nonassociative phase spaces. New types of star-deformed geometric objects such as symmetric and nonsymmetric metrics, nonlinear and linear connections, defining various types of differential and integral calculi and non-unique variational procedures are also defined. Such geometric constructions become effective local for parametric decompositions on the Planck and string constants which allows us to extend the principles of locality, deformation and homology on nonassociative phase spaces from [25, 26]. The first important result of our work consisted in a natural generalization of the classical BV formalism in the abstract geometric form [27, 12, 16] as an effective gauge gravity theory on phase spaces. For such models, a double de Sitter structure group is used in a form encoding consequent nonlinear extensions of the (double) affine structure group and the Poincaré group. The first structure group is for the base spacetime Lorentz manifold and the second one is for a typical fiber.

The second main result of this paper consisted in elaborating on nonholonomic frame geometric methods for quantizing physically important nonlinear systems of partial differential equations, PDEs, in GR and MGTs. The constructions are related via corresponding classes of generic off-diagonal solutions to perturbative and nonperturbative BV schemes with effective actions and Lagrangians encoding nonassociative data from string theory. Such nonassociative nonlinear systems of PDEs are characterized by certain types of nonlinear symmetries which are different from the prescribed gauge type symmetries. The nonlinear symmetries allow us to introduce effective cosmological constants and state certain well-defined physical conditions for applying the BV-scheme, with possible linearizations and definition of nonassociative classical BV operators and related master equations etc.

The third main result of this article was in relating the BV-scheme to the quantization of nonassociative gauge de Sitter gravity and analyzing the corresponding renormalization problems. That allowed to derive nonassociative versions of quantum BV operators and quantum master equations. As an example, we have shown how our generalization of the methods of BV quantization can be applied for quantizing in a non-perturbative way nonassociative 8-d modified BH solutions on phase spaces. That can be also considered as the fourth important result. We note that in our approach "non-perturbative" means that the scheme of quantization works for any type of prime or target (off-diagonal) metrics. It can be further performed in a perturbative way for respective parametric decompositions on physically important parameters by choosing respective classes of generating functions and generating data, and prescribed effective cosmological constants which result in effective asymptotic renormalizable theories.

Finally, we outline four perspectives (P1-P4) for developing the results of the work:

- P1: To elaborate on perturbation algebraic QFT, pAQFT, methods on phase spaces which will result in asymptotic safe (nonassociative) QG models with star R-flux products, Goroff-Sagnotti counter-terms [48], and renormalization flows.
- P2: To extend the BV quantization and pAQFT methods for quantizing nonassociative Einstein-Dirac-YM-Higgs systems studied in [47, 16].
- P3: To generalize and apply the BV and pAQFT methods for quantizing locally anisotropic (nonassociative) cosmological models with nonholonomic variables (A.7) and study quantum effects of off-diagonal solutions in MGTs with nonassociative, nonmetric, generalized Finsler-Lagrange-Hamilton configurations [33, 34, 35] modelled on phase spaces (i.e. on (co) tangent Lorentz bundles and their various possible star product deformations and respective models of deformation quantization).
- P4: To elaborate on nonassociative geometric and quantum information flow theories and BV quantization of generalized G. Perelman thermodynamic models (extending the results from [19, 13, 14, 15, 16, 47, 48] for AQFT and MGTs).

The author plans to report on progress P1-P4 in his (and co-authors') future works.

Acknowledgement: The author thanks for kind collaboration his co-authors (L. Bubuianu, J. O. Seti, D. Singleton, P. Stavrinos and E. V. Veliev) of partner works [12, 13, 14, 15] on classical nonassociative geometric flows, gravity and MGTs. The "nonassociative gravity and strings" research program were supported by Prof. Douglas Alexander Singleton, as a host of a Fulbright visit to the USA, and by Prof. Dieter Lüst, as a host of a scientist at risk fellowship at CAS LMU, Munich, Germany. This work is performed in the framework of author's volunteer research program in Ukraine; it is devoted to generalizations and applications of the BV formalism for nonassociative and noncommutative QG and classical matter fields, and QFT theories, elaborated in [28, 16, 47, 48].

A Off-diagonal quasi-stationary solutions on nonassociative phase spaces

In [12, 13, 15], the anholonomic frame and connection deformation method, AFCDM, was developed for decoupling and integration in general off-diagonal form various physically important systems of nonlinear systems of PDE in (nonassociative) geometric flow and MGTs. There were provided various examples and applications involving nonassociative black holes, BH, wormholes, WH, and locally anisotropic solitonic cosmological solutions were considered in [16]. Here, we show how applying the AFCDM we can generate quasi-stationary solutions for nonassociative YM equations (12). An example of regular phase space BH off-diagonal solutions is also provided.

A.1 Gravitational polarizations and nonlinear symmetries

The techniques of constructing quasi-stationary solutions for nonassociative modified Einstein equations (15) is outlined in Appendix B2, with constructions related to formula (B2), to [12]. Here we provide the formulas for a class of solution defined by nonlinear quadratic element in terms of η -polarization functions used in (18):

$$-\frac{[\ ^{|}\partial^{5}(\ ^{|}\eta^{6}\ ^{|}\mathring{g}^{6})]^{2}}{[\ ^{|}\int dp_{5}\ ^{|}_{3}\mathcal{J}^{\star}\ ^{|}\partial^{5}(\ ^{|}\eta^{6}\ ^{|}\mathring{g}^{6})]}{[\ ^{|}_{3}\mathcal{J}^{\star}\ ^{|}\partial^{5}(\ ^{|}\eta^{6}\ ^{|}\mathring{g}^{6})]}dx^{i_{2}}\}^{2}}$$
$$+(\ ^{|}\eta^{6}\ ^{|}\mathring{g}^{6})\{dp_{6}+[\ ^{|}_{1}n_{k_{2}}+\ ^{|}_{2}n_{k_{2}}\int dp_{5}\frac{[\ ^{|}\partial p_{5}\ ^{|}_{3}\mathcal{J}^{\star}\ ^{|}\partial^{5}(\ ^{|}\eta^{6}\ ^{|}\mathring{g}^{6})]^{2}}{[\ ^{|}\partial^{5}(\ ^{|}\eta^{6}\ ^{|}\mathring{g}^{6})]^{2}}dx^{k_{2}}\}$$

$$-\frac{[\ ^{|}\partial^{7}(\ ^{|}\eta^{8}\ ^{|}\mathring{g}^{8})]^{2}}{[\ \int dp_{7}\ ^{|}_{4}\mathcal{J}^{\star}\ ^{|}\partial^{8}(\ ^{|}\eta^{7}\ ^{|}\mathring{g}^{7})\ |\ (\ ^{|}\eta^{7}\ ^{|}\mathring{g}^{7})}\{dp_{7}+\frac{\ ^{|}\partial_{i_{3}}[\int dp_{7}\ ^{|}_{4}\mathcal{J}^{\star}\ ^{|}\partial^{7}(\ ^{|}\eta^{8}\ ^{|}\mathring{g}^{8})]}{[\ ^{|}_{4}\mathcal{J}^{\star}\ ^{|}\partial^{7}(\ ^{|}\eta^{8}\ ^{|}\mathring{g}^{8})}d\ ^{|}x^{i_{3}}\}^{2}$$
$$+(\ ^{|}\eta^{8}\ ^{|}\mathring{g}^{8})\{dE+[\ _{1}n_{k_{3}}+\ _{2}n_{k_{3}}\int dp_{7}\frac{[\ ^{|}\partial^{7}(\ ^{|}\eta^{8}\ ^{|}\mathring{g}^{8})]^{2}}{|\int dp_{7}\ ^{|}_{4}\mathcal{J}^{\star}[\ ^{|}\partial^{7}(\ ^{|}\eta^{8}\ ^{|}\mathring{g}^{8})]|\ [(\ ^{|}\eta^{8}\ ^{|}\mathring{g}^{8})]^{5/2}}]d\ ^{|}x^{k_{3}}\}.$$

The formulas (A.1) describe nonholonomic off-diagonal deformations of a prescribed prime metric into other families of target ones, $\dot{s} \mathbf{\hat{g}} = [\dot{g}_{\alpha_s}, \dot{N}_{i_{s-1}}^{a_s}] \rightarrow \dot{s} \mathbf{g}$ (17). Certain η -polarizations (involving a ψ as a solution of the Poisson equations, see bellow formula (A.6)), can be used as generating functions

$$\psi \simeq \psi(\hbar, \kappa, x^{k_1}), \eta_4 \simeq \eta_4(\hbar, \kappa, x^{k_1}, y^3), \ \ \eta^6 \simeq \ \ \eta^6(\hbar, \kappa, x^{i_2}, p_5), \ \ \eta^8 \simeq \ \ \eta^8(\hbar, \kappa, x^{i_2}, p_5, p_7).$$
(A.2)

Here we note that the generating functions can be prescribed in certain forms which allow to generate "small" (for instance, on parameter κ) off-diagonal deformations of some prime s-metrics into target ones. For such configurations, we have to choose (A.2) in such forms when ${}_{s}^{'}\eta {}_{g}^{'}\alpha_{s} \sim {}^{'}\zeta_{\alpha_{s}}(1+\kappa {}^{'}\chi_{\alpha_{s}}) {}^{'}g_{\alpha_{s}}$. Such solutions were studied in [12, 13, 15], for instance, with the aim to construct deformations of BH solutions into nonassociative black ellipsoid, BE, ones etc.

Off-diagonal solutions of type (A.1) posses important nonlinear symmetries which allow to change the generating functions and generating sources into another types of generating functions and effective cosmological constants on each shell, ${}^{'}_{s}\Lambda, [\eta_4, {}^{'}\eta^6, {}^{'}\eta^8, {}^{'}_{s}\mathcal{J}^*] \longleftrightarrow [{}_{s}\Psi, {}^{'}_{s}\mathcal{J}^*] \longleftrightarrow [{}_{s}\Phi^*, {}^{'}_{s}\Lambda]$. We put a star label on Φ to emphasize that such generating functions "absorb" nonlinearly the nonassociative data encoded in parametric forms in ${}^{'}_{s}\mathcal{J}^*$. By straightforward computations, we can check that such nonlinear transforms keep invariant the quasi-stationary configurations if different types of generating data are related by such differential, or integral, formulas:

$$\partial_{3}[({}_{2}\Psi)^{2}] = -\int dy^{3}({}_{2}\mathcal{J}^{\star})\partial_{3}g_{4} \simeq -\int dy^{3}({}_{2}\mathcal{J}^{\star})\partial_{3}({}^{\prime}\eta_{4} \, \mathring{g}_{4})$$

$$\simeq -\int dy^{3}({}_{2}\mathcal{J}^{\star})\partial_{3}[{}^{\prime}\zeta_{4}(1+\kappa \, \chi_{4}) \, \mathring{g}_{4}],$$

$$({}_{2}\Phi^{\star})^{2} = -4 \, {}_{2}\Lambda g_{4} \simeq -4 \, {}_{2}\Lambda \, {}^{\prime}\eta_{4} \, \mathring{g}_{4} \simeq -4 \, {}_{2}\Lambda \, {}^{\prime}\zeta_{4}(1+\kappa \, \chi_{4}) \, \mathring{g}_{4};$$
(A.3)

$$\begin{split} \left|\partial^{5}\left[\left(\begin{smallmatrix} {}^{+}_{3}\Psi\right)^{2}\right] &= -\int dp_{5}\left(\begin{smallmatrix} {}^{+}_{3}\mathcal{J}^{\star}\right)\left|\partial^{5}\right|g^{6}\simeq -\int dp_{5}\left(\begin{smallmatrix} {}^{+}_{3}\mathcal{J}^{\star}\right)\left|\partial^{5}\left(\begin{smallmatrix} {}^{+}\eta^{6}\right|\overset{\circ}{g}^{6}\right)\right.\\ &\simeq -\int dp_{5}\left(\begin{smallmatrix} {}^{+}_{3}\mathcal{J}^{\star}\right)\left|\partial^{5}\left[\begin{smallmatrix} {}^{+}\zeta^{6}(1+\kappa\right]\chi^{6}\right)\overset{\circ}{g}^{6}\right],\\ \left(\begin{smallmatrix} {}^{+}_{3}\Phi^{\star}\right)^{2} &= -4\begin{smallmatrix} {}^{+}_{3}\Lambda\right]g^{6}\simeq -4\begin{smallmatrix} {}^{+}_{3}\Lambda\right]\eta^{6}(\tau)\overset{\circ}{g}^{6}\simeq -4\begin{smallmatrix} {}^{+}_{3}\Lambda\right]\zeta^{6}(1+\kappa\right]\chi^{6}(1+\kappa)\chi^{6})\overset{\circ}{g}^{6}; \end{split}$$

$$\label{eq:product} \begin{split} ^{\scriptscriptstyle 1}\partial^7[(\begin{smallmatrix} {}^{\scriptscriptstyle 1}_4\Psi)^2] &= -\int dp_7(\begin{smallmatrix} {}^{\scriptscriptstyle 1}_4\mathcal{J}^\star) \, {}^{\scriptscriptstyle 1}\partial^7 \, {}^{\scriptscriptstyle 1}g^8 \simeq -\int dp_7(\begin{smallmatrix} {}^{\scriptscriptstyle 1}_4\mathcal{J}^\star) \, {}^{\scriptscriptstyle 1}\partial^7(\begin{smallmatrix} {}^{\scriptscriptstyle 1}\eta^8 \, {}^{\scriptscriptstyle 1}\mathring{g}^8) \\ &\simeq -\int dp_7(\begin{smallmatrix} {}^{\scriptscriptstyle 1}_4\mathcal{J}^\star(\tau)) \, {}^{\scriptscriptstyle 1}\partial^7[\begin{smallmatrix} {}^{\scriptscriptstyle 1}\zeta^8(1+\kappa \, {}^{\scriptscriptstyle 1}\chi^8) \, \mathring{g}^8], \\ (\begin{smallmatrix} {}^{\scriptscriptstyle 1}_4\Phi^\star)^2 &= -4\begin{smallmatrix} {}^{\scriptscriptstyle 1}_4\Lambda \, {}^{\scriptscriptstyle 1}g^8 \simeq \, -4\begin{smallmatrix} {}^{\scriptscriptstyle 1}_4\Lambda \, {}^{\scriptscriptstyle 1}\eta^8 \, {}^{\scriptscriptstyle 1}\mathring{g}^8 \simeq \, -4\begin{smallmatrix} {}^{\scriptscriptstyle 1}_4\Lambda \, {}^{\scriptscriptstyle 1}\zeta^8(1+\kappa \, {}^{\scriptscriptstyle 1}\chi^8) \, {}^{\scriptscriptstyle 1}\mathring{g}^8. \end{split}$$

The nonlinear symmetries (A.3) allows us to re-write the nonassociative equations $\widehat{\mathbf{R}}_{\gamma_s}^{\star\beta_s}(_{s}\Psi) = \delta_{\gamma_s}^{\beta_s} \mathcal{J}^{\star}(15)$ into a system of nonlinear functional parametric equations with effective cosmological constants:

$${}^{\mathsf{'}}\widehat{\mathbf{R}}^{\beta_{s}}_{\gamma_{s}}({}_{s}\Phi, {}^{\mathsf{'}}_{s}\mathcal{J}^{\star}) = \delta^{\beta_{s}}_{\gamma_{s}}{}^{\mathsf{'}}_{s}\Lambda.$$
(A.4)

For (A.4), the solution involve effective cosmological constants ${}_{s}^{\prime}\Lambda$ which are important for computing G. Perelman variables and elaborating (nonassociative) quantum geometric and information flow theories [13, 15]. Here we note that it is not possible completely to transform ${}_{s}^{\prime}\mathcal{J}^{\star} \rightarrow {}_{s}^{\prime}\Lambda$ because ${}_{s}^{\prime}\mathcal{J}^{\star}$ is always present in certain coefficients of the s-metric and N-connection, but in parametric form it is possible explicitly to find solutions for models with effective cosmological constants.

Quasi-stationary solutions (A.1) are characterized by such classes generating and integration functions related via nonlinear symmetries to generating sources and effective cosmological constants:

generating functions:

$$\begin{aligned} \psi \simeq \psi(\hbar, \kappa, x^{k_1}); \ _2\Psi \simeq \ _2\Psi(\hbar, \kappa, x^{k_1}, y^3); \quad (A.5) \\ & _{3}\Psi \simeq \ _{3}\Psi(\hbar, \kappa, x^{k_2}, p_5); \ _{4}\Psi \simeq \ _{4}\Psi(\hbar, \kappa, \ _{x}^{k_3}, p_7); \\ & _{1}\mathcal{J}^* \simeq \ _{1}\mathcal{J}^*(\hbar, \kappa, x^{k_1}); \ _{2}\mathcal{J}^* \simeq \ _{2}\mathcal{J}^*(\hbar, \kappa, x^{k_1}, y^3); \\ & _{3}\mathcal{J}^* \simeq \ _{3}\mathcal{J}^*(\hbar, \kappa, x^{k_2}, p_5); \ _{4}\mathcal{J}^* \simeq \ _{4}\mathcal{J}^*(\hbar, \kappa, \ _{x}^{k_3}, p_7); \\ & \text{integrating functions:} \ g_{4}^{[0]} \simeq \ g_{4}^{[0]}(\hbar, \kappa, x^{k_1}), \ _{1}n_{k_1} \simeq \ _{1}n_{k_1}(\hbar, \kappa, x^{j_1}), \ _{2}n_{k_2} \simeq \ _{2}n_{k_1}(\hbar, \kappa, x^{j_2}); \\ & -g_{6}^{[0]} \simeq \ -g_{6}^{[0]}(\hbar, \kappa, x^{k_2}), \ _{1}n_{k_2} \simeq \ _{1}n_{k_3}(\hbar, \kappa, \ _{x}^{j_3}), \ _{2}n_{k_3} \simeq \ _{2}n_{k_3}(\hbar, \kappa, \ _{x}^{j_3}). \end{aligned}$$

The generating functions $\psi(\hbar, \kappa, x^{k_1})$ are solutions of the 2-d Poisson equations,

$$\partial_{11}^2 \psi + \partial_{22}^2 \psi = 2 \, _1 \mathcal{J}^*(\hbar, \kappa, x^{k_1}). \tag{A.6}$$

For certain subclasses of generating and integration data (A.5) and (A.6), we can generate BH or WH solutions with polarization of physical constants, deformation of horizons (if they exist) and embedding in nonassociative gravitational vacuum, or subjected to nonassociative off-diagonal interactions with effective matter fields.

The quasi-stationary solutions (A.1) can be transformed into locally anisotropic cosmological solutions encoding in off-diagonal form nonassociative data. This is possible if we perform corresponding re-definitions of nonholonomic variables (and respective coordinate dependencies) on shells s = 1 and 2:

$$\begin{array}{rcl} \partial_4 &=& \partial_t \to \partial_3 = \partial/\partial y^3, \text{ Killing symmetry } ; x^3 = y^3 \to x^4 = y^4 = t, \text{ space into time coordinates;} \\ g_3(x^i, y^3) &\to& \underline{g}_4(x^i, t) \text{ and } g_4(x^i, y^3) \to \underline{g}_3(x^i, y); \\ N_k^3(x^i, y^3) &=& w_k(x^i, y^3) \to \underline{N}_k^4(x^i, t) = \underline{n}_k(x^i, t) \text{ and } N_k^4(x^i, y^3) = n_k(x^i, y^3) \to \underline{N}_k^3(x^i, t) = \underline{w}_k(x^i, t). \end{array}$$

In above formulas, we underlined the symbols with explicit dependence on t-coordinate. Such re-definitions of nonholonomic variables introduce time like dependencies into above formulas (A.2) - (A.5). Such nonholonomic parameterizations in nonassociative gauge gravity were used for deriving parametric solutions with cosmological solitonic hierarchies [16]. We can consider co-fiber redefinition of nonholonomic variables like (A.7), to transforms off-diagonal solutions with $p_8 = E_0$ into "rainbow" s-metrics with variable E, but, for instance, $p_7 = const$. In [12, 13, 15], there are studied examples of phase space parametric solutions encoding nonassociative data.

In this work, we restrict our approach only for quasi-stationary parametric configurations (A.1). Such results can be re-defined geometrically in certain dual forms (on time and energy type coordinates) which positively result in another type of locally anisotropic cosmological and nonassociative gravitationals models.

A.2 Nonassociative star product deformations of regular Dymnikova black holes

In GR, a very interesting BH regular solution was constructed by I. Dymnikova [30]. For recent higher dimension constructions, we cite [31] and references therein. In abstract geometric form, those constructions can be redefined in (nonassociative) nonholonomic forms on 8-d phase spaces ${}_{s}\mathcal{M}$. Similar details on applications of the AFCDM we provided in [14] for quasi-stationary solutions describing nonassociative star product deformations of a a d = 5 dimensional analog of the Reisner-Nordström AdS, RN. In this section, we consider a different type of primary metric using phase space coordinates on a 7-d phase space with signature (+ + + - + + +)trivially extended to a 8-d one with a diagonal quadratic element of the prime metric, we considered

$$d \,\breve{s}_{[7+1]}^2 = \,\, {}^{}\,\breve{g}_{\alpha_s}(\,\,{}^{}\,u^{\gamma_s})(\breve{\mathbf{e}}^{\alpha_s})^2 = \frac{d\breve{r}^2}{\breve{f}(\breve{r})} - \breve{f}(\breve{r})dt^2 + \breve{r}^2[(d^2\hat{x}^2)^2 + (d\hat{x}^3)^2 + (dp_5)^5 + (dp_6)^2 + (dp_7)^2] - dE^2.$$
(A.8)

In this formula, the spherical coordinates are for $\hat{x}^1 = \check{r} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (p_5)^2 + (p_6)^2 + (p_7)^2}$, when $\hat{x}^2 = \hat{x}^2(x^2, x^3, p_5, p_6, p_7)$, $\hat{x}^3 = \hat{x}^3(x^2, x^3, p_5, p_6, p_7)$, ... $\hat{x}^7 = \hat{x}^7(x^2, x^3, p_5, p_6, p_7)$ are chosen as coordinates for a diagonal metric on an effective 7-d Einstein phase space $V_{[7]}$. For details and physical motivations, we cite section II of [31] (with that difference that we work with higher dimension coordinates considered as momentum ones; when the dimension D = 6, i.e. d = 6 following our conventions). The 7-d phase space generalization of the Dymnikova BH is given by

$$\breve{f}(\breve{r}) = 1 - (\frac{\breve{r}_g}{\breve{r}})^4 \{ 1 - \exp[(\frac{\breve{r}}{\breve{r}_*})^6] \}, \text{ for constants } \breve{r}_*^6 = \breve{r}_0^2 \breve{r}_g^4, \text{ where } \breve{r}_0^2 = \frac{15}{\rho_0^2};$$
(A.9)

and when the nontrivial components of the energy-momentum tensors are

$$T_1^1 = T_4^4 = -\rho_0 \exp[-(\breve{r}/\breve{r}_*)^6] \text{ and } T_2^2 = T_3^3 = T_5^5 = T_6^6 = T_7^7 = [\frac{6}{5}(\frac{\breve{r}}{\breve{r}_*})^6 - 1] \exp[-(\breve{r}/\breve{r}_*)^6];$$
(A.10)

being used spherical coordinates of unit 5-d sphere, with constant energy density ρ_0 defining the vacuum energy of a phase space.

The diagonal prime metric (A.8) define a solution of the Einstein equations (15) on a commutative and associative 7-d phase space which yields a de Sitter solution for $\breve{r} \ll \breve{r}_*$ and a higher dimension Schwarzschild solution for $\breve{r} \gg \breve{r}_*$. Such regular BH solutions can be considered also for the gauge gravity equations (6) if the energy-momentum tensor (A.10) is used for defining the source (7).

The apply the AFCDM and construct nonassociative solutions we consider certain (s-adapted coordinate transforms ${}^{\prime}u^{\gamma_s} = {}^{\prime}u^{\gamma_s}({}^{\prime}\hat{u}^{\gamma_s})$ of (A.8) into certain data ${}^{\prime}_{s}\breve{g} = \{{}^{\prime}\breve{g}_{\alpha_s}\} = {}^{\prime}_{s}\breve{g} = [{}^{\prime}\mathring{g}_{\alpha_s}, {}^{\prime}\mathring{N}_{i_{s-1}}^{a_s}]$ as in (A.1), when nontrivial values ${}^{\prime}\mathring{g}_{\alpha_s}$ and ${}^{\prime}\mathring{N}_{i_{s-1}}^{a_s}$ allow to generate nonsingular off-diagonal solutions. For general star product deformations, it is not clear what physical interpretation could be provided for such nonassociative modifications of Dymnikova phase space BH solutions of (12) and (15). In principle, we can assume that certain stability can be achieved by corresponding nonholonomic constraints on η -polarizations as we considered in section 5.3 of [14]. Then, considering small parametric distortions of type ${}^{\prime}_{s}\eta{}^{\prime}\mathring{g}_{\alpha_s} \sim {}^{\prime}\zeta_{\alpha_s}(1 + \kappa{}^{\prime}\chi_{\alpha_s}){}^{\prime}\mathring{g}_{\alpha_s}$, we can model additional locally anisotropic polarization of the vacuum energy ρ_0 and respective horizons; and various effective source parameters encoding nonassociative data. For some nonholonomic configurations, we can model, for instance, ellipsoidal-type deformations of horizons and keep a standard interpretation of phase space space Dymnikova background, which are κ -deformed.

$$d_{\downarrow}^{\chi} s_{[7\subset 8d]}^{2} = e^{\psi_{0}} (1+\kappa^{\psi_{\downarrow}}\chi) [\breve{g}_{1}(\breve{r})d\breve{r}^{2}+\breve{g}_{2}(\breve{r})(d\mathring{x}^{2})] - \{\frac{4[\hat{\partial}_{3}(|\zeta_{4}\breve{g}_{4}(\breve{r})|^{1/2})]^{2}}{\breve{g}_{4}(\breve{r})|\int d\mathring{x}^{3}\{2\mathcal{J}^{\star}\hat{\partial}_{3}(\zeta_{4}\breve{g}_{4}(\breve{r}))\}|}$$

$$-\kappa [\frac{\hat{\partial}_{3}(\chi_{4}|\zeta_{4}\breve{g}_{4}(\breve{r})|^{1/2})}{4\hat{\partial}_{3}(|\zeta_{4}\breve{g}_{4}(\breve{r})|^{1/2})} - \frac{\int d\mathring{x}^{3}\{2\mathcal{J}^{\star}\hat{\partial}_{3}[(\zeta_{4}\breve{g}_{4}(\breve{r}))\chi_{4}]\}}{\int d\mathring{x}^{3}\{2\mathcal{J}^{\star}\hat{\partial}_{3}(\zeta_{4})\breve{g}_{4}(\breve{r}))\}}]\}\breve{g}_{3}(\mathbf{e}^{3})^{2} + \zeta_{4}(1+\kappa\chi_{4})\breve{g}_{4}(\breve{r})dt^{2}$$

$$(A.11)$$

$$-\{\frac{4[\hat{\partial}_{5}(|\ \zeta^{6}\ \breve{g}^{6}|^{1/2})]^{2}}{[\breve{g}_{5}(\breve{r})|\int d\hat{x}^{5}\{\ {}_{3}\ \mathcal{J}^{\star}\partial^{7}(\ (\zeta^{6}(\tau)\ \breve{g}^{6})\}|} -\kappa[\frac{\hat{\partial}_{5}(\ (\chi^{6}|\ \zeta^{6}\ \breve{g}^{6}|^{1/2})}{4\hat{\partial}_{5}(|\ \zeta^{6}\ \breve{g}^{6}|^{1/2})} -\frac{\int d\hat{x}^{5}\{\ {}_{3}\ \mathcal{J}^{\star}\ \hat{\partial}_{5}[(\ (\zeta^{6}\ \breve{g}^{6})\ \chi^{8}]\}}{\int d\hat{x}^{5}\{\ {}_{3}\ \mathcal{J}^{\star}\ \hat{\partial}_{5}[(\ (\zeta^{6}\ \breve{g}^{6})]\}}]\} \ \breve{g}_{5}(\breve{r})(\mathbf{e}^{5})^{2} + \zeta^{6}\ (1+\kappa\ \chi^{6})(dp_{6})^{2} + (dp_{7})^{2} - dE^{2},$$

where

$$\mathbf{e}^{3} = d\hat{x}^{3} + \left[\frac{\partial_{i_{1}}\int d\hat{x}^{3} \ _{2}\mathcal{J}^{\star} \ \partial_{3}\zeta_{4}}{\breve{N}_{i_{1}}^{3} \ _{2}\mathcal{J}^{\star} \partial_{3}\zeta_{4}} + \kappa \left(\frac{\partial_{i_{1}}\left[\int d\hat{x}^{3} \ _{2}\mathcal{J}^{\star} \partial_{3}(\zeta_{4}\chi_{4})\right]}{\partial_{3}\zeta_{4}}\right] - \frac{\partial_{3}(\zeta_{4}\chi_{4})}{\partial_{3}\zeta_{4}}\right] - \frac{\partial_{3}(\zeta_{4}\chi_{4})}{\partial_{3}\zeta_{4}}\right] \breve{N}_{i_{1}}^{3} dx^{i_{1}},$$

$$\mathbf{P}^{1} \mathbf{e}^{5} = d\hat{x}^{5} + \left[\frac{\partial_{i_{2}}}{\breve{N}_{i_{2}}}\int d\hat{x}^{5} \ _{3}\mathcal{J}^{\star} \ \partial_{5}(\ \zeta^{6})}{\dot{\delta}_{5}(\ \zeta^{6})} + \kappa \left(\frac{\partial_{i_{2}}\left[\int d\hat{x}^{5} \ _{3}\mathcal{J}^{\star} \ \partial_{5}(\ \zeta^{6})\right]}{\partial_{5}(\ \zeta^{6})}\right] - \frac{\partial_{5}(\ \zeta^{6} \ \breve{g}^{6})}{\partial_{5}(\ \zeta^{6})}\right] - \frac{\partial_{5}(\ \zeta^{6} \ \breve{g}^{6})}{\partial_{5}(\ \zeta^{6})}\right] - \breve{N}_{i_{2}}^{5} d^{4}x^{i_{2}}.$$

$$\mathbf{P}^{1} \mathbf{e}^{7} = d\hat{x}^{7} + \left[\frac{\partial_{i_{3}}}{d\hat{x}}\int d\hat{x}^{7} \ _{4}\mathcal{J}^{\star} \ \partial_{7}(\ \zeta^{8})}{\partial_{7}(\ \zeta^{8})} + \kappa \left(\frac{\partial_{i_{3}}\left[\int d\hat{x}^{7} \ _{4}\mathcal{J}^{\star} \ \partial_{7}(\ \zeta^{8} \ \breve{g}^{8})\right]}{\partial_{6}i_{3}}\left[\int d\hat{x}^{7} \ _{4}\mathcal{J}^{\star} \ \partial_{7}(\ \zeta^{8})\right]} - \frac{\partial_{7}(\ \zeta^{8} \ \breve{g}^{8})}{\partial_{7}(\ \zeta^{8})}\right] - \breve{N}_{i_{3}}^{7} d^{4}x^{i_{3}}.$$

Such solutions are similar to those defined by formulas (95) in [14]. Nevertheless, in this case they may not involve BH or black ellipsoid singularities (because in (A.11) we use regular Dymnikova type configurations). The nonassociative generating sources ${}_{s}^{*}\mathcal{J}^{*}$ are different in such cases being defined by star product deformations of certain primary energy-momentum components (A.10).

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